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# CALDERÓN-ZYGMUND ESTIMATES FOR PARABOLIC MEASURE DATA EQUATIONS

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ABSTRACT. We consider parabolic equations of the type $u_t - \operatorname{div} a(x,t,Du) = \mu$ having a Radon measure on the right–hand side and prove fractional integrability and differentiability results of Calderón-Zygmund type for weak solutions. We extend some of the integrability results for elliptic equations achieved by G. Mingione [Ann. SNS, 2007] to the parabolic setting and locally recover the integrability results of L. Boccardo, A. Dall'Aglio, T. Gallouët and L. Orsina [JFA, 1997].

## 1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we study inhomogeneous parabolic equations with a right–hand side being merely a Radon measure. Our aim is to establish quantified higher integrability properties of Calderón-Zygmund type for the spacial gradient of the weak solution to such problems. More precisely, we consider equations of the form

$$\begin{cases} \partial_t u - \operatorname{div} a(x,t,Du) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{1.1}$$

being $\mu$ a signed Radon measure with finite total mass, $|\mu|(\Omega_T) < \infty$. We denote with $\Omega_T$ the parabolic cylinder $\Omega \times (-T, 0)$, being $\Omega \subset \mathbb{R}^n$, $n \geq 2$ a bounded open set and $T > 0$, while $\partial_{\mathcal{P}}\Omega_T$ is its parabolic boundary. Furthermore $a : \Omega \times (-T, 0) \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field, fulfilling the following classical **monotonicity and continuity** conditions:

$$\begin{cases} \big\langle a(x,t,\xi_1) - a(x,t,\xi_2),\, \xi_1 - \xi_2 \big\rangle \geq \nu\, |\xi_1 - \xi_2|^2, \\ \big|a(x,t,\xi_1) - a(x,t,\xi_2)\big| \leq L|\xi_1 - \xi_2|, \\ |a(x,t,0)| \leq Ls, \\ \big|a(x_1,t,\xi) - a(x_2,t,\xi)\big| \leq L\,|x_1 - x_2|(s + |\xi|) \end{cases} \tag{1.2}$$

for all $(x,t) \in \Omega_T, x_1, x_2 \in \Omega, t \in (-T, 0)$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$, with constants $0 < \nu \leq L < \infty$, $s \geq 0$. In the case where the inhomogeneity $\mu$ belongs to the dual space $L^2(-T, 0; W^{-1,2}(\Omega))$, classical existence theory (see for example [22]) applies and provides a unique solution of (1.1) in the Sobolev space $L^2(-T, 0; W_0^{1,2}(\Omega))$. However, as in our setting $\mu$ is merely a Radon measure, or $\mu \in L^1(\Omega_T)$, the existence of a weak solution in the sense mentioned above in general fails; in this case one is lead to a different notion of "weak solution". For our setting, we adapt the following definition:

**Definition 1.1.** *A* **very weak solution** *to* $(1.1)_1$ *is a function* $u \in L^1(-T, 0; W^{1,1}(\Omega))$ *such that* $a(x,t,Du) \in L^1(\Omega_T; \mathbb{R}^n)$ *and*

$$\int_{\Omega_T} \Big[-u\,\varphi_t + \langle a(x,t,Du), D\varphi\rangle\Big]\,dz = \int_{\Omega_T} \varphi\,d\mu, \tag{1.3}$$

*for every* $\varphi \in \mathcal{C}^\infty(\Omega_T)$ *which is equal to zero in a neighborhood of* $\partial_{\mathcal{P}}\Omega_T$.

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The basic references for the existence of such solutions for the general nonlinear parabolic case are the works of Boccardo, Gallouët, Dall'Aglio & Orsina [5] and Boccardo & Gallouët [6], while [7] provides an analogue result in the elliptic shape. The approach to show existence of weak solutions followed by the authors in [5, 6, 7] consists in setting up an appropriate approximation scheme. I.e., one considers regular right–hand sides $f_k$ which converge in the weak sense of measures to $\mu$, and the weak solutions $u_k$ to the regularized problems (1.1) with $\mu$ replaced by $f_k$. Exploiting then the classical theory of parabolic equations with regular data (see for a complete overview [22, 27]) allows to establish a priori estimates for the solutions $u_k$, being stable when passing to the limit $k \to \infty$. Roughly speaking, this stability in the limit is guaranteed by showing that the a priori estimates merely involve $\|f_k\|_{L^1}$. Solutions obtained in such a way are called SOLAs (Solutions Obtained by Limits of Approximations). Using this approach, the authors in [6] prove the existence of at least one solution to (1.1) belonging to $L^q(-T, 0; W_0^{1,q}(\Omega))$ for every exponent $q$ satisfying

$$1 \le q < 2 - \frac{n}{n+1}, \tag{1.4}$$

while in [5] the result is refined in the following anisotropic sense: the solution is shown to belong to $L^r(-T, 0; W_0^{1,q}(\Omega))$, where the couple of exponents $(r, q)$ satisfies the following bounds:

$$\begin{cases} 1 \le q < \dfrac{n}{n-1}, \\ 1 \le r < 2, \\ \dfrac{2}{r} + \dfrac{n}{q} > n+1. \end{cases} \tag{1.5}$$

In this paper, we provide higher regularity results for the spatial gradient $Du$ of solutions to parabolic equations of the type (1.1), which are the natural "parabolic" extensions of the ones proved by Mingione in [24] in the elliptic setting, giving an explicit estimate of its fractional Sobolev norm. Although the basic idea in the parabolic setting is the same as in the elliptic one [24], a number of additional difficulties had to be overcome. A refined iteration scheme, involving finite difference operators in space as well as in time finally allows for fractional estimates of the spatial gradient $Du$ in space and time. Fundamental tools in improving, step by step, fractional regularity of the solution are the fractional Poincaré inequality Lemma 4.6 and classical regularity results for homogeneous problems, established in chapter 7, which lead by suitable comparison techniques to appropriate estimates in parabolic Nikolski spaces. Those, in turn, can be carried over to fractional Sobolev spaces by standard isomorphisms.

By now, fractional Sobolev spaces are an essential tool in providing precise estimates on the differentiability of solutions of elliptic and parabolic problems, in the sense that they provide a natural intermediate scale to state optimal regularity results. Moreover they provide a natural tool leading also to the Hausdorff dimension of the singular set, see [25, 13]. We refer the reader again to [23, 24] for interesting discussions about regularity and optimality in fractional order spaces for the elliptic case.

Coming back to the parabolic setting which is studied here, the main goal of this paper is to show the following theorem:

**Theorem 1.2** (Fractional regularity)**.** *Under the assumptions* (1.2) *on the vector field* $a$ *there exists a solution* $u \in L^1(-T, 0; W_0^{1,1}(\Omega))$ *of the equation* (1.1) *such that*

$$Du \in W^{\delta(q)-\varepsilon, \frac{\delta(q)-\varepsilon}{2}; q}_{\mathrm{loc}}(\Omega_T; \mathbb{R}^n) \tag{1.6}$$

*for all* $\varepsilon \in (0, \delta)$*, where*

$$1 \le q < 2 - \frac{n}{n+1} \qquad \textit{and} \qquad \delta \equiv \delta(q) := \frac{n+2}{q} - (n+1). \tag{1.7}$$

**Remark 1.3** (on the exponents)**.** *The above statement includes in particular that*

$$Du \in W^{1-\varepsilon,\frac{1-\varepsilon}{2};1}_{\mathrm{loc}}(\Omega_T;\mathbb{R}^n)$$

*for all* $\varepsilon \in (0,1)$*, which means that the solution* $u$ *has "almost" second derivatives in space and its spatial gradient* $Du$ *has "almost half a derivative" in time.*

Let us stress for a moment the analogies to the elliptic case [24]: assuming analogue hypotheses (1.2) on the continuity and monotonicity of the vector field, in the elliptic setting $u$ is "almost" twice differentiable, and more generally

$$Du \in W^{\tilde{\delta}(q)-\varepsilon,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^n), \qquad \text{where } \tilde{\delta}(q) := \frac{n}{q} - (n-1),$$

for $\varepsilon \in (0,\tilde{\delta})$, which is the analogue to (1.6), keeping in mind that, due to the structure of the parabolic metric,

$$d_{\mathcal{P}}(z_1,z_2) := \max\Big\{|x_1-x_2|, \sqrt{|t_1-t_2|}\Big\}. \tag{1.8}$$

for all $z_1,z_2 \in \Omega_T$, the "dimension" of the parabolic cylinders is $n+2$.

Having in mind (1.4), in [5] is shown the existence of a solution to (1.1) in the space $L^r(-T,0;W^{1,q}_0(\Omega))$ under the restrictions (1.5) on $(r,q)$. As a corollary of our result Theorem 1.2 we can recover this result, at least locally, and moreover we can show also some kind of "dual" integrability result.

**Corollary 1.4** (Local recovery of the result of [5])**.** *There exists a solution* $u$ *to problem* (1.1) *such that*

$$Du \in L^r_{\mathrm{loc}}(-T,0;L^q_{\mathrm{loc}}(\Omega)) \cap L^q_{\mathrm{loc}}(\Omega;L^r_{\mathrm{loc}}(-T,0))$$

*for all* $(r,q)$ *satisfying* (1.5)*.*

Moreover, we deduce the following local estimates of Calderón-Zygmund type:

**Theorem 1.5** (Local Calderón-Zygmund estimates)**.** *Under the assumptions of Theorem 1.2, let* $q$ *and* $\delta$ *be as in* (1.7)*, let* $\sigma(q) := \delta(q)q$ *and* $\sigma \in (0,\sigma(q))$*. Then there exists a constant* $c \equiv c(n,\nu,L,q,\sigma(q)-\sigma)$ *such that for every cylinder* $Q_\varrho \equiv B_\varrho \times I_\varrho \Subset \Omega_T$ *of radius* $\varrho > 0$ *it holds*

$$\begin{aligned}\int_{I_{\varrho/2}}\int_{B_{\varrho/2}}\int_{B_{\varrho/2}} &\frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\sigma}}\,dx\,dy\,dt \\ &+ \int_{B_{\varrho/2}}\int_{I_{\varrho/2}}\int_{I_{\varrho/2}} \frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\sigma/2}}\,dt\,ds\,dx \\ &\qquad \le c\,\varrho^{-\sigma}\int_{Q_\varrho}(s+|Du|)^q\,dz + c\,\varrho^{\sigma(q)-\sigma}|\mu|(\overline{Q_\varrho})^q.\end{aligned} \tag{1.9}$$

*Furthermore, for any open subset* $\Omega_T{}' \equiv \Omega' \times J' \Subset \Omega_T$ *the estimate*

$$\begin{aligned}\int_{\Omega_T{}'}|Du|^q\,dz &+ \int_{J'}\int_{\Omega'}\int_{\Omega'} \frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\sigma}}\,dx\,dy\,dt \\ &+ \int_{\Omega'}\int_{J'}\int_{J'} \frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\sigma/2}}\,dt\,ds\,dx \le c\left[s^q + |\mu|(\Omega_T)^q\right]\end{aligned} \tag{1.10}$$

*holds true with a constant* $c$ *depending on* $n$*,* $L/\nu$*,* $q$*,* $\mathrm{dist}(\Omega_T{}',\partial_{\mathcal{P}}\Omega_T)$*,* $|\Omega|$ *and* $T$*.*

Finally using standard immersion theorems between fractional Sobolev spaces we can deduce the following anisotropic regularity result

**Theorem 1.6.** *Let* $u \in L^1(-T,0;W^{1,1}_0(\Omega))$ *be a weak solution of the problem* (1.1)*. Then we have:*

(i) *for all* $(r,q)$ *satisfying* (1.5) *and the condition* $r<q$ *we have*

$$Du \in L^r_{\rm loc}(-T,0;W^{\delta,q}_{\rm loc}(\Omega))\cap W^{\delta,q}_{\rm loc}(\Omega;L^r_{\rm loc}(-T,0)) \qquad \textit{for all } \delta\in[0,\tilde\delta(r,q));$$

(ii) *for all* $(r,q)$ *satisfying* (1.5) *and the condition* $r>q$ *on the other hand*

$$Du \in L^q_{\rm loc}(\Omega;W^{\delta/2,r}_{\rm loc}(-T,0))\cap W^{\delta/2,r}_{\rm loc}(-T,0;L^q_{\rm loc}(\Omega)) \qquad \textit{for all } \delta\in[0,\tilde\delta(r,q)).$$

*In both cases* $\tilde\delta$ *denotes the function*

$$\tilde\delta(r,q):=\frac{n}{q}+\frac{2}{r}-(n+1)>0 \qquad \textit{for } (r,q) \textit{ satisfying } (1.5).$$

## 2. Notation

In the following we introduce some notation which will be used in the whole paper. In the sequel, the letter $c$ will denote a constant, larger or equal than one which will not necessarily be the same at different occurrences throughout the paper. In particular it may also change from line to line. For reasons of readability, dependencies of the constants will often be omitted within the chains of estimates, therefore stated after the estimate. We denote

$$B_\varrho(x_0)\equiv B(x_0,\varrho):=\{x\in\mathbb{R}^n:\,|x-x_0|<\varrho\}$$

the open ball in $\mathbb{R}^n$ with center $x_0\in\mathbb{R}^n$ and radius $\varrho>0$. If clear by the context, we will often leave out the center of the ball, just writing $B_\varrho$. Moreover we denote

$$I_\varrho(t_0):=(t_0-\varrho^2,t_0+\varrho^2),$$

again possibly dropping the dependence on $t_0$. Consequently we will denote the parabolic cylinder

$$Q_\varrho(z_0)\equiv Q(z_0,\varrho):=B_\varrho(x_0)\times I_\varrho(t_0),$$

with "center" at $z_0=(x_0,t_0)\in\mathbb{R}^{n+1}$ and radius $\varrho>0$. Furthermore we will denote by $B_1\equiv B_1(0)$ the unit ball in $\mathbb{R}^n$; analogously, $I_1\equiv I_1(0)$ and $Q_1\equiv Q_1(0)=B_1\times I_1$. Accordingly with the parabolic metric (1.8), for $\alpha>0$ we shall write $\alpha I_\varrho(t_0)=I_{\alpha\varrho}(t_0):=(t_0-\alpha^2\varrho^2,t_0+\alpha^2\varrho^2)$.

$\mathbb{R}^{n+1}$ will always be thought as $\mathbb{R}^n\times\mathbb{R}$, so a point $z\in\mathbb{R}^{n+1}$ will be often denoted as $(x,t)$, $z_0$ as $(x_0,t_0)$, and so on. Analogously our subsets $C\subset\Omega_T$ will always be a product of a spacial subset and a temporal one: $C=A\times J$, with $A\subset\Omega$ and $J=(t_1,t_2)\subset(-T,0)$. Hence by **parabolic boundary** of $C$ we will mean

$$\partial_{\mathcal{P}}C:=A\times\{t_1\}\cup\partial A\times J.$$

Moreover writing $C\Subset\Omega_T$ we will mean that $A\Subset\Omega$, $J\Subset(-T,0)$, eventually keeping implied the spacial and temporal sections.

Being $\widetilde{C}\in\mathbb{R}^m$ a measurable set with positive measure and $f:\widetilde{C}\to\mathbb{R}^k$ with $k\geq 1$ a measurable map, we denote with $(f)_{\widetilde{C}}$ the averaged integral

$$(f)_{\widetilde{C}}:=\fint_{\widetilde{C}} f(x)\,dx:=\frac{1}{|\widetilde{C}|}\int_{\widetilde{C}} f(x)\,dx.$$

In particular, when $\widetilde{C}=Q_\varrho(z_0)$

$$(f)_{Q_\varrho(z_0)}=:(f)_{\varrho,z_0}=\frac{1}{2|B_1|\varrho^{n+2}}\int_{Q_\varrho(z_0)} f(x)\,dx.$$

Concerning time derivatives, we will use different notations throughout of the paper. Most frequently we take use of $\partial_t u$ to express $\frac{\partial}{\partial t}u$, however in order to shorten the notation we alto write $u_t$ at several stages of the paper. All of these expressions have the same meaning. For the spatial gradient of $u$ we will always use the notation $Du$.

In the rest of the paper we shall always keep in mind the bound on $q$ defined in (1.4). Consequently, for such $q$, we will denote by $\sigma(q)$ the quantity

$$\sigma(q) := n + 2 - q(n + 1),$$

and by $\delta \equiv \delta(q)$ the quantity

$$\delta(q) := \frac{\sigma(q)}{q} = \frac{n+2}{q} - (n + 1). \tag{2.1}$$

Let's remark that $\sigma(q) > 0$ for all the numbers $q$ satisfying (1.4). Let's also stress that in that case we also have $\sigma(q) \leq q$, so that $\delta \leq 1$.

## 3. Preliminaries

Starting with a weak solution of the problem (1.1), with $a$ satisfying hypotheses (1.2), according to the Definition 1.1, we have to specify the meaning of $u = 0$ on $\partial_{\mathcal{P}} Q_T$. The fact that $u$ vanishes on the lateral boundary is prescribed by denoting $u(\cdot, t) \in W_0^{1,1}(\Omega)$ for a.e. $t$. However the initial boundary value $u(x, -T) = 0$ should be understood in the $L^1$ sense, which means that

$$\lim_{h \searrow 0} \frac{1}{h} \int_{-T}^{-T+h} \int_{\Omega} |u(x,t)| \, dx \, dt = 0.$$

In this paper, we will frequently use the following "slicewise" reformulation of (1.3): For $h > 0$ and $t \in (-T, 0)$ we define the so-called **Steklov average** of $u$ by

$$u_h(x,t) := \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} u(x, \tilde{t}) \, d\tilde{t} & \text{if } t \leq -h, \\ 0 & \text{if } t > -h. \end{cases} \tag{3.1}$$

This definition naturally extends to the case when $h$ is negative, averaging backward instead of forward. Being $u$ a weak solution of (1.1) with $\mu \in L^1(\Omega_T)$ and $u_h$ the Steklov average of $u$, the slicewise equality

$$\int_{\Omega} \Big[ \partial_t u_h \varphi + \langle [a(\cdot, t, Du)]_h, D\varphi \rangle \Big] \, dx = \int_{\Omega} \varphi \mu_h \, dx,$$

holds true for any $\varphi \in \mathcal{C}_c^{\infty}(\Omega)$ and for a.e. $t \in (-T, 0)$ (see [11, Chapter 2]).

Let us now specify what is the SOLA approach we will use in this paper to treat solutions to (1.1): we consider the regular problem

$$\begin{cases} \partial_t u - \operatorname{div} \, a(x,t,Du) = f & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}} \Omega_T, \end{cases} \tag{3.2}$$

with $f \in L^2(\Omega_T)$ and its unique solution $u \in L^2(-T, 0; W_0^{1,2}(\Omega)) \cap \mathcal{C}^0([-T, 0]; L^2(\Omega))$; such a solution exists via monotonicity methods, see for instance [22].

Then we consider a sequence of functions $\{f_k\}$ in $\mathcal{C}^{\infty}(\Omega_T)$ which converges weakly in the sense of the measures to $\mu$, eventually defined on the whole $\mathbb{R}^{n+1}$ in the trivial way $|\mu|(\mathbb{R}^{n+1} \setminus \Omega_T) := 0$, with the property that

$$\|f_k\|_{L^1(\Omega_T)} \leq |\mu|(\Omega_T) \qquad \text{and} \qquad \|f_k\|_{L^1(Q_\varrho)} \leq |\mu|(Q_{\varrho + 1/k}). \tag{3.3}$$

We shall denote by $u_k$ the solution to (3.2) with $f \equiv f_k$ and we deduce the regularity theorems first for the solutions $u_k$; finally, we obtain the regularity result for the solution $u$ of the original problem with measure data exploiting the fact that the properties are stable when passing to the limit. We finally stress that we shall only care about the regularity of a special kind of solution, namely a SOLA solution; in fact the distributional formulation (1.3) is not the unique notion of solution of (1.1) which could be approached; however, since our aim is to deduce *a priori regularity estimates*, we will confine ourselves to solutions defined as in (1.3), and moreover we will not discuss uniqueness problems at all (see [10]).

We finish this chapter with a fundamental technical Lemma: the following reverse Hölder type inequality allows to reduce the integral power on the right–hand side below the natural exponent.

**Lemma 3.1.** *Let $g : \Omega_T \to \mathbb{R}^n$ an integrable map such that*

$$\left[\fint_{Q_\varrho} |g|^{\chi_0}\, dz\right]^{1/\chi_0} \le c\left[\fint_{Q_{2\varrho}} \left(s+|g|\right)^2 dz\right]^{1/2}$$

*holds whenever $Q_{2\varrho} \Subset \Omega_T$, where $s \ge 0$, $\chi_0 > 2$ and $c > 0$. Then, for every $\sigma \in (0, 2]$, there exists a constant $c_0 = c_0(n, \sigma, c)$ such that*

$$\left[\fint_{Q_\varrho} |g|^2\, dz\right]^{1/2} \le c_0\left[\fint_{Q_{2\varrho}} \left(s+|g|\right)^\sigma dz\right]^{1/\sigma}$$

*for every $Q_{2\varrho} \Subset \Omega_T$.*

## 4. Banach valued, parabolic fractional Sobolev and Nikolski spaces

In this chapter we recall some definitions and basic facts about different spaces of functions we will use in the following. Our approach will mainly aim to the few (notation) concepts we need, so it will be not be as much general as possible; we refer however to the classical books [2, 29] for an exhaustive treatment.

First of all some general notation: whereas $E = E(\Omega)$ is a Banach space of integrable functions over $\Omega$, its local variant $E_{\mathrm{loc}}$ is defined in the usual way, that is $f \in E_{\mathrm{loc}}(\Omega)$ if $f \in E(\Omega')$ whenever $\Omega' \Subset \Omega$. The local variant with respect to time is defined similarly. We will lighten a bit notations writing $E(\Omega)$ for $E(\Omega; \mathbb{R}^k)$ when treating vectorial valued functions where no confusion shall arise. In this spirit, we restrict our description of the following spaces to the scalar case: the reader should however keep in mind that they have a trivial generalization for vector valued (and, as we will see, for Banach-valued) functions.

*Fractional Sobolev and Nikolski spaces.* For a domain $A \subset \mathbb{R}^N$ in space, the **elliptic fractional Sobolev space** $W^{\alpha,q}(A)$ is the subspace of $L^q(A)$ made up of all the functions $g$ whose fractional Sobolev seminorm

$$[g]^q_{W^{\alpha,q}(A)} := \int_A \int_A \frac{|g(x)-g(y)|^q}{|x-y|^{n+\alpha q}}\, dx\, dy$$

is finite. It is endowed with the norm $\|g\|_{W^{\alpha,q}(A)} := \|g\|_{L^q(A)} + [g]_{W^{\alpha,q}(A)}$. For the following embedding result see [21, Theorem 14.29] with minor changes, keeping in mind that $B^{s,p,p} \equiv W^{s,p}$, or also [2].

**Proposition 4.1** (Fractional Sobolev embedding)**.** *Let $A \subset \mathbb{R}^N$ a Lipschitz domain and let $g \in W^{\alpha,q}(A)$ with $\le q < \infty$ and $\alpha \in (0, 1)$ such that $\alpha q < N$. Then $g \in L^{Nq/(N-\alpha q)}(A)$ and there exists a constant $c \equiv c(N, \alpha, q, [\partial A]_{0,1})$ such that*

$$\|g\|_{L^{Nq/(N-\alpha q)}(A)} \le c\, \|g\|_{W^{\alpha,q}(A)}.$$

Moreover the following Proposition, which roughly says that we can increase integrability of a fractional Sobolev function up to lowering their fractional differentiability, will be fundamental to obtain our anisotropic regularity result. The proof is found in [21, Theorem 14.22], see also [2, Theorem 7.58].

**Proposition 4.2.** *Let $g \in W^{\tilde\theta,p}(A)$ for $\tilde\alpha \in (0, 1)$, $1 \le p < \infty$ and $A$ as in Proposition 4.1. Then for every $\alpha \in (0, \tilde\alpha)$ there exists a constant $c \equiv c(n, p, \tilde\alpha, \alpha, [\partial A]_{0,1})$ such that*

$$[g]_{W^{\alpha,q}(A)} \le c[g]_{W^{\tilde\alpha,p}(A)}$$

*if $q \in (p,\infty)$ satisfies*

$$\alpha - \frac{n}{q} = \tilde{\alpha} - \frac{n}{p}.$$

*In particular $W^{\tilde{\alpha},p}(A) \subset W^{\alpha,q}(A)$ for such $q$.*

For a function $g : \Omega \to \mathbb{R}$, any "small" real number $h \in \mathbb{R}$ and $i \in \{1,\dots,n\}$, we define the **spatial finite difference operator** $\tau_{i,h}$ as

$$\big[\tau_{i,h} g\big](x) = \tau_{i,h} g(x) := g(x + h\, e_i) - g(x),$$

being $e_i$ the $i$-th vector of the standard orthonormal basis of $\mathbb{R}^n$. This will make sense, for example, whenever $x \in A \Subset \Omega$, $A$ an open set and $0 < |h| < \operatorname{dist}(A, \partial\Omega)$, an assumption that will be always satisfied whenever we shall use this operator. Analogously, we define also the **finite difference operator in time** $\tau_h$ as

$$\big[\tau_h \tilde{g}\big](t) = \tau_h \tilde{g}(t) := \tilde{g}(t+h) - \tilde{g}(t),$$

again for $|h| > 0$ sufficiently small such that the definition makes sense.

For a set $A \Subset \Omega$, we define the **Nikolski space** $\mathcal{N}^{\alpha,q}(A)$ as the space of the $L^q(\Omega)$ functions $g$ such that their $\mathcal{N}^{\alpha,q}$ norm

$$\|g\|_{\mathcal{N}^{\alpha,q}(A)} := \|g\|_{L^q(A)} + [g]_{\mathcal{N}^{\alpha,q}(A)}\,,$$

with

$$[g]_{\mathcal{N}^{\alpha,q}(A)} := \sum_{i=1}^{n} \sup_{0<h<\operatorname{dist}(A,\partial\Omega)} |h|^{-\alpha} \|\tau_{i,h} g\|_{L^q(A)},$$

is finite. In the following we shall also let $W^{0,q}(A) = \mathcal{N}^{0,q}(A) = L^q(A)$. It is well known that there exists a precise chain of inclusions between fractional Sobolev and Nikolski spaces (see, among the others, [20, Lemma 2.3] or [9]), which reads as

$$W^{\alpha,q}(A) \subset \mathcal{N}^{\alpha,q}(A) \subset W^{\alpha-\varepsilon,q}(A) \qquad \text{for all } \varepsilon \in (0,\alpha). \tag{4.1}$$

*Banach-valued spaces.* Since we will treat various Banach valued spaces of functions, which are quite common in the parabolic setting, let's spend a couple of words about them. Notice that the treatment of Banach-valued spaces of functions requires additional cares (see again [2, 29]), but every time we will use them the assumptions needed will be largely satisfied. So let's fix a measurable function $g : A \times B \to \mathbb{R}^k$, where $A \subset \mathbb{R}^l$ and $B \subset \mathbb{R}^m$ are open bounded sets whose points are denoted respectively by $y_1$ and $y_2$. Let's moreover take two spaces of integrable functions $E$ and $F$, which could be defined over $A$ and $B$, with respective norms $\|\cdot\|_E$ and $\|\cdot\|_F$. By writing $g \in E(A; F(B))$ we will simply mean that the scalar function $\|g(y_1,\cdot)\|_{F(B)} : A \to \mathbb{R}$ belongs to $E(A)$.

In particular for this paper, $E$ and $F$ will always be or a Lebesgue space or one of the previously defined spaces, and the sets $A$ and $B$ will be, alternatively, a bounded interval of $\mathbb{R}$ and a bounded open subset of $\mathbb{R}^n$. For the particular choice $E \equiv L^r$, $A \equiv (-T,0)$, $F(B) \equiv W^{\alpha,q}(\Omega)$ we have

$$g \in L^r(-T,0;W^{\alpha,q}(\Omega)) \quad \text{if} \quad \int_{-T}^{0} \Big( \int_\Omega \int_\Omega \frac{|g(x,t)-g(y,s)|^q}{|x-y|^{n+\alpha q}}\, dx\, dy \Big)^{r/q} dt < \infty;$$

whereas with the choice $E \equiv W^{\alpha,r}$, $A \equiv (-T,0)$, $F(B) \equiv L^q(\Omega)$ we obtain

$$g \in W^{\alpha,r}(-T,0;L^q(\Omega)) \quad \text{if} \quad \int_{-T}^{0} \int_{-T}^{0} \frac{\big|\|g(\cdot,t)\|_{L^q(\Omega)} - \|g(\cdot,s)\|_{L^q(\Omega)}\big|^r}{|t-s|^{1+\alpha r}}\, dt\, ds < \infty;$$

similarly interchanging $\Omega$ and $(-T,0)$.

We shall lighten again notations denoting $E(T_1,T_2;F(A)) := E((T_1,T_2);F(A))$ and similarly, as we already did. Finally a straightforward inclusion in between some of these spaces is the following

**Remark 4.3.** *For $g \in L^q(-T,0;W^{\theta,q}(\Omega))$ we have the inequality*

$$\|g\|_{W^{\theta,q}(\Omega;L^q(-T,0))} \leq \|g\|_{L^q(-T,0;W^{\theta,q}(\Omega))},$$

*whose immediate consequence is the continuous immersion*

$$L^q(-T,0;W^{\theta,q}(\Omega)) \subset W^{\theta,q}(\Omega;L^q(-T,0)).$$

**Proof.** The proof is a straightforward consequence of triangle inequality and Fubini's theorem:

$$\begin{aligned}
\|g\|_{W^{\theta,q}(\Omega;L^q(-T,0))} &= \|g\|_{L^q(\Omega)} + \int_\Omega\int_\Omega \frac{\left|\|g(x,\cdot)\|_{L^q(-T,0)} - \|g(y,\cdot)\|_{L^q(-T,0)}\right|^q}{|x-y|^{n+\theta q}}\,dx\,dy \\
&\leq \|g\|_{L^q(\Omega)} + \int_\Omega\int_\Omega \frac{\|g(x,\cdot)-g(y,\cdot)\|^q_{L^q(-T,0)}}{|x-y|^{n+\theta q}}\,dx\,dy \\
&= \|g\|_{L^q(\Omega)} + \int_\Omega\int_\Omega\int_{-T}^0 \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+\theta q}}\,dt\,dx\,dy \\
&= \|g\|_{L^q(-T,0;W^{\theta,q}(\Omega))}.
\end{aligned}$$

□

Obviously the previous Lemma can be applied interchanging the sets $\Omega$ and $(-T,0)$ so that we also have the continuous immersion

$$L^q(\Omega;W^{\theta,q}(-T,0)) \subset W^{\theta,q}(-T,0;L^q(\Omega)). \tag{4.2}$$

*Parabolic spaces.* We say that a function $g \in L^q(\Omega_T)$ belongs to the **parabolic fractional Sobolev space** $W^{\theta,\tilde\theta;q}(\Omega_T)$, with $\theta,\tilde\theta \in (0,1)$ and $1 \leq q < \infty$, if it belongs to $L^q(-T,0;W^{\theta,q}(\Omega)) \cap L^q(\Omega;W^{\tilde\theta,q}(-T,0))$, which is the space consisting of all functions $u \in L^q(-T,0;L^q(\Omega))$ such that

$$\begin{aligned}
[g]^q_{W^{\theta,\tilde\theta;q}(\Omega_T)} :&= \int_{-T}^0 [g(\cdot,t)]^q_{W^{\theta,q}(\Omega)}\,dt + \int_\Omega [g(x,\cdot)]^q_{W^{\tilde\theta,q}(-T,0)}\,dx \\
&= \int_{-T}^0\int_\Omega\int_\Omega \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+\theta q}}\,dx\,dy\,dt \\
&\qquad + \int_\Omega\int_{-T}^0\int_{-T}^0 \frac{|g(x,t)-g(x,s)|^q}{|t-s|^{1+\tilde\theta q}}\,ds\,dt\,dx < \infty.
\end{aligned} \tag{4.3}$$

It is a Banach space if it is endowed with the norm, see [27],

$$\|g\|^q_{W^{\theta,\tilde\theta;q}(\Omega_T)} := \|g\|^q_{L^q(\Omega_T)} + [g]^q_{W^{\theta,\tilde\theta;q}(\Omega_T)}.$$

Also Nikolski spaces have a natural generalization when considered in parabolic shape (see [4]): precisely, we call the **parabolic Nikolski space** $\mathcal{N}^{\theta,\tilde\theta;q}(\Omega_T{}')$, for $\Omega_T{}' := A \times J$, $A \Subset \Omega$, $J \Subset (-T,0)$ and $\theta,\tilde\theta \in (0,1]$, the space of functions $\tilde g \in L^q(\Omega_T)$ such that

$$\begin{aligned}
[\tilde g]_{\mathcal{N}^{\theta,\tilde\theta,q}(\Omega_T{}')} :=& \sup_{0<|h|<\operatorname{dist}(J,\partial(-T,0))} |h|^{-\tilde\theta}\|\tau_h g\|_{L^q(\Omega_T{}')} \\
&+ \sum_{i=1}^n \sup_{0<h<\operatorname{dist}(A,\partial\Omega)} |h|^{-\theta}\|\tau_{i,h} g\|_{L^q(\Omega_T{}')} < \infty.
\end{aligned}$$

Obviously there is a chain of inclusion similar to (4.1) between the $W^{\theta,\tilde\theta;q}_{\text{loc}}$ and the $\mathcal{N}^{\theta,\tilde\theta;q}$ spaces, and this is specified in the following two results. The first one is the parabolic version of the second inclusion in (4.1) and its proof is a straightforward variation on the proof of the elliptic analogues, see [12, 19, 20]; for this parabolic formulation we refer to [13, Proposition 3.4], see also [4].

**Proposition 4.4.** *Let $g \in L^q(\Omega_T)$ with $1 \le q < \infty$ and assume that there exists $\bar\alpha \in (0,1]$, two open sets $\widetilde\Omega \Subset \Omega$ and $\tilde J \Subset (-T,0)$ such that*

$$\|\tau_{i,h} g\|_{L^q(\widetilde\Omega \times \tilde J)} \le S\,|h|^{\bar\alpha}, \tag{4.4}$$

*for some constant $S > 0$, for every $i \in \{1,\dots,n\}$ and every $h \in \mathbb{R}$ satisfying $0 < |h| < \mathcal{D}$, where $0 < \mathcal{D} \le \min\{1, \operatorname{dist}(\tilde\Omega, \partial\Omega)\}$. Then $g \in L^q(\tilde J; W^{\alpha,q}_{\mathrm{loc}}(\widetilde\Omega))$ for every $\alpha \in [0,\bar\alpha)$. In particular for each open set $\mathcal{O} \Subset \widetilde\Omega$ there exists a constant $c$ depending on $q, \bar\alpha - \alpha, \mathcal{D}, \operatorname{dist}(\widetilde\Omega, \partial\Omega), \operatorname{dist}(\mathcal{O}, \partial\widetilde\Omega)$, $|\Omega|$ such that*

$$\int_{\tilde J}\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy\,dt \le c\left[S^q + \|g\|^q_{L^q(\tilde\Omega\times\tilde J)}\right].$$

*Moreover if for some $\bar\beta \in (0,1]$ there holds*

$$\|\tau_h g\|_{L^q(\widetilde\Omega\times\tilde J)}\,dt \le \tilde S\,|h|^{\bar\beta}, \tag{4.5}$$

*for every $h \in \mathbb{R}$ satisfying $0 < |h| < \widetilde{\mathcal{D}}$ with $0 < \widetilde{\mathcal{D}} \le \min\{1, \operatorname{dist}(J, \partial(-T,0))\}$ and with a constant $\tilde S > 0$, then $g \in L^q(\tilde\Omega; W^{\beta,q}(\tilde J))$ for every $\beta \in [0,\bar\beta)$; moreover there exists a constant $\tilde c$ depending only on $q, \bar\beta - \beta, \widetilde{\mathcal{D}}$, $\operatorname{dist}(\tilde J, \partial(-T,0))$ and $T$ such that*

$$\int_{\tilde\Omega}\int_{\tilde J}\int_{\tilde J} \frac{|g(x,t)-g(x,s)|^q}{|t-s|^{1+\beta q}}\,dt\,ds\,dx \le \tilde c\left[S^q + \|g\|^q_{L^q(\tilde\Omega\times\tilde J)}\right].$$

We will always use the two results of the previous Proposition coupled together with the choice $\bar\beta \equiv \bar\alpha/2$; so we state explicitly the following Corollary

**Corollary 4.5.** *Let $g \in L^q(\Omega_T)$ satisfy the following estimate*

$$\|\tau_{h^2} g\|_{L^q(\widetilde\Omega\times\tilde J)} + \sum_{i=1}^n \|\tau_{i,h} g\|_{L^q(\widetilde\Omega\times\tilde J)} \le S\,|h|^{\bar\theta},$$

*for every $0 < |h| < \mathcal{D}$, with $\tilde\Omega, \tilde J$ as in the Proposition 4.4, $\bar\theta \in (0,1]$, $S > 0$ and $0 < \mathcal{D} \le \min\{1, \operatorname{dist}(\tilde\Omega, \partial\Omega), \operatorname{dist}(\tilde J, \partial(-T,0))\}$. Then $g \in W^{\theta,\theta/2;q}_{\mathrm{loc}}(\tilde\Omega \times \tilde J)$ for every $\theta \in [0,\bar\theta)$ with the explicit estimate*

$$[g]_{W^{\theta,\theta/2;q}(\mathcal{O}\times\mathcal{J})} \le c\left[S + \|g\|_{L^q(\tilde\Omega\times\tilde J)}\right]. \tag{4.6}$$

*for $\mathcal{O} \Subset \tilde\Omega$ and $\mathcal{J} \Subset \tilde J$. The constant $c$ depends on $q, \bar\theta - \theta, \mathcal{D}, \operatorname{dist}(\tilde\Omega, \partial\Omega)$, $\operatorname{dist}(\mathcal{O}, \partial\widetilde\Omega)$, $\operatorname{dist}(\tilde J, \partial(-T,0))$, $|\Omega|, T$.*

The final statement of this chapter is an appropriate version of the fractional Poincaré inequality. The proof is simple and follows widely the classical ones in the elliptic setting, see [12, 13], so we skip it.

**Lemma 4.6.** *Let $g \in W^{\theta,\theta/2;q}(Q_\varrho)$ for $\theta \in (0,1)$ and $q \ge 1$. Then there holds*

$$\fint_{Q_\varrho} |g - (g)_{Q_\varrho}|\,dz \le c\,\varrho^{\theta - \frac{n+2}{q}}\,[g]_{W^{\theta,\theta/2;q}(Q_\varrho)},$$

*with a constant $c \equiv c(n,q)$.*

## 5. A GLOBAL ESTIMATE

**Lemma 5.1** (Global estimate)**.** *Let $u \in L^2(-T,0; W^{1,2}_0(\Omega))$ be a weak solution to the problem (3.2) and let $q$ satisfy (1.4). Then we have the global estimate*

$$\|Du\|_{L^q(\Omega_T)} \le c\left[s + \|f\|_{L^1(\Omega_T)}\right],$$

*with $c \equiv c(n,\nu,L,q,|\Omega|,T)$.*

**Proof.** The proof is similar to the one appearing in [6], but for the convenience of the reader and in order to deduce the exact dependence upon the $L^1$-norm of $f$, we write it here.

We first suppose $\|f\|_{L^1(\Omega_T)} \le 1$ and $s \le 1$ and later show the statement for the general case by a scaling argument. We start with the Steklov formulation of (3.2): For a.e. $t \in (-T, 0)$ we have

$$\int_\Omega \Big[\partial_t u_h(\cdot,t)\varphi + \langle [a(\cdot,t,Du)]_h, D\varphi\rangle\Big]\,dx = \int_\Omega f_h(\cdot,t)\varphi\,dx, \tag{5.1}$$

for any test function $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and by density also for any $\varphi \in W_0^{1,2}(\Omega)$. $u_h$ denotes the Steklov average of $u$ defined in (3.1). The initial datum $u = 0$ on $\Omega \times \{-T\}$ is taken in the sense of $L^2$ which means that $u_h(\cdot,-T) \to 0$ in $L^2(\Omega)$. The proof is performed by applying a classical truncation technique (see [7, 5, 24]). For $k \in \mathbb{N}$, we define the truncation operators

$$T_k(\varsigma) := \max\big\{-k, \min\{k,\varsigma\}\big\}, \qquad \Phi_k(\varsigma) := T_1\big(\varsigma - T_k(\varsigma)\big), \tag{5.2}$$

for each $\varsigma \in \mathbb{R}$. Moreover we define

$$D_k := \{z \in \Omega_T : k < |u(z)| \le k+1\}. \tag{5.3}$$

Furthermore let $\Psi_k : \mathbb{R} \to \mathbb{R}$ be defined as $\Psi_k(\varsigma) := \int_0^\varsigma \Phi_k(\zeta)\,d\zeta$. An explicit calculation of $\Psi_k$ shows immediately (see [14]) that

$$\Psi_k(\varsigma) \ge 0 \qquad \text{for any } \varsigma \in \mathbb{R}. \tag{5.4}$$

We now test the Steklov formulation (5.1) with the function

$$\varphi(x,t) := \zeta(t)\,\Phi_k\big(u_h(x,t)\big), \quad x \in \Omega,$$

for a function $\zeta(t)$ in time. Note that $\varphi$ is admissible in (5.1) for a.e. $t \in (-T,0)$, i.e. $\varphi(\cdot,t) \in W_0^{1,2}(\Omega)$. Integrating the resulting equation over $(-T,0)$ with respect to $t$ gives

$$\begin{aligned}\int_{\Omega_T} \partial_t u_h \Phi_k(u_h)\zeta(t)\,dz + \int_{\Omega_T} \langle [a(\cdot,t,Du)]_h, D\Phi_k(u_h)\rangle \zeta(t)\,dz \\ = \int_{\Omega_T} \Phi_k(u_h) f_h \zeta(t)\,dz.\end{aligned}$$

For $\tau \in (-T,0)$ and $\varepsilon > 0$ let $\zeta \in W^{1,\infty}(\mathbb{R})$ be defined as

$$\zeta(t) := \begin{cases} 1 & \text{if } t \le \tau, \\ 1 - \dfrac{1}{\varepsilon}(t-\tau) & \text{if } \tau < t \le \tau+\varepsilon, \\ 0 & \text{if } t > \tau+\varepsilon. \end{cases} \tag{5.5}$$

Using this function in the previous identity and recalling the definition of $\Psi_k$ we obtain

$$\begin{aligned}\int_{\Omega_T} \partial_t u_h \Phi_k(u_h)\zeta(t)\,dz &= \int_{\Omega_T} \partial_t\left[\Psi_k(u_h)\zeta(t)\right]\,dz - \int_{\Omega_T} \Psi_k(u_h)\zeta'(t)\,dz \\ &= -\int_\Omega \Psi_k(u_h)(x,-T)\,dx - \int_{\Omega_T} \Psi_k(u_h)\zeta'(t)\,dz,\end{aligned}$$

for a.e. $\tau \in (-T,0)$. Now, the second integral on the right–hand side of the preceding equality converges, as $\varepsilon \searrow 0$, to $\int_\Omega \Psi_k(u)(x,\tau)\,dx$ for a.e. $\tau \in (-T,0)$, whereas the first integral converges to $0$ as $h \searrow 0$ , since $u_h(\cdot,-T) \to 0$ in the sense of $L^2$. Therefore, letting first $\varepsilon \searrow 0$ then $h \searrow 0$, we obtain for a.e. $\tau \in (-T, 0)$

$$\int_\Omega \Psi_k(u)(x,\tau)\,dx + \int_{-T}^{\tau}\int_\Omega \big\langle a(x,t,Du), D\Phi_k(u)\big\rangle\,dx\,dt = \int_{-T}^{\tau}\int_\Omega \Phi_k(u) f\,dx\,dt. \tag{5.6}$$

Now recalling the definition of $D_k$ and exploiting the explicit calculations of $\Phi_k(u)$, $\Psi_k(u)$ and $D\Phi_k(u)$ (we refer the reader to [14] for a detailed calculation) the terms of the previous identity can be treated as follows:

$$\int_{\Omega_T} \langle a(x,t,Du), D\Phi_k(u)\rangle\ dz = \int_{D_k} \langle a(x,t,Du), Du\rangle\ dz,$$

$$\left|\int_{\Omega_T} \Phi_k(u) f\,dz\right| \le \int_{\Omega_T} |f|\,dz,$$

$$\int_\Omega \Psi_k(u)(x,\tau)\,dx \ge 0 \qquad \text{for all } k \text{ and for every } \tau \in (-T,0),$$

since $u \in C^0([-T,0]; L^2(\Omega))$ and (5.4). Now exploiting the structure conditions $(1.2)_1$ and $(1.2)_3$, then (5.6) together with the previous estimates, and finally Young's inequality and the fact that $\|f\|_{L^1(\Omega_T)} \le 1$, we deduce

$$\begin{aligned}
\nu\int_{D_k} |Du|^2\,dz &\le \int_{D_k} \langle a(x,t,Du) - a(x,t,0), Du\rangle\ dz \\
&\le \int_\Omega \Psi_k(u)(x,0)\,dx + \int_{D_k} \langle a(x,t,Du), Du\rangle\ dz - \int_{D_k} \langle a(x,t,0), Du\rangle\ dz \\
&\le \int_{\Omega_T} |f|\,dz + Ls\int_{D_k} |Du|\,dz \\
&\le 1 + \varepsilon\int_{D_k} |Du|^2\,dz + \frac{L^2 s^2}{4\varepsilon}|D_k|.
\end{aligned}$$

Choosing $\varepsilon = \nu/2$ we therefore conclude

$$\int_{D_k} |Du|^2\,dz \le c(L/\nu)\Big(1 + s^2|D_k|\Big). \tag{5.7}$$

Secondly, writing (5.6) for $k = 0$ we get, writing for shortness $D_0(\tau) := D_0 \cap (\Omega \times (-T,\tau))$,

$$\begin{aligned}
\|f\|_{L^1(\Omega_T)} &\ge \int_{D_0(\tau)} \langle a(x,t,Du), Du\rangle\ dz + \int_\Omega \Psi_0(u)(x,\tau)\,dx \\
&= \int_{D_0(\tau)} \langle a(x,t,Du) - a(x,t,0), Du\rangle\ dz \\
&\qquad + \int_{D_0(\tau)} \langle a(x,t,0), Du\rangle\ dz + \int_\Omega \Psi_0(u)(x,\tau)\,dx \\
&\ge \int_\Omega \Psi_0(u)(x,\tau)\,dx - Ls\int_{D_0} |Du|\,dz,
\end{aligned}$$

keeping in mind the structure conditions (1.2) and discarding the positive term. Now, calculating $\Psi_0$ explicitely, we achieve

$$\int_\Omega \Psi_0(u)(x,\tau)\,dx \ge \int_\Omega |u(x,\tau)|\,dx - \frac{1}{2}|\Omega|.$$

Thus, merging this with the last estimate, the fact that $\|f\|_{L^1(\Omega_T)} \leq 1$ and $s \leq 1$, together with Young's inequality and (5.7), we finally conclude the $L^\infty$–$L^1$ estimate

$$\begin{aligned}\sup_{\tau\in(-T,0)} \int_\Omega |u(x,\tau)|\,dx &\leq 1 + Ls\int_{D_0} |Du|\,dz + \frac{1}{2}|\Omega| \\ &\leq 1 + L^2 s^2|\Omega_T| + \int_{D_0} |Du|^2\,dz + \frac{1}{2}|\Omega| \\ &\leq c(\nu,L)\Big(1 + s^2|\Omega_T|\Big) + \frac{1}{2}|\Omega| \leq c(\nu,L,|\Omega|,T). \end{aligned} \tag{5.8}$$

Let $\tilde q > 1$ be a free parameter, which will be chosen later. Using Hölder's inequality, (5.7) and the definition of $D_k$ in (5.3) we obtain for $1 \leq q < 2$ and for any $k \geq 1$

$$\begin{aligned}\int_{D_k} |Du|^q\,dz &\leq |D_k|^{1-\frac{q}{2}}\Big(\int_{D_k} |Du|^2\,dz\Big)^{\frac{q}{2}} \\ &\leq c|D_k|^{1-\frac{q}{2}} + c\,|D_k| \\ &\leq c\,k^{-\tilde q(1-\frac{q}{2})}\Big(\int_{D_k} |u|^{\tilde q}\,dz\Big)^{1-\frac{q}{2}} + c\,|D_k|, \end{aligned} \tag{5.9}$$

with $c \equiv c(L/\nu, q)$. Now we split in the following way, using also Hölder's inequality and (5.7) in order to deduce

$$\begin{aligned}\int_{\Omega_T} |Du|^q\,dz &= \int_{D_0} |Du|^q\,dz + \sum_{k=1}^\infty \int_{D_k} |Du|^q\,dz \\ &\leq c\Big[\sum_{k=1}^\infty |D_k| + \sum_{k=1}^\infty k^{-\tilde q(1-\frac{q}{2})}\Big(\int_{D_k} |u|^{\tilde q}\,dz\Big)^{1-\frac{q}{2}} |D_0|^{1-\frac{q}{2}}\Big(\int_{D_0} |Du|^2\,dz\Big)^{\frac{q}{2}} + \Big] \\ &\leq c\Big[1 + |\Omega_T| + \sum_{k=1}^\infty k^{-\tilde q(1-\frac{q}{2})}\Big(\int_{D_k} |u|^{\tilde q}\,dz\Big)^{1-\frac{q}{2}}\Big] \\ &\leq c\Big[1 + \Big(\sum_{k=1}^\infty k^{-\tilde q(\frac{2}{q}-1)}\Big)^{\frac{q}{2}}\Big(\int_\Omega |u|^{\tilde q}\,dz\Big)^{1-\frac{q}{2}}\Big], \end{aligned} \tag{5.10}$$

for a constant $c \equiv c(\nu, L, q, |\Omega|, T)$. To treat the integral on the right–hand side we remark that a well–known version of the Gagliardo–Nirenberg embedding (see for example [18, Chapter 7]), applied on time slices $t \in (-T, 0)$, gives us

$$\|u(\cdot,t)\|_{L^{\tilde q}(\Omega)} \leq c(n,q)\|Du(\cdot,t)\|^\theta_{L^q(\Omega)}\|u(\cdot,t)\|^{1-\theta}_{L^1(\Omega)},$$

for an interpolation parameter $0 \leq \theta \leq 1$ such that $\frac{1}{\tilde q} = \theta\Big(\frac{1}{q} - \frac{1}{n}\Big) + 1 - \theta$. If we choose $\tilde q \equiv q(n+1)/n$ an we keep in mind (5.8) it is easy to check that

$$\int_{\Omega_T} |u|^{\tilde q}\,dz \leq c(n,\nu,L,q,|\Omega|)\int_{\Omega_T} |Du|^q\,dz$$

and that $\tilde q\Big(\frac{2}{q} - 1\Big) > 1$, if $q$ satisfies (1.4), so the series appearing in (5.10) is convergent. Subsequently we can write

$$\int_{\Omega_T} |Du|^q\,dz \leq c\Big[1 + \Big(\int_{\Omega_T} |Du|^q\,dz\Big)^{1-q/2}\Big]$$

with $c \equiv c(n,\nu,L,q,|\Omega|,T)$, and finally conclude using Hölder's inequality, since $1-\frac{q}{2} < 1$, to re–absorb the right–hand side norm of $Du$:

$$u \in L^q(-T,0;W^{1,q}_0(\Omega)), \qquad \text{i.e.} \qquad \int_\Omega |Du|^q\,dz \leq c, \tag{5.11}$$

for all $q$ satisfying (1.4), with a constant $c$ that depends on $n, \nu, L, q, |\Omega|, T$. In a last step, it remains to eliminate the assumptions $\|f\|_{L^1(\Omega_T)} \le 1$ and $s \le 1$ by a scaling argument: Let $u \in L^2(-T, 0; W_0^{1,2}(\Omega))$ be as in the statement of the Lemma. We define $F := \|f\|_{L^1(\Omega_T)} + s > 0$ (otherwise the statement is trivial) and let

$$\bar{u} := \frac{1}{F}\,u, \quad \bar{f} := \frac{1}{F}\,f, \quad \bar{a}(x,t,z) := \frac{1}{F}\,a(x,t,Fz).$$

We therefore easily see that

$$\bar{u}_t - \operatorname{div} \bar{a}(x,t,D\bar{u}) = \bar{f} \quad \text{on } \Omega_T \quad \text{and} \quad \|\bar{f}\|_{L^1(\Omega_T)} \le 1.$$

Furthermore, $\bar{a}$ fulfills the conditions (1.2) with $s$ replaced by $\bar{s} := s/F$ and we have $\bar{s} = s/F \le 1$. Therefore estimate (5.11) holds for $\bar{u}$. Having in mind $\bar{u} = u/F$ we conclude

$$\int_\Omega |Du|^q \, dz \le c\Big[s + \|f\|_{L^1(\Omega_T)}\Big]^q,$$

with $c \equiv c(n, \nu, L, q, |\Omega|, T)$. The proof is now complete. □

## 6. Comparison Lemmata

A main tool of the proof of Theorems 1.2 and 1.5 is a series of comparison procedures. Let us first fix $z_0 \in \Omega_T$ and $0 < \varrho \le 1$ such that $Q_\varrho(z_0) \Subset \Omega_T$, and let $v \in u + L^2(I_\varrho(t_0); W_0^{1,2}(B_\varrho(x_0)))$ the unique weak solution to

$$\begin{cases} \partial_t v - \operatorname{div}\, a(x,t,Dv) = 0 & \text{in } Q_\varrho(z_0), \\ v = u & \text{on } \partial_{\mathcal{P}} Q_\varrho(z_0). \end{cases} \tag{6.1}$$

Existence and uniqueness directly follow from the structure conditions and can be referred from [22]. Since $v$ is the solution of a homogeneous problem, we have the following **higher integrability** property for $v$ (see [17, Theorem 2.1] or [26]):

**Lemma 6.1.** *Let $v \in u + L^2(I_\varrho(t_0); W_0^{1,2}(B_\varrho(x_0)))$ be the solution of (6.1), where the vector field $a$ satisfies the ellipticity and monotonicity assumptions $(1.2)_1$ and $(1.2)_2$. Then there exists $\chi_0 > 1$, depending on $n$ and $L/\nu$, such that $Dv \in L^{2\chi_0}_{\mathrm{loc}}(Q_\varrho(z_0))$. Furthermore there exists a constant $c \equiv c(n, L/\nu)$ such that for any $Q_{2\tilde{\varrho}} \Subset Q_\varrho(z_0)$ and any $\chi \le \chi_0$ the following estimate holds true:*

$$\left[\fint_{Q_{\tilde{\varrho}}} |Dv|^{2\chi} \, dz\right]^{1/\chi} \le c \fint_{Q_{2\tilde{\varrho}}} (s + |Dv|)^2 \, dz.$$

**Remark 6.2.** *The higher integrability statement in [17] is done for homogeneous parabolic systems of the special type $v_t - \operatorname{div}(a(z)Dv) = 0$ with bounded, measurable, continuous and elliptic coefficients $a(z)$. However, some minor modifications of the proof in [17], involving the growth and ellipticity conditions $(1.2)_1$ and $(1.2)_2$, also provide the result for equations (and systems) of the type (6.1).*

**Remark 6.3.** *Once having higher integrability in terms of Lemma 6.1 at hand, Lemma 3.1 allows to reduce the integral power in the sense of*

$$\left[\fint_{Q_{\tilde{\varrho}}} |Dv|^2 \, dz\right]^{1/2} \le c \fint_{Q_{2\tilde{\varrho}}} (s + |Dv|) \, dz,$$

*with a constant $c$ depending on $n, L/\nu$.*

A second step consists in considering the following homogeneous frozen Dirichlet problem on a smaller parabolic cylinder

$$\begin{cases} \partial_t v_0 - \operatorname{div}\, a(x_0,t,Dv_0) = 0 & \text{in } Q_{\varrho/4}(z_0), \\ v_0 = v & \text{on } \partial_{\mathcal{P}} Q_{\varrho/4}(z_0), \end{cases} \tag{6.2}$$

and its unique solution which belongs to $v + L^2(I_{\varrho/4}(t_0); W_0^{1,2}(B_{\varrho/4}(x_0)))$. Again, existence and uniqueness of such a solution can be referred from [22].

We now establish suitable **comparison estimates** between the solution $u$ of the original problem and the solution $v$ of the homogeneous one, respectively $v_0$ of the homogeneous frozen one. Note at this point that it is essential to involve nothing more than the $L^1$ norm of the inhomogeneity $f$ on the right–hand side. Therefore the proofs again involve certain truncation techniques. We start with comparison between $u$ and $v$:

**Lemma 6.4.** *Let $u \in L^2(-T, 0; W_0^{1,2}(\Omega)) \cap C^0([-T, 0]; L^2(\Omega))$ be the weak solution of problem (3.2) and $v \in u + L^2(I_\varrho(t_0); W_0^{1,2}(B_\varrho(x_0)))$ the solution of problem (6.1). Then the following comparison estimate holds true:*

$$\|Du - Dv\|_{L^q(Q_\varrho(z_0))} \le c\, \varrho^{\delta(q)} \|f\|_{L^1(Q_\varrho(z_0))},$$

*for all $q$ satisfying (1.4), with $c \equiv c(n, \nu, q)$.*

**Proof.** We first consider the case $Q_\varrho(z_0) = Q_1(0) \equiv Q \equiv B \times I$ and suppose $\|f\|_{L^1(Q)} = 1$. The general case will follow again by a scaling argument. We start with the Steklov formulations of the equations which write as

$$\int_B \Big[\partial_t u_h(\cdot, t)\varphi + \langle [a(\cdot, t, Du)]_h, D\varphi\rangle\Big]\, dx = \int_B f_h(\cdot, t)\varphi\, dx, \tag{6.3}$$

for all $\varphi \in W_0^{1,2}(B)$ and for a.e. $t \in I$, respectively

$$\int_B \Big[\partial_t v_h(\cdot, t)\varphi + \langle [a(\cdot, t, Dv)]_h, D\varphi\rangle\Big]\, dx = 0, \tag{6.4}$$

for all $\varphi \in W_0^{1,2}(B)$ and for a.e. $t \in I$.

Again we remark that the initial datum is taken in $L^2$, i.e. $v_h(\cdot, -1) \to u(\cdot, -1)$ in $L^2(B)$. Defining now the truncation operator $\Phi_k(\varsigma)$ as in (5.2), having again $\Psi_k(\varsigma) := \int_0^\varsigma \Phi_k(\zeta)\, d\zeta$ as in the proof of Lemma 5.1 and denoting

$$D_k := \{z \in Q_1 : k < |u(z) - v(z)| \le k + 1\},$$

we test the difference of (6.3) and (6.4) by $\varphi(x, t) := \Phi_k(u_h - v_h)(x, t)\zeta(t)$, $x \in B$, where $\zeta(\cdot)$ denotes a Lipschitz continuous function in time, and subsequently integrate over $I$ with respect to $t$ to achieve

$$\begin{aligned}
&\int_Q \partial_t(u_h - v_h)\Phi_k(u_h - v_h)\zeta\, dz \\
&\qquad + \int_Q \langle [a(\cdot, t, Du)]_h - [a(\cdot, t, Dv)]_h, D\Phi_k(u_h - v_h)\rangle\, \zeta\, dz \\
&\qquad\qquad = \int_Q f_h \Phi_k(u_h - v_h)\zeta\, dz.
\end{aligned}$$

Now choosing $\zeta(t)$ as in (5.5) and arguing exactly as in (5.6), letting $\varepsilon \searrow 0$, then $h \searrow 0$ and taking the supremum, we finally arrive at

$$\begin{aligned}
&\sup_{-1<\tau<1} \int_B \Psi_k(u - v)(x, \tau)\, dx \\
&\quad + \int_Q \langle a(x, t, Du) - a(x, t, Dv), D\Phi_k(u - v)\rangle\, \zeta\, dz \le \int_Q |f||\Phi_k(u - v)|\, dz.
\end{aligned} \tag{6.5}$$

Writing (6.5) for $k = 0$ and exploiting $(1.2)_2$ we immediately have

$$\sup_{-1<\tau<1} \int_B \Psi_0(u - v)(x, \tau)\, dx \le \int_Q |f|\, dz = 1.$$

On the other hand carefully exploiting Young's inequality and the explicit expression for $\Psi_0$ we have for a.e. $\tau \in I$

$$
\begin{aligned}
\int_B |u(\cdot,\tau)-v(\cdot,\tau)|\,dx &= \int_{B\cap\{|u-v|<1\}} |\dots|\,dx + \int_{B\cap\{|u-v|\geq 1\}} |\dots|\,dx \\
&\leq \frac{1}{2}\int_{B\cap\{|u-v|<1\}} |u(\cdot,\tau)-v(\cdot,\tau)|^2\,dx + \frac{1}{2}\,|B\cap\{|u-v|<1\}| \\
&\quad + \int_{B\cap\{|u-v|\geq 1\}} |u(\cdot,\tau)-v(\cdot,\tau)|\,dx \\
&= \frac{1}{2}\int_{B\cap\{|u-v|<1\}} |u(\cdot,\tau)-v(\cdot,\tau)|^2\,dx + \frac{1}{2}\,|B| - \frac{1}{2}\,|B\cap\{|u-v|\geq 1\}| \\
&\quad + \int_{B\cap\{|u-v|\geq 1\}} |u(\cdot,\tau)-v(\cdot,\tau)|\,dx \\
&= \int_B \Psi_0(u-v)(\cdot,\tau)\,dx + \frac{1}{2}|B|.
\end{aligned}
$$

Merging this estimate with the previous one, we arrive at

$$
u-v \in L^\infty(-1,1;L^1(B)) \qquad \text{and} \qquad \|u-v\|_{L^\infty(-1,1;L^1(B))} \leq c(n).
$$

Having again a look at (6.5) , keeping in mind that $D\Phi_k(u-v) = Du - Dv$ on the set $D_k$ and $D\Phi_k(u-v) = 0$ otherwise, subsequently exploiting $(1.2)_2$, $|\Phi_k| \leq 1$ and (5.4), we achieve

$$
\begin{aligned}
\nu \int_{D_k} |Du-Dv|^2\,dz &\leq \int_{D_k} \langle a(x,t,Du) - a(x,t,Dv), Du-Dv\rangle\; dz \\
&\leq \int_Q |f|\,dz = 1,
\end{aligned}
$$

and thus

$$
\int_{D_k} |Du-Dv|^2\,dx \leq \frac{1}{\nu}.
$$

Now further proceding exactly as in the proof of Lemma 5.1, here with the function $u-v$ instead of $u$, we finally conclude

$$
Du - Dv \in L^q(Q_1), \quad \|Du-Dv\|_{L^q(Q_1)} \leq c(n,\nu,q), \tag{6.6}
$$

for all $q$ satisfying (1.4) (cfr. (5.11)). The case $0 < F := \|f\|_{L^1(Q_1)} \neq 1$ (if $\|f\|_{L^1(Q_1)} = 0$ the thesis is trivial since $u = v$) is faced exactly as in the proof of Lemma 5.1, considering the functions $\bar{u} := u/F$ and $\bar{v} := v/F$; consequently we get

$$
\|Du-Dv\|_{L^q(Q_1)} \leq c\,\|f\|_{L^1(Q_1)}.
$$

Finally for the general case $Q_\varrho(z_0)$ we consider the rescaled functions, defined in $Q_1$:

$$
\begin{cases}
\tilde{u}(x,t) := \frac{1}{\varrho}\,u(\varrho x + x_0, \varrho^2 t + t_0), & \tilde{v}(x,t) := \frac{1}{\varrho}\,v(\varrho x + x_0, \varrho^2 t + t_0), \\
\tilde{a}(x,t,z) := a(\varrho x + x_0, \varrho^2 t + t_0, z), & \tilde{f}(x,t) := \varrho f(\varrho x + x_0, \varrho^2 t + t_0).
\end{cases}
$$

We observe that $\tilde{a}$ satisfies $(1.2)_1$, that $\tilde{u} - \tilde{v} = 0$ on $\partial_{\mathcal{P}} Q_1$ and that there holds

$$
\partial_t \tilde{u} - \operatorname{div}\, \tilde{a}(x,t,D\tilde{u}) = \tilde{f}, \quad \partial_t \tilde{v} - \operatorname{div}\, \tilde{a}(x,t,D\tilde{v}) = 0 \quad \text{in } Q_1.
$$

So by (6.6) we arrive at

$$
\begin{aligned}
\varrho^{-\frac{n+2}{q}}\,\|Du-Dv\|_{L^q(Q_\varrho(x_0))} &= \|D\tilde{u} - D\tilde{v}\|_{L^q(Q_1)} \\
&\leq c\,\big\|\tilde{f}\big\|_{L^1(Q_1)} \\
&= c\,\varrho^{-(n+1)}\big\|f\big\|_{L^1(Q_\varrho(z_0))},
\end{aligned}
$$

which is the desired estimate. Let us note that (1.4) ensures that the exponent of $\varrho$ is positive. The proof is complete. □

Subsequently, we establish a comparison estimate between the solution $v$ of the homogeneous problem and the solution $v_0$ of the frozen homogeneous one:

**Lemma 6.5.** *Let $v \in u + L^2(I_\varrho(t_0); W_0^{1,2}(B_\varrho(x_0)))$ be the unique weak solution to (6.1) and $v_0 \in v + L^2(I_{\varrho/4}(t_0); W_0^{1,2}(B_{\varrho/4}(x_0)))$ the one of (6.2). Then the following comparison estimate holds true:*

$$\|Dv - Dv_0\|_{L^q(Q_{\varrho/4}(z_0))} \le c\,\varrho^{\delta(q)} \left[ \int_{Q_\varrho(z_0)} \big(s + |Dv|\big)\,dz \right],$$

*with $c = c(n, L/\nu, q)$.*

**Proof.** To focus on the main aspects of the proof, the following argumentation is merely formal, since it would need time derivatives of both $v$ and $v_0$. On the other hand, the calculations can easily be made rigorous by again involving the Steklov formulation of the equations, thereafter passing to the limit. We test the difference of the equations

$$\int_{Q_{\varrho/4}(z_0)} \Big[ \partial_t(v - v_0)\varphi + \langle a(x,t,Dv) - a(x_0,t,Dv_0), D\varphi \rangle \Big]\,dz = 0,$$

by the function $\varphi := (v - v_0)\zeta$, with $\zeta$ as in (5.5), and proceed – with the help of the Steklov formulation – analogously to the argumentation in the proof of Lemma 6.4 to achieve (6.5), arriving at

$$\begin{aligned} \sup_{\tau \in I_{\varrho/4}(t_0)} &\int_{B_{\varrho/4}(x_0)} |v - v_0|^2(x,\tau)\,dx \\ &+ \int_{Q_{\varrho/4}(z_0)} \langle a(x,t,Dv) - a(x_0,t,Dv_0), Dv - Dv_0 \rangle\,dz \le 0, \end{aligned}$$

and therefore by $(1.2)_1$ also at

$$\begin{aligned} \nu \int_{Q_{\varrho/4}(z_0)} &|Dv - Dv_0|^2\,dz \\ &\le \int_{Q_{\varrho/4}(z_0)} \langle a(x_0,t,Dv) - a(x_0,t,Dv_0), Dv - Dv_0 \rangle\,dz \\ &\le \sup_{\tau \in I_{\varrho/4}(t_0)} \int_{B_{\varrho/4}(x_0)} |v - v_0|^2(x,\tau)\,dx \\ &\qquad + \int_{Q_{\varrho/4}(z_0)} \langle a(x,t,Dv) - a(x_0,t,Dv_0), Dv - Dv_0 \rangle\,dz \\ &\qquad + \int_{Q_{\varrho/4}(z_0)} \langle a(x_0,t,Dv) - a(x,t,Dv), Dv - Dv_0 \rangle\,dz \\ &\le \left| \int_{Q_{\varrho/4}(z_0)} \langle a(x_0,t,Dv) - a(x,t,Dv), Dv - Dv_0 \rangle\,dz \right|. \end{aligned}$$

Exploiting now $(1.2)_4$ and using Young's inequality we finally arrive at

$$\begin{aligned} \nu \int_{Q_{\varrho/4}(z_0)} &|Dv - Dv_0|^2\,dz \\ &\le Lc(\varepsilon)\,\varrho^2 \int_{Q_{\varrho/4}(z_0)} \big(s^2 + |Dv|^2\big)\,dz + L\varepsilon \int_{Q_{\varrho/4}(z_0)} |Dv - Dv_0|^2\,dz. \end{aligned}$$

Choosing $\varepsilon \equiv \nu/(2L)$ and reabsorbing the last term of the estimate, we get

$$\int_{Q_{\varrho/4}(z_0)} |Dv - Dv_0|^2\,dz \le c(L/\nu)\,\varrho^2 \int_{Q_{\varrho/4}(z_0)} \big(s^2 + |Dv|^2\big)\,dz. \tag{6.7}$$

Using now again Hölder's inequality, (6.7) and thereafter the reverse Hölder inequality of Remark 6.3, we deduce

$$\begin{aligned}\int_{Q_{\varrho/4}(z_0)} |Dv - Dv_0|^q \, dz &\le c\, \varrho^{(n+2)(1-\frac{q}{2})} \left[\int_{Q_{\varrho/4}(z_0)} |Dv - Dv_0|^2 \, dz\right]^{\frac{q}{2}} \\ &\le c\, \varrho^{n+2+q} \left[\fint_{Q_{\varrho/4}(z_0)} \big(s^2 + |Dv|^2\big) \, dz\right]^{\frac{q}{2}} \\ &\le c\, \varrho^{n+2-q(n+1)} \left[\int_{Q_{\varrho}(z_0)} \big(s + |Dv|\big) \, dz\right]^{q}\end{aligned}$$

with $c \equiv c(n, L/\nu, q)$, which is the desired comparison estimate. □

Finally we deduce an **energy estimate** for the $L^2$ norm of $Dv_0$ in terms of $L^q$ norm of $Du$, in the following sense:

**Lemma 6.6.** *Let $u$ be a weak solution to (3.2) with $f \in L^1(\Omega_T)$, $v$ and $v_0$ respectively as in (6.1) and (6.2). Then the following estimate holds true:*

$$\begin{aligned}&\left[\int_{Q_{\varrho/4}} (s^2 + |Dv_0|^2) \, dz\right]^{1/2} \\ &\qquad \le c\, \varrho^{1-\frac{2}{q}+n\frac{q-2}{2q}} \Big[\|s + |Du|\|_{L^q(Q_\varrho(z_0))} + \|f\|_{L^1(Q_\varrho(z_0))}\Big],\end{aligned}$$

*with $c \equiv c(n, L/\nu, q)$.*

**Proof.** We start, using the intermediate comparison estimate (6.7), reverse Hölder's inequality of Remark 6.3 and Hölder's inequality (note that $\varrho \le 1$), to deduce

$$\begin{aligned}\fint_{Q_{\varrho/4}} (s^2 + |Dv_0|^2) \, dz &\le 2 \fint_{Q_{\varrho/4}} (s^2 + |Dv|^2) \, dz + 2 \fint_{Q_{\varrho/4}} |Dv - Dv_0|^2 \, dz \\ &\le c(L/\nu) \fint_{Q_{\varrho/4}} \big(s^2 + |Dv|^2\big) \, dz \\ &\le c(n, L/\nu) \left[\fint_{Q_{\varrho/2}} \big(s + |Dv|\big) \, dz\right]^2 \\ &\le c(n, L/\nu, q) \left[\fint_{Q_{\varrho/2}} \big(s^q + |Dv|^q\big) \, dz\right]^{\frac{2}{q}}.\end{aligned}$$

Now exploiting Lemma 6.4 and recalling $\varrho \le 1$ we get

$$\begin{aligned}\left[\int_{Q_{\varrho/4}} (s^2 + |Dv_0|^2) \, dz\right]^{\frac{1}{2}} &\le c\, \varrho^{\frac{n+2}{2}} \left[\fint_{Q_{\varrho/2}} \big(s^q + |Du|^q\big) \, dz + \fint_{Q_{\varrho/2}} |Du - Dv|^q \, dz\right]^{\frac{1}{q}} \\ &\le c\, \varrho^{1-\frac{2}{q}+n\frac{q-2}{2q}} \Bigg[\|f\|_{L^1(Q_\varrho(z_0))} \\ &\qquad + \left(\int_{Q_\varrho} (s + |Du|)^q \, dz\right)^{\frac{1}{q}}\Bigg],\end{aligned}$$

where $c \equiv c(n, L/\nu, q)$. This finishes the proof.

□

## 7. FRACTIONAL ESTIMATES FOR THE REFERENCE PROBLEM

In this chapter we consider the reference problem (6.2) which is homogeneous and with no dependence of the vector field on the space variable, while the dependence on the time variable is merely measurable. We will show by approximation that the gradient $Dv_0$ of its solution $v_0$ is differentiable with respect to space and at least “almost” half differentiable with respect to time. This is the content of the following

**Lemma 7.1.** *Let $Q_\varrho(z_0) \subset \Omega_T$ be a parabolic cylinder and let furthermore $v_0 \in v + L^2(I_{\varrho/4}(t_0); W_0^{1,2}(B_{\varrho/4}(x_0)))$ be the solution of the frozen Dirichlet problem (6.2) on the cylinder $Q_{\varrho/4}(z_0)$, where the vector field $a$ is supposed to satisfy the hypotheses (1.2). Then for any $\theta \in (0, 1/2)$ we have*

$$Dv_0 \in L^2_{\rm loc}(I_{\varrho/4}(t_0); W^{1,2}_{\rm loc}(B_{\varrho/4})) \cap W^{\theta,2}_{\rm loc}(I_{\varrho/4}(t_0); L^2_{\rm loc}(B_{\varrho/4})).$$

*Moreover, there exists a constant $c \equiv c(n, L/\nu)$ such that for arbitrary $\eta \in \mathbb{R}^n$ the following estimates hold true:*

$$\left[ \fint_{Q_{\varrho/16}} |D^2 v_0|^2 \, dz \right]^{\frac{1}{2}} \le c\, \varrho^{-1} \fint_{Q_{\varrho/4}} |Dv_0 - \eta| \, dz \tag{7.1}$$

*and*

$$\left[ \fint_{Q_{\varrho/32}} \frac{|\tau_h Dv_0|^2}{|h|} \, dz \right]^{\frac{1}{2}} \le c\, \varrho^{-1} \fint_{Q_{\varrho/4}} |Dv_0 - \eta| \, dz, \tag{7.2}$$

*for any $h \in \mathbb{R}$ with $0 < |h| < (\varrho/32)^2$.*

**Proof.** The proof is done in firmly exploiting Lemma 9.4. of [13], see also [3, 4]. Since the vector field $a$ is not differentiable with respect to the variable $z$, we proceed analogously to [24, Lemma 3.2], regularizing $a$ in an appropriate way, showing the desired estimates for the solution of the regularized problem and finally passing to the limit, as in [1].

*1st step: Approximation by regularized vector fields.* Let us, for the whole proof, use the abbreviation $\tilde a(t,p) := a(x_0, t, p)$. We define a standard smooth, radial, nonnegative mollifier $\phi : \mathbb{R}^n \to \mathbb{R}$, such that $\phi \in \mathcal{C}_c^\infty(B_1)$, $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$ and impose the additional condition

$$\int_{B_1 \setminus B_{1/2}} \phi(\xi) \, d\xi \ge \frac{1}{1000},$$

which is a technical condition needed for this kind of approximation procedures (see also [24, 16]). For $k \in \mathbb{N}$ we set $\phi_k(\xi) := k^n \phi(k\xi)$ and define the smooth vector fields $\tilde a_k$ by convolution

$$\tilde a_k(t,p) := \big(\tilde a(t,\cdot) * \phi_k\big)(p) := \int_{B_1(0)} \tilde a(t, p + k^{-1}y)\phi_k(y) \, dy.$$

Proceeding analogously to [16, Lemma 3.1] and having in mind (1.2), defining $s_k := s + k^{-1}$, we find that the smoothened vector fields satisfy the following structure conditions

$$\begin{cases} |\tilde a_k(t,p)| \left(s_k^2 + |p|^2\right)^{-1/2} + |D_p \tilde a_k(t,p)| \le \tilde c, \\ \tilde c^{-1} |\lambda|^2 \le \big\langle D_p \tilde a_k(t,p)\lambda, \lambda \big\rangle, \\ |\tilde a(t,p) - \tilde a_k(t,p)| \le \tilde c k^{-1}, \end{cases} \tag{7.3}$$

for all $p, \lambda \in \mathbb{R}^n$, $t \in (-T, 0)$, with a constant $\tilde c \equiv \tilde c(n, L/\nu)$. Moreover each vector field $\tilde a_k$ satisfies the assumptions (1.2) with $s$ replaced by $s_k$, for different growth and ellipticity constants $\tilde \nu$, $\tilde L$ but still depending on the original ones and independent of $k$. Therefore the Dirichlet problem

$$\begin{cases} \partial_t v_k - \operatorname{div} \tilde a_k(t, Dv_k) = 0 & \text{in } Q_{\varrho/4}(z_0), \\ v_k = v_0 & \text{on } \partial_{\mathcal{P}} Q_{\varrho/4}(z_0). \end{cases} \tag{7.4}$$

has a unique solution $v_k \in v_0 + L^2(I_{\varrho/4}; W_0^{1,2}(B_{\varrho/4}))$.

*2nd step: Estimates for the regularized problems.* We start with the estimate corresponding to (7.1) for the **second spatial derivatives**. By Nash-Moser's theory (see [15]) we conclude that $v_k \in L^2_{\mathrm{loc}}(I_{\varrho/4}; W^{2,2}_{\mathrm{loc}}(B_{\varrho/4}))$; moreover $w_k := D_i v_k$ for $i \in \{1,\dots,n\}$ belongs to $C^0_{\mathrm{loc}}(I_{\varrho/4}; W^{1,2}_{\mathrm{loc}}(B_{\varrho/4}))$ and is a weak solution of the differentiated equation

$$\partial_t w_k - \operatorname{div}\,(\bar a_k(x,t) Dw_k) = 0, \tag{7.5}$$

with $\bar a_k(x,t) := D_p \tilde a_k(t, Dv_k(x,t))$. Furthermore $\bar a_k(x,t)$ has measurable entries and by (7.3) is elliptic and bounded by a constant which does not depend on $k$, i.e.

$$\tilde c^{-1}|\lambda|^2 \le \big\langle \bar a_k(x,t)\lambda, \lambda\big\rangle, \qquad \big|\bar a_k(x,t)\big| \le \tilde c,$$

for every $(x,t) \in Q_{\varrho/4}(z_0)$ and all $\lambda \in \mathbb{R}^n$, where $\tilde c \equiv \tilde c(n, L/\nu)$ is the constant from (7.3). Thus, [8, Lemma 2.10] provides for any $\eta_i \in \mathbb{R}$ the estimate

$$\int_{Q_{\varrho/16}} |DD_i v_k|^2\,dz \le \frac{c}{\varrho^2}\int_{Q_{\varrho/8}} |D_i v_k - \eta_i|^2\,dz,$$

with $c = c(n, L/\nu)$. Since $D_i v_k - \eta_i$ is a solution to (7.5), we can apply the higher integrability Lemma 6.1 and Remark 6.3, which hold – with $s = 0$ – also for equations like (7.5) (see Remark 6.2), getting

$$\left[\fint_{Q_{\varrho/16}} |DD_i v_k|^2\,dz\right]^{1/2} \le c\,\varrho^{-1}\fint_{Q_{\varrho/4}} |D_i v_k - \eta_i|\,dz. \tag{7.6}$$

To prove the existence of the **fractional time derivative** of $Dv_0$ we argue as follows: Taking the approximated problem (7.4) and having in mind that $w_k = D_i v_k$ solves the linear equation (7.5) in $Q_{\varrho/8}$, write the Steklov formulation of (7.5) at "level" $h$ (we consider only the case $h > 0$, the $h < 0$ one is very similar), noting that $\tau_h w_k = h\,\partial_t [w_k]_h$:

$$\int_{B_{\varrho/4}} \frac{\tau_h w_k}{h}\,\varphi + \langle [\bar a_k(x,t) Dw_k]_h, D\varphi\rangle\,dx = 0 \qquad \varphi \in W^{1,2}_0(B_{\varrho/8}).$$

Choosing as testing function $\varphi(x,t) := \xi^2(x)\tau_h w_k$, where $\xi \in C^\infty_c(B_{\varrho/8})$ denotes a cut-off function, $0 \le \xi \le 1$, $\xi \equiv 1$ on $B_{\varrho/32}$ and $\xi \equiv 0$ outside $B_{\varrho/16}$, with $|D\xi| \le c/\varrho$ and integrating with respect to time over $I_{\varrho/32}$, we deduce

$$\int_{I_{\varrho/32}}\int_{B_{\varrho/8}} \frac{|\tau_h w_k|^2}{h}\xi^2\,dx\,dt = -\int_{I_{\varrho/32}}\int_{B_{\varrho/8}} \big\langle [\bar a_k Dw_k]_h, D(\xi^2 \tau_h w_k)\big\rangle\,dx\,dt.$$

Now we take into account $(7.3)_1$, apply Young's inequality and use $|D\xi| \le c/\varrho$ to arrive at

$$\begin{aligned}\int_{I_{\varrho/32}}\int_{B_{\varrho/8}} \frac{|\tau_h w_k|^2}{h}\xi^2\,dx\,dt &\le \varepsilon \int_{I_{\varrho/32}}\int_{B_{\varrho/16}} \frac{|\tau_h w_k|^2}{\varrho^2}\xi^2\,dx\,dt\\ &\quad + \frac{\tilde c}{4\varepsilon}\int_{I_{\varrho/32}}\int_{B_{\varrho/16}} \Big[|Dw_k|^2 + \eta^2|\tau_h Dw_k|^2\Big]\,dx\,dt.\end{aligned}$$

Finally, estimating $|\tau_h Dw_k|^2 \le 2(|Dw_k(x,t)|^2 + |Dw_k(x,t+h)|^2)$ and exploiting that $h \le (\varrho/32)^2$ we may choose $\varepsilon = \frac{1}{2\cdot 32^2}$ to absorb the first term of the right–hand side on the left and conclude

$$\int_{Q_{\varrho/32}} \frac{|\tau_h D_i v_k|^2}{h}\,dz \le c\int_{Q_{\varrho/16}} |DD_i v_k|^2\,dz.$$

At this point we may exploit estimate (7.6) which we already derived before to achieve

$$\fint_{Q_{\varrho/32}} \frac{|\tau_h D_i v_k|^2}{h}\,dz \le c\,\varrho^{-1}\fint_{Q_{\varrho/4}} |D_i v_k - \eta_i|\,dz. \tag{7.7}$$

*3rd step: Passing to the limit.* We now prove the strong $L^2$-convergence of $\{Dv_k\}_k$. Since both $v_k$ and $v_0$ are solutions and coincide on the parabolic boundary, arguing analogously to Lemma 6.5, taking $(1.2)_1$ adapted for $\tilde a_k$, subsequently Young's inequality we achieve

$$\begin{aligned}\tilde{\nu}\int_{Q_{\varrho/4}}|Dv_k-Dv_0|^2\,dz &\le \int_{Q_{\varrho/4}}\big\langle \tilde{a}_k(t,Dv_k)-\tilde{a}_k(t,Dv_0),Dv_k-Dv_0\big\rangle\,dz\\ &\le \frac{\tilde{\nu}}{2}\int_{Q_{\varrho/4}}|Dv_k-Dv_0|^2\,dz\\ &\qquad + c\int_{Q_{\varrho/4}}\big|\tilde{a}(t,Dv_0)-\tilde{a}_k(t,Dv_0)\big|^2\,dz\end{aligned}$$

where $Q_{\varrho/4}\equiv Q_{\varrho/4}(z_0)$; hence absorbing the first term of the right–hand side on the left one, and noting that by $(7.3)_3$ the second integral on the right–hand side goes to zero as $k\to\infty$, we immediately deduce that $Dv_k\longrightarrow Dv_0$ strongly in $L^2(Q_{\varrho/4};\mathbb{R}^n)$ and also in $L^1(Q_{\varrho/4};\mathbb{R}^n)$. In consequence, using the strong convergence for the right–hand side of the inequalities (7.6) and (7.7) and lower semicontinuity for the left–hand sides, we may pass to the limit $k\to\infty$ and obtain both estimates for the limit function $v_0$. Summing over $i=1,\dots,n$ finally provides the desired inequalities (7.1) and (7.2). □

## 8. Proof of the main theorem

In this section we will take use of the previous Lemmata to construct the proof of Theorem 1.2. First, we recall the definition of $\delta$ in (2.1) and we define

$$\gamma(\kappa):=\frac{\delta}{\delta+1-\kappa}\qquad\text{for every }\kappa\in[0,\delta+1). \tag{8.1}$$

The strategy of the proof is now the following: In a first step, by comparison techniques, we show initial fractional differentiability of $Du$, i.e.

$$Du\in W^{\tilde{\kappa},\tilde{\kappa}/2;q}_{\text{loc}}(\Omega_T)\qquad\text{for some }\tilde{\kappa}>0,$$

(see (8.9) for $\gamma(0)=\delta/(\delta+1)$). This is the starting point of an iteration procedure: Once having fractional estimates to some quantified exponent (coupled with an explicit local estimate), one may exploit this information in order to increase the amount of differentiability in space and time. Thus, this procedure can be iterated to finally prove the desired result. Let us mention that for the whole proof, we argue on the finite differences of step $h$ in space and step $h^2$ in time, whereas the estimates are established on cylinders $Q$ of "radius" $|h|^\beta$. Thus, the step size of the finite differences is linked to the size of the radii of appearing parabolic cylinders.

8.1. **Uniform fractional estimates.** Let us first fix a notation: for subsets $A\subset\Omega$ and $J\subset(-T,0)$, with $C:=A\times J$, we denote with $\lambda_0[C]$ the quantity

$$\lambda_0[C]:=\|s+|Du|\|_{L^q(C)}+\|f\|_{L^1(C)}. \tag{8.2}$$

Moreover, for a cylinder $Q\equiv Q_\varrho(z_0)$ with $32Q\Subset\Omega_T$, let $v$ be the solution of the homogeneous problem (6.1) on the cylinder $32Q$ and $v_0$ the solution of the frozen homogeneous problem (6.2) on the cylinder $8Q$. Later in this chapter, $Q$ will be a cylinder of radius $\varrho\equiv|h|^\beta$ (see the definition in (8.14)), where $h\in\mathbb{R}$ denotes the step size of the finite differences in space and time. However, for the first Lemma, we leave step size and radius uncoupled.

Let us first recall the definitions of the finite difference operator of step $\xi\in\mathbb{R}$ in space

$$[\tau_{i,\xi}f](x,t):=f(x+\xi e_i,t)-f(x,t), \tag{8.3}$$

for $i\in\{1,\dots,n\}$ with $e_i$ denoting the unit vector in direction $i$, as well as the finite difference operator of step $\xi^2$ in time

$$[\tau_{\xi^2}f](x,t):=f(x,t+\text{sign}(\xi)\xi^2)-f(x), \tag{8.4}$$

both for $|\xi|$ small enough to assure that the expressions are well defined.

**Lemma 8.1.** *There exists a constant $c \equiv c(n, L/\nu, q)$ such that for any $\xi \in \mathbb{R}$ with $|\xi| \le \varrho$ and for any $\eta \in \mathbb{R}^n$ the following estimate holds true:*

$$\|\tau_{\xi^2} Du\|_{L^q(Q)} + \sum_{i=1}^{n} \|\tau_{i,\xi} Du\|_{L^q(Q)} \le c\,\varrho^{\delta(q)}\lambda_0[32Q] + c\,\varrho^{\frac{n+2}{q}-1}|\xi| \fint_{8Q} |Du-\eta|\,dz.$$

**Proof.** For the **finite difference operator in space** we argue as follows: For $i = 1, \dots, n$, keeping in mind that $|\xi| \le \varrho$, we obtain

$$\begin{aligned}
\|\tau_{i,\xi} Du\|_{L^q(Q)} &\le \|\tau_{i,\xi} Dv_0\|_{L^q(Q)} + \|Du - Dv\|_{L^q(Q)}\,dz + \|Dv - Dv_0\|_{L^q(Q)} \\
&\qquad + \Big(\int_Q |Du(x+\xi e_i, t) - Dv(x+\xi e_i, t)|^q\,dx\;dt\Big)^{1/q} \\
&\qquad + \Big(\int_Q |Dv(x+\xi e_i, t) - Dv_0(x+\xi e_i, t)|^q\,dx\;dt\Big)^{1/q} \\
&\le I + II + III,
\end{aligned}$$

where we define

$$\begin{aligned}
I &:= \|\tau_{i,\xi} Dv_0\|_{L^q(Q)}, \\
II &:= \|Du - Dv\|_{L^q(2Q)}, \\
III &:= \|Dv - Dv_0\|_{L^q(2Q)}.
\end{aligned}$$

Using Lemma 6.4 we **estimate $II$**:

$$II \le \|Du - Dv\|_{L^q(8Q)} \le c(n,\nu,q)\,\varrho^{\delta(q)}\|f\|_{L_1(8Q)}.$$

Secondly, we **estimate $III$** in the following way: using Lemma 6.5 and the estimate for $II$ we established before, always having in mind $|\xi| \le \varrho \le 1$, we deduce

$$\begin{aligned}
III &\le c\,\varrho^{\delta(q)} \int_{8Q} \big(s + |Dv|\big)\,dz \\
&\le c\,\varrho\Big[\|Du - Dv\|_{L^q(8Q)} + \|s + |Du|\|_{L^q(8Q)}\Big] \\
&\le c\,\varrho\Big[\|s + |Du|\|_{L^q(8Q)} + \|f\|_{L^1(8Q)}\Big]
\end{aligned}$$

where $c = c(n,\nu,L,q)$. Hence, summarizing the estimates for $II$ and $III$, taking into account $\delta \le 1$, we get

$$\begin{aligned}
II + III &\le c\,\big(\varrho + \varrho^{\delta(q)}\big)\Big[\|s + |Du|\|_{L^q(8Q)} + \|f\|_{L^1(8Q)}\Big] \\
&\le c\,\varrho^{\delta}\Big[\|s + |Du|\|_{L^q(8Q)} + \|f\|_{L^1(8Q)}\Big] \\
&= c\,\varrho^{\delta}\lambda_0[8Q],
\end{aligned}$$

with a constant depending on $n, \nu, L, q$.

To **estimate $I$**, we take use of Lemma 7.1. First, noting that $Dv_0(\cdot,t) \in W^{1,2}(B)$ for a.e. $t$, elementary properties of Sobolev functions together with $|\xi| \le \varrho$ provide that

$$\int_B |\tau_{i,\xi} Dv_0(\cdot,t)|^2\,dx \le c(n)\,|\xi|^2 \int_{2B} |D^2 v_0(\cdot,t)|^2\,dx.$$

Secondly, applying Lemma 7.1, equation (7.1) with $Q_{\varrho/16} \equiv 2Q$ we obtain

$$\Big[\int_{2Q} |D^2 v_0|^2\,dz\Big]^{1/2} \le c(n, L/\nu)\,\varrho^{\frac{n}{2}} \fint_{8Q} |Dv_0 - \eta|\,dz.$$

Merging the second last estimate (integrated with respect to time) and the last one, using twice Hölder's inequality, we therefore conclude

$$
\begin{aligned}
I=\Big(\fint_Q |\tau_{i,\xi} Dv_0|^q\,dz\Big)^{1/q} &\le c\,\varrho^{\frac{n+2}{q}(1-\frac{q}{2})}\Big[\fint_Q |\tau_{i,h}Dv_0|^2\,dz\Big]^{1/2}\\
&\le c\,\varrho^{\frac{n+2}{q}(1-\frac{q}{2})}|\xi|\Big[\fint_{2Q} \big|D^2 v_0\big|^2\,dz\Big]^{1/2}\\
&\le c\,\varrho^{\frac{n+2}{q}-1}|\xi|\fint_{8Q}|Dv_0-\eta|\,dz
\end{aligned}
$$

for any $\eta\in\mathbb{R}^n$, with a constant $c\equiv c(n,L/\nu,q)$. For the last term in the preceding inequality, we write, using again Hölder's inequality:

$$
\begin{aligned}
\fint_{8Q}|Dv_0-\eta|\,dz &\le \fint_{8Q}|Dv_0-Du|\,dz+\fint_{8Q}|Du-\eta|\,dz\\
&\le c\,\varrho^{-\frac{n+2}{q}}\|Dv_0-Du\|_{L^q(8Q)}+\fint_{8Q}|Du-\eta|\,dz\\
&\le c\,\varrho^{-\frac{n+2}{q}}\big[\widetilde{II}+\widetilde{III}\big]+\fint_{8Q}|Du-\eta|\,dz \qquad (8.5)
\end{aligned}
$$

with the definitions

$$
\widetilde{II}:=\|Du-Dv\|_{L^q(8Q)},\qquad \text{and }\ \widetilde{III}:=\|Dv-Dv_0\|_{L^q(8Q)}.
$$

Note that the quatities $\widetilde{II}$ and $\widetilde{III}$ similar to the expressions $II$ and $III$ which we defined before, just being integrated over the cylinder $8Q$ instead of $2Q$. However, the same argumentation which lead to the estimate of $II+III$ also applies here and gives

$$
\widetilde{II}+\widetilde{III}\le c\,\varrho^{\delta(q)}\lambda_0[32Q].
$$

Merging this estimate with the one before, which gives an estimate for $I$, combining this with the estimate we established for $II+III$, and having in mind that $|\xi|\le\varrho$, we finally conclude

$$
\|\tau_{i,\xi}Du\|_{L^q(Q)}\le c\,\varrho^{\frac{n+2}{q}-1}|\xi|\fint_{8Q}|Du-\eta|\,dz+c\,\varrho^{\delta}\lambda_0[32Q]. \qquad (8.6)
$$

Let us now have a look at the **finite difference operator in time**. We argue analogously, first writing

$$
\big\|\tau_{\xi^2}Du\big\|_{L^q(Q)}\le \tilde I+II+III,
$$

where we define

$$
\tilde I:=\big\|\tau_{\xi^2}Dv_0\big\|_{L^q(Q)},
$$

and $II$, $III$ are exactly as before. Consequently, it remains here to **estimate the quantity $\tilde{I}$**. We use Hölder's inequality, subsequently Lemma 7.1, estimate (7.2) with $h$ replaced by $\operatorname{sign}(\xi)\xi^2$, and Lemma 6.6 to conclude

$$
\tilde I\le c\,\varrho^{\frac{n+2}{q}}\Big[\fint_Q|\tau_{\xi^2}Dv_0|^2\,dz\Big]^{1/2}\le c\,\varrho^{\frac{n+2}{q}-1}|\xi|\fint_{8Q}|Dv_0-\eta|\,dz,
$$

with $c\equiv c(n,L/\nu,r,q)$. To replace $Dv_0$ in the last integral of the preceding estimate, we proceed again as in (8.5).

We conclude the proof of the Lemma by merging together the estimates for $\tilde I$, $II$ and $III$ with (8.6). □

The following Proposition is the key to the proof of Theorem 1.2. For the seek of brevity, we define for sets $C:=A\times J$ with subsets $A\subset\Omega$, $J\subset(-T,0)$ the mapping

$$
\lambda_\kappa[C]:=\lambda_0[C]+\chi(\kappa)[Du]_{W^{\kappa,\kappa/2;q}(C)}, \qquad (8.7)
$$

where $\chi(\kappa) = 0$, if $\kappa = 0$, and $\chi(\kappa) = 1$, whenever $\kappa > 0$; $\lambda_0$ is the function defined in (8.2). Note that $\lambda_\kappa$ is a true extension of $\lambda_0$. Let's also use the following notation, regarding the sets mentioned in the statement of the Proposition: for $i = 1, 2$ we denote

$$\Omega_{T,i} := \Omega_i \times J_i, \qquad {\Omega_T}' := \Omega' \times J' \qquad \text{and naturally} \qquad {\Omega_T}'' := \Omega'' \times J'',$$

and we recall the meaning of the compact inclusion for a product set.

Our aim is to prove the following estimates for the finite differences of step $h$, $h^2$ respectively, in space and time:

**Proposition 8.2.** *Let $u \in L^2(-T, 0; W_0^{1,2}(\Omega))$ be the unique weak solution to (3.2), under the assumptions (1.2) with $n \geq 2$ and let $q$ be as in (1.4). Assume that for some $\kappa \in [0, \delta)$, where $\delta$ is defined in (2.1), and that for any couple of subsets ${\Omega_T}' \Subset {\Omega_T}'' \Subset \Omega_T$, there exists a constant $c_1$ such that the estimate*

$$[Du]_{W^{\kappa,\kappa/2;q}({\Omega_T}')} \leq c_1 \lambda_0[{\Omega_T}''] \tag{8.8}$$

*holds true. Then*

$$Du \in W^{\tilde\kappa,\tilde\kappa/2;q}_{\text{loc}}(\Omega_T) \qquad \text{for all } \tilde\kappa \in [0, \gamma(\kappa)), \tag{8.9}$$

*where $\gamma(\cdot)$ is the function defined in (8.1). Moreover, for every couple of subsets $\Omega_{T,1} \Subset \Omega_{T,2} \Subset \Omega_T$ the following statements hold:*

(i) *There exists a constant $\mathcal{D} \in (0, 1)$ depending on $\delta - \kappa$, $\operatorname{dist}(\Omega_1, \partial\Omega_2)$, $\operatorname{dist}(J_1, \partial J_2)$ and a constant $c_2$ depending on $\mathcal{D}, c_1, n, L/\nu, q$ such that for any $0 < |h| < \mathcal{D}$ there holds*

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,1})} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(\Omega_{T,1})} \leq c_2 \, |h|^{\gamma(\kappa)} \lambda_0[\Omega_{T,2}]; \tag{8.10}$$

(ii) *There exists a constant $\tilde c_1$ depending on $c_1, n, q$, $\delta - \gamma(\kappa)$, $\gamma(\kappa) - \tilde\kappa$, $\operatorname{dist}(\Omega_2, \partial\Omega)$, $\operatorname{dist}(\Omega_1, \partial\Omega_2)$, $\operatorname{dist}(J_1, \partial J_2)$, $\operatorname{dist}(J_1, \partial J_2)$ such that*

$$[Du]_{W^{\tilde\kappa,\tilde\kappa/2,q}(\Omega_{T,1})} \leq \tilde c_1 \lambda_0[\Omega_{T,2}]. \tag{8.11}$$

**Proof.** *Step 1: Choice of suitable parabolic cylinders.* Let us take a parabolic cylinder $Q \equiv Q_R(z_0) \Subset \Omega_T$ of radius $R$ and center $z_0 = (x_0, t_0)$. We denote by $\mathcal{Q}_R$ the cuboid of the form

$$\mathcal{Q}_R(z_0) := \left\{(x,t) \in \mathbb{R}^{n+1} : \max\left\{\max_j \frac{|x_j - (x_0)_j|}{\sqrt{n}}, \sqrt{t - t_0}\right\} < R\right\},$$

which is the largest cuboid centered in $z_0 = (x_0, t_0)$ and contained in $Q_R$. Therefore we denote this cuboid also by $\mathcal{Q}_{\text{inn}} \equiv \mathcal{Q}_{\text{inn}}(Q)$. Analogously we denote by $\mathcal{Q}_{\text{out}} \equiv \mathcal{Q}_{\text{out}}(Q)$ the smallest cuboid containing $Q$. Denoting by $\hat Q \equiv 32Q$ the enlarged cylinder $\hat Q$, we denote $\mathcal{Q}_{\text{inn}} \equiv \mathcal{Q}_{\text{inn}}(Q)$ and $\hat{\mathcal{Q}}_{\text{out}} \equiv \mathcal{Q}_{\text{out}}(\hat Q)$ and finally have the following inclusions:

$$\mathcal{Q}_{\text{inn}} \subset Q \Subset 2Q \Subset 32Q = \hat Q \subset \hat{\mathcal{Q}}_{\text{out}}. \tag{8.12}$$

Now we fix arbitrary open sets $\Omega_{T,1} \Subset \Omega_{T,2} \Subset \Omega_T$, and find an intermediate subset $\Omega_{T,3} = \Omega_3 \times J_3$ such that $\Omega_{T,1} \Subset \Omega_{T,3} \Subset \Omega_{T,2}$. It is easy to see that

$$\begin{aligned}\Omega_{T,3} := \{z = (x,t) \in \Omega_{T,2} :& \operatorname{dist}(x, \partial\Omega_2) > \operatorname{dist}(\partial\Omega_2, \Omega_1)/2, \\ & \operatorname{dist}(t, \partial J_2) > \operatorname{dist}(\partial J_2, J_1)/2\}\end{aligned}$$

is an appropriate choice. Take $\beta \in (0, 1)$ to be chosen later, and let $h \in \mathbb{R}$ be a real number satisfying

$$0 < |h| < \min\left\{\left(\frac{\operatorname{dist}(\Omega_1, \partial\Omega_3)}{100\sqrt{n}}\right)^{\frac{1}{\beta}}, \left(\frac{\sqrt{\operatorname{dist}(J_1, \partial J_3)}}{100}\right)^{\frac{1}{\beta}}, 1\right\} =: \mathcal{D}. \tag{8.13}$$

We take $z_0 \in \Omega_{T,1}$ and fix a cylinder of radius $|h|^\beta$, i.e.

$$Q := Q(h) := Q_{|h|^\beta}(z_0) = B_{|h|^\beta}(x_0) \times (t_0 - |h|^{2\beta}, t_0 + |h|^{2\beta}). \tag{8.14}$$

Let us recall that for $\alpha > 0$ we write

$$\alpha Q := B_{\alpha|h|^\beta}(x_0) \times (t_0 - \alpha^2|h|^{2\beta}, t_0 + \alpha^2|h|^{2\beta}).$$

Note that by condition (8.13) we have that $\hat{\mathcal{Q}}_{\text{out}} \Subset \Omega_{T,3}$ and since $\beta \in (0,1)$ we moreover have $|h| \le |h|^\beta$.

Finally, let $v$ and $v_0$ respectively be the solutions of (6.1) and (6.2) with $\varrho = 32|h|^\beta$, which means that $v$ solves (6.1) on the cylinder $32Q \equiv Q_{32|h|^\beta}(z_0)$, whereas $v_0$ solves (6.2) on $8Q \equiv Q_{8|h|^\beta}(z_0)$.

*Step 2: Estimates on certain parabolic cylinders:* We start by Lemma 8.1, which we apply with $\varrho = |h|^\beta$ and $\xi = h$, to deduce

$$\begin{aligned}\|\tau_{h^2} Du\|_{L^q(Q)} &+ \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(Q)} \\ &\le c\,|h|^{\beta\delta}\lambda_0[32Q] + c\,|h|^{\beta[\frac{n+2}{q}-1]+1} \fint_{8Q} |Du - \eta|\, dz,\end{aligned} \tag{8.15}$$

with $c \equiv c(n, L/\nu, q)$ and where we recall the definition of $\lambda_0[C]$ in (8.2). Let us now distinguish two cases: In **case of $\boldsymbol{\kappa = 0}$** we choose $\eta \equiv 0$ and obtain by Hölder's inequality

$$\fint_{8Q} |Du|\, dz \le c(n,q)|h|^{-\beta\frac{n+2}{q}}\|Du\|_{L^q(8Q)},$$

and therefore

$$\|\tau_{h^2} Du\|_{L^q(Q)} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(Q)} \le c\left[|h|^{\beta\delta} + |h|^{(1-\beta)}\right]\lambda_0[32Q],$$

with a constant $c \equiv c(n, L/\nu, q)$. In **case of $\boldsymbol{\kappa > 0}$** we choose $\eta \equiv (Du)_{8Q}$ and apply the fractional Poincaré inequality in terms of Lemma 4.6 to deduce

$$\fint_{8Q} |Du - (Du)_{8Q}|\, dz \le c\,|h|^{\beta(\kappa - \frac{n+2}{q})}[Du]_{W^{\kappa,\kappa/2;q}(8Q)},$$

with $c \equiv c(n,q)$ and thus, merging this with (8.15), and having in mind the definition of $\lambda_\kappa$ in (8.7), we arrive at

$$\|\tau_{h^2} Du\|_{L^q(Q)} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(Q)} \le c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right]\lambda_\kappa[32Q],$$

for a constant depending on $n, L/\nu$ and $q$.

*Step 3: Covering argument:* Recalling the choice of the involved cylinders in (8.12), i.e. $\mathcal{Q}_{\text{inn}} \equiv \mathcal{Q}_{\text{inn}}(Q) \subset Q$ and $32Q \equiv \hat{Q} \subset \mathcal{Q}_{\text{out}}(\hat{Q}) \equiv \hat{\mathcal{Q}}_{\text{out}}$, we immediately have

$$\|\tau_{h^2} Du\|_{L^q(\mathcal{Q}_{\text{inn}})} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(\mathcal{Q}_{\text{inn}})} \le c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right]\lambda_\kappa[\hat{\mathcal{Q}}_{\text{out}}], \tag{8.16}$$

with a constant $c \equiv c(n, L/\nu, q)$.

Let's now observe that, even if the set function defined in (8.7) is not a measure – due to the presence of the term $[Du]_{W^{\kappa,\kappa/2;q}}$ – it is nevertheless countably super-additive, that is

$$\sum_j \lambda_\kappa[C_j] \le \lambda_\kappa\Big[\bigcup_j C_j\Big],$$

whenever $\{C_j\}$ is a countable family of mutually disjoint subsets. The covering argument is now the following: First, we recall that the sets $\mathcal{Q}$ involved here are cuboids with sides parallel to the coordinate axis. Then, for each $h \in \mathbb{R}$, satisfying the smallness condition

(8.13) we can find cylinders $Q_1 \equiv Q(z_1, |h|^\beta)$, ..., $Q_m \equiv Q(z_m, |h|^\beta)$ of the type considered in (8.14) such that the corresponding inner cuboids $\mathcal{Q}_{\rm inn}(Q_1)$, ..., $\mathcal{Q}_{\rm inn}(Q_m)$ are disjoint and cover $\Omega_{T,1}$ up to a negligible set, i.e.

$$\mathcal{L}^{n+1}\left(\Omega_{T,1} \setminus \cup \mathcal{Q}_{\rm inn}(Q_j)\right) = 0, \quad \mathcal{Q}_{\rm inn}(Q_k) \cap \mathcal{Q}_{\rm inn}(Q_j) = \emptyset \text{ for } k \neq j. \tag{8.17}$$

Precisely we proceed as follows: for the two sets $\Omega_{T,1}$ and $\Omega_{T,3}$, we first take cuboids $\{\mathcal{Q}_j\}$, all centered in $\Omega_{T,1}$, with sides parallel to the coordinate axes and side length comparable to $|h|^\beta$ in order to obtain (8.17). Then we see them as inner cuboids of the cylinders $Q(z_j, |h|^\beta)$, according to (8.12). Now, we sum up the inequalities (8.16) for $j \leq m$ and obtain

$$\sum_{j=1}^m \left[\|\tau_{h^2} Du\|_{L^q(\mathcal{Q}_{\rm inn}(Q_j))} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(\mathcal{Q}_{\rm inn}(Q_j))}\right] \leq c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right] \sum_{j=1}^m \lambda_\kappa[\mathcal{Q}_{\rm out}(\hat{Q}_j)]. \tag{8.18}$$

By construction, and in particular by (8.13) we have that $\mathcal{Q}_{\rm out}(\hat{Q}_j) \subset \Omega_{T,3}$ for any $j \leq m$. Moreover, each of the dilated cuboids $\mathcal{Q}_{\rm out}(\hat{Q}_k)$ intersects the similar ones $\mathcal{Q}_{\rm out}(\hat{Q}_j)$ less than $2^{n+1} \cdot 128^{n+2} = 2^{8n+15}$ times. Therefore, using these facts, i.e. (8.17), (8.18) together with the countably super-additivity of the set-function $\lambda_\kappa$ we end up with

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,1})} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(\Omega_{T,1})} \leq c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right] \lambda_\kappa[\Omega_{T,3}].$$

In a next step, we determine $\beta$ in order to minimize the right–hand side of the preceding inequalities with respect to $|h|$. I.e. we choose $\beta$ in such a way that $1 - \beta + \beta\kappa = \beta\delta$, that is $\beta = \gamma(\kappa)/\delta$, where we recall the definition of $\gamma(\kappa)$ in (8.1). Note at this point, that since $\kappa < \delta$ implies $\gamma(\kappa)/\delta < 1$, this choice of $\beta \in (0,1)$ is admissible. Therefore, for $h$ satisfying (8.13), the preceding estimate becomes

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,1})} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(\Omega_{T,1})} \leq c_0\, |h|^{\gamma(\kappa)} \lambda_\kappa[\Omega_{T,3}], \tag{8.19}$$

with a constant $c_0 \equiv c_0(n, L/\nu, q)$.

Now we are at the point to conclude the assertions of Proposition 8.2. First, we prove (8.10): In the case $\kappa = 0$, we have directly

$$\lambda_\kappa[\Omega_{T,3}] = \lambda_0[\Omega_{T,3}] \leq \lambda_0[\Omega_{T,2}],$$

whereas in the case $\kappa > 0$, we take (8.8) with $\Omega_{T,3}$ as inner subset, $\Omega_{T,2}$ as outer one, and achieve

$$\begin{aligned}\lambda_\kappa[\Omega_{T,3}] &= \lambda_0[\Omega_{T,3}] + [Du]_{W^{\kappa,\kappa/2;q}(\Omega_{T,3})} \\ &\leq \lambda_0[\Omega_{T,3}] + c_1[\Omega_{T,2}] \leq (1 + c_1)\, \lambda_0[\Omega_{T,2}].\end{aligned}$$

Merging these two estimates with (8.19), we conclude (8.10) for $0 < |h| < \mathcal{D}$ with $c_2 := c_0(1 + c_1)$ .

Having (8.19) at hand, the proof of (8.11) and (8.9) is performed via the Corollary 4.5: We retrace the proof in the previous lines in order to get the finite differences on the set $\Omega_{T,3}$ estimated by $\lambda_0[\Omega_{T,2}]$, using a further intermediate set. We hence have

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,3})} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(\Omega_{T,3})} \leq (1 + c_1)\, |h|^{\gamma(\kappa)} \lambda_0[\Omega_{T,2}], \tag{8.20}$$

for every $0 < |h| < \mathcal{D}$. This estimate enables us to apply Corollary 4.5 with $\tilde{J} \equiv J_3$, $\tilde{\Omega} \equiv \Omega_3$, $\mathcal{O} \equiv \Omega_1$, $\mathcal{J} \equiv J_1$, $\bar{\theta}$ replaced by $\gamma(\kappa)$ and $S \equiv (1 + c_1)\lambda_0[\Omega_{T,2}]$ in order to obtain

$$
\begin{aligned}
[Du]_{W^{\tilde\kappa,\tilde\kappa/2;q}(\Omega_{T,1})} &\le \tilde c_1 \lambda_0[\Omega_{T,2}] \qquad\qquad\qquad \text{for all } \tilde\kappa \in [0,\gamma(\kappa))\\
&= \tilde c_1 \Big[\|s+|Du|\|_{L^q(\Omega_{T,2})} + \|f\|_{L^1(\Omega_{T,2})}\Big], \qquad (8.21)
\end{aligned}
$$

with $\tilde c_1$ depending on $c_1, n, q, \mathcal{D}, \gamma(\kappa)-\tilde\kappa, \operatorname{dist}(\Omega_2,\partial\Omega), \operatorname{dist}(\Omega_1,\partial\Omega_2)$, $\operatorname{dist}(J_1,\partial J_2)$, $\operatorname{dist}(J_1,\partial J_2)$ so that, since all our subsets are arbitrary,

$$
Du \in W^{\tilde\kappa,\tilde\kappa/2;q}_{\mathrm{loc}}(\Omega_T) \qquad \text{for all } \tilde\kappa \in [0,\gamma(\kappa)).
$$

□

The main Theorem 1.2 is now proved for the approximate sequence by an iteration argument:

**Proposition 8.3** (Iteration). *Let $u \in L^2(-T,0;W^{1,2}_0(\Omega))$ the (unique) solution to (3.2) under the assumptions (1.2) and let $q$ satisfies (1.4). Then*

$$
Du \in W^{\kappa,\kappa/2;q}_{\mathrm{loc}}(\Omega_T) \qquad \textit{for every } \kappa \in [0,\delta) \tag{8.22}
$$

*where $\delta$ is as in (2.1). Furthermore, for every couple of subsets $\Omega_{T,1} \Subset \Omega_{T,2} \Subset \Omega_T$ there exists a constant $c$ depending only on $n, \nu, L, q$, $\delta-\kappa$, $\operatorname{dist}(\Omega_1,\partial\Omega)$, $\operatorname{dist}(\Omega_2,\partial\Omega_1)$, $\operatorname{dist}(J_1,\partial J_2)$, $\operatorname{dist}(J_1,\partial J_2)$ such that*

$$
[Du]_{W^{\kappa,\kappa/2;q}(\Omega_{T,1})} \le c\Big[\|s+|Du|\|_{L^q(\Omega_{T,2})} + \|f\|_{L^1(\Omega_{T,2})}\Big]. \tag{8.23}
$$

**Proof.** The proof of this lemma follows essentially the lines of the one in [24, Lemma 6.3]. However, for the convenience of the reader we sketch at least the argumentation: The function $\gamma(\cdot)$ in (8.1) is easily seen to be non-decreasing and to satisfy

$$
\kappa \in (0,\delta) \Rightarrow \gamma(\kappa) \in (\kappa,\delta) \qquad \text{and} \qquad \gamma(\delta)=\delta. \tag{8.24}
$$

Let's define by induction the two sequences $\{\ell_k\}$ and $\{\kappa_k\}$ as follows:

$$
\ell_1 := \frac{\delta}{4(\delta+1)},\ \kappa_1 := \frac{\delta}{2(\delta+1)},\ \ell_{k+1} := \gamma(\ell_k),\ \kappa_{k+1} := \frac{\gamma(\kappa_k)+\gamma(\ell_k)}{2}.
$$

From (8.24) it follows that $\ell_k \nearrow \delta$; since $\gamma(\cdot)$ is increasing we have $\ell_k < \kappa_k < \delta$, hence also $\kappa_k \nearrow \delta$. Applying in a first step Proposition 8.2 with $\kappa = 0$, we get that $Du \in W^{\tilde\kappa,\tilde\kappa/2;q}_{\mathrm{loc}}(\Omega_T)$, with corresponding estimates of the type (8.11) for $\tilde\kappa$, for any $\tilde\kappa \in [0,\gamma(0))$, where $\gamma(0) = \delta/(\delta+1)$. Since $\gamma(\cdot)$ is increasing, we have in particular that $Du \in W^{\kappa_1,\kappa_1/2;q}_{\mathrm{loc}}(\Omega_T)$, with corresponding estimate of the type (8.10) and (8.11) for $\tilde\kappa \equiv \kappa_1$. Having once at hand the estimates on level $\kappa_k$, we once again apply Proposition 8.2 with $\kappa=\kappa_k$ and we get that $Du \in W^{\tilde\kappa,\tilde\kappa/2;q}_{\mathrm{loc}}(\Omega_T)$ for all $\tilde\kappa < \gamma(\kappa_k)$ and in particular, since $\gamma(\cdot)$ is increasing and thus $\ell_k < \kappa_k$, we have $\kappa_{k+1} < \gamma(\kappa_k)$ and therefore also $Du \in W^{\kappa_{k+1},\kappa_{k+1}/2;q}_{\mathrm{loc}}(\Omega_T)$. Moreover, (8.23) holds for $\kappa = \kappa_{k+1}$. Then by induction we get both (8.22) and (8.23). □

**Remark 8.4.** *It can be proved in particular, exploiting the iterative process of the previous Proposition together with estimate (8.10), that*

$$
\|\tau_h Du\|_{L^q(\Omega_{T,1})} \le c\,|h|^{\kappa/2}\Big[\|s+|Du|\|_{L^q(\Omega_{T,2})} + \|f\|_{L^1(\Omega_{T,2})}\Big] \tag{8.25}
$$

*for every $\kappa \in [0,\delta/2)$ and $|h|$ small, with a constant depending essentially on $\delta$ and on the distance between $\Omega_{T,1}$ and the boundary of $\Omega_{T,2}$.*

*Proof of Theorem 1.2 and estimate* (1.10). Let's consider the approximation sequence $\{u_k\}$ built as solutions of (3.2) with data $f \equiv f_k$ as stated in Section 3. The strong convergence in $L^1(\Omega_T)$ of the sequence $u_k$ to $u$ can be deduced exactly as in [5], using the fact that from the equation $\partial_t u_k$ is uniformly bounded in $L^1(-T,0;W^{-1,1}(\Omega))$, and deducing the convergence by compactness arguments, see [28]. For the convergence of the gradients, our stronger estimates allow a simpler, indipendent proof. The global estimate in Lemma 5.1 applied to any $u_k$, together with (3.3), leads to

$$\|Du_k\|_{L^q(\Omega_T)} \le c\left[s + \|f_k\|_{L^1(\Omega_T)}\right] \le c\big[s + |\mu|(\Omega_T)\big], \tag{8.26}$$

which coupled with (8.23) and (8.25) gives the following two facts: for $J \Subset (-T,0)$, $\Omega_1 \Subset \Omega$, for every $\kappa < \delta$

$$\left(\int_J [Du_k(\cdot,t)]^q_{W^{\kappa,q}(\Omega_1)}\,dt\right)^{1/q} \le c\big[s + |\mu|(\Omega_T)\big]$$

and

$$\|\tau_h Du\|_{L^q(\Omega_1\times J)} \le c\,|h|^{\kappa/2}\big[s + |\mu|(\Omega_T)\big].$$

In particular, $\{Du_k\}$ is uniformly bounded in $L^1(J;W^{\kappa,q}(\Omega_1))$ and $\|\tau_h Du\|_{L^q(\Omega_1\times J)} \to 0$ as $h \to 0$ uniformly with respect to $k$. Hence we can apply the compactness result [28, Theorem 3] to deduce, after extracting a non relabeled subsequence, the convergence of $Du_k$ to $Du$ strongly in $L^1_{\rm loc}(\Omega_T)$ and almost everywhere. Note that we made the choice $X \equiv W^{\kappa,q}(\Omega_1)$ which is compactly (see [2]) embedded into $B \equiv L^q(\Omega_1)$.

Hence finally we can prove our theorem for the function $u$ which is, a SOLA. Indeed it is now easy to see, using Lipschitz continuity $(1.2)_2$ and the convergences just proved, that $u$ solves (1.3). Writing estimate (8.23) for $u_k$ in particular we find, for $\Omega_{T,1} \Subset \Omega_T$

$$[Du_k]_{W^{\kappa,\kappa/2;q}(\Omega'\times J')} \le c\left[s + |\mu|(\Omega_T)\right],$$

where $c$ depends on $n$, $L/\nu$, $q$, dist$(\Omega'_{T'}, \partial\Omega_T)$, $|\Omega|$ and $T$. Here we have used (3.3) and the previous global estimate (8.26). Now estimate (1.10) follows by treating the left-hand sides of the previous inequality with Fatou's Lemma. □

*Proof of local estimates on cylinders.* Finally in order to prove (1.9) we make use of a scaling argument. Fix $Q_\varrho(z_0) \Subset \Omega_T$ and take $u \in L^2(-T,0;W^{1,2}_0(\Omega))$ the unique solution to (3.2) for a fixed regular $f$; then restrict $u$ to $Q_\varrho(z_0)$ and then rescale it to $Q_1$, as in Lemma 6.4, in order to get $\tilde u \in L^2(-1,1;W^{1,2}(B_1))$. Now observe that we may apply Lemma 8.3 to $\tilde u$ since the whole argument is just local and no boundary information is needed. Hence we can deduce by estimate (8.23) applied to $\tilde u$ with $\Omega_{T,1} \equiv Q_{1/2}$ and $\Omega_{T,2} \equiv Q_1$, up to a little change in notation, for $\sigma < \delta q$:

$$[D\tilde u]^q_{W^{\sigma/q,\sigma/(2q);q}(Q_{1/2})} \le c\Big[\|s + |D\tilde u|\|^q_{L^q(Q_1)} + \|\tilde f\|^q_{L^1(Q_1)}\Big].$$

Scaling back to $Q_\varrho(z_0)$ yields to

$$\begin{aligned}\varrho^{\sigma-n-2}\int_{I_{\varrho/2}}\int_{B_{\varrho/2}}\int_{B_{\varrho/2}} &\frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\sigma}}\,dx\,dy\,dt\\ &+ \varrho^{\sigma-n-2}\int_{B_{\varrho/2}}\int_{I_{\varrho/2}}\int_{I_{\varrho/2}} \frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\sigma/2}}\,dt\,ds\,dx\\ &\qquad\le c\Big[\varrho^{-n-2}\|s+|Du|\|^q_{L^q(Q_1)} + \varrho^{-q(n+1)}\|f\|^q_{L^1(Q_1)}\Big],\end{aligned}$$

that is

$$[Du]^q_{W^{\sigma/q,\sigma/(2q);q}(Q_{1/2})} \le c\,\varrho^{-\sigma}\Big[\|s + |Du|\|^q_{L^q(Q_1)} + \varrho^{\sigma(q)}\|f\|^q_{L^1(Q_1)}\Big].$$

Now it's enough to write the latter estimate for $u \equiv u_k$, $u_k$ being the approximated solution described in the beginning of this proof, and follow again the scheme described just above, using also (3.3). □

*Proof of Corollary 1.4.* Recall that by the definition (4.3) of fractional Sobolev spaces we have (the reason of the changing in the notation from $q$ to $r$ will become clear in a moment)

$$Du \in L^r_{\rm loc}(-T,0;W^{\bar\delta,r}_{\rm loc}(\Omega)) \qquad \text{for all } \bar\delta \in (0,\delta), \tag{8.27}$$

with

$$r \in \Big[1, 2-\frac{n}{n+1}\Big) \qquad \text{and} \qquad \delta \equiv \delta(r) = \frac{n+2}{r} - (n+1).$$

Using fractional Sobolev embedding of Proposition 4.1 slicewise in space, after a simple computation we have

$$Du \in L^r_{\rm loc}(-T,0;L^q_{\rm loc}(\Omega)) \qquad \text{for all } q \in [1,q^*), \qquad q^* \equiv q^*(r) := \frac{nr}{r(n+1)-2}. \tag{8.28}$$

Moreover by immersion (4.2) we have that

$$Du \in W^{\bar\delta/2,q}_{\rm loc}(-T,0;L^q_{\rm loc}(\Omega)) \qquad \text{for all } \bar\delta \in (0,\delta), \tag{8.29}$$

with this time

$$q \in \Big[1, 2-\frac{n}{n+1}\Big) \qquad \text{and} \qquad \delta \equiv \delta(q) = \frac{n+2}{q} - (n+1).$$

Applying Proposition 4.1 this time slicewise in time (which in this case means applied to the function $\|Du(\cdot,t)\|_{L^q}$), with $N \equiv 1$, gives

$$Du \in L^r_{\rm loc}(-T,0;L^q_{\rm loc}(\Omega)) \qquad \text{for all } r \in [1,r^*), \qquad r^* \equiv r^*(q) := \frac{2q}{q(n+1)-n}. \tag{8.30}$$

It is easy to check that the bounds for $r$ and $q$ appearing in (8.28) and (8.30) fully recover the bounds in (1.5), since the images of $q = q^*(r)$ and $r = r^*(q)$ are the same arc of hyperbola in the $(r,q)$ plane.

Reasoning exactly as above, using the facts that

$$Du \in L^q_{\rm loc}(\Omega; W^{\bar\delta/2,q}_{\rm loc}(-T,0)), \qquad q \in \Big[1, 2-\frac{n}{n+1}\Big) \tag{8.31}$$

and

$$Du \in W^{\bar\delta,r}_{\rm loc}(\Omega; L^r_{\rm loc}(-T,0)) \qquad r \in \Big[1, 2-\frac{n}{n+1}\Big) \tag{8.32}$$

we obtain $Du \in L^q_{\rm loc}(\Omega; L^r_{\rm loc}(-T,0))$ for all $(r,q)$ satisfying (1.5). □

*Proof of anisotropic regularity Theorem 1.6.* We argue similarly as above. Since by Proposition 4.2

$$W^{\delta,r}_{\rm loc}(\Omega) \subset W^{\delta-n/r+n/q,q}_{\rm loc}(\Omega)$$

plugging last result slicewise into (8.27) gives

$$Du \in L^r_{\rm loc}(-T,0;W^{\delta,q}_{\rm loc}(\Omega)) \qquad \text{for all } q > r \text{ and } \delta < \tilde\delta(r,q)$$

with

$$\tilde\delta(r,q) := \frac{n}{q} + \frac{2}{r} - (n+1).$$

Applying Proposition 4.2 to the function $\|Du(x,\cdot)\|_{L^r(J)}$ with $J \Subset (-T,0)$ generic we get by (8.32)

$$\|Du(x,\cdot)\|_{L^r(J)} \in W^{\delta,q}_{\rm loc}(\Omega) \qquad \text{that is} \qquad Du \in W^{\delta,q}_{\rm loc}(\Omega; L^r_{\rm loc}(-T,0))$$

for $\delta \in (0,\tilde\delta(r,q))$ and $r < q$. Finally we get results involving time regularity ($r > q$) exactly in the same way, using Proposition 4.2 in dimension 1 together with inclusions (8.31) and (8.29). This finally finishes the proof. □

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