Existence and regularity of minimizers of a functional for unsupervised multiphase segmentation

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Abstract

We consider a variational model for image segmentation proposed in [20]. In such a model the image domain is partitioned into a finite collection of subsets denoted phases. The segmentation is unsupervised, i.e., the model finds automatically an optimal number of phases, which are not required to be connected subsets. Unsupervised segmentation is obtained by minimizing a functional of the Mumford-Shah type [19], but modifying the geometric part of the Mumford-Shah energy with the introduction of a suitable scale term. The results of computer experiments discussed in [20] show that the resulting variational model has several properties which are relevant for applications. In the present paper we investigate the theoretical properties of the model. We study the existence of minimizers of the corresponding functional, first looking for a weak solution in a class of phases constituted by sets of finite perimeter. Then we find various regularity properties of such minimizers, particularly we study the structure of triple junctions by determining their optimal angles.

Keywords: Computer vision, image segmentation, calculus of variations

1. Introduction

The segmentation problem in image analysis consists in looking for a decomposition of an image into homogeneous regions corresponding to meaningful parts of objects. In recent years a number of variational models have been proposed for the segmentation problem. Mumford and Shah [19] proposed to minimize the functional

$$\mathcal{E}_{ms}(u,\Gamma) = \alpha\mathcal{H}^1(\Gamma) + \beta\int_{\Omega\setminus\Gamma} |\nabla u|^2 dx + \int_\Omega |u-u_o|^2 dx, \tag{1}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz image domain, $u_o : \Omega \longrightarrow \mathbb{R}_+ \cup \{0\}$ is a bounded function representing the given image, $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure, and α, β are positive weights. The functional has to be minimized over all closed sets $\Gamma \subset \overline{\Omega}$ and all $u \in \mathcal{C}^1(\Omega \setminus \Gamma)$. The function u represents a piecewise smooth (i.e., denoised) approximation of the input image u_o, and the set Γ represents the union of boundaries of the regions constituting the segmentation. If Γ is regular enough, then the measure $\mathcal{H}^1(\Gamma)$ is simply the total length of the boundaries.

In [4, 5] Chan and Vese introduced a multiphase variational model for image segmentation based on the Mumford-Shah functional and the level set method. They considered both a piecewise constant and a piecewise smooth approximation of the image u_o. In the piecewise constant case the variational model looks for a local minimizer of the functional

$$\mathcal{E}_{cv}(u, \Gamma) = \alpha \mathcal{H}^1(\Gamma) + \sum_{i=1}^{K} \int_{\chi_i} |u - u_o|^2 dx, \tag{2}$$

where K is a given integer, χ_i, $i = 1, \ldots, K$, are open subsets that constitute a Borel partition of the image domain Ω, here Γ is the union of the part of the boundaries of the χ_i inside Ω, so that

$$\Gamma = \bigcup_{i=1}^{K} \partial\chi_i \cap \Omega, \qquad \Omega = \Gamma \bigcup_{i=1}^{K} \chi_i, \tag{3}$$

and the function u is constant on every subset χ_i. It is easy to see that, for a fixed Γ, the functional $\mathcal{E}_{cv}$ is minimized with respect to the function u by setting, for each χ_i, u equal to the mean value of u_o in χ_i. The subsets χ_i are denoted phases and are not required to be connected. The functional $\mathcal{E}_{cv}$ is a piecewise constant version of the Mumford-Shah functional, since $\nabla u(x) = 0$ in $\Omega \setminus \Gamma$. With a level-set formulation and implementation, the model is frequently referred to as the Chan-Vese model for either two-phase ($K = 2$), or multiphase ($K > 2$), segmentation.

Extending this idea, there is a number of region-based multiphase segmentation models introduced, such as [1, 2, 6, 12, 14, 15, 22, 25] for $K \geq 2$. However, except for the case of two-phase segmentation, the multiphase case can have some sensitivity issues. Typically the number of phases $K > 2$ is pre-determined and result can depend on the initial guess used in the local minimization of the functional.

The model proposed in [20] addresses these issues, that the model automatically chooses a reasonable number of phases K, as it segments the image via the minimum of the following functional:

$$\mathcal{E}(K, \chi_1, \ldots, \chi_K) = \mu \left(\sum_{i=1}^{K} \frac{P(\chi_i)}{|\chi_i|} \right) \mathcal{H}^1(\Gamma) + \sum_{i=1}^{K} \int_{\chi_i} |u_o - c_i|^2 dx. \tag{4}$$

Here $P(\chi_i)$ denotes the perimeter of a phase χ_i, $|\chi_i|$ denotes the 2-dimensional area of a phase χ_i, Γ is defined by (3), for any $i = 1, \ldots, K$, c_i is the mean value

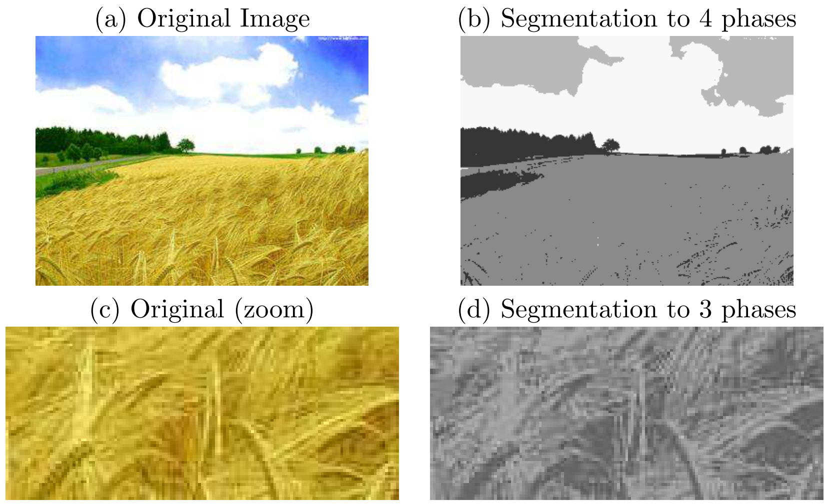

Figure 1: Figures from [20]. The original image (a) is automatically segmented to four phases in image (b) - each gray color represents different phases of the segmentation result. When zoomed into the field area and (c) is given as an original image, image (c) is automatically further segmented to three phases in image (d).

of u_o in χ_i, and μ is a positive parameter. Notice that both K and the χ_is are all unknown variables.

Compared to the piecewise constant Mumford-Shah model, one difference is the newly added weight $\sum_i \frac{P(\chi_i)}{|\chi_i|}$ in front of the length term. The ratio $\frac{P(\chi_i)}{|\chi_i|}$, called scale term, is related to Cheeger sets, which are widely studied in the Calculus of Variations [3, 9]. The Cheeger problem consists of finding a single subset of Ω minimizing the ratio perimeter/area, while in our problem this new weight is the summation of the scale terms for multiple phases. Such a weight gives an effective property of the model, allowing an unsupervised multiphase segmentation, as it has been discussed in [20]. Here, unsupervised segmentation means the automatic selection of the optimal number of phases K by functional minimization. The number of phases recovered in an optimal segmentation can be controlled by means of the parameter μ: large values of μ favor fewer phases with larger areas, while small values of μ prefer more phases with smaller area (see [20] for further details). Figure 1 shows an example from [20], where K is automatically selected depending on the focus of the image. In this example the value $\mu = 1$ has been used. This model has many properties in addition to being an automatic segmentation, such as giving balance among different phases (phases which are either extremely small or extremely large are disfavored), and good detail recovery [20]. This method can be additionally applied to image quantization [20], object identification for 3-dimensional Flash

Lidar Images [8], and scale segmentation and an extension to a regularized K-means [13].

In this paper, we study the analytic properties of the functional (4), such as the existence and regularity of minimizers. First, we prove the existence of a weak minimizer of the functional (4) in the class of sets with finite perimeter. Then we investigate some geometrical properties of optimal segmentations. It is known that sets Γ which minimize the Mumford-Shah functional can possess only very restricted types of singularities. Particularly, corners are not allowed in Γ. Moreover, if Γ is composed of regular arcs, then at most three arcs can meet at a single point, and they meet at such a point with 120^o angles. Such a point is called a triple junction. We find that, similarly to the Mumford-Shah case, corners are not allowed in sets Γ minimizing the functional (4). We also compute the optimal angles of triple junctions and we find different properties with respect to the Mumford-Shah functional. Moreover multiple junctions, where more than three arcs meet, are also allowed. Then we study the general regularity properties of minimizers of functional (4) by adapting to the present problem techniques developed for the Mumford-Shah functional by Tamanini, Congedo and Massari in [7, 16, 17, 19, 24]. We prove that the sets $\{\chi_1, \dots, \chi_K\}$ forming an optimal segmentation are open, and that the set Γ is constituted by smooth curves, except possibly a singular set of points which is locally finite.

In Section 2, mathematical notations are given, and the existence of a weak minimizer is proved in Section 3. In Section 4, we study the possible optimal angles of triple junctions and we motivate the regularity analysis in the following sections. In Section 5, we prove an elimination lemma which permits us to prove that the sets forming an optimal partition are open. Such a lemma will be crucial in order to prove further regularity properties. Then we show that the blow-up of an optimal set Γ only allows straight lines in Section 6. By using the result of the blow-up, together with the elimination lemma, regularity properties of a minimizer are proved in Section 7, followed by concluding remarks in Section 8.

2. Mathematical preliminaries and statement of the main result

For a given set $A \subset \mathbb{R}^2$ we denote by ∂A its topological boundary, by $|A|$ its two-dimensional Lebesgue measure and by $\mathcal{H}^1(A)$ its one-dimensional Hausdorff measure. We denote by $B_\rho(x)$ the open ball $\{y \in \mathbb{R}^2 : |y - x| < \rho\}$ with center $x \in \mathbb{R}^2$ and radius $\rho > 0$. When $x = 0$ we simply write B_ρ instead of $B_\rho(x)$. If A and B are open subsets of $\mathbb{R}^2$, by $A \subset\subset B$ we mean that $\overline{A}$ is compact and $\overline{A} \subset B$. We denote by 1_A the characteristic function of A, i.e., $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$.

For any set $A \subset \mathbb{R}^2$, we denote by $A(\alpha)$ the set of points of density $\alpha \in [0, 1]$ for A, which is defined by

$$A(\alpha) = \left\{ x \in \mathbb{R}^2 : \lim_{\rho \to 0^+} |A \cap B_\rho(x)|/|B_\rho(x)| = \alpha \right\}. \tag{5}$$

Let $\{A_h\}_h \subset \Omega$ be a sequence of measurable sets. If $1_{A_h} \to 1_A$ in $L^1(\Omega)$ as $h \to +\infty$, then we simply write $A_h \to A$ in $L^1(\Omega)$.

2.1. Sets of finite perimeter

We denote by $\Omega \subset \mathbb{R}^2$ the image domain, and we assume that Ω is an open rectangle. We say that $u \in L^1(\Omega)$ is a function of bounded variation in Ω, and we write $u \in BV(\Omega)$, if the distributional derivative Du of u is a vector-valued Radon measure with finite total variation in Ω. We denote by $|Du|$ the total variation of the measure Du.

We say that a Borel set $A \subset \mathbb{R}^2$ is a *set of finite perimeter* in Ω, if $1_A \in BV(\Omega)$. The *reduced boundary* $\partial^* A \cap \Omega$ of a set A of finite perimeter in Ω is defined as the set of points $x \in \Omega$ such that

$$\text{there exists} \quad \lim_{\rho \to 0} \frac{D1_A(B_\rho(x))}{|D1_A|\,(B_\rho(x))} := \nu_A(x) \quad \text{with} \quad |\nu_A(x)| = 1.$$

We notice that $\partial^* A \cap \Omega \subseteq \partial A \cap \Omega$. If A is a set with smooth boundary, then $\partial^* A \cap \Omega = \partial A \cap \Omega$. The perimeter of A in Ω is then defined by

$$P(A, \Omega) := |D1_A|(\Omega) = \mathcal{H}^1(\partial^* A \cap \Omega). \tag{6}$$

Properties of sets of finite perimeter can be found in [11]. If A is a set of finite perimeter in Ω, then the following properties hold:

$$\partial^* A \cap \Omega \subset A(1/2) \cap \Omega, \qquad \mathcal{H}^1\left(A(1/2) \cap \Omega \setminus \partial^* A\right) = 0. \tag{7}$$

The isoperimetric inequality and the relative isoperimetric inequality will be used.

Theorem 2.1. (Isoperimetric inequality) *Let A be a set of finite perimeter in Ω such that $|A \cap \Omega| \leq \frac{1}{2}|\Omega|$. Then there exists a positive constant C, independent of A, such that the following inequality holds:*

$$P(A, \Omega) \geq C \cdot \sqrt{|A \cap \Omega|}.$$

Moreover, if $\Omega = \mathbb{R}^2$ then $C = 2\sqrt{\pi}$.

The following further properties about sets of finite perimeter will be used (see [16], Section 2). If A and B are sets of finite perimeter in Ω such that $A \cap B \cap \Omega = \emptyset$, then

$$P(A, \Omega) + P(B, \Omega) - P(A \cup B, \Omega) = 2\mathcal{H}^1(\partial^* A \cap \partial^* B \cap \Omega). \tag{8}$$

If A is a set of finite perimeter in Ω and B is an open subset of Ω with locally Lipschitz boundary in Ω, then

$$P(A \cap B, \Omega) = P(A, B) + \int_{\partial B \cap \Omega} 1_A \, d\mathcal{H}^1. \tag{9}$$

2.2. Partitions in sets of finite perimeter

Let $K \in \mathbb{N}$ and let $\chi_i \subset \Omega$, $i \in \{1, \ldots, K\}$, be Borel sets. We say that the family of sets

$$\chi = \{\chi_1, \ldots, \chi_K\}$$

defines a Borel partition of Ω if

$$|\chi_i \cap \chi_j| = 0 \quad \forall i, j \in \{1, \ldots, K\},\ i \neq j, \qquad \left|\Omega \setminus \cup_{i=1}^K \chi_i\right| = 0.$$

In the following we assume that χ_i, $i \in \{1, \ldots, K\}$, are sets of finite perimeter in Ω. For any $i = 1, \ldots, K$, we choose a definite representation of the set χ_i by setting $\chi_i = \chi_i(1)$ according to (5), otherwise, the characteristic function 1_{χ_i} would be defined only almost everywhere. Note that by replacing each set χ_i by $\chi_i(1)$ we obtain an equivalent partition in sets of finite perimeter. This property will be used in order to prove that the sets of an optimal partition are open. We set

$$|\chi_m| = \min_{i=1,\ldots,K} |\chi_i| > 0, \qquad |\chi_M| = \max_{i=1,\ldots,K} |\chi_i|. \tag{10}$$

As it is shown in [7], see Lemma 1.4, if $\chi = \{\chi_1, \ldots, \chi_K\}$ is a Borel partition of Ω in sets of finite perimeter, then

$$\Omega = \left[\bigcup_{i=1}^K (\chi_i \cap \Omega)\right] \cup \left[\bigcup_{i \neq j}^K (\chi_i(1/2) \cap \chi_j(1/2) \cap \Omega)\right] \cup N, \tag{11}$$

with $N \subset \Omega$, $\mathcal{H}^1(N) = 0$. This is a structure property that says that Ω is constituted by points of density 1 (we will prove that such points are interior points of sets of an optimal partition), points of density 1/2 (boundary points from property (7)), and an exceptional set $N = \Omega \setminus \cup_{i=1}^K [\chi_i(1) \cup \chi_i(1/2)]$ having null $\mathcal{H}^1$ measure (null length). The boundary points form the interface between pairs of the partitioning sets $\{\chi_i(1/2) \cap \chi_j(1/2) \cap \Omega,\ i \neq j\}$, except at most a set of null length. Eventually, we will prove that the boundaries of sets of an optimal partition are regular curves in a neighborhood of points of density 1/2, so that singularities are contained in the set N.

2.3. The main result

Let $K \in \mathbb{N}$ and let $\chi = \{\chi_1, \ldots, \chi_K\}$ be a Borel partition of Ω. Let $u_o \in L^\infty(\Omega)$, μ be a positive number and c_i, $i \in \{1, \ldots, K\}$, be numbers defined by $c_i = \dfrac{1}{|\chi_i|} \displaystyle\int_{\chi_i} u_o(x)dx$.

We set $\Gamma = \cup_{i=1}^K \partial\chi_i \cap \Omega$. We prove the following result:

Theorem 2.2. (Main Theorem) *There exist $K \in \mathbb{N}$ and a Borel partition of Ω in open sets $\chi_1, \ldots, \chi_K$ which minimize the functional*

$$\mathcal{E}(K, \chi_1, \ldots, \chi_K) = \mu \left(\sum_{i=1}^K \frac{\mathcal{H}^1(\partial\chi_i \cap \Omega)}{|\chi_i|} \right) \mathcal{H}^1(\Gamma) + \sum_{i=1}^K \int_{\chi_i} |u_o - c_i|^2 dx. \tag{12}$$

Moreover, $\Gamma = \Gamma_{\rm reg} \cup \Gamma_{\rm sing}$*, where* $\Gamma_{\rm reg}$ *is a curve of class* $C^{1,1/2}$ *in* Ω*,* $\mathcal{H}^1(\Gamma_{\rm sing}) = 0$ *and the set* $\Gamma_{\rm sing}$ *is locally finite.*

2.4. The weak energy functional

Let $K \in \mathbb{N}$ and let $\chi = \{\chi_1, \dots, \chi_K\}$ be a partition of Ω in sets of finite perimeter. For a Borel subset $B \subseteq \Omega$, we introduce the following notations:

$$P(\chi, B) := \frac{1}{2} \sum_{i=1}^{K} P(\chi_i, B) \quad \text{and} \quad S(\chi, B) := \sum_{i=1}^{K} \frac{P(\chi_i, B)}{|\chi_i|}.$$

We have $P(\chi, B) = \mathcal{H}^1\left(\cup_{i=1}^{K} \partial^* \chi_i \cap B\right)$.

In order to avoid cumbersome formulas, in the sequel of the paper when $B = \Omega$, with an abuse of notation we simply write

$$\begin{aligned} &P(A) := P(A, \Omega) \quad \text{for any set } A \text{ of finite perimeter in } \Omega, \\ &P(\chi) := P(\chi, \Omega), \quad S(\chi) := S(\chi, \Omega). \end{aligned}$$

Then, a *weak* version of the functional (12) is defined by

$$E(K, \chi_1, \dots, \chi_K) = \mu S(\chi) P(\chi) + \sum_{i=1}^{K} \int_{\chi_i} |u_o - c_i|^2 dx. \tag{13}$$

The weak version E is obtained by replacing in the original functional $\mathcal{E}$ the topological boundary $\partial \chi_i \cap \Omega$ of each set of the partition with the corresponding reduced boundary $\partial^* \chi_i \cap \Omega$. Then, for any Borel partition $\chi = \{\chi_1, \dots, \chi_K\}$ of Ω, since $\partial^* \chi_i \cap \Omega \subseteq \partial \chi_i \cap \Omega$, then $\mathcal{H}^1(\partial^* \chi_i \cap \Omega) \leq \mathcal{H}^1(\partial \chi_i \cap \Omega)$ for any i, and it follows

$$E(K, \chi_1, \dots, \chi_K) \leq \mathcal{E}(K, \chi_1, \dots, \chi_K). \tag{14}$$

First we prove (Theorem 3.3) the existence of a partition in sets of finite perimeter minimizing the functional E, then from this result we derive the existence of a Borel partition minimizing the functional $\mathcal{E}$ (Theorem 5.3).

We also set

$$E_0(K, \chi_1, \dots, \chi_K) := S(\chi) P(\chi),$$

and we define a coefficient q_i for each phase χ_i, which in the sequel will have a suitable meaning of a weight of the length of boundaries,

$$q_i = \frac{1}{2} S(\chi) + \frac{1}{|\chi_i|} P(\chi), \qquad \forall i \in 1, \dots, K. \tag{15}$$

3. Existence of weak minimizers

In this section, we prove the existence of a minimizer of the functional (13) which is constituted by a finite number of sets of finite perimeter. We begin by proving the following compactness result.

Proposition 3.1. (Compactness) *Let $\{K_h\}_h \subset \mathbb{N}$ be a sequence of integers, and let $\{\chi_1^h, \ldots, \chi_{K_h}^h\}$ be a sequence of families of sets of finite perimeter such that the family $\{\chi_1^h, \ldots, \chi_{K_h}^h\}$ defines a partition of Ω for any $h \in \mathbb{N}$. Assume that*

$$E_0(K_h, \chi_1^h, \ldots, \chi_{K_h}^h) \leq M \quad \forall h \in \mathbb{N},$$

where M is a positive constant independent of h. Then there exist a finite integer $K \in \mathbb{N}$, a family of sets of finite perimeter $\{\chi_1, \ldots, \chi_K\}$ which defines a partition of Ω, a subsequence of integers $\{K_{h_k}\}_k$ such that $K_{h_k} = K$ for any $k \in \mathbb{N}$, and a subsequence of families of sets $\{\chi_1^{h_k}, \ldots, \chi_K^{h_k}\}$, such that

$$\chi_i^{h_k} \to \chi_i \quad \text{in } L^1(\Omega), \qquad \forall i \in \{1, \ldots, K\},$$

as k tends to infinity.

Proof. For any $h \in \mathbb{N}$, in the family $\{\chi_1^h, \ldots, \chi_{K_h}^h\}$ there is at most one set having Lebesgue measure strictly greater than $\frac{1}{2}|\Omega|$. We may arrange the family in such a way that such a set, if it exists, is the set $\chi_{K_h}^h$. For any $h \in \mathbb{N}$ we set

$$\Gamma_h = \bigcup_{i=1}^{K_h} \left(\partial^* \chi_i^h \cap \Omega\right).$$

Using the isoperimetric inequality, Theorem 2.1, we have

$$\mathcal{H}^1(\Gamma_h) \geq \frac{1}{2} \sum_{i=1}^{K_h - 1} P(\chi_i^h) \geq \frac{C}{2} \sum_{i=1}^{K_h - 1} \sqrt{|\chi_i^h|}. \tag{16}$$

Using again the isoperimetric inequality and (16), we find

$$\begin{aligned} M &\geq \left(\sum_{i=1}^{K_h - 1} \frac{P(\chi_i^h)}{|\chi_i^h|} \right) \mathcal{H}^1(\Gamma_h) \geq C \left(\sum_{i=1}^{K_h - 1} \frac{1}{\sqrt{|\chi_i^h|}} \right) \mathcal{H}^1(\Gamma_h) \\ &\geq \frac{C^2}{2} \sum_{i=1}^{K_h - 1} \sqrt{|\chi_i^h|} \frac{1}{\sqrt{|\chi_i^h|}} = \frac{C^2}{2}(K_h - 1). \end{aligned}$$

It follows that

$$K_h \leq \frac{2M}{C^2} + 1,$$

so that K_h is uniformly bounded with respect to h. Hence, for any h we may arrange the family of sets $\{\chi_1^h, \ldots, \chi_{K_h}^h\}$ in such a way that there exist finite integers $K, \widehat{K} \in \mathbb{N}$, with $K \leq \widehat{K}$, a subsequence of integers $\{K_{h_k}\}_k$ and a subsequence $\{\chi_1^{h_k}, \ldots, \chi_{K_{h_k}}^{h_k}\}$ of families of sets such that $K_{h_k} = \widehat{K}$ for any $k \in \mathbb{N}$, and

$$\lim_{k \to +\infty} |\chi_i^{h_k}| > 0 \quad \forall i = 1, \ldots, K, \qquad \lim_{k \to +\infty} |\chi_i^{h_k}| = 0 \quad \forall i = K+1, \ldots, \widehat{K}.$$

Let us denote by $\{\chi_1^k, \ldots, \chi_K^k\}$ the subsequence $\{\chi_1^{h_k}, \ldots, \chi_K^{h_k}\}$ of families of sets. For any k we assume that the only set having Lebesgue measure strictly greater than $\frac{1}{2}|\Omega|$, if it exists, is the set χ_K^k.

On such a subsequence, we have

$$M \geq \left(\sum_{i=1}^{K-1} \frac{P(\chi_i^k)}{|\chi_i^k|}\right) \frac{1}{2} \sum_{i=1}^{K-1} P(\chi_i^k) \geq \frac{1}{2} \frac{P(\chi_j^k)}{|\chi_j^k|} P(\chi_j^k) \geq \frac{C}{2} \frac{P(\chi_j^k)}{\sqrt{|\chi_j^k|}}$$

for any $k \in \mathbb{N}$ and any $j \in \{1, \ldots, K-1\}$, also using the isoperimetric inequality. From this we get

$$P(\chi_j^k) \leq \frac{2M}{C} \sqrt{|\Omega|} \tag{17}$$

for any $j \in \{1, \ldots, K-1\}$. Moreover, we have

$$P(\chi_K^k) \leq \sum_{i=1}^{K-1} P(\chi_i^k) \leq (K-1) \frac{2M}{C} \sqrt{|\Omega|}. \tag{18}$$

Then the perimeter of χ_i^k in Ω is uniformly bounded with respect to k, for any $i \in \{1, \ldots, K\}$. Using (6) it follows that the total variation $|D1_{\chi_i^k}|(\Omega)$ is uniformly bounded, so that the sequence of functions $\{1_{\chi_i^k}\}_k$ is uniformly bounded in BV with respect to k, for any $i \in \{1, \ldots, K\}$.

Using the BV compactness theorem [11], by extracting for any i a subsequence which we do not relabel for simplicity, there exist K sets $\{\chi_1, \ldots, \chi_K\}$ of finite perimeter such that

$$\chi_i^k \to \chi_i \quad \text{in } L^1(\Omega), \qquad \forall i \in \{1, \ldots, K\}, \tag{19}$$

as k tends to infinity.

Eventually, the property that the family of sets $\{\chi_1, \ldots, \chi_K\}$ defines a partition of Ω follows from Theorem 1.6 of [7]. $\square$

Now we prove a lower semicontinuity result.

Proposition 3.2. (Lower semicontinuity) *Let $K \in \mathbb{N}$ and let $\{\chi_1, \ldots, \chi_K\}$ be a family of sets of finite perimeter which defines a partition of Ω. Let $\{\chi_1^h, \ldots, \chi_K^h\}$ be a sequence of partitions of Ω in sets of finite perimeter such that*

$$\chi_i^h \to \chi_i \quad \textit{in } L^1(\Omega), \qquad \forall i \in \{1, \ldots, K\},$$

as h tends to infinity. Then

$$\liminf_{h \to +\infty} E(K, \chi_1^h, \ldots, \chi_K^h) \geq E(K, \chi_1, \ldots, \chi_K).$$

Proof. By using (6) and the lower semicontinuity of the total variation [11], we have

$$\liminf_{h \to +\infty} \mathcal{H}^1(\Gamma_h) = \liminf_{h \to +\infty} \frac{1}{2} \sum_{i=1}^{K} P(\chi_i^h) \geq \frac{1}{2} \sum_{i=1}^{K} \liminf_{h \to +\infty} P(\chi_i^h) \geq \frac{1}{2} \sum_{i=1}^{K} P(\chi_i) = P(\chi). \tag{20}$$

For any $i \in \{1, \dots, K\}$ the L^1-convergence of χ_i^h to χ_i implies that

$$\liminf_{h\to+\infty} \frac{P(\chi_i^h)}{|\chi_i^h|} \geq \frac{P(\chi_i)}{|\chi_i|}. \tag{21}$$

Since the lower limit of the product of two positive sequences is greater than equal to the product of the respective lower limits, using (20) and (21), we find

$$\liminf_{h\to+\infty} \frac{P(\chi_i^h)}{|\chi_i^h|}\mathcal{H}^1(\Gamma_h) \geq \frac{P(\chi_i)}{|\chi_i|}P(\chi).$$

It then follows

$$\begin{aligned} \liminf_{h\to+\infty} E_0(K, \chi_1^h, \dots, \chi_K^h) &\geq \sum_{i=1}^K \liminf_{h\to+\infty} \frac{P(\chi_i^h)}{|\chi_i^h|}\mathcal{H}^1(\Gamma_h) \\ &\geq \sum_{i=1}^K \frac{P(\chi_i)}{|\chi_i|}P(\chi) = E_0(K, \chi_1, \dots, \chi_K), \end{aligned}$$

hence the functional E_0 is lower semicontinuous.

Eventually, the statement of the proposition follows from the lower semicontinuity of E_0 and the continuity of the integrals $\int_{\chi_i} |u_o - c_i|^2 dx$. $\square$

Using the compactness and lower semicontinuity results, we get the existence of minimizers of the functional E by means of the direct method of the calculus of variations.

Theorem 3.3. (Existence of weak minimizers) *There exist a finite integer $K \in \mathbb{N}$ and a family of sets $\{\chi_1, \dots, \chi_K\}$, which minimize the functional E over all partitions of Ω in sets of finite perimeter.*

4. Corner smoothing and optimal angles of junctions

Minimizers of Mumford-Shah functional (1) are known to possess only restricted types of singularities. Corners are not allowed in optimal boundaries Γ, and arcs can meet at a triple junction only with 120^o angles. Moreover multiple junctions, where more than three arcs meet, are not allowed. In this section, we study the analogous properties for the unsupervised model (4). While we find that corners are still not allowed in our case, conversely optimal junctions exhibit very different properties. In this section we assume that the union Γ of the boundaries of a weak minimizer $\chi_1, \dots, \chi_K$ of the functional (4) is constituted, locally, by a finite number of regular arcs. Regularity properties of weak minimizers will be investigated in the subsequent sections: the openness of the sets constituting an optimal segmentation in Section 5, the blow-up of optimal boundaries in Section 6, and the regularity of optimal boundaries in Section 7. Particularly, it will be proved that an optimal set Γ is constituted by curves of class $C^{1,1/2}$, except possibly a singular set of points which is locally finite.

Corner smoothing. Let $\chi = \{\chi_1, \dots, \chi_K\}$ be a partition such that the common boundary $\partial\chi_1 \cap \partial\chi_2$ has a corner point P where two arcs meet at an

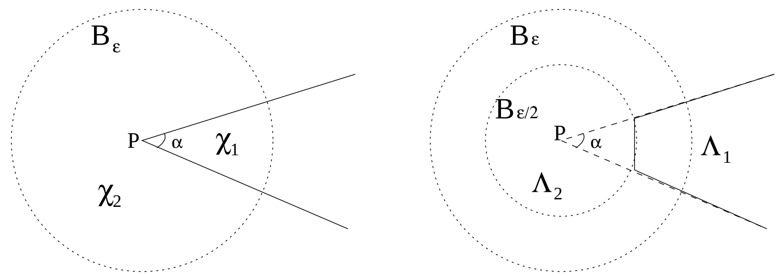

Figure 2: A partition χ and the new partition Λ.

angle α such that $0 < \alpha < \pi$, as it is shown in the left part of Figure 2. Let ε be the radius of a ball centered at P such that $\Gamma \cap B_\varepsilon(P) = \partial\chi_1 \cap \partial\chi_2 \cap B_\varepsilon(P)$. It is shown in [19] that a minimizer (u, Γ) of Mumford-Shah functional is such that the set Γ has no corner points. Such a result is obtained by comparing the energy $\mathcal{E}_{ms}(u, \Gamma)$ with $\mathcal{E}_{ms}$ evaluated on a modified pair $(\widehat{u}, \widehat{\Gamma})$. The modified pair $(\widehat{u}, \widehat{\Gamma})$ is obtained by cutting the corner P inside the ball $B_\varepsilon(P)$, as it is shown in the right part of Figure 2, see also [19]. Among the three terms in the functional (1), the change in the length term is of order ε, while the changes in the second and third term are of order $\varepsilon^{\frac{2\pi}{2\pi-\alpha}}$ and ε^2, respectively. Asymptotically, for small ε, the overall change of the energy can then be estimated by [19]

$$\mathcal{E}_{ms}(\widehat{u}, \widehat{\Gamma}) - \mathcal{E}_{ms}(u, \Gamma) \le c \left(\varepsilon^2 + \varepsilon^{\frac{2\pi}{2\pi-\alpha}} + \varepsilon \left(\sin \frac{\alpha}{2} - 1 \right) \right),$$

for some positive constant c. For a sufficiently small ε, the change of the energy is governed by $\varepsilon \left(\sin \frac{\alpha}{2} - 1 \right)$ which is always a negative value for any $0 < \alpha < \pi$, hence contradicting the minimality of the pair (u, Γ). It follows that the presence of a corner point is not allowed in a minimizer of Mumford-Shah functional.

In the same setting, the change in the energy for the model (4) can also be computed as follows. Analogously to Mumford-Shah case, we compare the energy $\mathcal{E}(K, \chi_1, \ldots, \chi_K)$ with $\mathcal{E}$ evaluated on a modified partition $\Lambda = \{\Lambda_1, \ldots, \Lambda_K\}$, which is obtained by cutting again the corner P inside the ball $B_\varepsilon(P)$, as it is shown in the right part of Figure 2. Hence we have $\Lambda_i = \chi_i$ for any $i > 2$. The change in the last term of functional (4) will be again of order ε^2. The change in the geometric terms of the functional is given by

$$S(\Lambda)P(\Lambda) - S(\chi)P(\chi) = (S(\Lambda) - S(\chi))P(\chi) + S(\Lambda)(P(\Lambda) - P(\chi)). \quad (22)$$

Here $S(\Lambda)$ can be evaluated using χ and we find

$$\begin{aligned}
S(\Lambda) &= \frac{P(\chi_1) - \varepsilon + \varepsilon \sin(\frac{\alpha}{2})}{|\chi_1| - \frac{\varepsilon^2}{8}\sin\alpha} + \frac{P(\chi_2) - \varepsilon + \varepsilon \sin(\frac{\alpha}{2})}{|\chi_2| + \frac{\varepsilon^2}{8}\sin\alpha} + \sum_{i=3}^{K} \frac{P(\chi_i)}{|\chi_i|} \\
&= \sum_{i=1}^{2} \frac{P(\chi_i) - \varepsilon + \varepsilon \sin(\frac{\alpha}{2})}{|\chi_i|} \left(1 + \mathcal{O}(\varepsilon^2)\right) + \sum_{i=3}^{K} \frac{P(\chi_i)}{|\chi_i|} \\
&= \varepsilon(-1 + \sin\frac{\alpha}{2}) \sum_{i=1}^{2} \frac{1}{|\chi_i|} + S(\chi) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Then, the terms with χ cancels to

$$(S(\Lambda) - S(\chi))P(\chi) = \varepsilon(-1 + \sin\frac{\alpha}{2}) \left(\frac{1}{|\chi_1|} + \frac{1}{|\chi_2|}\right) P(\chi) + \mathcal{O}(\varepsilon^2),$$

and the change in the length can be represented as

$$S(\Lambda)(P(\Lambda) - P(\chi)) = S(\Lambda)\varepsilon(-1 + \sin\frac{\alpha}{2}) + \mathcal{O}(\varepsilon^2).$$

Then, using the definition of the weights q_i in (15), the energy change becomes

$$\begin{aligned}
S(\Lambda)P(\Lambda) - S(\chi)P(\chi) &= \varepsilon(-1 + \sin\frac{\alpha}{2}) \left\{ \left(\frac{1}{|\chi_1|} + \frac{1}{|\chi_2|}\right) P(\chi) + S(\Lambda) \right\} + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon(-1 + \sin\frac{\alpha}{2}) \left\{ q_1 + q_2 + (S(\Lambda) - S(\chi)) \right\} + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon(-1 + \sin\frac{\alpha}{2})(q_1 + q_2) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

We used $(S(\Lambda) - S(\chi)) \approx \mathcal{O}(\varepsilon)$. Therefore, for any $0 < \alpha < \pi$, this is also negative, i.e., cutting a corner on Γ reduces the energy as in the case of Mumford-Shah model. If $q_1 = q_2 = 1$, this is exactly the case of Mumford-Shah model using a non-weighted length term $\mathcal{H}^1(\Gamma)$. Hence corners are not allowed along the boundaries of partitions minimizing functional (4).

Optimal angles of junctions. One of interesting properties of Mumford-Shah functional is the special structure of junctions that are allowed in a minimizer [19]. When arcs of Γ meet at a common endpoint, such an endpoint can only be a triple junction with equal angles of $\frac{2\pi}{3}$ radians. The main idea of the analysis in [19] is based on the case shown in Figure 3. If Γ has a triple junction with an angle α such that $\alpha < \frac{2\pi}{3}$, then the energy $\mathcal{E}_{ms}$ is reduced by modifying locally the set Γ in such a way that the angles of the junction become all equal to $\frac{2\pi}{3}$. More precisely, Γ is modified inside a ball B by extending $\partial\chi_2 \cap \partial\chi_3$ along the bisector of angle α, as it is shown in Figure 3. A new partition $\Lambda = \{\Lambda_1, \ldots, \Lambda_K\}$, with $\Lambda_i = \chi_i$ for any $i > 3$, is then obtained. Let the length of the segment $\overline{P_1P_2}$ be asymptotically (for a small ball B) equal to r. The original length $P(\chi, B)$ is equal to $2r$, while the new length $P(\Lambda, B)$ becomes

$$2(\frac{2}{\sqrt{3}} r \sin\frac{\alpha}{2}) + (r\cos\frac{\alpha}{2} - \frac{1}{\sqrt{3}} r \sin\frac{\alpha}{2}) = 2r\sin(\frac{\pi}{6} + \frac{\alpha}{2}).$$

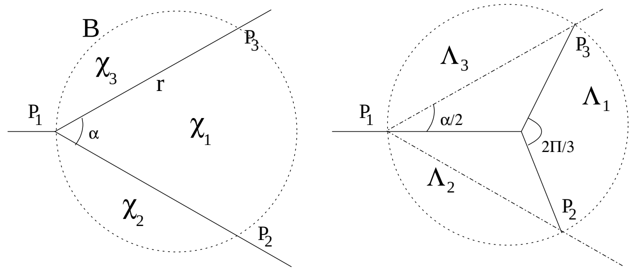

Figure 3: Optimal triple junction with equal angles for Mumford-Shah functional.

This new length is smaller than $2r$ for any $\alpha < \frac{2\pi}{3}$. Since the change in the other terms of the functional $\mathcal{E}_{ms}$ is again of a smaller order, then the energy is reduced. Therefore, a minimizer of Mumford-Shah functional allows only triple junctions with $\frac{2\pi}{3}$ equal angles. A similar argument leads to the conclusion that there are no junctions where four or more arcs meet at positive angles. In [19] it is shown how the energy is reduced by finding the smallest angle α and replacing a 4-fold intersection by two triple junctions.

Now we compute the energy change for the model (4) in the same case as in Figure 3. First, we write the difference of length as

$$\Delta_i = P(\Lambda_i, B) - P(\chi_i, B), \qquad i = 1, 2, 3.$$

In the energy change $S(\Lambda)P(\Lambda) - S(\chi)P(\chi)$, the length change $P(\Lambda) - P(\chi)$ is again given by $2r(-1 + \sin(\frac{\pi}{6} + \frac{\alpha}{2}))$, yet $S(\Lambda) - S(\chi)$ gives a weighted difference of lengths Δ_i. As before, the change in the last term of functional (4) is of order r^2, therefore negligible. With a similar computation as in the case of corner points, the energy change becomes

$$\begin{aligned}(S(\Lambda) \quad - \quad & S(\chi))P(\chi) + S(\Lambda)(P(\Lambda) - P(\chi)) \\ = \quad & \sum_{i=1}^{3} \frac{\Delta_i}{|\chi_i|} P(\chi) + \frac{1}{2} S(\Lambda) \sum_{i=1}^{3} \Delta_i + \mathcal{O}(r^2) \\ = \quad & \sum_{i=1}^{3} \frac{\Delta_i}{|\chi_i|} P(\chi) + \frac{1}{2}(S(\chi) + S(\Lambda) - S(\chi)) \sum_{i=1}^{3} \Delta_i + \mathcal{O}(r^2) \\ = \quad & \sum_{i=1}^{3} \left(\frac{P(\chi)}{|\chi_i|} + \frac{1}{2} S(\chi) \right) \Delta_i + \frac{1}{2}(S(\Lambda) - S(\chi)) \sum_{i=1}^{3} \Delta_i + \mathcal{O}(r^2) \\ = \quad & \sum_{i=1}^{3} q_i \Delta_i + \mathcal{O}(r^2). \end{aligned} \tag{23}$$

The energy decreases if $\sum_i q_i \Delta_i < 0$, therefore, the values of the optimal angles of the triple junction strongly depend on the values of q_i.

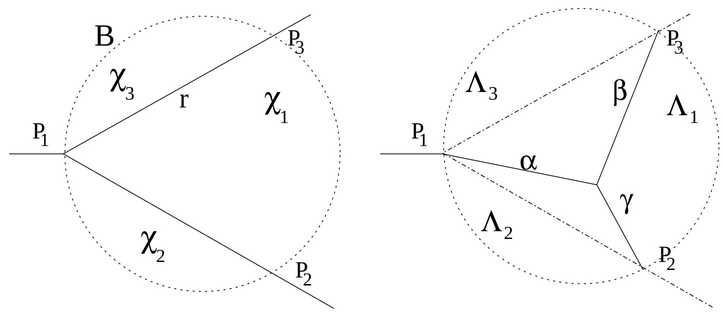

Figure 4: Definition of r, α, β and γ.

As a simple example, consider the case in Figure 4. The change in energy becomes

$$q_1(-2r + \beta + \gamma) + q_2(-r + \alpha + \gamma) + q_3(-r + \alpha + \beta) = q_1\Delta_1 + q_2\Delta_2 + q_3\Delta_3.$$

Here the length of $\partial\Lambda_1 \cap B$ is smaller than the length of $\partial\chi_1 \cap B$, i.e., $\Delta_1 < 0$, while the other two length changes are positive. Therefore, modifying the partition χ into the partition Λ, the energy is reduced only if

$$q_1 > -q_2(\Delta_2/\Delta_1) - q_3(\Delta_3/\Delta_1),$$

so that the result depends on the ratio of the length changes weighted by q_i. The optimal angles are strongly dependent on the values of the weights (15), which are $q_i = \frac{1}{2}S(\chi) + \frac{1}{|\chi_i|}P(\chi)$. Notice that in order for the weight q_1 to be large, the area χ_1 should be small. That is, if the area of χ_2 and χ_3 are big (i.e., q_2 and q_3 are small), the new partition Λ is preferable, i.e., increasing the area of χ_2 and χ_3 decreases the energy. This implies that the bigger the area of a phase is, the bigger that phase tries to become. The minimum of the functional (4) becomes a balance among these big phases while fitting the image datum u_o. This clearly shows the effect of the scale term.

A similar argument shows that minimizers of the model (4) can have multiple junctions, not only triple. This property is related to results obtained by Morgan and others [18, 10] about immiscible fluids in $\mathbb{R}^2$. Indeed, the energy of an immiscible fluid cluster is given by a linear combination of the lengths of the interfaces between fluids, where each length is weighted by a coefficient depending on which fluids the interface separates. Such coefficients play a role analogous to the weights q_i. In [10], the authors show that the interfaces of energy minimizing fluids can meet in any number around a junction, with the angles between segments determined by the weights.

5. Optimal segmentations by open sets

In the reminder of the paper $\chi = \{\chi_1, \ldots, \chi_K\}$ will denote a family of sets which minimizes the functional E over all partitions of Ω in sets of finite perime-

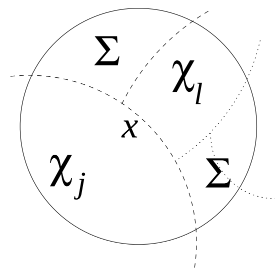

Figure 5: Setting of Elimination Lemma 5.1

ter. Moreover, without losing generality, we will assume $\mu = 1$. According to (11), the image domain Ω is partitioned in three different sets: the union of the sets $\chi_i(1)$ representing the phases, the union of the intersections $\{\chi_i(1/2) \cap \chi_j(1/2) \cap \Omega,\ i \neq j\}$ representing the boundaries of the segmentation, and the negligible set N.

A tool that we use in order to prove regularity properties of optimal segmentations is a suitable *elimination lemma* (Lemma 5.1 below). We adapt and modify the proof used by Tamanini and Congedo in [24] to prove an analogous result for the piecewise constant Mumford-Shah functional.

By means of the elimination lemma, first we prove that the sets $\chi_i(1)$ are open, then we will prove in Section 7 the regularity of the boundaries $\chi_i(1/2) \cap \chi_j(1/2) \cap \Omega$. We formulate a statement of the lemma that permits us to prove both regularity properties (see also Figure 5).

Lemma 5.1. (Elimination Lemma) *There exists a constant $\sigma > 0$ such that for any $x \in \Omega$ there exists a ball $B_{R_0}(x) \subset \Omega$ with the following property: if χ_j, χ_l, with $j, l \in \{1, \dots, K\}$, $j \neq l$, and $R \in (0, R_0)$ are such that*

$$\Sigma = \bigcup_{i=1, i \notin \{j,l\}}^{K} \chi_i, \qquad |\Sigma \cap B_R(x)| \leq \sigma R^2,$$

then $\left|\Sigma \cap B_{R/2}(x)\right| = 0$.

Proof. Let us assume $K > 2$, otherwise the result is trivial. Let us fix $x \in \Omega$ and assume for simplicity x is the origin of $\mathbb{R}^2$. We may assume that $j = 1$ and $l = 2$, so that $\Sigma = \bigcup_{i>2} \chi_i$. In order to prove that $|\Sigma \cap B_{R/2}| = 0$, we define $\alpha(r) = |\Sigma \cap B_r|$ and we assume that $\alpha(r) > 0$ in the interval $(R/2, R)$ (otherwise there is nothing to prove). By using the coarea formula we have

$$\alpha(r) = |\Sigma \cap B_r| = \int_0^r dt \int_{\partial B_t} 1_\Sigma d\mathcal{H}^1. \tag{24}$$

We may choose R_0 small enough in such a way that for any $r \in (R/2, R)$

$$|\chi_i \setminus B_r| \geq |\chi_i|/2, \quad |\chi_i \cup B_r| \leq 2|\chi_i|, \qquad \forall i = 1, \dots, K. \tag{25}$$

In addition, for almost all $r \in (R/2, R)$ we have

$$\mathcal{H}^1(\partial^* \chi_i \cap \partial B_r) = 0 \qquad \forall i = 1, \ldots, K. \tag{26}$$

Let us fix $r \in (R/2, R)$ in such a way that (26) holds. We may assume that the boundary of χ_1 has bigger intersection with the boundary of Σ in the ball B_r than χ_2 (otherwise it is enough to exchange the two sets):

$$\mathcal{H}^1(\partial^* \chi_1 \cap \partial^* \Sigma \cap B_r) \geq \mathcal{H}^1(\partial^* \chi_2 \cap \partial^* \Sigma \cap B_r). \tag{27}$$

Then, we define a perturbed partition Λ as follows:

$$\Lambda_1 = (\chi_1 \cup (\Sigma \cap B_r))\,(1), \qquad \Lambda_2 = \chi_2, \qquad \Lambda_i = \chi_i \setminus \overline{B_r}, \quad i = 3, \ldots, K. \tag{28}$$

Since c_i, $i = 1, \ldots, K$, are the optimal constants for the sets χ_i, we have

$$E_0(K, \Lambda_1, \ldots, \Lambda_K) + \sum_{i=1}^{K} \int_{\Lambda_i} |u_o - c_i|^2 dx \geq E(K, \Lambda_1, \ldots, \Lambda_K).$$

Since $\{\chi_1, \ldots, \chi_K\}$ is a minimizer of E, we have $E(K, \Lambda_1, \ldots, \Lambda_K) \geq E(K, \chi_1, \ldots, \chi_K)$, so that setting

$$\Delta E_0 = E_0(K, \Lambda_1, \ldots, \Lambda_K) - E_0(K, \chi_1, \ldots, \chi_K) + \sum_{i=1}^{K} \int_{\Lambda_i} |u_o - c_i|^2 dx - \sum_{i=1}^{K} \int_{\chi_i} |u_o - c_i|^2 dx, \tag{29}$$

we have $\Delta E_0 \geq 0$. We prove that, choosing R_0 small enough, this inequality implies the estimate

$$a_1 \int_{\partial B_r} 1_\Sigma \, d\mathcal{H}^1 - a_2 |\Sigma \cap B_r|^{1/2} \geq 0, \tag{30}$$

with $a_1, a_2 > 0$ independent of r and $a_2/4a_1 < \sqrt{\pi}$. Then, using (24), it follows

$$a_1 \frac{d\alpha}{dr} - a_2 \, [\alpha(r)]^{1/2} \geq 0,$$

which yields

$$\frac{d}{dr}[\alpha(r)]^{1/2} = \frac{1}{2} \, \frac{d\alpha/dr}{[\alpha(r)]^{1/2}} \geq \frac{a_2}{2a_1}.$$

Integrating on the interval $(R/2, R)$ we get

$$|\Sigma \cap B_R|^{1/2} \geq \frac{a_2}{4a_1} R + |\Sigma \cap B_{R/2}|^{1/2}.$$

Then, choosing the positive constant σ in the statement of the lemma in such a way that

$$\sigma = \left[\frac{a_2}{4a_1}\right]^2 < \pi,$$

it follows $|\Sigma \cap B_{R/2}| = 0$, proving the statement of the lemma. In the following, we show how the estimate (30) follows from the inequality $\Delta E_0 \geq 0$, for some a_1 and a_2 independent of r.

Let $E_0(K, \Lambda_1, \ldots, \Lambda_K) = S(\Lambda)P(\Lambda)$, and $P(\Lambda) = P(\chi) + \Delta P$. Then, as in (22), the difference of values of E_0 in (29) can be written as

$$E_0(K, \Lambda_1, \ldots, \Lambda_K) - E_0(K, \chi_1, \ldots, \chi_K) = (S(\Lambda) - S(\chi))P(\chi) + S(\Lambda)\Delta P.$$

The estimate (30) is obtained in four steps.

Step 1. *Estimate of the difference* $(S(\Lambda) - S(\chi))P(\chi)$.

Using (9) we have $P(\Sigma \cap B_r) = P(\Sigma, B_r) + \mathcal{H}^1(\Sigma \cap \partial B_r)$. Then, using the identity (8) and (27), it follows

$$\begin{aligned} P(\Lambda_1) &= P(\chi_1 \cup (\Sigma \cap B_r)) = P(\chi_1) + P(\Sigma \cap B_r) - 2\mathcal{H}^1(\partial^*\chi_1 \cap \partial^*(\Sigma \cap B_r)) \\ &= P(\chi_1) - \mathcal{H}^1(\partial^*\chi_1 \cap \partial^*\Sigma \cap B_r) + \mathcal{H}^1(\partial^*\chi_2 \cap \partial^*\Sigma \cap B_r) + \mathcal{H}^1(\Sigma \cap \partial B_r) \\ &\leq P(\chi_1) + \mathcal{H}^1(\Sigma \cap \partial B_r). \end{aligned}$$

Taking into account that $\Lambda_2 = \chi_2$ and using (26), we get

$$\begin{aligned} S(\Lambda) &= \frac{P(\Lambda_1)}{|\Lambda_1|} + \frac{P(\Lambda_2)}{|\Lambda_2|} + \sum_{i=3}^{K} \frac{P(\Lambda_i)}{|\Lambda_i|} \\ &\leq \frac{1}{|\Lambda_1|}\left\{P(\chi_1) + \mathcal{H}^1(\Sigma \cap \partial B_r)\right\} \\ &+ \frac{P(\chi_2)}{|\chi_2|} + \sum_{i=3}^{K} \frac{1}{|\Lambda_i|}\left\{P(\chi_i) - P(\chi_i, B_r) + \mathcal{H}^1(\chi_i \cap \partial B_r)\right\}. \end{aligned}$$

The term by term subtraction yields

$$S(\Lambda) - S(\chi) \leq \left(\frac{1}{|\Lambda_1|} - \frac{1}{|\chi_1|}\right) P(\chi_1) + \sum_{i=3}^{K} \left(\frac{1}{|\Lambda_i|} - \frac{1}{|\chi_i|}\right) P(\chi_i) \tag{31}$$

$$- \sum_{i=3}^{K} \frac{1}{|\Lambda_i|} P(\chi_i, B_r) \tag{32}$$

$$+ \frac{1}{|\Lambda_1|}\mathcal{H}^1(\Sigma \cap \partial B_r) + \sum_{i=3}^{K} \frac{1}{|\Lambda_i|}\mathcal{H}^1(\chi_i \cap \partial B_r). \tag{33}$$

Using (10) and (25) we have

$$\frac{1}{|\Lambda_i|} \leq \frac{2}{|\chi_m|}, \qquad -\frac{1}{|\Lambda_i|} \leq -\frac{1}{2|\chi_M|}, \qquad \forall i \in \{1, \ldots, K\}. \tag{34}$$

Notice the first term in the right-hand side of (31) is non-positive, so that it can be ignored. Using (34) for the other terms in (31) we have

$$\begin{aligned}\frac{1}{|\Lambda_i|}-\frac{1}{|\chi_i|} &= \frac{|\chi_i|-|\chi_i\setminus B_r|}{|\Lambda_i|\cdot|\chi_i|}=\frac{|\chi_i\cap B_r|}{|\Lambda_i|\cdot|\chi_i|}\le\frac{|\Sigma\cap B_r|}{|\Lambda_i|\cdot|\chi_i|}\\ &\le \frac{|\Sigma\cap B_r|^{1/2}|B_r|^{1/2}}{|\Lambda_i|\cdot|\chi_i|}\le 2\frac{|\Sigma\cap B_r|^{1/2}|B_r|^{1/2}}{|\chi_m|^2}\le\frac{2\sqrt{\pi}R}{|\chi_m|^2}|\Sigma\cap B_r|^{1/2}.\end{aligned}$$

Using $P(\chi)\le M$, where M is a positive constant that can be estimated arguing as in the proof of (17) and (18) of Proposition 3.1, the terms in (31) can be estimated by means of

$$\left(\frac{1}{|\Lambda_1|}-\frac{1}{|\chi_1|}\right)P(\chi_1)+\sum_{i=3}^{K}\left(\frac{1}{|\Lambda_i|}-\frac{1}{|\chi_i|}\right)P(\chi_i)\le\frac{4\sqrt{\pi}RM}{|\chi_m|^2}|\Sigma\cap B_r|^{1/2}.$$

Using (34), (9) and the isoperimetric inequality, the terms (32) (although negative) can be estimated as follows:

$$\begin{aligned}-\sum_{i=3}^{K}\frac{1}{|\Lambda_i|}P(\chi_i,B_r) &\le \frac{-1}{2|\chi_M|}\sum_{i=3}^{K}P(\chi_i,B_r)\le\frac{-1}{2|\chi_M|}P(\Sigma,B_r) \qquad (35)\\ &\le \frac{-1}{2|\chi_M|}\left\{P(\Sigma\cap B_r)-\int_{\partial B_r}1_\Sigma d\mathcal{H}^1\right\}\\ &\le \frac{-\sqrt{\pi}}{|\chi_M|}|\Sigma\cap B_r|^{1/2}+\frac{1}{2|\chi_M|}\int_{\partial B_r}1_\Sigma d\mathcal{H}^1.\end{aligned}$$

Using (34) the term (33) can be estimated as follows:

$$\begin{aligned}\frac{1}{|\Lambda_1|}\mathcal{H}^1(\Sigma\cap\partial B_r) &+ \sum_{i=3}^{K}\frac{1}{|\Lambda_i|}\mathcal{H}^1(\chi_i\cap\partial B_r) \qquad (36)\\ &\le \frac{1}{|\Lambda_1|}\mathcal{H}^1(\Sigma\cap\partial B_r)+\frac{2}{|\chi_m|}\mathcal{H}^1(\Sigma\cap\partial B_r)\le\frac{4}{|\chi_m|}\int_{\partial B_r}1_\Sigma d\mathcal{H}^1.\end{aligned}$$

Collecting all the estimates, and taking into account that $P(\chi)\le M$, we find

$$(S(\Lambda)-S(\chi))P(\chi)\le\left(\frac{-M\sqrt{\pi}}{|\chi_M|}+\frac{4\sqrt{\pi}RM^2}{|\chi_m|^2}\right)|\Sigma\cap B_r|^{1/2}+\left(\frac{M}{2|\chi_M|}+\frac{4M}{|\chi_m|}\right)\int_{\partial B_r}1_\Sigma d\mathcal{H}^1. \qquad (37)$$

Step 2. *Estimate of the difference* $S(\Lambda)\Delta P$.

Arguing as in Step 1 we have

$$P(\Lambda)\le\frac{1}{2}\left[P(\chi_1)+\mathcal{H}^1(\Sigma\cap\partial B_r)+P(\chi_2)+\sum_{i=3}^{K}\left\{P(\chi_i)-P(\chi_i,B_r)+\mathcal{H}^1(\chi_i\cap\partial B_r)\right\}\right].$$

Similar to above computations in (35) and (36),

$$\Delta P = P(\Lambda) - P(\chi) \leq -\sqrt{\pi}|\Sigma \cap B_r|^{1/2} + \frac{1}{2}\int_{\partial B_r} 1_\Sigma d\mathcal{H}^1 + \int_{\partial B_r} 1_\Sigma d\mathcal{H}^1.$$

$$\begin{aligned} S(\Lambda) &= \sum_{i=1}^{K} \frac{P(\Lambda_i)}{|\Lambda_i|} \leq \frac{2}{|\chi_m|}\left\{\sum_{i=1}^{K} P(\chi_i) - \sum_{i=3}^{K} P(\chi_i, B_r) + 2\mathcal{H}^1(\Sigma \cap \partial B_r)\right\} \\ &\leq \frac{2}{|\chi_m|}\left\{2P(\chi) + 2\mathcal{H}^1(\Sigma \cap \partial B_r)\right\} \leq \frac{4}{|\chi_m|}\{M + 2\pi R\}. \end{aligned}$$

Therefore,

$$S(\Lambda)\Delta P \leq -\frac{4M\sqrt{\pi}}{|\chi_m|}|\Sigma \cap B_r|^{1/2} + \frac{6}{|\chi_m|}(M + 2\pi R)\int_{\partial B_r} 1_\Sigma d\mathcal{H}^1. \tag{38}$$

***Step 3**. Estimate of the difference of the integral terms.*

Using (28) we have

$$\begin{aligned} & \int_{\Lambda_1} |u_o - c_1|^2 dx - \int_{\chi_1} |u_o - c_1|^2 dx + \sum_{i=3}^{K}\int_{\Lambda_i} |u_o - c_i|^2 dx - \sum_{i=3}^{K}\int_{\chi_i} |u_o - c_i|^2 dx \\ = & \int_{\Lambda_1 \cap B_r} |u_o - c_1|^2 dx - \int_{\chi_1 \cap B_r} |u_o - c_1|^2 dx - \sum_{i=3}^{K}\int_{\chi_i \cap B_r} |u_o - c_i|^2 dx \\ \leq & \int_{(\chi_1 \cup \Sigma) \cap B_r} |u_o - c_1|^2 dx - \int_{\chi_1 \cap B_r} |u_o - c_1|^2 dx = \int_{\Sigma \cap B_r} |u_o - c_1|^2 dx \\ \leq & M_1 |\Sigma \cap B_r| \leq M_1\, |\Sigma \cap B_r|^{1/2} |B_r|^{1/2}, \end{aligned}$$

where

$$M_1 = 2\|u_o\|^2_{L^\infty(\Omega)} + 2c_1^2.$$

Then we obtain

$$\sum_{i=1}^{K}\int_{\Lambda_i} |u_o - c_i|^2 dx - \sum_{i=1}^{K}\int_{\chi_i} |u_o - c_i|^2 dx \leq \sqrt{\pi} M_1\, R\, |\Sigma \cap B_r|^{1/2}. \tag{39}$$

***Step 4**. Collection of the inequalities.*

Collecting the estimates (37), (38) and (39), the quantity ΔE_0 defined in equation (29) is bounded by

$$0 \leq \Delta E_0 \leq a_1 \int_{\partial B_r} 1_\Sigma \, d\mathcal{H}^1 + (b_1 R - b_2)|\Sigma \cap B_r|^{1/2},$$

where

$$
\begin{aligned}
a_1 &= M\left(\frac{1}{2|\chi_M|}+\frac{10}{|\chi_m|}\right)+\frac{12\pi}{|\chi_m|}R_0,\\
b_1 &= \left(\frac{4M^2}{|\chi_m|^2}+M_1\right)\sqrt{\pi},\\
b_2 &= M\sqrt{\pi}\left(\frac{1}{|\chi_M|}+\frac{4}{|\chi_m|}\right).
\end{aligned}
$$

Now we choose $R_0 < b_2/(2b_1)$ so that, setting $a_2 = b_2/2$, we obtain inequality (30) with $a_1, a_2 > 0$ independent of r. Moreover, one can check that $a_2/4a_1 < \sqrt{\pi}$, hence completing the proof of the lemma. □

As a corollary of the elimination result, we obtain that the sets constituting an optimal segmentation are open.

Corollary 5.2. *For any $i = 1, \ldots, K$, the set χ_i is open.*

Proof. We prove that the set χ_1 is open, then it will be enough to repeat the argument for the sets $\chi_2, \ldots \chi_K$. We define $\Sigma = \bigcup_{i=2}^K \chi_i$. We fix a point of χ_1 and, without losing generality, we assume it is the origin of $\mathbb{R}^2$. Given $\sigma > 0$, let $R > 0$ be such that $B_R \subset \Omega$ and

$$
|B_R \cap \Sigma| = |B_R \setminus \chi_1| \leq \sigma R^2.
$$

This is possible since $\chi_1 = \chi_1(1)$, so that the origin of $\mathbb{R}^2$ is a point of density 0 for Σ.

With the same method of proof of Lemma 5.1 we find that $|\Sigma \cap B_{R/2}| = 0$ for a suitable value of σ. It follows immediately that $B_{R/2} \subset \chi_1$, thus proving that χ_1 is open. □

Now we can recover the existence of a minimizer of the functional $\mathcal{E}$.

Theorem 5.3. *There exist $K \in \mathbb{N}$ and a Borel partition of Ω in open sets $\chi_1, \ldots, \chi_K$ which minimize the functional $\mathcal{E}$.*

Proof. Let $K \in \mathbb{N}$ and let $\chi = \{\chi_1, \ldots, \chi_K\}$ be a partition of Ω in sets of finite perimeter which minimizes the functional E. The sets are open by Corollary 5.2. First we prove the following property:

$$
\mathcal{H}^1\left((\partial\chi_i \setminus \partial^*\chi_i) \cap \Omega\right) = 0, \qquad \forall\, i = 1, \ldots, K. \tag{40}
$$

Let us fix $i \in \{1, \ldots, K\}$ and let $x \in \partial\chi_i$. Hence $x \notin \chi_j(1)$ for any j, otherwise, since each set χ_j is open by Corollary 5.2, then x would be an interior point of χ_j, so that x could not belong to $\partial\chi_i$.

Moreover, $x \notin \chi_j(1/2) \cap \chi_l(1/2) \cap \Omega$ for any $j, l \neq i$ such that $j \neq l$. Indeed, if this is not the case, using (5) and Lemma 5.1, there exists $R \in (0, R_0)$ such that $|\Sigma \cap B_R(x)| \leq \sigma R^2$, so that $\left|\Sigma \cap B_{R/2}(x)\right| = 0$. It follows $\left|\chi_i \cap B_{R/2}(x)\right| = 0$, but this is in contradiction with the assumption $x \in \partial\chi_i$ which implies, the set χ_i being open, the existence of a ball $B_\rho(y)$ such that $B_\rho(y) \subset \chi_i \cap B_{R/2}(x)$.

Then, using the structure property (11) of partitions in sets of finite perimeter, it follows that $x \in (\chi_i(1/2) \cap \Omega) \cup N$, from which, using (7) and $\mathcal{H}^1(N) = 0$, the property (40) follows.

Using (40), then we have $\mathcal{E}(K, \chi_1, \ldots, \chi_K) = E(K, \chi_1, \ldots, \chi_K) < +\infty$. Now we prove that the Borel partition $\chi = \{\chi_1, \ldots, \chi_K\}$ minimizes the functional $\mathcal{E}$.

If this is not true, then there exist $\widehat{K} \in \mathbb{N}$ and another Borel partition $\Lambda = \{\Lambda_1, \ldots, \Lambda_{\widehat{K}}\}$ such that

$$\mathcal{E}(\widehat{K}, \Lambda_1, \ldots, \Lambda_{\widehat{K}}) < \mathcal{E}(K, \chi_1, \ldots, \chi_K) < +\infty.$$

Using property (14) we have $E(\widehat{K}, \Lambda_1, \ldots, \Lambda_{\widehat{K}}) \le \mathcal{E}(\widehat{K}, \Lambda_1, \ldots, \Lambda_{\widehat{K}})$. Moreover, arguing as in the proof of estimates (17) and (18) in Proposition 3.1, we have that the sets Λ_i have finite perimeter. Then, collecting all the above inequalities we get

$$E(\widehat{K}, \Lambda_1, \ldots, \Lambda_{\widehat{K}}) < E(K, \chi_1, \ldots, \chi_K),$$

which is a contradiction, since the integer K and the partition $\{\chi_1, \ldots, \chi_K\}$ minimize the functional E. The statement of the theorem then follows. □

In the following section, we further investigate properties of optimal partitions leading up to regularity properties of the boundaries.

6. Some results on optimal segmentations

We first prove some preliminary lemmas which will be useful to achieve a blow-up result of the boundaries of an optimal segmentation. Regularity properties of the boundaries will then be shown by combining the elimination lemma proved in the previous section with the blow-up result.

Lemma 6.1. *Let $x \in \Omega$ and $B_r = B_r(x) \subset\subset \Omega$. Let $\{\Lambda_1, \ldots, \Lambda_K\}$ be a family of sets of finite perimeter which define a partition of Ω satisfying*

$$\Lambda_i \setminus C = \chi_i \setminus C \quad \forall i \in 1, \ldots, K, \qquad C \subset B_r \text{ compact set.} \tag{41}$$

Then there exists $\delta > 0$ such that for any $r < \delta$ the following inequality holds:

$$\sum_{i=1}^{K} [\varphi(0) + \psi_i(r)]\, P(\chi_i, B_r) \le \sum_{i=1}^{K} \left[\varphi(r) + (1 + b_i r^2)\psi_i(r)\right] P(\Lambda_i, B_r) + a_1 r^2, \tag{42}$$

where a_1 is a positive constant independent of r and of the family $\{\Lambda_1, \ldots, \Lambda_K\}$, $b_i = 2\pi/|\chi_i|$, for $i = 1, \ldots, K$,

$$\varphi(r) = \frac{1}{2} \sum_{i=1}^{K} \frac{P(\Lambda_i)}{|\chi_i|}(1 + b_i r^2), \qquad \psi_i(r) = \frac{1}{|\chi_i|} P(\chi, \Omega \setminus B_r),$$

and $\varphi(0) = \lim_{r \to 0^+} \varphi(r)$.

Proof. Let $\delta > 0$ be such that

$$\frac{|B_\delta|}{|\chi_m|} \leq \frac{1}{2}, \tag{43}$$

and let $r < \delta$. Since c_i, $i = 1, \dots, K$, are the optimal constants for the sets χ_i, we have

$$E_0(K, \Lambda_1, \dots, \Lambda_K) + \sum_{i=1}^{K} \int_{\Lambda_i} |u_o - c_i|^2 dx \geq E(K, \Lambda_1, \dots, \Lambda_K). \tag{44}$$

For any $i \in \{1, \dots, K\}$, using (41) and (43), we have

$$\begin{aligned}
\frac{1}{|\Lambda_i|} &= \frac{1}{|\Lambda_i \setminus B_r| + |\Lambda_i \cap B_r|} = \frac{1}{|\chi_i \setminus B_r| + |\Lambda_i \cap B_r|} \\
&= \frac{1}{|\chi_i| + |\Lambda_i \cap B_r| - |\chi_i \cap B_r|} \leq \frac{1}{|\chi_i|} \frac{1}{1 - \xi_i},
\end{aligned}$$

where

$$\xi_i = \frac{\Big||\Lambda_i \cap B_r| - |\chi_i \cap B_r|\Big|}{|\chi_i|} \leq \frac{|B_r|}{|\chi_i|} \leq \frac{1}{2},$$

from which it follows

$$\frac{1}{|\Lambda_i|} \leq \frac{1}{|\chi_i|}(1 + 2\xi_i) \leq \frac{1}{|\chi_i|}(1 + b_i r^2), \quad b_i = \frac{2\pi}{|\chi_i|}. \tag{45}$$

Using (41), for any $i = 1, \dots, K$, we have

$$P(\Lambda_i) = P(\Lambda_i, B_r) + P(\chi_i, \Omega \setminus B_r). \tag{46}$$

Using (45) and (46) we have the following estimate for the energy of the partition $\{\Lambda_1, \dots, \Lambda_K\}$:

$$\begin{aligned}
E_0(K, \Lambda_1, \dots, \Lambda_K) &\leq P(\Lambda) \sum_{i=1}^{K} \frac{P(\Lambda_i)}{|\chi_i|}(1 + b_i r^2) \\
&= 2\varphi(r) P(\Lambda, B_r) + \sum_{i=1}^{K} (1 + b_i r^2) \psi_i(r) P(\Lambda_i, B_r) \quad (47) \\
&+ P(\chi, \Omega \setminus B_r) \sum_{i=1}^{K} \frac{P(\chi_i, \Omega \setminus B_r)}{|\chi_i|}(1 + b_i r^2).
\end{aligned}$$

Analogously, for the energy of the optimal partition $\{\chi_1, \dots, \chi_K\}$ we have

$$E_0(K, \chi_1, \dots, \chi_K) = 2\varphi(0) P(\chi, B_r) + \sum_{i=1}^{K} \psi_i(r) P(\chi_i, B_r) + P(\chi, \Omega \setminus B_r) S(\chi, \Omega \setminus B_r), \tag{48}$$

where, using (46),

$$\varphi(0) = \lim_{r\to 0^+} \varphi(r) = \frac{1}{2} S(\chi). \tag{49}$$

By the optimality of the partition $\{\chi_1, \ldots, \chi_K\}$, using (44), we have

$$E_0(K, \chi_1, \ldots, \chi_K) + \sum_{i=1}^{K} \int_{\chi_i} |u_o - c_i|^2 dx \le E_0(K, \Lambda_1, \ldots, \Lambda_K) + \sum_{i=1}^{K} \int_{\Lambda_i} |u_o - c_i|^2 dx, \tag{50}$$

from which, using (47) and (48), it follows

$$\begin{aligned} E_0(K, \Lambda_1, \ldots, \Lambda_K) - E_0(K, \chi_1, \ldots, \chi_K) \le \alpha r^2 - 2\varphi(0) P(\chi, B_r) \qquad (51)\\ - \sum_{i=1}^{K} \psi_i(r) P(\chi_i, B_r) + 2\varphi(r) P(\Lambda, B_r) + \sum_{i=1}^{K} (1 + b_i r^2) \psi_i(r) P(\Lambda_i, B_r), \end{aligned}$$

where, using $P(\chi) \le M$ and (45), the constant α is given by $\alpha = \dfrac{4\pi M^2}{\min_{i=1,\ldots,K} |\chi_i|^2}$.
Moreover, using (41) and taking into account that $|c_i| \le \|u_o\|_{L^\infty(\Omega)}$ for any $i = 1, \ldots, K$, we have

$$\begin{aligned} &\sum_{i=1}^{K} \int_{\Lambda_i} |u_o - c_i|^2 dx - \sum_{i=1}^{K} \int_{\chi_i} |u_o - c_i|^2 dx \\ &= \sum_{i=1}^{K} \int_{\Lambda_i \cap B_r} |u_o - c_i|^2 dx - \sum_{i=1}^{K} \int_{\chi_i \cap B_r} |u_o - c_i|^2 dx \\ &\le \sum_{i=1}^{K} \int_{\Lambda_i \cap B_r} |u_o - c_i|^2 dx \le \sum_{i=1}^{K} \left(2\|u_o\|_{L^\infty(\Omega)}\right)^2 |B_r| \le \beta r^2, \end{aligned} \tag{52}$$

where

$$\beta = \pi K \left(2\|u_o\|_{L^\infty(\Omega)}\right)^2.$$

Collecting (50), (51) and (52), we find the inequality (42), where $a_1 = \alpha + \beta$. □

Now we prove an estimate of the perimeter of an optimal partition in a ball.

Lemma 6.2. *Let $x \in \Omega$, $0 < r < 1$ and $B_r = B_r(x) \subset\subset \Omega$. Then the following estimate holds:*

$$P(\chi, B_r) \le a_2 r,$$

where a_2 is a positive constant independent of r.

Proof. Let $s \in (0, r)$ and let $B_s = B_s(x)$; we set

$$\Lambda_1 = \chi_1 \cup B_s, \qquad \Lambda_i = \chi_i \setminus \overline{B_s}, \quad i = 2, \ldots, K,$$

so that we have

$$\partial^* \Lambda_i \cap \Omega = (\partial^* \Lambda_i \cap \partial B_s) \bigcup \left(\partial^* \chi_i \cap (\Omega \setminus \overline{B_s})\right),$$

for any $i = 1, \dots, K$. Then we can write

$$\begin{aligned}
&E_0(K, \Lambda_1, \dots, \Lambda_K) \\
&= \left[P(\Lambda, \partial B_s) + P(\chi, \Omega \setminus \overline{B_s})\right] \cdot \left[S(\Lambda, \partial B_s) + \sum_{i=1}^{K} \frac{P(\chi_i, \Omega \setminus \overline{B_s})}{|\Lambda_i|}\right] \\
&\leq \left[2\pi s + P(\chi, \Omega \setminus \overline{B_s})\right] \cdot \left[\frac{4\pi s}{|\chi_m|}\left(1 + \frac{2\pi r^2}{|\chi_m|}\right) + \left(1 + \frac{2\pi r^2}{|\chi_m|}\right) S(\chi, \Omega \setminus \overline{B_s})\right] \\
&\leq \left(1 + \frac{2\pi r^2}{|\chi_m|}\right) S(\chi, \Omega \setminus \overline{B_s}) P(\chi, \Omega \setminus \overline{B_s}) + \alpha s + \beta s^2, \qquad (53)
\end{aligned}$$

where we have used the inequality $P(\Lambda, \partial B_s) \leq 2\pi s$, and (45) with $C = \overline{B_s}$; moreover, using $P(\chi) \leq M$, we set

$$\alpha = \frac{8\pi M}{|\chi_m|}\left(1 + \frac{2\pi}{|\chi_m|}\right), \qquad \beta = \frac{8\pi^2}{|\chi_m|}\left(1 + \frac{2\pi}{|\chi_m|}\right).$$

For the optimal partition $\{\chi_1, \dots, \chi_K\}$, by using the isoperimetric inequality, Theorem 2.1, we have

$$\begin{aligned}
E_0(K, \chi_1, \dots, \chi_K) &\geq S(\chi, \Omega \setminus \overline{B_s}) P(\chi, \Omega \setminus \overline{B_s}) + S(\chi) P(\chi, B_s) \\
&\geq S(\chi, \Omega \setminus \overline{B_s}) P(\chi, \Omega \setminus \overline{B_s}) + (K-1) \frac{C}{\sqrt{|\chi_M|}} P(\chi, B_s). \qquad (54)
\end{aligned}$$

Then taking the difference between the inequalities (53) and (54) we get

$$\begin{aligned}
E_0(K, \Lambda_1, \dots, \Lambda_K) &- E_0(K, \chi_1, \dots, \chi_K) \\
&\leq \alpha s + \beta s^2 + \frac{2\pi r^2}{|\chi_m|} S(\chi, \Omega \setminus \overline{B_s}) P(\chi, \Omega \setminus \overline{B_s}) - \eta P(\chi, B_s) \\
&\leq \alpha s + \beta s^2 + \gamma r^2 - \eta P(\chi, B_s),
\end{aligned}$$

where $\gamma = 4\pi M^2 / |\chi_m|^2$ and $\eta = C(K-1)/\sqrt{|\chi_M|}$.

Since $\{\chi_1, \dots, \chi_K\}$ is a minimizer of E we have $E(K, \Lambda_1, \dots, \Lambda_K) - E(K, \chi_1, \dots, \chi_K) \geq 0$, so that arguing as in the proof of Lemma 5.1 (Equation (29) and Step 3), we obtain

$$\eta P(\chi, B_s) \leq \alpha s + \beta s^2 + \gamma r^2 + \delta s^2,$$

where $\delta = 2\pi(\|u_o\|^2_{L^\infty(\Omega)} + c_1^2)$. Since $s \in (0, r)$ and $r < 1$, we have

$$P(\chi, B_s) \leq a_2 r, \qquad a_2 = (\alpha + \beta + \gamma + \delta)/\eta,$$

and the statement of the lemma follows by letting $s \to r$. $\square$

Let now $A \subset \Omega$ be an open set; we define the functionals

$$\begin{aligned}
I(\chi, A) &= \inf_{\{\Lambda_1, \dots, \Lambda_K\}} \left\{ \sum_{i=1}^{K} q_i P(\Lambda_i, A) : \ \Lambda_i \setminus C = \chi_i \setminus C \ \forall i, \ C \subset A \text{ compact} \right\}, \\
\Psi(\chi, A) &= -I(\chi, A) + \sum_{i=1}^{K} q_i P(\chi_i, A),
\end{aligned}$$

where the coefficients q_i have been defined in (15), and the infimum in $I(\chi, A)$ is taken over the families of sets $\{\Lambda_1, \dots, \Lambda_K\}$ of finite perimeter which define a partition of A.

Lemma 6.3. *Let $x \in \Omega$, $0 < r < 1$ and $B_r = B_r(x) \subset\subset \Omega$. Then the following estimate holds:*

$$\Psi(\chi, B_r) \leq a_3 r^2,$$

where a_3 is a positive constant independent of r.

Proof. For any η such that $0 < \eta < 1$ there exists a partition $\{\Lambda_1, \dots, \Lambda_K\}$ of Ω in sets of finite perimeter such that

$$\Lambda_i \setminus C = \chi_i \setminus C \quad \forall i \in 1, \dots, K, \qquad C \subset B_r \text{ compact set}, \tag{55}$$

and

$$\sum_{i=1}^{K} q_i P(\Lambda_i, B_r) \leq I(\chi, B_r) + \eta r. \tag{56}$$

Using Lemma 6.2 we have

$$I(\chi, B_r) \leq \sum_{i=1}^{K} q_i P(\chi_i, B_r) \leq 2q_M P(\chi, B_r) \leq 2q_M \ a_2 \ r,$$

where $q_M = \max_{i=1,\dots,K} q_i$. Then, using (56) it follows

$$P(\Lambda, B_r) = \frac{1}{2} \sum_{i=1}^{K} P(\Lambda_i, B_r) \leq \alpha_1 r, \tag{57}$$

where $\alpha_1 = \dfrac{\eta + 2q_M a_2}{2q_m}$ and $q_m = \min_{i=1,\dots,K} q_i$. Using inequality (42) of Lemma 6.1 we have:

$$\begin{aligned} \sum_{i=1}^{K} q_i P(\chi_i, B_r) \quad &\leq \quad \sum_{i=1}^{K} \left[q_i - \varphi(0) - \psi_i(r)\right] P(\chi_i, B_r) + \sum_{i=1}^{K} q_i P(\Lambda_i, B_r) \\ &+ \quad \sum_{i=1}^{K} \left[\varphi(r) + (1 + b_i r^2)\psi_i(r) - q_i\right] P(\Lambda_i, B_r) + a_1 r^2. \end{aligned} \tag{58}$$

Using the expression of $\psi_i(r)$ in Lemma 6.1, (49), and Lemma 6.2, we get

$$q_i - \varphi(0) - \psi_i(r) = \frac{1}{|\chi_i|} P(\chi) - \frac{1}{|\chi_i|} P(\chi, \Omega \setminus B_r) = \frac{1}{|\chi_i|} P(\chi, B_r) \leq \frac{a_2}{|\chi_m|} r.$$

Using again Lemma 6.2 we obtain

$$\sum_{i=1}^{K} \left[q_i - \varphi(0) - \psi_i(r)\right] P(\chi_i, B_r) \leq \frac{2a_2^2}{|\chi_m|} r^2. \tag{59}$$

Using (55) and the expressions of $\varphi(r)$ and $\psi_i(r)$ in Lemma 6.1, we have

$$\begin{aligned}
\varphi(r) \quad + & \quad (1+b_i r^2)\psi_i(r) - q_i \\
= & \quad \frac{1}{2}\sum_{j=1}^{K}\frac{P(\Lambda_j)}{|\chi_j|} - \frac{1}{2}S(\chi) + \frac{1}{|\chi_i|}P(\chi,\Omega\setminus B_r) - \frac{1}{|\chi_i|}P(\chi) \\
& \quad +r^2\left[\frac{1}{2}\sum_{j=1}^{K} b_j \frac{P(\Lambda_j)}{|\chi_j|} + b_i\psi_i(r)\right] \\
= & \quad \frac{1}{2}\sum_{j=1}^{K}\left(\frac{P(\Lambda_j)}{|\chi_j|} - \frac{P(\chi_j)}{|\chi_j|}\right) - \frac{1}{|\chi_i|}P(\chi,B_r) \\
& \quad +r^2\left[\sum_{j=1}^{K}\frac{\pi}{|\chi_j|}\frac{P(\chi_j,\Omega\setminus B_r)+P(\Lambda_j,B_r)}{|\chi_j|} + \frac{2\pi}{|\chi_i|^2}P(\chi,\Omega\setminus B_r)\right] \\
\le & \quad \frac{1}{|\chi_m|}P(\Lambda,B_r) + \frac{2\pi}{|\chi_m|^2}(2P(\chi,\Omega\setminus B_r)+P(\Lambda,B_r))r^2,
\end{aligned}$$

from which, using $P(\chi)\le M$, the inequality (57), and taking into account that $0<r<1$, it follows

$$\varphi(r) + (1+b_i r^2)\psi_i(r) - q_i \le \frac{1}{|\chi_m|}(1+\frac{2\pi r^2}{|\chi_m|})\alpha_1 r + \frac{4\pi M}{|\chi_m|^2}r^2 \le \alpha\, r,$$

where

$$\alpha = \frac{\alpha_1}{|\chi_m|}\left(1+\frac{2\pi}{|\chi_m|}\right) + \frac{4\pi M}{|\chi_m|^2}.$$

Then, using again the inequality (57), we obtain

$$\sum_{i=1}^{K}\left[\varphi(r) + (1+b_i r^2)\psi_i(r) - q_i\right]P(\Lambda_i,B_r) \le 2\alpha\alpha_1\, r^2. \tag{60}$$

Substituting (59) and (60) into (58) we have

$$\sum_{i=1}^{K} q_i P(\chi_i,B_r) \le \sum_{i=1}^{K} q_i P(\Lambda_i,B_r) + a_3 r^2 \quad \text{with} \quad a_3 = a_1 + \frac{2a_2^2}{|\chi_m|} + 2\alpha\,\alpha_1.$$

Then, using (56) we get

$$\sum_{i=1}^{K} q_i P(\chi_i,B_r) \le I(\chi,B_r) + \eta r + a_3 r^2,$$

from which it follows

$$\Psi(\chi,B_r) \le \eta r + a_3 r^2.$$

Since η is arbitrary, the statement of the lemma follows by letting $\eta\to 0^+$. $\square$

In the following we consider the blow-up of partitions: for $\varepsilon > 0$ and $A \subset \mathbb{R}^2$, we define

$$A_\varepsilon = \left\{ x \in \mathbb{R}^2 : \varepsilon x \in A \right\}. \tag{61}$$

For the partition $\chi = \{\chi_1, \ldots, \chi_K\}$ in the same way we define $\chi_\varepsilon = \{\chi_{1\varepsilon}, \ldots, \chi_{K\varepsilon}\}$. Then for any open set $A \subset \mathbb{R}^2$ we have

$$P(\chi_\varepsilon, A_\varepsilon) = \frac{1}{\varepsilon} P(\chi, A), \quad \Psi(\chi_\varepsilon, A_\varepsilon) = \frac{1}{\varepsilon} \Psi(\chi, A), \tag{62}$$

moreover if $B_\delta \subset A$, then $B_t \subset A_\varepsilon$ for every $t > 0$ and $\varepsilon \in (0, \delta/t)$.

The proof of the following proposition is essentially the same as in Theorem 7 of [16].

Proposition 6.4. (Blow-up) *Let $x \in \Omega$ and $B_1 = B_1(x)$, and let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero as $h \to +\infty$. Let $\chi_{\varepsilon_h} = \{\chi_{1\varepsilon_h}, \ldots, \chi_{K\varepsilon_h}\}$ be the corresponding sequence of families of dilated sets. Then the following properties hold.*

(i) *There exists a family of sets of finite perimeter $\chi^\infty = \{\chi_1^\infty, \ldots, \chi_K^\infty\}$ which defines a partition of B_1 such that, up to the extraction of a subsequence, we have*

$$\chi_{i\varepsilon_h} \to \chi_i^\infty \quad \text{in } L^1(B_1), \qquad \forall i \in \{1, \ldots, K\},$$

as h tends to infinity;

(ii) *for any $i = 1, \ldots, K$, if the set $\partial\chi_i^\infty \cap B_1$ is not empty, then it is the intersection of B_1 with a finite number of half-lines issuing from the center point x.*

Proof. Using (62) and Lemma 6.2 we have

$$\sum_{i=1}^K q_i P(\chi_{i\varepsilon_h}, B_1) \leq 2q_M P(\chi_{\varepsilon_h}, B_1) = \frac{2q_M}{\varepsilon_h} P(\chi, B_{\varepsilon_h}) \leq 2q_M\, a_2,$$

from which the compactness property (i) follows as in Proposition 3.1. Moreover, by the lower semicontinuity of the perimeter we have $P(\chi_i^\infty, B_1) < +\infty$ for any $i = 1, \ldots, K$.

The proof of property (ii) follows from Lemma 6.3 and (62) exactly in the same way as in the proof of Theorem 7 of [16]: it is enough to replace everywhere in the proof

$$\text{the perimeter} \quad \frac{1}{2} \sum_{i=1}^K P(\chi_i, B_{\varepsilon_h}) \qquad \text{with} \qquad \sum_{i=1}^K q_i P(\chi_i, B_{\varepsilon_h}),$$

and to replace

$$\text{the perimeter} \quad \frac{1}{2} \sum_{i=1}^K P(\chi_i^\infty, B_1) \qquad \text{with} \qquad \sum_{i=1}^K q_i P(\chi_i^\infty, B_1).$$

Then, for almost all $r, s \in (0,1)$ we find

$$\sum_{i=1}^{K} \left(\int_{\partial B_1} |1_{\chi_i^\infty}(r\omega) - 1_{\chi_i^\infty}(s\omega)| d\mathcal{H}^1(\omega) \right)^2 = 0.$$

Hence each characteristic function $1_{\chi_i^\infty}$ is homogeneous of degree 0, so that, taking into account that $P(\chi_i^\infty, B_1) < +\infty$ for any i, property (ii) follows. $\square$

7. Regularity properties of optimal segmentations

The following theorem shows that the boundaries of an optimal segmentation are constituted by smooth curves, except possibly for a singular set having null one-dimensional Hausdorff measure. Moreover, the singular set is locally finite. The proof of the theorem follows from the results of sections 5 and 6 as in [16, 19]. For the sake of completeness we give here the proof since it is short.

Theorem 7.1. (Regularity of boundaries) *The set $\Gamma = \cup_{i=1}^{K}(\partial\chi_i \cap \Omega)$ has the following properties: $\Gamma = \Gamma_{\rm reg} \cup \Gamma_{\rm sing}$, where $\Gamma_{\rm reg}$ is a curve of class $C^{1,1/2}$ in Ω and $\mathcal{H}^1(\Gamma_{\rm sing}) = 0$.*

Moreover, for any compact subset $C \subset \Omega$ the set $\Gamma_{\rm sing} \cap C$ is a finite set of points.

Proof. Let $j, l \in 1, \ldots, K$, with $j \neq l$, and let $x \in \chi_j(1/2) \cap \chi_l(1/2) \cap \Omega$. Using (5) and Lemma 5.1, there exists $R \in (0, R_0)$ such that $|\Sigma \cap B_R(x)| \leq \sigma R^2$, so that $\left|\Sigma \cap B_{R/2}(x)\right| = 0$. Then we have

$$\partial^*\chi_i \cap B_{R/2}(x) = \emptyset \quad \forall i \neq j, l, \tag{63}$$

from which, for any $s < R/2$ it follows

$$\sum_{i=1}^{K} q_i P(\chi_i, B_s(x)) = (q_j + q_l) P(\chi_j, B_s(x)) = (q_j + q_l) P(\chi_l, B_s(x)).$$

Using Lemma 6.3 we find

$$(q_j + q_l) P(\chi_j, B_s(x)) = I(\chi, B_s(x)) + \Psi(\chi, B_s(x)) \leq I(\chi, B_s(x)) + a_3 s^2. \tag{64}$$

Let now $\{\Lambda_1, \ldots, \Lambda_K\}$ be a partition of $B_s(x)$ such that $\Lambda_i \cap B_s(x) = \emptyset$ for any $i \neq j, l$, and

$$\Lambda_j \setminus C = \chi_j \setminus C, \quad \Lambda_l \setminus C = \chi_l \setminus C, \qquad C \subset B_s(x) \text{ compact set.} \tag{65}$$

Then it follows

$$I(\chi, B_s(x)) \leq (q_j + q_l) P(\Lambda_j, B_s(x)),$$

from which, using (64), for any pair of sets Λ_j, Λ_l satisfying (65) we have

$$P(\chi_j, B_s(x)) \leq P(\Lambda_j, B_s(x)) + c_3 s^2,$$

where $c_3 = a_3/(q_j+q_l)$. We deduce from Theorem 1 of [23] that $\partial\chi_j \cap B_{R/2}(x) = \partial\chi_l \cap B_{R/2}(x)$ is a curve of class $C^{1,1/2}$. Then $x \in \Gamma_{\mathrm{reg}}$, and from property (11) of partitions in sets of finite perimeter it follows that $\mathcal{H}^1(\Gamma_{\mathrm{sing}}) = 0$.

Let now $C \subset \Omega$ be a compact set and let us assume that $\Gamma_{\mathrm{sing}} \cap C$ is not a finite set of points. Then there exist a point $x \in C$ and a sequence of points $\{x_h\}_h \subseteq \Gamma_{\mathrm{sing}} \cap C$ converging to x. Let $B_1 = B_1(x)$ and $t \in (0,1)$; for any $h \in \mathbb{N}$ we set

$$\varepsilon_h = \frac{|x_h - x|}{t}, \qquad y_h = x + \frac{x_h - x}{|x_h - x|}\, t,$$

so that $y_h \in \partial B_t(x)$ for any h. Then there exists $y \in \partial B_t(x)$ such that the sequence $\{y_h\}_h$ converges to y as $h \to +\infty$, up to the extraction of a subsequence.

Let $\{\chi_{1\varepsilon_h}, \ldots, \chi_{K\varepsilon_h}\}$ be the sequence of families of sets defined in Proposition 6.4; according to property (i) of Proposition 6.4 such a sequence converges to $\{\chi_1^\infty, \ldots, \chi_K^\infty\}$ in $[L^1(B_1)]^K$. Using property (ii) of Proposition 6.4 there exist $\rho > 0$ and $j, l \in \{1, \ldots, K\}$ such that $B_\rho(y) \subset B_1$ and

$$\Sigma^\infty = \bigcup_{i=1, i\notin\{j,l\}}^{K} \chi_i^\infty, \qquad \Sigma^\infty \cap B_\rho(y) = \emptyset.$$

Then, for any $\sigma > 0$ and for h large enough we have

$$\Sigma^h = \bigcup_{i=1, i\notin\{j,l\}}^{K} \chi_{i\varepsilon_h}, \qquad \left|\Sigma^h \cap B_\rho(y)\right| \le \sigma\rho^2.$$

Using (61) we find

$$\Sigma = \bigcup_{i=1, i\notin\{j,l\}}^{K} \chi_i, \qquad \left|\Sigma \cap B_{\varepsilon_h\rho}(x + \varepsilon_h y - \varepsilon_h x)\right| \le \sigma\,(\varepsilon_h\rho)^2.$$

Then, using Lemma 5.1 for h large enough we have

$$\left|\Sigma \cap B_{\varepsilon_h\rho/2}(x + \varepsilon_h y - \varepsilon_h x)\right| = 0,$$

from which it follows $\left|\Sigma^h \cap B_{\rho/2}(y)\right| = 0$.

For h large enough $y_h \in B_{\rho/2}(y)$, so that y_h is an exterior point of $\chi_{i\varepsilon_h}$ for any $i \ne j, l$. Hence x_h is an exterior point of χ_i for any $i \ne j, l$.

Then there exists $\delta > 0$ small enough such that $\partial^*\chi_i \cap B_\delta(x_h) = \emptyset$ for any $i \ne j, l$, so that, arguing again as in the proof after formula (63), it follows that either x_h is an interior point of one of the sets χ_j, χ_l, or $x_h \in \Gamma_{\mathrm{reg}}$. Hence $x_h \notin \Gamma_{\mathrm{sing}}$ and we have a contradiction. We conclude that $\Gamma_{\mathrm{sing}} \cap C$ is a finite set of points. □

Eventually, the statement of the main result, Theorem 2.2, follows collecting the results stated in Theorem 5.3 and Theorem 7.1.

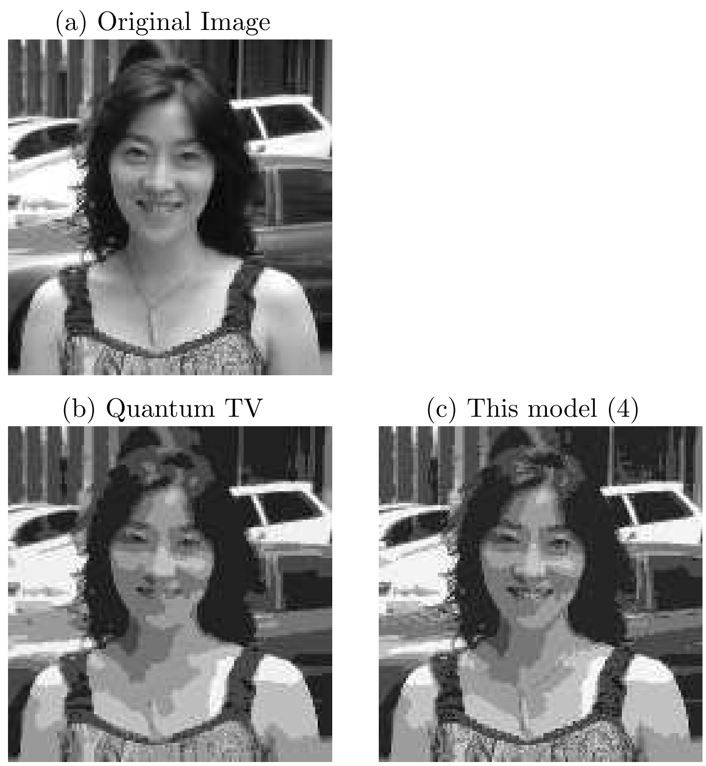

Figure 6: Figures from [20]. The original image (a) is automatically segmented to six phases in image (c). Image (b) is using Total Variation based quantization [21]. The result in (c) keeps more finer details, especially the necklace.

8. Concluding Remarks

We explored the analytical properties of the variational model (4). Following the existence of minimizers, we showed this model does not allow corners in an optimal segmentation, hence corners are smoothed. Yet, differently from Mumford-Shah, the model allows different angles for multiple junctions. In [20], it is also noticed by numerical experiments that the recovered boundaries are more detailed compared to models involving only the total length of the boundaries such as [4, 19]. Figure 6 shows an application to image quantization showing sharper details. Though the model (4) prefers to smooth corners and denoise the image, yet, if the image datum u_o can be approximated by means of a phase with a big area, more details can be kept and more oscillations in the boundaries are allowed.

Considering the computation of optimal angles in Section 4, the difference in energy (23) implies that the bigger the area of a phase is, the bigger that phase tries to become. Therefore, the minimum of the model (4) becomes a balance among these big phases, trying to fit the image datum u_o. In [20], the stopping criterion of the method of discrete numerical minimization (when to stop adding new phases), also has similar terms as in (22):

$$\mu \left\{ (S(\Lambda) - S(\chi))P(\chi) + S(\Lambda)(P(\Lambda) - P(\chi)) \right\} (1 - 1/n_l) < (u_o - c_l)^2,$$

where c_l is the mean value of u_o in the phase χ_l, and n_l is the number of pixels in such a phase. A new phase is created inside the phase χ_l only if the inequality is satisfied. This formula shows the effects of scale term and total length term together. It only allows the new region to be created when the image datum is significantly non-homogeneous, enough to overcome increasing the total length and handle the scale change among all the phases. Once each phase is big enough, it gets harder to add new phases, which gives the automatic stopping of the algorithm and determines the number of phases K.

On the other hand, in the proof of the regularity results, we have also proved that the boundaries are guaranteed to be not very complicated: boundaries are smooth curves, except possibly a singular set of points which is locally finite, i.e., a result similar to that of Mumford-Shah functional. Nevertheless, there is a difference in the properties of blow-up of the two functionals, which is due to the presence of the weights q_i which were defined in (15). Indeed, consider a ball of radius ε centered at a point of an optimal Γ. Arguing as in Section 4, the value of ε at which the energy is locally minimized by cutting a corner, or modifying a junction, may have to be smaller compared to Mumford-Shah model. In this case details of the boundaries can be recovered at a finer spatial scale, in agreement with numerical experiments.

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