Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds

Luigi Ambrosio * Nicola Gigli † Giuseppe Savaré ‡

September 20, 2012

Abstract

The aim of the present paper is to bridge the gap between the Bakry-Émery and the Lott-Sturm-Villani approaches to provide synthetic and abstract notions of lower Ricci curvature bounds.

We start from a strongly local Dirichlet form $\mathcal{E}$ admitting a *Carré du champ* Γ in a Polish measure space $(X, \mathfrak{m})$ and a canonical distance $\mathsf{d}_{\mathcal{E}}$ that induces the original topology of X. We first characterize the distinguished class of *Riemannian Energy measure spaces*, where $\mathcal{E}$ coincides with the Cheeger energy induced by $\mathsf{d}_{\mathcal{E}}$ and where every function f with $\Gamma(f) \le 1$ admits a continuous representative.

In such a class we show that if $\mathcal{E}$ satisfies a suitable weak form of the *Bakry-Émery curvature dimension condition* $\mathrm{BE}(K, \infty)$ then the metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the Riemannian Ricci curvature bound $\mathrm{RCD}(K, \infty)$ according to [5], thus showing the equivalence of the two notions.

Two applications are then proved: the tensorization property for Riemannian Energy spaces satisfying the Bakry-Émery condition $\mathrm{BE}(K, N)$ (and thus the corresponding one for $\mathrm{RCD}(K, \infty)$ spaces without assuming nonbranching) and the stability of $\mathrm{BE}(K, N)$ with respect to Sturm-Gromov-Hausdorff convergence.

Contents

*Scuola Normale Superiore, Pisa. email: l.ambrosio@sns.it. Supported by ERC ADG GeMeThNES

†Nice University. email: nicola.gigli@unice.fr

‡Università di Pavia. email: giuseppe.savare@unipv.it. Partially supported by PRIN08 grant from MIUR for the project *Optimal transport theory, geometric and functional inequalities, and applications.*

1 Introduction

The aim of the present paper is to bridge the gap between the Bakry-Émery and the Lott-Sturm-Villani approaches to provide synthetic and abstract notions of lower Ricci curvature bounds.

The Bakry-Émery condition $\mathrm{BE}(K,N)$: Dirichlet forms and Γ-calculus

The first approach is based on the functional Γ-calculus developed by Bakry-Émery since [8], see [7, 9].

A possible starting point is a local and symmetric Dirichlet form $\mathcal{E}$ on the measure space $(X,\mathcal{B},\mathfrak{m})$ with dense domain $D(\mathcal{E})\subset L^2(X,\mathfrak{m})$, and the associated Markov semigroup $(\mathsf{P}_t)_{t\geq 0}$ on $L^2(X,\mathfrak{m})$ with generator $\Delta_\mathcal{E}$ (general references are [18, 29, 12]). In a suitable algebra $\mathcal{A}$ of functions dense in the domain $D(\Delta_\mathcal{E})$ of $\Delta_\mathcal{E}$ one introduces the *Carré du champ*

$$\Gamma\big(f,g\big):=\frac{1}{2}\Big(\Delta_\mathcal{E}(fg)-f\Delta_\mathcal{E}g-g\Delta_\mathcal{E}f\Big),\qquad f,g\in\mathcal{A},$$

related to $\mathcal{E}$ by the local representation formula

$$\mathcal{E}(f,g)=\int_X\Gamma\big(f,g\big)\,\mathrm{d}\mathfrak{m}\qquad\text{for every } f,g\in\mathcal{A}. \tag{1.1}$$

One also assumes that $\Delta_\mathcal{E}$ is a diffusion operator, i.e., with the notation $\Gamma\big(f\big):=\Gamma\big(f,f\big)$, it holds

$$\Delta_\mathcal{E}\phi(f)=\phi'(f)\Delta_\mathcal{E}f+\phi''(f)\Gamma\big(f\big)\quad\text{for every } f\in\mathcal{A},\quad \phi\in\mathrm{C}^2_b(\mathbb{R}).$$

The model example is provided by a smooth Riemannian manifold $(\mathbb{M}^d,\mathsf{g})$ endowed with the measure $\mathfrak{m}:=\mathrm{e}^{-V}\mathrm{Vol}_\mathsf{g}$ for a given smooth potential $V:\mathbb{M}^d\to\mathbb{R}$. In this case one typically chooses $\mathcal{A} = \mathrm{C}_c^\infty(\mathbb{M}^d)$ and

$$\mathcal{E}(f,g) = \int_{\mathbb{M}^d} \langle \nabla f, \nabla g\rangle_{\mathsf{g}}\,\mathrm{d}\mathfrak{m}, \qquad \text{so that } \Gamma\big(f\big) = |\nabla f|_{\mathsf{g}}^2 \text{ and } \Delta_{\mathcal{E}} = \Delta_{\mathsf{g}} - \langle \nabla V, \nabla \cdot\rangle_{\mathsf{g}}, \tag{1.2}$$

where Δ_{g} is the usual Laplace-Beltrami operator on $\mathbb{M}$. This fundamental example shows that Γ carries the metric information of $\mathbb{M}^d$, since one can recover the Riemannian distance d_{g} in $\mathbb{M}^d$ by the formula

$$\mathsf{d}_{\mathsf{g}}(x,y) = \sup\Big\{\psi(y) - \psi(x) : \psi \in \mathcal{A},\ \Gamma\big(\psi\big) \le 1\Big\} \quad \text{for every } x, y \in \mathbb{M}^d. \tag{1.3}$$

A further iteration yields the Γ_2 operator, defined by

$$2\Gamma_2(f) = \Delta_{\mathcal{E}}\Gamma\big(f\big) - 2\Gamma\big(f, \Delta_{\mathcal{E}} f\big) \qquad f \in \mathcal{A}. \tag{1.4}$$

In the above example Bochner's formula yields

$$\Gamma_2(f) = \|\mathrm{Hess}_{\mathsf{g}} f\|_{\mathsf{g}}^2 + \big(\mathrm{Ric}_{\mathsf{g}} + \mathrm{Hess}_{\mathsf{g}} V\big)(\nabla f, \nabla f), \tag{1.5}$$

and one obtains the fundamental inequality

$$\Gamma_2(f) \ge K\,\Gamma\big(f\big) + \frac{1}{N}(\Delta_{\mathcal{E}} f)^2 \quad \text{for every } f \in \mathcal{A}, \tag{1.6}$$

if the quadratic form associated to the tensor $\mathrm{Ric}_{\mathsf{g}} + \mathrm{Hess}_{\mathsf{g}} V$ is bounded from below by $K\mathsf{g} + \frac{1}{N-d}\nabla V \otimes \nabla V$ for some $K \in \mathbb{R}$ and $N > d$. When $V \equiv 0$ it is possible to show that $(\mathbb{M}^d, \mathsf{g})$ has Ricci curvature bounded from below by K iff (1.6) is satisfied for $N \ge d$.

It is then natural to use (1.6) as a definition of curvature-dimension bounds even in the abstract setting: it is the so-called *Bakry-Émery curvature-dimension condition*, that we denote here by $\mathrm{BE}(K,N)$.

One of the most remarkable applications of (1.6) is provided by pointwise gradient estimates for the Markov semigroup (see e.g. [7, 9] for relevant and deep applications). Considering here only the case $N = \infty$, (1.6) yields

$$\Gamma\big(\mathsf{P}_t f\big) \le \mathrm{e}^{-2Kt}\,\mathsf{P}_t\big(\Gamma\big(f\big)\big) \quad \text{for every } f \in \mathcal{A}, \tag{1.7}$$

a property that is essentially equivalent to $\mathrm{BE}(K,\infty)$ (we refer to [43] for other formulations of $\mathrm{BE}(K,N)$ for Riemannian manifolds, see also the next Section 2.2) and involves only first order "differential" operators.

Up to the choice of an appropriate functional setting (in particular the algebra $\mathcal{A}$ and the distance d associated to Γ as in (1.3) play a crucial role), Γ-calculus and curvature-dimension inequalities provide a very powerful tool to establish many functional inequalities and geometric properties, often in sharp form.

Lower Ricci curvature bounds by optimal transport: the $\mathrm{CD}(K, \infty)$ condition

A completely different approach to lower Ricci bounds has been recently proposed by Sturm [39, 40] and Lott-Villani [28]: here the abstract setting is provided by metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a separable, complete and length metric space and $\mathfrak{m}$ is a nonnegative σ-finite Borel measure. Just for simplicity, in this Introduction we also assume $\mathfrak{m}(X) < \infty$, but the theory covers the case of a measure satisfying the exponential growth condition $\mathfrak{m}(B_r(x)) \le M \exp(c\, r^2)$ for some constants $M, c \ge 0$.

The Lott-Sturm-Villani theory (LSV in the following) is based on the notion of displacement interpolation [30], a powerful tool of optimal transportation that allows to extend the notion of geodesic interpolation from the state space X to the space of Borel probability measures $\mathscr{P}_2(X)$ with finite quadratic moment. Considering here only the case $N = \infty$, a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the LSV lower Ricci curvature bound $\mathrm{CD}(K, \infty)$ if the relative entropy functional

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \int_X f \log f \, \mathrm{d}\mathfrak{m}, \qquad \mu = f\mathfrak{m}, \tag{1.8}$$

is displacement K-convex in the Wasserstein space $(\mathscr{P}_2(X), W_2)$ (see [42, 2] and the next §3.1). This definition is consistent with the Riemannian case [41] and thus equivalent to $\mathrm{BE}(K, \infty)$ in such a smooth framework.

Differently from the Bakry-Émery's approach, the LSV theory does not originally involve energy functionals or Markov semigroups but it is intimately connected to the metric d (through the notion of displacement interpolation) and to the measure $\mathfrak{m}$ (through the entropy functional (1.8)). Besides many useful geometric and functional applications of this notion [27, 34, 19], one of its strongest features is its stability under measured Gromov-Hausdorff convergence [17], also in the weaker transport-formulation proposed by Sturm [39].

Starting from the $\mathrm{CD}(K, \infty)$ assumption, one can then construct an evolution semigroup $(\mathsf{H}_t)_{t \ge 0}$ on the convex subset of $\mathscr{P}_2(X)$ given by probability measures with finite entropy [19]: it is the metric gradient flow of the entropy functional in $\mathscr{P}_2(X)$ [2]. Since also Finsler geometries (as in the flat case of $\mathbb{R}^d$ endowed with a non-euclidean norm) can satisfy the $\mathrm{CD}(K, \infty)$ condition, one cannot hope in such a general setting that H_t are linear operators. Still, $(\mathsf{H}_t)_{t \ge 0}$ can be extended to a continuous semigroup of contractions in $L^2(X, \mathfrak{m})$ (and in any $L^p(X, \mathfrak{m})$-space), which can also be characterized as the $L^2(X, \mathfrak{m})$-gradient flow $(\mathsf{P}_t)_{t \ge 0}$ of a convex and 2-homogeneous functional, the Cheeger energy [14], [3, §4.1, Rem. 4.7]

$$\mathsf{Ch}(f) := \inf \Big\{ \liminf_{n \to \infty} \frac{1}{2} \int_X |\mathrm{D} f_n|^2 \, \mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_b(X), \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}) \Big\} \tag{1.9}$$

(here $|\mathrm{D} f_n|$ is the local Lipschitz constant, or slope, of the Lipschitz function f, see §3.1).

The remarkable identification between $(\mathsf{H}_t)_{t \ge 0}$ and $(\mathsf{P}_t)_{t \ge 0}$ has been firstly proposed and proved in Euclidean spaces by a seminal paper of Jordan-Kinderleher-Otto [23] and then extended to Riemannian manifolds [16, 42], Hilbert spaces [6], Finsler spaces [31], Alexandrov spaces [21] and eventually to $\mathrm{CD}(K, \infty)$ metric measure spaces [3].

Spaces with Riemannian Ricci curvature bounded from below: the $\mathrm{RCD}(K,\infty)$ condition

Having the energy functional (1.9) and the contraction semigroup $(\mathsf{P}_t)_{t\geq 0}$ at our disposal, it is then natural to investigate when LSV spaces satisfy $\mathrm{BE}(K,\infty)$. In order to attack this question, one has of course to clarify when the Cheeger energy (1.9) is a Dirichlet (thus *quadratic*) form on $L^2(X,\mathfrak{m})$ (or, equivalently, when $(\mathsf{P}_t)_{t\geq 0}$ is a semigroup of *linear* operators) and when this property is also stable under Sturm-Gromov-Hausdorff convergence.

One of the most important results of [5] (see also [1] for general σ-finite measures) is that $\mathrm{CD}(K,\infty)$ spaces with a quadratic Cheeger energy can be equivalently characterized as those metric measure spaces where there exists the Wasserstein gradient flow $(\mathsf{H}_t)_{t\geq 0}$ of the entropy functional (1.8) in the EVI_K-sense. This condition means that for all initial data $\mu\in\mathscr{P}_2(X)$ with $\operatorname{supp}\mu\subset\operatorname{supp}\mathfrak{m}$ there exists a locally Lipschitz curve $t\mapsto\mathsf{H}_t\mu\in\mathscr{P}_2(X)$ satisfying the Evolution Variational Inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{W_2^2(\mathsf{H}_t\mu,\nu)}{2}+\frac{K}{2}W_2^2(\mathsf{H}_t\mu,\nu)+\mathrm{Ent}_{\mathfrak{m}}(\mathsf{H}_t\mu)\leq\mathrm{Ent}_{\mathfrak{m}}(\nu)\quad\text{for a.e. } t\in(0,\infty) \tag{1.10}$$

for all $\nu\in\mathscr{P}_2(X)$ with $\mathrm{Ent}_{\mathfrak{m}}(\nu)<\infty$.

Such a condition is denoted by $\mathrm{RCD}(K,\infty)$ and it is stronger than $\mathrm{CD}(K,\infty)$, since the existence of an EVI_K flow solving (1.10) yields both the geodesic K-convexity of the entropy functional $\mathrm{Ent}_{\mathfrak{m}}$ [15] and the linearity of $(\mathsf{H}_t)_{t\geq 0}$ [5, Thm. 5.1], but it is still stable under Sturm-Gromov-Hausdorff convergence. When it is satisfied, the metric measure space $(X,\mathsf{d},\mathfrak{m})$ is called in [5] a space with *Riemannian* Ricci curvature bounded from below by K.

In $\mathrm{RCD}(K,\infty)$-spaces the Cheeger energy is associated to a strongly local Dirichlet form $\mathcal{E}_{\mathsf{Ch}}(f,f):=2\mathsf{Ch}(f)$ admitting a Carré du champ Γ. With the calculus tools developed in [5], it can be proved that Γ has a further equivalent representation $\Gamma(f)=|\mathrm{D}f|_w^2$ in terms of the *minimal weak gradient* $|\mathrm{D}f|_w$ of f. The latter is the element of minimal L^2-norm among all the possible weak limits of $|\mathrm{D}f_n|$ in the definition (1.9).

It follows that $\mathcal{E}_{\mathsf{Ch}}$ can also be expressed by $\mathcal{E}_{\mathsf{Ch}}(f,f)=\int_X|\mathrm{D}f|_w^2\,\mathrm{d}\mathfrak{m}$ and the set of Lipschitz functions f with $\int_X|\mathrm{D}f|^2\,\mathrm{d}\mathfrak{m}<\infty$ is strongly dense in the domain of $\mathcal{E}_{\mathsf{Ch}}$. In fact, the Dirichlet form $\mathcal{E}_{\mathsf{Ch}}$ enjoys a further upper-regularity property, common to every Cheeger energy [4, §8.3]:

(a) for every $f\in D(\mathcal{E})$ there exist $f_n\in D(\mathcal{E})\cap\mathrm{C}_b(X)$ and upper semicontinuous bounded functions $g_n:X\to\mathbb{R}$ such that

$$\Gamma(f_n)\leq g_n^2,\qquad f_n\to f\text{ in }L^2(X,\mathfrak{m}),\qquad\limsup_{n\to\infty}\int_X g_n^2\,\mathrm{d}\mathfrak{m}\leq\mathcal{E}(f,f). \tag{1.11}$$

From $\mathrm{RCD}(K,\infty)$ to $\mathrm{BE}(K,\infty)$

The previous properties of the Cheeger energy show that the investigation of Bakry-Émery curvature bounds makes perfectly sense in $\mathrm{RCD}(K,\infty)$ spaces. One of the main results of [5] connecting these two approaches shows in fact that $\mathrm{RCD}(K,\infty)$ yields $\mathrm{BE}(K,\infty)$ in the gradient formulation (1.7) for every $f \in D(\mathcal{E}_{\mathsf{Ch}})$.

In fact, an even more refined result holds [5, Thm. 6.2], since it is possible to control the slope of $\mathsf{P}_t f$ in terms of the minimal weak gradient of f

$$|\mathrm{D}\mathsf{P}_t f|^2 \le \mathrm{e}^{-2Kt}\mathsf{P}_t\big(|\mathrm{D}f|_w^2\big) \quad \text{whenever } f \in D(\mathcal{E}_{\mathsf{Ch}}),\ |\mathrm{D}f|_w \in L^\infty(X,\mathfrak{m}), \tag{1.12}$$

an estimate that has two useful geometric-analytic consequences:

(b) d coincides with the intrinsic distance associated to the Dirichlet form $\mathcal{E}_{\mathsf{Ch}}$ (introduced in Biroli-Mosco [10], see also [37, 38] and [36]), namely

$$\mathsf{d}(x,y) = \sup\Big\{\psi(y)-\psi(x) : \psi \in D(\mathcal{E}_{\mathsf{Ch}}) \cap \mathrm{C}_b(X),\ \Gamma\big(\psi\big) \le 1\Big\} \qquad x,y \in X. \tag{1.13}$$

(c) Every function $\psi \in D(\mathcal{E}_{\mathsf{Ch}})$ with $\Gamma\big(\psi\big) \le 1$ $\mathfrak{m}$-a.e. admits a continuous (in fact 1-Lipschitz) representative $\tilde\psi$.

From $\mathrm{BE}(K,\infty)$ **to** $\mathrm{RCD}(K,\infty)$

In the present paper we provide necessary and sufficient conditions for the validity of the converse implication, i.e. $\mathrm{BE}(K,\infty) \Rightarrow \mathrm{RCD}(K,\infty)$.

In order to state this result in a precise way, one has first to clarify how the metric structure should be related to the Dirichlet one. Notice that this problem is much easier from the point of view of the metric measure setting, since one has the canonical way (1.9) to construct the Cheeger energy.

Since we tried to avoid any local compactness assumptions on X as well as doubling or Poincaré conditions on $\mathfrak{m}$, we used the previous structural properties (a,b,c) as a guide to find a reasonable set of assumptions for our theory; notice that they are in any case necessary conditions to get a $\mathrm{RCD}(K,\infty)$ space.

We thus start from a strongly local and symmetric Dirichlet form $\mathcal{E}$ on a Polish topological space (X,τ) endowed with its Borel σ-algebra and a finite (for the scope of this introduction) Borel measure $\mathfrak{m}$. In the algebra $\mathbb{V}_\infty := D(\mathcal{E}) \cap L^\infty(X,\mathfrak{m})$ we consider the subspace $\mathbb{G}_\infty$ of functions f admitting a Carré du champ $\Gamma(f) \in L^1(X,\mathfrak{m})$: they are characterized by the identity

$$\mathcal{E}(f, f\varphi) - \frac{1}{2}\mathcal{E}(f^2,\varphi) = \int_X \Gamma(f)\varphi\,\mathrm{d}\mathfrak{m} \qquad \text{for every } \varphi \in \mathbb{V}_\infty. \tag{1.14}$$

We can therefore introduce the intrinsic distance $\mathsf{d}_\mathcal{E}$ as in (b)

$$\mathsf{d}_\mathcal{E}(x,y) := \sup\Big\{\psi(y)-\psi(x) : \psi \in \mathbb{G}_\infty \cap \mathrm{C}(X),\ \Gamma\big(\psi\big) \le 1\Big\} \qquad x,y \in X, \tag{1.15}$$

and, following the standard approach, we will assume that $\mathsf{d}_\mathcal{E}$ is a complete distance on X and the topology induced by $\mathsf{d}_\mathcal{E}$ coincides with τ.

In this way we end up with *Energy measure spaces* $(X, \tau, \mathfrak{m}, \mathcal{E})$ and in this setting we prove in Theorem 3.12 that $\mathcal{E} \le \mathcal{E}_{\mathsf{Ch}}$, where $\mathcal{E}_{\mathsf{Ch}}$ is the Cheeger energy associated to $\mathsf{d}_{\mathcal{E}}$; moreover, Theorem 3.14 shows that $\mathcal{E} = \mathcal{E}_{\mathsf{Ch}}$ if and only if (a) holds (see [24, §5] for a similar result in the case of doubling spaces satisfying a local Poincaré condition and for interesting examples where $\mathcal{E}_{\mathsf{Ch}}$ is not quadratic and $\mathcal{E} \neq \mathcal{E}_{\mathsf{Ch}}$). It is also worth mentioning (Theorem 3.10) that for this class of spaces $(X, \mathsf{d}_{\mathcal{E}})$ is always a length metric space, a result previously known in a locally compact framework [38, 36].

The Bakry-Émery condition $\mathrm{BE}(K,\infty)$ can then be stated in a weak integral form (strongly inspired by [7, 9, 43]) just involving the Markov semigroup $(\mathsf{P}_t)_{t\ge 0}$ (see (2.55) of Corollary 2.3 and (2.37), (2.38) for relevant definitions) by asking that the differential inequality

$$\frac{\partial^2}{\partial s^2}\int_X (\mathsf{P}_{t-s}f)^2\mathsf{P}_s\varphi\,\mathrm{d}\mathfrak{m} \ge 4K\frac{\partial}{\partial s}\int_X (\mathsf{P}_{t-s}f)^2\mathsf{P}_s\varphi\,\mathrm{d}\mathfrak{m}, \qquad 0<s<t, \tag{1.16}$$

is fulfilled for any $f \in L^2(X,\mathfrak{m})$ and any nonnegative $\varphi \in L^2 \cap L^\infty(X,\mathfrak{m})$. Notice that in the case $K = 0$ (1.16) is equivalent to the convexity in $(0,t)$ of the map $s \mapsto \int_X (\mathsf{P}_{t-s}f)^2\mathsf{P}_s\varphi\,\mathrm{d}\mathfrak{m}$.

If we also assume that $\mathrm{BE}(K,\infty)$ holds, it turns out that (c) is in fact equivalent to a weak-Feller condition on the semigroup $(\mathsf{P}_t)_{t\ge0}$, namely P_t maps $\mathrm{Lip}_b(X)$ in $\mathrm{C}_b(X)$. Moreover, (c) implies the upper-regularity (a) of $\mathcal{E}$ and the fact that every $f \in D(\mathcal{E}) \cap L^\infty(X,\mathfrak{m})$ admits a Carré du champ Γ satisfying (1.14).

Independently of $\mathrm{BE}(K,\infty)$, when properties (a) and (c) are satisfied, we call $(X, \tau, \mathfrak{m}, \mathcal{E})$ a *Riemannian Energy measure space*, since these space seem appropriate non-smooth versions of Riemannian manifolds. It is also worth mentioning that in this class of spaces $\mathrm{BE}(K,\infty)$ is equivalent to an (exponential) contraction property for the semigroup $(\mathsf{H}_t)_{t\ge0}$ with respect to the Wasserstein distance W_2 (see Corollary 3.18), in analogy with [25].

Our main equivalence Theorem 4.17 shows that a $\mathrm{BE}(K,\infty)$ Riemannian Energy measure space satisfies the $\mathrm{RCD}(K,\infty)$ condition: thus, in view of the converse implication proved in [5], $\mathrm{BE}(K,\infty)$ is essentially equivalent to $\mathrm{RCD}(K,\infty)$. A more precise formulation of our result, in the simplified case when the measure $\mathfrak{m}$ is finite, is:

Theorem 1.1 (Main result)**.** *Let (X,τ) be a Polish space and let $\mathfrak{m}$ be a finite Borel measure in X. Let $\mathcal{E} : L^2(X,\mathfrak{m}) \to [0,\infty]$ be a strongly local, symmetric Dirichlet form generating a mass preserving Markov semigroup $(\mathsf{P}_t)_{t\ge0}$ in $L^2(X,\mathfrak{m})$, let $\mathsf{d}_{\mathcal{E}}$ be the intrinsic distance defined by* (1.15) *and assume that:*

(i) $\mathsf{d}_{\mathcal{E}}$ is a complete distance on X inducing the topology τ and any function $f \in \mathbb{G}_\infty$ with $\Gamma(f) \le 1$ admits a continuous representative;

(ii) the Bakry-Émery $\mathrm{BE}(K,\infty)$ *condition* (1.16) *is fulfilled by* $(\mathsf{P}_t)_{t\ge0}$.

Then $(X, \mathsf{d}_{\mathcal{E}}, \mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$ *space.*

We believe that this equivalence result, between the "Eulerian" formalism of the Bakry-Émery $\mathrm{BE}(K,\infty)$ theory and the "Lagrangian" formalism of the $\mathrm{CD}(K,\infty)$ theory, is conceptually important and that it could be a first step for a better understanding of curvature conditions in metric measure spaces. Also, this equivalence is technically useful. Indeed, in the last section of this paper we prove the tensorization of $\mathrm{BE}(K,N)$ spaces. Then, in the case $N=\infty$, we can use the implication from $\mathrm{BE}(K,\infty)$ to $\mathrm{RCD}(K,\infty)$ to read this property in terms of tensorization of $\mathrm{RCD}(K,\infty)$ spaces: this was previously known, see [5], only under an apriori nonbranching assumption on the bases spaces (notice that the $\mathrm{CD}(K,N)$ theory, even with $N=\infty$, suffers at this moment the same limitation). On the other hand, we use the implication from $\mathrm{RCD}(K,\infty)$ to $\mathrm{BE}(K,\infty)$, (1.17) below and the strong stability properties which follow by the EVI_K formulation to provide stability of the $\mathrm{BE}(K,N)$ condition under a very weak convergence, the Sturm-Gromov-Hausdorff convergence.

Plan of the paper

Section 2 collects notation and preliminary results on Dirichlet forms, Markov semigroups and functional Γ-calculus, following the presentation of [12], which avoids any topological assumption. A particular attention is devoted to various formulations of the $\mathrm{BE}(K,N)$ condition: they are discussed in §2.2, trying to present an intrinsic approach that does not rely on the introduction of a distinguished algebra of functions $\mathcal{A}$ and extra assumptions on the Dirichlet form $\mathcal{E}$, besides locality. In its weak formulation (see (2.55) of Corollary 2.3 and (2.37), (2.38))

$$\frac{1}{4}\frac{\partial^2}{\partial s^2}\int_X(\mathsf{P}_{t-s}f)^2\mathsf{P}_s\varphi\,\mathrm{d}\mathfrak{m}\ge\frac{K}{2}\frac{\partial}{\partial s}\int_X(\mathsf{P}_{t-s}f)^2\mathsf{P}_s\varphi\,\mathrm{d}\mathfrak{m}+\frac{1}{N}\int_X(\Delta_{\mathcal{E}}\mathsf{P}_{t-s}f)^2\mathsf{P}_s\varphi\,\mathrm{d}\mathfrak{m},\tag{1.17}$$

which is well suited to study stability issues, $\mathrm{BE}(K,N)$ does not even need a densely defined Carré du Champ Γ, because only the semigroup $(\mathsf{P}_t)_{t\ge0}$ is involved.

Section 3 is devoted to study the interaction between energy and metric structures. A few metric concepts are recalled in §3.1, whereas §3.2 shows how to construct a dual semigroup $(\mathsf{H}_t)_{t\ge0}$ in the space of probability measures $\mathscr{P}(X)$ under suitable Lipschitz estimates on $(\mathsf{P}_t)_{t\ge0}$. By using refined properties of the Hopf-Lax semigroup, we also extend some of the duality results proved by Kuwada [25] to general complete and separable metric measure spaces, avoiding any doubling or Poincaré condition.
§3.3 presents a careful analysis of the intrinsic distance $\mathsf{d}_{\mathcal{E}}$ (1.15) associated to a Dirichlet form and of Energy measure structures $(X,\tau,\mathfrak{m},\mathcal{E})$. We will thoroughly discuss the relations between the Dirichlet form $\mathcal{E}$ and the Cheeger energy Ch induced by a distance d, possibly different from the intrinsic distance $\mathsf{d}_{\mathcal{E}}$ and we will obtain a precise characterization of the distinguished case when $\mathsf{d}=\mathsf{d}_{\mathcal{E}}$ and $\mathcal{E}=2\mathsf{Ch}$: here conditions *(a,b)* play a crucial role.
A further investigation when $\mathrm{BE}(K,\infty)$ is also assumed is carried out in §3.4, leading to the class of *Riemannian Energy measure spaces*.

Section 4 contains the proof of the main equivalence Theorem 1.1 between $\mathrm{BE}(K,\infty)$ and $\mathrm{RCD}(K,\infty)$. Apart the basic estimates of §4.1, the argument is split in two main steps: §4.2 proves a first $\mathrm{L}\log\mathrm{L}$ regularization estimate for the semigroup $(\mathsf{H}_t)_{t\ge0}$, starting from arbitrary measures in $\mathscr{P}_2(X)$ (here we follow the approach of [43]). §4.3 contains the crucial action estimates to prove the EVI_K inequality (1.10). Even if the strategy of the proof has been partly inspired by the geometric heuristics discussed in [15] (where the Eulerian approach of [32] to contractivity of gradient flows has been extended to cover also convexity and evolutions in the EVI_K sense) this part is completely new and it uses in a subtle way all the refined technical issues discussed in the previous sections of the paper.

In the last Section 5 we discuss the above mentioned applications of the equivalence between $\mathrm{BE}(K,\infty)$ and $\mathrm{RCD}(K,\infty)$.

Acknowledgement. We would like to thank Michel Ledoux for useful discussions on various aspects of the theory of Dirichlet forms and Γ-calculus.

2 Dirichlet forms, Markov semigroups, Γ-calculus

2.1 Dirichlet forms and Γ-calculus

Let $(X,\mathcal{B})$ be a measurable space, let $\mathfrak{m}:\mathcal{B}\to[0,\infty]$ σ-additive and let $L^p(X,\mathfrak{m})$ be the Lebesgue spaces (for notational simplicity we omit the dependence on $\mathcal{B}$). Possibly enlarging $\mathcal{B}$ and extending $\mathfrak{m}$ we assume that $\mathcal{B}$ is $\mathfrak{m}$-complete. In the next sections 3, 4 we will typically consider the case when $\mathcal{B}$ is the $\mathfrak{m}$-completion of the Borel σ-algebra generated by a Polish topology τ on X.

In all this paper we will assume that

$$\begin{gathered}\mathcal{E}:L^2(X,\mathfrak{m})\to[0,\infty]\text{ is a strongly local, symmetric Dirichlet form}\\ \text{generating a Markov semigroup }(\mathsf{P}_t)_{t\ge0}\text{ in }L^2(X,\mathfrak{m});\end{gathered}\tag{2.1}$$

Let us briefly recall the precise meaning of this statement.

A **symmetric Dirichlet form** $\mathcal{E}$ is a $L^2(X,\mathfrak{m})$-lower semicontinuous quadratic form satisfying the Markov property

$$\mathcal{E}(\eta\circ f)\le\mathcal{E}(f)\qquad\text{for every normal contraction }\eta:\mathbb{R}\to\mathbb{R},\tag{2.2}$$

i.e. a 1-Lipschitz map satisfying $\eta(0)=0$. We refer to [12, 18] for equivalent formulations of (2.2). We also define

$$\mathbb{V}:=D(\mathcal{E})=\{f\in L^2(X,\mathfrak{m}):\ \mathcal{E}(f)<\infty\},\qquad\mathbb{V}_\infty:=D(\mathcal{E})\cap L^\infty(X,\mathfrak{m}).\tag{2.3}$$

We also assume that $\mathbb{V}$ is dense in $L^2(X,\mathfrak{m})$.

We still denote by $\mathcal{E}(\cdot,\cdot):\mathbb{V}\to\mathbb{R}$ the associated continuous and symmetric bilinear form

$$\mathcal{E}(f,g):=\frac14\Big(\mathcal{E}(f+g)-\mathcal{E}(f-g)\Big).\tag{2.4}$$

We will assume **strong locality of $\mathcal{E}$,** namely

$$\mathcal{E}(f,g)=0\qquad\text{whenever }f,\ g\in\mathbb{V}\text{ and }(f+a)g=0\ \mathfrak{m}\text{-a.e. in }X\text{ for some }a\in\mathbb{R}.\tag{2.5}$$

It is possible to prove that $\mathbb{V}_\infty$ is an algebra with respect to pointwise multiplication, so that for every $f \in \mathbb{V}_\infty$ the linear form on $\mathbb{V}_\infty$

$$\mathbf{\Gamma}[f;\varphi] := \mathcal{E}(f, f\varphi) - \frac{1}{2}\mathcal{E}(f^2, \varphi), \quad \varphi \in \mathbb{V}_\infty, \tag{2.6}$$

is well defined and for every normal contraction $\eta : \mathbb{R} \to \mathbb{R}$ it satisfies [12, Prop. 2.3.3]

$$0 \le \mathbf{\Gamma}[\eta \circ f;\varphi] \le \mathbf{\Gamma}[f;\varphi] \le \|\varphi\|_\infty \, \mathcal{E}(f) \quad \text{for every } f, \varphi \in \mathbb{V}_\infty,\ \varphi \ge 0. \tag{2.7}$$

(2.7) shows that for every nonnegative $\varphi \in \mathbb{V}_\infty$ $f \mapsto \mathbf{\Gamma}[f;\varphi]$ is a quadratic form in $\mathbb{V}_\infty$ which satisfies the Markov property and can be extended by continuity to $\mathbb{V}$. We call $\mathbb{G}$ the set of functions $f \in \mathbb{V}$ such that the linear form $\varphi \mapsto \mathbf{\Gamma}[f;\varphi]$ can be represented by a an absolutely continuous measure w.r.t. $\mathfrak{m}$ with density $\Gamma\big(f\big) \in L^1_+(X, \mathfrak{m})$:

$$f \in \mathbb{G} \ \Leftrightarrow\ \mathbf{\Gamma}[f;\varphi] = \int_X \Gamma\big(f\big)\varphi \,\mathrm{d}\mathfrak{m} \quad \text{for every } \varphi \in \mathbb{V}_\infty. \tag{2.8}$$

Since $\mathcal{E}$ is strongly local, [12, Thm. 6.1.1] yields the representation formula

$$\mathcal{E}(f, f) = \int_X \Gamma\big(f\big) \,\mathrm{d}\mathfrak{m} \quad \text{for every } f \in \mathbb{G}. \tag{2.9}$$

It is not difficult to check that $\mathbb{G}$ is a closed vector subspace of $\mathbb{V}$, the restriction of $\mathcal{E}$ to $\mathbb{G}$ is still a strongly local Dirichlet form admitting the ***Carré du champ*** Γ defined by (2.8) (see e.g. [12, Def. 4.1.2]): Γ is a quadratic continuous map defined in $\mathbb{G}$ with values in $L^1_+(X, \mathfrak{m})$. We will see in the next Section 2.2 that if $\mathcal{E}$ satisfies the $\mathrm{BE}(K, \infty)$ condition then $\mathbb{G}$ coincides with $\mathbb{V}$ and $\mathcal{E}$ admits a functional Γ-calculus on the whole space $\mathbb{V}$.

Since we are going to use Γ-calculus techniques, we use the Γ notation also for the symmetric, bilinear and continuous map

$$\Gamma\big(f, g\big) := \frac{1}{4}\Big(\Gamma(f + g) - \Gamma(f - g)\Big) \in L^1(X, \mathfrak{m}) \qquad f,\, g \in \mathbb{G}, \tag{2.10}$$

which, thanks to (2.9), represents the bilinear form $\mathcal{E}$ by the formula

$$\mathcal{E}(f, g) = \int_X \Gamma\big(f, g\big) \,\mathrm{d}\mathfrak{m} \quad \text{for every } f, g \in \mathbb{G}. \tag{2.11}$$

Because of Markovianity and locality $\Gamma\big(\cdot,\cdot\big)$ satisfies the chain rule [12, Cor. 7.1.2]

$$\Gamma\big(\eta(f), g\big) = \eta'(f)\Gamma\big(f, g\big) \quad \text{for every } f,\, g \in \mathbb{G}, \quad \eta \in \mathrm{Lip}(\mathbb{R}),\ \eta(0) = 0, \tag{2.12}$$

and the Leibnitz rule:

$$\Gamma\big(fg, h\big) = f\Gamma\big(g, h\big) + g\Gamma\big(f, h\big) \quad \text{for every } f,\, g,\, h \in \mathbb{G}_\infty := \mathbb{G} \cap L^\infty(X, \mathfrak{m}). \tag{2.13}$$

Notice that by [12, Theorem 7.1.1] (2.12) is well defined since for every Lebesgue measurable set $N \subset \mathbb{R}$ (as the set where ϕ is not differentiable)

$$\mathscr{L}^1(N) = 0 \quad \Rightarrow \quad \Gamma\big(f\big) = 0 \quad \mathfrak{m}\text{-a.e. on } f^{-1}(N). \tag{2.14}$$

Among the most useful consequences of (2.14) and (2.12) that we will repeatedly use in the sequel, we recall that for every $f, g \in \mathbb{G}$

$$\Gamma\big(f - g\big) = 0 \quad \mathfrak{m}\text{-a.e. on } \{f = g\}, \tag{2.15}$$

and the following identities hold $\mathfrak{m}$-a.e.:

$$\Gamma\big(f \wedge g\big) = \begin{cases} \Gamma\big(f\big) & \text{on } \{f \le g\}, \\ \Gamma\big(g\big) & \text{on } \{f \ge g\}, \end{cases} \qquad \Gamma\big(f \vee g\big) = \begin{cases} \Gamma\big(f\big) & \text{on } \{f \ge g\}, \\ \Gamma\big(g\big) & \text{on } \{f \le g\}. \end{cases} \tag{2.16}$$

We conclude this section by stating the following lower semicontinuity result:

$$f_n, f \in \mathbb{G}, \quad f_n \rightharpoonup f, \ \sqrt{\Gamma\big(f_n\big)} \rightharpoonup G \text{ in } L^2(X, \mathfrak{m}) \quad \Rightarrow \quad \Gamma\big(f\big) \le G^2 \quad \mathfrak{m}\text{-a.e. in } X. \tag{2.17}$$

It can be easily proved by using Mazur's Lemma and the $\mathfrak{m}$-a.e. convexity of $f \mapsto \sqrt{\Gamma\big(f\big)}$ namely

$$\sqrt{\Gamma\big((1-t)f + tg\big)} \le (1-t)\sqrt{\Gamma\big(f\big)} + t\sqrt{\Gamma\big(g\big)} \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } t \in [0,1],$$

which follows since Γ is quadratic and nonnegative.

The Markov semigroup and its generator

The Dirichlet form $\mathcal{E}$ induces a densely defined selfadjoint operator $\Delta_{\mathcal{E}} : D(\Delta_{\mathcal{E}}) \subset \mathbb{V} \to L^2(X, \mathfrak{m})$ defined by the integration by parts formula $\mathcal{E}(f, g) = -\int_X g\, \Delta_{\mathcal{E}} f \,\mathrm{d}\mathfrak{m}$ for all $g \in \mathbb{V}$.

When $\mathbb{G} = \mathbb{V}$ the operator $\Delta_{\mathcal{E}}$ is of "diffusion" type, since it satisfies the following chain rule for every $\eta \in \mathrm{C}^2(\mathbb{R})$ with $\eta(0) = 0$ and bounded first and second derivatives (see [12, Corollary 6.1.4] and the next (2.26)): if $f \in D(\Delta_{\mathcal{E}})$ with $\Gamma\big(f\big) \in L^2(X, \mathfrak{m})$ then $\eta(f) \in D(\Delta_{\mathcal{E}})$ with

$$\Delta_{\mathcal{E}} \eta(f) = \eta'(f) \Delta_{\mathcal{E}} f + \eta''(f) \Gamma\big(f\big). \tag{2.18}$$

The heat flow P_t associated to $\mathcal{E}$ is well defined starting from any initial condition $f \in L^2(X, \mathfrak{m})$. Recall that in this framework the heat flow $(\mathsf{P}_t)_{t \ge 0}$ is an analytic Markov semigroup and $f_t = \mathsf{P}_t f$ can be characterized as the unique C^1 map $f : (0, \infty) \to L^2(X, \mathfrak{m})$, with values in $D(\Delta_{\mathcal{E}})$, satisfying

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} f_t = \Delta_{\mathcal{E}} f_t & \text{for } t \in (0, \infty), \\ \lim\limits_{t \downarrow 0} f_t = f & \text{in } L^2(X, \mathfrak{m}). \end{cases} \tag{2.19}$$

Because of this, $\Delta_{\mathcal{E}}$ can equivalently be characterized in terms of the strong convergence $(\mathsf{P}_t f - f)/t \to \Delta_{\mathcal{E}} f$ in $L^2(X, \mathfrak{m})$ as $t \downarrow 0$.

One useful consequence of the Markov property is the L^p contraction of $(\mathsf{P}_t)_{t\geq 0}$ from $L^p \cap L^2$ to $L^p \cap L^2$. Because of the density of $L^p \cap L^2$ in L^p when $p \in [1, \infty)$, this allows to extend uniquely P_t to a strongly continuous semigroup of linear contractions in $L^p(X, \mathfrak{m})$, $p \in [1, \infty)$, for which we retain the same notation. Furthermore, $(\mathsf{P}_t)_{t\geq 0}$ is sub-Markovian (cf. [12, Prop. 3.2.1]), since it preserves one-sided essential bounds, namely $f \leq C$ (resp. $f \geq C$) $\mathfrak{m}$-a.e. in X for some $C \geq 0$ (resp. $C \leq 0$) implies $\mathsf{P}_t f \leq C$ (resp. $\mathsf{P}_t f \geq C$) $\mathfrak{m}$-a.e. in X for all $t \geq 0$.

We will mainly be concerned with the mass-preserving case i.e.

$$\int_X \mathsf{P}_t f \, \mathrm{d}\mathfrak{m} = \int_X f \, \mathrm{d}\mathfrak{m} \quad \text{for every } f \in L^1(X, \mathfrak{m}), \tag{2.20}$$

a property which is equivalent to $1 \in D(\mathcal{E})$ when $\mathfrak{m}(X) < \infty$. In the next session (see Theorem 3.14) we will discuss a metric framework, which will imply (2.20).

The semigroup $(\mathsf{P}_t)_{t\geq 0}$ can also be extended by duality to a weakly*-continuous semigroup of contractions in $L^\infty(X, \mathfrak{m})$, so that

$$\int_X \mathsf{P}_t f \, \varphi \, \mathrm{d}\mathfrak{m} = \int_X f \, \mathsf{P}_t \varphi \, \mathrm{d}\mathfrak{m} \quad \text{for every } f \in L^\infty(X, \mathfrak{m}),\ \varphi \in L^1(X, \mathfrak{m}). \tag{2.21}$$

It is easy to show that if $f_n \in L^2 \cap L^\infty(X, \mathfrak{m})$ weakly* converge to f in $L^\infty(X, \mathfrak{m})$ then $\mathsf{P}_t f_n \stackrel{\star}{\rightharpoonup} \mathsf{P}_t f$ in $L^\infty(X, \mathfrak{m})$.

The generator of the semigroup in $L^1(X, \mathfrak{m})$

Sometimes it will also be useful to consider the generator $\Delta^{(1)}_{\mathcal{E}} : D(\Delta^{(1)}_{\mathcal{E}}) \subset L^1(X, \mathfrak{m}) \to L^1(X, \mathfrak{m})$ of $(\mathsf{P}_t)_{t\geq 0}$ in $L^1(X, \mathfrak{m})$ [33, §1.1]:

$$f \in D(\Delta^{(1)}_{\mathcal{E}}),\ \Delta^{(1)}_{\mathcal{E}} f = g \quad \Leftrightarrow \quad \lim_{t\downarrow 0} \frac{1}{t}\big(\mathsf{P}_t f - f\big) = g \quad \text{strongly in } L^1(X, \mathfrak{m}). \tag{2.22}$$

Thanks to (2.22) it is easy to check that

$$f \in D(\Delta^{(1)}_{\mathcal{E}}) \quad \Rightarrow \quad \mathsf{P}_t f \in D(\Delta^{(1)}_{\mathcal{E}}), \quad \Delta^{(1)}_{\mathcal{E}} \mathsf{P}_t f = \mathsf{P}_t \Delta^{(1)}_{\mathcal{E}} f \quad \text{for all } t \geq 0, \tag{2.23}$$

and, when (2.20) holds,

$$\int_X \Delta^{(1)}_{\mathcal{E}} f \, \mathrm{d}\mathfrak{m} = 0 \quad \text{for every } f \in D(\Delta^{(1)}_{\mathcal{E}}). \tag{2.24}$$

The operator $\Delta^{(1)}_{\mathcal{E}}$ is m-accretive and coincides with the smallest closed extension of $\Delta_{\mathcal{E}}$ to $L^1(X, \mathfrak{m})$: [12, Prop. 2.4.2]

$$g = \Delta^{(1)}_{\mathcal{E}} f \quad \Leftrightarrow \quad \begin{cases} \exists f_n \in D(\Delta_{\mathcal{E}}) \cap L^1(X, \mathfrak{m}) \text{ with } g_n = \Delta_{\mathcal{E}} f_n \in L^1(X, \mathfrak{m}) : \\ f_n \to f, \quad g_n \to g \quad \text{strongly in } L^1(X, \mathfrak{m}). \end{cases} \tag{2.25}$$

Whenever $f \in D(\Delta_{\mathcal{E}}^{(1)}) \cap L^2(X, \mathfrak{m})$ and $\Delta_{\mathcal{E}}^{(1)} f \in L^2(X, \mathfrak{m})$ one can recover $f \in D(\Delta_{\mathcal{E}})$ by (2.23), the integral formula $\mathsf{P}_t f - f = \int_0^t \mathsf{P}_r \Delta_{\mathcal{E}}^{(1)} f \, \mathrm{d}r$ and the contraction property of $(\mathsf{P}_t)_{t \geq 0}$ in every $L^p(X, \mathfrak{m})$, thus obtaining

$$f \in D(\Delta_{\mathcal{E}}^{(1)}) \cap L^2(X, \mathfrak{m}), \ \Delta_{\mathcal{E}}^{(1)} f \in L^2(X, \mathfrak{m}) \quad \Rightarrow \quad f \in D(\Delta_{\mathcal{E}}), \quad \Delta_{\mathcal{E}} f = \Delta_{\mathcal{E}}^{(1)} f. \quad (2.26)$$

Semigroup mollification

A useful tool to prove the above formula is given by the mollified semigroup: we fix a

$$\text{nonnegative kernel } \kappa \in \mathrm{C}_c^\infty(0, \infty) \text{ with } \int_0^\infty \kappa(r) \, \mathrm{d}r = 1, \quad (2.27)$$

and for every $f \in L^p(X, \mathfrak{m})$, $p \in [1, \infty]$, we set

$$\mathfrak{h}^\varepsilon f := \frac{1}{\varepsilon} \int_0^\infty P_r f \, \kappa(r/\varepsilon) \, \mathrm{d}r, \quad \varepsilon > 0, \quad (2.28)$$

where the integral should be intended in the Bochner sense whenever $p < \infty$ and by taking the duality with arbitrary $\varphi \in L^1(X, \mathfrak{m})$ when $p = \infty$.

Since $\Delta_{\mathcal{E}}$ is the generator of $(\mathsf{P}_t)_{t \geq 0}$ in $L^2(X, \mathfrak{m})$ it is not difficult to check [33, Proof of Thm. 2.7] that if $f \in L^2 \cap L^p(X, \mathfrak{m})$ for some $p \in [1, \infty]$ then

$$\Delta_{\mathcal{E}}(\mathfrak{h}^\varepsilon f) = \frac{1}{\varepsilon^2} \int_0^\infty P_r f \, \kappa'(r/\varepsilon) \, \mathrm{d}r \in L^2 \cap L^p(X, \mathfrak{m}). \quad (2.29)$$

The same holds for $\Delta_{\mathcal{E}}^{(1)}$ if $f \in L^1(X, \mathfrak{m})$:

$$\Delta_{\mathcal{E}}^{(1)}(\mathfrak{h}^\varepsilon f) = \frac{1}{\varepsilon^2} \int_0^\infty P_r f \, \kappa'(r/\varepsilon) \, \mathrm{d}r \in L^1(X, \mathfrak{m}). \quad (2.30)$$

2.2 On the functional Bakry-Émery condition

We will collect in this section various equivalent characterizations of the Bakry-Émery condition $\mathrm{BE}(K, N)$ given in (1.6) for the Γ_2 operator operator (1.4). We have been strongly inspired by [7, 9, 43]: even if the essential estimates are well known, here we will take a particular care in establishing all the results in a weak form, under the minimal regularity assumptions on the functions involved. We consider here the case of finite dimension as well, despite the fact that the next sections 3 and 4 will be essentially confined to the case $N = \infty$. Applications of $\mathrm{BE}(K, N)$ with $N < \infty$ will be considered in the last Section 5.

Let us denote by $\mathbf{\Gamma} : (\mathbb{V}_\infty)^3 \to \mathbb{R}$ the multilinear map

$$\mathbf{\Gamma}[f, g; \varphi] := \frac{1}{2} \Big(\mathcal{E}(f, g\varphi) + \mathcal{E}(g, f\varphi) - \mathcal{E}(fg, \varphi) \Big), \qquad \mathbf{\Gamma}[f; \varphi] = \mathbf{\Gamma}[f, f; \varphi]; \quad (2.31)$$

Recalling (2.7), one can easily prove the uniform continuity property

$$\mathbb{V}_\infty \ni f_n \to f,\ \varphi_n \to \varphi \text{ strongly in } \mathbb{V},\ \sup_n \|\varphi_n\|_\infty < \infty \quad \Rightarrow \quad \exists \lim_{n\to\infty} \mathbf{\Gamma}[f_n; \varphi_n] \in \mathbb{R}, \tag{2.32}$$

which allows to extend $\mathbf{\Gamma}$ to a real multilinear map defined in $\mathbb{V} \times \mathbb{V} \times \mathbb{V}_\infty$, for which we retain the same notation. The extension $\mathbf{\Gamma}$ satisfies

$$\mathbf{\Gamma}[f, g; \varphi] = \int_X \Gamma\big(f, g\big)\varphi \,\mathrm{d}\mathfrak{m} \quad \text{if } f, g \in \mathbb{G}. \tag{2.33}$$

We also set

$$\mathbf{\Gamma}_2[f; \varphi] := \frac{1}{2}\mathbf{\Gamma}[f; \Delta_{\mathcal{E}}\varphi] - \mathbf{\Gamma}[f, \Delta_{\mathcal{E}} f; \varphi], \quad (f, \varphi) \in D(\mathbf{\Gamma}_2) \tag{2.34}$$

where

$$D(\mathbf{\Gamma}_2) := \big\{(f, \varphi) \in D(\Delta_{\mathcal{E}}) \times D(\Delta_{\mathcal{E}}) : \Delta_{\mathcal{E}} f \in \mathbb{V}, \quad \varphi, \Delta_{\mathcal{E}}\varphi \in L^\infty(X, \mathfrak{m})\big\}. \tag{2.35}$$

As for (2.33), we have

$$\mathbf{\Gamma}_2[f; \varphi] = \int_X \Big(\frac{1}{2}\Gamma\big(f\big)\,\Delta_{\mathcal{E}}\varphi - \Gamma\big(f, \Delta_{\mathcal{E}} f\big)\varphi\Big)\,\mathrm{d}\mathfrak{m} \quad \text{if } (f, \varphi) \in D(\mathbf{\Gamma}_2),\ f, \Delta_{\mathcal{E}} f \in \mathbb{G}. \tag{2.36}$$

Since $(\mathsf{P}_t)_{t\geq 0}$ is an analytic semigroup in $L^2(X, \mathfrak{m})$, for a given $f \in L^2(X, \mathfrak{m})$ and $\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$, we can consider the functions

$$\mathsf{A}_t[f; \varphi](s) := \frac{1}{2}\int_X \big(\mathsf{P}_{t-s} f\big)^2\,\mathsf{P}_s\varphi \,\mathrm{d}\mathfrak{m} \qquad t > 0,\ s \in [0, t], \tag{2.37}$$

$$\mathsf{A}^\Delta_t[f; \varphi](s) := \frac{1}{2}\int_X \big(\Delta_{\mathcal{E}}\mathsf{P}_{t-s} f\big)^2\,\mathsf{P}_s\varphi \,\mathrm{d}\mathfrak{m} \qquad t > 0,\ s \in [0, t), \tag{2.38}$$

$$\mathsf{B}_t[f; \varphi](s) := \mathbf{\Gamma}[\mathsf{P}_{t-s} f; \mathsf{P}_s\varphi] \qquad t > 0,\ s \in [0, t), \tag{2.39}$$

and, whenever $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$,

$$\mathsf{C}_t[f; \varphi](s) := \mathbf{\Gamma}_2[\mathsf{P}_{t-s} f; \mathsf{P}_s\varphi], \qquad t > 0,\ s \in [0, t). \tag{2.40}$$

Notice that whenever $\Delta_{\mathcal{E}} f \in L^2(X, \mathfrak{m})$

$$\mathsf{A}^\Delta_t[f; \varphi](s) = \mathsf{A}_t[\Delta_{\mathcal{E}} f; \varphi](s) \qquad t > 0,\ s \in [0, t). \tag{2.41}$$

Lemma 2.1. *For every $f \in L^2(X, \mathfrak{m}), \varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$ and every $t > 0$, we have:*

(i) *the function $s \mapsto \mathsf{A}_t[f; \varphi](s)$ belongs to $\mathrm{C}^0([0, t]) \cap \mathrm{C}^1((0, t))$;*

(ii) *the function $s \mapsto \mathsf{A}^\Delta_t[f; \varphi](s)$ belongs to $\mathrm{C}^1([0, t))$;*

(iii) *the function* $s \mapsto \mathsf{B}_t[f;\varphi](s)$ *belongs to* $\mathrm{C}^0((0,t)) \cap L^\infty(0,t)$ *and*

$$\frac{\partial}{\partial s}\mathsf{A}_t[f;\varphi](s) = \mathsf{B}_t[f;\varphi](s) \quad \textit{for every } s \in (0,t). \tag{2.42}$$

Equation (2.42) *and the regularity of* A *and* B *extend to* $s = t$ *if* $f \in \mathbb{V}$ *and to* $s = 0$ *if* $\varphi \in \mathbb{V}_\infty$.

(iv) *If* φ *is nonnegative,* $s \mapsto \mathsf{A}_t[f;\varphi](s)$ *and* $s \mapsto \mathsf{A}_t^\Delta[f;\varphi](s)$ *are nondecreasing.*

(v) *If* $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X,\mathfrak{m})$ *then* C *belongs to* $\mathrm{C}^0([0,t))$*,* B *belongs to* $\mathrm{C}^1([0,t))$*, and*

$$\frac{\partial}{\partial s}\mathsf{B}_t[f;\varphi](s) = 2\mathsf{C}_t[f;\varphi](s) \quad \textit{for every } s \in [0,t). \tag{2.43}$$

In particular $\mathsf{A} \in \mathrm{C}^2([0,t))$.

Proof. The continuity of A is easy to check, since $s \mapsto (\mathsf{P}_{t-s}f)^2$ is strongly continuous with values in $L^1(X,\mathfrak{m})$ and $s \mapsto \mathsf{P}_s\varphi$ is weakly* continuous in $L^\infty(X,\mathfrak{m})$. Analogously, the continuity of B follows from the fact that $s \mapsto \mathsf{P}_{t-s}f$ is a continuous curve in $\mathbb{V}$ whenever $s \in [0,t)$ thanks to the regularizing effect of the heat flow and (2.32). The continuity of C follows by a similar argument, recalling the definition (2.34) and the fact that the curves $s \mapsto \Delta_{\mathcal{E}}\mathsf{P}_{t-s}f$ and $s \mapsto \Delta_{\mathcal{E}}\mathsf{P}_s\varphi$ are continuous with values in $\mathbb{V}$ in the interval $[0,t)$.

In order to prove (2.42) and (2.43), let us first assume that $\varphi \in D(\Delta_{\mathcal{E}})$ with $\Delta_{\mathcal{E}}\varphi \in L^\infty(X,\mathfrak{m})$ and $f \in L^2 \cap L^\infty(X,\mathfrak{m})$. Since

$$\lim_{h\to 0}\frac{\mathsf{P}_{t-(s+h)}f - \mathsf{P}_{t-s}f}{h} = -\Delta_{\mathcal{E}}\mathsf{P}_{t-s}f \quad \text{strongly in } \mathbb{V} \text{ for } s \in [0,t),$$
$$\lim_{h\to 0}\frac{\mathsf{P}_{s+h}\varphi - \mathsf{P}_s\varphi}{h} = \Delta_{\mathcal{E}}\mathsf{P}_s\varphi \quad \text{weakly}^* \text{ in } L^\infty(X,\mathfrak{m}) \text{ for } s \in [0,t),$$

we easily get

$$\begin{aligned}\frac{\partial}{\partial s}\mathsf{A}_t[f;\varphi](s) &= \int_X \Big(-\mathsf{P}_{t-s}f\,\Delta_{\mathcal{E}}\mathsf{P}_{t-s}f\,\mathsf{P}_s\varphi + \frac{1}{2}\big(\mathsf{P}_{t-s}f\big)^2\Delta_{\mathcal{E}}\mathsf{P}_s\varphi\Big)\,\mathrm{d}\mathfrak{m}\\ &= \mathcal{E}(\mathsf{P}_{t-s}f, \mathsf{P}_{t-s}f\,\mathsf{P}_s\varphi) - \frac{1}{2}\mathcal{E}((\mathsf{P}_{t-s}f)^2, \mathsf{P}_s\varphi) = \mathsf{B}_t[f;\varphi](s),\end{aligned}$$

by the very definition (2.6) of $\mathbf{\Gamma}$, since $\mathsf{P}_{t-s}f$ is essentially bounded and therefore $(\mathsf{P}_{t-s}f)^2 \in \mathbb{V}_\infty$. A similar computation yields (2.43).

In order to extend the validity of (2.42) and (2.43) to general $f \in L^2(X,\mathfrak{m})$ we can approximate f by truncation setting $f_n := -n \vee f \wedge n$, $n \in \mathbb{N}$, and we pass to the limit in the integrated form

$$\mathsf{A}_t[f_n;\varphi](s_2) - \mathsf{A}_t[f_n;\varphi](s_1) = \int_{s_1}^{s_2}\mathsf{B}_t[f_n;\varphi](s)\,\mathrm{d}s \quad \text{for every } 0 \le s_1 < s_2 < t,$$

observing that $\mathsf{P}_{t-s}f_n$ converge strongly to $\mathsf{P}_{t-s}f$ in $\mathbb{V}$ as $n \to \infty$ for every $s \in [0,t)$, so that (2.32) yields the pointwise convergence of the integrands in the previous identity. A similar argument holds for (2.43), since $\Delta_{\mathcal{E}}\mathsf{P}_t f_n$ converges strongly to $\Delta_{\mathcal{E}}\mathsf{P}_t f$ in $\mathbb{V}$.

Eventually we extend (2.42) to arbitrary $\varphi \in L^2 \cap L^\infty(X,\mathfrak{m})$ by approximating φ with $\mathfrak{h}^\varepsilon\varphi$ given by (2.28), (2.27). It is not difficult to check that $\mathsf{P}_s(\mathfrak{h}^\varepsilon\varphi) \to \mathsf{P}_s\varphi$ in $\mathbb{V}$ as $\varepsilon \downarrow 0$ with uniform L^∞ bound if $s > 0$ (and also when $s = 0$ if $\varphi \in \mathbb{V}_\infty$). □

Lemma 2.2. *Let us consider functions* $\mathsf{a} \in \mathrm{C}^1([0,t))$, $\mathsf{g} \in \mathrm{C}^0([0,t))$ *and a parameter* $\nu \geq 0$. *The following properties are equivalent:*

(i) a, g *satisfy the differential inequality*

$$\mathsf{a}'' \geq 2K\,\mathsf{a}' + \nu\,\mathsf{g} \quad \textit{in } \mathscr{D}'(0,t), \tag{2.44}$$

and pointwise in $[0,t)$, *whenever* $\mathsf{a} \in \mathrm{C}^2([0,t))$.

(ii) a', g *satisfy the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{e}^{-2Ks}\mathsf{a}'(s)\Big) \geq \nu\,\mathrm{e}^{-2Ks}\mathsf{g}(s) \quad \textit{in } \mathscr{D}'(0,t). \tag{2.45}$$

(iii) *For every* $0 \leq s_1 < s_2 < t$ *and every test function* $\zeta \in \mathrm{C}^2([s_1,s_2])$ *we have*

$$\int_{s_1}^{s_2} \mathsf{a}(\zeta'' + 2K\zeta')\,\mathrm{d}s + \Big[\mathsf{a}'\,\zeta\Big]_{s_1}^{s_2} - \Big[\mathsf{a}\,(\zeta' + 2K\zeta)\Big]_{s_1}^{s_2} \geq \nu \int_{s_1}^{s_2} \mathsf{g}\,\zeta\,\mathrm{d}s. \tag{2.46}$$

(iv) *For every* $0 \leq s_1 < s_2$ *we have*

$$\mathrm{e}^{-2K(s_2-s_1)}\mathsf{a}'(s_2) \geq \mathsf{a}'(s_1) + \nu \int_{s_1}^{s_2} \mathrm{e}^{-2K(s-s_1)}\mathsf{g}(s)\,\mathrm{d}s. \tag{2.47}$$

The *proof* is straightforward; we only notice that (2.46) holds also for $s_1 = 0$ since $\mathsf{a} \in \mathrm{C}^1([0,t))$.

The inequality (2.46) has two useful consequences, that we make explicit in terms of the functions I_K and $\mathrm{I}_{K,2}$ defined by

$$\mathrm{I}_K(t) = \int_0^t \mathrm{e}^{Ks}\,\mathrm{d}s = \frac{\mathrm{e}^{Kt} - 1}{K}, \quad \mathrm{I}_{K,2}(t) = \int_0^t \mathrm{I}_K(s)\,\mathrm{d}s = \frac{\mathrm{e}^{Kt} - Kt - 1}{K^2}, \tag{2.48}$$

with the obvious definition for $K = 0$: $I_0(t) = t,\ I_{0,2}(t) = t^2/2$.

Choosing $s_1 = 0,\ s_2 = \tau$ and

$$\zeta(s) := \mathrm{I}_{2K}(\tau - s) = \frac{\mathrm{e}^{2K(\tau-s)} - 1}{2K}, \qquad \text{so that } \zeta' + 2K\zeta = -1, \quad \zeta(\tau) = 0, \tag{2.49}$$

we obtain

$$\mathrm{I}_{2K}(\tau)\,\mathsf{a}'(0) + \nu \int_0^\tau \mathrm{I}_{2K}(\tau - s)\,\mathsf{g}(s)\,\mathrm{d}s \leq \mathsf{a}(\tau) - \mathsf{a}(0) \quad \text{for every } \tau \in [0,t]. \tag{2.50}$$

Choosing

$$\zeta(s) := \mathrm{I}_{-2K}(s) = \frac{1-\mathrm{e}^{-2Ks}}{2K}, \qquad \text{so that} \quad \zeta' + 2K\zeta = 1, \quad \zeta(0) = 0, \tag{2.51}$$

we obtain

$$\mathsf{a}(\tau) - \mathsf{a}(0) + \nu \int_0^\tau \mathrm{I}_{-2K}(s)\, \mathsf{g}(s)\,\mathrm{d}s \le \mathsf{a}'(\tau)\,\mathrm{I}_{-2K}(\tau) \quad \text{for every } \tau \in [0,t). \tag{2.52}$$

Corollary 2.3. *Let $\mathcal{E}$ be a Dirichlet form in $L^2(X,\mathfrak{m})$ as in (2.1), and let $K \in \mathbb{R}$ and $\nu \ge 0$. The following conditions are equivalent:*

(i) *For every $(f,\varphi) \in D(\mathbf{\Gamma}_2)$, with $\varphi \ge 0$, we have*

$$\mathbf{\Gamma}_2[f;\varphi] \ge K\,\mathbf{\Gamma}[f;\varphi] + \nu \int_X (\Delta_{\mathcal{E}} f)^2\,\varphi\,\mathrm{d}\mathfrak{m}. \tag{2.53}$$

(ii) *For every $f \in L^2(X,\mathfrak{m})$ and every nonnegative $\varphi \in D(\Delta_{\mathcal{E}}) \cap L^\infty(X,\mathfrak{m})$ with $\Delta_{\mathcal{E}}\varphi \in L^\infty(X,\mathfrak{m})$ we have*

$$\mathsf{C}_t[f;\varphi](s) \ge K\mathsf{B}_t[f;\varphi](s) + 2\nu\,\mathsf{A}_t^\Delta[f;\varphi](s) \quad \textit{for every } 0 \le s < t. \tag{2.54}$$

(iii) *For every $f \in L^2(X,\mathfrak{m})$, every nonnegative $\varphi \in L^2 \cap L^\infty(X,\mathfrak{m})$, and $t > 0$*

$$\frac{\partial^2}{\partial s^2}\mathsf{A}_t[f;\varphi](s) \ge 2K\frac{\partial}{\partial s}\mathsf{A}_t[f;\varphi](s) + 4\nu\,\mathsf{A}_t^\Delta[f;\varphi](s) \quad \textit{in } \mathscr{D}'(0,t), \tag{2.55}$$

(or, equivalently, the inequality (2.55) holds pointwise in $[0,t)$ for every nonnegative $\varphi \in L^2 \cap L^\infty(X,\mathfrak{m})$ with $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X,\mathfrak{m})$.)

(iv) *For every $f \in L^2(X,\mathfrak{m})$ and $t > 0$ we have $\mathsf{P}_t f \in \mathbb{G}$ and*

$$\mathrm{I}_{2K}(t)\Gamma\big(\mathsf{P}_t f\big) + 2\nu\,\mathrm{I}_{2K,2}(t)\big(\Delta_{\mathcal{E}}\mathsf{P}_t f\big)^2 \le \frac{1}{2}\mathsf{P}_t\big(f^2\big) - \frac{1}{2}(\mathsf{P}_t f)^2 \quad \mathfrak{m}\textit{-a.e. in } X. \tag{2.56}$$

(v) *$\mathbb{G} = \mathbb{V}$ and for every $f \in \mathbb{V}$*

$$\frac{1}{2}\mathsf{P}_t\big(f^2\big) - \frac{1}{2}(\mathsf{P}_t f)^2 + 2\nu\,\mathrm{I}_{-2K,2}(t)\big(\Delta_{\mathcal{E}}\mathsf{P}_t f\big)^2 \le \mathrm{I}_{-2K,2}(t)\,\mathsf{P}_t\Gamma\big(f\big) \quad \mathfrak{m}\textit{-a.e. in } X. \tag{2.57}$$

(vi) *$\mathbb{G}$ is dense in $L^2(X,\mathfrak{m})$ and for every $f \in \mathbb{G}$ and $t > 0$ $\mathsf{P}_t f$ belongs to $\mathbb{G}$ with*

$$\Gamma\big(\mathsf{P}_t f\big) + 2\nu\,\mathrm{I}_{-2K}(t)\big(\Delta_{\mathcal{E}}\mathsf{P}_t f\big)^2 \le \mathrm{e}^{-2Kt}\mathsf{P}_t\Gamma\big(f\big) \quad \mathfrak{m}\textit{-a.e. in } X. \tag{2.58}$$

If one of these equivalent properties hods, then $\mathbb{G} = \mathbb{V}$ (i.e. $\mathcal{E}$ admits the Carré du Champ Γ in $\mathbb{V}$).

Proof. The implication (i)⇒(ii) is obvious. The converse implication is also true under the regularity assumption of (i): it is sufficient to pass to the limit in (2.54) as $s \uparrow t$ and then as $t \downarrow 0$.

(ii)⇒(iii) by (2.43) when $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$; the general case follows by approximation by the same argument we used in the proof of Lemma 2.1.

(iii)⇒(iv): by applying (2.50) (with obvious notation) we get

$$\mathrm{I}_{2K}(t)\mathbf{\Gamma}[\mathsf{P}_t f;\varphi] + 2\nu\,\mathrm{I}_{2K,2}(t)\int_X \big(\Delta_{\mathcal{E}}\mathsf{P}_t f\big)^2\varphi\,\mathrm{d}\mathfrak{m} \le \frac{1}{2}\int_X \Big(\mathsf{P}_t(f^2) - (\mathsf{P}_t f)^2\Big)\varphi\,\mathrm{d}\mathfrak{m}$$

for every nonnegative $\varphi \in \mathbb{V}_\infty$. Thus setting $h := \mathsf{P}_t(f^2) - (\mathsf{P}_t f)^2 \in L^1_+(X, \mathfrak{m})$, the linear functional ℓ on $\mathbb{V}_\infty$ defined by $\ell(\varphi) := \mathbf{\Gamma}[\mathsf{P}_t f;\varphi]$ satisfies

$$0 \le \ell(\varphi) \le \int_X h\,\varphi\,\mathrm{d}\mathfrak{m} \quad \text{for every } \varphi \in \mathbb{V}_\infty,\ \varphi \ge 0. \tag{2.59}$$

Since $\mathbb{V}_\infty$ is a lattice of functions satisfying the Stone property $\varphi \in \mathbb{V}_\infty \ \Rightarrow\ \varphi \wedge 1 \in \mathbb{V}_\infty$ and clearly (2.59) yields $\ell(\varphi_n) \to 0$ whenever $(\varphi_n)_{n\ge 0} \subset \mathbb{V}_\infty$ is a sequence of functions pointwise decreasing to 0, Daniell construction [11, Thm. 7.8.7] and Radon-Nykodim Theorem yields $\mathbf{\Gamma}[\mathsf{P}_t f;\varphi] = \int_X g\varphi\,\mathrm{d}\mathfrak{m}$ for some $g \in L^1_+(X, \mathfrak{m})$, so that $\mathsf{P}_t f \in \mathbb{G}$ and (2.56) holds.

This argument also shows that $\mathbb{G}$ is invariant under the action of $(\mathsf{P}_t)_{t\ge 0}$ and dense in $L^2(X, \mathfrak{m})$. A standard approximation argument yields the density in $\mathbb{V}$ (see, e.g. [5, Lemma 4.9]) and therefore $\mathbb{G} = \mathbb{V}$ (since $\mathbb{G}$ is closed in $\mathbb{V}$; see also [12, Prop. 4.1.3]).

Analogously, (iii)⇒(v) by (2.52) and (iii)⇒(vi) by (2.47).

Let us now show that (vi)⇒(iii). Since $\mathbb{G}$ is dense in $L^2(X, \mathfrak{m})$ and invariant with respect to $(\mathsf{P}_t)_{t\ge 0}$, we already observed that $\mathbb{G} = \mathbb{V}$. Let us now write (2.58) with $h > 0$ instead of t and with $f := \mathsf{P}_{t-s}v$ for some $0 < h < s < t$. Multiplying by $\mathsf{P}_{s-h}\varphi$ and integrating with respect to $\mathfrak{m}$, we obtain

$$\mathsf{B}_t[v;\varphi](s-h) + 4\nu\,\mathrm{I}_{-2K}(h)\mathsf{A}^\Delta_t[v;\varphi](s-h) \le \mathrm{e}^{-2Kh}\mathsf{B}_t[v;\varphi](s).$$

It is not restrictive to assume $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$ so that B is of class C^1 in $(0, t)$. We subtract $\mathsf{B}_t[v;\varphi](s)$ from both sides of the inequality, we divide by $h > 0$ and let $h \downarrow 0$ obtaining

$$\frac{\partial}{\partial s}\mathsf{B}_t[v;\varphi](s) - 2K\mathsf{B}_t[v;\varphi](s) \ge 4\nu\,\mathsf{A}^\Delta_t[v;\varphi](s)$$

i.e. (2.55).

To show that (iv)⇒(iii) we first write (2.56) at $t = h > 0$ in the form

$$\mathrm{I}_{2K,2}(h)\Big(K\,\Gamma\big(\mathsf{P}_h f\big) + 2\nu\,\big(\Delta_{\mathcal{E}}\mathsf{P}_h f\big)^2\Big) \le \frac{1}{2}\mathsf{P}_h(f^2) - \frac{1}{2}\big(\mathsf{P}_h f\big)^2 - h\Gamma\big(\mathsf{P}_h f\big),$$

obtaining by subtracting $h\Gamma\big(\mathsf{P}_h f\big)$ from both sides of the inequality. Then we choose $f = \mathsf{P}_{t-s-h}v$ and we multiply the inequality by $\mathsf{P}_s\varphi$, with $\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$ nonnegative and $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$. We obtain

$$\mathrm{I}_{2K,2}(h)\Big(2K\,\mathsf{B}_t[v;\varphi](s) + 4\nu\,\mathsf{A}^\Delta_t[v;\varphi](s)\Big) \le \mathsf{A}_t[v;\varphi](s+h) - \mathsf{A}_t[v;\varphi](s) - h\mathsf{B}_t[v;\varphi](s).$$

Since A is of class C^2 and $\mathsf{A}' = \mathsf{B}$, dividing by $h^2 > 0$ and passing to the limit as $h \downarrow 0$ a simple Taylor expansion yields

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}\mathsf{A}_t[v;\varphi](s) \geq \frac{1}{2}\Big(2K\,\mathsf{B}_t[v;\varphi](s) + 4\nu\,\mathsf{A}_t^{\Delta}[v;\varphi](s)\Big).$$

A similar argument shows the last implication (v)⇒(iii). □

Definition 2.4 (The condition BE(K,N))**.** Let $K \in \mathbb{R}$ and $\nu \geq 0$. We say that a Dirichlet form $\mathcal{E}$ in $L^2(X,\mathfrak{m})$ as in (2.1) satisfies a functional BE(K,N) condition if one of the equivalent properties in Corollary 2.3 holds with $N := 1/\nu$.

Notice that

$$\mathrm{BE}(K,N) \quad \Rightarrow \quad \mathrm{BE}(K',N') \quad \text{for every } K' \leq K,\ N' \geq N, \tag{2.60}$$

in particular BE$(K,N) \Rightarrow$ BE(K,∞).

Remark 2.5 (Carré du Champ in the case $N=\infty$). If a strongly local Dirichlet form $\mathcal{E}$ satisfies BE(K,∞) for some $K \in \mathbb{R}$, then it admits a Carré du Champ Γ on $\mathbb{V}$, i.e. $\mathbb{G} = \mathbb{V}$, by (v) of Corollary 2.3; moreover the spaces

$$\mathbb{V}^1_\infty := \big\{\varphi \in \mathbb{V}_\infty : \Gamma\big(\varphi\big) \in L^\infty(X,\mathfrak{m})\big\}, \quad \mathbb{V}^2_\infty := \big\{\varphi \in \mathbb{V}^1_\infty,\ \Delta_{\mathcal{E}}\varphi \in L^\infty(X,\mathfrak{m})\big\}. \tag{2.61}$$

are dense in $\mathbb{V}$: in fact (2.58) shows that they are invariant under the action of $(\mathsf{P}_t)_{t\geq 0}$ and (2.56) (possibly combined with a further mollification as in (2.28) in the case of $\mathbb{V}^2_\infty$) shows that their closure in $L^2(X,\mathfrak{m})$ contains $L^2 \cap L^\infty(X,\mathfrak{m})$.

3 Energy metric measure structures

In this section, besides the standing assumptions we made on $\mathcal{E}$, we shall study the relation between the measure/energetic structure of X and an additional metric structure. Our main object will be the canonical distance $\mathsf{d}_{\mathcal{E}}$ associated to the Dirichlet form $\mathcal{E}$, that we will introduce and study in the next §3.3. Before doing that, we will recall the metric notions that will be useful in the following. Since many properties will just depend of a few general compatibility conditions between the metric and the energetic structure, we will try to enucleate such a conditions and state the related theorems in full generality..

Our first condition just refers to the measure $\mathfrak{m}$ and a distance d and does not involve the Dirichlet form $\mathcal{E}$:

Condition (**MD**: Measure-Distance interaction)**.** d is a distance on $X \times X$ such that:
(**MD.a**) (X,d) is a complete and separable metric space, $\mathcal{B}$ coincides with the completion of the Borel σ-algebra of (X,d) with respect to $\mathfrak{m}$, and $\mathrm{supp}(\mathfrak{m}) = X$;
(**MD.b**) $\mathfrak{m}(B_r(x)) < \infty$ for every $x \in X$, $r > 0$.

Besides the finiteness condition (**MD.b**), we will often assume a further exponential growth condition on the measures of the balls of (X, d), namely that there exist $x_0 \in X,\ M > 0,\ c \geq 0$ such that

$$\mathfrak{m}(B_r(x_0)) \leq M \exp(c\, r^2) \quad \text{for every } r \geq 0; \tag{\textbf{MD.exp}}$$

in this case we will collectively refer to the above conditions (**MD**) and (**MD.exp**) as (**MD+exp**).

3.1 Metric notions

In this section we recall a few basic definitions and results which are related to a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfying (**MD**).

Absolutely continuous curves, Lipschitz functions and slopes

$\mathrm{AC}^p([a,b]; X)$, $1 \leq p \leq \infty$, is the collection of all the absolutely continuous curves $\gamma : [a,b] \to X$ with finite p-energy: $\gamma \in \mathrm{AC}^p([a,b]; X)$ if there exists $v \in L^p(a,b)$ such that

$$\mathsf{d}(\gamma(s), \gamma(t)) \leq \int_s^t v(r)\,\mathrm{d}r \quad \text{for every } a \leq s \leq t \leq b. \tag{3.1}$$

The metric velocity of γ, defined by

$$|\dot\gamma|(r) := \lim_{h \to 0} \frac{\mathsf{d}(\gamma(r+h), \gamma(r))}{|h|}, \tag{3.2}$$

exists for $\mathscr{L}^1$-a.e. $r \in (a,b)$, belongs to $L^p(a,b)$, and provides the minimal function v such that (3.1) holds. The length of an absolutely continuous curve γ is then defined by $\int_a^b |\dot\gamma|(r)\,\mathrm{d}r$.

(X, d) is a *length space* if

$$\mathsf{d}(x_0, x_1) = \inf\Big\{ \int_0^1 |\dot\gamma|(r)\,\mathrm{d}r : \gamma \in \mathrm{AC}([0,1]; X),\ \gamma(i) = x_i \Big\} \quad \text{for every } x_0, x_1 \in X. \tag{3.3}$$

We denote by $\mathrm{Lip}_b(X)$ the space of all Lipschitz and bounded function $\varphi : X \to \mathbb{R}$ and by $\mathrm{Lip}^1(X)$ the subset of functions with Lipschitz constant less than 1. Every Lipschitz function φ is absolutely continuous along any absolutely continuous curve; we say that a bounded Borel function $g : X \to [0, \infty)$ is an *upper gradient* of $\varphi \in \mathrm{Lip}_b(X)$ if for any curve $\gamma \in \mathrm{AC}([a,b]; X)$ the absolutely continuous map $\varphi \circ \gamma$ satisfies

$$\Big| \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\gamma(t)) \Big| \leq g(\gamma(t))\, |\dot\gamma|(t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (a,b). \tag{3.4}$$

Among the upper gradients of a function $\varphi \in \mathrm{Lip}_b(X)$ its *slopes* and its *local Lipschitz constant* play a crucial role: they are defined by

$$|\mathrm{D}^\pm\varphi|(x) := \limsup_{y\to x} \frac{\big(\varphi(y)-\varphi(x)\big)_\pm}{\mathsf{d}(y,x)}, \quad |\mathrm{D}\varphi|(x) := \limsup_{y\to x} \frac{\varphi(y)-\varphi(x)}{\mathsf{d}(y,x)}, \tag{3.5}$$

$$|\mathrm{D}^*\varphi|(x) := \limsup_{\substack{y,z\to x\\ y\neq z}} \frac{\varphi(y)-\varphi(z)}{\mathsf{d}(y,z)}, \tag{3.6}$$

and, whenever (X,d) is a length space

$$|\mathrm{D}^*\varphi|(x) = \limsup_{y\to x} |\mathrm{D}\varphi|(y), \qquad \mathrm{Lip}(\varphi) = \sup_{x\in X} |\mathrm{D}\varphi(x)| = \sup_{x\in X} |\mathrm{D}^*\varphi(x)|. \tag{3.7}$$

In fact, (3.4) written for $g := |\mathrm{D}\varphi|$ and the length condition (3.3) easily yield

$$|\varphi(y)-\varphi(z)| \le \mathsf{d}(y,z) \sup_{B_{2r}(x)} |\mathrm{D}\varphi| \quad \text{if } y,z\in B_r(x) \tag{3.8}$$

and provide the inequality $|\mathrm{D}^*\varphi| \le \limsup_{y\to x} |\mathrm{D}\varphi|(y)$. The proof of the converse inequality is trivial and a similar argument shows the last identity in (3.7).

The Hopf-Lax evolution formula

Let us suppose that (X,d) is a metric space; the Hopf-Lax evolution map $Q_t : \mathrm{C}_b(X) \to \mathrm{C}_b(X)$, $t\ge 0$, is defined by

$$Q_t f(x) := \inf_{y\in X} f(y) + \frac{\mathsf{d}^2(y,x)}{2t}. \tag{3.9}$$

We introduce as in [3, §3] the maps

$$\mathsf{D}^+(x,s) := \sup_{(y_n)} \limsup_n \mathsf{d}(x,y_n), \qquad \mathsf{D}^-(x,s) := \inf_{(y_n)} \liminf_n \mathsf{d}(x,y_n), \tag{3.10}$$

where the supremum and the infimum run among minimizing sequences for (3.9). We recall that D^+ and D^- are respectively upper and lower semicontinuous, nondecreasing w.r.t. s, and that $\mathsf{D}^+(x,r) \le \mathsf{D}^-(x,s) \le \mathsf{D}^+(x,s)$ whenever $0<r<s$. These properties imply $\mathsf{D}^-(x,s) = \sup_{r<s} \mathsf{D}^+(x,r)$. We shall need the inequality

$$Q_{s'}f(x) - Q_s f(x) \le \frac{(\mathsf{D}^+(x,s))^2}{2}\Big(\frac{1}{s'}-\frac{1}{s}\Big) \qquad s'>s, \tag{3.11}$$

as well as the pointwise properties

$$-\frac{\mathrm{d}^\pm}{\mathrm{d}s} Q_s f(x) = \frac{(\mathsf{D}^\pm(x,s))^2}{2s^2}, \quad |\mathrm{D}Q_s f|(x) \le \frac{\mathsf{D}^+(x,s)}{s}, \tag{3.12}$$

(these are proved in Proposition 3.3 and Proposition 3.4 of [3]). Since

$$\mathsf{d}(y,x) > 2s\,\mathrm{Lip}(f) \quad \Rightarrow \quad f(y) + \frac{\mathsf{d}^2(x,y)}{2s} > f(x) \ge Q_s f(x),$$

we immediately find $\mathsf{D}^+(x,s) \le 2s\,\mathrm{Lip}(f)$.

Since by (3.11) the map $s \mapsto Q_s f(x)$ is locally Lipschitz in $(0,\infty)$, integrating the first identity of (3.14) in (ε, t), $\varepsilon > 0$, and then letting $\varepsilon \downarrow 0$ we get

$$f(x) - Q_t f(x) = \int_0^t \frac{(\mathsf{D}^+(x,s))^2}{2s^2}\,\mathrm{d}s = \frac{1}{2}\int_0^1 \Big(\frac{\mathsf{D}^+(x,tr)}{tr}\Big)^2\,\mathrm{d}r.$$

Combining the above identity with the formula expressing the descending slope (see [2, Lemma 3.1.5], which holds without coercivity assumptions on f)

$$|\mathrm{D}^- f|^2(x) = 2\limsup_{t\downarrow 0} \frac{f(x) - Q_t f(x)}{t},$$

we end up with

$$|\mathrm{D}f|^2(x) \ge |\mathrm{D}^- f|^2(x) = \limsup_{t\downarrow 0} \int_0^1 \Big(\frac{\mathsf{D}^+(x,tr)}{tr}\Big)^2\,\mathrm{d}r. \tag{3.13}$$

When (X,d) is a length space $(Q_t)_{t\ge 0}$ is a semigroup and we have the refined identity [3, Thm. 3.6]

$$\frac{\mathrm{d}^+}{\mathrm{d}s} Q_s f(x) = -\frac{1}{2}|\mathrm{D}Q_s f|^2(x) = -\frac{(\mathsf{D}^+(x,s))^2}{2s^2}. \tag{3.14}$$

(3.12) and the length property of X yield the a priori bounds

$$\mathrm{Lip}(Q_s f) \le 2\,\mathrm{Lip}(f), \qquad \mathrm{Lip}\big(Q_\cdot f(x)\big) \le 2\,\big[\mathrm{Lip}(f)\big]^2. \tag{3.15}$$

The Cheeger energy

The Cheeger energy of a function $f \in L^2(X,\mathfrak{m})$ is defined as

$$\mathsf{Ch}(f) := \inf\Big\{ \liminf_{n\to\infty} \frac{1}{2}\int_X |\mathrm{D}f_n|^2\,\mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_b(X), \quad f_n \to f \text{ in } L^2(X,\mathfrak{m})\Big\}. \tag{3.16}$$

If $f \in L^2(X,\mathfrak{m})$ with $\mathsf{Ch}(f) < \infty$, then there exists a unique function $|\mathrm{D}f|_w \in L^2(X,\mathfrak{m})$, called *minimal weak gradient of* f, satisfying the two conditions

$$\begin{gathered}\mathrm{Lip}_b(X) \cap L^2(X,\mathfrak{m}) \ni f_n \rightharpoonup f,\ |\mathrm{D}f_n| \rightharpoonup G \quad \text{in } L^2(X,\mathfrak{m}) \quad \Rightarrow \quad |\mathrm{D}f|_w \le G \\ \mathsf{Ch}(f) = \frac{1}{2}\int_X |\mathrm{D}f|_w^2\,\mathrm{d}\mathfrak{m}.\end{gathered} \tag{3.17}$$

In the next section 3.3 we will also use a further approximation result proved in [4, §8.3]: for every $f \in L^2(X,\mathfrak{m})$ with $\mathsf{Ch}(f) < \infty$

$$\exists\, f_n \in \mathrm{Lip}_b(X) \cap L^2(X,\mathfrak{m}) : \quad f_n \to f, \quad |\mathrm{D}^* f_n| \to |\mathrm{D}f|_w \quad \text{strongly in } L^2(X,\mathfrak{m}). \tag{3.18}$$

Wasserstein distances

The metric structure allows us to introduce the corresponding spaces $\mathscr{P}(X)$ of Borel probability measures and $\mathscr{P}_p(X)$ of Borel probability measures with finite p-th moment

$$\int_X \mathsf{d}^p(x,x_0)\,\mathrm{d}\mu(x)<\infty,\quad x_0 \text{ given in } X. \tag{3.19}$$

The L^p-Wasserstein transport (extended) distance W_p on $\mathscr{P}(X)$ is defined by

$$W_p^p(\mu_1,\mu_2) := \inf\Big\{\int_{X\times X}\mathsf{d}^p(x_1,x_2)\,\mathrm{d}\boldsymbol{\mu} : \boldsymbol{\mu}\in\mathscr{P}(X\times X),\ \pi^i_\sharp\boldsymbol{\mu}=\mu_i\Big\}, \tag{3.20}$$

where $\pi^i : X\times X \ni (x_1,x_2)\to x_i$ is the coordinate map and for a Borel measure $\mu\in\mathscr{P}(Y)$ on a metric space Y and every Borel map $\boldsymbol{r}: Y\to X$, the push-forward measure $\boldsymbol{r}_\sharp\mu\in\mathscr{P}(X)$ can be characterized by

$$\boldsymbol{r}_\sharp\mu(B) := \mu(\boldsymbol{r}^{-1}B)\quad\text{for every Borel set } B\subset X.$$

In particular, the competing measures $\boldsymbol{\mu}\in\mathscr{P}(X\times X)$ in (3.20) have marginals μ_1 and μ_2 respectively.

We also introduce a family of bounded distances on $\mathscr{P}(X)$ associated to a

$$\begin{gathered}\text{continuous, concave and bounded modulus of continuity } \beta:[0,\infty)\to[0,\infty),\\ \text{with } 0=\beta(0)<\beta(r) \text{ for every } r>0.\end{gathered} \tag{3.21}$$

As in (3.20) we set

$$W_{(\beta)}(\mu_1,\mu_2) := \inf\Big\{\int_{X\times X}\beta(\mathsf{d}(x_1,x_2))\,\mathrm{d}\boldsymbol{\mu} : \boldsymbol{\mu}\in\mathscr{P}(X\times X),\ \pi^i_\sharp\boldsymbol{\mu}=\mu_i\Big\}; \tag{3.22}$$

$W_{(\beta)}$ is thus the L^1-Wasserstein distance induced by the bounded distance $\mathsf{d}_\beta(x_1,x_2) := \beta(\mathsf{d}(x_1,x_2))$. $(\mathscr{P}(X), W_{(\beta)})$ is then a complete and separable metric space, whose topology coincides with the topology of weak convergence of probability measures.

Entropy and $\mathrm{RCD}(K,\infty)$ spaces

In the following we will fix $x_0\in X,\ z>0,\ c\geq 0$ such that

$$\tilde{\mathfrak{m}} = \frac{1}{z}\,\mathrm{e}^{-V^2}\mathfrak{m}\in\mathscr{P}(X),\qquad V(x) := \sqrt{c}\,\mathsf{d}(x,x_0). \tag{3.23}$$

Notice that in the case $\mathfrak{m}(X)<\infty$ we can always take $V\equiv c=0$ with $z=\mathfrak{m}(X)$. When $\mathfrak{m}(X)=\infty$, the possibility to choose $x_0\in X,\ z>0,\ c\geq 0$ satisfying (3.23) follows from (**MD.exp**) (possibly with a different constant c; it is in fact equivalent to (**MD.exp**)).

If $\mathfrak{n} : \mathcal{B} \to [0, \infty]$ is σ-additive, the *relative entropy* $\mathrm{Ent}_{\mathfrak{n}}(\rho)$ of a probability measure $\rho : \mathcal{B} \to [0, 1]$ with respect to $\mathfrak{n}$ is defined by

$$\mathrm{Ent}_{\mathfrak{n}}(\rho) := \begin{cases} \int_X f \log f \, \mathrm{d}\mathfrak{n} & \text{if } \rho = f\mathfrak{n}; \\ +\infty & \text{otherwise.} \end{cases} \tag{3.24}$$

The expression makes sense if $\mathfrak{n}$ is a probability measure, and thanks to Jensen's inequality defines a nonnegative functional. More generally we recall (see [3, Lemma 7.2] for the simple proof) that, when $\mathfrak{n} = \mathfrak{m}$ and (3.23) holds, the formula above makes sense on measures $\rho = f\mathfrak{m} \in \mathscr{P}_2(X)$ thanks to the fact that the negative part of $f \log f$ is $\mathfrak{m}$-integrable. Precisely, defining $\tilde{\mathfrak{m}} \in \mathscr{P}(X)$ as in (3.23) above, the following formula for the change of reference measure will be useful to estimate the negative part of $\mathrm{Ent}_{\mathfrak{m}}(\rho)$:

$$\mathrm{Ent}_{\mathfrak{m}}(\rho) = \mathrm{Ent}_{\tilde{\mathfrak{m}}}(\rho) - \int_X V^2(x) \, \mathrm{d}\rho(x) - \log z. \tag{3.25}$$

Definition 3.1 (RCD(K, ∞) spaces)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space satisfying (**MD+exp**) and the length property (3.3). We say that $(X, \mathsf{d}, \mathfrak{m})$ has Riemannian curvature bounded from below by $K \in \mathbb{R}$ if for all $\rho \in \mathscr{P}_2(X)$ there exists a solution $(\mathsf{H}_t\rho)_{t \geq 0} \subset \mathscr{P}_2(X)$ of the EVI_K-differential inequality starting from ρ, namely
$\mathsf{H}_t\rho \to \rho$ as $t \downarrow 0$ and (denoting by $\frac{\mathrm{d}^+}{\mathrm{d}t}$ the upper right derivative)

$$\frac{\mathrm{d}^+}{\mathrm{d}t} \frac{W_2^2(\mathsf{H}_t\rho, \nu)}{2} + \frac{K}{2} W_2^2(\mathsf{H}_t\rho, \nu) + \mathrm{Ent}_{\mathfrak{m}}(\mathsf{H}_t\rho) \leq \mathrm{Ent}_{\mathfrak{m}}(\nu) \quad \text{for every } t \in (0, \infty) \tag{3.26}$$

for all $\nu \in \mathscr{P}_2(X)$ with $\mathrm{Ent}_{\mathfrak{m}}(\nu) < \infty$.

As we already quoted in the Introduction, among the properties of RCD(K, ∞) spaces proved in [5] we recall that the Cheeger energy

$$\begin{gathered} \mathsf{Ch} \text{ is quadratic, i.e. } \mathsf{Ch}(f) = \frac{1}{2}\mathcal{E}_{\mathsf{Ch}}(f) \text{ for a Dirichlet form } \mathcal{E}_{\mathsf{Ch}} \text{ as in (2.1),} \\ \text{with } |\mathrm{D}f|_w^2 = \Gamma\big(f\big) \quad \text{for every } f \in D(\mathsf{Ch}) = \mathbb{V}, \end{gathered} \tag{QCh}$$

(in particular $\mathbb{G} = \mathbb{V}$ and $\mathcal{E}_{\mathsf{Ch}}$ admits the Carré du Champ Γ in $\mathbb{V}$) and $\mathcal{E}$ satisfies the BE(K, ∞) condition. A further crucial property will be recalled in Section 3.3 below, see Condition (**ED**) and Remark 3.8.

3.2 The dual semigroup and its contractivity properties

In this section we study the contractivity property of the dual semigroup of $(\mathsf{P}_t)_{t \geq 0}$ in the spaces of Borel probability measures.

Thus $\mathcal{E}$ is a strongly local Dirichlet form as in (2.1), $(\mathsf{P}_t)_{t \geq 0}$ satisfies the mass-preserving property (2.20) and d is a distance on X satisfying condition (**MD**) (assumption (**MD.exp**) is not needed here).

We see how, under the mild contractivity property

$$\mathsf{P}_t f \in \mathrm{Lip}_b(X) \text{ and } \mathrm{Lip}(\mathsf{P}_t f) \le C(t)\mathrm{Lip}(f) \text{ for all } f \in \mathrm{Lip}_b(X) \cap L^2(X, \mathfrak{m}), \tag{3.27}$$

with C bounded on all intervals $[0,T]$, $T>0$, a dual semigroup H_t in $\mathscr{P}(X)$ can be defined, satisfying the contractivity property (3.31) below w.r.t. $W_{(\beta)}$ and to W_1. This yields also the fact that P_t has a (unique) pointwise defined version $\tilde{\mathsf{P}}_t$, canonically defined also on bounded Borel functions, and mapping $\mathrm{C}_b(X)$ to $\mathrm{C}_b(X)$ (we will always identify $\mathsf{P}_t f$ with $\tilde{\mathsf{P}}_t f$ whenever $f \in \mathrm{C}_b(X)$). Then we shall prove, following the lines of [25], that in length metric spaces the pointwise Bakry-Émery-like assumption

$$|\mathrm{D}\,\mathsf{P}_t f|^2(x) \le C^2(t)\,\tilde{\mathsf{P}}_t |\mathrm{D} f|^2(x) \qquad \text{for all } x \in X,\ f \in \mathrm{Lip}_b(X) \cap L^2(X, \mathfrak{m}), \tag{3.28}$$

with C bounded on all intervals $[0,T]$, $T>0$, provides contractivity of H_t even w.r.t. W_2. Notice that formally (3.28) implies (3.27), but one has to take into account that (3.28) involves a pointwise defined version of the semigroup, which depends on (3.27).

A crucial point here is that we want to avoid doubling or local Poincaré assumptions on the metric measure space. For the aim of this section we introduce the following notation:

$$\begin{gathered} \mathcal{Z} \text{ is the collection of probability densities } f \in L^1_+(X, \mathfrak{m}), \\ \mathcal{K} \text{ is the set of nonnegative bounded functions with bounded support.} \end{gathered} \tag{3.29}$$

Proposition 3.2. *Let $\mathcal{E}$ and $(\mathsf{P}_t)_{t\ge 0}$ be as in* (2.1) *and* (2.20) *and let d be a distance on X satisfying the condition* **(MD)**. *If* (3.27) *holds then*

(i) The mapping $\mathsf{H}_t(f\mathfrak{m}) := (\mathsf{P}_t f)\mathfrak{m}$, $f \in \mathcal{Z}$, uniquely extends to a $W_{(\beta)}$-Lipschitz map $\mathsf{H}_t : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfying

$$W_{(\beta)}(\mathsf{H}_t\mu, \mathsf{H}_t\nu) \le \big(C(t) \vee 1\big)\, W_{(\beta)}(\mu, \nu) \qquad \text{for every } \mu,\ \nu \in \mathscr{P}(X), \tag{3.30}$$

$$W_1(\mathsf{H}_t\mu, \mathsf{H}_t\nu) \le C(t) W_1(\mu, \nu) \qquad \text{for every } \mu,\ \nu \in \mathscr{P}(X), \tag{3.31}$$

with $C(t)$ given by (3.27).

(ii) Defining $\tilde{\mathsf{P}}_t f(x) := \int_X f\,\mathrm{d}\mathsf{H}_t\delta_x$ on bounded or nonnegative Borel functions, $\tilde{\mathsf{P}}_t$ maps $\mathrm{C}_b(X)$ to $\mathrm{C}_b(X)$ and $\tilde{\mathsf{P}}_t$ is a version of P_t for all Borel functions f with $\int_X |f|\,\mathrm{d}\mathfrak{m} < \infty$, namely $\tilde{\mathsf{P}}_t f(x)$ is everywhere defined and $\mathsf{P}_t f(x) = \tilde{\mathsf{P}}_t f(x)$ for $\mathfrak{m}$-a.e. $x \in X$. In addition, $\tilde{\mathsf{P}}_t f$ is $\mathfrak{m}$-a.e. defined for every Borel function semi-integrable w.r.t. $\mathfrak{m}$.

(iii) H_t is dual to $\tilde{\mathsf{P}}_t$ in the following sense:

$$\int_X f\,\mathrm{d}\mathsf{H}_t\mu = \int_X \tilde{\mathsf{P}}_t f\,\mathrm{d}\mu \qquad \text{for all } f : X \to \mathbb{R} \text{ bounded Borel, } \mu \in \mathscr{P}(X). \tag{3.32}$$

(iv) For every $f \in \mathrm{C}_b(X)$ and $x \in X$ we have $\lim_{t\downarrow 0} \tilde{\mathsf{P}}_t f(x) = f(x)$. In particular, for every $\mu \in \mathscr{P}(X)$ the map $t \mapsto \mathsf{H}_t\mu$ is weakly continuous in $\mathscr{P}(X)$.

Proof. The concavity of β yields that β is subadditive, so that d_β is a distance. Let us first prove that P_t maps d_β-Lipschitz functions in d_β-Lipschitz functions.

We use the envelope representation

$$\beta(r) = \inf_{(a,b)\in \mathrm{B}} a + br, \quad \mathrm{B} = \big\{(a,b) \in [0,\infty) \times [0,\infty) : \beta(s) \le a + bs \text{ for every } s \ge 0\big\},$$

and the fact that a function $\varphi : X \to \mathbb{R}$ is ℓ-Lipschitz with respect to a distance d on X if and only if

$$\varphi(x) \le R_\ell\varphi(x) := \inf_{y\in X} \varphi(y) + \ell\,\mathsf{d}(x,y) \quad \text{for every } x \in X.$$

It is easy to check that if φ is bounded, then $R_\ell\varphi$ is bounded and satisfies

$$\inf_X \varphi \le R_\ell\varphi(x) \le \varphi(x) \quad \text{for every } x \in X,\ \ell \ge 0, \tag{3.33}$$

so that in particular R_ℓ maps $\mathcal{K}$ in $\mathcal{K}$. (3.27) then yields for every $\varphi \in \mathrm{Lip}_b(X) \cap L^2(X,\mathfrak{m})$ with $\mathrm{Lip}(\varphi) \le b$

$$\mathsf{P}_t\varphi \le R_{C(t)b}(\mathsf{P}_t\varphi).$$

Let us now suppose that $\varphi \in \mathcal{K}$ is d_β-Lipschitz, with Lipschitz constant less than 1, so that for every $(a,b) \in \mathrm{B}$

$$\varphi(x) \le \inf_{y\in X} \varphi(y) + \beta(\mathsf{d}(x,y)) \le \inf_{y\in X} \varphi(y) + a + b\,\mathsf{d}(x,y) = a + R_b\varphi(x).$$

Since $(\mathsf{P}_t)_{t\ge 0}$ is order preserving, we get for $\varphi \in \mathcal{K}$

$$\mathsf{P}_t\varphi \le a + \mathsf{P}_t(R_b\varphi) \le a + R_{C(t)b}\big(\mathsf{P}_t(R_b\varphi)\big) \le a + R_{C(t)b}\big(\mathsf{P}_t\varphi\big)$$

where we used the right inequality of (3.33) and the fact that $\mathrm{Lip}(R_b\varphi) \le b$. It follows that for every $x, y \in X$ and every $(a,b) \in \mathrm{B}$

$$\mathsf{P}_t\varphi(x) \le \mathsf{P}_t\varphi(y) + a + C(t)b\mathsf{d}(x,y), \quad \text{i.e.} \quad \mathsf{P}_t\varphi(x) - \mathsf{P}_t\varphi(y) \le \beta(C(t)\mathsf{d}(x,y)).$$

By Kantorovich duality, for $f,\ g \in \mathcal{Z}$ we get

$$\begin{aligned} W_{(\beta)}(\mathsf{P}_t f\,\mathfrak{m}, \mathsf{P}_t g\,\mathfrak{m}) &= \sup\left\{\int_X \varphi\,\mathsf{P}_t f\,\mathrm{d}\mathfrak{m} - \int_X \varphi\,\mathsf{P}_t g\,\mathrm{d}\mathfrak{m} :\ \varphi \in \mathcal{K},\ \mathrm{Lip}_{\mathsf{d}_\beta}(\varphi) \le 1\right\} \\ &= \sup\left\{\int_X f\,\mathsf{P}_t\varphi\,\mathrm{d}\mathfrak{m} - \int_X g\,\mathsf{P}_t\varphi\,\mathrm{d}\mathfrak{m} :\ \varphi \in \mathcal{K},\ \mathrm{Lip}_{\mathsf{d}_\beta}(\varphi) \le 1\right\} \\ &\le (C(t) \vee 1) W_{(\beta)}(f\mathfrak{m}, g\mathfrak{m}). \end{aligned}$$

Hence, (3.30) holds when $\mu = f\mathfrak{m}$, $\nu = g\mathfrak{m}$. By the density of $\{f\mathfrak{m} : f \in \mathcal{Z}\}$ in $\mathscr{P}(X)$ w.r.t. $W_{(\beta)}$ we get (3.30) for arbitrary $\mu, \nu \in \mathscr{P}(X)$. A similar argument yields (3.31).

(ii) Continuity of $x \mapsto \tilde{\mathsf{P}}_t f(x)$ when $f \in \mathrm{C}_b(X)$ follows directly by the continuity of $x \mapsto \mathsf{H}_t\delta_x$. The fact that $\tilde{\mathsf{P}}_t f$ is a version of P_t when f is Borel and $\mathfrak{m}$-integrable is a simple consequence of the fact that P_t is selfadjoint, see [5] for details.

(iii) When $f,\, g \in \mathrm{C}_b(X) \cap L^2(X,\mathfrak{m})$ and $\mu = g\mathfrak{m}$, the identity (3.32) reduces to the fact that P_t is selfadjoint. The general case can be easily achieved using a monotone class argument.

(iv) In the case of $\varphi \in \mathrm{Lip}_b(X) \cap L^2(X,\mathfrak{m})$ it is easy to prove that $\tilde{\mathsf{P}}_t\varphi(x) \to \varphi(x)$ for all $x \in X$ as $t \downarrow 0$, since $\tilde{\mathsf{P}}_t\varphi$ are equi-Lipschitz, converge in $L^2(X,\mathfrak{m})$ to φ and $\operatorname{supp}\mathfrak{m} = X$. By (3.32) it follows that

$$\lim_{t\downarrow 0}\int_X \varphi\,\mathrm{d}\mathsf{H}_t\mu = \lim_{t\downarrow 0}\int_X \tilde{\mathsf{P}}_t\varphi\,\mathrm{d}\mu = \int_X \varphi\,\mathrm{d}\mu \quad \text{for every } \varphi \in \mathrm{Lip}_b(X) \cap L^2(X,\mathfrak{m}).$$

By a density argument we obtain that the same holds on $\mathrm{Lip}_b(X)$, so that $t \mapsto \mathsf{H}_t\mu$ is weakly continuous. Since $\tilde{\mathsf{P}}_t f(x) = \int_X f\,\mathrm{d}\mathsf{H}_t\delta_x$, we conclude that $\tilde{\mathsf{P}}_t f(x) \to f(x)$ for arbitrary $f \in \mathrm{C}_b(X)$. □

Writing $\mu = \int_X \delta_x\,\mathrm{d}\mu(x)$ and recalling the definition of $\tilde{\mathsf{P}}_t$, we can also write (3.32) in the form

$$\mathsf{H}_t\mu = \int_X \mathsf{H}_t\delta_x\,\mathrm{d}\mu(x) \qquad \forall \mu \in \mathscr{P}(X). \tag{3.34}$$

In order to prove that (3.28) yields the contractivity property

$$W_2(\mathsf{H}_t\mu, \mathsf{H}_t\nu) \le C(t)\,W_2(\mu,\nu) \quad \text{for every } \mu,\, \nu \in \mathscr{P}(X),\ t \ge 0, \tag{W_2-cont}$$

we need the following auxiliary results.

Lemma 3.3. *Assume that $(\mu_n) \subset \mathscr{P}(X)$ weakly converges to $\mu \in \mathscr{P}(X)$, and that f_n are equibounded Borel functions satisfying*

$$\limsup_{n\to\infty} f_n(x_n) \le f(x) \qquad \textit{whenever } x_n \to x$$

for some Borel function f. Then $\limsup_n \int_X f_n\,\mathrm{d}\mu_n \le \int_X f\,\mathrm{d}\mu$.

Proof. Possibly adding a constant, we can assume that all functions f_n are nonnegative. For all integers k and $t > 0$ it holds

$$\mu\Big(\overline{\bigcup_{m=k}^{\infty}\{f_m > t\}}\Big) \ge \limsup_{n\to\infty}\mu_n\Big(\overline{\bigcup_{m=k}^{\infty}\{f_m > t\}}\Big) \ge \limsup_{n\to\infty}\mu_n(\{f_n > t\}).$$

Taking the intersection of the sets in the left hand side and noticing that it is contained, by assumption, in $\{f \ge t\}$, we get $\limsup_n \mu_n(\{f_n > t\}) \le \mu(\{f \ge t\})$. By Cavalieri's formula and Fatou's lemma we conclude. □

Lemma 3.4. *Assume* (3.27), (3.28) *and the length property* (3.3). *For all $f \in \mathrm{Lip}_b(X)$ nonnegative and with bounded support $Q_t f$ is Lipschitz, nonnegative with bounded support and it holds*

$$|\mathsf{P}_t\, Q_1 f(x) - \mathsf{P}_t f(y)| \le \frac{1}{2}C^2(t)\mathsf{d}^2(x,y) \qquad \textit{for every } t \ge 0,\ x,\, y \in X. \tag{3.35}$$

Proof. It is immediate to check that $Q_s f(x) = 0$ if $f(x) = 0$, so that the support of all functions $Q_s f$, $s \in [0,1]$, are contained in a given ball and $Q_s f$ are also equi-bounded.

The stated inequality is trivial for $t = 0$, so assume $t > 0$. By (3.14) for every $s > 0$, setting $r_k := s - 1/k \uparrow s$, the sequence $r_k^2 |\mathrm{D}Q_{r_k} f|^2(x)$ monotonically converges to the function $\mathsf{D}^-(x,s)$; we can thus pass to the limit in the upper gradient inequality (which is a consequence of (3.28))

$$|\mathsf{P}_t Q_{r_k} f(\gamma_1) - \mathsf{P}_t Q_{r_k} f(\gamma_0)| \le C(t) \int_0^1 \sqrt{\tilde{\mathsf{P}}_t |\mathrm{D}Q_{r_k} f|^2(\gamma_s)} |\dot\gamma_s| \, \mathrm{d}s$$

to get that the function $C(t) G_s$, with $G_s(x) := s^{-1} \sqrt{\tilde{\mathsf{P}}_t \left(\mathsf{D}^-(x,s)^2\right)}$, is an upper gradient for $\mathsf{P}_t(Q_s f)$. Moreover, combining (3.11) and (3.9) we obtain

$$\limsup_{h \downarrow 0} \frac{Q_{s+h} f(x_h) - Q_s f(x_h)}{h} \le \frac{1}{2s^2} \limsup_{h \downarrow 0} -(\mathsf{D}^+(x_h, s))^2 \le -\frac{1}{2s^2} (\mathsf{D}^-(x,s))^2 \qquad (3.36)$$

along an arbitrary sequence $x_h \to x$.

Let γ be a Lipschitz curve with $\gamma_1 = x$ and $\gamma_0 = y$. We interpolate with a parameter $s \in [0,1]$, setting $g(s) := \mathsf{P}_t Q_s f(\gamma_s)$. Using (3.15) and (3.27) we obtain that g is absolutely continuous in $[0,1]$, so that we need only to estimate $g'(s)$. For $h > 0$, we write

$$\frac{g(s+h) - g(s)}{h} = \int_X \frac{Q_{s+h} f - Q_s f}{h} \, \mathrm{d}\mathsf{H}_t \delta_{\gamma_{s+h}} + \frac{\mathsf{P}_t Q_s f(\gamma_{s+h}) - \mathsf{P}_t Q_s f(\gamma_s)}{h}$$

and estimate the two terms separately. The first term can be estimated as follows:

$$\limsup_{h \downarrow 0} \int_X \frac{Q_{s+h} f - Q_s f}{h} \, \mathrm{d}\mathsf{H}_t \delta_{\gamma_{s+h}} \le -\frac{1}{2s^2} \int_X \mathsf{D}^-(\cdot, s)^2 \, \mathrm{d}\mathsf{H}_t \delta_{\gamma_s} = -\frac{1}{2} G_s^2(\gamma_s). \qquad (3.37)$$

Here we applied Lemma 3.3 with $f_h(x) = (Q_{s+h} f(x) - Q_s f(x))/h$, $\mu_h = \mathsf{H}_t \delta_{\gamma_{s+h}}$ and $\mu = \mathsf{H}_t \delta_{\gamma_s}$, taking (3.36) into account.

The second term can be estimated as follows. By the upper gradient property of $C(t) G_s$ for $\mathsf{P}_t(Q_s f)$ we get

$$\limsup_{h \downarrow 0} \frac{|\mathsf{P}_t Q_s f(\gamma_{s+h}) - \mathsf{P}_t Q_s f(\gamma_s)|}{h} \le G_s(\gamma_s) C(t) |\dot\gamma_s| \qquad (3.38)$$

for a.e. $s \in (0,1)$, more precisely at any Lebesgue point of $|\dot\gamma|$ and of $s \mapsto G_s(\gamma_s)$. Combining (3.37) and (3.38) and using the Young inequality we get $|\mathsf{P}_t Q_1 f(x) - \mathsf{P}_t f(y)| \le C^2(t) \frac{1}{2} \int_0^1 |\dot\gamma_s|^2 \, \mathrm{d}s$. Minimizing with respect to γ gives the result. $\square$

Theorem 3.5. *Let $\mathcal{E}$ and $(\mathsf{P}_t)_{t \ge 0}$ be as in* (2.1) *and* (2.20)*, and let d be a distance on X under the assumptions* **(MD)** *and* (3.3)*. Then* (3.27) *and* (3.28) *are satisfied by $(\mathsf{P}_t)_{t \ge 0}$ if and only if* (W_2-cont) *holds.*

Proof. We only prove the W_2 contraction assuming that (3.27) and (3.28) hold, since the converse implication have been already proved in [25] (see also [5, Theorem 6.2]) and it does not play any role in this paper.

We first notice that Kantorovich duality provides the identity

$$\frac{1}{2}W_2^2(\mathsf{H}_t\delta_x, \mathsf{H}_t\delta_y) = \sup |\mathsf{P}_t Q_1 f(x) - \mathsf{P}_t f(y)|$$

where the supremum runs in the class of bounded, nonnegative Lipschitz functions f with bounded support. Therefore, Lemma 3.4 gives

$$W_2^2(\mathsf{H}_t\delta_x, \mathsf{H}_t\delta_y) \leq C^2(t)\mathsf{d}^2(x,y). \tag{3.39}$$

Now, given $\mu, \nu \in \mathscr{P}(X)$ with $W_2(\mu,\nu) < \infty$ and a corresponding optimal plan $\boldsymbol{\gamma}$, we may use a measurable selection theorem (see for instance [11, Theorem 6.9.2]) to select in a $\boldsymbol{\gamma}$-measurable way optimal plans $\boldsymbol{\gamma}_{xy}$ from $\mathsf{H}_t\delta_x$ to $\mathsf{H}_t\delta_y$. Then, we define

$$\boldsymbol{\gamma}_0 := \int_{X\times X} \boldsymbol{\gamma}_{xy}\,\mathrm{d}\boldsymbol{\gamma}(x,y).$$

and notice that, because of (3.34), $\boldsymbol{\gamma}_0$ is an admissible plan from $\mathsf{H}_t\mu$ to $\mathsf{H}_t\nu$. Since (3.39) provides the inequality $\int \mathsf{d}^2\,\mathrm{d}\boldsymbol{\gamma}_0 \leq C^2(t)\int \mathsf{d}^2\,\mathrm{d}\boldsymbol{\gamma}$ we conclude. □

3.3 Energy measure spaces

In this section we want to study more carefully the interaction between the energy and the metric structures, particularly in the case when the initial structure is not provided by a distance, but rather by a Dirichlet form $\mathcal{E}$.

Given a Dirichlet form $\mathcal{E}$ in $L^2(X,\mathfrak{m})$ as in (2.1), assume that $\mathcal{B}$ is the $\mathfrak{m}$-completion of the Borel σ-algebra of (X,τ), where τ is a given topology in X. Then, under these structural assumptions, we define a first set of "locally 1-Lipschitz" functions as follows:

$$\mathcal{L} := \Big\{\psi \in \mathbb{G} : \Gamma\big(\psi\big) \leq 1 \ \mathfrak{m}\text{-a.e. in } X\Big\}, \quad \mathcal{L}_{\mathrm{C}} := \mathcal{L} \cap \mathrm{C}(X). \tag{3.40}$$

With this notion at hand we can generate canonically the intrinsic (possibly infinite) pseudo-distance [10]:

$$\mathsf{d}_{\mathcal{E}}(x_1,x_2) := \sup_{\psi\in\mathcal{L}_{\mathrm{C}}} |\psi(x_2) - \psi(x_1)| \quad \text{for every } x_1,x_2 \in X. \tag{3.41}$$

We also introduce truncation functions $\mathrm{S}_k \in \mathrm{C}^1(\mathbb{R})$ satisfying

$$\mathrm{S}(r) = \begin{cases} 1 & \text{if } |r| \leq 1, \\ 0 & \text{if } |r| \geq 3, \end{cases} \qquad |\mathrm{S}'(r)| \leq 1; \quad \mathrm{S}_k(r) := k\mathrm{S}(r/k), \quad r \in \mathbb{R},\ k > 0. \tag{3.42}$$

We have now all the ingredients to define the following structure.

Definition 3.6 (Energy measure space)**.** Let (X,τ) be a Polish space, let $\mathfrak{m}$ be a Borel measure with full support, let $\mathcal{B}$ be the $\mathfrak{m}$-completion of the Borel σ-algebra and let $\mathcal{E}$ be a Dirichlet form in $L^2(X,\mathfrak{m})$ satisfying (2.1) of Section 2.1. We say that $(X,\tau,\mathfrak{m},\mathcal{E})$ is a Energy measure space if

(a) There exists a function

$$\theta \in \mathrm{C}(X),\ \theta \geq 0, \text{ such that } \theta_k := \mathrm{S}_k \circ \theta \text{ belongs to } \mathcal{L} \text{ for every } k > 0. \tag{3.43}$$

(b) $\mathsf{d}_{\mathcal{E}}$ is a finite distance in X which induces the topology τ and $(X,\mathsf{d}_{\mathcal{E}})$ is complete.

Notice that if $\mathfrak{m}(X) < \infty$ and $1 \in D(\mathcal{E})$ then (3.43) is always satisfied by choosing $\theta \equiv 0$. In the general case condition (a) is strictly related to the finiteness property of the measure of balls (**MD.b**). In fact, we shall see in Theorem 3.9 that $(X,\mathsf{d}_{\mathcal{E}},\mathfrak{m})$ satisfies the measure distance condition **(MD)**.

Remark 3.7 (Completeness and length property)*.* Whenever $\mathsf{d}_{\mathcal{E}}$ induces the topology τ (and thus $(X,\mathsf{d}_{\mathcal{E}})$ is a separable space), completeness is not a restrictive assumption, since it can always be obtained by taking the abstract completion $\bar{X}$ of X with respect to $\mathsf{d}_{\mathcal{E}}$. Since (X,τ) is a Polish space, X can be identified with a Borel subset of $\bar{X}$ [11, Thm. 6.8.6] and $\mathfrak{m}$ can be easily extended to a Borel measure $\bar{\mathfrak{m}}$ on $\bar{X}$ by setting $\bar{\mathfrak{m}}(B) := \mathfrak{m}(B \cap X)$; in particular $\bar{X} \setminus X$ is $\bar{\mathfrak{m}}$-negligible and $\mathcal{E}$ can be considered as a Dirichlet form on $L^2(\bar{X},\bar{\mathfrak{m}})$ as well. Finally, once completeness is assumed, the length property is a consequence of the definition of the intrinsic distance $\mathsf{d}_{\mathcal{E}}$, see [38, 36] in the locally compact case and the next Corollary 3.10 in the general case.

In many cases τ is already induced by a distance d satisfying the compatibility condition (**MD**), so that we are actually dealing with a structure $(X,\mathsf{d},\mathfrak{m},\mathcal{E})$. In this situation it is natural to investigate under which assumptions the identity $\mathsf{d} = \mathsf{d}_{\mathcal{E}}$ holds: this in particular guarantees that $(X,\tau,\mathfrak{m},\mathcal{E})$ is an Energy measure space according to Definition 3.6. In the following remark we examine the case when $\mathcal{E}$ is canonically generated starting from d and $\mathfrak{m}$, and then we investigate possibly more general situations.

Remark 3.8 (The case of a quadratic Cheeger energy)*.* Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space satisfying (**MD**) and let us assume that the Cheeger energy is quadratic (i.e. $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian according to [20]), $\mathcal{E}_{\mathsf{Ch}} := 2\mathsf{Ch}$. Then it is clear that any 1-Lipschitz function $f \in L^2(X,\mathfrak{m})$ belongs to $\mathcal{L}_{\mathrm{C}}$, hence $\mathsf{d}_{\mathcal{E}} \geq \mathsf{d}$. It follows that $\mathsf{d} = \mathsf{d}_{\mathcal{E}}$ if and only if every continuous function $f \in D(\mathsf{Ch})$ with $|\mathrm{D}f|_w \leq 1$ is 1-Lipschitz w.r.t. d. In particular this is the case of $\mathrm{RCD}(K,\infty)$ spaces.
If $X = [0,1]$ endowed with the Lebesgue measure and the Euclidean distance, and if $\mathfrak{m} = \sum_n 2^{-n}\delta_{q_n}$, where (q_n) is an enumeration of $\mathbb{Q} \cap [0,1]$, then it is easy to check that $\mathsf{Ch} \equiv 0$ (see [3] for details), hence $\mathsf{d}_{\mathcal{E}}(x,y) = \infty$ whenever $x \neq y$.

If d is a distance on $X \times X$ satisfying (**MD**), in order to provide links between the Dirichlet form $\mathcal{E}$ and the distance d, we can also introduce a new set of Lipschitz functions

$$\mathrm{L} := \Big\{\psi \in \mathrm{Lip}(X,\mathsf{d}) : \mathrm{supp}(\psi) \text{ is bounded},\ |\mathrm{D}\psi| \leq 1\Big\}, \tag{3.44}$$

and the following condition:

Condition (**ED**: Energy-Distance interaction)**.** d is a distance on $X \times X$ such that **(ED.a)** every function $\psi \in \mathcal{L}_{\mathrm{C}}$ is 1-Lipschitz with respect to d;
(ED.b) every function $\psi \in \mathrm{L}$ belongs to $\mathcal{L}_{\mathrm{C}}$.

Theorem 3.9. *If $(X, \tau, \mathfrak{m}, \mathcal{E})$ is an Energy measure space according to Definition 3.6 then the canonical distance $\mathsf{d}_{\mathcal{E}}$ satisfies conditions* (**MD**, **ED**)*.*

Conversely, if $\mathcal{E}$ is a Dirichlet form as in (2.1) *and d is a distance on $X \times X$ inducing the topology τ and satisfying conditions* (**MD**, **ED**)*, then $(X, \tau, \mathfrak{m}, \mathcal{E})$ is an Energy measure space and*

$$\mathsf{d}(x_1, x_2) = \mathsf{d}_{\mathcal{E}}(x_1, x_2) \quad \textit{for every } x_1, x_2 \in X. \tag{3.45}$$

Proof. Let us first assume that $(X, \tau, \mathfrak{m}, \mathcal{E})$ is an Energy measure space. (**MD.a**) is immediate since $\mathsf{d}_{\mathcal{E}}$ is complete by assumption and τ is separable. (**ED.a**) is also a direct consequence of the definition of $\mathcal{E}$, since

$$|\psi(x_2) - \psi(x_1)| \leq \mathsf{d}_{\mathcal{E}}(x_1, x_2) \quad \text{for every } \psi \in \mathcal{L}_{\mathrm{C}},\ x_1, x_2 \in X. \tag{3.46}$$

Let us now prove **(MD.b) and (ED.b)**.

We first observe that the function θ of (3.43) is bounded on each ball $B_r(y)$, $y \in X$ and $r > 0$, otherwise we could find a sequence of points $y_k \in B_r(y)$, $k \in \mathbb{N}$, such that $\theta(y_k) \geq 3k$ and therefore $\theta_k(y_k) - \theta_k(y) \geq k$ whenever $\theta(y) \leq k$. This contradicts the fact that θ_k is 1-Lipschitz by (3.46). As a consequence, for every $y \in X$ and $r > 0$ there exists $k_{y,r} \in \mathbb{N}$ such that

$$\theta_k(x) \equiv k \quad \text{for every } x \in B_r(y),\ k \geq k_{r,y}. \tag{3.47}$$

In particular, since $\theta_k \in L^2(X, \mathfrak{m})$, we get that all the sets $B_r(y)$ with $y \in X$ and $r > 0$ have finite measure, so that (**MD.b**) holds.

We observe that by the separability of $X \times X$ we can find a countable family $(\psi_n) \subset \mathcal{L}_{\mathrm{C}}$ such that

$$\mathsf{d}_{\mathcal{E}}(x_1, x_2) = \sup_n \big|\psi_n(x_2) - \psi_n(x_1)\big| \quad \text{for every } x_1, x_2 \in X. \tag{3.48}$$

We set

$$\mathsf{d}_{k,N}(x_1, x_2) := \Big(\sup_{n \leq N} \big|\psi_n(x_2) - \psi_n(x_1)\big|\Big) \wedge \theta_k(x_2),$$

observing that for every $y \in X$ the map $x \mapsto \mathsf{d}_{k,N}(y, x)$ belongs to $\mathcal{L}_{\mathrm{C}}$. Passing to the limit as $N \to \infty$, it is easy to check that $\mathsf{d}_{k,N}(y, \cdot) \to \mathsf{d}_k(y, \cdot) = \mathsf{d}(y, \cdot) \wedge \theta_k$ pointwise in X and therefore in $L^2(X, \mathfrak{m})$, since $\theta_k \in L^2(X, \mathfrak{m})$. We deduce that $\mathsf{d}_k(y, \cdot) \in \mathcal{L}$ for every $y \in X$ and $k \in \mathbb{N}$.

Let us now prove that every map $f \in \mathrm{L}$ belongs to $\mathcal{L}$; it is not restrictive to assume f nonnegative. Since f is 1-Lipschitz it is easy to check that, setting $f_k = f \wedge \theta_k$, it holds

$$\begin{aligned} f_k(x) &= \Big(\inf_{z \in X} \big(f(z) + \mathsf{d}_{\mathcal{E}}(z, x)\big)\Big) \wedge \theta_k(x) = \inf_{z \in X} \Big(\big(f_k(z) + \mathsf{d}_{\mathcal{E}}(z, x)\big) \wedge \theta_k(x)\Big) \\ &= \Big(\inf_{z \in X} \big(f_k(z) + \mathsf{d}_{\mathcal{E}}(z, x) \wedge \theta_k(x)\big)\Big) \wedge \theta_k(x) = \Big(\inf_{z \in X} \big(f_k(z) + \mathsf{d}_k(z, x)\big)\Big) \wedge \theta_k(x). \end{aligned}$$

Let (z_i) be a countable dense set of X. The functions

$$f_{k,n}(x) := \Big(\min_{1\le i\le n} f_k(z_i) + \mathsf{d}_k(z_i,x)\Big) \wedge \theta_k(x)$$

belong to $\mathcal{L}$, are nonincreasing with respect to n, and satisfy $0 \le f_{k,n} \le \theta_k$. Since

$$\{x \in X : \theta_k(x) > 0\} \subset \{x \in X : \theta_{3k}(x) = 3k\},$$

we easily see that $\mathfrak{m}\big(\operatorname{supp}(\theta_k)\big) < \infty$. Passing to the limit as $n \uparrow \infty$, since $z \mapsto \mathsf{d}_k(z,x)$ is continuous, they converge monotonically to

$$\Big(\inf_{z\in X}\big(f_k(z) + \mathsf{d}_k(z,x)\big)\Big) \wedge \theta_k(x) = f_k(x),$$

and their energy is uniformly bounded by $\mathfrak{m}\big(\operatorname{supp}(\theta_k)\big)$. This shows that $f_k \in \mathcal{L}$. Eventually, letting $k \uparrow \infty$ and recalling that $\operatorname{supp}(f_k) \subset \operatorname{supp}(f)$ and $\mathfrak{m}(\operatorname{supp}(f)) < \infty$ by (**MD.a**), we obtain $f \in \mathcal{L}$.

The converse implication is easier: it is immediate to check that **(ED.a)** is equivalent to

$$\mathsf{d}(x_1,x_2) \ge \mathsf{d}_{\mathcal{E}}(x_1,x_2) \quad \text{for every } x_1,x_2 \in X; \tag{3.49}$$

if **(ED.b)** holds and balls have finite measure according to (**MD.b**), we have $x \mapsto \mathrm{T}_k(\mathsf{d}(y,x)) \in \mathcal{L}$ for every $y \in X$, where $\mathrm{T}_k(r) := r \wedge \mathrm{S}_k(r)$. Since

$$\mathsf{d}(x_1,x_2) = \mathrm{T}_k(\mathsf{d}(x_2,x_1)) - \mathrm{T}_k(\mathsf{d}(x_2,x_2)) \quad \text{whenever } k > \mathsf{d}(x_1,x_2),$$

we easily get the converse inequality to (3.49), and therefore (3.45) and property (b) of Definition (3.6). In order to get also (a) it is sufficient to take $\theta(x) := \mathsf{d}(x,x_0)$ for an arbitrary $x_0 \in X$. □

Theorem 3.10 (Length property of $\mathsf{d}_{\mathcal{E}}$)**.** *If* $(X,\tau,\mathfrak{m},\mathcal{E})$ *is an Energy measure space then* $(X,\mathsf{d}_{\mathcal{E}})$ *is a length metric space, i.e. it also satisfies* (3.3)*.*

Proof. We follow the same argument as in [38, 36]. Since (X,d) is complete, it is well known (see e.g. [13, Thm. 2.4.16]) that the length condition is equivalent to show that for every couple of points $x_0,x_1 \in X$ and $\varepsilon \in (0,r)$ with $r := \mathsf{d}_{\mathcal{E}}(x_0,x_1)$ there exists an ε-midpoint $y \in X$ such that

$$\mathsf{d}_{\mathcal{E}}(y,x_i) < \frac{r}{2} + \varepsilon, \quad i = 0,1.$$

We argue by contradiction assuming that $B_{r/2+\varepsilon}(x_0) \cap B_{r/2+\varepsilon}(x_1) = \emptyset$ and we introduce the function

$$\psi(x) := \Big(\frac{1}{2}(r+\varepsilon) - \mathsf{d}_{\mathcal{E}}(x,x_0)\Big)_+ - \Big(\frac{1}{2}(r+\varepsilon) - \mathsf{d}_{\mathcal{E}}(x,x_1)\Big)_+.$$

ψ is Lipschitz, has bounded support and it is easy to check that $|\mathrm{D}\psi|(x) \le 1$ for every $x \in X$ since $B_{r/2+\varepsilon}(x_0)$ and $B_{r/2+\varepsilon}(x_1)$ are disjoint. It turns out that $\psi \in \mathrm{L}$ and therefore it is 1-Lipschitz by (**ED**). On the other hand $\psi(x_0) - \psi(x_1) = r + \varepsilon > \mathsf{d}_{\mathcal{E}}(x_0,x_1)$. □

We now examine some additional properties of Energy measure spaces.

Proposition 3.11. *Let $(X, \tau, \mathfrak{m}, \mathcal{E})$ be an Energy measure space. Let $f \in \mathbb{G} \cap \mathrm{C}_b(X)$ and let $\zeta : X \to [0, \infty)$ be a bounded upper semicontinuous function such that $\Gamma\big(f\big) \le \zeta^2$ $\mathfrak{m}$-a.e. in X. Then f is Lipschitz (with respect to the induced distance $\mathsf{d}_{\mathcal{E}}$) and $|\mathrm{D}^* f| \le \zeta$. In particular ζ is an upper gradient of f.*

Proof. We know that (**ED**) holds, by the previous theorem, and we set for simplicity $\mathsf{d} := \mathsf{d}_{\mathcal{E}}$. Since ζ is bounded, f is Lipschitz by (**ED.a**). We fix $x \in X$ and for every $\varepsilon > 0$ we set $G_\varepsilon := \sup_{B_\varepsilon(x)} \zeta$. The Lipschitz function

$$\psi(y) := \Big[|f(y) - f(x)| \vee \big(G_\varepsilon\, \mathrm{S}_{3\varepsilon}(\mathsf{d}(y, x))\big)\Big] \wedge \big(G_\varepsilon \mathrm{S}_\varepsilon(\mathsf{d}(x, y))\big) \tag{3.50}$$

belongs to $\mathbb{V}_\infty$; moreover

$$\psi(y) = G_\varepsilon \mathrm{S}_\varepsilon(\mathsf{d}(x, y)) \quad \text{if } \mathsf{d}(y, x) \ge \varepsilon$$

so that $\Gamma\big(\psi\big) \le G_\varepsilon$ $\mathfrak{m}$-a.e. in X. It follows that ψ is G_ε-Lipschitz and $\psi(y) \le G_\varepsilon\, \mathsf{d}(y, x)$ for every $y \in X$ since $\psi(x) = 0$, so that

$$|\mathrm{D} f(x)| \le \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)} \le \limsup_{y \to x} \frac{\psi(y)}{\mathsf{d}(y, x)} \le G_\varepsilon. \tag{3.51}$$

Since $\varepsilon > 0$ is arbitrary and $\lim_{\varepsilon \downarrow 0} G_\varepsilon = \zeta(x)$ we obtain $|\mathrm{D} f(x)| \le \zeta(x)$. Since ζ is upper semicontinuous and X is a length space, we also get $|\mathrm{D}^* f| \le \zeta$. □

The following result provides a first inequality between $\mathcal{E}$ and Ch, in the case when a priori the distances d and $\mathsf{d}_{\mathcal{E}}$ are different, and we assume only **(ED.b)**.

Theorem 3.12. *$\mathcal{E}$ be a Dirichlet form in $L^2(X, \mathfrak{m})$ satisfying* (2.1) *of Section 2.1 and let d be a distance on $X \times X$ satisfying condition* **(MD)**. *Then condition* **(ED.b)** *is satisfied if and only if for every Lipschitz function $f \in \mathrm{Lip}(X)$ with bounded support we have*

$$f \in \mathbb{G}, \quad |\mathrm{D} f|^2 \ge \Gamma\big(f\big) \quad \mathfrak{m}\text{-a.e. in } X. \tag{3.52}$$

In particular, if **(MD)** *and* **(ED.b)** *hold, we have*

$$\begin{gathered} 2\mathsf{Ch}(g) \ge \mathcal{E}(g) \quad \text{for every } g \in L^2(X, \mathfrak{m}), \\ D(\mathsf{Ch}) \subset \mathbb{G} \subset \mathbb{V}, \quad |\mathrm{D} g|_w^2 \ge \Gamma\big(g\big) \quad \text{for every } g \in D(\mathsf{Ch}). \end{gathered} \tag{3.53}$$

Proof. The implication (3.52)⇒(**ED.b**) is trivial; let us consider the converse one.

Up to replacing f with $(f + c) \wedge \mathrm{S}_k(\mathsf{d}(x_0, \cdot))$ with $c = \sup_{B_{3k}(x_0)}(-f \vee 0)$ (notice that f is bounded) we can assume that f is nonnegative, bounded, with support contained in $B_{3k}(x_0)$ for some $k > 0$ and $x_0 \in X$.

Recall the Hopf-Lax formula (3.9) for the map Q_t; if (z_i) is a countable dense subset of X we define

$$Q_t^n f(x) = \Big(\min_{1\le i\le n} f(z_i) + \frac{1}{2t}\mathsf{d}^2(z_i,x)\Big) \wedge \mathrm{S}_k(\mathsf{d}(x_0,x)), \tag{3.54}$$

and we set $I_n(x) = \big\{ i\in\{1,\dots,n\} : z_i \text{ minimizes (3.54)}\big\}$.

The locality property and the fact that $\big(f(z_i)+\mathsf{d}^2(z_i,\cdot)/2t\big)\wedge \mathrm{S}_k(\mathsf{d}(x_0,\cdot))\in\mathbb{V}$ yield

$$\sqrt{\Gamma\big(Q_t^n f\big)(x)} \le A_n(x) := \frac{1}{t}\max_{i\in I_n(x)} \mathsf{d}(x,z_i) \quad \mathfrak{m}\text{-a.e. in } X.$$

If we define $z_n(x)$ as the value z_i that realizes the maximum for $A_n(x)$ with the lowest index $i\in I_n(x)$, the previous formula yields

$$A_n(x) = \frac{1}{t}\mathsf{d}(x,z_n(x)) \quad \text{for every } x\in X, \tag{3.55}$$

and it is not difficult to see that $\big(z_n(x)\big)_{n\ge 0}$ is a minimizing sequence for $Q_t f(x)$: in fact, since $Q_t f(x)\le f(x)\le \mathrm{S}_k(\mathsf{d}(x_0,x))$, we have

$$0\le Q_t f(x) \le Q_t^n f(x) \le \mathrm{S}_k(\mathsf{d}(x_0,x)),\quad Q_t^n f(x)\downarrow Q_t f(x) \quad \text{as } n\uparrow\infty,$$

and

$$\frac{1}{2t}\mathsf{d}^2(x,z_n(x)) + f(z_n(x)) \to Q_t f(x) \quad \text{as } n\to\infty.$$

It follows that

$$\limsup_{n\to\infty} A_n(x) = \limsup_{n\to\infty} \frac{1}{t}\mathsf{d}(x,z_n(x)) \le \mathsf{D}^+(t,x).$$

Since $Q_t^n f$ is supported in a bounded set, it is uniformly bounded, and it pointwise converges to $Q_t f$. Considering any weak limit point G of $\sqrt{\Gamma\big(Q_t^n f\big)(x)}$ in $L^2(X)$ we obtain by (2.17)

$$\Gamma\big(Q_t f\big)(x) \le G^2(x) \le \frac{\big(\mathsf{D}^+(x,t)\big)^2}{t^2} \quad \mathfrak{m}\text{-a.e.}$$

Since f is Lipschitz, it follows that $\mathsf{D}^+(x,t)/t$ is uniformly bounded. Integrating (3.13) on an arbitrary bounded Borel set A and applying Fatou's Lemma, we get

$$\begin{aligned}\int_A |\mathrm{D}f|^2(x)\,\mathrm{d}\mathfrak{m}(x) &\ge \int_A \limsup_{t\downarrow 0}\int_0^1 \Big(\frac{\mathsf{D}^+(x,tr)}{tr}\Big)^2\mathrm{d}r\,\mathrm{d}\mathfrak{m}\\ &\ge \limsup_{t\downarrow 0}\int_0^1\int_A \Big(\frac{\mathsf{D}^+(x,tr)}{tr}\Big)^2\mathrm{d}\mathfrak{m}\,\mathrm{d}r \ge \limsup_{t\downarrow 0}\int_0^1\int_A \Gamma\big(Q_{tr}f\big)(x)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r\\ &\ge \int_0^1 \liminf_{t\downarrow 0}\Big(\int_A \Gamma\big(Q_{tr}f\big)(x)\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}r \ge \int_A \Gamma\big(f\big)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where in the last inequality we applied (2.17) once more. Since A is arbitrary we conclude.
□

In order to conclude our analysis of the relations between $\mathcal{E}$ and Ch for Energy measure spaces $(X, \tau, \mathfrak{m}, \mathcal{E})$ we introduce a further property.

Definition 3.13 (Upper regularity)**.** Let $(X, \tau, \mathfrak{m}, \mathcal{E})$ be an Energy measure space. We say that the Dirichlet form $\mathcal{E}$ is upper-regular if for every f in a dense subset of $\mathbb{V}$ there exist $f_n \in \mathbb{G} \cap \mathrm{C}_b(X)$ and $g_n : X \to \mathbb{R}$ bounded and upper semicontinuous such that

$$\sqrt{\Gamma\big(f_n\big)} \le g_n \ \mathfrak{m}\text{-a.e.}, \quad f_n \to f \text{ strongly in } L^2(X, \mathfrak{m}), \quad \limsup_{n\to\infty} \int_X g_n^2 \, \mathrm{d}\mathfrak{m} \le \mathcal{E}(f). \tag{3.56}$$

Theorem 3.14. *Let $(X, \tau, \mathfrak{m}, \mathcal{E})$ be an Energy measure space. Then the Cheeger energy associated to $(X, \mathsf{d}_{\mathcal{E}}, \mathfrak{m})$ coincides with $\mathcal{E}$ i.e.*

$$\mathcal{E}(f) = 2\mathsf{Ch}(f) \quad \textit{for every } f \in L^2(X, \mathfrak{m}), \tag{3.57}$$

if and only if $\mathcal{E}$ is upper-regular. In this case $\mathbb{G} = \mathbb{V}$, $\mathcal{E}$ admits a Carré du Champ Γ and

$$\Gamma\big(f\big) = |\mathrm{D}f|_w^2 \quad \mathfrak{m}\textit{-a.e. in } X \textit{ for every } f \in \mathbb{V}. \tag{3.58}$$

In particular, the space $\mathbb{V} \cap \mathrm{Lip}_b(X)$ is dense in $\mathbb{V}$. If moreover (**MD.exp**) *holds, then $(\mathsf{P}_t)_{t\ge 0}$ satisfies the mass preserving property* (2.20)*.*

Proof. Since Ch is always upper-regular by (3.18), the condition is clearly necessary. In order to prove its sufficiency, by (3.53) of Theorem 3.12 we have just to prove that every $f \in \mathbb{V}$ satisfies the inequality $2\mathsf{Ch}(f) \le \mathcal{E}(f)$. If f_n, g_n are sequences as in (3.56), Proposition 3.11 yields that f_n are Lipschitz and

$$|\mathrm{D}f_n| \le g_n, \quad \mathsf{Ch}(f_n) \le \frac{1}{2} \int_X |\mathrm{D}f_n|^2 \, \mathrm{d}\mathfrak{m} \le \frac{1}{2} \int_X g_n^2 \, \mathrm{d}\mathfrak{m}.$$

Passing to the limit as $n \to \infty$ we obtain the desired inequality thanks to the lower-semicontinuity of Ch in $L^2(X, \mathfrak{m})$. The last statement of the Theorem follows by [3, Thm. 4.20]. □

3.4 Riemannian Energy measure spaces and the $\mathrm{BE}(K, \infty)$ condition

In this section we will discuss various consequences of the Energy measure space axiomatization in combination with $\mathrm{BE}(K, \infty)$. From now on it will be always be implicitly assumed that an Energy measure space $(X, \tau, \mathfrak{m}, \mathcal{E})$ is metrized by its canonical distance $\mathsf{d}_{\mathcal{E}}$.

Taking into account the previous section the Bakry-Émery condition $\mathrm{BE}(K, N)$ as stated in Definition 2.4 makes perfectly sense for a Energy measure space $(X, \tau, \mathfrak{m}, \mathcal{E})$. In the next result we will show that under a weak-Feller property on the semigroup $(\mathsf{P}_t)_{t\ge 0}$ we gain upper-regularity of $\mathcal{E}$, the identifications $\mathcal{E} = 2\mathsf{Ch}$ of Theorem 3.14 and $\mathcal{L} = \mathcal{L}_{\mathrm{C}}$.

Theorem 3.15. *Let $(X, \tau, \mathfrak{m}, \mathcal{E})$ be a Energy measure space satisfying the* $\mathrm{BE}(K, \infty)$ *condition. Then, its Markov semigroup $(\mathsf{P}_t)_{t\geq 0}$ satisfies the weak-Feller property*

$$f \in \mathrm{L} \quad \Rightarrow \quad \mathsf{P}_t f \in \mathrm{C}_b(X) \quad \text{for every } t > 0 \tag{w-Feller}$$

if and only if

$$\mathcal{L} = \mathcal{L}_{\mathrm{C}}, \tag{3.59}$$

i.e. if every function $f \in \mathcal{L}$ admits a continuous representative. In this case $\mathcal{E}$ is upper-regular and, as a consequence, (3.57), (3.58) *hold.*

Proof. The implication (3.59)⇒(w-Feller) is easy, since the Bakry-Émery condition $\mathrm{BE}(K, \infty)$ (i.e. (2.58) with $\nu = 0$) and the bound $\Gamma\big(f\big) \leq 1$ given by (**ED.b**) yields $\mathrm{e}^{Kt}\mathsf{P}_t f \in \mathcal{L}$.

Now we prove the converse implication, from (w-Feller) to (3.59). The Bakry-Émery condition $\mathrm{BE}(K, \infty)$ in conjunction with **(ED.b)** and (w-Feller) yield $\mathrm{e}^{Kt}\mathsf{P}_t f \in \mathcal{L}_{\mathrm{C}}$ for every $f \in \mathrm{L}$ and $t > 0$, thus in particular $\mathrm{e}^{Kt}\mathsf{P}_t f$ is 1-Lipschitz by **(ED.a)**. Let us now fix $f \in \mathcal{L} \cap L^\infty(X, \mathfrak{m})$ and let us consider a sequence of uniformly bounded functions $f_n \in \mathrm{Lip}_b(X) \cap L^2(X, \mathfrak{m})$ with bounded support converging to f in $L^2(X, \mathfrak{m})$. By the previous step we know that $\mathsf{P}_t f_n \in \mathrm{Lip}_b(X)$ and the estimate (2.56) shows that $\Gamma\big(\mathsf{P}_t f_n\big) \leq C/t$ for a constant C independent of n. (**ED**) then shows that $\mathrm{Lip}(\mathsf{P}_t f_n) \leq C/t$; passing to the limit as $n \to \infty$, we can find a subsequence $n_k \to \infty$ such that $\lim_{k\to\infty} \mathsf{P}_t f_{n_k}(x) = \mathsf{P}_t f(x)$ for every $x \in X \setminus \mathcal{N}$ with $\mathfrak{m}(\mathcal{N}) = 0$. Since $\mathsf{P}_t f_n$ are uniformly Lipschitz functions, also $\mathsf{P}_t f$ is Lipschitz in $X \setminus \mathcal{N}$ so that it admits a Lipschitz representative $\tilde{f}_t$ in X.

On the other hand, $\mathrm{BE}(K, \infty)$ and (**ED**) show that $\mathrm{Lip}(f_t) \leq \mathrm{e}^{-Kt}$. Passing to the limit along a suitable sequence $t_n \downarrow 0$ and repeating the previous argument we obtain that f admits a Lipschitz representative.

Let us prove now that $\mathcal{E}$ is upper regular, by checking (3.56) for every $f \in \mathbb{V}^1_\infty$, which is dense in $\mathbb{V}$. Observe that the estimate (2.56), (3.59) and **(ED.a)** yield that , for every $t > 0$ and every function $g \in L^2 \cap L^\infty(X, \mathfrak{m})$, $\mathsf{P}_t g$ admits a Lipschitz, thus continuous, and bounded representative g_t. Choosing in particular $g := \sqrt{\Gamma\big(f\big)}$ we obtain by (2.58)

$$\Gamma\big(\mathsf{P}_t f\big) \leq \mathrm{e}^{-2Kt} g_t^2, \quad \mathsf{P}_t f \to f \text{ in } L^2(X, \mathfrak{m}), \quad \lim_{t\downarrow 0} \int_X \mathrm{e}^{-2Kt} g_t^2 \,\mathrm{d}\mathfrak{m} = \int_X g^2 \,\mathrm{d}\mathfrak{m} = \mathcal{E}(f). \qquad \square$$

According to the previous Theorem we introduce the natural, and smaller, class of Energy measure spaces $(X, \tau, \mathfrak{m}, \mathcal{E})$, still with no curvature bound, but well adapted to the Bakry-Émery condition. In such a class, that we call *Riemannian Energy measure spaces*, the Dirichlet form $\mathcal{E}$ coincides with the Cheeger energy Ch associated to the intrinsic distance $\mathsf{d}_\mathcal{E}$ and every function in $\mathcal{L}$ admits a continuous (thus 1-Lipschitz, by the Energy measure space axiomatization) representative.

Definition 3.16 (Riemannian Energy measure spaces)**.** $(X, \tau, \mathfrak{m}, \mathcal{E})$ is a Riemannian Energy measure space if the following properties hold:

(a) $(X,\tau,\mathfrak{m},\mathcal{E})$ is a Energy measure space;

(b) $\mathcal{E}$ is upper regular according to Definition 3.13;

(c) every function in $\mathcal{L}$ admits a continuous representative.

The next Theorem presents various equivalent characterizations of Riemannian Energy measure spaces in connection with $\mathrm{BE}(K,\infty)$.

Theorem 3.17. *The following conditions are equivalent:*

(i) $(X,\tau,\mathfrak{m},\mathcal{E})$ *is a Riemannian Energy measure space satisfying* $\mathrm{BE}(K,\infty)$.

(ii) $(X,\tau,\mathfrak{m},\mathcal{E})$ *is a Energy measure space satisfying* (w-Feller) *and* $\mathrm{BE}(K,\infty)$.

(iii) $(X,\tau,\mathfrak{m},\mathcal{E})$ *is a Energy measure space satisfying* $\mathcal{L}=\mathcal{L}_{\mathrm{C}}$ *and* $\mathrm{BE}(K,\infty)$.

(iv) $(X,\tau,\mathfrak{m},\mathcal{E})$ *is a Energy measure space with* $\mathcal{E}$ *upper regular, and for every function* $f\in L^2(X,\mathfrak{m})\cap\mathrm{Lip}_b(X)$ *with* $|\mathrm{D}f|\in L^2(X,\mathfrak{m})$

$$\mathsf{P}_t f\in\mathrm{Lip}(X),\quad |\mathrm{D}\mathsf{P}_t f|^2\le \mathrm{e}^{-2Kt}\mathsf{P}_t\big(|\mathrm{D}f|^2\big)\quad \mathfrak{m}\text{-a.e. in } X. \tag{3.60}$$

(v) $\mathcal{E}$ *is a Dirichlet form in* $L^2(X,\mathfrak{m})$ *as in* (2.1), d *is a distance on* $X\times X$ *inducing the topology* τ *and satisfying conditions* **(MD,ED.b)**, *and for every* $f\in\mathcal{L}_{\mathrm{C}}\cap L^\infty(X)$ *and* $t>0$

$$\mathsf{P}_t f\in\mathrm{Lip}_b(X),\quad |\mathrm{D}\mathsf{P}_t f|^2\le \mathrm{e}^{-2Kt}\mathsf{P}_t\Gamma\big(f\big)\quad \mathfrak{m}\text{-a.e. in } X. \tag{3.61}$$

If one of the above equivalent conditions holds with (**MD.exp**), *then* (3.27) *holds, the semigroups* $(\tilde{\mathsf{P}}_t)_{t\ge0}$ *and* $(\mathsf{H}_t)_{t\ge0}$ *are well defined according to Proposition 3.2,* $\mathsf{H}_t(\mu)\ll\mathfrak{m}$ *for every* $t>0$ *and* $\mu\in\mathscr{P}(X)$*, and the strong Feller property*

$$\tilde{\mathsf{P}}_t \text{ maps } L^2\cap L^\infty(X,\mathfrak{m}) \text{ into } \mathrm{Lip}_b(X) \tag{S-Feller}$$

holds with

$$|\mathrm{D}\tilde{\mathsf{P}}_t f|^2=\Gamma\big(\mathsf{P}_t f\big)\quad \mathfrak{m}\text{-a.e. in } X\quad \text{for every } t>0,\ f\in L^2\cap L^\infty(X,\mathfrak{m}). \tag{3.62}$$

Eventually (recall (2.48) *for the definition of* $\mathrm{I}_{2K}(t)$*)*

$$2\mathrm{I}_{2K}(t)\,|\mathrm{D}\tilde{\mathsf{P}}_t f|^2\le\tilde{\mathsf{P}}_t f^2\qquad \text{for every } t\in(0,\infty),\ f\in L^\infty(X,\mathfrak{m}), \tag{3.63}$$

and in particular

$$\sqrt{2\mathrm{I}_{2K}(t)}\,\mathrm{Lip}(\mathsf{P}_t f)\le\|f\|_{L^\infty(X,\mathfrak{m})}\quad \text{for every } t\in(0,\infty). \tag{3.64}$$

Proof. The equivalence (i)⇔(ii)⇔(iii) is just the statement of Theorem 3.15.

Let us first prove the implication (v) ⇒ (ii). Since L is contained in $\mathcal{L}_{\mathrm{C}}$ we immediately get (w-Feller). (3.61) also yields the density of $\mathbb{V}_\infty$ in $\mathbb{V}$ and, thanks to (3.52), condition (2.58) for BE(K,∞). Since $(\mathsf{P}_t)_{t\geq 0}$ is order preserving, (3.52) and (v) yield **(ED.a)**: if $f \in \mathcal{L}_{\mathrm{C}}$ then (3.61) yields $\mathsf{P}_t f$ Lipschitz with constant less then e^{-Kt}. Since along a suitable vanishing sequence $t_n \downarrow 0$ $\mathsf{P}_{t_n} f \to f$ $\mathfrak{m}$-a.e. as $n \uparrow \infty$, arguing as in the proof of Theorem 3.15 it is easy to check that f is 1-Lipschitz. We can thus apply Theorem 3.17 to get that $(X,\tau,\mathfrak{m},\mathcal{E})$ is an Energy measure space with $\mathsf{d} \equiv \mathsf{d}_{\mathcal{E}}$. Since (ii) in particular shows that $\mathcal{E}$ is upper-regular, we proved that (v) ⇒ (iv) as well.

The implication (iv) ⇒ (ii) is also immediate: by the density of $\mathbb{V}\cap\mathrm{Lip}_b(X)$ in $D(\mathsf{Ch}) = \mathbb{V}$ stated in Theorem 3.14 and the upper bound (3.52) we get (2.58) which is one of the equivalent characterizations of BE(K,∞). Moreover, (3.60) clearly yields (w-Feller).

Let us now assume (i) and observe that the estimate (2.56) and the property $\mathcal{L} \subset \mathrm{Lip}(X)$ yield that for every $t > 0$ and every function $f \in L^2 \cap L^\infty(X,\mathfrak{m})$ $\mathsf{P}_t f$ admits a Lipschitz representative satisfying (3.64). Moreover, if f is also Lipschitz, (2.58) yields the estimate (3.27) with $C(t) := \mathrm{e}^{-Kt}$. We can then apply proposition 3.2 and conclude that when $f \in \mathrm{C}_b(X) \cap L^2(X,\mathfrak{m})$ the Lipschitz representative of $\mathsf{P}_t f$ coincides with $\tilde{\mathsf{P}}_t f$. Since by definition

$$\tilde{\mathsf{P}}_t f(x) = \int_X f \,\mathrm{d}\mathsf{H}_t\delta_x \qquad \text{for all Borel } f \text{ bounded from below}$$

we can use a monotone class argument to prove the identification of $\tilde{\mathsf{P}}_t$ with the continuous version of P_t in the general case of bounded, Borel and square integrable functions. Notice that we use (3.64) to convert monotone equibounded convergence of f_n into pointwise convergence on X of (the continuous representative of) $\mathsf{P}_t f_n$, $t > 0$.

Another immediate application of (3.64) is the absolute continuity of $\mathsf{H}_t\mu$ w.r.t. $\mathfrak{m}$ for all $\mu \in \mathscr{P}(X)$ and $t > 0$. Indeed, if A is a Borel and $\mathfrak{m}$-negligible set, then $\tilde{\mathsf{P}}_t\chi_A$ is identically null (being equal to $\mathsf{P}_t\chi_A$, hence continuous, and null $\mathfrak{m}$-a.e. in X), hence (3.32) gives $\mathsf{H}_t\mu(A) = 0$. As a consequence, we can also compute $\tilde{\mathsf{P}}_t f(x)$ for $\mathfrak{m}$-measurable functions f, provided f is semi-integrable with respect to $\mathsf{H}_t\delta_x$.

If now $f \in \mathcal{L}$ (2.58) then yields

$$\Gamma\big(\mathsf{P}_t f\big) \leq \mathrm{e}^{-2Kt}\tilde{\mathsf{P}}_t\Gamma\big(f\big) \quad \mathfrak{m}\text{-a.e. in } X, \tag{3.65}$$

and Proposition 3.11 yields (3.61) since $\tilde{\mathsf{P}}_t\Gamma\big(f\big)$ is continuous and bounded. This concludes the proof of the implication (i) ⇒ (v). A similar argument shows (3.63), starting from (2.56).

Let us eventually prove (3.62). Since the inequality ≥ is true by assumption, let us see why (3.61) provides the converse one: we start from

$$|\mathrm{D}\tilde{\mathsf{P}}_t f|^2 \leq \mathrm{e}^{-2K\varepsilon}\tilde{\mathsf{P}}_\varepsilon\big(\Gamma\big(\mathsf{P}_{t-\varepsilon} f\big)\big) \quad \text{for every } \varepsilon \in (0,t).$$

Recalling that $\Gamma\big(\mathsf{P}_{t-\varepsilon} f\big)$ converges strongly in $L^2(X,\mathfrak{m})$ to $\Gamma\big(\mathsf{P}_t f\big)$ as $\varepsilon \downarrow 0$, we get (3.62). □

Recalling Theorem 3.17, Theorem 3.5, the characterization (3.60), and the notation (3.29), we immediately have

Corollary 3.18. *Let* $(X, \tau, \mathfrak{m}, \mathcal{E})$ *be a Energy measure space satisfying the upper-regularity property* (3.56) *(in particular a Riemannian one) and* (**MD.exp**).
Then $\mathrm{BE}(K, \infty)$ *holds if and only if the semigroup* $(\mathsf{P}_t)_{t\geq 0}$ *satisfies the contraction property*

$$W_2\big((\mathsf{P}_t f)\mathfrak{m}, (\mathsf{P}_t g)\mathfrak{m}\big) \leq \mathrm{e}^{-Kt}\, W_2(f\mathfrak{m}, g\mathfrak{m}) \quad \text{for every } f, g \in \mathcal{Z}. \tag{3.66}$$

4 Proof of the equivalence result

In this section we assume that $(X, \tau, \mathfrak{m}, \mathcal{E})$ is a Riemannian Energy measure space satisfying (**MD.exp**) (relative to $\mathsf{d}_\mathcal{E}$) and the $\mathrm{BE}(K, \infty)$ condition, as discussed in Section 3.4. In particular all results of the previous sections on existence of the dual semigroup $(\mathsf{H}_t)_{t\geq 0}$, its W_2-contractivity and regularizing properties of $(\mathsf{P}_t)_{t\geq 0}$ are applicable. Furthermore, by Theorem 3.14, the Dirichlet form setup described in Section 2 and the metric setup described in Section 3.1 are completely equivalent. In particular, we can apply the results of [3].

4.1 Entropy, Fisher information, and moment estimates

Let us first recall that the *Fisher information functional* $\mathsf{F} : L^1_+(X, \mathfrak{m}) \to [0, \infty]$ is defined by

$$\mathsf{F}(f) := 4\mathcal{E}(\sqrt{f}) \qquad \sqrt{f} \in \mathbb{V}, \tag{4.1}$$

set equal to $+\infty$ if $\sqrt{f} \in L^2(X, \mathfrak{m}) \setminus \mathbb{V}$. Since $f_n \to f$ in $L^1_+(X, \mathfrak{m})$ implies $\sqrt{f_n} \to \sqrt{f}$ in $L^2(X, \mathfrak{m})$, F is L^1-lower semicontinuous.

Proposition 4.1. *Let* $f \in L^1_+(X, \mathfrak{m})$. *Then* $\sqrt{f} \in \mathbb{V}$ *if and only if* $f_N := \min\{f, N\} \in \mathbb{V}$ *for all* N *and* $\int_X \Gamma\big(f\big)/f \,\mathrm{d}\mathfrak{m} < \infty$. *If this is the case,*

$$\mathsf{F}(f) = \int_{\{f>0\}} \frac{\Gamma\big(f\big)}{f} \,\mathrm{d}\mathfrak{m}. \tag{4.2}$$

In addition F *is convex in* $L^1_+(X, \mathfrak{m})$.

We refer to [3, Lemma 4.10] for the *proof.*

By applying the results of [3] we can prove that $(\mathsf{H}_t)_{t\geq 0}$ is a continuous semigroup in $\mathscr{P}_2(X)$ and we can calculate the dissipation rate of the entropy functional along it. Some of the results below are very simple in the case $\mathfrak{m}(X) < \infty$.

Lemma 4.2 (Estimates on moments, Fisher information, metric derivative)**.** *For every* $\bar\mu \in \mathscr{P}_2(X)$ *the map* $t \mapsto \mu_t := \mathsf{H}_t\bar\mu$ *is a continuous curve in* $\mathscr{P}_2(X)$ *with respect to* W_2. *Moreover, for every* $T > 0$ *there exists* $C_T > 0$ *such that*

$$\int_0^T \mathsf{F}(\mathsf{P}_s f)\,\mathrm{d}s + \int_0^T \int_X V^2\, \mathsf{P}_t f \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \leq C_T\Big(\mathrm{Ent}_{\tilde{\mathfrak{m}}}(\bar\mu) + \int_X V^2 \,\mathrm{d}\bar\mu\Big) \tag{4.3}$$

for all $\bar\mu = f\mathfrak{m}$ *with* $\mathrm{Ent}_{\mathfrak{m}}(\bar\mu) < \infty$*. In addition, if* $f \in L^2(X,\mathfrak{m})$*, it holds*

$$|\dot\mu_t|^2 \le \mathsf{F}(\mathsf{P}_t f) \quad \textit{for a.e. } t > 0. \tag{4.4}$$

Proof. The estimate (4.3) follows [3, Thm. 4.20], thanks to the integrability condition (3.23), when $f \in L^2(X,\mathfrak{m})$. In the general case it can be recovered by a truncation argument, using the lower semicontinuity of F. The estimate (4.4) follows by [3, Lemma 6.1], which can be applied here since the Dirichlet form $\mathcal{E}$ coincides with the Cheeger energy (Theorem 3.14).

Concerning the continuity of the map $t \mapsto \mathsf{H}_t\bar\mu$ with respect to W_2 for every $\bar\mu \in \mathscr{P}_2(X)$, it is a standard consequence of contractivity and existence of a dense set of initial conditions (namely the set $D(\mathrm{Ent}_{\mathfrak{m}}) := \{\nu \in \mathscr{P}_2(X) : \mathrm{Ent}_{\mathfrak{m}}(\nu) < \infty\}$) for which continuity holds up to $t = 0$. □

Integration by parts for probability densities

We shall see now that assuming the Bakry-Émery $\mathrm{BE}(K,\infty)$ condition, integration by parts formulae for the $\Delta_{\mathcal{E}}^{(1)}$ operator can be extended to probability densities with finite Fisher information, provided that the set of test functions φ is restricted to the spaces $\mathbb{V}^1_\infty, \mathbb{V}^2_\infty$ defined in (2.61). Recall that $\mathbb{V}^1_\infty, \mathbb{V}^2_\infty$ are strongly dense in $\mathbb{V}$ and that

$$\Gamma\big(\varphi_\varepsilon - \varphi\big) \overset{\star}{\rightharpoonup} 0 \quad \text{in } L^\infty(X,\mathfrak{m}) \text{ as } \varepsilon \downarrow 0 \quad \text{for every } \varphi \in \mathbb{V}^1_\infty, \tag{4.5}$$

where φ_ε are defined as in (2.28), since $\Gamma\big(\varphi_\varepsilon - \varphi\big)$ is uniformly bounded and converges to 0 in $L^1(X,\mathfrak{m})$.

In the sequel we introduce an extension of the bilinear form $\Gamma\big(f,g\big)$, denoted $\tilde\Gamma\big(f,g\big)$, which is particularly appropriate to deal with probability densities f with finite Fisher information and test functions $\varphi \in \mathbb{V}^1_\infty$.

Definition 4.3 (Extension of $\Gamma\big(f,\varphi\big)$)**.** Let $f = g^2 \in L^1_+(X,\mathfrak{m})$ with $\mathsf{F}(f) = 4\mathcal{E}(g) < \infty$. For all $\varphi \in \mathbb{V}^1_\infty$ we define

$$\tilde\Gamma\big(f,\varphi\big) := 2g\Gamma\big(g,\varphi\big). \tag{4.6}$$

The definition is well posed, consistent with the case when $f \in \mathbb{V}$, and it holds

$$\tilde\Gamma\big(f,\varphi\big) = \lim_{N\to\infty} \Gamma\big(f_N,\varphi\big) \qquad \text{in } L^1(X,\mathfrak{m}), \tag{4.7}$$

thanks to (2.12) and to the fact that if $\mathsf{F}(f) < \infty$ then $f_N = (g_N)^2 \in \mathbb{V}$, where $g_N := g \wedge \sqrt{N}$; it follows that $\Gamma\big(f_N,\varphi\big) = 2g\chi_N\Gamma\big(g,\varphi\big)$, χ_N being the characteristic function of the set $\{f < N\}$ and

$$\int_X \big|\tilde\Gamma\big(f,\varphi\big) - \Gamma\big(f_N,\varphi\big)\big|\,\mathrm{d}\mathfrak{m} = 2\int_X |1-\chi_N| g\Gamma\big(g,\varphi\big)\,\mathrm{d}\mathfrak{m} \le \Big(\|\Gamma\big(\varphi\big)\|_\infty\,\mathsf{F}(f)\int_{\{f\ge N\}} f\,\mathrm{d}\mathfrak{m}\Big)^{\frac12}$$

thus proving the limit in (4.7). The same argument provides the estimate

$$\int_X \psi\big|\tilde\Gamma\big(f,\varphi\big)\big|\,\mathrm{d}\mathfrak{m} \le \sqrt{\mathsf{F}(f)}\Big(\int_X \psi^2\Gamma\big(\varphi\big)\,f\,\mathrm{d}\mathfrak{m}\Big)^{1/2} \qquad \varphi \in \mathbb{V}^1_\infty,\ \psi \ge 0. \tag{4.8}$$

Theorem 4.4 (Integration by partsof $\Delta_{\mathcal{E}}^{(1)}$)**.** *If* $\mathrm{BE}(K,\infty)$ *holds, then for every* $f \in L^1_+(X,\mathfrak{m})$ *with* $\mathsf{F}(f) < \infty$ *we have*

$$\int_X \tilde\Gamma\big(f,\varphi\big)\,\mathrm{d}\mathfrak{m} = -\int_X f\Delta_{\mathcal{E}}\varphi\,\mathrm{d}\mathfrak{m} \quad \textit{for every } \varphi\in\mathbb{V}^2_\infty. \tag{4.9}$$

In addition, if $f\in D(\Delta_{\mathcal{E}}^{(1)})$ *it holds*

$$\int_X \tilde\Gamma\big(f,\varphi\big)\,\mathrm{d}\mathfrak{m} = -\int_X \Delta_{\mathcal{E}}^{(1)} f\,\varphi\,\mathrm{d}\mathfrak{m} \qquad \forall\varphi\in\mathbb{V}^1_\infty. \tag{4.10}$$

Proof. Formula (4.9) follows by the limit formula in (4.7) simply integrating by parts before passing to the limit as $N\to\infty$. Assuming now $f\in D(\Delta_{\mathcal{E}}^{(1)})$ we have

$$\begin{aligned}-\int_X \Delta_{\mathcal{E}}^{(1)} f\,\varphi\,\mathrm{d}\mathfrak{m} &= \lim_{t\downarrow 0}\frac{1}{t}\int_X (f-\mathsf{P}_t f)\varphi\,\mathrm{d}\mathfrak{m} = \lim_{t\downarrow 0}\frac{1}{t}\int_X f(\varphi-\mathsf{P}_t\varphi)\,\mathrm{d}\mathfrak{m}\\ &= \lim_{t\downarrow 0}\int_0^t\int_X f\,\Delta_{\mathcal{E}}(\mathsf{P}_s\varphi)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s = \lim_{t\downarrow 0}\int_0^t\int_X \tilde\Gamma\big(f,\mathsf{P}_s\varphi\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s = \int_X\tilde\Gamma\big(f,\varphi\big),\end{aligned}$$

where the last limit follows by (4.8) and the fact that

$$\Gamma\Big(\frac{1}{t}\int_0^t \mathsf{P}_s\varphi\,\mathrm{d}s-\varphi\Big)\le\frac{1}{t}\int_0^t\Gamma\big(\mathsf{P}_s\varphi-\varphi\big)\,\mathrm{d}s\overset{\star}{\rightharpoonup}0\quad\text{in } L^\infty(X,\mathfrak{m})\quad\text{as } t\downarrow 0.$$ □

4.2 Log-Harnack and L logL estimates

Lemma 4.5. *Let* $\omega:[0,\infty)\to\mathbb{R}$ *be a function of class* C^2*, let* $f\in\mathrm{Lip}_b(X,\mathfrak{m})\cap\mathbb{V}$ *and let* $\mu\in\mathscr{P}(X)$*. The function*

$$G(s) := \int_X \omega\big(\mathsf{P}_{t-s}f\big)\,\mathrm{d}\mathsf{H}_s\mu \qquad s\in[0,t], \tag{4.11}$$

belongs to $C^0([0,t])\cap C^1(0,t)$ *and for every* $s\in(0,t)$ *it holds*

$$G'(s) = \int_X \omega''(\mathsf{P}_{t-s}f)\Gamma\big(\mathsf{P}_{t-s}f\big)\,\mathrm{d}\mathsf{H}_s\mu. \tag{4.12}$$

Proof. Since $\mathsf{H}_s\mu$ are all probability measures is not restrictive to assume $\omega(0)=0$. Continuity of G is obvious, since $\mathsf{P}_{t-s}f$ are equi-Lipschitz, equi-bounded and the semigroup H_t is weakly-continuous. Let us first consider the case $\mu=\zeta\mathfrak{m}$ with $\zeta\in L^1\cap L^\infty(X,\mathfrak{m})$ (in particular $\zeta\in L^2(X,\mathfrak{m})$). Setting $f_{t-s}:=\mathsf{P}_{t-s}f$ and $\zeta_s:=\mathsf{P}_s\zeta$, we observe that a.e. in the open interval $(0,t)$ the following properties hold:
- $s\mapsto\zeta_s$ is differentiable in $L^2(X,\mathfrak{m})$;

- $s \mapsto \omega(f_{t-s})$ is differentiable in $L^2(X,\mathfrak{m})$, with derivative $-\omega'(f_{t-s})\Delta_{\mathcal{E}} f_{t-s}$.
Therefore the chain rule (2.18) gives

$$\begin{aligned} G'(s) &= -\int_X \omega'(f_{t-s})\zeta_s \Delta_{\mathcal{E}} f_{t-s}\,\mathrm{d}\mathfrak{m} + \int_X \omega(f_{t-s})\Delta_{\mathcal{E}}\zeta_s\,\mathrm{d}\mathfrak{m} \\ &= \int_X \Big(\Gamma\big(\omega'(f_{t-s})\zeta_s, f_{t-s}\big) - \Gamma\big(\omega(f_{t-s}),\zeta_s\big)\Big)\,\mathrm{d}\mathfrak{m} = \int_X \omega''(f_{t-s})\Gamma\big(f_{t-s}\big)\,\zeta_s\,\mathrm{d}\mathfrak{m} \end{aligned}$$

for a.e. $s \in (0,t)$. But, since the right hand side is continuous, the formula holds pointwise in $(0,t)$. The formula for arbitrary measures $\mu = \zeta\mathfrak{m} \in \mathscr{P}(X)$ then follows by monotone approximation (i.e. considering $\zeta_n = \min\{\zeta, n\}/c_n$ with $c_n \uparrow 1$ normalizing constants), by using the uniform L^∞ bounds on $\omega''(f_{t-s})$ and on $\Gamma\big(f_{t-s}\big)$ (and the formula (4.12) for G' still provides a continuous function). Finally, if $\mu \in \mathscr{P}(X)$ we approximate μ by the absolutely continuous measures $\mu_\varepsilon = \mathsf{H}_\varepsilon\mu$ and, passing to the limit in the formula (where we use the fact that P_t is also selfadjoint in the canonical pairing between L^1 and L^∞ and the absolute continuity of $\mathsf{H}_s\mu$)

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \omega(f_{t-s})\,\mathrm{d}\mathsf{H}_s\mu_\varepsilon = \int_X \omega''(f_{t-s})\Gamma\big(f_{t-s}\big)\,\mathrm{d}\mathsf{H}_s\mu_\varepsilon = \int_X \mathsf{P}_\varepsilon\big(\omega''(f_{t-s})\Gamma\big(f_{t-s}\big)\big)\,\mathrm{d}\mathsf{H}_s\mu$$

we conclude. □

In order to prove the LlogL regularization we use the next lemma, which follows by a careful adaptation to our more abstract context of a result by Wang [43, Theorem 1.1(6)].

Lemma 4.6 (Log-Harnack inequality)**.** *For every nonnegative $f \in L^1(X,\mathfrak{m}) + L^\infty(X,\mathfrak{m})$, $t > 0$, $\varepsilon \in [0,1]$, and x, $y \in X$ we have $\log(1+f) \in L^1(X,\mathsf{H}_t\delta_y)$ with*

$$\tilde{\mathsf{P}}_t(\log(f+\varepsilon))(y) \le \log\big(\tilde{\mathsf{P}}_t f(x) + \varepsilon\big) + \frac{\mathsf{d}^2(x,y)}{4\mathrm{I}_{2K}(t)}. \tag{4.13}$$

Proof. In the following we set $\omega_\varepsilon(r) := \log(r+\varepsilon)$, for $r \ge 0$ and $\varepsilon \in (0,1]$. Let us first assume in addition that $f \in \mathrm{Lip}_b(X) \cap L^1(X,\mathfrak{m}) \cap \mathbb{V}$, let $\gamma : [0,1] \to X$ be a Lipschitz curve connecting x to y in X, and, recalling the definition (2.48) of I_K, let

$$\tilde{\gamma}_r = \gamma_{\theta(r)}, \quad \text{with} \quad \theta(r) = \frac{\mathrm{I}_{2K}(r)}{\mathrm{I}_{2K}(t)}, \quad r \in [0,t].$$

We set $f_{t-s} := \mathsf{P}_{t-s}f$ and, for $r \in [0,t]$ and $s \in (0,t)$, we consider the functions

$$G(r,s) := \int_X \omega_\varepsilon(f_{t-s})\,\mathrm{d}\mathsf{H}_s\delta_{\tilde{\gamma}_r} = F_s(\tilde{\gamma}_r) \quad \text{with} \quad F_s(x) := \tilde{\mathsf{P}}_s\Big(\omega_\varepsilon(f_{t-s})\Big)(x). \tag{4.14}$$

Notice that Lemma 4.5 with $\mu = \delta_{\tilde{\gamma}_r}$ ensures that for every $r \in [0,t]$ the function $s \mapsto G(r,s)$ is continuous in $[0,t]$ and continuously differentiable in $(0,t)$, with

$$\frac{\partial}{\partial s}G(r,s) = \int_X \omega_\varepsilon''(f_{t-s})\Gamma\big(f_{t-s}\big)\,\mathrm{d}\mathsf{H}_s\delta_{\tilde{\gamma}_r}. \tag{4.15}$$

This gives immediately that $G(r,\cdot)$ are uniformly Lipschitz in $[0,t]$ for $r \in [0,t]$. On the other hand, since $\tilde\gamma$ is Lipschitz, the map $r \mapsto \mathsf{P}_s\delta_{\tilde\gamma_r}$ is Lipschitz in $[0,t]$ with respect to the L^1-Wasserstein distance W_1 uniformly w.r.t $s \in [0,t]$. Hence, taking also the fact that $\omega(f_{t-s})$ are equi-Lipschitz into account, it follows that also the maps $G(\cdot,s)$ are Lipschitz continuous in $[0,t]$ with Lipschitz constant uniform w.r.t. $s \in [0,t]$. These properties imply that the map $s \mapsto G(s,s)$ is Lipschitz in $[0,t]$.

Since the chain rule and (3.61) (which can be applied since we can subtract the constant $\omega_\varepsilon(0)$ from F_s without affecting the calculation of its slope) give

$$|\mathrm{D}F_s|^2 \le \mathrm{e}^{-2Ks}\,\mathsf{P}_s\Big((\omega_\varepsilon'(f_{t-s}))^2\Gamma(f_{t-s})\Big),$$

we can use $\theta'(r) = \mathrm{e}^{2Kr}/\mathrm{I}_{2K}(t)$ to get the pointwise estimate

$$\begin{aligned}\mathrm{e}^{K(s-r)}\limsup_{h\downarrow 0}\frac{|G(r+h,s)-G(r,s)|}{h} &\le \sqrt{\theta'(r)}\frac{|\dot\gamma_{\theta(r)}|}{\sqrt{\mathrm{I}_{2K}(t)}}\mathrm{e}^{Ks}\,|\mathrm{D}F_s|(\tilde\gamma_r)\\ &\le \theta'(r)\frac{|\dot\gamma_{\theta(r)}|^2}{4\mathrm{I}_{2K}(t)} + \int_X (\omega_\varepsilon'(f_{t-s}))^2\Gamma(f_{t-s})\,d\mathsf{P}_s\delta_{\tilde\gamma_r}.\end{aligned}$$

Applying the calculus lemma [2, Lemma 4.3.4] and using the identity $\omega_\varepsilon'' = -(\omega_\varepsilon')^2$, the previous inequality with $r = s$ in combination with (4.15) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}G(s,s) &\le \lim_{h\downarrow 0}\frac{G(s,s-h)-G(s,s)}{h} + \limsup_{h\downarrow 0}\frac{|G(s+h,s)-G(s,s)|}{h}\\ &\le \theta'(s)\frac{|\dot\gamma_{\theta(s)}|^2}{4\mathrm{I}_{2K}(t)} \quad \text{a.e. in } (0,t).\end{aligned}$$

An integration in $(0,t)$ and a minimization w.r.t. γ yield

$$\int_X \omega_\varepsilon(f)\,\mathrm{d}\mathsf{H}_t\delta_y \le \omega_\varepsilon(f_t)(x) + \frac{\mathsf{d}^2(x,y)}{4\mathrm{I}_{2K}(t)}. \tag{4.16}$$

If $f \in L^\infty(X,\mathfrak{m})$ we consider a uniformly bounded sequence (f_n) contained in $\mathrm{Lip}_b(X,\mathfrak{m}) \cap L^1(X,\mathfrak{m}) \cap \mathbb{V}$ converging to f pointwise $\mathfrak{m}$-a.e. Since $\omega_\varepsilon \ge \log(\varepsilon)$ and $\tilde{\mathsf{P}}_t f_n$ converges to $\tilde{\mathsf{P}}_t f$ pointwise, Fatou's Lemma yields (4.16) also in this case. Finally, a truncation argument extends the validity of (4.16) and (4.13) to arbitrary nonnegative $f \in L^1(X,\mathfrak{m}) + L^\infty(X,\mathfrak{m})$. Passing to the limit as $\varepsilon \downarrow 0$ we get (4.13) also in the case $\varepsilon = 0$.

Finally, notice that (4.16) for $\varepsilon = 1$ and the fact that $\tilde{\mathsf{P}}_t f(x)$ is finite for $\mathfrak{m}$-a.e. x yield the integrability of $\log_+ f$ with respect to $\mathsf{H}_t\delta_y$. □

In the sequel we set

$$\mathsf{H}_t\delta_y = u_t[y]\,\mathfrak{m}, \quad \text{so that} \quad \tilde{\mathsf{P}}_t f(y) = \int_X f u_t[y]\,\mathrm{d}\mathfrak{m} \tag{4.17}$$

for every $\mathfrak{m}$-measurable and semi-integrable function f.

Corollary 4.7. *For every $t > 0$ and $y \in X$ we have*

$$\int_X u_t[y] \log(u_t[y]) \, \mathrm{d}\mathfrak{m} \le \log\big(u_{2t}[y](x)\big) + \frac{\mathsf{d}^2(x,y)}{4\mathrm{I}_{2K}(t)} \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{4.18}$$

In particular, when $\mathfrak{m}$ is a probability measure,

$$u_{2t}[y](x) \ge \exp\Big(-\frac{\mathsf{d}^2(x,y)}{4\mathrm{I}_{2K}(t)}\Big) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{4.19}$$

Proof. Simply take $f = u_t[y]$ in (4.13) and notice that $\tilde{\mathsf{P}}_t f(x) = u_{2t}[y](x)$ for $\mathfrak{m}$-a.e. $x \in X$ by the semigroup property. □

In the next crucial result, we will show that (4.18) yields $\mathrm{Ent}_{\mathfrak{m}}(\mathsf{H}_t\mu) < \infty$ for every measure $\mu \in \mathscr{P}_2(X)$.

Theorem 4.8 (L logL regularization)**.** *Let $\mu \in \mathscr{P}_2(X)$ and let $f_t \in L^1(X,\mathfrak{m})$ be the densities of $\mathsf{H}_t\mu \in \mathscr{P}_2(X)$. Then*

$$\int_X f_t \log f_t \, \mathrm{d}\mathfrak{m} \le \frac{1}{2\mathrm{I}_{2K}(t)}\Big(r^2 + \int_X \mathsf{d}^2(x,x_0)\,\mathrm{d}\mu(x)\Big) - \log\big(\mathfrak{m}(B_r(x_0))\big) \tag{4.20}$$

for every $x_0 \in X$ and r, $t > 0$.

Proof. By approximation, it suffices to consider the case when $\mu = f\mathfrak{m}$ with $f \in L^2(X,\mathfrak{m})$. Let us fix $x_0 \in X$, $r > 0$ and set $\mathsf{z} = \mathfrak{m}(B_r(x_0))$ and $\nu = \mathsf{z}^{-1}\mathfrak{m} \llcorner B_r(x_0)$. Notice first that we have the pointwise inequality

$$\tilde{\mathsf{P}}_t f(z) \log(\tilde{\mathsf{P}}_t f(z)) = \Big(\int_X u_t[z]\,\mathrm{d}\mu\Big) \log\Big(\int_X u_t[z]\,\mathrm{d}\mu\Big) \le \int_X u_t[z](y) \log\big(u_t[z](y)\big)\,\mathrm{d}\mu(y).$$

Since $\tilde{\mathsf{P}}_t f = f_t$ $\mathfrak{m}$-a.e., integrating with respect to $\mathfrak{m}$ and using the symmetry property of u_t, (4.18), and Jensen's inequality, we get

$$\begin{aligned}
\int_X f_t \log(f_t)\,\mathrm{d}\mathfrak{m} &\le \int_X \Big(\int_X u_t[y](z) \log u_t[y](z)\,\mathrm{d}\mathfrak{m}(z)\Big)\,\mathrm{d}\mu(y) \\
&= \int_{X\times X} \Big(\int_X u_t[y](z) \log u_t[y](z)\,\mathrm{d}\mathfrak{m}(z)\Big)\,\mathrm{d}\nu(x)\mathrm{d}\mu(y) \\
&\le \int_{X\times X} \Big(\log\big(u_{2t}[y](x)\big) + \frac{\mathsf{d}^2(x,y)}{4\mathrm{I}_{2K}(t)}\Big)\,\mathrm{d}\nu(x)\mathrm{d}\mu(y) \\
&\le \log\Big(\int_X \int_X u_{2t}[y](x)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(y)\Big) + \frac{1}{2\mathrm{I}_{2K}(t)}\Big(r^2 + \int_X \mathsf{d}^2(x,x_0)\,\mathrm{d}\mu(x)\Big) \\
&\le -\log \mathsf{z} + \frac{1}{2\mathrm{I}_{2K}(t)}\Big(r^2 + \int_X \mathsf{d}^2(x,x_0)\,\mathrm{d}\mu(x)\Big),
\end{aligned}$$

where we used the inequality

$$\int_X u_{2t}[y](x)\,\mathrm{d}\nu(x) = \frac{1}{\mathsf{z}} \int_{B_r(x_0)} u_{2t}[y](x)\,\mathrm{d}\mathfrak{m}(x) \le \frac{1}{\mathsf{z}}. \qquad \square \tag{4.21}$$

We conclude with a further regularization and an integration by parts formula for $\Delta^{(1)}_{\mathcal{E}}$ in a special case. Notice that thanks to the regularizing effect of H_t we can extend $\mathfrak{h}^\varepsilon$ to measures $\mu \in \mathscr{P}_2(X)$, i.e. we set

$$\mathfrak{h}^\varepsilon \mu := \frac{1}{\varepsilon}\int_0^\infty f_r\, \kappa(r/\varepsilon)\,\mathrm{d}r, \quad f_r\mathfrak{m} = \mathsf{H}_r\mu \quad \text{for } r > 0, \tag{4.22}$$

obtaining a map $\mathfrak{h}^\varepsilon : \mathscr{P}_2(X) \to D(\Delta^{(1)}_{\mathcal{E}})$.

Lemma 4.9. *Let $\tilde\mu \in \mathscr{P}_2(X)$ and let $f = \mathfrak{h}^\varepsilon\tilde\mu$ as in (4.22). Then for every $\varepsilon, T > 0$ there exists a constant $C(\varepsilon, T)$ such that*

$$\mathsf{F}(\mathsf{P}_t f) \le C(\varepsilon, T)\Big(1 + \int_X V^2\,\mathrm{d}\mu\Big) \tag{4.23}$$

and, writing $\mu_t = \mathsf{P}_t f\mathfrak{m}$,

$$|\dot\mu_t|^2 \le C(\varepsilon, T)\Big(1 + \int_X V^2\,\mathrm{d}\mu\Big) \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0, T]. \tag{4.24}$$

Moreover, for every bounded and nondecreasing Lipschitz function $\omega : [0,\infty) \to \mathbb{R}$ such that $\sup_r r\omega'(r) < \infty$, we have

$$\int_X \omega(f)\Delta^{(1)}_{\mathcal{E}} f\,\mathrm{d}\mathfrak{m} + 4\int_X f\omega'(f)\,\Gamma\big(\sqrt{f}\big)\,\mathrm{d}\mathfrak{m} \le 0. \tag{4.25}$$

Proof. Combining (4.3), (4.20), the commutation identity $\mathsf{P}_t\mathfrak{h}^\varepsilon = \mathfrak{h}^\varepsilon\mathsf{P}_t$, and the convexity of F we get (4.23). We obtain immediately the Lipschitz estimate (4.24) from (4.23) and (4.4) when $f \in L^2(X,\mathfrak{m})$. The general case follows by a truncation argument. Concerning (4.25), if $\tilde\mu = \tilde f\mathfrak{m}$ with $\tilde f \in L^2(X,\mathfrak{m})$, then $f \in L^2(X,\mathfrak{m})$, $\Delta^{(1)}_{\mathcal{E}} f = \Delta_{\mathcal{E}} f \in L^2(X,\mathfrak{m})$ and the stated inequality is an equality, by the chain rule $\Gamma\big(f, \omega(f)\big) = \omega'(f)\Gamma\big(f\big) = 4f\omega'(f)\,\Gamma\big(\sqrt{f}\big)$. In the general case we approximate $\tilde\mu$ in $\mathscr{P}_2(X)$ by a sequence of measures $\tilde\mu_n = \tilde f_n\mathfrak{m}$ with $f_n \in L^2(X,\mathfrak{m})$ and we consider $f_n = \mathfrak{h}^\varepsilon\mu_n$. By (2.30) we obtain that $\Delta^{(1)}_{\mathcal{E}} f_n \to \Delta^{(1)}_{\mathcal{E}} f$ in $L^1(X,\mathfrak{m})$ while, setting $\phi(s) = \int_0^s \sqrt{r^2\omega'(r^2)}\,\mathrm{d}r$, the lower semicontinuity of $g \mapsto \int \Gamma\big(g\big)\,\mathrm{d}\mathfrak{m}$ and the strong convergence of $\sqrt{f_n}$ to $\sqrt{f}$ in $L^2(X,\mathfrak{m})$ give

$$\begin{aligned} 4\int_X f\omega'(f)\,\Gamma\big(\sqrt{f}\big)\,\mathrm{d}\mathfrak{m} &= \int_X \Gamma\big(\phi(\sqrt{f})\big)\,\mathrm{d}\mathfrak{m} \le \liminf_{n\to\infty}\int_X \Gamma\big(\phi(\sqrt{f_n})\big)\,\mathrm{d}\mathfrak{m} \\ &= \liminf_{n\to\infty}\int_X f_n\omega'(f_n)\,\Gamma\big(\sqrt{f_n}\big)\,\mathrm{d}\mathfrak{m}. \qquad \square \end{aligned}$$

Motivated by the regularity assumptions needed in the next section, we give the following definition.

Definition 4.10 (Regular curve)**.** Let $\rho_s = f_s\mathfrak{m} \in \mathscr{P}(X)$, $s \in [0,1]$. We say that ρ is regular if:

(a) $\rho \in \mathrm{AC}^2([0,1]; (\mathscr{P}_2(X), W_2))$;

(b) $\mathrm{Ent}_{\mathfrak{m}}(\rho_s)$ is bounded;

(c) $f \in \mathrm{C}^1([0,1]; L^1(X, \mathfrak{m}))$;

(d) There exists $\eta > 0$ such that for all $s \in [0,1]$ the function f_s is representable in the form $\mathfrak{h}^\eta \tilde{f}_s$ for some $\tilde{f}_s \in L^1(X, \mathfrak{m})$; in addition $\Delta_{\mathcal{E}}^{(1)} f \in \mathrm{C}([0,1]; L^1(X, \mathfrak{m}))$ and

$$\sup \{\mathsf{F}(\mathsf{P}_t f_s) : \ s \in [0,1], \ t \in [0,T]\} < \infty \qquad \forall T > 0. \tag{4.26}$$

In particular, if $\rho_s = f_s \mathfrak{m}$ is a regular curve, for every $T > 0$ there exist positive constants M_T, E_T, F_T such that

$$\int_X V^2 \,\mathrm{d}\mathsf{H}_t \rho_s \le M_T, \ \mathrm{Ent}_{\mathfrak{m}}(\mathsf{P}_t \rho_s) \le E_T, \ \mathsf{F}(\mathsf{P}_t f_s) \le F_T \qquad s \in [0,1], \ t \in [0,T]. \tag{4.27}$$

Proposition 4.11 (Approximation by regular curves)**.** *For all $\rho \in \mathrm{AC}^2([0,1]; (\mathscr{P}_2(X), W_2))$ there exist regular curves ρ^n such that $\rho^n_s \to \rho_s$ in $\mathscr{P}_2(X)$ for all $s \in [0,1]$ and*

$$\limsup_n \int_0^1 |\dot{\rho}^n_s|^2 \,\mathrm{d}s \le \int_0^1 |\dot{\rho}_s|^2 \,\mathrm{d}s. \tag{4.28}$$

Proof. First we extend ρ by continuity and with constant values in $(-\infty, 0) \cup (1, \infty)$. Then, we define $\rho^{n,1}_s := \mathsf{H}_{\tau_n} \rho_s$, with $\tau_n^{-1} \in [n, 2n]$. By the contractivity properties of H_t, we see that $\rho^{n,1}$ fulfills the first two requirements of the lemma and (4.28), but we need to check regularity. Indeed, obviously condition (a) is fulfilled, while we gain absolute continuity of $\rho^{n,1}_s$ and $\sup_s \mathrm{Ent}_{\mathfrak{m}}(\rho^{n,1}_s) < \infty$ by Theorem 4.8. In order to achieve condition (c) we do an additional regularization, by averaging w.r.t. the s variable: precisely, denoting by $f^{n,1}_s$ the densities of $\rho^{n,1}_s$, we set $\rho^{n,2}_s := f^{n,2}_s \mathfrak{m}$, where

$$f^{n,2}_s := \int_{\mathbb{R}} f^{n,1}_{s-s'} \chi_n(s') \,\mathrm{d}s' \tag{4.29}$$

and $\chi_n \in C^\infty_c(\mathbb{R})$ are standard convolution kernels convergent to the identity. By the convexity properties of squared Wasserstein distance and entropy we see that properties (a), (b) are retained and that the action does not increase. In addition, we clearly gain property (c). In the last step we mollify using the heat semigroup, setting $\rho^n_s := f^n_s \mathfrak{m}$, where $f^n_s = \mathfrak{h}^{\varepsilon_n} f^{n,2}_s$ and $\varepsilon_n \downarrow 0$. By the same reasons used for $\rho^{n,2}$, properties (a), (b) are retained by ρ^n and the action does not increase. In addition, (c) is retained as well since $\mathfrak{h}^\varepsilon$ is a continuous linear map from $L^1(X, \mathfrak{m})$ to $D(\Delta_{\mathcal{E}}^{(1)})$. Finally, (4.23) provides the sup bound on Fisher information. $\square$

4.3 Action estimates

This section contains the core of the arguments leading to the equivalence Theorem 4.17. We refer to [15] for the underlying geometric ideas in a smooth Riemannian context and the role of the Bochner identity. Here we had to circumvent many technical difficulties related to regularity issues, to the lack of ultracontractivity properties of the semigroup $(\mathsf{P}_t)_{t\geq 0}$ (i.e. regularization from L^1 to L^∞), and to the weak formulation of the Bakry-Émery condition.

Since we shall often consider regular curves of measures $\rho \in \mathrm{AC}^2([0,1];(\mathscr{P}_2(X),W_2))$, representable in the form $\rho = f\mathfrak{m}$ with $f \in C^1([0,1];L^1(X,\mathfrak{m}))$, we shall denote by $\dot f \in C([0,1];L^1(X,\mathfrak{m}))$ the functional derivative in $L^1(X,\mathfrak{m})$, retaining the notation $|\dot\rho_t|$ for the metric derivative w.r.t. W_2.

We begin with a simple estimate of the oscillation of $s \mapsto \int_X \varphi \,\mathrm{d}\rho_s$ along absolutely continuous or C^1 curves.

Lemma 4.12. *For all $\rho \in \mathrm{AC}^2([0,1];(\mathscr{P}_2(X),W_2))$ it holds*

$$\left|\int_X \varphi \,\mathrm{d}\rho_1 - \int_X \varphi \,\mathrm{d}\rho_0\right| \leq \int_0^1 |\dot\rho_s| \sqrt{\int_X |\mathrm{D}\varphi|^2 \,\mathrm{d}\rho_s}\,\mathrm{d}s \qquad \text{for every } \varphi \in \mathbb{V}^1_\infty. \tag{4.30}$$

If moreover $\rho = f\mathfrak{m}$ with $f \in \mathrm{C}^1\big([0,1];L^1(X,\mathfrak{m})\big)$, for all $\varphi \in \mathbb{V}^1_\infty$ it holds

$$\left|\int_X \dot f_s \varphi \,\mathrm{d}\mathfrak{m}\right| \leq |\dot\rho_s| \bigg(\int_X |\mathrm{D}\varphi|^2 \,\mathrm{d}\rho_s\bigg)^{1/2} \text{for } \mathscr{L}^1\text{-a.e. } s \in (0,1). \tag{4.31}$$

Proof. It is easy to check that (4.30) can be obtained using the representation of ρ_s given by Lisini's theorem [26] (see [3, Lemma 5.15]).

Choosing now a Lebesgue point $\bar s$ both for $s \mapsto |\dot\rho_s|^2$ and $\int_X |\mathrm{D}\varphi|^2 \,\mathrm{d}\rho_s$, for all $a > 0$ we can pass to the limit as $h \downarrow 0$ in the inequality

$$\frac{1}{h}\left|\int_X \varphi \,\mathrm{d}\rho_{\bar s + h} - \int_X \varphi \,\mathrm{d}\rho_{\bar s}\right| \leq \frac{1}{2h}\int_{\bar s}^{\bar s + h} \bigg(a|\dot\rho_s|^2 + \frac{1}{a}\int_X |\mathrm{D}\varphi|^2 \,\mathrm{d}\rho_s\bigg)\,\mathrm{d}s$$

and then minimize w.r.t. a, obtaining (4.31). □

Lemma 4.13. *Let $\rho = f\mathfrak{m} \in \mathrm{AC}^2([0,1];(\mathscr{P}_2(X),W_2))$ be a regular curve according to Definition 4.10, and let $\vartheta : [0,1] \to [0,1]$ be a C^1 function with $\vartheta(i) = i$, $i = 0,1$. Define*

$$\rho_{s,t} := \mathsf{H}_{st}\rho_{\vartheta(s)} = f_{s,t}\mathfrak{m}, \quad s \in [0,t],\ t \geq 0.$$

Then, for every $t \geq 0$ the curve $s \mapsto \rho_{s,t}$ belongs to $\mathrm{AC}^2([0,1];(\mathscr{P}_2(X),W_2))$ and $\mathsf{F}(f_{s,t})$ is uniformly bounded. Moreover, for any $\varphi \in \mathrm{Lip}_b(X)$ with bounded support, setting $\varphi_s := Q_s\varphi$, the map $s \mapsto \int_X \varphi_s \,\mathrm{d}\rho_{s,t}$ is absolutely continuous in $[0,1]$ and

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s \,\mathrm{d}\rho_{t,s} = \dot\vartheta(s)\int_X \dot f_s \mathsf{P}_{st}\varphi_s \,\mathrm{d}\mathfrak{m} - \frac{1}{2}\int_X |\mathrm{D}\varphi_s|^2 \,\mathrm{d}\rho_{s,t} - t\int_X \tilde\Gamma\big(f_{s,t},\varphi_s\big)\,\mathrm{d}\mathfrak{m} \tag{4.32}$$

for $\mathscr{L}^1$-a.e. $s \in (0,1)$.

Proof. We only consider the case $t > 0$ and we set $\tilde\rho_s := \rho_{\vartheta(s)} = \tilde f_s \mathfrak{m}$. Notice that $f_{s,t} = \mathsf{P}_{st}\tilde f_s$ and $\tilde\rho_s$ satisfies the same assumptions than ρ_s. Since H_t is a Wasserstein K-contraction

$$W_2(\rho_{s_0,t},\rho_{s_1,t}) \le \mathrm{e}^{-Ks_0t} W_2(\tilde\rho_{s_0}, \mathsf{H}_{(s_1-s_0)t}\tilde\rho_{s_1}) \le \mathrm{e}^{-Ks_0t}\Big(W_2(\tilde\rho_{s_0},\tilde\rho_{s_1}) + W_2(\tilde\rho_{s_1}, \mathsf{H}_{(s_1-s_0)t}\tilde\rho_{s_1})\Big),$$

and (4.24) and the regularity of ρ give

$$W_2(\tilde\rho_{s_1}, \mathsf{H}_{(s_1-s_0)t}\tilde\rho_{s_1}) \le C(\rho,T)(s_1-s_0)t \qquad \text{whenever } (s_1-s_0)t \le T.$$

We conclude that $s \mapsto \rho_{s,t}$ belongs to $\mathrm{AC}^2([0,1];(\mathscr{P}_2(X),W_2))$. Moreover, using the splitting

$$\begin{aligned}\int_X \varphi_{s_1}\,\mathrm{d}\rho_{s_1,t} - \int_X \varphi_{s_0}\,\mathrm{d}\rho_{s_0,t} &= \int_X \varphi_{s_1}\,\mathrm{d}\rho_{s_1,t} - \int_X \varphi_{s_0}\,\mathrm{d}\rho_{s_1,t} + \int_X \varphi_{s_0}\,\mathrm{d}\rho_{s_1,t} - \int_X \varphi_{s_0}\,\mathrm{d}\rho_{s_1,t}\\ &\le \|\varphi_{s_1}-\varphi_{s_0}\|_\infty + \mathrm{Lip}(\varphi_{s_0}) W_2(\tilde\rho_{s_1},\tilde\rho_{s_0})\end{aligned}$$

we immediately see that also $s \mapsto \int_X \varphi_s\,\mathrm{d}\rho_{s,t}$ is absolutely continuous. In order to compute its derivative we write

$$\begin{aligned}\int \varphi_{s+h}\,\mathrm{d}\rho_{s+h,t} - \int_X \varphi_s\,\mathrm{d}\rho_{s,t} &= \int \varphi_{s+h}\,\mathrm{d}\rho_{s+h,t} - \int_X \varphi_s\,\mathrm{d}\rho_{s+h,t}\\ &\quad + \int \mathsf{P}_{(s+h)t}\varphi_s\,\mathrm{d}(\tilde\rho_{s+h}-\tilde\rho_s)\\ &\quad + \int (\mathsf{P}_{ht}\varphi_s - \varphi_s)\,\mathrm{d}\mathsf{H}_{st}\tilde\rho_s.\end{aligned}$$

Now, the Hopf-Lax formula (3.14) and the strong convergence of $f_{s+h,t}$ to $f_{s,t}$ in $L^1(X,\mathfrak{m})$ yield

$$\lim_{h\downarrow 0}\frac{1}{h}\Big(\int \varphi_{s+h}\,\mathrm{d}\rho_{s+h,t} - \int_X \varphi_s\,\mathrm{d}\rho_{s+h,t}\Big) = -\frac{1}{2}\int_X |\mathrm{D}\varphi_s|^2\,\mathrm{d}\rho_{s,t}.$$

The differentiability of ρ_s in $L^1(X,\mathfrak{m})$ yields

$$h^{-1}\int \mathsf{P}_{(s+h)t}\varphi\,\mathrm{d}(\tilde\rho_{s+h}-\tilde\rho_s) \to \dot\vartheta(s)\int_X \mathsf{P}_{st}\varphi\,\dot f_s\,\mathrm{d}\mathfrak{m}. \tag{4.33}$$

Finally, the next lemma yields

$$h^{-1}\int (\mathsf{P}_{ht}\varphi_s - \varphi_s)\,\mathrm{d}\mathsf{P}_{st}\rho_s \to -t\int_X \tilde\Gamma\big(f_{s,t},\varphi_s\big)\,\mathrm{d}\mathfrak{m}. \qquad \square \tag{4.34}$$

Lemma 4.14. *For all $\varphi \in \mathbb{V}^1_\infty$ and all $\rho = f\mathfrak{m} \in \mathscr{P}(X)$ with $\mathsf{F}(f) < \infty$ it holds*

$$\lim_{h\downarrow 0}\int_X \frac{\mathsf{P}_h\varphi - \varphi}{h} f\,\mathrm{d}\mathfrak{m} = -\int_X \tilde\Gamma\big(f,\varphi\big)\mathrm{d}\mathfrak{m}$$

Proof. We argue as in [5, Lemma 4.2], proving first that

$$\int_X \frac{\mathsf{P}_h\varphi - \varphi}{h} f \,\mathrm{d}\mathfrak{m} = -\int_0^1 \int_X \tilde{\Gamma}\big(f, \mathsf{P}_{rh}\varphi\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r. \tag{4.35}$$

Notice first that, possibly approximating φ with the functions $\varphi^\varepsilon := \mathfrak{h}^\varepsilon\varphi$ whose laplacian is in $L^\infty(X,\mathfrak{m})$, in the proof of (4.35) we can assume with no loss of generality that $\Delta_{\mathcal{E}}\varphi \in L^\infty(X,\mathfrak{m})$. Indeed, (4.8) and the strong convergence in Γ norm of $\mathsf{P}_{rh}\varphi^\varepsilon$ to $\mathsf{P}_{rh}\varphi$ ensure the dominated convergence of the integrals in the right hand sides, while the convergence of the left hand sides is obvious.

Assuming $\Delta_{\mathcal{E}}\varphi \in L^\infty(X,\mathfrak{m})$, since

$$\int_X \frac{\mathsf{P}_h\varphi - \varphi}{h} g \,\mathrm{d}\mathfrak{m} = \int_0^1 \int_X g\Delta_{\mathcal{E}}\mathsf{P}_{rh}\varphi\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r$$

for all $g \in L^2(X,\mathfrak{m})$ we can consider the truncated functions $g_N = \min\{f, N\}$ and pass to the limit as $N \to \infty$ to get that f satisfies the same identity. Since $\Delta_{\mathcal{E}}\mathsf{P}_{rh}\varphi = \mathsf{P}_{rh}\Delta_{\mathcal{E}}\varphi \in L^\infty(X,\mathfrak{m})$ we can use (4.9) to obtain (4.35).

Having established (4.35), the statement follows using once more (4.8) and the strong convergence of $\mathsf{P}_{rh}\varphi$ to φ in Γ norm. □

Under the same assumptions of Lemma 4.13, the same computation leading to (4.32) (actually with a simplification, due to the fact that φ is independent on s) and (4.10) give

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi\,\mathrm{d}\rho_{s,t} = \int_X (\dot{\vartheta}(s)\mathsf{P}_{st}\dot{f}_s + t\mathsf{P}_{st}\Delta^{(1)}_{\mathcal{E}} f_s)\varphi\,\mathrm{d}\mathfrak{m}.$$

for $\mathscr{L}^1$-a.e. $s \in (0,1)$ and all $\varphi \in \mathbb{V}^1_\infty$. Here we used the fact that $\Delta^{(1)}_{\mathcal{E}}(P_r g) = P_r\Delta^{(1)}_{\mathcal{E}} g$ whenever $g \in D(\Delta^{(1)}_{\mathcal{E}})$. Since $\Delta^{(1)}_{\mathcal{E}} f \in \mathrm{C}([0,1]; L^1(X,\mathfrak{m}))$, the right hand side is a continuous function of s, hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi\,\mathrm{d}\rho_{s,t} = \int_X (\dot{\vartheta}_s\,\mathsf{P}_{st}\dot{f}_s + t\,\mathsf{P}_{st}\Delta^{(1)}_{\mathcal{E}} f_s)\varphi\,\mathrm{d}\mathfrak{m} \qquad \text{for every } s \in (0,1). \tag{4.36}$$

For $\varepsilon > 0$, let us now consider the regularized entropy functionals

$$E_\varepsilon(\rho) := \int_X e_\varepsilon(f)\,\mathrm{d}\mathfrak{m}, \quad \text{where} \quad e'_\varepsilon(r) := \log(\varepsilon + r \wedge \varepsilon^{-1}) \in \mathrm{Lip}([0,\infty)),\ e_\varepsilon(0) = 0. \tag{4.37}$$

Since we will mainly consider functions f with finite Fisher information, we will also introduce the function

$$p_\varepsilon(r) := e'_\varepsilon(r^2) - \log\varepsilon = \log(\varepsilon + r^2 \wedge \varepsilon^{-1}) - \log\varepsilon.$$

Since p_ε is also Lipschitz and $p_\varepsilon(0) = 0$, we have

$$f \in L^1_+(X,\mathfrak{m}),\ \mathsf{F}(f) < \infty \quad \Rightarrow \quad e'_\varepsilon(f) - \log\varepsilon = p_\varepsilon(\sqrt{f}) \in \mathbb{V}.$$

Lemma 4.15 (Derivative of E_ε). *With the same notation of Lemma 4.13, if ρ is regular and $t > 0$ we have for $g^\varepsilon_{s,t} := p_\varepsilon(\sqrt{f_{s,t}})$*

$$E_\varepsilon(\rho_{1,t}) - E_\varepsilon(\rho_{0,t}) \le \int_0^1 \Big(- t \int_X f_{s,t}\Gamma\big(g^\varepsilon_{s,t}\big)\,\mathrm{d}\mathfrak{m} + \dot\vartheta_s \int_X \mathsf{P}_{st}\big(g^\varepsilon_{s,t}\big)\dot f_s\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s. \tag{4.38}$$

Proof. The weak differentiability of $s \mapsto f_{s,t}$ (namely, in duality with functions in $\mathbb{V}^1_\infty$) given in (4.36) can, thanks to the continuity assumption made on $\Delta^{(1)}_{\mathcal{E}} f_s$, turned into strong $L^1(X,\mathfrak{m})$ differentiability, so that

$$\frac{\mathrm{d}}{\mathrm{d}s} f_{s,t} = \dot\vartheta_s\,\mathsf{P}_{st}\dot f_s + t\,\mathsf{P}_{st}(\Delta^{(1)}_{\mathcal{E}} f_s) \qquad \text{in } L^1(X,\mathfrak{m}), \text{ for all } s \in (0,1). \tag{4.39}$$

Since e_ε is of class $\mathrm{C}^{1,1}$, it is easy to check that this implies the absolute continuity of $s \mapsto E_\varepsilon(\rho_{s,t})$. In addition, the mean value theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}s} E_\varepsilon(\rho_{s,t}) = \lim_{h\to 0}\int_X e'_\varepsilon(f_{s,t})\frac{f_{s+h,t}-f_{s,t}}{h}\,\mathrm{d}\mathfrak{m} \qquad \forall s \in (0,1).$$

Notice also that Lemma 4.9 with $f = f_{s,t}$ and $\omega = e'_\varepsilon - \log\varepsilon$ gives (since $e'_\varepsilon(f_{s,t}) - \log\varepsilon = p_\varepsilon(\sqrt{f_{s,t}})$ is nonnegative and integrable and $\mathsf{P}_{st}(\Delta^{(1)}_{\mathcal{E}} f_s) = \Delta^{(1)}_{\mathcal{E}}(\mathsf{P}_{st} f_s)$ has null mean)

$$\int_X \mathsf{P}_{st}(\Delta^{(1)}_{\mathcal{E}} f_s) e'_\varepsilon(f_{s,t})\,\mathrm{d}\mathfrak{m} \le -4\int_X f_{s,t}e''_\varepsilon(f_{s,t})\,\Gamma\big(\sqrt{f_{s,t}}\big)\,\mathrm{d}\mathfrak{m}. \tag{4.40}$$

Now we use (4.39), (4.40) and conclude

$$\frac{\mathrm{d}}{\mathrm{d}s} E_\varepsilon(\rho_{s,t}) \le -t\int_X 4 f_{s,t}e''_\varepsilon(f_{s,t})\Gamma\big(\sqrt{f_{s,t}}\big)\,\mathrm{d}\mathfrak{m} + \dot\vartheta_s\int_X (e'_\varepsilon(f_{s,t}) - \log\varepsilon)\mathsf{P}_{s,t}\dot f_s\,\mathrm{d}\mathfrak{m}.$$

On the other hand, since $4re''_\varepsilon(r) \ge 4r^2\big(e''_\varepsilon(r)\big)^2 = r\big(p'_\varepsilon(\sqrt{r})\big)^2$, we get

$$-4f_{s,t}e''_\varepsilon(f_{s,t})\Gamma\big(\sqrt{f_{s,t}}\big) \le -f_{s,t}\big(p'_\varepsilon(\sqrt{f_{s,t}})\big)^2\Gamma\big(\sqrt{f_{s,t}}\big) = -f_{s,t}\Gamma\big(p_\varepsilon(\sqrt{f_{s,t}})\big)$$

and an integration with respect to s and the definition of $g^\varepsilon_{s,t}$ yield (4.38). □

Theorem 4.16 (Action and entropy estimate on regular curves). *Let $\rho_s = f_s\mathfrak{m}$ be a regular curve. Then, setting $\rho_{1,t} = \mathsf{H}_t\rho_1$, it holds*

$$W_2^2(\rho_0,\rho_{1,t}) + 2t\mathrm{Ent}_{\mathfrak{m}}(\rho_{1,t}) \le \mathrm{R}^2_K(t)\int_0^1 |\dot\rho_s|^2\,\mathrm{d}s + 2t\mathrm{Ent}_{\mathfrak{m}}(\rho_0). \tag{4.41}$$

where

$$\mathrm{R}_K(t) := \frac{t}{\mathrm{I}_K(t)} = \frac{Kt}{\mathrm{e}^{Kt}-1} \quad \textit{if } K \ne 0, \qquad \mathrm{R}_0(t) \equiv 1.$$

Proof. Set $\rho_{s,t}, f_{s,t}$ as in Lemma 4.13, $p_\varepsilon(r) = e'_\varepsilon(r^2) - \log\varepsilon$, $g^\varepsilon_{s,t} = p_\varepsilon(\sqrt{f_{s,t}})$ as in Lemma 4.15, $q_\varepsilon(r) := \sqrt{r}(2 - \sqrt{r}p'_\varepsilon(\sqrt{r}))$, and $\varphi_s := Q_s\varphi$ for a Lipschitz function φ with bounded support.

Notice that by (4.6)

$$\tilde\Gamma\big(f_{s,t}, \varphi_s\big) = 2\sqrt{f_{s,t}}\Gamma\big(\sqrt{f_{s,t}}, \varphi_s\big) = f_{s,t}\Gamma\big(g^\varepsilon_{s,t}, \varphi_s\big) + q_\varepsilon(f_{s,t})\Gamma\big(\sqrt{f_{s,t}}, \varphi_s\big).$$

Applying (4.32), (4.38) in the weaker form

$$tE_\varepsilon(\rho_{1,t}) - tE_\varepsilon(\rho_{0,t}) \le \int_0^1 \Big(-\frac{t^2}{2}\int_X f_{s,t}\Gamma\big(g^\varepsilon_{s,t}\big)\,\mathrm{d}\mathfrak{m} + t\dot\vartheta_s\int_X \mathsf{P}_{st}\big(g^\varepsilon_{s,t}\big)\dot f_s\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s.$$

and eventually the Young inequality $2xy \le ax^2 + y^2/a$ in (4.31) with $a := \dot\vartheta_s \mathrm{e}^{-2Kst}$, we obtain

$$\begin{aligned}
&\int_X \varphi_1\,\mathrm{d}\rho_{1,t} - \int_X \varphi_0\,\mathrm{d}\rho_{0,t} + t\Big(E_\varepsilon(\rho_{1,t}) - E_\varepsilon(\rho_{0,t})\Big)\\
&\quad\le \int_0^1 \Big(\dot\vartheta_s\int_X \dot f_s\mathsf{P}_{st}\Big(\varphi_s + tg^\varepsilon_{s,t}\Big)\,\mathrm{d}\mathfrak{m} - \frac12\int_X\Big(|\mathrm{D}\varphi_s|^2 + t^2\Gamma\big(g^\varepsilon_{s,t}\big)\Big)\,\mathrm{d}\rho_{s,t}\\
&\qquad - t\int_X \Gamma\big(g^\varepsilon_{s,t}, \varphi_s\big)\,\mathrm{d}\rho_{s,t} - t\int_X q_\varepsilon(f_{s,t})\Gamma\big(\sqrt{f_{s,t}}, \varphi_s\big)\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s\\
&\quad\le \int_0^1 \Big(\dot\vartheta_s\int_X \dot f_s\mathsf{P}_{st}\Big(\varphi_s + tg^\varepsilon_{s,t}\Big)\,\mathrm{d}\mathfrak{m} - \frac12\int_X \Gamma\big(\varphi_s + tg^\varepsilon_{s,t}\big)\,\mathrm{d}\rho_{s,t}\\
&\qquad - t\int_X q_\varepsilon(f_{s,t})\Gamma\big(\sqrt{f_{s,t}}, \varphi_s\big)\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s\\
&\quad\le \int_0^1 \Big(\dot\vartheta_s\int_X \dot f_s\mathsf{P}_{st}\Big(\varphi_s + tg^\varepsilon_{s,t}\Big)\,\mathrm{d}\mathfrak{m} - \frac12\mathrm{e}^{2Kst}\int_X \Gamma\big(\mathsf{P}_{st}\big(\varphi_s + tg^\varepsilon_{s,t}\big)\big)\,\mathrm{d}\rho_s\\
&\qquad + t\int_X |q_\varepsilon(f_{s,t})|\,\big|\Gamma\big(\sqrt{f_{s,t}}, \varphi_s\big)\big|\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s\\
&\quad\le \int_0^1 \Big(\frac12(\dot\vartheta_s)^2\mathrm{e}^{-2Kst}|\dot\rho_s|^2 + \frac{t}{8}\delta\mathsf{F}(\rho_{s,t}) + \frac{t}{2\delta}\int_X q^2_\varepsilon(f_{s,t})|\mathrm{D}\varphi_s|^2\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s.
\end{aligned}$$

Now we pass first to the limit as $\varepsilon \downarrow 0$, observing that $p'_\varepsilon(r) = 2r(\varepsilon + r^2)^{-1}\chi_{r^2<\varepsilon^{-1}}$ gives

$$q^2_\varepsilon(r) = 4r\Big(1 - \frac{r}{\varepsilon + r}\Big)^2\chi_{r<\varepsilon^{-1}} \le 4r, \qquad \lim_{\varepsilon\downarrow 0} q^2_\varepsilon(r) = 0,$$

and then as $\delta \downarrow 0$; choosing

$$\vartheta(s) := \frac{\mathrm{I}_K(st)}{\mathrm{I}_K(t)}, \quad \text{so that} \quad \dot\vartheta(s) = \mathrm{R}_K(t)\mathrm{e}^{Kst},$$

we obtain

$$\int_X \varphi_1\,\mathrm{d}\rho_{1,t} - \int_X \varphi_0\,\mathrm{d}\rho_{0,t} + t\Big(\mathrm{Ent}_{\mathfrak{m}}(\rho_{1,t}) - \mathrm{Ent}_{\mathfrak{m}}(\rho_{0,t})\Big) \le \frac12\mathrm{R}^2_K(t)\int_0^1 |\dot\rho_s|^2\,\mathrm{d}s.$$

Eventually we take the supremum with respect to φ, obtaining

$$\frac{1}{2}W_2^2(\rho_{1,t},\rho_0)+t\Big(\mathrm{Ent}_{\mathfrak m}(\rho_{1,t})-\mathrm{Ent}_{\mathfrak m}(\rho_{0,t})\Big)\le\frac{1}{2}\mathrm{R}_K^2(t)\int_0^1|\dot\rho_s|^2\,\mathrm{d}s.\qquad\square$$

Theorem 4.17 (BE(K,∞) is equivalent to RCD(K,∞))**.** *If $(X,\tau,\mathfrak m,\mathcal E)$ is a Riemannian Energy measure space satisfying* (**MD.exp**) *relative to $\mathsf d_{\mathcal E}$ and* BE(K,∞)*, then $(X,\mathsf d_{\mathcal E},\mathfrak m)$ is a* RCD(K,∞) *space.*
Conversely, if $(X,\mathsf d,\mathfrak m)$ is a RCD(K,∞) *space then, denoting by τ the topology induced by $\mathsf d$ and by $\mathcal E=2\mathsf{Ch}$ the Cheeger energy, $(X,\tau,\mathfrak m,\mathcal E)$ is a Riemannian Energy measure space satisfying $\mathsf d_{\mathcal E}=\mathsf d$,* (**MD.exp**)*, and* BE(K,∞)*.*

Proof. Let ρ, $\nu\in\mathscr P_2(X)$ with finite entropy. We have to show that (3.26) holds with $\mathsf H_t\rho$ precisely given by the dual semigroup. By the semigroup property, it is sufficient to prove (3.26) at $t=0$. For any $\rho\in\mathrm{AC}^2([0,1];(\mathscr P_2(X),W_2))$ joining $\rho_0:=\nu$ to $\rho_1:=\rho$ we find regular curves ρ^n as in Proposition 4.11 and apply the action estimate (4.41) to the curves $\rho^n_{s,t}=\mathsf H_{st}\rho^n_s$ to obtain

$$W_2^2(\mathsf H_t\rho^n_1,\rho^n_0)+2t\mathrm{Ent}_{\mathfrak m}(\mathsf H_t\rho^n_1)\le\mathrm{R}_K^2(t)\int_0^1|\dot\rho^n_s|^2\,\mathrm{d}s+2t\mathrm{Ent}_{\mathfrak m}(\rho^n_0).\tag{4.42}$$

We pass to the limit as $n\to\infty$ and use the lower semicontinuity of W_2 and of the entropy, to get

$$W_2^2(\mathsf H_t\rho,\nu)+2t\mathrm{Ent}_{\mathfrak m}(\mathsf P_t\rho)\le\mathrm{R}_K^2(t)\int_0^1|\dot\rho_s|^2\,\mathrm{d}s+2t\mathrm{Ent}_{\mathfrak m}(\nu).$$

We can now minimize w.r.t. ρ and use the fact that $(\mathscr P_2(X),W_2)$ is a length space because $(X,\mathsf d)$ is (this can be obtained starting from an optimal Kantorovich plan $\boldsymbol\pi$, choosing in a $\boldsymbol\pi$-measurable way a ε-optimal geodesic with constant speed as in the proof of Theorem 3.5), getting

$$W_2^2(\mathsf H_t\rho,\nu)+2t\mathrm{Ent}_{\mathfrak m}(\mathsf P_t\rho)\le\mathrm{R}_K^2(t)W_2^2(\rho,\nu)+2t\mathrm{Ent}_{\mathfrak m}(\nu).$$

After dividing by $t>0$, letting $t\downarrow0$ and using $\mathrm{R}_K(t)=1-\frac{K}{2}t+o(t)$ we obtain (3.26).

The converse implication, from RCD(K,∞) to BE(K,∞) has been proved in [5, Section 6]. $\square$

We conclude with an immediate application of the previous result to metric measure spaces: it follows by Theorem 3.17 and Corollary 3.18. Notice that for the Cheeger energy condition (**ED.b**) and upper-regularity are always true.

Corollary 4.18. *Let $(X,\mathsf d,\mathfrak m)$ be a metric measure space satisfying* (**MD+exp**) *with a quadratic Cheeger energy* Ch *defining the Dirichlet form $\mathcal E=2\mathsf{Ch}$ as in* (QCh)*. $(X,\mathsf d,\mathfrak m)$ is a* RCD(K,∞)*-space if (and only if) at least one of the following properties hold:*

(i) $(\mathsf P_t)_{t\ge0}$ *satisfies property* (3.61)*, i.e. for every function $f\in D(\mathsf{Ch})$ with $|\mathrm Df|_w\le1$ and every $t>0$,*

$$\mathsf P_tf\in\mathrm{Lip}_b(X),\quad|\mathrm D\mathsf P_tf|^2\le\mathrm e^{-2Kt}\,\mathsf P_t\big(|\mathrm Df|_w^2\big)\quad\mathfrak m\text{-}a.e.\ in\ X.\tag{4.43}$$

(ii) *Conditions* (**ED.a**), (w-Feller) *(or* $\mathcal{L} = \mathcal{L}_{\mathrm{C}}$*), and* $\mathrm{BE}(K, \infty)$ *hold.*

(iii) *Condition* (**ED.a**) *holds and* $(\mathsf{H}_t)_{t\geq 0}$ *satisfies the contraction property* (3.66) *(or* $(\mathsf{P}_t)_{t\geq 0}$ *satisfies the Lipschitz bound* (3.60)*).*

5 Applications of the equivalence result

In this section we present two applications of our equivalence result: in one direction we can use it to prove that the $\mathrm{RCD}(K, \infty)$ condition is stable under tensorization, a property proved in [5] only under a non branching assumption on the base spaces. We will also prove the same property for Riemannian Energy measure spaces satisfying the $\mathrm{BE}(K, N)$ condition, obtaining in particular the natural bound on the dimension of the product.

In the other direction, we shall prove a stability result for Riemannian Energy measure spaces satisfying a uniform $\mathrm{BE}(K, N)$ condition under Sturm-Gromov-Hausdorff convergence.

5.1 Tensorization

Let $(X, \mathsf{d}_X, \mathfrak{m}_X)$, $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ be $\mathrm{RCD}(K, \infty)$ metric measure spaces.

We may define a product space $(Z, \mathsf{d}, \mathfrak{m})$ by

$$Z := X \times Y, \qquad \mathsf{d}\big((x,y),(x',y')\big) := \sqrt{\mathsf{d}_X^2(x,x') + \mathsf{d}_Y^2(y,y')}, \qquad \mathfrak{m} := \mathfrak{m}_X \times \mathfrak{m}_Y. \tag{5.1}$$

Notice that also $\mathfrak{m}$ satisfies the quantitative σ-finiteness condition (**MD.exp**).

Denoting by $\mathcal{E}^X, \mathcal{E}^Y$ the Dirichlet forms associated to the respective (quadratic) Cheeger energies with domains $\mathbb{V}^X, \mathbb{V}^Y$, we consider the cartesian Dirichlet form

$$\mathcal{E}(f) := \int_Y \mathcal{E}^X(f^y)\,\mathrm{d}\mathfrak{m}_Y(y) + \int_X \mathcal{E}^X(f^x)\,\mathrm{d}\mathfrak{m}_X(x) \qquad f \in L^2(Z, \mathfrak{m}), \tag{5.2}$$

where for every $f \in L^2(Z, \mathfrak{m})$ and $z = (x,y) \in Z$ we set $f^x = f(x, \cdot)$, $f^y(\cdot) = f(\cdot, y)$. By [5, Thm. 6.18] the proper domain $\mathbb{V}$ of $\mathcal{E}$ in $L^2(Z, \mathfrak{m})$ is the Hilbert space

$$\begin{aligned}\mathbb{V} := \Big\{ f \in L^2(Z, \mathfrak{m}) : & f^x \in \mathbb{V}^Y \text{ for } \mathfrak{m}_X\text{-a.e. } x \in X,\ f^y \in \mathbb{V}^X \text{ for } \mathfrak{m}_Y\text{-a.e. } y \in Y \\ & |\mathrm{D}f^x|_w(y),\ |\mathrm{D}f^y|_w(x) \in L^2(Z, \mathfrak{m}) \Big\},\end{aligned} \tag{5.3}$$

$\frac{1}{2}\mathcal{E}$ coincides with the Cheeger energy Ch in $(Z, \mathsf{d}, \mathfrak{m})$, and

$$|\mathrm{D}f|_w^2(x,y) = \Gamma(f)(x,y) = |\mathrm{D}f^x|_w^2(y) + |\mathrm{D}f^y|_w(x)^2 \quad \mathfrak{m}\text{-a.e. in } Z. \tag{5.4}$$

Even though the result in [5] is stated for finite metric measure spaces, the proof extends with no difficulty to the σ-finite case. Also, it is worthwhile to mention that the curvature assumption on the base spaces plays almost no role in the proof, it only used to build, via the product semigroup, an operator with good regularization properties (specifically from L^∞ to C_b), see [5, Lemma 6.13]

It will be convenient, as in [5] and in the previous sections, to work with a pointwise defined version of the semigroups in the base spaces, namely

$$\mathsf{P}^X_t u(x) := \int u(x')\,\mathrm{d}\mathsf{H}^X_t\delta_x(x'), \qquad \mathsf{P}^Y_t v(y) := \int v(y')\,\mathrm{d}\mathsf{H}^Y_t\delta_y(y')$$

for $u : X \to \mathbb{R}$ and $v : Y \to \mathbb{R}$ bounded Borel, where $(\mathsf{H}^X_t)_{t\ge 0}$ and $(\mathsf{H}^Y_t)_{t\ge 0}$ denote the Wasserstein semigroups on the base spaces. These pointwise defined semigroups also provide the continuous versions of (S-Feller) for the base spaces, see [5, Theorem 6.1(iii)].

Since the heat flows are linear, the tensorization (5.4) implies a corresponding tensorization of the heat flows, namely for all $g : Z \to \mathbb{R}$ bounded and Borel, $\mathfrak{m}$-a.e. in Z the following identities hold:

$$\mathsf{P}_t g(z) = \int_X \mathsf{P}^Y_t g(x',\cdot)(y)\,\mathrm{d}\mathsf{H}^X_t\delta_x(x'), \qquad \mathsf{P}_t g(z) = \int_Y \mathsf{P}^X_t g(\cdot,y')(x)\,\mathrm{d}\mathsf{H}^Y_t\delta_y(y'). \tag{5.5}$$

With these ingredients at hand, we can now prove the main tensorization properties.

Theorem 5.1. *With the above notation, if $(X,\mathsf{d}_X,\mathfrak{m}_X)$ and $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ are* $\mathrm{RCD}(K,\infty)$ *spaces then the space $(Z,\mathsf{d},\mathfrak{m})$ is* $\mathrm{RCD}(K,\infty)$ *as well.*

Proof. According to the characterization of $\mathrm{RCD}(K,\infty)$ given in point (i) of Corollary 4.18, since the Cheeger energy in Z satisfies (QCh) by the above mentioned result of [5], it suffices to show that the length space property and (4.43) are stable under tensorization.
Stability of the length space property. This is simple to check, one obtains an almost minimizing geodesic $\gamma : [0,1] \to Z$ combining almost minimizing geodesics on the base spaces with constant speed and parameterized on $[0,1]$.
Stability of (4.43). Let us first notice that P_t maps bounded and Borel functions into continuous ones, thanks of any of the two identities in (5.5) and (S-Feller).

Let $f \in \mathrm{Lip}_b(Z) \cap L^2(Z,\mathfrak{m})$. Keeping y initially fixed, the second identity in (5.5) tells us that $x \mapsto \mathsf{P}_t f(x,y) = (\mathsf{P}_t f)^y(x)$ is the mean w.r.t. y', weighted with $\mathsf{H}^Y_t\delta_y$, of the functions $\mathsf{P}^X_t f^{y'}(x)$. Hence, the convexity of the slope gives

$$|\mathrm{D}(\mathsf{P}_t f)^y|(x) \le \int_Y |\mathrm{D}\mathsf{P}^X_t f^{y'}|(x)\,\mathrm{d}\mathsf{H}^Y_t\delta_y(y'),$$

where gradients are understood with respect to the first variable. We can thus use the Hölder inequality to get

$$|\mathrm{D}(\mathsf{P}_t f)^y|^2(x) \le \int_Y |\mathrm{D}\mathsf{P}^X_t f^{y'}|^2(x)\,\mathrm{d}\mathsf{H}^Y_t\delta_y(y'). \tag{5.6}$$

Now, for $\mathfrak{m}_Y$-a.e. $y' \in Y$ we apply (4.43) in the space X to the functions $f^{y'}$ and use Fubini's theorem to get

$$|\mathrm{D}\mathsf{P}^X_t f^{y'}|^2(x) \le \mathrm{e}^{-2Kt} P^X_t \left|\mathrm{D}f^{y'}\right|^2_w(x) \qquad \text{for } \mathfrak{m}_Y\text{-a.e. } y' \in Y. \tag{5.7}$$

Combining (5.6) and (5.7) and using once more (5.5) with $g(x,y) = |\mathrm{D}f^y|_w^2(x)$ we get

$$|\mathrm{D}(\mathsf{P}_t f)^y|^2(x) \le \mathrm{e}^{-2Kt} \int_Y \mathsf{P}_t^X \left|\mathrm{D}f^{y'}\right|_w^2 (x)\,\mathrm{d}\mathsf{H}_t\delta_y(y') = \mathrm{e}^{-2Kt}\mathsf{P}_t\,|\mathrm{D}f^y|_w^2(x).$$

Repeating a similar argument with the first identity in (5.5) and adding the two inequalities we obtain

$$|\mathrm{D}(\mathsf{P}_t f)^y|^2(x) + |\mathrm{D}(\mathsf{P}_t f)^x|^2(y) \le \mathrm{e}^{-2Kt}\mathsf{P}_t\,|\mathrm{D}f|_w^2(x,y).$$

We conclude that (4.43) holds using the calculus lemma [5, Lemma 6.2], which provides the information that the square root of $|\mathrm{D}(\mathsf{P}_t f)^y|^2(x) + |\mathrm{D}(\mathsf{P}_t f)^x|^2(y)$ is an upper gradient of $\mathsf{P}_t f$. It follows that $\mathrm{e}^{-Kt}\sqrt{\mathsf{P}_t\,|\mathrm{D}f|_w^2}$ is an upper gradient as well; being continuous, it provides a pointwise upper bound for the slope. □

Let us now consider the corresponding version of the tensorization theorem for Riemannian Energy measure spaces.

Theorem 5.2. *Let* $(X,\tau_X,\mathcal{E}^X,\mathfrak{m}_X)$, $(Y,\tau_Y,\mathcal{E}^Y,\mathfrak{m}_Y)$ *be Riemannian Energy measure spaces satisfying the Bakry-Émery conditions* $\mathrm{BE}(K,N_X)$ *and* $\mathrm{BE}(K,N_Y)$ *respectively, and let us consider the cartesian Dirichlet form* $\mathcal{E}$ *defined by* (5.2) *on* $Z = X\times Y$ *endowed with the product topology* $\tau = \tau_X\otimes\tau_Y$ *and the product measure* $\mathfrak{m}$ *as in* (5.1).

Then $(Z,\tau,\mathfrak{m},\mathcal{E})$ *is a Riemannian Energy measure space, it satisfies the Bakry-Émery condition* $\mathrm{BE}(K,N_X+N_Y)$ *and the induced distance* $\mathsf{d}_\mathcal{E}$ *on* Z *coincides with the product distance defined in* (5.1).

Proof. It is not restrictive to assume that $N_X, N_Y < \infty$; by the previous Theorem we already know that $(Z,\tau,\mathfrak{m},\mathcal{E})$ is a Riemannian energy measure space satisfying $\mathrm{BE}(K,\infty)$ whose induced distance $\mathsf{d}_\mathcal{E}$ is given by (5.1); we want to prove that (2.58) holds with $\nu_Z := \nu_X\nu_Y/(\nu_X+\nu_Y)$ where $\nu_X := N_X^{-1}$ and $\nu_Y := N_Y^{-1}$. We argue as in (5.7), observing that for $\mathfrak{m}_Y$-a.e. $y'\in Y$ (2.58) and (3.62) yield

$$|\mathrm{D}\mathsf{P}_t^X f^{y'}|^2 + 2\nu_X \mathrm{I}_{2K,2}(t)\big(\Delta_X\mathsf{P}_t^X f^{y'}\big)^2 \le \mathrm{e}^{-2Kt}\mathsf{P}_t^X|\mathrm{D}f^{y'}|_w^2. \tag{5.8}$$

Integrating with respect to the measure $\mathsf{H}_t^Y\delta_y$ in y' and recalling (5.5) and (5.6), we get

$$|\mathrm{D}(\mathsf{P}_t^X f)^y|^2(x) + 2\nu_X\mathrm{I}_{2K,2}(t)(\Delta_X(\mathsf{P}_t f)^y)^2(x) \le \mathrm{e}^{-2Kt}\mathsf{P}_t\big(|\mathrm{D}f^y|_w^2\big)(x) \quad \mathfrak{m}\text{-a.e. in } Z, \tag{5.9}$$

where we also used the Hölder inequality

$$\int_Y \big(\Delta_X\mathsf{P}_t^X f^{y'}\big)^2(x)\,\mathrm{d}\mathsf{H}_t\delta_y(y') \ge \Big(\int_Y \Delta_X\mathsf{P}_t^X f^{y'}(x)\,\mathrm{d}\mathsf{H}_t\delta_y(y')\Big)^2 = (\Delta_X\mathsf{P}_t f^y)^2(x).$$

By repeating a similar argument inverting the role of X and Y we get

$$|\mathrm{D}(\mathsf{P}_t^Y f)^x|_w^2(y) + 2\nu_Y\mathrm{I}_{2K,2}(t)(\Delta_Y(\mathsf{P}_t f)^x)^2(y) \le \mathrm{e}^{-2Kt}\mathsf{P}_t\big(|\mathrm{D}f^x|_w^2\big)(y) \quad \mathfrak{m}\text{-a.e. in } Z. \tag{5.10}$$

Summing up (5.9) and (5.10), and recalling the elementary inequality

$$\nu_X a^2 + \nu_Y b^2 \ge \frac{\nu_X\nu_Y}{\nu_X+\nu_Y}(a+b)^2 \quad \text{for every } a,b\ge 0,$$

we conclude thanks to the next simple Lemma. □

Lemma 5.3. *If $f \in \mathbb{V}$ satisfies*

$$f^y \in D(\Delta_X), \quad f^x \in D(\Delta_Y) \quad \text{for } \mathfrak{m}\text{-a.e. } (x,y) \in Z, \quad \Delta_X f^y,\ \Delta_Y f^x \in L^2(Z,\mathfrak{m}). \tag{5.11}$$

then $f \in D(\Delta_Z)$ and $\Delta_Z f(x,y) = \Delta_X f^y(x) + \Delta_Y f^x(y)$ $\mathfrak{m}$-a.e. in Z.

Proof. If (5.11) holds, Fubini's theorem and the very definition of Δ_X, Δ_Y yield for every $\varphi \in \mathbb{V}$

$$\begin{aligned}
\mathcal{E}(f,\varphi) &= \int_Y \mathcal{E}^X(f^y,\varphi^y)\,\mathrm{d}\mathfrak{m}_Y(y) + \int_X \mathcal{E}^Y(f^x,\varphi^x)\,\mathrm{d}\mathfrak{m}_X(x) \\
&= -\int_Y \Big(\int_X \Delta_X f^y \varphi^y\,\mathrm{d}\mathfrak{m}_X\Big)\,\mathrm{d}\mathfrak{m}_Y - \int_X \Big(\int_Y \Delta_Y f^x \varphi^x\,\mathrm{d}\mathfrak{m}_Y\Big)\,\mathrm{d}\mathfrak{m}_X \\
&= -\int_Z \Big(\Delta_X f^y + \Delta_Y f^x\Big)\,\varphi\,\mathrm{d}\mathfrak{m}. \qquad \square
\end{aligned}$$

5.2 Stability

We refer to [39, § 3.1], [5, §2.3] for the definition of Sturm-Gromov-Hausdorff convergence of metric measure spaces. For the sake of simplicity, we restrict here to the case when $\mathfrak{m}_n \in \mathscr{P}_2(X_n)$. The general case of σ-finite measures satisfying (**MD.exp**) could be attacked by the techniques developed in [22], assuming that (**MD.exp**) holds uniformly along the sequence.

Theorem 5.4. *Let $(X_n, \tau_n, \mathcal{E}_n, \mathfrak{m}_n)$ be Riemannian energy measure spaces satisfying* $\mathrm{BE}(K,N)$ *with $\mathfrak{m}_n \in \mathscr{P}_2(X_n)$ and let us suppose that, denoting by d_n the corresponding distances $\mathsf{d}_{\mathcal{E}_n}$, $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ converge to $(X, \mathsf{d}, \mathfrak{m})$ in the Sturm-Gromov-Hausdorff sense.*

If $\frac{1}{2}\mathcal{E}$ is the Cheeger energy in the limit space and τ the limit topology, then $(X, \tau, \mathfrak{m}, \mathcal{E})$ is a Riemannian Energy measure space satisfying $\mathrm{BE}(K,N)$.

The *proof* is based on a general criterium of convergence, strongly related to the theory of Young measures, for sequences of functions defined in L^2-spaces associated to different measures (see e.g. [2, §5.4]). We first make precise this notion of convergence and a few simple properties.

Let us recall that by [5, Prop. 2.6] it is not restrictive to assume that

$$X_n = X,\ \mathsf{d}_n = \mathsf{d} \text{ and } \mathfrak{m}_n \text{ are converging to } \mathfrak{m} \text{ in } \mathscr{P}_2(X). \tag{5.12}$$

(one has just to take care that in general $\mathfrak{m}_n, \mathfrak{m}$ could be not fully supported). We denote by $(\mathsf{P}^n_t)_{t\geq 0}$ the Markov semigroups in $L^2(X, \mathfrak{m}_n)$ with generators $\Delta_n := \Delta_{\mathcal{E}^n}$.

Consider a sequence of vector valued functions $\boldsymbol{f}_n \in L^2(X, \mathfrak{m}_n; \mathbb{R}^k)$, $k \in \mathbb{N}$, and a candidate limit $\boldsymbol{f} \in L^2(X, \mathfrak{m}; \mathbb{R}^k)$. We say that $\boldsymbol{f}_n$ converges to $\boldsymbol{f}$ as $n \to \infty$ if

$$(\boldsymbol{i} \times \boldsymbol{f}_n)_\sharp \mathfrak{m}_n \to (\boldsymbol{i} \times \boldsymbol{f})_\sharp \mathfrak{m} \quad \text{in } \mathscr{P}_2(X \times \mathbb{R}^k). \tag{5.13}$$

We will use three properties:

(i) (5.13) is equivalent to the convergence of each component f_n^j, $j = 1, \dots, k$, to f^j (e.g. by applying [2, Lemma 5.3.2]).

(ii) In the scalar case $k = 1$, if f_n satisfy

$$\begin{aligned} &\lim_{n\to\infty} \int_X f_n \varphi \, \mathrm{d}\mathfrak{m}_n = \int_X f \varphi \, \mathrm{d}\mathfrak{m} \quad \text{for every } \varphi \in \mathrm{C}_b(X), \\ &\lim_{n\to\infty} \int_X f_n^2 \, \mathrm{d}\mathfrak{m}_n = \int_X f^2 \, \mathrm{d}\mathfrak{m}, \end{aligned} \tag{5.14}$$

then f_n converges to f according to (5.13). The proof follows the same arguments of [2, Thm. 5.4.4] (the fact that the base space X is a general metric space instead of an Hilbert space is not relevant here). A similar result holds if $f_n \in L^1_+(X, \mathfrak{m}_n)$ are uniformly bounded probability densities satisfying

$$f_n \mathfrak{m}_n \rightharpoonup f\mathfrak{m} \quad \text{in } \mathscr{P}(X), \quad \lim_{n\to\infty} \int_X f_n \log f_n \, \mathrm{d}\mathfrak{m}_n = \int_X f \log f \, \mathrm{d}\mathfrak{m}. \tag{5.15}$$

(iii) Finally, if $\boldsymbol{r} : \mathbb{R}^k \to \mathbb{R}^h$ is a continuous map with linear growth, and $\boldsymbol{f}_n$ converge to $\boldsymbol{f}$ according to (5.13) then $\boldsymbol{r} \circ \boldsymbol{f}_n$ converge to $\boldsymbol{r} \circ \boldsymbol{f}$.

Property (i) follows by the fact that a probability measure in $\mathbb{R}^h$ is a Dirac mass if and only if its coordinate projections are Dirac masses, while property (ii) follows by the fart that, for strictly convex functions, equality holds in Jensen's inequality only when the measure is a Dirac mass. The proof of (iii) is straightforward.

Lemma 5.5. *Under the same assumptions of Theorem 5.4 and* (5.12)*, let us assume that* $f_n \in L^\infty(X, \mathfrak{m}_n)$ *converge to* $f \in L^\infty(X, \mathfrak{m})$ *according to* (5.13)*, with uniformly bounded* L^∞ *norm. Then* $\mathsf{P}^n_t f_n$ *converge to* $\mathsf{P}_t f$ *for every* $t \geq 0$ *and* $\Delta_n \mathsf{P}^n_t f_n$ *converge to* $\Delta_{\mathcal{E}} \mathsf{P}_t f$ *as* $n \to \infty$ *for every* $t > 0$.

Proof. When f_n are probability densities the statement follows by applying (ii) and the convergence results of [5, Theorem 6.11] (which shows that $\mathsf{P}^n_t f_n \mathfrak{m}_n$ converges to $\mathsf{P}_t f \mathfrak{m}$ in $\mathscr{P}_2(X)$) and [22] (which yields the convergence of the entropies $\mathrm{Ent}_{\mathfrak{m}_n}(f_n \mathfrak{m}_n) \to \mathrm{Ent}_{\mathfrak{m}}(f\mathfrak{m})$).

The case $f_n \in L^1_+(X, \mathfrak{m})$ can be easily reduced to the previous one by a rescaling, since $\int_X f_n \, \mathrm{d}\mathfrak{m}_n \to \int_X f \, \mathrm{d}\mathfrak{m}$ by (5.13) and $(\mathsf{P}^n_t)_{t\geq 0}$ is mass preserving.

The general case can be proved by decomposing each f_n into the difference $f_n^+ - f_n^-$ of its positive and negative part, observing that $f_n^\pm$ converge to $f^\pm$ thanks to (iii). Thus, by (i) it follows that $(\mathsf{P}^n_t f_n^+, \mathsf{P}^n_t f_n^-)$ converge to $(\mathsf{P}_t f^+, \mathsf{P}_t f^-)$ and a further application of (iii) yields the convergence result by the linearity of the semigroups.

In order to prove the convergence of $\Delta_n \mathsf{P}^n_t f_n$ we still apply (ii): recall that P^n_t are analytic semigroups in $L^2(X, \mathfrak{m}_n)$, $\Delta_n \mathsf{P}^n_t f_n = \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{P}^n_t f_n$, and the uniform estimates (see e.g. [35, Page 75, step 2])

$$t^j \left\| \frac{\mathrm{d}^j}{\mathrm{d}t^j} \mathsf{P}^n_t f_n \right\|_{L^2(X,\mathfrak{m}_n)} \leq A_j \|f_n\|_{L^2(X,\mathfrak{m}_n)} \quad \text{for every } t > 0, \ n \in \mathbb{N} \tag{5.16}$$

hold with universal constants A_j for every $t > 0$. Since we just proved that for every $\varphi \in C_b(X)$ the sequence of functions $\zeta_n(t) := \int_X \mathsf{P}^n_t f_n \, \varphi \, \mathrm{d}\mathfrak{m}_n$ converge pointwise to the corresponding ζ as $n \to \infty$, (5.16) yields that

$$\lim_{n\to\infty} \zeta_n'(t) = \lim_{n\to\infty} \int_X \Delta_n \mathsf{P}^n_t f_n \, \varphi \, \mathrm{d}\mathfrak{m}_n = \zeta'(t) = \int_X \Delta_{\mathcal{E}} \mathsf{P}_t f \, \varphi \, \mathrm{d}\mathfrak{m} \quad \text{for every } t > 0.$$

The same argument holds for

$$t \mapsto \int_X (\Delta_n \mathsf{P}^n_t f_n)^2 \, \mathrm{d}\mathfrak{m}_n = \frac{1}{4} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_X (\mathsf{P}^n_t f_n)^2 \, \mathrm{d}\mathfrak{m}_n. \qquad \square$$

Proof of Theorem 5.4. The case $N = \infty$ follows by the identification Theorem 4.17 and [5, Thm. 6.10]. In particular, the limit space endowed with the Cheeger energy and the limit topology is a Riemannian Energy measure space.

We can thus consider the case $N < \infty$. By [5, Lemma 6.12] and the previous point (ii), for every $f, \varphi \in L^\infty(X, \mathfrak{m})$, φ nonnegative, we can find sequences $f_n, \varphi_n \in L^\infty(X, \mathfrak{m}_n)$, φ_n nonnegative, converging to f, φ according to (5.13). We can also suppose that f_n, φ_n are uniformly bounded by some constant $C > 0$.

Applying the previous Lemma 5.5 we get that $\mathsf{P}^n_{t-s} f_n$ converge to $\mathsf{P}_{t-s} f$ and $\mathsf{P}^n_s \varphi_n$ converge to $\mathsf{P}_s \varphi$ as $n \to \infty$ for every $t \geq 0$ and $s \in [0, t]$. Applying (i) to the function $\boldsymbol{f}_n := (\mathsf{P}^n_{t-s} f_n, \mathsf{P}^n_s \varphi_n)$ and choosing the test function $\psi(x, r_1, r_2) = r_1^2 r_2 \mathrm{S}_C(r_2)$, $(x, r_1, r_2) \in X \times \mathbb{R}^2$ (with S defined in (3.42)), we obtain

$$\lim_{n\to\infty} \int_X (\mathsf{P}^n_{t-s} f_n)^2 \mathsf{P}^n_s \varphi_n \, \mathrm{d}\mathfrak{m}_n = \lim_{n\to\infty} \int \psi \, \mathrm{d}(\boldsymbol{i} \times \boldsymbol{f}_n)_\sharp \mathfrak{m}_n = \int \psi \, \mathrm{d}(\boldsymbol{i} \times \boldsymbol{f})_\sharp \mathfrak{m} = \int_X (\mathsf{P}_{t-s} f)^2 \mathsf{P}_s \varphi \, \mathrm{d}\mathfrak{m}.$$

A similar argument yields

$$\lim_{n\to\infty} \int_X (\Delta^n_{\mathcal{E}} P^n_{t-s} f_n)^2 P^n_s \varphi_n \, \mathrm{d}\mathfrak{m}_n = \int_X (\Delta_{\mathcal{E}} \mathsf{P}_{t-s} f)^2 \mathsf{P}_s \varphi \, \mathrm{d}\mathfrak{m} \quad \text{for every } t > 0, \ s \in [0, t).$$

We can then pass to the limit in the distributional characterization (2.55) of $\mathrm{BE}(K, N)$. The case of general $f \in L^2(X, \mathfrak{m})$ can then be recovered by standard approximation. $\square$

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure*, Preprint, (2012).

[2] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, second ed., 2008.

[3] ——, *Calculus and heat flow in metric measure spaces and applications to spaces with ricci bounds from below*, Arxiv 1106.2090, (2011), pp. 1–74.

[4] ——, *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces*, Arxiv 1111.3730, (2011), pp. 1–28.

[5] ——, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Arxiv 1109.0222, (2011), pp. 1–60.

[6] L. Ambrosio, G. Savaré, and L. Zambotti, *Existence and stability for Fokker-Planck equations with log-concave reference measure.*, Probab. Theory Relat. Fields, 145 (2009), pp. 517–564.

[7] D. Bakry, *Functional inequalities for Markov semigroups*, in Probability measures on groups: recent directions and trends, Tata Inst. Fund. Res., Mumbai, 2006, pp. 91–147.

[8] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de probabilités, XIX, 1983/84, vol. 1123, Springer, Berlin, 1985, pp. 177–206.

[9] D. Bakry and M. Ledoux, *A logarithmic Sobolev form of the Li-Yau parabolic inequality*, Rev. Mat. Iberoamericana, 22 (2006), p. 683.

[10] M. Biroli and U. Mosco, *A Saint-Venant principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl., 169 (1995), pp. 125–181.

[11] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007.

[12] N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener sapces*, vol. 14 of De Gruyter studies in Mathematics, De Gruyter, 1991.

[13] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, vol. 33 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.

[14] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9 (1999), pp. 428–517.

[15] S. Daneri and G. Savaré, *Eulerian calculus for the displacement convexity in the Wasserstein distance*, SIAM J. Math. Anal., 40 (2008), pp. 1104–1122.

[16] M. Erbar, *The heat equation on manifolds as a gradient flow in the Wasserstein space*, Annales de l'Institut Henri Poincaré - Probabilités et Statistiques, 46 (2010), pp. 1–23.

[17] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Inventiones mathematicae, 87 (1987), pp. 517–547.

[18] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, vol. 19 of de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, extended ed., 2011.

[19] N. Gigli, *On the heat flow on metric measure spaces: existence, uniqueness and stability*, Calc. Var. Partial Differential Equations, 39 (2010), pp. 101–120.

[20] N. Gigli, *On the differential structure of metric measure spaces and applications*, (2012).

[21] N. Gigli, K. Kuwada, and S. Ohta, *Heat flow on Alexandrov spaces*, (2010).

[22] N. Gigli, A. Mondino, and G. Savaré, *On the notion of convergence of non-compact metric measure spaces and applications*, Preprint, (2012).

[23] R. Jordan, D. Kinderlehrer, and F. Otto, *The variational formulation of the Fokker-Planck equation*, SIAM J. Math. Anal., 29 (1998), pp. 1–17 (electronic).

[24] P. Koskela, N. Shanmugalingam, and Y. Zhou, *Geometry and analysis of Dirichlet forms (ii)*, Preprint, (2012).

[25] K. Kuwada, *Duality on gradient estimates and Wasserstein controls*, Journal of Functional Analysis, 258 (2010), pp. 3758–3774.

[26] S. Lisini, *Characterization of absolutely continuous curves in Wasserstein spaces*, Calc. Var. Partial Differential Equations, 28 (2007), pp. 85–120.

[27] J. Lott and C. Villani, *Weak curvature bounds and functional inequalities*, J. Funct. Anal., 245 (2007), pp. 311–333.

[28] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2), 169 (2009), pp. 903–991.

[29] Z.-M. Ma and M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms*, Springer, New York, 1992.

[30] R. J. McCann, *A convexity principle for interacting gases*, Adv. Math., 128 (1997), pp. 153–179.

[31] S.-I. Ohta and K.-T. Sturm, *Heat flow on Finsler manifolds*, Comm. Pure Appl. Math., 62 (2009), pp. 1386–1433.

[32] F. Otto and M. Westdickenberg, *Eulerian calculus for the contraction in the Wasserstein distance*, SIAM J. Math. Anal., 37 (2005), pp. 1227–1255 (electronic).

[33] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[34] T. Rajala, *Local Poincaré inequalities from stable curvature conditions on metric spaces*, arXiv.org, math.DG (2011).

[35] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory.*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970.

[36] P. Stollmann, *A dual characterization of length spaces with application to Dirichlet metric spaces*, Polska Akademia Nauk. Instytut Matematyczny. Studia Mathematica, 198 (2010), pp. 221–233.

[37] K.-T. Sturm, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka Journal of Mathematics, 32 (1995), pp. 275–312.

[38] ——, *The geometric aspect of Dirichlet forms*, in New directions in Dirichlet forms, vol. 8 of AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 1998, pp. 233–277.

[39] ——, *On the geometry of metric measure spaces. I*, Acta Math., 196 (2006), pp. 65–131.

[40] ——, *On the geometry of metric measure spaces. II*, Acta Math., 196 (2006), pp. 133–177.

[41] K.-T. Sturm and M.-K. von Renesse, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math., 58 (2005), pp. 923–940.

[42] C. Villani, *Optimal transport. Old and new*, vol. 338 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2009.

[43] F.-Y. Wang, *Equivalent semigroup properties for the curvature-dimension condition*, Bull. Sci. Math., 135 (2011), pp. 803–815.