Analysis of a temperature-dependent model for adhesive contact with friction

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Abstract

We propose a model for (unilateral) contact with adhesion between a viscoelastic body and a rigid support, encompassing thermal *and* frictional effects. Following Frémond's approach, adhesion is described in terms of a surface damage parameter χ. The related equations are the momentum balance for the vector of small displacements, and parabolic-type evolution equations for χ and for the absolute temperatures of the body and of the adhesive substance on the contact surface. All of the constraints on the internal variables, as well as the contact and the friction conditions, are rendered by means of subdifferential operators. Furthermore, the temperature equations, derived from an *entropy* balance law, feature singular functions. Therefore, the resulting PDE system has a highly nonlinear character.

The main result of the paper states the existence of global-in-time solutions to the associated Cauchy problem. It is proved by passing to the limit in a carefully tailored approximate problem, via variational techniques.

Key words: contact, adhesion, friction, thermoviscoelasticity, entropy balance.

AMS (MOS) Subject Classification: 35K55, 74A15, 74M15.

1 Introduction

In this paper we propose and analyze a PDE system modelling thermal effects in adhesive contact with friction. More specifically, we focus on the phenomenon of contact between a thermoviscoelastic body and a *rigid* support. We suppose that the body adheres to the support on a part of its boundary.

Following M. Frémond's approach [25, 26], we describe adhesion in terms of a surface damage parameter χ (and its gradient $\nabla\chi$), related to the state of the bonds responsible for the adherence of the body to the support. Further, we consider small displacements $\mathbf{u}$ and possibly different temperatures in the body and on the contact surface. Internal constraints, such as unilateral conditions, are ensured by the presence of non-smooth monotone operators, also generalizing the Signorini conditions for unilateral contact to the case when adhesion is active. Frictional effects are encompassed in the model through a regularization of the well-known Coulomb law, here generalized to the case of adhesive contact and assuming thermal dependence of the friction coefficient.

The mathematics of (unilateral) contact problems, possibly with friction, has received notable attention lately, as attested, among others, by the recent monographs on the topic [22] and [31]. The analysis of Frémond's model for contact with adhesion has been developed over the last years in a series of papers: in [7]–[8] we have focused on existence, uniqueness, and long-time behavior

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results on the isothermal system for (frictionless) adhesive contact. Thermal effects both on the contact surface, and in the bulk domain, have been included in the later papers [9]–[10], respectively tackling the well-posedness and large-time behavior analysis. In [11] we have dealt with a (isothermal) unilateral contact problem taking into account both friction and adhesion. As far as we know, the present contribution is the first one addressing a model which combines adhesion, frictional contact, and thermal effects.

The PDE system. Before stating the PDE system under investigation, let us fix the notation for the normal and tangential component of vectors and tensors that will be used in what follows.

Notation 1.1 Given a vector $\mathbf{v} \in \mathbb{R}^3$, we denote by v_{N} and $\mathbf{v}_{\mathrm{T}}$ its normal component and its tangential part, defined on the contact surface by

$$v_{\mathrm{N}} := \mathbf{v} \cdot \mathbf{n}, \quad \mathbf{v}_{\mathrm{T}} := \mathbf{v} - v_{\mathrm{N}}\mathbf{n}, \tag{1.1}$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary. Analogously, the normal component and the tangential part of the stress tensor $\boldsymbol{\sigma}$ are denoted by σ_{N} and $\boldsymbol{\sigma}_{\mathrm{T}}$, and defined by

$$\sigma_{\mathrm{N}} := \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n}, \quad \boldsymbol{\sigma}_{\mathrm{T}} := \boldsymbol{\sigma}\mathbf{n} - \sigma_{\mathrm{N}}\mathbf{n}. \tag{1.2}$$

We address the PDE system

$$\partial_t(\ln(\vartheta)) - \mathrm{div}(\partial_t\mathbf{u}) - \Delta\vartheta = h \qquad \text{in } \Omega \times (0,T), \tag{1.3}$$

$$\partial_{\mathbf{n}}\vartheta = \begin{cases} 0 & \text{in } (\partial\Omega \setminus \Gamma_c) \times (0,T), \\ -k(\chi)(\vartheta - \vartheta_s) - \mathfrak{c}'(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t\mathbf{u}_{\mathrm{T}}| & \text{in } \Gamma_c \times (0,T), \end{cases} \tag{1.4}$$

$$\partial_t(\ln(\vartheta_s)) - \partial_t(\lambda(\chi)) - \Delta\vartheta_s = k(\chi)(\vartheta - \vartheta_s) + \mathfrak{c}'(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t\mathbf{u}_{\mathrm{T}}| \quad \text{in } \Gamma_c \times (0,T), \tag{1.5}$$

$$\partial_{\mathbf{n}_s}\vartheta_s = 0 \qquad \text{in } \partial\Gamma_c \times (0,T), \tag{1.6}$$

$$-\mathrm{div}\boldsymbol{\sigma} = \mathbf{f} \text{ with } \boldsymbol{\sigma} = K_e\varepsilon(\mathbf{u}) + K_v\varepsilon(\partial_t\mathbf{u}) + \vartheta\mathbf{1} \quad \text{in } \Omega \times (0,T), \tag{1.7}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_1 \times (0,T), \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_2 \times (0,T), \tag{1.8}$$

$$\sigma_{\mathrm{N}} \in -\chi u_{\mathrm{N}} - \partial I_{(-\infty,0]}(u_{\mathrm{N}}) \quad \text{in } \Gamma_c \times (0,T), \tag{1.9}$$

$$\boldsymbol{\sigma}_{\mathrm{T}} \in -\chi\mathbf{u}_{\mathrm{T}} - \mathfrak{c}(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\mathrm{N}})\mathbf{d}(\partial_t\mathbf{u}) \quad \text{in } \Gamma_c \times (0,T), \tag{1.10}$$

$$\partial_t\chi + \delta\partial I_{(-\infty,0]}(\partial_t\chi) - \Delta\chi + \partial I_{[0,1]}(\chi) + \sigma'(\chi) \ni -\lambda'(\chi)(\vartheta_s - \vartheta_{\mathrm{eq}}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_c \times (0,T), \tag{1.11}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{in } \partial\Gamma_c \times (0,T) \tag{1.12}$$

with $\delta \geq 0$, where

$$\mathbf{d}(\mathbf{v}) = \begin{cases} \frac{\mathbf{v}_{\mathrm{T}}}{|\mathbf{v}_{\mathrm{T}}|} & \text{if } \mathbf{v}_{\mathrm{T}} \neq \mathbf{0} \\ \{\mathbf{w}_{\mathrm{T}} \; : \; \mathbf{w} \in \overline{B}_1\} & \text{if } \mathbf{v}_{\mathrm{T}} = \mathbf{0}, \end{cases} \tag{1.13}$$

and $\overline{B}_1$ is the closed unit ball in $\mathbb{R}^3$. Note that $\mathbf{d} : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ is the subdifferential of the function $j : \mathbb{R}^3 \to [0,+\infty)$ defined by $j(\mathbf{v}) = |\mathbf{v}_{\mathrm{T}}|$, cf. (2.5) later on. In what follows, Ω is a (sufficiently smooth) bounded domain in $\mathbb{R}^3$, in which the body is located, with $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_c$ and Γ_c the contact surface between the body and the rigid support. From now on, we will suppose that Γ_{c} is a smooth bounded domain of $\mathbb{R}^2$ (one may think of a flat surface), and we denote by $\mathbf{n}_s$ the outward unit normal vector to $\partial\Gamma_{\mathrm{c}}$. Let us point out that $\mathbf{u}$ is the vector of small displacements, $\boldsymbol{\sigma}$ the stress tensor, whereas χ is the so-called *adhesion parameter*, which denotes the fraction of active microscopic bonds on the contact surface. As for the thermal variables, ϑ is the absolute temperature of the body Ω whereas ϑ_s is the absolute temperature of the adhesive substance on Γ_{c}. Here and in what follows, we shall write v, in place of $v|_{\Gamma_c}$, for the trace on Γ_c of a function v defined in Ω.

While postponing to Section 2 the rigorous derivation of the PDE system (1.3–1.12) based on the laws of Thermomechanics, let us briefly comment on the nonlinearities therein involved. First, we observe that the singular terms $\ln(\vartheta)$ and $\ln(\vartheta_s)$ in (1.3) and (1.5) force the temperatures ϑ and ϑ_s

to remain strictly positive, which is necessary to get thermodynamical consistency. The multivalued operator $\partial I_C : \mathbb{R} \rightrightarrows \mathbb{R}$ (with C the interval $[0,1]$ or the half-line $(-\infty, 0]$)) is the subdifferential (in the sense of convex analysis) of the indicator function of the convex set C. We recall that $\partial I_C(y) \neq \emptyset$ if and only if $y \in C$, with $\partial I_C(y) = \{0\}$ if y is in the interior of C, while $\partial I_C(y)$ is the normal cone to the boundary of C if $y \in \partial C$. By means of indicator functions, we enforce the unilateral condition $u_{\mathrm{N}} \leq 0$, as well as the constraint $\chi \in [0,1]$ and, if $\delta > 0$, the irreversibility of the damage process, viz. $\partial_t \chi \leq 0$. Then, the subdifferential terms occurring in (1.4), (1.5), (1.9), (1.10), and (1.11) represent internal forces which activate to prevent u_{N}, χ, and $\partial_t \chi$ from taking values outside the physically admissible range. Finally, K_e and K_v are positive-definite tensors, k, λ, σ, and $\mathfrak{c}$ are sufficiently smooth functions, whereas ϑ_{eq} is a critical temperature and h, $\mathbf{f}$, and $\mathbf{g}$ are given data: we refer to Section 2 for all details on their physical meaning, and to Section 3.2 for the precise assumptions on them.

As for conditions (1.9)–(1.10), let us observe that (1.9) can be rephrased as

$$u_{\mathrm{N}} \leq 0, \quad \sigma_{\mathrm{N}} + \chi u_{\mathrm{N}} \leq 0, \quad u_{\mathrm{N}}(\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}) = 0 \quad \text{in } \Gamma_c \times (0,T), \tag{1.14}$$

which, in the case $\chi = 0$, reduce to the classical Signorini conditions for unilateral contact. Conversely, when $0 < \chi \leq 1$, σ_{N} can be positive, namely the action of the adhesive substance on Γ_c prevents separation when a tension is applied. Moreover, in view of (1.9), (1.10) can be expressed by

$$\begin{aligned}
&|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| \leq \mathfrak{c}(\vartheta - \vartheta_s)|\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}|, \\
&|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| < \mathfrak{c}(\vartheta - \vartheta_s)|\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}| \Longrightarrow \partial_t \mathbf{u}_{\mathrm{T}} = \mathbf{0}, \\
&|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| = \mathfrak{c}(\vartheta - \vartheta_s)|\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}| \Longrightarrow \exists\, \nu \geq 0 : \ \partial_t \mathbf{u}_{\mathrm{T}} = -\nu(\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}),
\end{aligned} \tag{1.15}$$

which generalize the *dry friction* Coulomb law, to the case when adhesion effects are taken into account. Note that the *positive* function $\mathfrak{c}$ in (1.15) is the *friction coefficient.*

Related literature. We refer to [22] and [31], for a general survey of the literature on models of (unilateral) contact, as well as to the references in [7, 8] for *isothermal* models of contact with adhesion, possibly with friction (see also [11]). Here we will just focus on (a partial review of) temperature-dependent models for frictional contact: as previously said, to the best of our knowledge the PDE system (1.3–1.12) is the first one modelling *adhesive contact* with frictional *and* thermal effects.

The major difficulty in the analysis of unilateral contact problems with friction is the presence of constraints on both the (normal) displacement u_{N} and the tangential velocity $\partial_t \mathbf{u}_{\mathrm{T}}$. It can be overcome only by resorting to suitable simplifications in the related PDE systems. Over the years, several options have bee explored in this direction, without affecting the physical consistency of the underlying models.

Since Duvaut's pioneering work [20], a commonly accepted approximation of the dry friction Coulomb law (1.15), which we will also adopt, involves the usage of a nonlocal regularizing operator $\mathcal{R}$, cf. (1.16) below. An alternative regularization of the classical Coulomb law is the so-called *SJK-Coulomb* law of friction, which is for example considered in [28] and in [1], dealing with *quasistatic* models for thermoviscoelastic contact with friction.

An alternative possibility is to replace the Signorini contact conditions (1.14) with a *normal compliance* condition, which allows for the interpenetration of the surface asperities and thus for dispensing with the unilateral constraint on u_{N}. Analytically, the normal compliance law corresponds to a penalization of the subdifferential operator $\partial I_{(-\infty,0]}$ in (1.9)–(1.10). In this connection, we refer e.g. to [2], analyzing a *dynamic* model for frictional contact of a thermoviscoelastic body with a rigid obstacle, with the power law normal compliance condition for unilateral contact, and the corresponding generalization of Coulomb's law of dry friction. Unilateral contact is modelled by a normal compliance condition in [3] as well, where a dynamic contact problem for a thermoviscoelastic body, with frictional and wear effects on the contact surface, is investigated. A wide class of dynamic frictional contact problems in thermoelasticity and thermoviscoelasticity is also tackled in [24], with contact rendered by means of a normal compliance law.

In an extensive series of papers (cf. the references in the monograph [22]), C. Eck & J. Jarušek developed a different approach, which enabled them to prove existence results for dynamic contact problems, coupling *dry* friction and Signorini contact, *without* recurring to any regularizing operator. However, they used a different form of Signorini conditions, expressed not in terms of $\mathbf{u}$ but of $\partial_t\mathbf{u}$. They observed that this different law can be interpreted as a first-order approximation with respect to the time variable, realistic for a short time interval and for a vanishing initial gap between the body and the obstacle, and hence it is physically interesting. Within this modelling approach, existence results for contact problems with friction and thermal effects were for instance obtained in [23] and in [21].

Let us stress that, in all of these contributions the existence of solutions is proved for a weak formulation of the related PDE systems, which involves a variational inequality, and not an evolutionary differential inclusion, for the displacement.

Analytical difficulties. From now on, we will take $\delta = 0$ and thus confine ourselves to the investigation of the *reversible* case. We postpone the analysis of the *irreversible* case $\delta > 0$ to the forthcoming paper [12], where we will address also slightly different equations for ϑ and ϑ_s obtained by a different choice of the bulk and surface entropy fluxes (here given by $-\nabla\vartheta$ and $-\nabla\vartheta_s$, respectively). For more details, we refer to the upcoming Remark 2.2.

Still, the analysis of system (1.3–1.12) is fraught with obstacles:

1) First of all, the highly nonlinear character of the equations, due to the presence of several singular and multivalued operators. In particular, since Coulomb friction is included in the model, a multivalued operator occurs even in the coupling terms between the equations for ϑ, ϑ_s, and $\mathbf{u}$, and (1.10) features the product of two subdifferentials.
2) A second analytical difficulty is given by the coupling of *bulk* and *(contact) surface* equations, which requires sufficient regularity of the bulk variables ϑ and $\mathbf{u}$ for their traces on Γ_{c} to make sense.
3) In turn, the mixed boundary conditions on $\mathbf{u}$ do not allow for elliptic regularity estimates which could significantly enhance the spatial regularity of $\mathbf{u}$ and $\partial_t\mathbf{u}$. In particular, we are not in the position to obtain the $H^2(\Omega;\mathbb{R}^3)$-regularity for $\mathbf{u}$ and $\partial_t\mathbf{u}$. The same problem arises for ϑ, due to the third type boundary condition (1.4).
4) Additionally, the temperature equations (1.3) and (1.5) need to be carefully handled because of the singular character of the terms $\partial_t \ln(\vartheta)$ and $\partial_t \ln(\vartheta_s)$ therein occurring. In particular, they do not allow but for poor time-regularity of ϑ and ϑ_s.

Let us stress that the presence of several multivalued operators in system (1.3–1.12) is due to the fact that, all the constraints on the internal variables, as well as the unilateral contact conditions and the friction law, are rendered by means of subdifferential operators. Therefore, in our opinion the resulting formulation of the problem provides more complete information than those formulations based on variational inequalities. Indeed, it enables us to clearly identify the internal forces and reactions which derive by the enforcement of the physical constraints.

In particular, let us dwell on the coupling of unilateral contact and the dry friction Coulomb law: as already mentioned, it introduces severe mathematical difficulties, unresolved even in the case without adhesion. These problems are mainly related to the fact that we cannot control σ_{N} and $\boldsymbol{\sigma}_{\mathrm{T}}$ in (1.9) and (1.10) pointwise, due to the presence of two different non-smooth operators. For this reason, as done in [11] and following Duvaut's work [20], we are led to regularize (1.10) by resorting to a *nonlocal version* of the Coulomb law. The idea is to replace the nonlinearity in (1.10) involving friction by the term

$$\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\mathbf{d}(\partial_t\mathbf{u}). \tag{1.16}$$

Accordingly, the coupling term $\mathfrak{c}'(\vartheta-\vartheta_s)\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t\mathbf{u}_{\mathrm{T}}|$ in (1.4) and (1.5) is replaced by

$$\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))||\partial_t\mathbf{u}_{\mathrm{T}}|. \tag{1.17}$$

In (1.16) and (1.17), $\mathcal{R}$ is a regularization operator, taking into account nonlocal interactions on the contact surface. We refer to [11] for further comments and references to several items in the literature on frictional contact problems, where this regularization is adopted. In the forthcoming Remark 2.3, we hint at the modeling derivation of the PDE system (1.3–1.12) with the terms (1.16) and (1.17), while in Example 3.2 below we give the construction of an operator $\mathcal{R}$ complying with the conditions we will need to impose for our existence result.

The difficulties attached to the coupling between thermal and frictional effects are apparent in the dependence of the friction coefficient $\mathfrak{c}$ on the thermal gap $\vartheta - \vartheta_s$. In order to deal with this, and to tackle the (passage to the limit in the approximation of the) terms (1.16) and (1.17), it is crucial to prove strong compactness for (the sequences approximating) ϑ and ϑ_s in suitable spaces. A key step in this direction is the derivation of an estimate for ϑ in $\mathrm{BV}(0,T;W^{1,3+\epsilon}(\Omega)')$ and for ϑ_s in $\mathrm{BV}(0,T;W^{1,2+\epsilon}(\Gamma_c)')$ for some $\epsilon > 0$. Combining this information with a suitable version of the Lions-Aubin compactness theorem generalized to BV spaces (see, e.g., [29]), we will deduce the desired compactness for (the sequences approximating) ϑ and ϑ_s.

Our results. The main result of this paper (cf. Theorem 1 in Section 3.3) states the existence of (global-in-time) solutions to the Cauchy problem for system (1.3–1.12).

In order to prove it, we introduce a carefully tailored approximate problem for system (1.3–1.12), in which we replace some of the nonlinearities therein involved by suitable Moreau-Yosida-type regularizations. The existence of solutions for the approximate problem is obtained via a Schauder fixed point argument, yielding a local-in-time existence result. The latter is combined with a series of *global* a priori estimates, holding with constants independent of the regularization parameter, which enable us to extend the local solution to a global one. These very same estimates are then exploited in the passage to the limit in the approximate problem, together with techniques from maximal monotone operator theory and refined compactness results. In this way, we carry out the proof of Thm. 1.

Plan of the paper. In Section 2 we outline the rigorous derivation of system (1.3–1.12) and we comment on its thermodynamical consistency. Hence, in Section 3, after enlisting all of the assumptions on the nonlinearities featured in (1.3–1.12) and on the problem data, we set up the variational formulation of the problem and state our main existence result (cf. Theorem 1 below). Section 4 is devoted to the proof of Thm. 4.1, ensuring the existence of (global-in-time) solutions to the approximate problem. Finally, in Section 5 we pass to the limit with the approximation parameter and conclude the proof of Thm. 1.

2 The model

In this section we sketch the modeling approach underlying the derivation of our phase-transition type system for a contact problem, in which adhesion and friction are taken into account in a non-isothermal framework. The equations, written in the bulk domain and on the contact surface, are recovered by the general laws of Thermomechanics with energies and dissipation potentials defined in Ω and on Γ_c. They can be considered as balance equations, based on a generalization of the principle of virtual powers. Such a generalization accounts for micro-movements and micro-forces, which are responsible for the breaking of the adhesive bonds on the contact surface. We will not derive the PDE system (1.3–1.12) in full detail, referring the reader to the discussions in [9] on the modeling of thermal effects in contact with adhesion, and in [11] for a (isothermal) model of adhesive contact with friction. Here, we will rather briefly outline the main ingredients of the derivation, and just focus on the modeling novelty of this paper.

The phenomenon of adhesive contact is described by state and dissipative variables, defining the equilibrium and the evolution of the system, respectively. The state variables are the symmetric strain $\varepsilon(\mathbf{u})$, the trace of the vector of small displacements $\mathbf{u}$ on the contact surface, the surface adhesion parameter χ, its gradient $\nabla\chi$, and the absolute temperatures ϑ and ϑ_s of the bulk domain and of the contact surface, respectively. The evolution is derived in terms of a pseudo-potential of dissipation, depending on the dissipative variables $\partial_t \mathbf{u}$, $\partial_t \chi$, $\nabla\vartheta$, $\nabla\vartheta_s$, and the thermal gap on the contact surface $\vartheta - \vartheta_s$.

The free energy and the dissipation potential. The free energy of our system is written as follows

$$\Psi = \Psi_\Omega + \Psi_\Gamma,$$

Ψ_Ω being defined in Ω and Ψ_Γ on Γ_c. The bulk contribution Ψ_Ω is given by

$$\Psi_\Omega = \vartheta(1 - \ln(\vartheta)) + \vartheta \mathrm{tr}(\varepsilon(\mathbf{u})) + \frac{1}{2}\varepsilon(\mathbf{u})K_e\varepsilon(\mathbf{u}), \tag{2.1}$$

with K_e the elasticity tensor and $\mathrm{tr}(\varepsilon(\mathbf{u}))$ the trace of $\varepsilon(\mathbf{u})$. Note that, for the sake of simplicity, we have taken both the specific heat and the thermal expansion coefficient equal to 1. The free energy on the contact surface is defined by

$$\begin{aligned}\Psi_\Gamma =&\vartheta_s(1 - \ln(\vartheta_s)) + \lambda(\chi)(\vartheta_s - \vartheta_{\mathrm{eq}}) + I_{[0,1]}(\chi) + \sigma(\chi) \\ &+ \frac{c_{\mathrm{N}}}{2}\chi(u_{\mathrm{N}})^2 + \frac{c_{\mathrm{T}}}{2}\chi|\mathbf{u}_{\mathrm{T}}|^2 + I_{(-\infty,0]}(u_{\mathrm{N}}) + \frac{\kappa_s}{2}|\nabla\chi|^2,\end{aligned} \tag{2.2}$$

where c_{N}, c_{T}, κ_s are positive constants. Note that c_{N} and c_{T} (which are the adhesive coefficients for the normal and tangential components, respectively) a priori may be different, due to possible anisotropy in the response of the material to stresses. However, for the sake of simplicity in what follows we let $c_{\mathrm{N}} = c_{\mathrm{T}} = \kappa_s = 1$. In (2.2), σ is a sufficiently smooth function accounting for some internal properties of the adhesive substance on Γ_c, such as cohesion: the simplest form for the cohesive contribution to the energy is $\sigma(\chi) = w_s(1-\chi)$ for some $w_s > 0$. Let us now briefly comment on the internal constraints. The energy is defined for any value of the state variables, but it is set equal to $+\infty$ if the variables assume values which are not physically consistent. Indeed, the indicator function $I_{(-\infty,0]}$ enforces the internal constraint $u_{\mathrm{N}} \leq 0$, i.e. it renders the impenetrability condition between the body and the support. Finally, the term $I_{[0,1]}(\chi)$ forces χ to take values in the interval $[0,1]$. The function λ provides the latent heat λ', while ϑ_{eq} is a critical temperature, which governs the evolution of the cohesion of the adhesive substance with respect to the temperature.

As far as dissipation is concerned, we consider in particular dissipative effects on the boundary due to friction. The pseudo-potential is

$$\Phi = \Phi_\Omega + \Phi_\Gamma$$

Φ_Ω being defined in Ω and Φ_Γ on Γ_c. For the bulk contribution Φ_Ω we have

$$\Phi_\Omega := \frac{1}{2}\varepsilon(\partial_t\mathbf{u})K_v\varepsilon(\partial_t\mathbf{u}) + \frac{\alpha(\vartheta)}{2}|\nabla\vartheta|^2, \tag{2.3}$$

while the contact surface contribution Φ_Γ reads

$$\begin{aligned}\Phi_\Gamma :=\, &\mathfrak{c}(\vartheta - \vartheta_s)\,|-R_N + u_{\mathrm{N}}\chi|\, j(\partial_t\mathbf{u}) + \frac{1}{2}|\partial_t\chi|^2 \\ &+ \delta I_{(-\infty,0]}(\partial_t\chi) + \frac{\alpha_s(\vartheta_s)}{2}|\nabla\vartheta_s|^2 + \frac{1}{2}k(\chi)(\vartheta - \vartheta_s)^2,\end{aligned} \tag{2.4}$$

where the positive function $\mathfrak{c}$ has the meaning of a friction coefficient, R_N will be specified later (see (2.15)), the function j is

$$j(\mathbf{v}) = |\mathbf{v}_{\mathrm{T}}| \quad \text{for all } \mathbf{v} \in \mathbb{R}^3. \tag{2.5}$$

$\delta \geq 0$, and α and α_s are the thermal diffusion coefficients in the bulk domain and on the contact surface, respectively. The positive (and sufficiently smooth) function k is also a contact surface thermal diffusion coefficient, accounting for the heat exchange between the body and the adhesive substance on Γ_c. The assumptions on all of these functions have to guarantee that the pseudo-potential of dissipation is a non-negative function, convex w.r.t. the dissipative variables (note that

ϑ and ϑ_s are not dissipative variables), and that dissipation is zero once the dissipative variables are equal to zero (see also the upcoming discussion on the *thermodynamical consistency* of the model). Observe that, if $\delta > 0$, the model encompasses an *irreversible* evolution for the damage-type variable χ, as it enforces $\partial_t \chi \leq 0$. Furthermore, the 1-homogeneity of the function j in (2.5) reflects the rate-independent character of frictional dissipation.

Remark 2.1 Note that the functions σ and λ in the free energy (2.2) are related to the cohesion of the adhesive substance, as it results from (1.11). Indeed, the term $-\sigma'(\chi) - \lambda'(\chi)(\vartheta_s - \vartheta_{\rm eq})$ is a (generalized) cohesion of the material, depending on the temperature. It represents a threshold for $\frac{1}{2}|\mathbf{u}|^2$ to produce damage.

The main novelty of this paper in comparison with [11] is the fact that the friction coefficient $\mathfrak{c}$ depends on the thermal gap $\vartheta - \vartheta_s$ (cf. [31, p. 12]). This relies on the modeling ansatz that *friction generates heat* (cf. [23]), so that ultimately we have the contribution $\mathfrak{c}'(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\rm N})|\partial_t \mathbf{u}_{\rm T}|$ as a source of heat on the contact surface $\Gamma_{\rm c}$, both in the boundary condition (1.4) for ϑ and in equation (1.5) for ϑ_s.

The balance equations. The equations for the evolution of the temperature variables are recovered from entropy balance equations in Ω and Γ_c (cf. [15] for a detailed motivation of this approach and [9] for the application to contact problems), i.e.

$$\partial_t s + \operatorname{div} \mathbf{Q} = h \text{ in } \Omega \times (0,T), \quad \mathbf{Q} \cdot \mathbf{n} = F \text{ on } \partial\Omega \times (0,T) \tag{2.6}$$

with h an external volume source, F the entropy flux through the boundary, s the entropy, $\mathbf{Q}$ the entropy flux in the bulk domain, and

$$\partial_t s_s + \operatorname{div} \mathbf{Q}_s = F \text{ in } \Gamma_c \times (0,T), \quad \mathbf{Q}_s \cdot \mathbf{n}_s = 0 \text{ on } \partial\Gamma_c \times (0,T), \tag{2.7}$$

where we have denoted by s_s and $\mathbf{Q}_s$ the entropy and the entropy flux on the contact surface, respectively. Note that, on Γ_c the term F (involved in the boundary condition for (2.6)) is the entropy provided by the adhesive substance to the body. We couple (2.6) and (2.7) with the generalized momentum equations for macroscopic motions in Ω and micro-movements in Γ_c, viz.

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} \text{ in } \Omega \times (0,T), \tag{2.8}$$

$$\boldsymbol{\sigma}\mathbf{n} = -\mathbf{R} \text{ on } \Gamma_c \times (0,T), \quad \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \times (0,T), \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_2 \times (0,T) \tag{2.9}$$

where recall that $\boldsymbol{\sigma}$ is the macroscopic stress tensor, $\mathbf{f}$ is a volume applied force, $\mathbf{g}$ is a known traction, $-\mathbf{R}$ is the action of the obstacle on the solid, and

$$B - \operatorname{div} \mathbf{H} = 0 \text{ in } \Gamma_c \times (0,T), \quad \mathbf{H} \cdot \mathbf{n}_s = 0 \text{ on } \partial\Gamma_c \times (0,T), \tag{2.10}$$

with $\mathbf{H}$ and B microscopic internal stresses, responsible for the damage of the adhesive bonds between the body and the support.

The constitutive equations. To recover the PDE system (1.3–1.12), we have to combine (2.6)–(2.7) with suitable constitutive relations, for the involved physical quantities, in terms of Ψ and Φ. We have

$$s = -\frac{\partial \Psi_\Omega}{\partial \vartheta}, \quad s_s = -\frac{\partial \Psi_\Gamma}{\partial \vartheta_s} \tag{2.11}$$

and

$$\mathbf{Q} = -\frac{\partial \Phi_\Omega}{\partial \nabla \vartheta}, \qquad \mathbf{Q}_s = -\frac{\partial \Phi_\Gamma}{\partial \nabla \vartheta_s}. \tag{2.12}$$

As for as the flux through the boundary F, we impose

$$F = \frac{\partial \Phi_\Gamma}{\partial(\vartheta - \vartheta_s)} \text{ on } \Gamma_c, \quad F = 0 \text{ on the remaning part.} \tag{2.13}$$

Then, we prescribe for (the dissipative and non-dissipative contributions to) the stress tensor

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{nd} + \boldsymbol{\sigma}^{d} = \frac{\partial \Psi_\Omega}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \Phi_\Omega}{\partial \varepsilon(\partial_t \mathbf{u})}, \tag{2.14}$$

and the reaction $\mathbf{R} = R_N \mathbf{n} + \mathbf{R}_T$ reads

$$R_N = R_N^{nd} = \frac{\partial \Psi_\Gamma}{\partial u_{\mathrm{N}}} \tag{2.15}$$

$$\mathbf{R}_T = \mathbf{R}_T^{nd} + \mathbf{R}_T^{d} = \frac{\partial \Psi_\Gamma}{\partial \mathbf{u}_{\mathrm{T}}} + \frac{\partial \Phi_\Gamma}{\partial(\partial_t \mathbf{u}_{\mathrm{T}})}. \tag{2.16}$$

Finally, B and $\mathbf{H}$ are given by

$$B = B^{nd} + B^{d} = \frac{\partial \Psi_\Gamma}{\partial \chi} + \frac{\partial \Phi_\Gamma}{\partial(\partial_t \chi)}, \tag{2.17}$$

$$\mathbf{H} = \mathbf{H}^{nd} + \mathbf{H}^{d} = \frac{\partial \Psi_\Gamma}{\partial \nabla \chi} + \frac{\partial \Phi_\Gamma}{\partial(\nabla(\partial_t \chi))}. \tag{2.18}$$

Remark 2.2 Let us briefly comment on the entropy flux laws for $\mathbf{Q}$ and $\mathbf{Q}_s$: as shown by (2.3) and (2.4), they depend on the choice of the thermal diffusion coefficients α and α_s. In general the latter are functions of the temperature variables ϑ and ϑ_s (cf. e.g. [22]). If $\alpha(\vartheta) = \alpha > 0$ we get $\mathbf{Q} = -\alpha\nabla\vartheta$. In this case, for the heat flux $\mathbf{q} = \vartheta\mathbf{Q}$ we obtain $\mathbf{q} = -\alpha\vartheta\nabla\vartheta = -\frac{\alpha}{2}\nabla\vartheta^2$. Analogously, if $\alpha(\vartheta) = \frac{1}{\vartheta}$ we get $\mathbf{Q} = -\nabla\ln(\vartheta)$ and for $\mathbf{q}$ the standard Fourier law $\mathbf{q} = -\nabla\vartheta$. Hence, in the case, e.g., $\alpha(\vartheta) = \vartheta$ (which is admissible for thermodynamics) we have $\mathbf{Q} = -\vartheta\nabla\vartheta = -\frac{1}{2}\nabla\vartheta^2$. Analogous considerations hold for $\mathbf{Q}_s$.

In what follows, we fix as heat flux laws

$$\alpha(\vartheta) \equiv \alpha = 1, \quad \alpha_s(\vartheta_s) \equiv \alpha_s = 1. \tag{2.19}$$

Furthermore, as already mentioned, we will confine our analysis to the case of a *reversible* evolution for the damage-type variable χ, taking

$$\delta = 0 \text{ in } \Phi_\Gamma.$$

With these choices, combining (2.11)–(2.18) with (2.6)–(2.10) we derive the PDE system (1.3–1.12).

Remark 2.3 Let us point out that, the PDE system tackled in this paper, featuring the non-local regularization for the Coulomb law $\mathcal{R}$, can be derived following the very same procedure described above. Indeed, it is sufficient to replace Φ_Γ by

$$\begin{aligned}\Phi_\Gamma := \mathfrak{c}(\vartheta - \vartheta_s)\,|\mathcal{R}(-R_N + u_{\mathrm{N}}\chi)|\, j(\partial_t \mathbf{u}) + \frac{1}{2}|\partial_t \chi|^2 \\ + \delta I_{(-\infty,0]}(\partial_t \chi) + \frac{\alpha_s(\vartheta_s)}{2}|\nabla\vartheta_s|^2 + \frac{1}{2}k(\chi)(\vartheta - \vartheta_s)^2.\end{aligned} \tag{2.20}$$

Thermodynamical consistency. Let us now briefly comment on the derivation of the model in relation to its thermodynamical consistency. In particular we discuss the validity of the Clausius-Duhem inequality for the whole thermomechanical system we are dealing with. The latter consists of the body Ω and the contact adhesive surface Γ_c, encompassing the dissipative heat exchange between the body and the adhesive substance on Γ_c, due to the difference of the temperatures. Recall that we have derived the evolution equations for the temperature variables from an entropy balance in the domain Ω and on Γ_c. To prove thermodynamical consistency (which corresponds to showing that dissipation is positive), we have to discuss the entropy balance for the whole system, i.e. also including the interactions between the bulk domain and the adhesive substance.

As far as the domain Ω is concerned, this property is fairly standard (the reader may refer to [26]): it is ensured once Φ_Ω has the characteristics of a pseudo-potential of dissipation. In particular, this requires that, in (2.3) the tensor K_v is positive definite, and the thermal diffusion coefficient α is non-negative. This leads to positivity of the dissipation, viz.

$$\varepsilon(\partial_t \mathbf{u}) K_v \varepsilon(\partial_t \mathbf{u}) + \alpha(\vartheta)|\nabla\vartheta|^2 \geq 0.$$

Then, the Clausius-Duhem inequality is obtained by writing the energy balance and exploiting the already specified constitutive relations.

Second, the internal energy balance on Γ_c is recovered by the first principle, in which the effective power of the internal forces responsible for the damage process is included. Again using the constitutive relations, it yields

$$\theta_s \partial_t s_s + \theta_s \mathrm{div}\, \mathbf{Q}_s = \theta_s Q_s + B^d \partial_t \chi + \mathbf{H}^d \cdot \nabla \partial_t \chi - \mathbf{Q}_s \cdot \nabla \theta_s, \tag{2.21}$$

where we denote by Q_s the entropy received by the adhesive substance from the solid and $B^d, \mathbf{H}^d, \mathbf{Q}_s$ are defined by (2.17), (2.18), and (2.12). Note that Q_s actually represents an entropy source. In our model we have prescribed that (see (2.6)–(2.7))

$$-Q = Q_s = F, \tag{2.22}$$

Q denoting the entropy received by the solid from the adhesive substance. Note that by (2.6) we have

$$\mathbf{Q} \cdot \mathbf{n} + Q = 0. \tag{2.23}$$

To deduce the conditions on F which ensure the thermodynamical consistency, we write an entropy equality on the contact surface, accounting for the interaction between the body and the adhesive substance, i.e.

$$\bar{\vartheta} \partial_t s_{int} = -Q\vartheta - Q_s \vartheta_s, \tag{2.24}$$

where s_{int} is the entropy exchange, defined in terms of the variable $\bar{\vartheta} = \frac{1}{2}(\vartheta + \vartheta_s)$ (for any further details see [26]). Now, the Clausius-Duhem inequality on the contact surface (combining the adhesive substance and its interaction with the body) reads

$$\partial_t s_s + \mathrm{div}\, \mathbf{Q}_s + \partial_t s_{int} \geq \mathbf{Q} \cdot \mathbf{n}. \tag{2.25}$$

Thus, exploiting (2.21), (2.22), (2.23), and (2.24), inequality (2.25) follows once it is ensured that (recall that the absolute temperatures are strictly positive)

$$F(\theta - \theta_s) + B^d \partial_t \chi + \mathbf{H}^d \cdot \nabla \partial_t \chi - \mathbf{Q}_s \cdot \nabla \theta_s \geq 0.$$

Observe that this corresponds to prescribing in (2.4) that

$$\alpha_s \geq 0, \quad \mathfrak{c} \geq 0, \quad k \geq 0, \quad \mathfrak{c}'(\vartheta - \vartheta_s)(\vartheta - \vartheta_s) \geq 0.$$

Indeed, all of the above conditions are ensured by (2.19) and by the forthcoming Hypotheses (I) and (V) on $\mathfrak{c}$ and k.

Remark 2.4 Note that, in the case no heat is exchanged between the body and the adhesive substance, the Clausius-Duhem inequality on the contact surface is granted upon requiring that

$$B^d \partial_t \chi + \mathbf{H}^d \cdot \nabla \partial_t \chi - \mathbf{Q}_s \cdot \nabla \theta_s \geq 0.$$

This follows from (2.21) taking $Q_s = 0$.

3 Main results

3.1 Setup

Throughout the paper we shall assume that

$$\begin{gathered}\Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^3, \text{ with}\\ \partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_c, \quad \Gamma_i,\ i = 1,2,c, \text{ open disjoint subsets in the relative topology of } \partial\Omega, \text{ such that}\\ \mathcal{H}^2(\Gamma_1),\, \mathcal{H}^2(\Gamma_c) > 0, \qquad \text{and } \Gamma_c \subset \mathbb{R}^2 \text{ a sufficiently smooth } \textit{flat} \text{ domain.}\end{gathered} \tag{3.1}$$

More precisely, by *flat* we mean that Γ_c is a subset of a hyperplane of $\mathbb{R}^2$ and $\mathcal{H}^2(\Gamma_c) = \mathcal{L}^2(\Gamma_c)$, $\mathcal{L}^d$ and $\mathcal{H}^d$ denoting the d-dimensional Lebesgue and Hausdorff measures. As for smoothness, we require that Γ_c has a C^2-boundary: thanks to, e.g., [27, Thm. 8.12], this will justify the elliptic regularity estimates we will perform on the solution component χ.

Notation 3.1 Given a Banach space X, we denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between its dual space X' and X itself and by $\|\cdot\|_X$ the norm in X. In particular, we shall use the following short-hand notation for function spaces

$$\begin{gathered} H := L^2(\Omega), \quad V := H^1(\Omega), \quad \mathbf{H} := L^2(\Omega;\mathbb{R}^3), \quad \mathbf{V} := H^1(\Omega;\mathbb{R}^3),\\ H_{\Gamma_c} := L^2(\Gamma_c), \quad V_{\Gamma_c} := H^1(\Gamma_c), \quad Y_{\Gamma_c} := H^{1/2}_{00,\Gamma_1}(\Gamma_c),\\ \mathbf{W} := \{\mathbf{v} \in \mathbf{V} \ :\ \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\}, \quad \mathbf{H}_{\Gamma_c} := L^2(\Gamma_c;\mathbb{R}^3), \quad \mathbf{Y}_{\Gamma_c} := H^{1/2}_{00,\Gamma_1}(\Gamma_c;\mathbb{R}^3), \end{gathered}$$

where we recall that

$$H^{1/2}_{00,\Gamma_1}(\Gamma_c) = \left\{ w \in H^{1/2}(\Gamma_c) \ :\ \exists\, \tilde{w} \in H^{1/2}(\Gamma) \text{ with } \tilde{w} = w \text{ in } \Gamma_c,\ \tilde{w} = 0 \text{ in } \Gamma_1 \right\}$$

and $H^{1/2}_{00,\Gamma_1}(\Gamma_c;\mathbb{R}^3)$ is analogously defined. We will also use the space $H^{1/2}_{00,\Gamma_1}(\Gamma_2;\mathbb{R}^3)$. The space $\mathbf{W}$ is endowed with the natural norm induced by $\mathbf{V}$. We will make use of the operator

$$A : V_{\Gamma_c} \to V'_{\Gamma_c} \qquad \langle A\chi, w\rangle_{V_{\Gamma_c}} := \int_{\Gamma_c} \nabla\chi\, \nabla w \,\mathrm{d}x \quad \text{for all } \chi,\, w \in V_{\Gamma_c} \tag{3.2}$$

and of the notation

$$m(w) := \frac{1}{\mathcal{L}^d(A)} \int_A w \,\mathrm{d}x \quad \text{for } w \in L^1(A). \tag{3.3}$$

Useful inequalities. We are going to exploit the following trace results and continuous embeddings

$$V \subset L^4(\Gamma_\mathrm{c}), \qquad \mathbf{W} \subset L^4(\Gamma_\mathrm{c};\mathbb{R}^3), \qquad V_{\Gamma_c} \subset L^q(\Gamma_\mathrm{c}) \text{ for all } 1 \le q < \infty, \tag{3.4}$$

and the fact that, by Poincaré's inequality, for every Lipschitz domain $A \subset \mathbb{R}^d$

$$\exists\, C > 0\ \forall v \in W^{1,2}(A)\ : \qquad \|v\|_{W^{1,2}(A)} \le C(\|v\|_{L^1(A)} + \|\nabla v\|_{L^2(A)}), \tag{3.5}$$

where $\|v\|_{L^1(A)}$ can be replaced by $|m(v)|$.

Linear viscoelasticity. We recall the definition of the standard bilinear forms of linear viscoelasticity, which are involved in the variational formulation of equation (1.7). Dealing with an anisotropic and inhomogeneous material, we assume that the fourth-order tensors $K_e = (a_{ijkh})$ and $K_v = (b_{ijkh})$, denoting the elasticity and the viscosity tensor, respectively, satisfy the classical symmetry and ellipticity conditions

$$\begin{aligned} & a_{ijkh} = a_{jikh} = a_{khij}\,, \qquad b_{ijkh} = b_{jikh} = b_{khij} \quad \text{for } i,j,k,h = 1,2,3\\ & \exists\, \alpha_0 > 0\ : \qquad a_{ijkh}\xi_{ij}\xi_{kh} \ge \alpha_0 \xi_{ij}\xi_{ij} \quad \forall\, \xi_{ij} : \xi_{ij} = \xi_{ji} \quad \text{for } i,j = 1,2,3\,,\\ & \exists\, \beta_0 > 0\ : \qquad b_{ijkh}\xi_{ij}\xi_{kh} \ge \beta_0 \xi_{ij}\xi_{ij} \quad \forall\, \xi_{ij} : \xi_{ij} = \xi_{ji} \quad \text{for } i,j = 1,2,3\,, \end{aligned}$$

where the usual summation convention is used. Moreover, we require

$$a_{ijkh}, b_{ijkh} \in L^\infty(\Omega), \quad i,j,k,h = 1,2,3.$$

By the previous assumptions on the elasticity and viscosity coefficients, the following bilinear forms $a, b : \mathbf{W} \times \mathbf{W} \to \mathbb{R}$, defined by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega a_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{W},$$
$$b(\mathbf{u},\mathbf{v}) := \int_\Omega b_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{W},$$

turn out to be continuous and symmetric. In particular, we have

$$\exists\,\bar{C} > 0 : \ |a(\mathbf{u},\mathbf{v})| + |b(\mathbf{u},\mathbf{v})| \le \bar{C}\|\mathbf{u}\|_{\mathbf{W}}\|\mathbf{v}\|_{\mathbf{W}} \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{W}. \tag{3.6}$$

Moreover, since Γ_1 has positive measure, by Korn's inequality we deduce that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are $\mathbf{W}$-elliptic, i.e., there exist $C_a, C_b > 0$ such that

$$a(\mathbf{u},\mathbf{u}) \ge C_a\|\mathbf{u}\|^2_{\mathbf{W}} \qquad \text{for all } \mathbf{u} \in \mathbf{W}, \tag{3.7}$$
$$b(\mathbf{u},\mathbf{u}) \ge C_b\|\mathbf{u}\|^2_{\mathbf{W}} \qquad \text{for all } \mathbf{u} \in \mathbf{W}. \tag{3.8}$$

3.2 Assumptions

We specify all of the assumptions on the nonlinearities in system (1.3–1.12).

Hypothesis (I). As for the *friction coefficient* $\mathfrak{c} : \mathbb{R} \to (0,+\infty)$, we require that

$$\mathfrak{c} \in \mathrm{C}^1(\mathbb{R}), \qquad \exists\, c_1, c_2 > 0\ \forall\, x \in \mathbb{R} : \ \mathfrak{c}(x) \ge c_1, \ \ |\mathfrak{c}'(x)| \le c_2, \qquad \mathfrak{c}'(x)x \ge 0. \tag{3.9}$$

For instance, the function $\mathfrak{c}(x) = \int_0^x \arctan(t)\,\mathrm{d}t + c_1 = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c_1$ complies with (3.9).

Hypothesis (II). We generalize the operator $\partial I_{(-\infty,0]}$ by replacing $I_{(-\infty,0]}$ in (2.2) with a function

$$\phi : \mathbb{R} \to [0,+\infty] \ \text{ proper, convex and lower semicontinuous, with } \phi(0) = 0 \tag{3.10}$$

and effective domain $\mathrm{dom}(\phi)$; let us emphasize that the *physical case*, in which the constraint $u_{\mathrm{N}} \le 0$ on Γ_{c} is enforced, occurs when $\mathrm{dom}(\phi) \subset (-\infty,0]$ (and it is included in our analysis). Then, we define

$$\varphi : Y_{\Gamma_c} \to [0,+\infty] \ \text{ by } \ \varphi(v) := \begin{cases} \int_{\Gamma_c} \phi(v)\,\mathrm{d}x & \text{if } \phi(v) \in L^1(\Gamma_c), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.11}$$

Hence, we introduce

$$\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_c} \to [0,+\infty], \quad \text{defined by} \ \ \boldsymbol{\varphi}(\mathbf{u}) := \varphi(u_{\mathrm{N}}) \quad \text{for all } \mathbf{u} \in \mathbf{Y}_{\Gamma_c}. \tag{3.12}$$

Since $\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_c} \to [0,+\infty]$ is a proper, convex and lower semicontinuous functional on $\mathbf{Y}_{\Gamma_c}$, its subdifferential $\partial\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_c} \rightrightarrows \mathbf{Y}'_{\Gamma_c}$ is a maximal monotone operator.

Hypothesis (III). Concerning the regularizing operator $\mathcal{R}$, in the lines of [11] we require that there exists $\nu > 0$ such that

$$\begin{aligned} &\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_c}) \to L^\infty(0,T;L^{2+\nu}(\Gamma_{\mathrm{c}};\mathbb{R}^3)) \text{ is weakly-strongly continuous, viz.} \\ &\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \ \text{ in } L^2(0,T;\mathbf{Y}'_{\Gamma_c}) \quad \Rightarrow \quad \mathcal{R}(\boldsymbol{\eta}_n) \to \mathcal{R}(\boldsymbol{\eta}) \ \text{ in } L^\infty(0,T;L^{2+\nu}(\Gamma_{\mathrm{c}};\mathbb{R}^3)) \end{aligned} \tag{3.13}$$

for all $(\boldsymbol{\eta}_n)$, $\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}'_{\Gamma_c})$. It is not difficult to check that (3.13) implies that $\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_c}) \to L^\infty(0,T;L^{2+\nu}(\Gamma_{\mathrm{c}};\mathbb{R}^3))$ is bounded. In Example 3.2 at the end of this section we are going to exhibit an operator $\mathcal{R}$ complying with (3.13).

Hypothesis (IV). We generalize $\partial I_{[0,1]}$ in (1.11) by considering the multivalued operator $\partial\widehat{\beta}$: $[0,+\infty) \rightrightarrows \mathbb{R}$, with

$$\begin{gathered}\widehat{\beta}:\mathbb{R}\to(-\infty,+\infty] \text{ proper, convex and lower semicontinuous,}\\ \text{such that } \operatorname{dom}(\widehat{\beta})\subset[0,+\infty).\end{gathered} \tag{H4}$$

In what follows, we use the notation $\beta := \partial\widehat{\beta}$.

Hypothesis (V). We assume that the nonlinearities k in (1.4)–(1.5), λ in (1.5) and (1.11), and σ in (1.11) fulfill

$$k:\mathbb{R}\to[0,+\infty) \text{ is Lipschitz continuous,} \tag{3.14}$$

$$\lambda\in \mathrm{C}^1(\mathbb{R}), \text{ with } \lambda':\mathbb{R}\to\mathbb{R} \text{ Lipschitz continuous,} \tag{3.15}$$

$$\sigma\in \mathrm{C}^1(\mathbb{R}), \text{ with } \sigma':\mathbb{R}\to\mathbb{R} \text{ Lipschitz continuous.} \tag{3.16}$$

Assumptions on the problem and on the initial data. We suppose that

$$h\in L^2(0,T;V')\cap L^1(0,T;H)\,, \tag{3.17}$$

$$\mathbf{f}\in L^2(0,T;\mathbf{W}')\,, \tag{3.18}$$

$$\mathbf{g}\in L^2(0,T;H^{1/2}_{00,\Gamma_1}(\Gamma_2;\mathbb{R}^3)'). \tag{3.19}$$

It follows from (3.18)–(3.19) that, the function $\mathbf{F}:(0,T)\to\mathbf{W}'$ defined by

$$\langle\mathbf{F}(t),\mathbf{v}\rangle_{\mathbf{W}} := \langle\mathbf{f}(t),\mathbf{v}\rangle_{\mathbf{W}}+\langle\mathbf{g}(t),\mathbf{v}\rangle_{H^{1/2}_{00,\Gamma_1}(\Gamma_2;\mathbb{R}^3)} \quad \text{for all } \mathbf{v}\in\mathbf{W} \text{ and almost all } t\in(0,T),$$

satisfies

$$\mathbf{F}\in L^2(0,T;\mathbf{W}')\,. \tag{3.20}$$

Finally, we require that the initial data fulfill

$$\vartheta_0\in L^1(\Omega) \quad\text{and}\quad \ln(\vartheta_0)\in H\,, \tag{3.21}$$

$$\vartheta_s^0\in L^1(\Gamma_c) \quad\text{and}\quad \ln(\vartheta_s^0)\in H_{\Gamma_c}\,, \tag{3.22}$$

$$\mathbf{u}_0\in\mathbf{W} \text{ and } \mathbf{u}_0\in\operatorname{dom}(\boldsymbol{\varphi})\,, \tag{3.23}$$

$$\chi_0\in V_{\Gamma_c},\quad \widehat{\beta}(\chi_0)\in L^1(\Gamma_c)\,. \tag{3.24}$$

We conclude with the example, partially mutuated from [11, Ex. 2.4], of an operator $\mathcal{R}$ complying with Hypothesis (III).

Example 3.2 Fix $\ell:\Gamma_c\times\Gamma_c\to\mathbb{R}^3$ such that $\ell\in L^{2+\nu}(\Gamma_c;\mathbf{Y}_{\Gamma_c})$ for some $\nu>0$, and for all $\boldsymbol{\eta}\in L^2(0,T;\mathbf{Y}'_{\Gamma_c})$ set

$$\mathcal{R}(\boldsymbol{\eta})(x,t) := \left(\int_0^t \langle\boldsymbol{\eta}(\cdot,s),\ell(x,\cdot)\rangle_{\mathbf{Y}_{\Gamma_c}}\,\mathrm{d}s\right)\mathbf{w} \quad \text{for a.a. } (x,t)\in\Gamma_c\times(0,T),$$

where $\mathbf{w}$ is a fixed vector, e.g. with unit norm, in $\mathbb{R}^3$. Then, for almost all $(x,t)\in\Gamma_c\times(0,T)$ there holds

$$|\mathcal{R}(\boldsymbol{\eta})(x,t)| \le \int_0^t |\langle\boldsymbol{\eta}(\cdot,s),\ell(x,\cdot)\rangle_{\mathbf{Y}_{\Gamma_c}}|\,\mathrm{d}s \le t^{1/2}\|\ell(x,\cdot)\|_{\mathbf{Y}_{\Gamma_c}}\|\boldsymbol{\eta}\|_{L^2(0,t;\mathbf{Y}'_{\Gamma_c})}. \tag{3.25}$$

Integrating (3.25) over Γ_c, we easily conclude that for all $t\in(0,T)$

$$\|\mathcal{R}(\boldsymbol{\eta})(\cdot,t)\|_{L^{2+\nu}(\Gamma_c;\mathbb{R}^3)} \le T^{1/2}\|\boldsymbol{\eta}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})}\|\ell\|_{L^{2+\nu}(\Gamma_c;\mathbf{Y}_{\Gamma_c})},$$

hence $\mathcal{R}:L^2(0,T;\mathbf{Y}'_{\Gamma_c})\to L^\infty(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))$ is a linear and bounded operator. Furthermore, it fulfills (3.13). Indeed, let $\boldsymbol{\eta}_n\rightharpoonup\boldsymbol{\eta}$ in $L^2(0,T;\mathbf{Y}'_{\Gamma_c})$: for almost all $(x,t)\in\Gamma_c\times(0,T)$ we have

$$\mathcal{R}(\boldsymbol{\eta}_n)(x,t)=\int_0^T\langle\boldsymbol{\eta}_n(\cdot,s),\mathbf{1}_{(0,t)}\ell(x,\cdot)\rangle_{\mathbf{Y}_{\Gamma_c}}\mathbf{w}\,\mathrm{d}s\to\int_0^T\langle\boldsymbol{\eta}(\cdot,s),\mathbf{1}_{(0,t)}\ell(x,\cdot)\rangle_{\mathbf{Y}_{\Gamma_c}}\mathbf{w}\,\mathrm{d}s=\mathcal{R}(\boldsymbol{\eta})(x,t)$$

as $n \to \infty$. Then, estimate (3.25) and the dominated convergence theorem yield $\mathcal{R}(\boldsymbol{\eta}_n) \to \mathcal{R}(\boldsymbol{\eta})$ in $L^q(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))$ for all $1 \le q < \infty$. In order to conclude that $\mathcal{R}(\boldsymbol{\eta}_n) \to \mathcal{R}(\boldsymbol{\eta})$ in $L^\infty(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))$, it is sufficient to observe that the sequence $(\mathcal{R}(\boldsymbol{\eta}_n))_n$ is in fact compact in $C^0([0,T];L^{2+\nu}(\Gamma_c;\mathbb{R}^3))$. This follows from the Ascoli-Arzelà theorem, since, arguing as for (3.25), it is not difficult to see that, if $(\boldsymbol{\eta}_n)_n \subset L^2(0,T;\mathbf{Y}'_{\Gamma_c})$ is bounded, then $(\mathcal{R}(\boldsymbol{\eta}_n))_n$ fulfills the equicontinuity condition for all $0 \le s \le t \le T$

$$\|\mathcal{R}(\boldsymbol{\eta}_n)(\cdot,t) - \mathcal{R}(\boldsymbol{\eta}_n)(\cdot,s)\|_{L^{2+\nu}(\Gamma_c;\mathbb{R}^3)} \le (t-s)^{1/2}\|\boldsymbol{\eta}_n\|_{L^2(s,t;\mathbf{Y}'_{\Gamma_c})}\|\ell\|_{L^{2+\nu}(\Gamma_c;\mathbf{Y}_{\Gamma_c})} \le C(t-s)^{1/2}.$$

3.3 Statement of the main result

We now specify the variational formulation of system (1.3–1.12).

Problem 3.3 Given a quadruple of initial data $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ fulfilling (3.21)–(3.24), find a seven-tuple $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \boldsymbol{\mu}, \xi)$, with

$$\vartheta \in L^2(0,T;V) \cap L^\infty(0,T;L^1(\Omega)), \tag{3.26}$$
$$\ln(\vartheta) \in L^\infty(0,T;H) \cap H^1(0,T;V'), \tag{3.27}$$
$$\vartheta_s \in L^2(0,T;V_{\Gamma_c}) \cap L^\infty(0,T;L^1(\Gamma_c)), \tag{3.28}$$
$$\ln(\vartheta_s) \in L^\infty(0,T;H_{\Gamma_c}) \cap H^1(0,T;V'_{\Gamma_c}), \tag{3.29}$$
$$\mathbf{u} \in H^1(0,T;\mathbf{W}), \tag{3.30}$$
$$\chi \in L^2(0,T;H^2(\Gamma_c)) \cap L^\infty(0,T;V_{\Gamma_c}) \cap H^1(0,T;H_{\Gamma_c}), \tag{3.31}$$
$$\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}'_{\Gamma_c}), \tag{3.32}$$
$$\boldsymbol{\mu} \in L^2(0,T;\mathbf{H}_{\Gamma_c}), \tag{3.33}$$
$$\xi \in L^2(0,T;H_{\Gamma_c}), \tag{3.34}$$

satisfying the initial conditions

$$\vartheta(0) = \vartheta_0 \quad \text{a.e. in } \Omega, \tag{3.35}$$
$$\vartheta_s(0) = \vartheta_s^0 \quad \text{a.e. in } \Gamma_c, \tag{3.36}$$
$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega, \tag{3.37}$$
$$\chi(0) = \chi_0 \quad \text{a.e. in } \Gamma_c, \tag{3.38}$$

and

$$\begin{aligned}\langle \partial_t \ln(\vartheta), v\rangle_V &- \int_\Omega \operatorname{div}(\partial_t \mathbf{u})\, v\, dx + \int_\Omega \nabla\vartheta\,\nabla v\, dx + \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s)v\, dx \\ &+ \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})||\partial_t \mathbf{u}_{\mathrm{T}}|v\, dx = \langle h, v\rangle_V \quad \forall\, v \in V \ \text{ a.e. in } (0,T),\end{aligned} \tag{3.39}$$

$$\begin{aligned}\langle \partial_t \ln(\vartheta_s), v\rangle_{V_{\Gamma_c}} &- \int_{\Gamma_c} \partial_t \lambda(\chi)\, v\, dx + \int_{\Gamma_c} \nabla\vartheta_s\,\nabla v\, dx \\ &= \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s)v\, dx + \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})||\partial_t \mathbf{u}_{\mathrm{T}}|v\, dx \quad \forall\, v \in V_{\Gamma_c} \ \text{ a.e. in } (0,T),\end{aligned} \tag{3.40}$$

$$\begin{aligned}b(\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u},\mathbf{v}) &+ \int_\Omega \vartheta \operatorname{div}(\mathbf{v})\, dx + \int_{\Gamma_c} \chi \mathbf{u}\cdot\mathbf{v}\, dx \\ &+ \langle \boldsymbol{\eta}, \mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}} + \int_{\Gamma_c} \mathfrak{c}(\vartheta-\vartheta_s)\boldsymbol{\mu}\cdot\mathbf{v}\, dx = \langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0,T),\end{aligned} \tag{3.41}$$

$$\boldsymbol{\eta} \in \partial\boldsymbol{\varphi}(\mathbf{u}) \text{ in } \mathbf{Y}'_{\Gamma_c}, \ \text{ a.e. in } (0,T), \tag{3.42}$$
$$\boldsymbol{\mu} = |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \ \text{ with } \ \mathbf{z} \in \mathbf{d}(\partial_t \mathbf{u}) \text{ a.e. in } \Gamma_c \times (0,T), \tag{3.43}$$
$$\partial_t \chi + A\chi + \xi + \sigma'(\chi) = -\lambda'(\chi)\vartheta_s - \frac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_c \times (0,T), \tag{3.44}$$
$$\xi \in \beta(\chi) \text{ a.e. in } \Gamma_c \times (0,T). \tag{3.45}$$

Note that, to simplify notation we have incorporated the contribution $-\lambda'(\chi)\vartheta_{\mathrm{eq}}$ occurring in (1.11) into the term $\sigma'(\chi)$ in (3.44).

Remark 3.4 Observe that the subdifferentials in (3.43) and (3.45) are multivalued operators on $\mathbb{R}^3$ with values in $\mathbb{R}^3$, and on $\mathbb{R}$ with values in $\mathbb{R}$, respectively. Hence the related subdifferential inclusions for $\mathbf{z}$ and ξ hold a.e. in $\Gamma_{\mathrm{c}} \times (0,T)$. Instead, (3.42) features the (abstract) operator $\partial\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_{\mathrm{c}}} \rightrightarrows \mathbf{Y}'_{\Gamma_{\mathrm{c}}}$, thus (3.42) holds in $\mathbf{Y}'_{\Gamma_{\mathrm{c}}}$. Note that, in the case we further assume (for physical consistency) that $\mathrm{dom}\,\phi \subset (-\infty, 0]$, it still follows from the definition (3.12) of $\boldsymbol{\varphi}$ that, if $\mathbf{u} \in \mathrm{dom}(\partial\boldsymbol{\varphi})$, then $\mathbf{u}$ complies with the constraint $u_{\mathrm{N}} \leq 0$ a.e. in $\Gamma_{\mathrm{c}} \times (0,T)$.

We are now in the position to state our existence theorem for Problem 3.3. Observe that we obtain enhanced regularity in time for ϑ and ϑ_s thanks to some refined BV-estimates (cf. Remark 4.10 later on). Relying on (3.13) in Hypothesis (III), we also find (3.48) for $\boldsymbol{\mu}$.

Theorem 1
In the framework of (3.1), *under Hypotheses (I)–(V) and conditions* (3.17)–(3.19) *on the data* h, $\mathbf{f}$, $\mathbf{g}$, *and* (3.21)–(3.24) *on* ϑ_0, ϑ_s^0, $\mathbf{u}_0$, *and* χ_0, *Problem 3.3 admits at least a solution* $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \boldsymbol{\mu}, \xi)$, *which in addition satisfies*

$$\vartheta \in \mathrm{BV}(0,T; W^{1,q}(\Omega)') \text{ for every } q > 3, \tag{3.46}$$

$$\vartheta_s \in \mathrm{BV}(0,T; W^{1,\sigma}(\Gamma_{\mathrm{c}})') \text{ for every } \sigma > 2, \tag{3.47}$$

$$\boldsymbol{\mu} \in L^\infty(0,T; L^{2+\nu}(\Gamma_c; \mathbb{R}^3)) \qquad \text{with } \nu > 0 \text{ from } (3.13). \tag{3.48}$$

The *proof* will be developed throughout Sections 4–5. First, we will analyze a suitable approximation of Problem 3.3, for which we will obtain the existence of solutions in Thm. 4.1. In Section 5 we will then pass to the limit and show that (up to a subsequence) the approximate solutions converge to a solution of Problem 3.3.

Notation 3.5 In what follows, we will denote most of the positive constants occurring in the calculations by the symbols c, c', C, C', whose meaning may vary even within the same line. Furthermore, the symbols I_i, $i = 0, 1, \ldots$, will be used as place-holders for several integral terms popping in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, the symbol I_1 will appear several times with different meanings.

4 Approximation

Here we focus on the approximation of the PDE system (3.39–3.45) through Yosida-type regularization of some of the (maximal monotone) nonlinearities therein involved. For the related definitions and results we refer to the classical monographs [5, 16].

In Sec. 4.1 we justify the regularizations we will perform, giving raise to the approximate Problem (P_ε). The existence of solutions is proved in two steps: first, in Sec. 4.2 we show that Problem (P_ε) admits local-in-time solutions by means of a Schauder fixed point argument. We then extend them to solutions on the whole $[0,T]$ relying on the *global* a priori estimates which we derive in Sec. 4.3. In fact, we will obtain estimates *independent* of the approximation parameter ε: they will be the starting point for the passage to the limit as $\varepsilon \downarrow 0$ developed in Section 5.

4.1 The approximate problem

In order to motivate the way we are going to approximate Problem 3.3, let us discuss in advance some of the *global* a priori estimates to be performed on system (3.39–3.45). Clearly, these estimates correspond to (some of) the summability and regularity properties (3.26)–(3.34) required for solutions to Problem 3.3. As we will see, the related calculations can be developed on system (3.39–3.45) only on a *formal* level: they can be made rigorous by means of the Yosida-type regularizations we will consider (cf. the global a priori estimates on the approximate solutions performed in Section 4.3).

Heuristics for the approximate problem. The basic *energy estimate* for system (3.39–3.45) consists in testing (3.39) by ϑ, (3.40) by ϑ_s, (3.41) by $\partial_t \mathbf{u}$, (3.44) by $\partial_t \chi$, adding the resulting relations, and integrating in time. Taking into account the *formal* identities

$$\begin{aligned} &\int_0^t \langle \partial_t \ln(\vartheta), \vartheta \rangle_V \, \mathrm{d}r = \|\vartheta(t)\|_{L^1(\Omega)} - \|\vartheta_0\|_{L^1(\Omega)} \,, \\ &\int_0^t \langle \partial_t \ln(\vartheta_s), \vartheta_s \rangle_{V_{\Gamma_c}} \, \mathrm{d}r = \|\vartheta_s(t)\|_{L^1(\Gamma_c)} - \|\vartheta_s^0\|_{L^1(\Gamma_c)} \,, \end{aligned} \tag{4.1}$$

this estimate leads, among others, to a bound for ϑ in $L^\infty(0,T;L^1(\Omega))$ and, correspondingly, for ϑ_s in $L^\infty(0,T;L^1(\Gamma_c))$ (cf. (3.26) and (3.28)). As a first step towards making (4.1) rigorous, following [13, 10] we will

1. replace the logarithm $\ln$ in equations (3.39) and (3.40) by its approximation $\mathcal{L}_\varepsilon : \mathbb{R} \to \mathbb{R}$

   $$\mathcal{L}_\varepsilon(r) := \varepsilon r + \ln_\varepsilon(r), \tag{4.2}$$

   where for $\varepsilon > 0$ $\ln_\varepsilon$ denotes the Yosida regularization of the logarithm $\ln$. Therefore, $\mathcal{L}_\varepsilon$ is differentiable, strictly increasing and Lipschitz continuous, see also Lemma 4.2 below.

As pointed out in Remark 4.12 below, this procedure is not sufficient to justify (4.1) completely. In order to do so, following [9, 10] we should also add a viscosity term both in (3.39) and (3.40), modulated by a second parameter ν. We choose to overlook this point for the sake of simplicity. Anyhow, let us stress that approximating the logarithm $\ln$ by $\mathcal{L}_\varepsilon$ makes the test of (3.39) by $\ln(\vartheta)$ rigorous (and, respectively, it justifies the test of (3.40) by $\ln(\vartheta_s)$). This gives raise to an estimate for $\ln(\vartheta)$ in $L^\infty(0,T;H)$ (for $\ln(\vartheta_s)$ in $L^\infty(0,T;H_{\Gamma_c})$, respectively).

A consequence of the aforementioned *energy estimate* and of a comparison argument in the momentum equation (3.41) is the following estimate

$$\|\mathfrak{c}(\vartheta - \vartheta_s)\boldsymbol{\mu} + \boldsymbol{\eta}\|_{L^2(0,T;\mathbf{Y}_{\Gamma_c}')} \le C. \tag{4.3}$$

Using the *formal* observation that $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ are *orthogonal*, from (4.3) one concludes the crucial information

$$\|\mathfrak{c}(\vartheta - \vartheta_s)\boldsymbol{\mu}\|_{L^2(0,T;\mathbf{Y}_{\Gamma_c}')} + \|\boldsymbol{\eta}\|_{L^2(0,T;\mathbf{Y}_{\Gamma_c}')} \le C. \tag{4.4}$$

In order to justify this argument, we need to suitably approximate the maximal monotone operator $\partial\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_c} \rightrightarrows \mathbf{Y}_{\Gamma_c}'$ in such a way as to replace $\boldsymbol{\eta} \in \mathbf{Y}_{\Gamma_c}'$ in (3.41) with a term $\boldsymbol{\eta}_\varepsilon$ having *null tangential component*, cf. (4.5) below. Following [11], we will thus

2. replace the function ϕ, which enters in the definition of the functional $\boldsymbol{\varphi}$ through (3.11) and (3.12), by its Yosida approximation $\phi_\varepsilon : \mathbb{R} \to [0,+\infty)$. We recall that ϕ_ε is convex, differentiable, and such that ϕ_ε' is the Yosida regularization of the subdifferential $\partial\phi : \mathbb{R} \rightrightarrows \mathbb{R}$.

Therefore, in this way we will replace the constraint (3.42) by its regularized version

$$\boldsymbol{\eta}_\varepsilon := \phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n} \quad \text{a.e. in } \Gamma_c \times (0,T). \tag{4.5}$$

Furthermore, we introduce the function $\mathcal{I}_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{I}_\varepsilon(x) := \int_0^x s\mathcal{L}_\varepsilon'(s)\,\mathrm{d}s \tag{4.6}$$

which we will use in the proof of the existence of solutions to Problem (P_ε), and in particular in the derivation of suitable a priori estimates. In the two following lemmas, we collect some useful properties of the functions $\ln_\varepsilon$, $\mathcal{L}_\varepsilon$, and $\mathcal{I}_\varepsilon$.

Lemma 4.1 *The following inequalities hold:*

$$\exists\, \varepsilon_* > 0 : \quad \forall\, \varepsilon \in (0, \varepsilon_*) \quad \forall\, x > 0 \quad \ln'_\varepsilon(x) \le \frac{2}{x}\,, \tag{4.7a}$$

$$\forall\, \varepsilon > 0 \quad \forall\, x \in \mathbb{R} \quad \ln'_\varepsilon(x) \ge \frac{1}{|x| + 2 + \varepsilon}\,. \tag{4.7b}$$

As a consequence, the function $\mathcal{J}_\varepsilon : \mathbb{R} \to \mathbb{R}$ (4.6) *satisfies*

$$\exists\, \varepsilon_* > 0 : \quad \forall\, \varepsilon \in (0, \varepsilon_*) \quad \forall\, x \ge 0 \quad \mathcal{J}_\varepsilon(x) \le \frac{\varepsilon}{2} x^2 + 2x\,; \tag{4.8a}$$

$$\exists\, C_1,\, C_2 > 0 : \quad \forall\, \varepsilon > 0 \quad \forall\, x \in \mathbb{R} \quad \mathcal{J}_\varepsilon(x) \ge \frac{\varepsilon}{2} x^2 + C_1|x| - C_2\,. \tag{4.8b}$$

Proof. Estimates (4.7a) and (4.7b) have been derived in [10, Lemma 4.1] (cf. also [13, Lemma 4.2]). Inequalities (4.8a) and (4.8b) can be deduced from (4.7a)–(4.7b) by arguing in a completely analogous way as in the proof of [10, Lemma 4.1], to which the reader is referred. ∎

Lemma 4.2 *The function* $\mathcal{L}_\varepsilon : \mathbb{R} \to \mathbb{R}$ *satisfies:*

$$\varepsilon < \mathcal{L}'_\varepsilon(x) \le \varepsilon + \frac{2}{\varepsilon} \quad \textit{for all } x \in \mathbb{R}\,, \tag{4.9}$$

$$\left| \frac{1}{\mathcal{L}'_\varepsilon(x)} - \frac{1}{\mathcal{L}'_\varepsilon(y)} \right| \le |x - y| \quad \textit{for all } x, y \in \mathbb{R}\,. \tag{4.10}$$

Proof. The left-hand side inequality in (4.9) directly follows from the definition (4.2) of $\mathcal{L}_\varepsilon$ and from estimate (4.7b) for $\ln'_\varepsilon$. In order to prove the right-hand side estimate, it is sufficient to show that

$$|\mathcal{L}_\varepsilon(x) - \mathcal{L}_\varepsilon(y)| \le \left(\varepsilon + \frac{2}{\varepsilon}\right) |x - y| \quad \text{for all } x, y \in \mathbb{R}\,. \tag{4.11}$$

This can be checked by recalling that the Yosida-regularization $\ln_\varepsilon$ is defined by

$$\ln_\varepsilon(x) := \frac{1}{\varepsilon}(x - \rho_\varepsilon(x)) \tag{4.12}$$

where $\rho_\varepsilon = (\mathrm{Id}+\varepsilon \ln)^{-1} : \mathbb{R} \to (0, +\infty)$ is the ε-resolvent of $\ln$. Plugging (4.12) into the definition (4.2) of $\mathcal{L}_\varepsilon$ and using the well-known fact that ρ_ε is a contraction, we immediately deduce (4.11).

Observe that, by the first of (4.9) the function $x \mapsto \frac{1}{\mathcal{L}'_\varepsilon(x)}$ is well-defined on $\mathbb{R}$. In order to show that it is itself a contraction, we use the formula

$$\ln'_\varepsilon(x) = \frac{1}{\rho_\varepsilon(x) + \varepsilon} \qquad \text{for all } x \in \mathbb{R} \tag{4.13}$$

(cf. [13, Lemma 4.2]). Therefore, for every $x,\, y \in \mathbb{R}$

$$\begin{aligned} \left| \frac{1}{\mathcal{L}'_\varepsilon(x)} - \frac{1}{\mathcal{L}'_\varepsilon(y)} \right| &= \left| \frac{\rho_\varepsilon(x) + \varepsilon}{\varepsilon^2 + \varepsilon\rho_\varepsilon(x) + 1} - \frac{\rho_\varepsilon(y) + \varepsilon}{\varepsilon^2 + \varepsilon\rho_\varepsilon(y) + 1} \right| \\ &= \frac{|\rho_\varepsilon(x) - \rho_\varepsilon(y)|}{(\varepsilon^2 + \varepsilon\rho_\varepsilon(x) + 1)(\varepsilon^2 + \varepsilon\rho_\varepsilon(y) + 1)} \le |\rho_\varepsilon(x) - \rho_\varepsilon(y)| \end{aligned}$$

and (4.10) ensues, taking into account that ρ_ε is a contraction. ∎

Furthermore, we will supplement Problem (P_ε) by the approximate initial data $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon}, \mathbf{u}_0, \chi_0)$, where the family $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon})_\varepsilon$ fulfills

$$(\vartheta_0^\varepsilon)_\varepsilon \subset H, \qquad (\vartheta_s^{0,\varepsilon})_\varepsilon \subset H_{\Gamma_c}, \tag{4.14a}$$

and it approximates the data $(\vartheta_0, \vartheta_s^0)$ from (3.21)–(3.22) in the sense that

$$(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon}) \to (\vartheta_0, \vartheta_s^0) \text{ in } L^1(\Omega) \times L^1(\Gamma_c) \text{ as } \varepsilon \downarrow 0\,, \tag{4.14b}$$

$$\| \ln_\varepsilon(\vartheta_0^\varepsilon)\|_H \leq \| \ln(\vartheta_0)\|_H\,, \qquad \| \ln_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{H_{\Gamma_c}} \leq \| \ln(\vartheta_s^0)\|_{H_{\Gamma_c}} \quad \text{for all}\, \varepsilon > 0\,, \tag{4.14c}$$

$$\exists\, \bar{S} > 0 : \quad \forall\, \varepsilon > 0 \quad \varepsilon(\|\vartheta_0^\varepsilon\|_H + \|\vartheta_s^{0,\varepsilon}\|_{H_{\Gamma_c}}) \leq \bar{S}\,, \tag{4.14d}$$

$$\exists\, \bar{S}_1 > 0 : \quad \forall\, \varepsilon > 0 \quad \int_\Omega \mathcal{J}_\varepsilon(\vartheta_0^\varepsilon(x))\, \mathrm{d}x \leq \bar{S}_1(1 + \|\vartheta_0\|_{L^1(\Omega)})\,, \tag{4.14e}$$

$$\exists\, \bar{S}_2 > 0 : \quad \forall\, \varepsilon > 0 \qquad \int_{\Gamma_c} \mathcal{J}_\varepsilon(\vartheta_s^{0,\varepsilon})\, \mathrm{d}x \leq \bar{S}_2(1 + \|\vartheta_s^0\|_{L^1(\Gamma_c)}). \tag{4.14f}$$

Indeed, (4.14a) reflects the enhanced regularity (4.19) and (4.21) required of solutions (ϑ, ϑ_s) to Problem (P_ε). In what follows, we give an example of construction of a sequence $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon})_\varepsilon$ fulfilling properties (4.14).

Example 4.3 We will carry out the construction of the sequence $(\vartheta_0^\varepsilon)_\varepsilon$ only, the argument for $(\vartheta_s^{0,\varepsilon})_\varepsilon$ being completely analogous. For all $\varepsilon > 0$ and a.e. in Ω, let us define

$$\vartheta_0^\varepsilon := \min\{\vartheta_0, \varepsilon^{-\alpha}\} \qquad \text{for some } \alpha > 0\,. \tag{4.15}$$

Observe that $\vartheta_0^\varepsilon > 0$ a.e. in Ω (being $\vartheta_0 > 0$ a.e. in Ω thanks to the second of (3.21)), $\vartheta_0^\varepsilon \in L^\infty(\Omega)$, and moreover

$$\|\vartheta_0^\varepsilon\|_{L^1(\Omega)} \leq \|\vartheta_0\|_{L^1(\Omega)}\,. \tag{4.16}$$

Hence, (4.14b) is an immediate consequence of the dominated convergence theorem. Next, noting that $\|\vartheta_0^\varepsilon\|_H^2 \leq |\Omega|\, \varepsilon^{-2\alpha}$ and choosing $\alpha = \frac{1}{2}$, we can deduce

$$\exists\, \bar{S} > 0 : \quad \forall\, \varepsilon > 0 \quad \varepsilon^{1/2}\|\vartheta_0^\varepsilon\|_H \leq \bar{S} \tag{4.17}$$

and also (4.14d) follows. Moreover, (4.8a), (4.16), and (4.17) give (4.14e). Finally, relying on the definition (4.15) and on well-known properties of the Yosida regularization $\ln_\varepsilon$, we find

$$|\ln_\varepsilon(\vartheta_0^\varepsilon)| \leq |\ln_\varepsilon(\vartheta_0)| \leq |\ln(\vartheta_0)| \quad \text{a.e. in } \Omega \tag{4.18}$$

whence (4.14c).

All in all, the approximation of Problem 3.3 reads:

Problem 4.4 (P_ε) Let us a consider a quadruple of initial data $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon}, \mathbf{u}_0, \chi_0)$ satisfying (3.23)–(3.24) and (4.14). Find a sextuple $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\mu}, \xi)$, fulfilling

$$\vartheta \in L^2(0,T;V) \cap \mathrm{C}^0([0,T];H), \tag{4.19}$$

$$\mathcal{L}_\varepsilon(\vartheta) \in L^2(0,T;V) \cap \mathrm{C}^0([0,T];H) \cap H^1(0,T;V'), \tag{4.20}$$

$$\vartheta_s \in L^2(0,T;V_{\Gamma_c}) \cap \mathrm{C}^0([0,T];H_{\Gamma_c}), \tag{4.21}$$

$$\mathcal{L}_\varepsilon(\vartheta_s) \in L^2(0,T;V_{\Gamma_c}) \cap \mathrm{C}^0([0,T];H_{\Gamma_c}) \cap H^1(0,T;V'_{\Gamma_c}), \tag{4.22}$$

and such that $(\mathbf{u}, \chi, \boldsymbol{\mu}, \xi)$ comply with (3.30)–(3.31), (3.48), (3.34), satisfy the initial conditions

$$\vartheta(0) = \vartheta_0^\varepsilon \quad \text{a.e. in } \Omega\,, \tag{4.23}$$

$$\vartheta_s(0) = \vartheta_s^{0,\varepsilon} \quad \text{a.e. in } \Gamma_c\,, \tag{4.24}$$

as well as (3.37)–(3.38), and the equations

$$\begin{aligned}&\langle \partial_t \mathcal{L}_\varepsilon(\vartheta), v\rangle_V - \int_\Omega \mathrm{div}(\partial_t \mathbf{u})\, v \,\mathrm{d}x + \int_\Omega \nabla\vartheta\, \nabla v \,\mathrm{d}x + \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s) v \,\mathrm{d}x \\ &\quad + \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n})||\partial_t \mathbf{u}_{\mathrm{T}}| v \,\mathrm{d}x = \langle h, v\rangle_V \quad \forall\, v\in V \text{ a.e. in } (0,T)\,,\end{aligned} \tag{4.25}$$

$$\begin{aligned}&\langle \partial_t \mathcal{L}_\varepsilon(\vartheta_s), v\rangle_{V_{\Gamma_c}} - \int_{\Gamma_c} \partial_t \lambda(\chi)\, v \,\mathrm{d}x + \int_{\Gamma_c} \nabla\vartheta_s\, \nabla v \,\mathrm{d}x \\ &= \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s) v \,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n})||\partial_t \mathbf{u}_{\mathrm{T}}| v \,\mathrm{d}x \quad \forall\, v\in V_{\Gamma_c} \text{ a.e. in } (0,T)\,,\end{aligned} \tag{4.26}$$

$$\begin{aligned}&b(\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_\Omega \vartheta \,\mathrm{div}(\mathbf{v}) \,\mathrm{d}x + \int_{\Gamma_c} \chi \mathbf{u}\cdot\mathbf{v} \,\mathrm{d}x \\ &\quad + \int_{\Gamma_c} \phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n}\cdot\mathbf{v} \,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}(\vartheta-\vartheta_s)\boldsymbol{\mu}\cdot\mathbf{v} \,\mathrm{d}x = \langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v}\in\mathbf{W} \text{ a.e. in } (0,T)\,,\end{aligned} \tag{4.27}$$

$$\boldsymbol{\mu} = |\mathcal{R}(\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n})|\mathbf{z} \text{ with } \mathbf{z}\in \mathbf{d}(\partial_t \mathbf{u}) \text{ a.e. in } \Gamma_c\times(0,T), \tag{4.28}$$

as well as relations (3.44)–(3.45).

Observe that, in (4.25), (4.26), and (4.28) the term $\mathcal{R}(\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n})$ needs to be understood as $\mathcal{R}(J(\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n}))$, where J denotes the embedding operator from $L^2(0,T;L^4(\Gamma_c;\mathbb{R}^3))$ to $L^2(0,T;\mathbf{Y}_{\Gamma_c}')$.

In the next two sections, we address the existence of solutions to Problem (P_ε) for $\varepsilon>0$ fixed. That is why, for notational convenience we will not specify the dependence of such solutions on the parameter ε.

4.2 The approximate problem: local existence

The main result of this section is the forthcoming Proposition 4.9, stating the existence of a local-in-time solution to Problem (P_ε). We prove it by means of a Schauder fixed point argument, which relies on intermediate well-posedness results for the single equations in Problem (P_ε).

Fixed point setup. In view of Hypothesis (III), we may choose $\delta\in(0,1)$ such that

$$\mathcal{R}: L^2(0,T;\mathbf{Y}_{\Gamma_c}') \to L^\infty(0,T;L^{\frac{2}{1-\delta}}(\Gamma_c;\mathbb{R}^3)) \text{ is weakly-strongly continuous} \tag{4.29}$$

(and therefore bounded). For a fixed $\tau>0$ and a fixed constant $M>0$, we consider the set

$$\begin{aligned}\mathcal{Y}_\tau = \{&(\vartheta,\vartheta_s,\mathbf{u},\chi)\in L^2(0,\tau;H^{1-\delta}(\Omega))\times L^2(0,\tau;H^{1-\delta}(\Gamma_c))\times L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3))\times X_\tau \,: \\ &\|\vartheta\|_{L^2(0,\tau;H^{1-\delta}(\Omega))} + \|\vartheta_s\|_{L^2(0,\tau;H^{1-\delta}(\Gamma_c))} + \|\mathbf{u}\|_{L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3))} + \|\chi\|_{L^2(0,\tau;H_{\Gamma_c})} \le M,\end{aligned} \tag{4.30}$$

with the topology of $L^2(0,\tau;H^{1-\delta}(\Omega))\times L^2(0,\tau;H^{1-\delta}(\Gamma_c))\times L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3))\times L^2(0,\tau;H_{\Gamma_c})$, where

$$X_\tau := \{\chi\in L^2(0,\tau;H_{\Gamma_c}) : \quad \chi\in\mathrm{dom}(\widehat{\beta}) \text{ a.e. on } \Gamma_c\times(0,\tau)\}.$$

We are going to construct an operator $\mathcal{T}$ mapping $\mathcal{Y}_{\widehat{T}}$ into itself for a suitable time $0\le\widehat{T}\le T$, in such a way that any fixed point of $\mathcal{T}$ yields a solution to Problem (P_ε) on the interval $(0,\widehat{T})$. In the proof of Proposition 4.9, we will then proceed to show that $\mathcal{T}:\mathcal{Y}_{\widehat{T}}\to\mathcal{Y}_{\widehat{T}}$ does admit a fixed point.

Notation 4.5 In the following lines, we will denote by S_i, $i=1,2,3$, a positive constant depending on the problem data, on $\bar{S}_1$ and $\bar{S}_2$ in (4.14e)-(4.14f), on $M>0$ in (4.30), but *independent* of $\varepsilon>0$, and by $S_4(\varepsilon)$ a constant depending on the above quantities and on $\varepsilon>0$ as well. Furthermore, with the symbols $\pi_i(A)$, $\pi_{i,j}(A),\ldots$, we will denote the projection of a set A on its i-, or (i,j)-component, ...

Step 1: As a first step in the construction of $\mathcal{T}$, we fix $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}) \in \mathcal{Y}_\tau$ and prove a well-posedness result for (the Cauchy problem for) the PDE system (4.27–4.28), with $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\chi})$ in place of $(\vartheta, \vartheta_s, \chi)$, and

$$\mathfrak{c}(\widehat{\vartheta} - \widehat{\vartheta}_s)|\mathcal{R}(\phi'_\varepsilon(\widehat{u}_{\mathrm{N}})\mathbf{n})| \quad \text{replacing} \quad \mathfrak{c}(\vartheta - \vartheta_s)|\mathcal{R}(\phi'_\varepsilon(u_{\mathrm{N}})\mathbf{n})|.$$

Lemma 4.6 *Assume* (3.1)*, Hypotheses (I)–(III), and suppose that* $\mathbf{f}$, $\mathbf{g}$, $\mathbf{u}_0$ *comply with* (3.18)–(3.19)*, and* (3.23)*.*

Then, there exists a constant $S_1 > 0$ *such that for all* $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}) \in \mathcal{Y}_\tau$ *there exists a unique pair* $(\mathbf{u}, \boldsymbol{\mu}) \in H^1(0,\tau;\mathbf{W}) \times L^\infty(0,\tau;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))$ *fulfilling the initial condition* (3.37) *and*

$$\begin{aligned}
& b(\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_\Omega \widehat{\vartheta}\,\mathrm{div}(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_c} \widehat{\chi}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x \\
& + \int_{\Gamma_c} \phi'_\varepsilon(u_{\mathrm{N}})\mathbf{n}\cdot\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}(\widehat{\vartheta} - \widehat{\vartheta}_s)\boldsymbol{\mu}\cdot\mathbf{v}\,\mathrm{d}x = \langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in \mathbf{W} \quad \text{a.e. in } (0,T), \\
& \boldsymbol{\mu} = |\mathcal{R}(\phi'_\varepsilon(\widehat{u}_{\mathrm{N}})\mathbf{n})|\mathbf{z} \quad \text{with} \quad \mathbf{z} \in \mathbf{d}(\partial_t\mathbf{u}) \ \text{a.e. in } \Gamma_c \times (0,T),
\end{aligned} \tag{4.31}$$

and the estimate

$$\|\mathbf{u}\|_{H^1(0,\tau;\mathbf{W})} + \|\mathbf{z}\|_{L^\infty(\Gamma_c\times(0,\tau))} + \|\boldsymbol{\mu}\|_{L^\infty(0,\tau;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))} \le S_1. \tag{4.32}$$

Proof. Observe that (4.31) has the very same structure of the momentum equation in the isothermal model for adhesive contact with friction tackled in [11], for which the existence of solutions was proved by passing to the limit in a time-discretization scheme. The arguments from [11] can be easily adapted to the present setting, also taking into account that the term

$$\widehat{\mathcal{R}} := \mathfrak{c}(\widehat{\vartheta} - \widehat{\vartheta}_s)|\mathcal{R}(\phi'_\varepsilon(\widehat{u}_{\mathrm{N}})\mathbf{n})| \text{ is in } L^2(0,\tau;L^{4/3}(\Gamma_c)). \tag{4.33}$$

Indeed, (4.33) can be checked by observing that $\mathfrak{c}(\widehat{\vartheta} - \widehat{\vartheta}_s) \in L^2(0,\tau;L^{4/(1+2\delta)}(\Gamma_c))$ by trace theorems and the Lipschitz continuity of $\mathfrak{c}$, and combining this with (4.29). It follows from (4.33) that the term $\int_{\Gamma_c} \widehat{\mathcal{R}}\mathbf{z}\cdot\mathbf{v}\,\mathrm{d}x$ in the momentum equation (4.31), with a selection $\mathbf{z} \in L^\infty(\Gamma_c \times (0,\tau))$ from $\mathbf{d}(\partial_t\mathbf{u})$ as in the second of (4.31), is well-defined for every $\mathbf{v} \in \mathbf{W}$. Observe that the $L^\infty(0,\tau;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))$-regularity of $\boldsymbol{\mu}$ derives from (3.13).

To prove uniqueness, we proceed as in [11, Sec. 5]: given two solution pairs $(\mathbf{u}_1, \boldsymbol{\mu}_1)$ and $(\mathbf{u}_2, \boldsymbol{\mu}_2)$ to the Cauchy problem for (4.31), we test the equation fulfilled by $\tilde{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2$ and $\tilde{\boldsymbol{\mu}} := \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$, $\boldsymbol{\mu}_i = |\mathcal{R}(\phi'_\varepsilon(\widehat{u}_{\mathrm{N}})\mathbf{n})|\mathbf{z}_i$ for some $\mathbf{z}_i \in \mathbf{d}(\partial_t\mathbf{u}_i)$, $i = 1,2$, with $\mathbf{v} := \partial_t\tilde{\mathbf{u}}$ and integrate in time. With straightforward calculations, setting $\tilde{\mathbf{z}} = \mathbf{z}_1 - \mathbf{z}_2$ and using the place-holder $\widehat{\mathcal{R}}$ from (4.33), we obtain

$$\begin{aligned}
& \int_0^t b(\partial_t\tilde{\mathbf{u}}, \partial_t\tilde{\mathbf{u}})\,\mathrm{d}s + \frac{1}{2}a(\tilde{\mathbf{u}}(t), \tilde{\mathbf{u}}(t)) + \int_0^t \int_{\Gamma_c} \widehat{\mathcal{R}}\,\tilde{\mathbf{z}}\cdot\partial_t\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}s \\
& \le \left| \int_0^t \int_{\Gamma_c} \widehat{\chi}\tilde{\mathbf{u}}\cdot\partial_t\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}s \right| + \left| \int_0^t \int_{\Gamma_c} (\phi'_\varepsilon((u_1)_{\mathrm{N}}) - \phi'_\varepsilon((u_2)_{\mathrm{N}}))\mathbf{n}\cdot\partial_t\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}s \right|.
\end{aligned}$$

Now, the third term on the left-hand side is positive by monotonicity of $\mathbf{d}$ and positivity of $\mathfrak{c}$ (cf. (3.9)), whereas the second integral on the right-hand side can be estimated relying on the Lipschitz continuity of ϕ'_ε. All in all, the desired contraction estimate for $\tilde{\mathbf{u}}$ ensues from applying the Gronwall lemma, with calculations analogous to those in [11, Sec. 5]. Clearly, from this we also deduce that there exists a unique $\boldsymbol{\mu}$ complying with (4.31).

Finally, to prove estimate (4.32) we test (4.31) by $\mathbf{v} = \partial_t \mathbf{u}$ and integrate on $(0,t)$ with $t \in (0,\tau]$. We obtain the following estimate

$$
\begin{aligned}
& C_b \int_0^t \|\partial_t \mathbf{u}\|_{\mathbf{W}}^2 \, \mathrm{d}s + \frac{C_a}{2} \|\mathbf{u}(t)\|_{\mathbf{W}}^2 + \int_0^t \int_{\Gamma_c} \mathfrak{c}(\widehat{\vartheta} - \widehat{\vartheta}_s) \boldsymbol{\mu} \cdot \partial_t \mathbf{u} \, \mathrm{d}x \, \mathrm{d}s + \int_{\Gamma_c} \phi_\varepsilon(u_{\mathrm{N}}(t)) \, \mathrm{d}x \\
& \leq c \left(\|\mathbf{u}_0\|_{\mathbf{W}}^2 + \int_{\Gamma_c} \phi_\varepsilon(u_{\mathrm{N}}(0)) \, \mathrm{d}x + \int_0^t (\|\widehat{\vartheta}\|_H + \|\mathbf{F}\|_{\mathbf{W}'}) \|\partial_t \mathbf{u}\|_{\mathbf{W}} \, \mathrm{d}s + \int_0^t \|\widehat{\chi}\|_{H_{\Gamma_c}} \|\mathbf{u}\|_{\mathbf{W}} \|\partial_t \mathbf{u}\|_{\mathbf{W}} \, \mathrm{d}s \right) \\
& \leq c \left(1 + M^2 + \int_0^t \|\widehat{\chi}\|_{H_{\Gamma_c}}^2 \|\mathbf{u}\|_{\mathbf{W}}^2 \, \mathrm{d}s \right) + \frac{C_b}{2} \|\partial_t \mathbf{u}\|_{\mathbf{W}}^2,
\end{aligned}
\tag{4.34}
$$

where for the left-hand side we have used the chain-rule identities

$$
\begin{aligned}
& \int_0^t a(\mathbf{u}(s), \partial_t \mathbf{u}(s)) \, \mathrm{d}s = \frac{1}{2} a(\mathbf{u}(t), \mathbf{u}(t)) - \frac{1}{2} a(\mathbf{u}_0, \mathbf{u}_0) \\
& \int_0^t \int_{\Gamma_c} \phi_\varepsilon'(u_{\mathrm{N}}) \mathbf{n} \cdot \partial_t \mathbf{u} \, \mathrm{d}x \, \mathrm{d}s = \int_{\Gamma_c} \phi_\varepsilon(u_{\mathrm{N}}(t)) \, \mathrm{d}x - \int_{\Gamma_c} \phi_\varepsilon(u_{\mathrm{N}}(0)) \, \mathrm{d}x,
\end{aligned}
\tag{4.35}
$$

(cf. [16, Lemma 3.3], [18, Lemma 4.1]) the coercivity properties (3.7) and (3.8) of the forms a and b, and ultimately that $\int_{\Gamma_c} \phi_\varepsilon(u_{\mathrm{N}}(0)) \, \mathrm{d}x \leq \boldsymbol{\varphi}(\mathbf{u}_0) < \infty$ by (3.23). By the monotonicity of $\mathbf{d}$ and the positivity of $\mathfrak{c}$ (cf. (3.9)), we also infer that

$$
\int_0^t \int_{\Gamma_c} \mathfrak{c}(\widehat{\vartheta} - \widehat{\vartheta}_s) \boldsymbol{\mu} \cdot \partial_t \mathbf{u} \, \mathrm{d}x \, \mathrm{d}s \geq 0. \tag{4.36}
$$

For the estimates on the right-hand side of (4.34), we rely on Young's inequality and trace theorems. Therefore, (4.32) ensues from (4.34) by the Gronwall lemma. Clearly, the estimate for $\mathbf{z}$ follows from the fact that $|\mathbf{z}| \leq 1$ a.e. on $\Gamma_c \times (0,\tau)$, whence the estimate for $\boldsymbol{\mu}$, in view of (3.13). ■

Thanks to Lemma 4.6 we may define an operator

$$
\mathcal{T}_1 : \mathcal{Y}_\tau \to \mathcal{U}_\tau := \{ \mathbf{u} \in H^1(0,\tau; \mathbf{W}) \, : \, \|\mathbf{u}\|_{H^1(0,\tau;\mathbf{W})} \leq S_1 \} \tag{4.37}
$$

mapping every quadruple $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}) \in \mathcal{Y}_\tau$ into the unique solution $\mathbf{u}$ (together with $\boldsymbol{\mu} \in |\mathcal{R}(\phi_\varepsilon'(\widehat{u}_{\mathrm{N}})\mathbf{n})| \mathbf{d}(\partial_t \mathbf{u})$) of the Cauchy problem for (4.31).

Step 2: As a second step, we solve (the Cauchy problem for) (3.44)–(3.45), with $\widehat{\vartheta}_s \in \pi_2(\mathcal{Y}_\tau)$ and $\mathbf{u}$ from Lemma 4.6 on the right-hand side of (3.44).

Lemma 4.7 *Assume* (3.1)*, Hypotheses (IV)–(V), and suppose that* χ_0 *complies with* (3.24).

Then, there exists a constant $S_2 > 0$ *such that for all* $(\widehat{\vartheta}_s, \mathbf{u}) \in \pi_2(\mathcal{Y}_\tau) \times \mathcal{U}_\tau$ *there exists a unique pair* $(\chi, \xi) \in (L^2(0,\tau; H^2(\Gamma_c)) \cap L^\infty(0,\tau; V_{\Gamma_c}) \cap H^1(0,\tau; H_{\Gamma_c})) \times L^2(0,\tau; H_{\Gamma_c})$ *fulfilling the initial condition* (3.38)*, the relations*

$$
\begin{aligned}
& \partial_t \chi + A\chi + \xi + \sigma'(\chi) = -\lambda'(\chi) \widehat{\vartheta}_s - \frac{1}{2} |\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_c \times (0,T), \\
& \xi \in \beta(\chi) \text{ a.e. in } \Gamma_c \times (0,T),
\end{aligned}
\tag{4.38}
$$

and the estimate

$$
\|\chi\|_{L^2(0,\tau;H^2(\Gamma_c)) \cap L^\infty(0,\tau;V_{\Gamma_c}) \cap H^1(0,\tau;H_{\Gamma_c})} + \|\xi\|_{L^2(0,\tau;H_{\Gamma_c})} \leq S_2. \tag{4.39}
$$

Proof. The well-posedness of the Cauchy problem for (4.38) follows from standard results in the theory of parabolic equations with maximal monotone operators, after observing that, by Sobolev embeddings and trace theorems, $\widehat{\vartheta}_s$ and $1/2|\mathbf{u}|^2$ are respectively estimated in $L^2(0,\tau; L^{2/\delta}(\Gamma_c))$ (with $\delta \in (0,1)$ as in (4.29)), and in $L^2(0,\tau; H_{\Gamma_c})$.

In order to obtain estimate (4.39), we test (4.38) by $\partial_t \chi$ and integrate in time. Exploiting the chain rule for $\widehat{\beta}$ from [16, Lemma 3.3]

$$\int_0^t \int_{\Gamma_c} \xi \partial_t \chi \, \mathrm{d}x \, \mathrm{d}s = \int_{\Gamma_c} \widehat{\beta}(\chi(t)) \, \mathrm{d}x - \int_{\Gamma_c} \widehat{\beta}(\chi_0) \, \mathrm{d}x \,, \tag{4.40}$$

we obtain the estimate

$$\begin{aligned}
&\int_0^t \int_{\Gamma_c} |\partial_t \chi|^2 \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_{\Gamma_c} |\nabla \chi(t)|^2 \, \mathrm{d}x + \int_{\Gamma_c} \widehat{\beta}(\chi(t)) \, \mathrm{d}x \\
&\leq \frac{1}{2} \int_{\Gamma_c} |\nabla \chi_0|^2 \, \mathrm{d}x + \int_{\Gamma_c} \widehat{\beta}(\chi_0) \, \mathrm{d}x \\
&\qquad + \left| \int_0^t \int_{\Gamma_c} \sigma'(\chi) \partial_t \chi \, \mathrm{d}x \, \mathrm{d}s \right| + \left| \int_0^t \int_{\Gamma_c} \lambda'(\chi) \widehat{\vartheta}_s \partial_t \chi \, \mathrm{d}x \, \mathrm{d}s \right| + \left| \int_0^t \int_{\Gamma_c} |\mathbf{u}|^2 \partial_t \chi \, \mathrm{d}x \, \mathrm{d}s \right| \\
&\leq C(\|\chi_0\|_{V_{\Gamma_c}}^2 + 1) + I_1 + I_2 + I_3.
\end{aligned} \tag{4.41}$$

Now, the last inequality ensues from (3.24), and we estimate the integral terms I_i, $i = 1, 2, 3$, as follows:

$$\begin{aligned}
I_1 &\leq \frac{1}{4} \int_0^t \int_{\Gamma_c} |\partial_t \chi|^2 \, \mathrm{d}x \, \mathrm{d}s + C \int_0^t \int_{\Gamma_c} (|\chi|^2 + 1) \, \mathrm{d}x \, \mathrm{d}s + C' \\
&\leq \frac{1}{4} \int_0^t \int_{\Gamma_c} |\partial_t \chi|^2 \, \mathrm{d}x \, \mathrm{d}s + C \int_0^t \|\partial_t \chi\|_{L^2(0,s;H_{\Gamma_c})}^2 \, \mathrm{d}s + C \|\chi_0\|_{H_{\Gamma_c}}^2 + C' \\
I_2 &\leq c \int_0^t (\|\chi\|_{L^{2/(1-\delta)}(\Gamma_c)} + 1) \|\widehat{\vartheta}_s\|_{L^{2/\delta}(\Gamma_c)} \|\partial_t \chi\|_{H_{\Gamma_c}} \, \mathrm{d}s \\
&\leq \frac{1}{4} \int_0^t \|\partial_t \chi\|_{H_{\Gamma_c}}^2 \, \mathrm{d}s + C \int_0^t \|\widehat{\vartheta}_s\|_{L^{2/\delta}(\Gamma_c)}^2 (\|\chi\|_{V_{\Gamma_c}}^2 + 1) \, \mathrm{d}s, \\
I_3 &\leq \frac{1}{8} \int_0^t \|\partial_t \chi\|_{H_{\Gamma_c}}^2 \, \mathrm{d}s + c \int_0^t \|\mathbf{u}\|_{\mathbf{W}}^4 \, \mathrm{d}s,
\end{aligned} \tag{4.42}$$

where for I_1 we have used that σ' is Lipschitz continuous by virtue of (3.16), whereas the estimate for I_2 follows from the Lipschitz continuity of λ' and from observing that $H^1(\Gamma_c)$ embeds continuously in $L^{2/(1-\delta)}(\Gamma_c)$ for any $\delta \in (0,1)$, and for I_3 we have exploited the trace result (3.4). As for the left-hand side of (4.41), it remains to observe that, by convexity,

$$\exists\, c',\, C' > 0 \ \text{ such that for every } \ t \in (0, \tau] \ : \ \int_{\Gamma_c} \widehat{\beta}(\chi(t)) \, \mathrm{d}x \geq -c' \int_{\Gamma_c} |\chi(t)| \, \mathrm{d}x - C'. \tag{4.43}$$

Plugging (4.42) and (4.43) into (4.41) and using the Gronwall Lemma we conclude an estimate for χ in $L^\infty(0, \tau; V_{\Gamma_c}) \cap H^1(0, \tau; H_{\Gamma_c})$. A comparison argument in (4.38) (cf. also the forthcoming *Seventh a priori estimate* in Sec. 4.3), then yields an estimate in $L^2(0, \tau; H_{\Gamma_c})$ for $A\chi + \xi$, hence for $A\chi$ and ξ separately in $L^2(0, \tau; H_{\Gamma_c})$ by monotonicity of β. Therefore, by elliptic regularity results we get the desired bound for χ in $L^2(0, \tau; H^2(\Gamma_c))$, which concludes the proof of (4.39). ∎

It follows from Lemma 4.7 that we may define an operator

$$\begin{aligned}
\mathcal{T}_2 : \pi_2(\mathcal{Y}_\tau) \times \mathcal{U}_\tau \to \mathcal{X}_\tau := \{ &\chi \in L^2(0, \tau; H^2(\Gamma_c)) \cap L^\infty(0, \tau; H^1(\Gamma_c)) \cap H^1(0, \tau; H_{\Gamma_c}) \cap X_\tau \ : \\
&\|\chi\|_{L^2(0,\tau;H^2(\Gamma_c))} + \|\chi\|_{L^\infty(0,\tau;V_{\Gamma_c})} + \|\chi\|_{H^1(0,\tau;H_{\Gamma_c})} \leq S_2 \}
\end{aligned} \tag{4.44}$$

mapping $(\widehat{\vartheta}_s, \mathbf{u}) \in \pi_2(\mathcal{Y}_\tau) \times \mathcal{U}_\tau$ into the unique solution χ of the Cauchy problem for (4.38) (together with $\xi \in \beta(\chi)$).

Step 3: Eventually, we solve the Cauchy problem for the system (4.25, 4.26) with fixed $(\widehat{\vartheta}, \widehat{\vartheta}_s) \in \pi_{1,2}(\mathcal{Y}_\tau)$ and $(\mathbf{u}, \chi)$ from Lemmas 4.6 and 4.7, respectively. In particular, we set

$$\widehat{\mathcal{F}} := k(\chi)(\widehat{\vartheta} - \widehat{\vartheta}_s) + \mathfrak{c}'(\widehat{\vartheta} - \widehat{\vartheta}_s) |\mathcal{R}(\phi'_\varepsilon(u_{\mathrm{N}}) \mathbf{n})| |\partial_t \mathbf{u}_{\mathrm{T}}| \tag{4.45}$$

and plug it into the boundary integral on the left-hand side of (4.25) and on the right-hand side of (4.26). Observe that, due to (4.30), (4.32), (4.39), and to the Lipschitz continuity of $\mathfrak{c}$ and k, there holds

$$\widehat{\mathcal{F}} \in L^2(0,\tau; L^{4/3+s}(\Gamma_c)) \quad \text{for some } s = s(\delta) > 0. \tag{4.46}$$

We mention in advance that, relying the very fact that the boundary term $\widehat{\mathcal{F}}$ in (4.47) below does not depend on the unknown ϑ, we will be able to prove uniqueness of solutions for (the Cauchy problem for) (4.47).

Lemma 4.8 *Assume* (3.1)*, Hypotheses (I), (II), (III), and (V), suppose that h complies with* (3.17)*, and let $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon})$ fulfill* (4.14)*.*

Then, there exist positive constants S_3 and $S_4(\varepsilon)$ such that for all $(\widehat{\vartheta}, \widehat{\vartheta}_s, \mathbf{u}, \chi) \in \pi_{1,2}(\mathcal{Y}_\tau) \times \mathcal{U}_\tau \times \mathcal{X}_\tau$ there exists a unique couple of functions (ϑ, ϑ_s) complying with (4.19)–(4.22)*, fulfilling the initial conditions* (4.23)–(4.24)*, the equations a.e. in* $(0,T)$

$$\langle \partial_t \mathcal{L}_\varepsilon(\vartheta), v \rangle_V - \int_\Omega \operatorname{div}(\partial_t \mathbf{u})\, v \,\mathrm{d}x + \int_\Omega \nabla\vartheta \,\nabla v \,\mathrm{d}x + \int_{\Gamma_c} \widehat{\mathcal{F}} v \,\mathrm{d}x = \langle h, v \rangle_V \quad \forall\, v \in V\,, \tag{4.47}$$

$$\langle \partial_t \mathcal{L}_\varepsilon(\vartheta_s), v \rangle_{V_{\Gamma_c}} - \int_{\Gamma_c} \partial_t \lambda(\chi)\, v \,\mathrm{d}x + \int_{\Gamma_c} \nabla\vartheta_s \,\nabla v \,\mathrm{d}x = \int_{\Gamma_c} \widehat{\mathcal{F}} v \,\mathrm{d}x \quad \forall\, v \in V_{\Gamma_c}\,, \tag{4.48}$$

and the estimates

$$\|\vartheta\|_{L^2(0,\tau;V)\cap L^\infty(0,\tau;L^1(\Omega))} + \|\vartheta_s\|_{L^2(0,\tau;V_{\Gamma_c})\cap L^\infty(0,\tau;L^1(\Gamma_c))} \le S_3, \tag{4.49}$$

$$\|\partial_t \mathcal{L}_\varepsilon(\vartheta)\|_{L^2(0,\tau;V')} + \|\partial_t \mathcal{L}_\varepsilon(\vartheta_s)\|_{L^2(0,\tau;V'_{\Gamma_c})} \le S_3, \tag{4.50}$$

$$\|\vartheta\|_{L^\infty(0,\tau;H)} + \|\vartheta_s\|_{L^\infty(0,\tau;H_{\Gamma_c})} \le S_4(\varepsilon). \tag{4.51}$$

Proof. Observe that system (4.47, 4.48) is decoupled, hence we will tackle equations (4.47) and (4.48) separately.

Also taking into account (4.46), the well-posedness for the Cauchy problem for the doubly nonlinear equation (4.47) follows from standard results, cf. [19, Thm. 1] (see also [9, Lemma 3.5]): in particular, uniqueness for (4.47) is trivial, since the terms $\operatorname{div}(\partial_t \mathbf{u})$ and $\widehat{\mathcal{F}}$ are fixed. In order to conclude estimates (4.49)–(4.51) for ϑ, we test (4.47) by ϑ and integrate on $(0,t)$ with $t \in (0,\tau]$. Recalling the definition (4.6) of $\mathcal{J}_\varepsilon$, we exploit the formal identity (cf. Remark 4.12)

$$\langle \partial_t \mathcal{L}_\varepsilon(\vartheta), \vartheta \rangle_V = \int_\Omega \mathcal{L}'_\varepsilon(\vartheta)\partial_t\vartheta\vartheta \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}_\varepsilon(\vartheta)\,\mathrm{d}x \tag{4.52}$$

and thus infer

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega |\vartheta(t)|^2 \,\mathrm{d}x + C_1 \int_\Omega |\vartheta(t)| \,\mathrm{d}x - C_2 + \int_0^t \int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\le \int_\Omega \mathcal{J}_\varepsilon(\vartheta(t)) \,\mathrm{d}x + \int_0^t \int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\le \int_\Omega \mathcal{J}_\varepsilon(\vartheta_0^\varepsilon) \,\mathrm{d}x + \int_0^t \int_\Omega \operatorname{div}(\partial_t\mathbf{u})(\vartheta - m(\vartheta)) \,\mathrm{d}x \,\mathrm{d}s - \int_0^t \int_{\Gamma_c} \widehat{\mathcal{F}}(\vartheta - m(\vartheta)) \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + \int_0^t \langle h, \vartheta - m(\vartheta) \rangle_V \,\mathrm{d}s + \int_0^t \int_\Omega \operatorname{div}(\partial_t\mathbf{u}) m(\vartheta) \,\mathrm{d}x \,\mathrm{d}s \\
&\quad - \int_0^t \int_{\Gamma_c} \widehat{\mathcal{F}} m(\vartheta) \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_\Omega h m(\vartheta) \,\mathrm{d}x \,\mathrm{d}s \\
&\le \bar{S}_1(1 + \|\vartheta_0\|_{L^1(\Omega)}) + I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned} \tag{4.53}$$

where the first inequality is due to (4.8b), and the estimate for $\int_\Omega \mathcal{J}_\varepsilon(\vartheta_0^\varepsilon)\,\mathrm{d}x$ follows from (4.14e). As for the terms I_i, $i = 1, \dots, 6$, by the Sobolev embeddings and trace results (3.4), joint with

Poincaré's inequality (3.5), we easily have

$$\begin{aligned}
I_1 + I_2 + I_3 &\le \int_0^t \Big(\|\partial_t \mathbf{u}\|_{\mathbf{W}} \|\vartheta - m(\vartheta)\|_H + \|\widehat{\mathcal{F}}\|_{L^{4/3}(\Gamma_c)} \|\vartheta - m(\vartheta)\|_{L^4(\Gamma_c)} + \|h\|_{V'} \|\vartheta - m(\vartheta)\|_V \Big) \,\mathrm{d}s \\
&\le \frac{1}{4} \int_0^t \|\nabla \vartheta\|_H^2 \,\mathrm{d}s + \int_0^t \Big(\|\partial_t \mathbf{u}\|_{\mathbf{W}}^2 + \|\widehat{\mathcal{F}}\|_{L^{4/3}(\Gamma_c)}^2 + \|h\|_{V'}^2 \Big) \,\mathrm{d}s, \\
I_4 + I_5 + I_6 &\le C \int_0^t \Big(\|\partial_t \mathbf{u}\|_{\mathbf{W}} + \|\widehat{\mathcal{F}}\|_{L^{4/3}(\Gamma_c)} + \|h\|_{L^2(\Omega)} \Big) \|\vartheta\|_{L^1(\Omega)} \,\mathrm{d}s.
\end{aligned}$$

We plug the above estimates into (4.53) and use (3.17), estimate (4.32), and (4.46). Relying on the Gronwall lemma, we conclude estimates (4.49) and (4.51) for ϑ. Estimate (4.50) for $\mathcal{L}_\varepsilon(\vartheta)$ follows from a comparison in (4.47).

Since the calculations related to the analysis of equation (4.48) are completely analogous, we choose to omit them. ■

Thanks to Lemma 4.8, we may define an operator

$$\begin{aligned}
&\mathcal{T}_3 : \pi_{1,2}(\mathcal{Y}_\tau) \times \mathcal{U}_\tau \times \mathcal{X}_\tau \to \\
&\mathcal{W}_\tau := \{(\vartheta, \vartheta_s) \in (L^2(0,\tau;V) \cap L^\infty(0,\tau;H)) \times (L^2(0,\tau;V_{\Gamma_c}) \cap L^\infty(0,\tau;H_{\Gamma_c})) : \\
&\qquad \|\vartheta\|_{L^2(0,\tau;V) \cap L^\infty(0,\tau;L^1(\Omega))} + \|\vartheta_s\|_{L^2(0,\tau;V_{\Gamma_c}) \cap L^\infty(0,\tau;L^1(\Gamma_c))} \le S_3, \\
&\qquad \|\vartheta\|_{L^\infty(0,\tau;H)} + \|\vartheta_s\|_{L^\infty(0,\tau;H_{\Gamma_c})} \le S_4(\varepsilon)\}
\end{aligned} \tag{4.54}$$

mapping $(\widehat{\vartheta}, \widehat{\vartheta}_s, \mathbf{u}, \chi) \in \pi_{1,2}(\mathcal{Y}_\tau) \times \mathcal{U}_\tau \times \mathcal{X}_\tau$ into the unique solution (ϑ, ϑ_s) of the Cauchy problem for system (4.47, 4.48).

We are now in the position to prove the existence of local-in-time solutions to Problem (P_ε), defined on some interval $[0, \widehat{T}]$ with $0 < \widehat{T} \le T$. Note that $\widehat{T}$ in fact does not depend on the parameter $\varepsilon > 0$.

Proposition 4.9 (Local existence for Problem (P_ε)) *Assume* (3.1), *Hypotheses (I)–(V), and conditions* (3.17)–(3.19) *on the data* h, $\mathbf{f}$, $\mathbf{g}$, (3.23)–(3.24) *on* $\mathbf{u}_0$, χ_0, *and* (4.14) *on* ϑ_0^ε, $\vartheta_s^{0,\varepsilon}$.

Then, there exists $\widehat{T} \in (0, T]$ *such that for every* $\varepsilon > 0$ *Problem* (P_ε) *admits a solution* $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\mu}, \xi)$ *on the interval* $(0, \widehat{T})$.

Proof. Let the operator $\mathcal{T} : \mathcal{Y}_\tau \to \mathcal{W}_\tau \times \mathcal{U}_\tau \times \mathcal{X}_\tau$ be defined by

$$\mathcal{T}(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}) := (\vartheta, \vartheta_s, \mathbf{u}, \chi) \quad \text{with} \quad \begin{cases} \mathbf{u} := \mathcal{T}_1(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}), \\ \chi := \mathcal{T}_2(\widehat{\vartheta}_s, \mathbf{u}), \\ (\vartheta, \vartheta_s) := \mathcal{T}_3(\widehat{\vartheta}, \widehat{\vartheta}_s, \mathbf{u}, \chi). \end{cases} \tag{4.55}$$

In what follows, we will show that there exists $\widehat{T} \in (0, T]$ such that for every $\varepsilon > 0$

$$\mathcal{T} \text{ maps } \mathcal{Y}_{\widehat{T}} \text{ into itself,} \tag{4.56}$$

$$\begin{aligned}
&\mathcal{T} : \mathcal{Y}_{\widehat{T}} \to \mathcal{Y}_{\widehat{T}} \quad \text{is compact and continuous w.r.t. the topology of} \\
&L^2(0,\tau;H^{1-\delta}(\Omega)) \times L^2(0,\tau;H^{1-\delta}(\Gamma_c)) \times L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3)) \times L^2(0,\tau;H_{\Gamma_c}).
\end{aligned} \tag{4.57}$$

Ad (4.56)**.** In order to show (4.56), let $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}) \in \mathcal{Y}_\tau$ be fixed, and let $(\vartheta, \vartheta_s, \mathbf{u}, \chi) := \mathcal{T}(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}} \widehat{\chi})$. We use the interpolation inequality

$$\|\vartheta(t)\|_{H^{1-\delta}(\Omega)} \le c \|\vartheta(t)\|_{H^1(\Omega)}^{1-\delta} \|\vartheta(t)\|_{L^2(\Omega)}^{\delta} \qquad \text{for a.a. } t \in (0,\tau) \tag{4.58}$$

(cf. e.g. [17, Cor. 3.2]). Now, a further interpolation between the spaces $L^2(0,\tau;V)$ and $L^\infty(0,\tau;L^1(\Omega))$ and estimate (4.49) also yield the bound $\|\vartheta\|_{L^{10/3}(0,\tau;L^2(\Omega))} \le \bar{C} S_3$ for some interpolation constant $\bar{C}$. Integrating (4.58) in time and using Hölder's inequality we therefore have

$$\begin{aligned}
\|\vartheta\|_{L^2(0,t;H^{1-\delta}(\Omega))}^2 &\le c \int_0^t \|\vartheta(s)\|_{H^1(\Omega)}^{2(1-\delta)} \|\vartheta(s)\|_{L^2(\Omega)}^{2\delta} \,\mathrm{d}s \\
&\le c \|\vartheta\|_{L^2(0,\tau;H^1(\Omega))}^{2(1-\delta)} t^{(2\delta)/5} \|\vartheta\|_{L^{10/3}(0,\tau;L^2(\Omega))}^{2\delta} \le C S_3^2 t^{(2\delta)/5}.
\end{aligned} \tag{4.59}$$

We use (4.58) for ϑ_s and $\mathbf{u}$ to perform calculations analogous to (4.59), whereas for χ we trivially have $\|\chi\|^2_{L^2(0,t;H_{\Gamma_c})} \leq t\|\chi\|^2_{L^\infty(0,t;H_{\Gamma_c})} \leq tS_2^2$. Combining all of these estimates, we conclude that there exists a sufficiently small $\widehat{T} > 0$ for which (4.56) holds.

Ad (4.57)**: compactness.** Exploiting estimates (4.32), (4.39), and the compactness results [30, Thm. 4, Cor. 5], it is immediate to check compactness of the operator $\mathcal{T}$ as far as the $(\mathbf{u}, \chi)$-component is concerned. As for the (ϑ, ϑ_s)-component, from (4.49)–(4.51) and the Lipschitz continuity of $\mathcal{L}_\varepsilon$ we deduce an estimate (with a constant depending on ε) for $\mathcal{L}_\varepsilon(\vartheta)$ in $L^2(0,\widehat{T};V) \cap L^\infty(0,\widehat{T};H) \cap H^1(0,\widehat{T};V')$ (for $\mathcal{L}_\varepsilon(\vartheta_s)$ in $L^2(0,\widehat{T};V_{\Gamma_c}) \cap L^\infty(0,\widehat{T};H_{\Gamma_c}) \cap H^1(0,\widehat{T};V'_{\Gamma_c})$, resp.), whence compactness for $\mathcal{L}_\varepsilon(\vartheta)$ in $L^2(0,\widehat{T};H^{1-\delta}(\Omega))$ (for $\mathcal{L}_\varepsilon(\vartheta_s)$ in $L^2(0,\widehat{T};H^{1-\delta}(\Gamma_c))$, resp.), hence for $\vartheta = \mathcal{L}_\varepsilon^{-1}(\mathcal{L}_\varepsilon(\vartheta))$ in the same space (and analogously for ϑ_s). Observe that the latter argument relies on the bi-Lipschitz continuity (4.9) of $\mathcal{L}_\varepsilon$.

Ad (4.57)**: continuity.** In order to prove that $\mathcal{T}$ (4.55) is continuous, we will check that the operators $\mathcal{T}_i$, $i = 1,2,3$ defined by (4.37), (4.44), and (4.54) are continuous w.r.t. to suitable topologies.

First of all, we fix a sequence $\{(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\mathbf{u}}_n, \widehat{\chi}_n)\}_n \subset \mathcal{Y}_{\widehat{T}}$ converging to a $(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\mathbf{u}}_\infty, \widehat{\chi}_\infty) \in \mathcal{Y}_{\widehat{T}}$, with

$$\begin{aligned} &\widehat{\vartheta}_n \to \widehat{\vartheta}_\infty \quad \text{in } L^2(0,\widehat{T};H^{1-\delta}(\Omega)), && \widehat{\vartheta}_{s,n} \to \widehat{\vartheta}_{s,\infty} \quad \text{in } L^2(0,\widehat{T};H^{1-\delta}(\Gamma_c)), \\ &\widehat{\mathbf{u}}_n \to \widehat{\mathbf{u}}_\infty \quad \text{in } L^2(0,\widehat{T};H^{1-\delta}(\Omega;\mathbb{R}^3)), && \widehat{\chi}_n \to \widehat{\chi}_\infty \quad \text{in } L^2(0,\widehat{T};H_{\Gamma_c}) \end{aligned} \tag{4.60}$$

as $n \to \infty$. We let $\mathbf{u}_n := \mathcal{T}_1(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \widehat{\mathbf{u}}_n, \widehat{\chi}_n)$, and denote by $(\boldsymbol{\mu}_n)_n$ the associated sequence such that $(\mathbf{u}_n, \boldsymbol{\mu}_n)$ fulfill (4.31). Due to estimate (4.32), there exist a (not relabeled) subsequence and a pair $(\mathbf{u}_\infty, \mathbf{z}_\infty)$ such that as $n \to \infty$

$$\mathbf{u}_n \rightharpoonup \mathbf{u}_\infty \quad \text{in } H^1(0,\widehat{T};\mathbf{W}), \qquad \boldsymbol{\mu}_n \rightharpoonup^* \boldsymbol{\mu}_\infty \quad \text{in } L^\infty(0,\widehat{T};L^{2+\nu}(\Gamma_c;\mathbb{R}^3)). \tag{4.61}$$

Hence, by well-known compactness results, $(\mathbf{u}_n)_n$ strongly converges to $\mathbf{u}$ in $C^0([0,\widehat{T}];H^{1-\delta}(\Omega;\mathbb{R}^3))$ for all $\delta \in (0,1]$. Now, combining convergences (4.60) and (4.61) and arguing in the very same way as in the proof of Thm. 1 (cf. the forthcoming Section 5), we manage to pass to the limit as $n \to \infty$ in (4.31), concluding that the pair $(\mathbf{u}_\infty, \boldsymbol{\mu}_\infty)$ fulfill equation (4.31) with $(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\mathbf{u}}_\infty \widehat{\chi}_\infty)$. Therefore, we have that

$$\mathbf{u}_\infty = \mathcal{T}_1(\widehat{\vartheta}_\infty, \widehat{\vartheta}_{s,\infty}, \widehat{\mathbf{u}}_\infty, \widehat{\chi}_\infty), \text{ and convergences (4.61) hold for the } \textit{whole } \{(\mathbf{u}_n, \boldsymbol{\mu}_n)\}_n, \tag{4.62}$$

the latter fact by uniqueness of the limit.

Secondly, we consider the sequence $\chi_n := \mathcal{T}_2(\widehat{\vartheta}_{s,n}, \mathbf{u}_n)$ with $(\mathbf{u}_n)_n$ from the previous step, and let $(\xi_n)_n$ be the associated sequence of selections in $\beta(\chi_n)$, such that (χ_n, ξ_n) fulfill (4.38). Thanks to estimate (4.39), we have that $(\chi_n, \xi_n)_n$ is bounded in $(L^2(0,\widehat{T};H^2(\Gamma_c)) \cap L^\infty(0,\widehat{T};V_{\Gamma_c}) \cap H^1(0,\widehat{T};H_{\Gamma_c})) \times L^2(0,\widehat{T};H_{\Gamma_c})$. Therefore, there exists $(\chi_\infty, \xi_\infty)$ such that, up to a subsequence, as $n \to \infty$

$$\begin{aligned} &\chi_n \rightharpoonup^* \chi_\infty \quad \text{in } L^2(0,\widehat{T};H^2(\Gamma_c)) \cap L^\infty(0,\widehat{T};V_{\Gamma_c}) \cap H^1(0,\widehat{T};H_{\Gamma_c}), \\ &\xi_n \rightharpoonup \xi_\infty \quad \text{in } L^2(0,\widehat{T};H_{\Gamma_c}), \end{aligned} \tag{4.63}$$

and $(\chi_n)_n$ strongly converges to χ_∞ in $L^2(0,T;H^{2-\rho}(\Gamma_c)) \cap C^0([0,T];H^{1-\delta}(\Gamma_c))$ for all $\rho \in (0,2]$ and $\delta \in (0,1]$ by [30, Thm. 4, Cor. 5]. Relying on convergence (4.60) for $(\widehat{\vartheta}_{s,n})_n$, (4.61) for $(\mathbf{u}_n)_n$, and (4.63), and arguing as in the passage to the limit developed in Sec. 5, it can be shown that the functions (χ, ξ) solve (4.38) with $(\widehat{\vartheta}_{s,\infty}, \mathbf{u}_\infty)$, i.e.

$$\chi_\infty = \mathcal{T}_2(\widehat{\vartheta}_{s,\infty}, \mathbf{u}_\infty), \text{ and convergences (4.63) hold for the } \textit{whole } (\chi_n, \xi_n)_n. \tag{4.64}$$

Thirdly, we let $(\vartheta_n, \vartheta_{s,n}) := \mathcal{T}_3(\widehat{\vartheta}_n, \widehat{\vartheta}_{s,n}, \mathbf{u}_n, \chi_n)$ with $(\mathbf{u}_n)_n$ and $(\chi_n)_n$ from the previous steps. Estimates (4.49)–(4.51) imply that there exist $(\vartheta_\infty, \vartheta_{s,\infty})$ such that, along a (not relabeled) subsequence, the following weak convergences hold as $n \to \infty$

$$\begin{aligned} &\vartheta_n \rightharpoonup^* \vartheta_\infty \quad && \text{in } L^2(0,\widehat{T};V) \cap L^\infty(0,\widehat{T};H), \\ &\vartheta_{s,n} \rightharpoonup^* \vartheta_{s,\infty} \quad && \text{in } L^2(0,\widehat{T};V_{\Gamma_c}) \cap L^\infty(0,\widehat{T};H_{\Gamma_c}). \end{aligned} \tag{4.65}$$

Furthermore, taking into account the that $(\mathcal{L}_\varepsilon(\vartheta_n))_n$ ($(\mathcal{L}_\varepsilon(\vartheta_{s,n}))_n$, respectively), is bounded in $L^2(0,\widehat{T};V)\cap L^\infty(0,\widehat{T};H)\cap H^1(0,\widehat{T};V')$ (in $L^2(0,\widehat{T};V_{\Gamma_c})\cap L^\infty(0,\widehat{T};H_{\Gamma_c})\cap H^1(0,\widehat{T};V'_{\Gamma_c})$, resp.) and relying on [30, Thm. 4, Cor. 5], we find that $\mathcal{L}_\varepsilon(\vartheta_n)\to\mathcal{L}_\varepsilon(\vartheta)$ in $L^2(0,\widehat{T};H^{1-\delta}(\Omega))\cap \mathrm{C}^0([0,\widehat{T}];H)$, and analogously for $(\mathcal{L}_\varepsilon(\vartheta_{s,n}))_n$. Therefore, thanks to the bi-Lipschitz continuity of $\mathcal{L}_\varepsilon$ we conclude that

$$\begin{aligned} &\vartheta_n\to\vartheta_\infty && \text{in } L^2(0,\widehat{T};H^{1-\delta}(\Omega))\cap \mathrm{C}^0([0,\widehat{T}];H),\\ &\vartheta_{s,n}\to\vartheta_{s,\infty} && \text{in } L^2(0,\widehat{T};H^{1-\delta}(\Gamma_c))\cap \mathrm{C}^0([0,\widehat{T}];H_{\Gamma_c}). \end{aligned} \tag{4.66}$$

Convergences (4.60) for $(\widehat{\vartheta}_n,\widehat{\vartheta}_{s,n})_n$, (4.61) for $(\mathbf{u}_n)_n$, (4.63) for $(\chi_n)_n$, and (4.65)–(4.66), combined with the arguments of Sec. 5, allow us to pass to the limit as $n\to\infty$ in system (4.47, 4.48). Therefore, we conclude that

$$\begin{gathered} (\vartheta_\infty,\vartheta_{s,\infty})=\mathcal{T}_3(\widehat{\vartheta}_\infty,\widehat{\vartheta}_{s,\infty},\mathbf{u}_\infty,\chi_\infty) \text{ and}\\ \text{convergences (4.65)–(4.66) hold for the } whole\ (\vartheta_n,\vartheta_{s,n})_n. \end{gathered} \tag{4.67}$$

Ultimately, the continuity of $\mathcal{T}$ ensues from (4.62), (4.64), and (4.67). ∎

Remark 4.10 A key trick to prove compactness and continuity of the operator $\mathcal{T}$ (4.55) in the (ϑ,ϑ_s)-component has been:

- to prove compactness and continuity in $(\mathcal{L}_\varepsilon(\vartheta),\mathcal{L}_\varepsilon(\vartheta_s))$ (exploiting the estimates on the pair $(\partial_t\mathcal{L}_\varepsilon(\vartheta),\partial_t\mathcal{L}_\varepsilon(\vartheta_s))$ deduced from a comparison in the temperature equations (4.25) and (4.26)),
- then to infer compactness and continuity in (ϑ,ϑ_s), relying on the fact that $\mathcal{L}_\varepsilon$ is bi-Lipschitz, cf. Lemma 4.2.

Obviously we will not be in the position to use such an argument any longer, when taking the limit as $\varepsilon\to 0$. Indeed, for such a passage to the limit we will rely on new, BV-type estimates on (ϑ,ϑ_s) (cf. the *Fifth* and *Sixth a priori estimate* in the forthcoming Sec. 4.3). Exploiting such bounds and a version of the Aubin-Lions theorem for the case of time derivatives as measures (see, e.g., [29, Chap. 7, Cor. 7.9]), we will conclude the desired compactness for ϑ and ϑ_s.

4.3 The approximate problem: global existence

We now prove the following

Theorem 4.1 (Global existence for Problem (P_ε)**)** *Assume* (3.1)*, Hypotheses (I)–(V), conditions* (3.17)*–*(3.19) *on the data* h, $\mathbf{f}$, $\mathbf{g}$, (3.23)*–*(3.24) *on* $\mathbf{u}_0$*, and* χ_0*, and* (4.14) *on the approximate data* ϑ_0^ε *and* $\vartheta_s^{0,\varepsilon}$*. Then,*

1. *for all* $\varepsilon>0$ *Problem* (P_ε) *admits a solution* $(\vartheta,\vartheta_s,\mathbf{u},\chi,\boldsymbol{\mu},\xi)$ *on the* whole *interval* $(0,T)$*;*
2. *there exists a constant* $C>0$ *such that for every* $\varepsilon>0$ *and for* any *global-in-time solution* $(\vartheta,\vartheta_s,\mathbf{u},\chi,\boldsymbol{\mu},\xi)$ *to Problem* (P_ε) *the following estimates hold:*

$$\varepsilon^{1/2}\|\vartheta\|_{L^\infty(0,T;H)}+\varepsilon^{1/2}\|\vartheta_s\|_{L^\infty(0,T;H_{\Gamma_c})}\le C, \tag{4.68}$$

$$\|\vartheta\|_{L^2(0,T;V)\cap L^\infty(0,T;L^1(\Omega))}+\|\vartheta_s\|_{L^2(0,T;V_{\Gamma_c})\cap L^\infty(0,T;L^1(\Gamma_c))}\le C, \tag{4.69}$$

$$\|\mathbf{u}\|_{H^1(0,T;\mathbf{W})}+\|\chi\|_{L^\infty(0,T;V_{\Gamma_c})\cap H^1(0,T;H_{\Gamma_c})}\le C, \tag{4.70}$$

$$\|\phi'_\varepsilon(u_{\mathrm{N}})\mathbf{n}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})}+\|\boldsymbol{\mu}\|_{L^\infty(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))}\le C \qquad \textit{with } \nu>0 \textit{ from } (3.13), \tag{4.71}$$

$$\|\partial_t\mathcal{L}_\varepsilon(\vartheta)\|_{L^2(0,T;V')}+\|\partial_t\mathcal{L}_\varepsilon(\vartheta_s)\|_{L^2(0,T;H^1(\Gamma_c)')}\le C, \tag{4.72}$$

$$\|\vartheta\|_{\mathrm{BV}(0,T;W^{1,q}(\Omega)')}\le C \qquad \textit{for any } q>3, \tag{4.73}$$

$$\|\vartheta_s\|_{\mathrm{BV}(0,T;W^{1,\sigma}(\Gamma_c)')}\le C \qquad \textit{for any } \sigma>2, \tag{4.74}$$

$$\|\chi\|_{L^2(0,T;H^2(\Gamma_c))}+\|\xi\|_{L^2(0,T;H_{\Gamma_c})}\le C, \tag{4.75}$$

$$\|\mathcal{L}_\varepsilon(\vartheta)\|_{L^\infty(0,T;H)}+\|\mathcal{L}_\varepsilon(\vartheta_s)\|_{L^\infty(0,T;H_{\Gamma_c})}\le C\,. \tag{4.76}$$

In order to prove Thm. 4.1, in what follows we establish a priori estimates on the $(\vartheta,\vartheta_s,\mathbf{u}\,\chi)$-component of *any* given solution to Problem (P_ε), *independent* of the time-interval on which such solution is defined. Exploiting these *global-in-time* estimates and a standard prolongation argument, we will conclude that the local solution to Problem (P_ε) from Proposition 4.9 extends to a *global-in-time* solution. In this way we will obtain the first part of the statement.

In fact, it will be clear from the calculations below that such global estimates hold for a constant *independent* of the parameter $\varepsilon > 0$, whence (4.68)–(4.76), which will provide the starting point for the passage to the limit as $\varepsilon \to 0$ in Sec. 5.

We mention in advance, the *First, Fifth, Sixth a priori estimates* below are only formally derived: in Remark 4.12 later on we will clarify how they can be made fully rigorous.

Notation 4.11 We stress that, from now on the symbols c, c', C, C', shall denote a generic constant possibly depending on the problem data but not on ε, which we let vary, say, in $(0,1)$. Since the estimates below are not going to depend on the final time, we will perform all the related calculations directly on the interval $(0,T)$.

First a priori estimate. We test (4.25) by ϑ, (4.26) by ϑ_s, (4.27) by $\partial_t \mathbf{u}$, and (3.44) by $\partial_t \chi$, add the resulting relations, and integrate on $(0,t)$, $t \in (0,T]$. Recalling (4.52), we *formally* have

$$\begin{aligned}\int_0^t \langle \partial_t \mathcal{L}_\varepsilon(\vartheta), \vartheta \rangle_V \, \mathrm{d}s &= \int_\Omega \mathfrak{I}_\varepsilon(\vartheta(t))\, \mathrm{d}x - \int_\Omega \mathfrak{I}_\varepsilon(\vartheta_0^\varepsilon)\, \mathrm{d}x \\ &\geq \frac{\varepsilon}{2}\|\vartheta(t)\|_H^2 + C_1 \|\vartheta(t)\|_{L^1(\Omega)} - \bar{S}_1(1 + \|\vartheta_0\|_{L^1(\Omega)}) \\ &\geq \frac{\varepsilon}{2}\|\vartheta(t)\|_H^2 + C_1 \|\vartheta(t)\|_{L^1(\Omega)} - C\,,\end{aligned} \tag{4.77}$$

the first inequality due to (4.8b) and (4.14e), and the last one to the first of (3.21). In the same way, we have

$$\int_0^t \langle \partial_t \mathcal{L}_\varepsilon(\vartheta_s), \vartheta_s \rangle_{V_{\Gamma_c}} \, \mathrm{d}s \geq \frac{\varepsilon}{2}\|\vartheta_s(t)\|_{H_{\Gamma_c}}^2 + C_1 \|\vartheta_s(t)\|_{L^1(\Gamma_c)} - C\,. \tag{4.78}$$

We take into account the cancellation of some terms, the chain-rule identities (4.35) and (4.40), as well as

$$\int_0^t \int_{\Gamma_c} \left(\chi \mathbf{u} \cdot \partial_t \mathbf{u} + \frac{1}{2}|\mathbf{u}|^2 \partial_t \chi \right) \mathrm{d}x\, \mathrm{d}r = \frac{1}{2}\int_{\Gamma_c} \chi(t)|\mathbf{u}(t)|^2 \, \mathrm{d}x - \frac{1}{2}\int_{\Gamma_c} \chi_0 |\mathbf{u}_0|^2 \, \mathrm{d}x.$$

Therefore, with easy calculations we arrive at

$$\begin{aligned}&\frac{\varepsilon}{2}\|\vartheta(t)\|_H^2 + C_1\|\vartheta(t)\|_{L^1(\Omega)} + \int_0^t \int_\Omega |\nabla \vartheta|^2 \, \mathrm{d}x\, \mathrm{d}r + \frac{\varepsilon}{2}\|\vartheta_s(t)\|_{H_{\Gamma_c}}^2 + C_1 \|\vartheta_s(t)\|_{L^1(\Gamma_c)} \\ &\quad + \int_0^t \int_{\Gamma_c} |\nabla \vartheta_s|^2 \, \mathrm{d}x\, \mathrm{d}r + \int_0^t \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s)^2 \, \mathrm{d}x\, \mathrm{d}r \\ &\quad + \int_0^t \int_{\Gamma_c} \mathfrak{c}'(\vartheta - \vartheta_s)(\vartheta - \vartheta_s)|\mathcal{R}(\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n})||\partial_t \mathbf{u}_{\mathrm{T}}| \, \mathrm{d}x\, \mathrm{d}r + \int_0^t b(\partial_t \mathbf{u}, \partial_t \mathbf{u})\, \mathrm{d}r \\ &\quad + \frac{1}{2}a(\mathbf{u}(t), \mathbf{u}(t)) + \frac{1}{2}\int_{\Gamma_c} \chi(t)|\mathbf{u}(t)|^2 \, \mathrm{d}x + \int_0^t \int_{\Gamma_c} \mathfrak{c}(\vartheta - \vartheta_s)\boldsymbol{\mu} \cdot \partial_t \mathbf{u} \, \mathrm{d}x\, \mathrm{d}r \\ &\quad + \int_{\Gamma_c} \phi_\varepsilon(u_{\mathrm{N}}(t))\, \mathrm{d}x + \int_0^t \int_{\Gamma_c} |\partial_t \chi|^2 \, \mathrm{d}x\, \mathrm{d}r + \frac{1}{2}\int_{\Gamma_c} |\nabla \chi(t)|^2 \, \mathrm{d}x \\ &\leq C + \frac{1}{2}a(\mathbf{u}_0, \mathbf{u}_0) + \frac{1}{2}\int_{\Gamma_c} \chi_0 |\mathbf{u}_0|^2 \, \mathrm{d}x + \int_{\Gamma_c} \phi_\varepsilon(u_{\mathrm{N}}(0))\, \mathrm{d}x + \frac{1}{2}\|\nabla \chi_0\|_{H_{\Gamma_c}}^2 + I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{4.79}$$

Now, observe that, due to the positivity of k in (3.14), the seventh integral term on the left-hand side is positive, and so are the eighth term, thanks to (3.9), the eleventh, since $\chi \in \mathrm{dom}(\widehat{\beta}) \subset [0, +\infty)$, and the twelfth, by (3.9) and the fact that $\boldsymbol{\mu} \cdot \partial_t \mathbf{u} \geq 0$ a.e. in $\Gamma_c \times (0,T)$. As for the right-hand side of (4.79), it holds $\int_{\Gamma_c} \phi_\varepsilon(u_{\rm N}(0))\,{\rm d}x \leq \boldsymbol{\varphi}(\mathbf{u}_0) < \infty$ by (3.23). Moreover, we have that

$$I_1 = \int_0^t \langle \mathbf{F}, \partial_t \mathbf{u}\rangle_{\mathbf{W}} \,{\rm d}r \leq \varrho \int_0^t \|\partial_t \mathbf{u}\|^2_{\mathbf{W}} \,{\rm d}r + C\|\mathbf{F}\|^2_{L^2(0,T;\mathbf{W}')} \leq \frac{\varrho}{C_b}\int_0^t b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,{\rm d}r + C\|\mathbf{F}\|^2_{L^2(0,T;\mathbf{W}')}$$

$$I_2 = \int_0^t \langle h, \vartheta\rangle_V \,{\rm d}r = \int_0^t \langle h, \vartheta - m(\vartheta)\rangle_V \,{\rm d}r + \int_0^t\int_\Omega h m(\vartheta)\,{\rm d}x\,{\rm d}r$$
$$\leq \frac{1}{2}\int_0^t\int_\Omega |\nabla\vartheta|^2\,{\rm d}x\,{\rm d}r + C\int_0^t \|h\|_H \|\vartheta\|_{L^1(\Omega)}\,{\rm d}r + C\int_0^t \|h\|^2_{V'}\,{\rm d}r,$$

where the second inequality in the first line is due to (3.8), and we choose $\varrho = C_b/2$ in order to absorb the term $\int_0^t b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,{\rm d}r$ into the corresponding term on the left-hand side. In the estimate for I_2, the second passage follows from Poincaré's inequality. Finally, exploiting the convexity of $\widehat{\beta}$ and the fact that σ' has at most a linear growth, (cf. also (4.42) and (4.43)), we have

$$I_3 = -\int_{\Gamma_c} \widehat{\beta}(\chi(t))\,{\rm d}x \leq C\int_{\Gamma_c} |\chi(t)|\,{\rm d}x + C' \leq \frac{1}{4}\int_0^t\int_{\Gamma_c} |\partial_t\chi|^2\,{\rm d}x\,{\rm d}s + C\|\chi_0\|_{L^1(\Omega)} + C'.$$

$$I_4 = -\int_0^t\int_{\Gamma_c} \sigma'(\chi)\partial_t\chi\,{\rm d}x\,{\rm d}s \leq \frac{1}{8}\int_0^t\int_{\Gamma_c} |\partial_t\chi|^2\,{\rm d}x\,{\rm d}s + C\int_0^t \||\partial_t\chi\|^2_{L^2(0,s;H_{\Gamma_c})}\,{\rm d}s + C'\|\chi_0\|^2_{H_{\Gamma_c}} + C'.$$

We plug the above calculations into the right-hand side of (4.79). Relying on assumptions (3.17) for h, (3.20) for $\mathbf{F}$, and on (3.23)–(3.24) for the data $\mathbf{u}_0$ and χ_0, applying the Gronwall Lemma we immediately deduce estimates (4.68)–(4.70).

Second a priori estimate. It follows from estimate (4.70) and the continuous embeddings (3.4) that the term $\chi\mathbf{u}$ on the left-hand side of (4.27) is bounded in $L^2(0,T;L^{4-\epsilon}(\Gamma_{\rm c};\mathbb{R}^3))$ for every $\epsilon \in (0,3]$. Therefore, taking into account the previously obtained estimates on $\mathbf{u}$ and ϑ, and arguing by comparison (4.27), we also obtain the bound

$$\|\mathfrak{c}(\vartheta-\vartheta_s)\,\boldsymbol{\mu} + \phi_\varepsilon'(u_{\rm N})\mathbf{n}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})} \leq C. \tag{4.80}$$

Exploiting the fact that $\boldsymbol{\mu}$ and $\phi_\varepsilon'(u_{\rm N})\mathbf{n}$ are *orthogonal* and arguing as in [11, Sec. 4], from (4.80) we conclude that

$$\|\mathfrak{c}(\vartheta-\vartheta_s)\,\boldsymbol{\mu}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})} + \|\phi_\varepsilon'(u_{\rm N})\mathbf{n}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})} \leq C, \tag{4.81}$$

whence (4.71): the estimate for $\boldsymbol{\mu} = |\mathcal{R}(\phi_\varepsilon'(u_{\rm N})\mathbf{n})|\mathbf{z}$ follows from the fact that $\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_c}) \to L^\infty(0,T;L^{2+\nu}(\Gamma_{\rm c};\mathbb{R}^3))$ is bounded thanks to (3.13), and from the fact that

$$|\mathbf{z}| \leq 1 \qquad \text{a.e. on } \Gamma_{\rm c}\times(0,T) \tag{4.82}$$

by the definition (1.13) of $\mathbf{d}$.

Third a priori estimate. We argue by comparison in the temperature equation (4.25). It follows from estimates (4.69)–(4.70), the continuous embeddings (3.4), and the Lipschitz continuity (3.14) of k, that

$$\|k(\chi)(\vartheta-\vartheta_s)\|_{L^2(0,T;L^{4-\epsilon}(\Gamma_{\rm c}))} \leq C \quad \text{for every } \epsilon\in(0,3]. \tag{4.83}$$

Now, taking into account estimates (4.70) and (4.71), again (3.4), the fact that $\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_c}) \to L^\infty(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3))$ is bounded by (3.13), and the boundedness of $\mathfrak{c}'$ by (3.9), we have at least

$$\|\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi_\varepsilon'(u_{\rm N})\mathbf{n})||\partial_t\mathbf{u}_{\rm T}|\|_{L^2(0,T;L^{4/3}(\Gamma_{\rm c}))} \leq C. \tag{4.84}$$

Therefore, in view of (3.17) on h, by comparison in (4.25) we obtain estimate (4.72) for $\partial_t\mathcal{L}_\varepsilon(\vartheta)$.

Fourth a priori estimate. We rely on (4.83), (4.84), and estimate (4.70) which, combined with (3.15) for λ, in particular yields

$$\|\partial_t\lambda(\chi)\|_{L^2(0,T;L^{3/2}(\Gamma_{\rm c}))}\,. \tag{4.85}$$

Therefore, a comparison in the temperature equation (4.26) yields the second of (4.72).

Fifth a priori estimate. Let us *formally* rewrite (3.39) as

$$\begin{aligned}\int_\Omega \mathcal{L}'_\varepsilon(\vartheta)\partial_t\vartheta\cdot v\,\mathrm{d}x = \int_\Omega \operatorname{div}(\partial_t\mathbf{u})\,v\,\mathrm{d}x - \int_\Omega \nabla\vartheta\,\nabla v\,\mathrm{d}x - \int_{\Gamma_c} k(\mathcal{X})(\vartheta-\vartheta_s)v\,\mathrm{d}x \\ - \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi'_\varepsilon(u_{\mathrm{N}})\mathbf{n})||\partial_t\mathbf{u}_{\mathrm{T}}|v\,\mathrm{d}x + \langle h, v\rangle_V \quad \forall\, v\in V \ \text{ a.e. in } (0,T)\end{aligned} \tag{4.86}$$

and choose in (4.86) a test function $v\in V$ of the form

$$v = \frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w, \qquad \text{with } w\in W^{1,q}(\Omega) \text{ and } q>3. \tag{4.87}$$

Taking the contraction property (4.10) of $\frac{1}{\mathcal{L}'_\varepsilon}$ into account and considering that $\vartheta\in V\subset L^6(\Omega)$, we have that $\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}\in V$ and therefore $\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\in V$. Furthermore, it follows from (4.10) that

$$\left|\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}\right| \le \frac{1}{\varepsilon+\ln'_\varepsilon(1)} + |\vartheta-1| = \frac{\rho_\varepsilon(1)+\varepsilon}{1+\varepsilon\rho_\varepsilon(1)+\varepsilon^2} + |\vartheta-1| \le |\vartheta|+2 \qquad \text{a.e. in}\,\Omega\times(0,T), \tag{4.88}$$

where we have also used formula (4.13) involving the resolvent ρ_ε of $\ln$, which satisfies $\rho_\varepsilon(1)=1$. Therefore again exploiting (4.10) we find

$$\left\|\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}\right\|_H \le \|\vartheta\|_H + c, \qquad \left\|\nabla\left(\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}\right)\right\|_H \le \|\nabla\vartheta\|_H\,. \tag{4.89}$$

Now, we have

$$\int_\Omega \mathcal{L}'_\varepsilon(\vartheta)\partial_t\vartheta\cdot\left(\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\right)\,\mathrm{d}x = \int_\Omega \partial_t\vartheta w\,\mathrm{d}x. \tag{4.90}$$

Moreover, in view of (4.10), (4.88), (4.89), the previously obtained estimates, as well as (3.17) on h, we see that

$$\begin{aligned}&\left|\int_\Omega \operatorname{div}(\partial_t\mathbf{u})\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\,\mathrm{d}x\right| \le \|\partial_t\mathbf{u}\|_{\mathbf{W}}\|(\|\vartheta\|_H+c)\|w\|_{L^\infty(\Omega)} \doteq f_1\in L^2(0,T),\\ &\left|\int_\Omega \nabla\vartheta\,\nabla\left(\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\right)\,\mathrm{d}x\right| \le \|\nabla\vartheta\|_H^2\|w\|_{L^\infty(\Omega)} + (\|\vartheta\|_{L^6(\Omega)}+c)\|\nabla\vartheta\|_H\|\nabla w\|_{L^q(\Omega)} \doteq f_2\in L^1(0,T),\\ &\left|\int_{\Gamma_c} k(\mathcal{X})(\vartheta-\vartheta_s)\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\,\mathrm{d}x\right| \le \|k(\mathcal{X})(\vartheta-\vartheta_s)\|_{H_{\Gamma_c}}(\|\vartheta\|_{L^4(\Gamma_c)}+c)\|w\|_{L^4(\Gamma_c)} \doteq f_3\in L^2(0,T),\\ &\left|\int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi'_\varepsilon(u_{\mathrm{N}})\mathbf{n})||\partial_t\mathbf{u}_{\mathrm{T}}|\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\,\mathrm{d}x\right|\\ &\quad \le \|\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi'_\varepsilon(u_{\mathrm{N}})\mathbf{n})||\partial_t\mathbf{u}_{\mathrm{T}}|\|_{L^{4/3}(\Gamma_c)}(\|\vartheta\|_{L^4(\Gamma_c)}+1)\|w\|_{L^\infty(\Gamma_c)} \doteq f_4\in L^1(0,T),\\ &\left\langle h, \frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\right\rangle_V \le \|h\|_{V'}\left\|\frac{1}{\mathcal{L}'_\varepsilon(\vartheta)}w\right\|_V\\ &\qquad\qquad \le \|h\|_{V'}\big((\|\vartheta\|_V+c)\|w\|_{L^\infty(\Omega)} + (\|\vartheta\|_{L^6(\Omega)}+c)\|\nabla w\|_{L^3(\Omega)}\big) \doteq f_5\in L^1(0,T),\end{aligned}$$

where we have also used the continuous embeddings (3.4), $W^{1,q}(\Omega)\subset L^\infty(\Omega)$, $V\subset L^6(\Omega)$, as well as the trace result $W^{1,q}(\Omega)\subset L^\infty(\Gamma_c)$. All in all, we conclude that

$$\begin{aligned}&\exists\, f\in L^1(0,T) \ \text{ for a.a.}\,t\in(0,T) \ \ \forall\, w\in W^{1,q}(\Omega)\ :\\ &\left|\int_\Omega \partial_t\vartheta(t)w\,\mathrm{d}x\right| = \left|\langle\partial_t\vartheta(t), w\rangle_{W^{1,q}(\Omega)}\right| \le f(t)\|w\|_{W^{1,q}(\Omega)}.\end{aligned} \tag{4.91}$$

Hence, we have

$$\|\partial_t\vartheta\|_{L^1(0,T;W^{1,q}(\Omega)')} \le C, \tag{4.92}$$

yielding (4.73).

Sixth a priori estimate. We proceed in an analogous way with (3.40) and test it by

$$v = \frac{1}{\mathcal{L}_\varepsilon'(\vartheta_s)} w, \qquad \text{with } w \in W^{1,\sigma}(\Gamma_c) \text{ and } \sigma > 2. \tag{4.93}$$

The analogues of estimates (4.88) and (4.89) hold, therefore we obtain

$$\begin{aligned}\int_{\Gamma_c} \partial_t \vartheta_s w \, dx = & \int_{\Gamma_c} \partial_t \lambda(\chi) \frac{1}{\mathcal{L}_\varepsilon'(\vartheta_s)} w \, dx - \int_{\Gamma_c} \nabla \vartheta_s \nabla \left(\frac{1}{\mathcal{L}_\varepsilon'(\vartheta_s)} w \right) dx \\ & + \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s) \frac{1}{\mathcal{L}_\varepsilon'(\vartheta_s)} w \, dx + \int_{\Gamma_c} \mathfrak{c}'(\vartheta - \vartheta_s) |\mathcal{R}(\phi_\varepsilon'(u_N)\mathbf{n})| |\partial_t \mathbf{u}_T| \frac{1}{\mathcal{L}_\varepsilon'(\vartheta_s)} w \, dx \\ & \doteq I_5 + I_6 + I_7 + I_8. \end{aligned} \tag{4.94}$$

Using that $\frac{1}{\mathcal{L}_\varepsilon'}$ and λ' are Lipschitz and relying on the continuous embedding $W^{1,\sigma}(\Gamma_c) \subset L^\infty(\Gamma_c)$ we have

$$\begin{aligned} |I_5| & \leq \|\lambda'(\chi)\|_{L^4(\Gamma_c)} \|\partial_t \chi\|_{L^2(\Gamma_c)} (\|\vartheta_s\|_{L^4(\Gamma_c)} + c) \|w\|_{L^\infty(\Gamma_c)} \\ & \leq C(1 + \|\chi\|_{L^4(\Gamma_c)}) \|\partial_t \chi\|_{L^2(\Gamma_c)} (\|\vartheta_s\|_{L^4(\Gamma_c)} + c) \|w\|_{L^\infty(\Gamma_c)} \doteq f_6 \in L^1(0,T) \end{aligned}$$

in view of estimates (4.69)–(4.70). Analogously, we have

$$\begin{aligned} |I_6| & \leq \|\nabla \vartheta_s\|_{H_{\Gamma_c}}^2 \|w\|_{L^\infty(\Gamma_c)} + (\|\vartheta_s\|_{L^\rho(\Gamma_c)} + c) \|\nabla \vartheta_s\|_{H_{\Gamma_c}} \|\nabla w\|_{L^\sigma(\Gamma_c)} \doteq f_7 \in L^1(0,T) \\ |I_7| & \leq (1 + \|\chi\|_{H_{\Gamma_c}}) \|\vartheta - \vartheta_s\|_{L^4(\Gamma_c)} (\|\vartheta_s\|_{L^4(\Gamma_c)} + c) \|w\|_{L^\infty(\Gamma_c)} \doteq f_8 \in L^1(0,T), \\ |I_8| & \leq \|\mathfrak{c}'(\vartheta - \vartheta_s) |\mathcal{R}(\phi_\varepsilon'(u_N)\mathbf{n})| |\partial_t \mathbf{u}_T|\|_{L^{4/3}(\Gamma_c)} (\|\vartheta_s\|_{L^4(\Gamma_c)} + c) \|w\|_{L^\infty(\Gamma_c)} \doteq f_9 \in L^1(0,T) \end{aligned}$$

where in the estimate of I_6 we choose ρ in such a way that $1/\rho + 1/2 + 1/\sigma = 1$, exploiting (3.4), whereas to deal with I_7 we have used the fact that k is Lipschitz, and finally for I_8 we have relied on (4.84). All in all, we conclude that

$$\begin{aligned} & \exists\, f \in L^1(0,T) \ \text{ for a.a. } t \in (0,T) \ \ \forall\, w \in W^{1,\sigma}(\Gamma_c) \ : \\ & \left| \int_{\Gamma_c} \partial_t \vartheta_s(t) w \, dx \right| = \left| \langle \partial_t \vartheta_s(t), w \rangle_{W^{1,\sigma}(\Gamma_c)} \right| \leq f(t) \|w\|_{W^{1,\sigma}(\Gamma_c)}. \end{aligned} \tag{4.95}$$

Hence, we have

$$\|\partial_t \vartheta_s\|_{L^1(0,T;W^{1,\sigma}(\Gamma_c)')} \leq C, \tag{4.96}$$

whence (4.74).

Seventh a priori estimate. We test (3.44) by $A\chi + \xi$ and integrate in time. Taking into account the chain-rule identity (4.40), we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Gamma_c} |\nabla \chi(t)|^2 \, dx + \int_{\Gamma_c} \widehat{\beta}(\chi(t)) \, dx + \int_0^t \int_{\Gamma_c} |A\chi + \xi|^2 \, dx \, dr \\ & = \frac{1}{2} \int_{\Gamma_c} |\nabla \chi_0|^2 \, dx + \int_{\Gamma_c} \widehat{\beta}(\chi_0) \, dx + I_9 + I_{10} + I_{11}, \end{aligned}$$

and estimate

$$\begin{aligned} I_9 = - \int_0^t \int_{\Gamma_c} \sigma'(\chi)(A\chi + \xi) \, dx \, dr & \leq C \int_0^t (1 + \|\chi\|_{H_{\Gamma_c}}) \|A\chi + \xi\|_{H_{\Gamma_c}} \, dr \\ & \leq \frac{1}{4} \int_0^t \|A\chi + \xi\|_{H_{\Gamma_c}}^2 \, dr + C \int_0^t \|\chi\|_{H_{\Gamma_c}}^2 \, dr + C', \\ I_{10} = - \int_0^t \int_{\Gamma_c} \lambda'(\chi) \vartheta_s (A\chi + \xi) \, dx \, dr & \leq C \int_0^t (1 + \|\chi\|_{L^\rho(\Gamma_c)}) \|\vartheta_s\|_{L^\nu(\Gamma_c)} \|A\chi + \xi\|_{H_{\Gamma_c}} \, dr \\ & \leq C(1 + \|\chi\|_{L^\infty(0,T;V_{\Gamma_c})})^2 \int_0^t \|\vartheta_s\|_{V_{\Gamma_c}}^2 \, dr + \frac{1}{4} \int_0^t \|A\chi + \xi\|_{H_{\Gamma_c}}^2 \, dr \\ I_{11} = -\frac{1}{2} \int_0^t \int_{\Gamma_c} |\mathbf{u}|^2 (A\chi + \xi) \, dx \, dr & \leq \frac{1}{8} \int_0^t \|A\chi + \xi\|_{H_{\Gamma_c}}^2 \, dr + C \int_0^t \|\mathbf{u}\|_{\mathbf{W}}^4 \, dr, \end{aligned}$$

where for I_9 we have used the Lipschitz continuity of σ', for I_{10} chosen ρ and ν in such a way that $1/\rho + 1/\nu + 1/2 = 1$ and then used the continuous embedding (3.4), and analogously for I_{11}. Applying the Gronwall Lemma and taking into account estimates (4.69)–(4.70), we then conclude that $\int_0^t \|A\chi + \xi\|^2_{H_{\Gamma_c}} \,\mathrm{d}r \leq C$. A monotonicity argument yields

$$\|A\chi\|_{L^2(0,T;H_{\Gamma_c})} + \|\xi\|_{L^2(0,T;H_{\Gamma_c})} \leq C, \tag{4.97}$$

whence (4.75) by elliptic regularity.

Eighth estimate. We test (4.25) by $\mathcal{L}_\varepsilon(\vartheta) \in V$ and (4.26) by $\mathcal{L}_\varepsilon(\vartheta_s) \in V_{\Gamma_c}$, add the resulting relations and integrate in time. Observe that, by (4.9) we have

$$\int_0^t \int_\Omega \nabla \mathcal{L}_\varepsilon(\vartheta) \nabla \vartheta \,\mathrm{d}x \,\mathrm{d}s = \int_0^t \int_\Omega \mathcal{L}'_\varepsilon(\vartheta) |\nabla \vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s \geq \varepsilon \int_0^t \int_\Omega |\nabla \vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s,$$

and analogously for the term involving $\nabla \mathcal{L}_\varepsilon(\vartheta_s)$. Therefore, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\mathcal{L}_\varepsilon(\vartheta(t))|^2 \,\mathrm{d}x + \varepsilon \int_0^t \int_\Omega |\nabla \vartheta|^2 \,\mathrm{d}x \,\mathrm{d}r + \frac{1}{2}\int_{\Gamma_c} |\mathcal{L}_\varepsilon(\vartheta_s(t))|^2 \,\mathrm{d}x + \varepsilon \int_0^t \int_{\Gamma_c} |\nabla \vartheta_s|^2 \,\mathrm{d}x \,\mathrm{d}r \\
&\quad + \int_0^t \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s)(\mathcal{L}_\varepsilon(\vartheta) - \mathcal{L}_\varepsilon(\vartheta_s)) \,\mathrm{d}x \,\mathrm{d}r \\
&\quad + \int_0^t \int_{\Gamma_c} \mathfrak{c}'(\vartheta - \vartheta_s) |\mathcal{R}(\phi'_\varepsilon(u_\mathrm{N})\mathbf{n})| |\partial_t \mathbf{u}_\mathrm{T}| (\mathcal{L}_\varepsilon(\vartheta) - \mathcal{L}_\varepsilon(\vartheta_s)) \,\mathrm{d}x \,\mathrm{d}r \\
&\leq \frac{1}{2}\int_\Omega |\mathcal{L}_\varepsilon(\vartheta_0^\varepsilon)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_c} |\mathcal{L}_\varepsilon(\vartheta_s^{0,\varepsilon})|^2 \,\mathrm{d}x + I_{12} + I_{13} + I_{14}.
\end{aligned} \tag{4.98}$$

Now, due to the monotonicity of $\mathcal{L}_\varepsilon$ the fourth term on the left-hand side is non-negative, and so is the fifth one, as

$$\mathfrak{c}'(y-z)(\mathcal{L}_\varepsilon(y) - \mathcal{L}_\varepsilon(z)) = \mathfrak{c}'(y-z)(y-z)\frac{\mathcal{L}_\varepsilon(y) - \mathcal{L}_\varepsilon(z)}{y-z} \geq 0 \quad \text{for all } y \neq z$$

also in view of (3.9). On the other hand, by the very definition (4.2) of $\mathcal{L}_\varepsilon$, there holds

$$\|\mathcal{L}_\varepsilon(\vartheta_0^\varepsilon)\|_H^2 \leq 2\varepsilon^2 \|\vartheta_0^\varepsilon\|_H^2 + 2\|\ln_\varepsilon(\vartheta_0^\varepsilon)\|_H^2 \leq C \tag{4.99}$$

thanks to (4.14c)-(4.14d), and we have an analogous bound for $\|\mathcal{L}_\varepsilon(\vartheta_s^{0,\varepsilon})\|^2_{H_{\Gamma_c}}$. Moreover, we estimate

$$\begin{aligned}
I_{12} &= -\int_0^t \int_\Omega \operatorname{div}(\partial_t \mathbf{u}) \mathcal{L}_\varepsilon(\vartheta) \,\mathrm{d}x \,\mathrm{d}r \leq \int_0^t \|\partial_t \mathbf{u}\|^2_{\mathbf{W}} \,\mathrm{d}s + \frac{1}{4}\int_0^t \|\mathcal{L}_\varepsilon(\vartheta)\|_H^2 \,\mathrm{d}r \\
I_{13} &= -\int_0^t \int_\Omega h \mathcal{L}_\varepsilon(\vartheta) \,\mathrm{d}x \,\mathrm{d}r \leq \int_0^t \|h\|_H \|\mathcal{L}_\varepsilon(\vartheta)\|_H \,\mathrm{d}r \\
I_{14} &= -\int_0^t \int_{\Gamma_c} \lambda'(\chi) \partial_t \chi \mathcal{L}_\varepsilon(\vartheta_s) \,\mathrm{d}x \,\mathrm{d}s \leq C \int_0^t (1 + \|\chi\|_{L^\infty(\Gamma_c)}) \|\partial_t \chi\|_{H_{\Gamma_c}} \|\mathcal{L}_\varepsilon(\vartheta_s)\|_{H_{\Gamma_c}} \,\mathrm{d}r.
\end{aligned}$$

We plug the above estimates into the r.h.s. of (4.98), and use the previously proved bounds (4.70), (4.75) (yielding a bound for χ in $L^2(0,T;L^\infty(\Gamma_c))$), and (3.17) for h. Applying a generalized version of the Gronwall Lemma (see, e.g., [6]), we conclude

$$\|\mathcal{L}_\varepsilon(\vartheta)\|_{L^\infty(0,T;H)} + \|\mathcal{L}_\varepsilon(\vartheta_s)\|_{L^\infty(0,T;H_{\Gamma_c})} \leq C \tag{4.100}$$

whence, in view of (4.68),

$$\|\ln_\varepsilon(\vartheta)\|_{L^\infty(0,T;H)} + \|\ln_\varepsilon(\vartheta_s)\|_{L^\infty(0,T;H_{\Gamma_c})} \leq C. \tag{4.101}$$

Remark 4.12 As previously mentioned, the *First, Fifth, Sixth estimates* should be performed on a further approximate version of Problem (P_ε). In fact, identities (4.77), (4.78), (4.86), and (4.94) are just formal since the $\partial_t \mathcal{L}_\varepsilon(\vartheta)$ only belongs to $L^2(0,T;V')$ (analogously, $\partial_t \mathcal{L}_\varepsilon(\vartheta_s)$ only belongs to $L^2(0,T;(H^1(\Gamma_c))')$. These calculations can be rigorously justified in a framework where equations (4.25) and (4.26) are further regularized by adding viscosity contributions modulated by a second parameter $\nu > 0$, that is

$$\begin{aligned}&\int_\Omega \partial_t \mathcal{L}_\varepsilon(\vartheta) v \,\mathrm{d}x - \int_\Omega \mathrm{div}(\partial_t \mathbf{u})\, v \,\mathrm{d}x + \int_\Omega \nabla\vartheta \,\nabla v \,\mathrm{d}x + \nu \int_\Omega \nabla(\partial_t \vartheta)\nabla v \,\mathrm{d}x \\ &+ \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s) v \,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi'_\varepsilon(u_\mathrm{N})\mathbf{n})||\partial_t \mathbf{u}_\mathrm{T}| v \,\mathrm{d}x = \langle h, v\rangle_V \quad \forall\, v \in V \ \text{ a.e. in } (0,T)\,,\end{aligned} \tag{4.102}$$

$$\begin{aligned}&\int_{\Gamma_c} \partial_t \mathcal{L}_\varepsilon(\vartheta_s) v \,\mathrm{d}x - \int_{\Gamma_c} \partial_t \lambda(\chi)\, v \,\mathrm{d}x + \int_{\Gamma_c} \nabla\vartheta_s \,\nabla v \,\mathrm{d}x + \nu \int_{\Gamma_c} \nabla(\partial_t \vartheta_s)\,\nabla v \,\mathrm{d}x \\ &= \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s) v \,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\phi'_\varepsilon(u_\mathrm{N})\mathbf{n})||\partial_t \mathbf{u}_\mathrm{T}| v \,\mathrm{d}x \quad \forall\, v \in V_{\Gamma_c} \ \text{ a.e. in } (0,T)\,,\end{aligned} \tag{4.103}$$

The presence of these additional viscosity terms in (4.102) and (4.103) implies that the solution to the PDE system of Problem (P_ε), with (4.102) in place of (4.25) and (4.103) in place of (4.26), and supplemented by natural initial conditions, satisfies in addition

$$\mathcal{L}_\varepsilon(\vartheta), \vartheta \in H^1(0,T;V), \qquad \mathcal{L}_\varepsilon(\vartheta_s), \vartheta_s \in H^1(0,T;V_{\Gamma_c}) \tag{4.104}$$

and hence the formal identities (4.77), (4.78), (4.86), and (4.94) can be rigorously revised. Such an approximation of Problem 3.3 was considered in [9], (see also [10]), to which we refer the reader. Let us just mention here that, for technical reasons (see [9, Remark 3.2]) the viscosity parameter ν has to be kept distinct from Yosida parameter ε for the logarithm. Hence it is necessary to derive global a priori estimates independent of ε and/or ν, and then perform the passage to the limit procedure in two steps, first as $\nu \downarrow 0$ and subsequently as $\varepsilon \downarrow 0$.

To avoid overburdening the paper, we have preferred to omit this further vanishing viscosity regularization, at the price of developing the calculations for the *First, Fifth*, and *Sixth estimates* only on a formal level.

5 Proof of Theorem 1

In this section, we detail the passage to the limit in the approximate Problem (P_ε) as ε tends to 0 and we achieve the proof of Theorem 1 showing that the approximate solutions converge (up to a subsequence) to a solution of Problem 3.3. Hereafter we make explicit the dependence of the approximate solutions on the parameter ε and use the place-holder

$$\boldsymbol{\eta}_\varepsilon := \phi'_\varepsilon(u_\mathrm{N}^\varepsilon)\mathbf{n}\,.$$

We split the proof in some steps.

Compactness. Combining estimates (4.68)–(4.72), (4.75)–(4.76), and (4.82) with the Ascoli-Arzelà theorem, the well-known [30, Thm. 4, Cor. 5], and standard weak and weak*-compactness results, we find that there exists an nine-uple $(\vartheta, w, \vartheta_s, w_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \mathbf{z}, \xi)$ such that, along a suitable (not relabeled) subsequence, the following convergences hold

$$\begin{aligned} &\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } H^1(0,T;\mathbf{W}), \\ &\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{in } \mathrm{C}^0([0,T];H^{1-\delta}(\Omega;\mathbb{R}^3)) \quad \text{for all } \delta\in(0,1], \end{aligned} \tag{5.1}$$

$$\begin{aligned} &\chi_\varepsilon \rightharpoonup^* \chi \quad \text{in } L^2(0,T;H^2(\Gamma_c))\cap L^\infty(0,T;V_{\Gamma_c})\cap H^1(0,T;H_{\Gamma_c}), \\ &\chi_\varepsilon \to \chi \quad \text{in } L^2(0,T;H^{2-\rho}(\Gamma_c))\cap \mathrm{C}^0([0,T];H^{1-\delta}(\Gamma_c)) \quad \text{for all } \rho\in(0,2] \text{ and } \delta\in(0,1], \end{aligned} \tag{5.2}$$

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{in } L^2(0,T;H_{\Gamma_c}), \tag{5.3}$$

$$\boldsymbol{\eta}_\varepsilon \rightharpoonup \boldsymbol{\eta} \quad \text{in } L^2(0,T;\mathbf{Y}'_{\Gamma_c}), \tag{5.4}$$

$$\mathbf{z}_\varepsilon \rightharpoonup^* \mathbf{z} \quad \text{in } L^\infty(\Gamma_c\times(0,T);\mathbb{R}^3), \tag{5.5}$$

$$\boldsymbol{\mu}_\varepsilon = |\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon \rightharpoonup^* \boldsymbol{\mu} \quad \text{in } L^\infty(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3)) \qquad \text{with } \nu>0 \text{ from (3.13)}, \tag{5.6}$$

$$\vartheta_\varepsilon \rightharpoonup \vartheta \quad \text{in } L^2(0,T;V), \qquad \varepsilon\vartheta_\varepsilon \to 0 \quad \text{in } L^\infty(0,T;H), \tag{5.7}$$

$$\vartheta_{s,\varepsilon} \rightharpoonup \vartheta_s \quad \text{in } L^2(0,T;V_{\Gamma_c}), \qquad \varepsilon\vartheta_{s,\varepsilon} \to 0 \quad \text{in } L^\infty(0,T;H_{\Gamma_c}), \tag{5.8}$$

$$\begin{aligned} &\mathcal{L}_\varepsilon(\vartheta_\varepsilon) \rightharpoonup^* w \quad \text{in } L^\infty(0,T;H)\cap H^1(0,T;V'), \\ &\mathcal{L}_\varepsilon(\vartheta_\varepsilon) \to w \quad \text{in } \mathrm{C}^0([0,T];V'), \end{aligned} \tag{5.9}$$

$$\begin{aligned} &\mathcal{L}_\varepsilon(\vartheta_{s,\varepsilon}) \rightharpoonup^* w_s \quad \text{in } L^\infty(0,T;H_{\Gamma_c})\cap H^1(0,T;V'_{\Gamma_c}), \\ &\mathcal{L}_\varepsilon(\vartheta_{s,\varepsilon}) \to w_s \quad \text{in } \mathrm{C}^0([0,T];V'_{\Gamma_c}) \end{aligned} \tag{5.10}$$

as $\varepsilon\downarrow 0$. In addition, in view of condition (3.13) on $\mathcal{R}$, we have

$$\mathcal{R}(\boldsymbol{\eta}_\varepsilon) \to \mathcal{R}(\boldsymbol{\eta}) \quad \text{in } L^\infty(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3)), \text{ so that } \quad \boldsymbol{\mu} = |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}, \tag{5.11}$$

and (5.6) improves to

$$|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon \rightharpoonup^* |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \quad \text{in } L^\infty(0,T;L^{2+\nu}(\Gamma_c;\mathbb{R}^3)). \tag{5.12}$$

Next, applying a generalized version of the Aubin-Lions theorem for the case of time derivatives as measures (see, e.g., [29, Chap. 7, Cor. 7.9]), from (4.69), and estimates (4.92) and (4.96) we deduce that

$$\begin{aligned} &\vartheta_\varepsilon \to \vartheta \quad \text{in } L^2(0,T;H^{1-\delta}(\Omega)) \quad \text{for all } \delta\in(0,1], \\ &\vartheta_\varepsilon \to \vartheta \quad \text{in } L^2(0,T;L^\delta(\Gamma_c)) \quad \text{for all } \delta\in[1,4), \end{aligned} \tag{5.13}$$

$$\vartheta_{s,\varepsilon} \to \vartheta_s \quad \text{in } L^2(0,T;L^\delta(\Gamma_c)) \quad \text{for all } \delta\in[1,+\infty). \tag{5.14}$$

Moreover, taking into account the Lipschitz continuity and the C^1- regularity of $\mathfrak{c}$ (cf. (3.9)), from (5.13)–(5.14), we have

$$\begin{aligned} &\mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon}) \to \mathfrak{c}(\vartheta-\vartheta_s) \quad \text{in } L^2(0,T;L^\delta(\Gamma_c)) \quad \text{for all } \delta\in[1,4), \\ &\mathfrak{c}'(\vartheta_\varepsilon-\vartheta_{s,\varepsilon}) \to \mathfrak{c}'(\vartheta-\vartheta_s) \quad \text{in } L^q(0,T;L^q(\Gamma_c)) \quad \text{for all } q\in[1,\infty). \end{aligned} \tag{5.15}$$

Passage to the limit in (3.44)**.** Now, we consider (3.44)–(3.45) written for the approximate solutions $(\mathbf{u}_\varepsilon,\chi_\varepsilon,\vartheta_{s,\varepsilon},\xi_\varepsilon)_\varepsilon$. Taking into account convergences (5.1)–(5.3), (5.14), and the Lipschitz continuity of λ' and σ' (cf. (3.15), (3.16)), we easily conclude that the limit quadruple $(\mathbf{u},\chi,\vartheta_s,\xi)$ satisfies equation (3.44). Combining the weak convergence (5.3) with the strong one specified in (5.2), and taking into account the strong-weak closedness in $L^2(0,T;H_{\Gamma_c})$ of the graph of (the operator induced by) β, we conclude that $\xi\in\beta(\chi)$ a.e. on $\Gamma_c\times(0,T)$, i.e. (3.45) holds.

Passage to the limit in (4.27)**.** Owing to convergences (5.1)–(5.2), (5.4), (5.6)–(5.7) and (5.15), we can pass to the limit in (4.27). We get

$$b(\partial_t\mathbf{u},\mathbf{v})+a(\mathbf{u},\mathbf{v})+\int_\Omega \vartheta\,\mathrm{div}(\mathbf{v})\,\mathrm{d}x+\int_{\Gamma_c}\chi\mathbf{u}\mathbf{v}\,\mathrm{d}x+\langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}}+\int_{\Gamma_c}\mathfrak{c}(\vartheta-\vartheta_s)\,\boldsymbol{\mu}\cdot\mathbf{v}\,\mathrm{d}s = \langle\mathbf{F},\mathbf{v}\rangle_{\mathbf{W}}, \tag{5.16}$$

for all $\mathbf{v} \in \mathbf{W}$. Now we have to identify $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ as elements of $\partial\boldsymbol{\varphi}(\mathbf{u})$ and $|\mathcal{R}(\boldsymbol{\eta})|\mathbf{d}(\partial_t\mathbf{u})$, respectively, i.e. to show that (3.42) and (3.43) hold.

First, we test (4.27) by $\mathbf{u}_\varepsilon$. For every $t \in [0,T]$ we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_0^t \int_{\Gamma_c} \boldsymbol{\eta}_\varepsilon \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\
&= -\liminf_{\varepsilon\to 0} \Big(b(\mathbf{u}_\varepsilon(t),\mathbf{u}_\varepsilon(t)) - b(\mathbf{u}_0,\mathbf{u}_0) + \int_0^t \big(a(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon) + \int_\Omega \vartheta_\varepsilon \operatorname{div}(\mathbf{u}_\varepsilon)\,\mathrm{d}x\big)\,\mathrm{d}s \\
&\quad + \int_0^t \big(\int_{\Gamma_c} \chi_\varepsilon |\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}(\vartheta_\varepsilon - \vartheta_{s,\varepsilon})\boldsymbol{\mu}_\varepsilon \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x\big)\,\mathrm{d}s - \int_0^t \langle \mathbf{F}, \mathbf{u}_\varepsilon \rangle_{\mathbf{W}} \Big) \\
&\le -\int_0^t \Big(b(\partial_t \mathbf{u},\mathbf{u}) + a(\mathbf{u},\mathbf{u}) + \int_\Omega \vartheta \operatorname{div}(\mathbf{u})\,\mathrm{d}x + \int_{\Gamma_c} \chi |\mathbf{u}|^2\,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}(\vartheta - \vartheta_s)\boldsymbol{\mu}\cdot\mathbf{u} - \langle \mathbf{F},\mathbf{u}\rangle_{\mathbf{W}} \Big)\,\mathrm{d}s \\
&= \int_0^t \int_{\Gamma_c} \boldsymbol{\eta}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

where the $\le$ follows from exploiting (5.1), (5.2), (5.4), (5.6), (5.7), and (5.15), combined with lower semicontinuity arguments, and the last equality is due to (5.16). We use the above inequality and to show that for all $\mathbf{v} \in \mathbf{Y}_{\Gamma_c}$ and $t \in [0,T]$ there holds

$$\begin{aligned}
\int_0^t \langle \boldsymbol{\eta}, \mathbf{v}-\mathbf{u}\rangle_{\mathbf{Y}_{\Gamma_c}}\,\mathrm{d}s \le \liminf_{\varepsilon\to 0}\int_0^t\int_{\Gamma_c} \boldsymbol{\eta}_\varepsilon\cdot(\mathbf{v}-\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s &\le \liminf_{\varepsilon\to 0}\int_0^t\int_{\Gamma_c} (\phi_\varepsilon(v_{\mathrm{N}}) - \phi_\varepsilon(u_{\mathrm{N}}^\varepsilon))\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^t\int_{\Gamma_c} (\phi(v_{\mathrm{N}}) - \phi(u_{\mathrm{N}}))\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t (\boldsymbol{\varphi}(\mathbf{v}) - \boldsymbol{\varphi}(\mathbf{u}))\,\mathrm{d}s,
\end{aligned}$$

where the second inequality follows from the fact that $\boldsymbol{\eta}_\varepsilon = \phi_\varepsilon'(u_{\mathrm{N}}^\varepsilon)\mathbf{n}$, the third one from the Mosco-convergence (see, e.g., [4]) of ϕ_ε to ϕ, and the last one from the definition (3.12) of $\boldsymbol{\varphi}$. All in all, we conclude (3.42).

Let us now show (3.43). Preliminarily, for every fixed $\vartheta \in L^2(0,T;V)$, $\vartheta_s \in L^2(0,T;V_{\Gamma_c})$, and $\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}_{\Gamma_c}')$, we introduce the functional $\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})} : L^2(0,T;L^4(\Gamma_c;\mathbb{R}^3)) \to [0,+\infty)$ defined for all $\mathbf{v} \in L^2(0,T;L^4(\Gamma_c;\mathbb{R}^3))$ by

$$\begin{aligned}
\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{v}) :&= \int_0^T\int_{\Gamma_c} \mathfrak{c}(\vartheta(x,t)-\vartheta_s(x,t))|\mathcal{R}(\boldsymbol{\eta})(x,t)|j(\mathbf{v}(x,t))\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\int_{\Gamma_c} \mathfrak{c}(\vartheta(x,t)-\vartheta_s(x,t))|\mathcal{R}(\boldsymbol{\eta})(x,t)||\mathbf{v}_{\mathrm{T}}(x,t)|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Clearly, $\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}$ is a convex and lower semicontinuous functional on $L^2(0,T;L^4(\Gamma_c;\mathbb{R}^3))$. It can be easily verified that the subdifferential $\partial\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})} : L^2(0,T;L^4(\Gamma_c;\mathbb{R}^3)) \rightrightarrows L^2(0,T;L^{4/3}(\Gamma_c;\mathbb{R}^3))$ of $\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}$ is given at every $\mathbf{v} \in L^2(0,T;L^4(\Gamma_c;\mathbb{R}^3))$ by

$$\mathbf{h} \in \partial\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{v}) \;\Leftrightarrow\; \begin{cases} \mathbf{h} \in L^2(0,T;L^{4/3}(\Gamma_c;\mathbb{R}^3)), \\ \mathbf{h}(x,t) \in \mathfrak{c}(\vartheta(x,t)-\vartheta_s(x,t))|\mathcal{R}(\boldsymbol{\eta})(x,t)|\mathbf{d}(\mathbf{v}(x,t)) \end{cases} \tag{5.17}$$

for almost all $(x,t) \in \Gamma_c \times (0,T)$, where $\mathbf{d} = \partial j$ is given by (1.13). We shall prove that

$$\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{w}) - \mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\partial_t\mathbf{u}) \ge \int_0^T\int_{\Gamma_c} \mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot(\mathbf{w}-\partial_t\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t \tag{5.18}$$

for all $\mathbf{w} \in L^2(0,T;L^4(\Gamma_c;\mathbb{R}^3))$. From (5.18) we will conclude that $\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \in \partial\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\partial_t\mathbf{u})$, hence the desired (3.43) by (5.17), the strict positivity (3.9) of $\mathfrak{c}$, and (5.11). In order to show (5.18), we first observe that

$$\limsup_{\varepsilon\to0}\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\le\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot\partial_t\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t,\qquad(5.19)$$

which can be checked by testing (4.27) by $\partial_t\mathbf{u}_\varepsilon$ and passing to the limit via convergences (5.1)–(5.2), (5.4)–(5.6), (5.11), (5.13), lower semicontinuity arguments, and again the Mosco convergence of ϕ_ε. Therefore, we have

$$\begin{aligned}\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot(\mathbf{w}-\partial_t\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t&\le\liminf_{\varepsilon\to0}\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot(\mathbf{w}-\partial_t\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\&\le\liminf_{\varepsilon\to0}\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|(|\mathbf{w}_\mathrm{T}|-|(\partial_t\mathbf{u}_\varepsilon)_\mathrm{T}|)\,\mathrm{d}x\,\mathrm{d}t\\&\le\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|(|\mathbf{w}_\mathrm{T}|-|(\partial_t\mathbf{u})_\mathrm{T}|)\,\mathrm{d}x\,\mathrm{d}t\qquad(5.20)\end{aligned}$$

where the first inequality follows from (5.19) and convergences (5.12) and (5.15), the second one from the fact that $|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\in|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{d}(\partial_t\mathbf{u}_\varepsilon)$, and the last one from combining the *weak* convergence (5.1) with the *strong* convergences (5.11) and (5.15). Then, (5.18) ensues. Furthermore, arguing as in the derivation of (5.20), relying on (5.1), (5.11), (5.15), and (3.43), and using that, indeed, $|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon=|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)||(\partial_t\mathbf{u}_\varepsilon)_\mathrm{T}|$ a.e. in $\Gamma_c\times(0,T)$, we deduce

$$\begin{aligned}&\liminf_{\varepsilon\to0}\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\&=\liminf_{\varepsilon\to0}\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)||(\partial_t\mathbf{u}_\varepsilon)_\mathrm{T}|\,\mathrm{d}x\,\mathrm{d}t\\&\ge\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})||(\partial_t\mathbf{u})_\mathrm{T}|\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot\partial_t\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}\qquad(5.21)$$

Ultimately, from (5.19) and (5.21) we conclude

$$\lim_{\varepsilon\to0}\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\Gamma_c}\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot\partial_t\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\,.\qquad(5.22)$$

Now, in addition to (5.1), we prove the following strong convergence

$$\partial_t\mathbf{u}_\varepsilon\to\partial_t\mathbf{u}\quad\text{in }L^2(0,T;\mathbf{W}),\qquad(5.23)$$

which is crucial in order to pass to the limit in the frictional contribution $\int_{\Gamma_c}\mathfrak{c}'(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\phi'_\varepsilon(\mathbf{u}_{\varepsilon_N}\mathbf{n}))||(\partial_t\mathbf{u}_\varepsilon)_T|v\,\mathrm{d}x$ in (4.25) and (4.26). To this aim, we first observe that

$$\limsup_{\varepsilon\to0}\int_0^T b(\partial_t\mathbf{u}_\varepsilon,\partial_t\mathbf{u}_\varepsilon)\,\mathrm{d}t\le\int_0^T b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,\mathrm{d}t\qquad(5.24)$$

arguing in a similar way as in the derivation of (5.19): we test (4.27) by $\partial_t\mathbf{u}_\varepsilon$ and we pass to the limit exploiting convergences (5.1)–(5.2), (5.4)–(5.6), (5.13), (5.22) and the Mosco convergence of ϕ_ε. Since the converse inequality for the $\liminf_{\varepsilon\to0}$ holds by the first of (5.1), we then conclude that

$$\lim_{\varepsilon\to0}\int_0^T b(\partial_t\mathbf{u}_\varepsilon,\partial_t\mathbf{u}_\varepsilon)\,\mathrm{d}t=\int_0^T b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,\mathrm{d}t.$$

This gives (5.23), by the $\mathbf{W}$-ellipticity of b (cf. (3.8)).

Passage to the limit in (4.25) **and** (4.26)**.** We pass to the limit in (4.25) and (4.26) relying on the above convergences (5.1)–(5.2), (5.4), (5.7)–(5.11), (5.13)–(5.15), (5.23), and the following additional convergences for the nonlinear terms in (4.25) and (4.26). Indeed, conditions (3.14)–(3.15) on k and λ and convergences (5.2), (5.13)–(5.14) yield

$$k(\chi_\varepsilon)(\vartheta_\varepsilon - \vartheta_{s,\varepsilon}) \to k(\chi)(\vartheta - \vartheta_s) \quad \text{in } L^2(0,T;H_{\Gamma_c}), \tag{5.25}$$

as well as

$$\lambda(\chi_\varepsilon) \rightharpoonup \lambda(\chi) \quad \text{in } H^1(0,T;L^{3/2}(\Gamma_c)). \tag{5.26}$$

Exploiting all of the above convergences, we get

$$\begin{aligned} &\langle \partial_t w, v\rangle_V - \int_\Omega \mathrm{div}(\partial_t \mathbf{u})\, v \,\mathrm{d}x + \int_\Omega \nabla\vartheta\,\nabla v \,\mathrm{d}x \\ &+ \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s) v\,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})||\partial_t \mathbf{u}_{\mathrm{T}}|v\,\mathrm{d}x = \langle h, v\rangle_V \quad \forall\, v\in V \ \text{ a.e. in } (0,T)\,, \end{aligned} \tag{5.27}$$

$$\begin{aligned} &\langle \partial_t w_s, v\rangle_{V_{\Gamma_c}} - \int_{\Gamma_c} \partial_t\lambda(\chi)\, v \,\mathrm{d}x + \int_{\Gamma_c} \nabla\vartheta_s\,\nabla v \,\mathrm{d}x \\ &= \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s) v\,\mathrm{d}x + \int_{\Gamma_c} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})||\partial_t \mathbf{u}_{\mathrm{T}}|v\,\mathrm{d}x \quad \forall\, v\in V_{\Gamma_c} \ \text{ a.e. in } (0,T)\,. \end{aligned} \tag{5.28}$$

It remains to show that

$$w(x,t) = \ln(\vartheta(x,t)) \quad \text{ for a.a. } (x,t)\in\Omega\times(0,T), \tag{5.29}$$

$$w_s(x,t) = \ln(\vartheta_s(x,t)) \quad \text{ for a.a. } (x,t)\in\Gamma_c\times(0,T)\,. \tag{5.30}$$

We argue just for (5.29), the procedure for (5.30) being completely analogous. First, recalling the definition of $\mathcal{L}_\varepsilon$ (cf. (4.2)), we observe that (5.9) and the second of (5.7) give

$$\ln_\varepsilon(\vartheta_\varepsilon) \rightharpoonup^* w \quad \text{in } L^\infty(0,T;H)\,. \tag{5.31}$$

Thus, by relying on well-known properties of Yosida regularizations (cf. [5, Lemma 1.3, p. 42]), to conclude (5.29) it is sufficient to check that

$$\limsup_{\varepsilon\searrow 0} \int_0^T \int_\Omega \ln_\varepsilon(\vartheta_\varepsilon)\vartheta_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega w\vartheta\,\mathrm{d}x\,\mathrm{d}t\,. \tag{5.32}$$

The latter follows combining the weak convergence (5.31) with the strong convergence (5.13). This concludes the proof. □

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