# AN OPTIMAL TRANSPORTATION PROBLEM WITH A COST GIVEN BY THE EUCLIDEAN DISTANCE PLUS IMPORT/EXPORT TAXES ON THE BOUNDARY

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Abstract. In this paper we analyze a mass transportation problem in a bounded domain in which there is the possibility of import/export mass across the boundary paying a tax fee in addition to the transport cost that is assumed to be given by the Euclidean distance. We show a general duality argument and for the dual problem we find a Kantorovich potential as the limit as $p \to \infty$ of solutions to $p-$Laplacian type problems with non linear boundary conditions. In addition, we show that this limit encodes all the relevant information for our problem, it provides the masses that are exported and imported from the boundary and also allows us the construction of an optimal transport plan. Finally we show that the arguments can be adapted to deal with the case in which the amount of mass that can be exported/imported is bounded by prescribed functions.

## 1. Introduction.

Mass transport problems have been widely considered in the literature recently. This is due not only to its relevance for applications but also for the novelty of the methods needed for its solution. The origin of such problems dates back to a work from 1781 by Gaspard Monge, *Mémoire su la théorie des déblais et des remblais*, where he formulated a natural question in economics which deals with the optimal way of moving points from one mass distribution to another so that the total work done is minimized. Here the cost of moving one unit of mass from $x$ to $y$ is measured with the Euclidean distance and the total work done is the sum (integral) of the transport cost, $|x-y|$, times the mass that is moved from $x$ to $y$. Evans and Gangbo in [8] used a PDE approach to find a proof of the existence of an optimal transport map for the classical Monge problem, different to the first one given by Sudakov in 1979 by means of probability methods ([12], see also [2] and [4]). For a general reference on transport problems we refer to [14] and [15].

The result by Evans and Gangbo was our first motivation for the present work. Let us be more precise. Let $\Omega$ be an open bounded domain of $\mathbb{R}^N$, that from now on we will assume convex although we do not need it in order to get the mathematical results,

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but it is more convenient for the transport interpretation. Alternatively, one can use the geodesic distance inside the domain as transport cost, but we prefer to restrict ourselves to the Euclidean distance to avoid technicalities that may hide with the main arguments. Let $f \in L^\infty(\Omega)$ and $N < p < +\infty$. Given $g_i \in C(\partial\Omega)$, with $g_1 \le g_2$ on $\partial\Omega$, we set $W^{1,p}_{g_1,g_2}(\Omega) = \{u \in W^{1,p}(\Omega) \ : \ g_1 \le u \le g_2 \ \text{ on } \partial\Omega\}$ and consider the functional

$$\Psi_p(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p}\, dx - \int_\Omega f(x)u(x)\, dx.$$

Since $W^{1,p}_{g_1,g_2}(\Omega)$ is a closed convex subset of $W^{1,p}(\Omega)$ and the functional $\Psi_p$ is convex, lower semi-continuous and coercive, the variational problem

$$\min_{u \in W^{1,p}_{g_1,g_2}(\Omega)} \Psi_p(u) \tag{1.1}$$

has a minimizer $u_p$ in $W^{1,p}_{g_1,g_2}(\Omega)$, which is a least energy solution of the obstacle problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ g_1 \le u \le g_2 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This minimizer is unique in case $\int_\Omega f \neq 0$; in the case $\int_\Omega f = 0$, we can have different minimizers, but two of them differ by a constant and this can happen only in case there are two different constant functions between $g_1$ and $g_2$. Note that when every minimizer coincides with $g_1$ on some part of the boundary and with $g_2$ on another part of the boundary, then the minimizer is unique.

Let us also assume that $g_1, g_2$ satisfy the following condition:

$$g_1(x) - g_2(y) \le |x - y| \qquad \forall\, x, y \in \partial\Omega. \tag{1.3}$$

Under this assumption it holds that we can take the limit as $p \to \infty$, see Theorem 3.1, and obtain that $u_p \to u_\infty$ uniformly, and that the limit $u_\infty$ is a maximizer of the variational problem

$$\max \left\{ \int_\Omega w(x) f(x)\, dx \colon w \in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)} \le 1 \right\}. \tag{1.4}$$

At this point it is natural to ask the following question, that constitutes the main problem to address in this paper:

**Main Problem:** *Consider $f_+$ and $f_-$ are the positive and negative part of $f$, that is, $f = f_+ - f_-$, with $f_\pm$ $L^\infty$-masses. Can $u_\infty$, the limit of the sequence $u_p$, that is a maximizer for* (1.4)*, be interpreted as a kind of Kantorovich potential for some transport problem involving $f_+$ and $f_-$ ?*

The answer to this question is yes. Let us explain briefly and informally the mass transport problem that is related to this limit procedure, see §2.2 for more details. We have to transport some amount of material represented by $f_+$ in $\Omega$ ($f_+$ encodes the amount of material and its location) to a hole with a distribution given by $f_-$ also defined in $\Omega$. The goal is to transport all the mass (there is an infinity cost penalization for not transporting the total mass) $f_+$ to $f_-$ or to the boundary (exporting the mass out of $\Omega$). In doing this, we pay the transport costs given by the Euclidean distance $c(x,y) = |x-y|$ and when a unit of mass is left on a point $y \in \partial\Omega$ an additional cost given by $T_e(y)$, the export taxes. We also have the constraint of filling the hole completely (there is also an infinity cost penalization for not covering all the mass in $f_-$), that is, we have to import product, if necessary, from the exterior of $\Omega$ (paying the transport costs plus the extra cost $T_i(x)$, the import taxes, for each unit of mass that enters $\Omega$ at the point $x \in \partial\Omega$). We have the freedom to chose to export or import mass provided we transport all the mass in $f_+$ and covers all the mass of $f_-$. The main goal here is to minimize the total cost of this operation, that is given by the transport cost plus export/import taxes. Note that in this transport problem there appear two masses on $\partial\Omega$ that are unknowns (the ones that encode the mass that is exported and the mass that is imported). Also note that the usual mass balance condition

$$\int_\Omega f_+(x)\,dx = \int_\Omega f_-(y)\,dy,$$

is not asked to be satisfied since we can import or export mass through the boundary if necessary. This means that we can use $\partial\Omega$ as an infinite reserve/repository, we can take as mass as we wish from the boundary, or send back as mass as we want, provided we pay the transportation cost plus the import/export taxes.

Our ideas can be adapted to deal with a more realistic situation. With a cost as the described above (the Euclidean distance plus import/export taxes) we can impose the restriction of not exceeding the punctual quantity of $M_e(x)$ when we export some mass through $x \in \partial\Omega$ and also a punctual limitation of $M_i(x)$ for import from $\partial\Omega$. Thus, in this case we do not assume that the boundary is an infinite reserve/repository, but we bound the quantities that can be exported or imported. For doing this the natural constraint that must be verified is given by

$$-\int_{\partial\Omega} M_e \le -\int_\Omega (f_+ - f_-) \le \int_{\partial\Omega} M_i.$$

This says that one can transport all the positive mass in $f_+$ and also that one can satisfy all the demand of the consumers (covering the whole of $f_-$). Hence the mass transport problem is possible, and the problem becomes, as before, to minimize the cost. Two limit situations are as follows: when $T_i = T_e = 0$ we have limited importation/exportation but without taxes. On the other hand, by assuming $M_i(x) = 0$ or $M_e(x) = 0$ on certain zones of the boundary, we do not allow importation or exportation in those zones; if we impose those conditions on the whole boundary (assuming $\int_\Omega f_+ = \int_\Omega f_-$) we obtain again a solution to the classical Monge-Kantorovich mass transport problem solved by Evans and Gangbo but taking limits of Newmann problems for the $p-$Laplacian in a fixed domain $\Omega$ (instead of taking Dirichlet boundary conditions in a large ball). For simplicity and since the main mathematical difficulties are present without restricting the mass that can be exported/imported through the boundary we present the details for this case and at the end of the article we sketch the necessary changes and adaptations that are needed to deal with the more realistic case.

A variant of this transport problem (allowing the possibility of import/export mass from/to $\partial\Omega$) was recently proposed in [10]. In that reference the transport cost is given by $|x-y|^2$ (which is strictly convex) and with zero taxes on the boundary. The authors use this transport problem to define a new distance between measures and study the gradient flow of a particular entropy that coincides with the heat equation, with Dirichlet boundary condition equal to 1 (see [1], [3] for related results concerning the relation between flows and transport problems). Here we deal with the cost given by the Euclidean distance $|x-y|$ (which is not strictly convex) and allow for nontrivial import/export taxes. In addition, we perform an approximation procedure using the $p-$Laplacian (as was done by Evans and Gangbo), something that is not needed for a quadratic cost. See also [7], [9] for regularity results for a partial mass transport problem in which there is no boundary involved but the amount of mass that has to be transported is prescribed (here it is also considered a quadratic transport cost, $|x-y|^2$).

Let us briefly summarize the contents of this paper. In §2 we recall some well known facts, terminology and notations concerning the usual Monge-Kantorovich problem and its dual formulation, and, in §2.2, we describe the mass transport problem in which we are interested and study its dual formulation. The next section is devoted to obtain the Kantorivich potential as limit of the solutions of some obstacle problems associated with the $p$-Laplacian operator, to give a complete proof of the duality, to obtain the import/export masses from that $p$-Laplacian problems, and to show how to construct optimal transport plans via optimal transport maps. In §4 we give some simple examples in which the solution to the mass transport problem described in §2.2 can be explicitly computed just as an illustration of our results. Finally in §5 we deal with the case of limited importation/exportation.

## 2. Statement of the mass transport problem.

To state the problem more precisely we need some notation. Given a Borel subset $X \subset \mathbb{R}^N$, let $\mathcal{M}(X)$ denote the spaces of non-negatives Borel measures on $X$ with finite total mass. A measure $\mu \in \mathcal{M}(X)$ and a Borel map $T : X \to \mathbb{R}^N$ induce a Borel measure $T\#\mu$, the pushforward measure of $\mu$ through $T$, defined by $(T\#\mu)[B] = \mu[T^{-1}(B)]$. When we write $T\#f = g$, where $f$ and $g$ are nonnegative functions, this means that the measure having density $f$ is pushed-forward to the measure having density $g$.

### 2.1. **Mass transport Theory.**

**The Monge problem.** *Given two measures $\mu, \nu \in \mathcal{M}(X)$ satisfying the mass balance condition*

$$\int_{\mathbb{R}^N} d\mu = \int_{\mathbb{R}^N} d\nu, \tag{2.1}$$

*is the infimum*

$$\inf_{T\#\mu=\nu} \int_X |x - T(x)| d\mu(x)$$

*attained among mappings $T$ which push $\mu$ forward to $\nu$? In the case that $\mu$ and $\nu$ represent the distribution for production and consumption of some commodity, the problem is then to decide which producer should supply each consumer minimizing the total transport cost.*

In general, the Monge problem is ill-posed. To overcome the difficulties of the Monge problem, in 1942, L. V. Kantorovich ([11]) proposed to study a relaxed version of the Monge problem and, what is more relevant here, introduced a dual variational principle.

We will use the usual convention of denoting by $\pi_i : \mathbb{R}^N \times \mathbb{R}^N$ the projections, $\pi_1(x, y) := x$, $\pi_2(x, y) := y$. Given a Radon measure $\gamma$ in $X \times X$, its marginals are defined by $proj_x(\gamma) := \pi_1\#\gamma$, $proj_y(\gamma) := \pi_2\#\gamma$.

**The Monge-Kantorovich problem.** *Fix two measures $\mu, \nu \in \mathcal{M}(X)$ satisfying the mass balance condition* (2.1)*. Let $\Pi(\mu, \nu)$ the set of transport plans between $\mu$ and $\nu$, that is, the set of nonnegative Radon measures $\gamma$ in $X \times X$ such that $proj_x(\gamma) = \mu$ and $proj_y(\gamma) = \nu$. The Monge-Kantorovich problem is to find a measure $\gamma^* \in \Pi(\mu, \nu)$ which minimizes the cost functional*

$$\mathcal{K}(\gamma) := \int_{X \times X} |x - y| \, d\gamma(x, y),$$

*in the set $\Pi(\mu, \nu)$. A minimizer $\gamma^*$ is called an optimal transport plan between $\mu$ and $\nu$.*

Linearity makes the Monge-Kantorovich problem simpler than Monge original problem; a continuity-compactness argument at least guarantees the existence of an optimal transport plan.

It is well-known that linear minimization problems as the Monge-Kantorovich problem admit a dual formulation. In the context of optimal mass transportation, this was introduced by Kantorovich in 1942 ([11]), who established the following result.

**Kantorovich duality.** *Fix two measures $\mu, \nu \in \mathcal{M}(X)$ satisfying the mass balance condition* (2.1)*. For $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$, define*

$$J(\varphi, \psi) := \int_X \varphi \, d\mu + \int_X \psi \, d\nu,$$

*and let* $\Phi$ *be the set of all measurable functions* $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$ *satisfying*

$$\varphi(x) + \psi(y) \leq |x - y| \qquad \text{for } \mu \times \nu - \text{almost all } (x, y) \in X \times X.$$

*Then*

$$\inf_{\gamma \in \Pi(\mu,\nu)} \mathcal{K}(\gamma) = \sup_{(\varphi,\psi)\in\Phi} J(\varphi, \psi).$$

The above result is true for more general cost functions than the corresponding to the Euclidean distance $|x-y|$. Now, for cost functions associated with lower semi-continuous distances there is a more precise result (see for instance [14, Theorem 1.14]), which for the Euclidean distance can be written as follows:

**Kantorovich-Rubinstein Theorem.** *Let* $\mu, \nu \in \mathcal{M}(X)$ *be two measures satisfying the mass balance condition* (2.1). *Then,*

$$\min\{\mathcal{K}(\gamma) \ : \ \gamma \in \Pi(\mu, \nu)\} = \sup \left\{ \int_X u \, d(\mu - \nu) \ : \ u \in K_1(X) \right\}, \tag{2.2}$$

*where*

$$K_1(X) := \{u : X \to \mathbb{R} \ : \ |u(x) - u(y)| \leq |x - y| \ \ \forall \, x, y \in X\}$$

*is the set of 1-Lipschitz functions in* $X$.

The maximizers $u^*$ of the right hand side of (2.2) are called *Kantorovich (transport) potentials.*

In the particular case of $\mu = f_+\mathcal{L}^N$ and $\nu = f_-\mathcal{L}^N$, for adequate Lebesgue integrable functions $f_+$ and $f_-$, Evans and Gangbo in [8] find a Kantorovich potential as a limit, as $p \to \infty$, of solutions to the $p-$Laplace equation with Dirichlet boundary conditions in a sufficiently large ball $B(0, R)$:

$$\begin{cases} -\Delta_p u_p = f_+ - f_- & \text{in } \ B(0, R), \\ u_p = 0 & \text{on } \ \partial B(0, R). \end{cases} \tag{2.3}$$

Moreover, they characterize the Kantorovich potential by means of a PDE.

**Evans-Gangbo Theorem.** *Let* $f_+, f_- \in L^1(\Omega)$ *be two non-negative Borel function satisfying the mass balance condition* (2.1). *Assume additionally that* $f_+$ *and* $f_-$ *are Lipschitz continuous functions with compact support such that* $supp(f_+) \cap supp(f_-) = \emptyset$. *Set* $u_p$ *the solutions of* (2.3). *Then* $u_p$ *converges uniformly to* $u^* \in K_1(\Omega)$ *as* $p \to \infty$. *The limit* $u^*$ *verifies*

$$\int_\Omega u^*(x)(f_+(x) - f_-(x)) \, dx = \max \left\{ \int_\Omega u(x)(f_+(x) - f_-(x)) \, dx \ : \ u \in K_1(\Omega) \right\},$$

*and moreover, there exists* $0 \leq a \in L^\infty(\Omega)$ *such that*

$$f_+ - f_- = -\mathrm{div}(a\nabla u^*) \quad \text{in } \ \mathcal{D}'(\Omega).$$

*Furthermore* $|\nabla u^*| = 1$ *a.e. in the set* $\{a > 0\}$.

The function $a$ that appears in the previous result is the Lagrange multiplier corresponding to the constraint $|\nabla u^*| \leq 1$, and it is called the *transport density.* Moreover, what is very important from the point of view of mass transport, Evans and Gangbo used this PDE to find a proof of the existence of an optimal transport map for the classical Monge problem, different to the first one given by Sudakov in 1979 by means of probability methods ([12], see also [2] and [4]).

2.2. **The mass transport problem with import/export taxes.**

Assume (following [14]) that a business man produces some product in some factories represented by $f_+$ in $\Omega$ (note that $f_+$ encodes the amount of product and its location). There are also some consumers of the product in $\Omega$ with a distribution given by $f_-$ also defined in $\Omega$. The goal of the business man is to transport all the mass (there is an infinity cost penalization for not transporting the total mass) $f_+$ to $f_-$ (to satisfy the consumers) or to the boundary (to export the product). In doing this, he pays the transport costs (given by the Euclidean distance) and when a unit of mass is left on a point $y \in \partial\Omega$ an additional cost given by $T_e(y)$, the export taxes. He also has the constraint of satisfying all the demand of the consumers (there is also an infinity cost penalization for not covering all the demand), that is, he has to import product, if necessary, from the exterior (paying the transport costs plus the extra cost $T_i(x)$, the import taxes, for each unit of mass that enters $\Omega$ at the point $x \in \partial\Omega$). He has the freedom to chose to export or import mass provided he transports all the mass in $f_+$ and covers all the mass of $f_-$, and of course the transport must verify the natural equilibrium of masses. Observe also that, by assuming that $\Omega$ is convex, we prevent the fact that the boundary of $\Omega$ is crossed when the mass is transported inside $\Omega$. This is the only point where we use the fact that $\Omega$ is convex.

His main goal is to minimize the total cost of this operation (distribution of his production to satisfy the consumers with export/import paying if convenient, minimizing the total cost, that is given by the transport cost plus export/import taxes). In other words, the main goal is, given the set

$$\mathcal{A}(f_+, f_-) := \left\{ \mu \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega}) \ : \ \pi_1 \# \mu \llcorner \Omega = f_+ \mathcal{L}^N \llcorner \Omega \text{ and } \pi_2 \# \mu \llcorner \Omega = f_- \mathcal{L}^N \llcorner \Omega \right\},$$

to obtain

$$\min \left\{ \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\mu + \int_{\partial\Omega} T_i d(\pi_1 \# \mu) + \int_{\partial\Omega} T_e d(\pi_2 \# \mu) \ : \ \mu \in \mathcal{A}(f_+, f_-) \right\}.$$

As we will see in the next section, this is the description of the mass transport problem involved in the above maximization problem obtained by the limiting procedure on (1.1) as $p \to \infty$.

To clarify this relation, let us give the following argument: a clever fellow proposes the business man to leave him the planing and offers him the following deal: to pick up the product $f_+(x)$, $x \in \Omega$, he will charge him $\varphi(x)$, and to pick it up at $x \in \partial\Omega$, $T_i(x)$ (paying the taxes by himself); and to leave the product at the consumer's location $f_-(y)$, $y \in \Omega$, he will charge him $\psi(y)$, and for leaving it at $y \in \partial\Omega$, $T_e(y)$ (paying again the taxes himself); moreover, he proposes that he will do all this in such a way that:

$$\varphi(x) + \psi(y) \le |x - y| \tag{2.4}$$

and

$$-T_i \le \varphi \quad \text{and} \quad -T_e \le \psi \quad \text{on } \partial\Omega. \tag{2.5}$$

In addition he guarantees some compensation (assuming negative payments if necessary, that is, $\varphi$ and $\psi$ are not necessarily non negative). Observe that in this case (2.4) and (2.5) imply

$$-T_i(x) - T_e(y) \le |x - y| \qquad \forall\, x, y \in \partial\Omega, \tag{2.6}$$

which is a natural condition because it says that if one imports some mass from $x$ and exports it to $y$, he does not get benefit. This condition, for $x = y \in \partial\Omega$, says that

$$T_i(x) + T_e(x) \ge 0,$$

i.e., in the same point the sum of exportation and importation taxes is non negative.

Let us now introduce the operator $J : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}$, defined by

$$J(\varphi, \psi) := \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy,$$

and let

$$\mathcal{B}(T_i, T_e) := \left\{ \begin{array}{l} (\varphi, \psi) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \varphi(x) + \psi(y) \le |x - y|, \\ \hfill -T_i \le \varphi,\ -T_e \le \psi \text{ on } \partial\Omega \end{array} \right\}.$$

The aim of the fellow that helps the business man is to obtain

$$\sup \left\{ J(\varphi, \psi) \ : \ (\varphi, \psi) \in \mathcal{B}(T_i, T_e) \right\}.$$

Now, given $(\varphi, \psi) \in \mathcal{B}(T_i, T_e)$ and $\mu \in \mathcal{A}(f_+, f_-)$ we have

$$\begin{aligned} J(\varphi, \psi) = & \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy \\ & = \int_{\overline{\Omega}} \varphi(x) d\pi_1 \# \mu - \int_{\partial\Omega} \varphi d\pi_1 \# \mu + \int_{\overline{\Omega}} \psi(y) d\pi_2 \# \mu - \int_{\partial\Omega} \psi d\pi_2 \# \mu \\ & \le \int_{\overline{\Omega} \times \overline{\Omega}} |x - y|\, d\mu + \int_{\partial\Omega} T_i d(\pi_1 \# \mu) + \int_{\partial\Omega} T_e d(\pi_2 \# \mu). \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup \left\{ J(\varphi, \psi) \ : \ (\varphi, \psi) \in \mathcal{B}(T_i, T_e) \right\} \\ & \le \inf \left\{ \int_{\overline{\Omega} \times \overline{\Omega}} |x - y|\, d\mu + \int_{\partial\Omega} T_i d(\pi_1 \# \mu) + \int_{\partial\Omega} T_e d(\pi_2 \# \mu) \ : \ \mu \in \mathcal{A}(f_+, f_-) \right\}. \end{aligned} \tag{2.7}$$

This inequality will imply that the business man accept the offer. But, in fact, there is no gap between both costs as we will see in the next duality result whose proof uses ideas from [14].

**Theorem 2.1.** *Assume that $T_i$ and $T_e$ satisfy*

$$-T_i(x) - T_e(y) < |x-y| \quad \forall\, x,y \in \partial\Omega. \tag{2.8}$$

*Then,*

$$\begin{aligned} &\sup\left\{ J(\varphi,\psi) \ : \ (\varphi,\psi)\in \mathcal{B}(T_i,T_e)\right\} \\ &= \min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu + \int_{\partial\Omega} T_i d\pi_1\#\mu + \int_{\partial\Omega} T_e d\pi_2\#\mu \ : \ \mu\in\mathcal{A}(f_+,f_-)\right\}. \end{aligned} \tag{2.9}$$

*Proof.* By (2.8), there exist $\varphi_0,\psi_0\in C(\overline{\Omega})$ such that $\varphi_{0|\partial\Omega}\le T_i$ and $\psi_{0|\partial\Omega}\le T_e$ satisfying

$$-\varphi_0(x) - \psi_0(y) < |x-y| \quad \forall\, x,y\in\Omega. \tag{2.10}$$

In fact, let $L<1$ (close to 1) be such that

$$-T_i(x) - T_e(y) < L|x-y| \quad \forall\, x,y\in\partial\Omega,$$

and let

$$\varphi_0(x) = \min_{y\in\partial\Omega}\left\{T_i(y) + L|x-y|\right\}.$$

Taking $x=y\in\partial\Omega$ we obtain

$$\varphi_0(x) = \min_{y\in\partial\Omega}\left\{T_i(y) + L|x-y|\right\} \le T_i(x).$$

Now, we have

$$-T_i(x) - L|x-y| < T_e(y) \quad \forall\, x,y\in\partial\Omega,$$

and there exists $\varepsilon>0$ small such that

$$-T_i(x) - L|x-y| + \varepsilon < T_e(y) \quad \forall\, x,y\in\partial\Omega,$$

hence

$$\psi_0(y) = -\min_{x\in\partial\Omega}(T_i(x) + L|x-y| - \varepsilon) < T_e(y).$$

Finally, we have,

$$\begin{aligned} \varphi_0(x)+\psi_0(y) &= \min_{z\in\partial\Omega}\left\{T_i(z) + L|x-z|\right\} - \min_{z\in\partial\Omega}(T_i(z)+L|z-y|) + \varepsilon \\ &\ge -L|x-y| + \varepsilon > -|x-y|. \end{aligned}$$

Now, the proof follows the ideas of the proof of the Kantorovich duality Theorem given in [14]. Let us introduce the operators

$$\Theta,\Psi : C(\overline{\Omega}\times\overline{\Omega}) \to [0,+\infty]$$

defined by

$$\Theta(u) := \begin{cases} 0 & \text{if } \; u(x,y) \geq -|x-y| \\ +\infty & \text{else.} \end{cases}$$

To define the operator $\Psi$, observe first that if for $u \in C(\overline{\Omega} \times \overline{\Omega})$, we define

$$\mathcal{A}(u) := \left\{ (\varphi, \psi) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \; : \; u(x,y) = \varphi(x) + \psi(y), \; T_i \geq \varphi, \; T_e \geq \psi \text{ on } \partial\Omega \right\},$$

then, in the case $\mathcal{A}(u) \neq \emptyset$, we have that there exists

$$\min_{(\varphi,\psi)\in\mathcal{A}(u)} \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy. \tag{2.11}$$

In fact, fix $(\tilde{\varphi}, \tilde{\psi}) \in \mathcal{A}(u)$. Then, for any $(\varphi, \psi) \in \mathcal{A}(u)$, we have

$$\varphi(x) - \tilde{\varphi}(x) = \tilde{\psi}(y) - \psi(y) = \alpha \quad \forall\, x, y \in \overline{\Omega},$$

consequently

$$\int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy = \int_\Omega \tilde{\varphi}(x) f_+(x)\, dx + \int_\Omega \tilde{\psi}(y) f_-(y)\, dy + \alpha \int_\Omega (f_+(x) - f_-(x))\, dx.$$

Moreover,

$$\alpha = \varphi(x) - \tilde{\varphi}(x) \leq T_i(x) - \tilde{\varphi}(x) \leq k_1$$

and

$$\alpha = \tilde{\psi}(y) - \psi(y) \geq \tilde{\psi}(y) - T_e(y) \geq -k_2.$$

Therefore, given a minimizing sequence $\{(\varphi_n, \psi_n)\}$ of (2.11), since

$$\alpha_n = \varphi_n(x) - \tilde{\varphi}(x) = \tilde{\psi}(y) - \psi_n(y),$$

with $-k_2 \leq \alpha_n \leq k_1$, if we let

$$\alpha := \lim_{n\to\infty} \alpha_n$$

and

$$\varphi = \tilde{\varphi} + \alpha, \quad \psi = \tilde{\psi} - \alpha,$$

we have that $(\varphi, \psi)$ is a minimizer of (2.11). So we can define the operator $\Psi$ as

$$\Psi(u) := \begin{cases} \displaystyle\min_{(\varphi,\psi)\in\mathcal{A}(u)} \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy & \text{if } \; \mathcal{A}(u) \neq \emptyset, \\ +\infty & \text{if } \; \mathcal{A}(u) = \emptyset. \end{cases}$$

Clearly, $\Theta, \Psi$ are convex functionals on $C(\overline{\Omega} \times \overline{\Omega})$. By (2.10), for $u_0(x,y) := \varphi_0(x) + \psi_0(y)$, $\mathcal{A}(u_0) \neq \emptyset$ and $\Psi(u_0) < \infty$. Moreover, since $u_0(x,y) > -|x-y|$, we have $\Theta(u_0) = 0$

and $\Theta$ is continuous at $u_0$. Then, we can apply the Fenchel-Rocafellar duality Theorem (see for instance [14, Theorem 1.9.]) to get

$$\max_{\mu\in\mathcal{M}(\overline{\Omega}\times\overline{\Omega})} \left[-\Theta^*(-\mu)-\Psi^*(\mu)\right] = \inf_{u\in C(\overline{\Omega}\times\overline{\Omega})} \left[\Theta(u)+\Psi(u)\right], \tag{2.12}$$

where $\Phi^*$ and $\Theta^*$ are the Legendre-Frenchel transform of the operator $\Phi$ and $\Theta$, respectively.

Now, we compute both sides of (2.12). For the right hand side we obtain,

$$\begin{aligned}\inf_{u\in C(\overline{\Omega}\times\overline{\Omega})} \left[\Theta(u)+\Psi(u)\right] &= \inf_{\substack{u\in C(\overline{\Omega}\times\overline{\Omega}):\\ u(x,y)\geq -|x-y|\\ \mathcal{A}(u)\neq\emptyset}} \Psi(u)\\ &= \inf_{\substack{\varphi,\psi\in C(\overline{\Omega}):\\ \varphi(x)+\psi(y)\geq -|x-y|\\ T_i\geq\varphi,\ T_e\geq\psi \text{ on } \partial\Omega}} \int_\Omega \varphi(x) f_+(x)\,dx + \int_\Omega \psi(y) f_-(y)\,dy,\end{aligned}$$

from where it follows that

$$\inf_{u\in C(\overline{\Omega}\times\overline{\Omega})} \left[\Theta(u)+\Psi(u)\right] = -\sup\left\{J(\varphi,\psi)\ :\ (\varphi,\psi)\in\mathcal{B}(T_i,T_e)\right\}. \tag{2.13}$$

For the left hand side of (2.12) we first compute the Legendre-Frenchel transforms of the operators $\Theta$ and $\Psi$. For $\mu\in\mathcal{M}(\overline{\Omega}\times\overline{\Omega})$, we have

$$\begin{aligned}\Theta^*(-\mu) &= \sup_{u\in C(\overline{\Omega}\times\overline{\Omega})} \left(-\int_{\overline{\Omega}\times\overline{\Omega}} u(x,y)\,d\mu(x,y) - \Theta(u)\right)\\ &= \sup_{\substack{u\in C(\overline{\Omega}\times\overline{\Omega})\\ u(x,y)\geq -|x-y|}} -\int_{\overline{\Omega}\times\overline{\Omega}} u(x,y)\,d\mu(x,y).\end{aligned}$$

Hence,

$$\Theta^*(-\mu) = \begin{cases} \displaystyle\int_{\overline{\Omega}\times\Omega} |x-y|\,d\mu(x,y) & \text{if } \mu\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})\\ +\infty & \text{else.}\end{cases}$$

On the other hand,

$$\begin{aligned}
&\Psi^*(\mu)\\
&= \sup_{\substack{u \in C(\overline{\Omega} \times \overline{\Omega})\\ \mathcal{A}(u) \neq \emptyset}} \int_{\overline{\Omega}\times\overline{\Omega}} u(x,y)\, d\mu(x,y) - \min_{(\varphi,\psi)\in\mathcal{A}(u)} \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy\\
&= \sup_{\substack{\varphi,\psi \in C(\overline{\Omega}):\\ T_i \geq \varphi,\ T_e \geq \psi \text{ on } \partial\Omega}} \int_{\overline{\Omega}\times\overline{\Omega}} (\varphi(x) + \psi(y))\, d\mu(x,y) - \int_\Omega \varphi(x) f_+(x)\, dx - \int_\Omega \psi(y) f_-(y)\, dy\\
&= \sup_{\substack{\varphi,\psi \in C(\overline{\Omega})\\ T_i \geq \varphi,\ T_e \geq \psi \text{ on } \partial\Omega}} \int_{\overline{\Omega}} (\varphi d\pi_1\#\mu + \psi d\pi_1\#\mu) - \int_\Omega \varphi(x) f_+(x)\, dx - \int_\Omega \psi(y) f_-(y)\, dy.
\end{aligned}$$

Hence,

$$\Psi^*(\mu) = \begin{cases} \displaystyle\sup_{\substack{\varphi,\psi \in C(\overline{\Omega})\\ T_i \geq \varphi,\ T_e \geq \psi \text{ on } \partial\Omega}} \int_{\partial\Omega} (\varphi d\pi_1\#\mu + \psi d\pi_1\#\mu) & \text{if } \mu \in \mathcal{A}(f_+, f_-)\\ +\infty. & \text{else.} \end{cases}$$

Therefore,

$$\begin{aligned}
\max_{\mu\in\mathcal{M}(\overline{\Omega}\times\overline{\Omega})} [-\Theta^*(-\mu) - \Psi^*(\mu)] &= - \min_{\mu\in\mathcal{M}(\overline{\Omega}\times\overline{\Omega})} [\Theta^*(-\mu) + \Psi^*(\mu)]\\
&= - \min_{\substack{\mu \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})\\ \mu \in \mathcal{A}(f_+, f_-)}} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu(x,y) + \int_{\partial\Omega} T_i d\pi_1\#\mu + \int_{\partial\Omega} T_e d\pi_1\#\mu.
\end{aligned}$$

Then, putting together the above expression and (2.13) into (2.12) we get (2.9). □

**Remark 2.2.** Let us point out that we can prove the above result for any lower semi-continuous cost function. Nevertheless, we have restricted ourselves to the Euclidean distance for the sake of clarity.

## 3. Duality, Kantorovich potentials and optimal export/import masses

In this section we obtain the main results of this paper. As we have mentioned in the introduction, our approach is based in taking limit as $p$ goes to infinity to solutions of the $p-$Laplacian problems (1.2) of least energy. This will give a complete proof of the duality theorem (note that this proof is different from the previous one), but this approach also gives more detailed information for the transport problem under consideration. It provides an explicit approximation of the Kantorovich potential $u_\infty$ and describes the required import/export masses on the boundary. In addition, we obtain transport plans constructed via transport maps.

First, let us present a proof of the fact that we can take limit of the functions $u_p$, solutions to the minimization problem (1.1), as $p \to \infty$.

**Theorem 3.1.** *Assume that $g_1, g_2$ verifies (1.3). Then, up to subsequence, $u_p \to u_\infty$ uniformly as $p \to \infty$, and $u_\infty$ is a maximizer of the variational problem*

$$\max \left\{ \int_\Omega w(x) f(x)\, dx \colon w \in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)} \le 1 \right\}. \tag{3.1}$$

*Proof.* Assume $p > N$. Let us see that there exists $w \in W^{1,\infty}_{g_1,g_2}(\Omega)$ with $\|\nabla w\|_{L^\infty(\Omega)} \le 1$. In fact, we define

$$w(x) = \max_{y \in \partial\Omega} \left\{ g_1(y) - |x-y| \right\}.$$

We have that $|w(x) - w(y)| \le |x-y|$ and (taking $x = y \in \partial\Omega$ in the definition of $w$)

$$w(x) \ge g_1(x) \qquad \text{for } x \in \partial\Omega.$$

Moreover, as (1.3) holds, we have

$$g_1(y) - |x-y| \le g_2(x) \qquad \forall\, x, y \in \partial\Omega,$$

and hence we obtain

$$w(x) = \max_{y \in \partial\Omega} \left\{ g_1(y) - |x-y| \right\} \le g_2(x) \qquad \forall\, x \in \partial\Omega.$$

Therefore, $w \in W^{1,\infty}_{g_1,g_2}(\Omega)$ with $\|\nabla w\|_{L^\infty(\Omega)} \le 1$. Now for any of such functions,

$$-\int_\Omega f u_p \le \frac{1}{p} \int_\Omega |\nabla u_p|^p - \int_\Omega f u_p \le \frac{1}{p} \int_\Omega |\nabla w|^p - \int_\Omega f w \le \frac{|\Omega|}{p} - \int_\Omega f w. \tag{3.2}$$

In [13, Theorem 2.E] it is proved that the Morrey's inequality

$$\|u\|_{L^\infty(\Omega)} \le C_{\Omega,p} \|\nabla u\|_{L^p(\Omega)} \quad \text{for any} \ \ u \in W^{1,p}_0(\Omega),\ p > N, \tag{3.3}$$

holds with constant

$$C_{\Omega,p} = \frac{1}{N^{\frac{1}{p}} |B_1(0)|^{\frac{1}{N}}} \left( \frac{p-1}{p-N} \right)^{1-\frac{1}{p}} |\Omega|^{\frac{1}{N}-\frac{1}{p}}.$$

Since the functions $(u_p - \max_{\partial\Omega} g_2)^+, (u_p - \min_{\partial\Omega} g_1)^+ \in W^{1,p}_0(\Omega)$, applying inequality (3.3), we get

$$\|u_p\|_{L^\infty(\Omega)} \le C_{\Omega,p} \|\nabla u_p\|_{L^p(\Omega)} + \max_{\partial\Omega} g_2,$$

and

$$\|u_p\|_{L^\infty(\Omega)} \le C_{\Omega,p} \|\nabla u_p\|_{L^p(\Omega)} + \min_{\partial\Omega} g_1.$$

Hence, we have

$$\|u_p\|_{L^\infty(\Omega)} \le C_{\Omega,p} \|\nabla u_p\|_{L^p(\Omega)} + \|g_1\|_{L^\infty(\partial\Omega)} + \|g_2\|_{L^\infty(\partial\Omega)}$$

Then, since

$$\lim_{p\to\infty} C_{\Omega,p} = \left(\frac{|\Omega|}{|B_1(0)|}\right)^{\frac{1}{N}},$$

we arrive to

$$\|u_p\|_{L^p(\Omega)} \le C_1 \|\nabla u_p\|_{L^p(\Omega)} + C_2, \tag{3.4}$$

where the constants $C_i$ are independent of $p$. Moreover, from (3.2), using Hölder's inequality and having in mind (3.4), we get

$$\frac{1}{p}\int_\Omega |\nabla u_p|^p \le C_3(\|u_p\|_{L^p(\Omega)} + 1) \le C_4(\|\nabla u_p\|_{L^p(\Omega)} + 1),$$

from where it follows that

$$\|\nabla u_p\|_{L^p(\Omega)}^{p-1} \le p\, C_5 \qquad \forall p > N. \tag{3.5}$$

From (3.4) and (3.5), we obtain that $\{u_p\}_{p>N}$ is bounded in $W^{1,p}(\Omega)$. We have in fact (see [5]) that,

$$|u_p(x) - u_p(y)| \le C_\Omega |x-y|^{1-\frac{N}{p}} \|\nabla u_p\|_p\,,$$

with $C_\Omega$ not depending on $p$. Then, by the Morrey-Sobolev's embedding and Arzela-Ascoli compactness criterion we can extract a sequence $p_i \to \infty$ such that

$$u_{p_i} \rightrightarrows u_\infty \quad \text{uniformly in } \overline{\Omega}.$$

Moreover, by (3.5), we obtain that

$$\|\nabla u_\infty\|_\infty \le 1.$$

Finally, passing to the limit in (3.2), we get

$$\int_\Omega u_\infty(x) f(x)\,dx = \max\left\{ \int_\Omega w(x) f(x)\,dx \colon w \in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)} \le 1 \right\},$$

as we wanted to prove. □

Now we present the general duality result that proves that the salesman and his fellow pay the same total cost under the natural condition (2.6) (note that the strict inequality is not necessary), giving a positive answer to the **Main Problem** stated in the Introduction.

**Theorem 3.2.** *If $T_i = -g_1$ and $T_e = g_2$ satisfy (2.6), or equivalently (1.3), then*

$$\begin{aligned} \int_\Omega u_\infty(x)(f_+(x) - f_-(x))dx &= \sup\left\{ J(\varphi,\psi) \ : \ (\varphi,\psi) \in \mathcal{B}(-g_1,g_2)\right\} \\ &= \min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu - \int_{\partial\Omega} g_1 d\pi_1\#\mu + \int_{\partial\Omega} g_2 d\pi_2\#\mu \ : \ \mu \in \mathcal{A}(f_+,f_-)\right\}. \end{aligned} \tag{3.6}$$

*where $u_\infty$ is the maximizer given in Theorem 3.1.*

Before proving this result we would like to pay attention to the following observation.

**Remark 3.3.** Fix $\mu \in \mathcal{A}(f_+, f_-)$ a measure where the minimum in (3.6) is taken. If $\mu_i := \pi_i \# \mu$, $i = 1, 2$, by the Kantorovich-Rubinstein Theorem, we have

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, d\nu : \ \nu \in \Pi(\mu_1, \mu_2) \right\} = \max \left\{ \int_{\overline{\Omega}} u d(\mu_1 - \mu_2) : \ u \in K_1(\overline{\Omega}) \right\}. \tag{3.7}$$

Let us see that $\mu$ is an optimal transport plan for (3.7). Indeed, if $\nu_\mu \in \Pi(\mu_1, \mu_2)$ is an optimal transport plan for (3.7), then, as $\mu \in \Pi(\mu_1, \mu_2)$,

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_\mu \leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\mu.$$

Now, since

$$\int_{\partial\Omega} g_1 d\pi_1 \# \nu_\mu - \int_{\partial\Omega} g_2 d\pi_2 \# \nu_\mu = \int_{\partial\Omega} g_1 d\mu_1 - \int_{\partial\Omega} g_2 d\mu_2,$$

we have

$$\begin{aligned} &\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_\mu - \int_{\partial\Omega} g_1 d\pi_1 \# \nu_\mu + \int_{\partial\Omega} g_2 d\pi_2 \# \nu_\mu \\ &\qquad \leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\mu - \int_{\partial\Omega} g_1 d\mu_1 + \int_{\partial\Omega} g_2 d\mu_2. \end{aligned}$$

On the other hand, since $\nu_\mu \in \mathcal{A}(f_+, f_-)$,

$$\begin{aligned} &\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\mu - \int_{\partial\Omega} g_1 d\mu_1 + \int_{\partial\Omega} g_2 d\mu_2 \\ &\qquad \leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_\mu - \int_{\partial\Omega} g_1 d\pi_1 \# \nu_\mu + \int_{\partial\Omega} g_2 d\pi_2 \# \nu_\mu. \end{aligned}$$

Therefore, the above inequality is an equality and then

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\mu = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_\mu,$$

and consequently $\mu$ is an optimal transport plan for (3.7).

Let $u^*$ be a Kantorovich potential in (3.7), then

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, d\mu = \int_{\overline{\Omega}} u^* d(\mu_1 - \mu_2).$$

Hence,

$$\begin{aligned} \int_\Omega u_\infty (f_+ - f_-) dx &= \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, d\mu - \int_{\partial\Omega} g_1 d\mu_1 + \int_{\partial\Omega} g_2 d\mu_2 \\ &= \int_{\overline{\Omega}} u^* d(\mu_1 - \mu_2) - \int_{\partial\Omega} g_1 d\mu_1 + \int_{\partial\Omega} g_2 d\mu_2, \end{aligned}$$

and, then,

$$\int_{\overline{\Omega}} u_\infty \, d(\mu_1 - \mu_2) \geq \int_\Omega u_\infty (f_+ - f_-) dx + \int_{\partial\Omega} g_1 d\mu_1 - \int_{\partial\Omega} g_2 \mu_2 = \int_{\overline{\Omega}} u^* d(\mu_1 - \mu_2),$$

that is, $u_\infty$ is also a Kantorovich potential for (3.7). From this last expression we also deduce that

$$u_\infty = g_i \quad \text{on} \;\; \operatorname{supp}(\mu_i \llcorner \partial\Omega), \;\; i = 1, 2.$$

Let us point out that there is an important difference between the problem we are studying and the classical transport problem: there are masses, the ones that appear on the boundary, that are unknown variables. We will see in the next result that by taking limit of $u_p$, minimizers of (1.1), we obtain, not only the potential $u_\infty$, but also such masses. This result also provides the proof of Theorem 3.2. This is an alternative proof of Theorem 2.1 (remark that in the previous proof the Kantorovich potentials are not used). Note that we first assume the more restrictive condition (3.8), that is, a strict inequality in (1.3), and then we obtain the result under (1.3) by an approximation argument (see below).

**Theorem 3.4.** *Assume that $g_1, g_2$ verifies*

$$g_1(x) - g_2(y) < |x - y| \qquad \forall \, x, y \in \partial\Omega. \tag{3.8}$$

*Let $u_p$ be a sequence of minimizers to Problem* (1.1) *and $u_\infty$ its uniform limit stated at Theorem 3.1. Then, up to a subsequence,*

$$\mathcal{X}_p := |Du_p|^{p-2} Du_p \to \mathcal{X} \quad \textit{weakly* in the sense of measures,}$$

$$-div(\mathcal{X}) = f \quad \textit{in the sense of distributions in } \Omega,$$

*Moreover, the distributions $\mathcal{X}_p \cdot \eta$ defined as*

$$\langle \mathcal{X}_p \cdot \eta, \varphi \rangle := \int_\Omega \mathcal{X}_p \cdot \nabla\varphi - \int_\Omega f\varphi \quad \textit{for } \varphi \in C_0^\infty(\mathbb{R}^N), \tag{3.9}$$

*are Radon measures supported on $\partial\Omega$, that, up to a subsequence,*

$$\mathcal{X}_p \cdot \eta \to \mathcal{V} \quad \textit{weakly* in the sense of measures;}$$

*and $u_\infty$ is a Kantorovich potential for the classical transport problem for the measures*

$$f_+ \mathcal{L}^N \llcorner \Omega + \mathcal{V}^+ \qquad \textit{and} \qquad f_- \mathcal{L}^N \llcorner \Omega + \mathcal{V}^-.$$

*Proof.* Consider $p > N$. Let $u_p$ be a minimizer of problem (1.1), which is a solution of the obstacle problem (1.2), and consider $\mathcal{X}_p = |Du_p|^{p-2} Du_p$. Then, we know that

$$-\mathrm{div}(\mathcal{X}_p) = f \quad \text{in the sense of distributions.} \tag{3.10}$$

Which implies that $\mathcal{X}_p \cdot \eta$ is a distribution supported on $\partial\Omega$. Let us see that, in fact,

$$\operatorname{supp}(\mathcal{X}_p \cdot \eta) \subset \{x \in \partial\Omega : u_p(x) = g_1(x)\} \cup \{x \in \partial\Omega : u_p(x) = g_2(x)\}. \tag{3.11}$$

Take $\varphi$ a smooth function such that

$$\operatorname{supp}(\varphi) \cap (\{x \in \partial\Omega : u_p(x) = g_1(x)\} \cup \{x \in \partial\Omega : u_p(x) = g_2(x)\}) = \emptyset,$$

then there exists $\delta > 0$ such that $u_p + t\varphi \in W^{1,p}_{g_1,g_2}(\Omega)$ for all $|t| < \delta$. Hence, since $u_p$ is a minimizer of the problem (1.1), we have

$$\int_\Omega f(u_p + t\varphi)\, dx - \int_\Omega f u_p\, dx \leq \int_\Omega \frac{|\nabla u_p + t\nabla\varphi|^p}{p}\, dx - \int_\Omega \frac{|\nabla u_p|^p}{p}\, dx.$$

Then, dividing by $t$ and taking limit as $t \to 0$, we get

$$\int_\Omega \mathcal{X}_p \cdot \nabla\varphi\, dx = \int_\Omega f\varphi\, dx,$$

from where it follows, having in mind (3.10), that

$$\langle \mathcal{X}_p \cdot \eta, \varphi \rangle = 0.$$

Consequently (3.11) holds.

On the other hand, if $\varphi$ is a positive smooth function whose support does not touch $\{x \in \partial\Omega : u_p(x) = g_2(x)\}$ (which is separated from $\{x \in \partial\Omega : u_p(x) = g_1(x)\}$ by the continuity of $u_p$ and the strict inequality in (3.8)) then there exists $\delta > 0$ such that $u_p + t\varphi \in W^{1,p}_{g_1,g_2}(\Omega)$ for all $0 \leq t < \delta$. Working as above we get

$$\langle \mathcal{X}_p \cdot \eta, \varphi \rangle \geq 0.$$

And similarly, if $\varphi$ is a positive smooth function whose support does not touch $\{x \in \partial\Omega : u_p(x) = g_1(x)\}$,

$$\langle \mathcal{X}_p \cdot \eta, \varphi \rangle \leq 0,$$

Consequently, $\mathcal{X}_p \cdot \eta$ is a Radon measure. The proof of this fact follows by writing $\mathcal{X}_p \cdot \eta = T_1 + T_2$ with $\langle T_i, \varphi \rangle = \langle \mathcal{X}_p \cdot \eta, \varphi\varphi_i \rangle$ with $\varphi_i \in \mathcal{C}^\infty_0(\mathbb{R}^N)$ such that

$$\varphi_1(x) = \begin{cases} 1, & x \in \partial\Omega,\ u_p(x) = g_1(x), \\ 0, & x \in \partial\Omega,\ u_p(x) = g_2(x), \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 1, & x \in \partial\Omega,\ u_p(x) = g_2(x), \\ 0, & x \in \partial\Omega,\ u_p(x) = g_1(x), \end{cases}$$

and noticing that the above arguments show that $T_1$ and $-T_2$ are nonnegative distributions and so Radon measures. Moreover,

$$\operatorname{supp}((\mathcal{X}_p \cdot \eta)^+) \subset \{x \in \partial\Omega : u_p(x) = g_1(x)\}, \tag{3.12}$$

and

$$\operatorname{supp}((\mathcal{X}_p \cdot \eta)^-) \subset \{x \in \partial\Omega : u_p(x) = g_2(x)\}. \tag{3.13}$$

In addition, we have that (3.9) is satisfied for test functions $\varphi \in W^{1,p}(\Omega)$ and we can rewrite it as

$$\int_\Omega \mathcal{X}_p \cdot \nabla\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} \varphi \, d\left(\mathcal{X}_p \cdot \eta\right). \tag{3.14}$$

Using (3.8), there is $0 < L < 1$ such that

$$g_1(x) - g_2(y) < L|x-y| \quad \forall\, x, y \in \partial\Omega.$$

Therefore, if we define

$$w(x) := \inf_{y\in\partial\Omega} (g_2(y) + L|x-y|),$$

we have $w$ is a $L$-Lipschitz function in $\overline{\Omega}$ satisfying

$$g_1(x) < w(x) \leq g_2(x) \quad \forall\, x \in \partial\Omega.$$

By (3.14), (3.11), (3.12) and (3.13), we have

$$\begin{aligned}\int_\Omega (u_p - w) f &= \int_\Omega \mathcal{X}_p \cdot \nabla(u_p - w) - \int_{\partial\Omega} (u_p - w) d\left(\mathcal{X}_p \cdot \eta\right) \\ &= \int_\Omega \mathcal{X}_p \cdot \nabla(u_p - w) - \int_{\{g_1 = u_p\}} (g_1 - w) d\left(\mathcal{X}_p \cdot \eta\right)^+ + \int_{\{g_2 = u_p\}} (g_2 - w) d\left(\mathcal{X}_p \cdot \eta\right)^-.\end{aligned}$$

Then, since $g_1 - w \leq -c$, with $c > 0$, and $g_2 - w \geq 0$, by Hölder's and Young inequalities, it follows that

$$\begin{aligned}\int_\Omega |\nabla u_p|^p + c\int_{\partial\Omega} d\left(\mathcal{X}_p \cdot \eta\right)^+ &\leq \int_\Omega (u_p - w) f + \int_\Omega \mathcal{X}_p \cdot \nabla w \\ &\leq C + \left(\int_\Omega |\nabla u_p|^p\right)^{\frac{1}{p'}} L|\Omega|^{\frac{1}{p'}} \leq C + \frac{L}{p'}\int_\Omega |\nabla u_p|^p + \frac{1}{p}|\Omega|.\end{aligned}$$

Hence,

$$\left(1 - \frac{L^{p'}}{p'}\right)\int_\Omega |\nabla u_p|^p + c\int_{\partial\Omega} d\left(\mathcal{X}_p \cdot \eta\right)^+ \leq C + \frac{1}{p}|\Omega|.$$

Therefore, since $0 < L < 1$ and $c > 0$, we obtain that there exist positive constants $A_1$, $A_2$, such that

$$\int_\Omega |\nabla u_p|^p \leq A_1, \qquad \forall\, p \geq N+1, \tag{3.15}$$

and

$$\int_{\partial\Omega} d\left(\mathcal{X}_p \cdot \eta\right)^+ \leq A_2, \qquad \forall\, p \geq N+1. \tag{3.16}$$

Moreover, working similarly, changing the function $w$ by the function

$$\tilde{w}(x) = \sup_{y\in\partial\Omega} (g_1(y) - L|x-y|),$$

we get

$$\int_{\partial\Omega} d\,(\mathcal{X}_p \cdot \eta)^- \leq A_3, \qquad \forall\, p \geq N+1. \tag{3.17}$$

As consequence of (3.15), we have that

$$\text{the measures } \ \mathcal{X}_p \mathcal{L}^N \llcorner \Omega \quad \text{are equi-bounded in } \Omega, \tag{3.18}$$

and from (3.16) and (3.17), we have that

$$\text{the measures } \ \mathcal{X}_p \cdot \eta\, \mathcal{H}^{N-1} \llcorner \partial\Omega \quad \text{are equi-bounded on } \partial\Omega. \tag{3.19}$$

Moreover, by Theorem 3.1, we know that

$$\{u_p\} \quad \text{are equi-bounded and equicontinuous in } \overline{\Omega}. \tag{3.20}$$

From (3.18), (3.19) and (3.20), we obtain that there exists a sequence $p_i \to \infty$ such that

$$\begin{gathered} u_{p_i} \rightrightarrows u_\infty \quad \text{uniformly in } \overline{\Omega}, \quad \text{with } \|\nabla u_\infty\|_\infty \leq 1, \\ \mathcal{X}_{p_i} \rightharpoonup \mathcal{X} \quad \text{weakly}^* \text{ as measures in } \Omega, \end{gathered} \tag{3.21}$$

and

$$\mathcal{X}_{p_i} \cdot \eta \rightharpoonup \mathcal{V} \quad \text{weakly}^* \text{ as measures on } \partial\Omega. \tag{3.22}$$

Moreover, we have that

$$\int_\Omega \nabla\varphi \, d\mathcal{X} = \int_\Omega f\varphi\, dx + \int_{\partial\Omega} \varphi d\mathcal{V} \quad \forall\, \varphi \in C^1(\overline{\Omega}). \tag{3.23}$$

That is, formally,

$$\begin{cases} -\mathrm{div}(\mathcal{X}) = f & \text{in } \Omega \\ \mathcal{X} \cdot \eta = \mathcal{V} & \text{on } \partial\Omega. \end{cases}$$

Set $\varphi = u_\infty$ in (3.14). Then taking limit as $i \to \infty$ and having in mind (3.22), we get

$$\lim_{i\to\infty} \int_\Omega \mathcal{X}_{p_i} \cdot \nabla u_\infty = \int_\Omega f u_\infty + \int_{\partial\Omega} u_\infty\, d\mathcal{V}. \tag{3.24}$$

Let $v_\epsilon$ be smooth functions uniformly converging to $u_\infty$ as $\epsilon \searrow 0$, with $\|\nabla v_\epsilon\|_\infty \leq 1$. By (3.14), we have

$$\int_\Omega \mathcal{X}_{p_i} \cdot \nabla u_\infty = \int_\Omega f(u_\infty - v_\epsilon) + \int_{\partial\Omega} (u_\infty - v_\epsilon)\, d\,(\mathcal{X}_{p_i} \cdot \eta) + \int_\Omega \nabla v_\epsilon\, d\,(\mathcal{X}_{p_i} \cdot \eta)\,.$$

Then, by (3.21), (3.22) and (3.24), taking limit in the above equality as $i \to \infty$, we obtain

$$\int_\Omega f u_\infty + \int_{\partial\Omega} u_\infty\, d\mathcal{V} = \int_\Omega f(u_\infty - v_\epsilon) + \int_{\partial\Omega} (u_\infty - v_\epsilon)\, d\mathcal{V} + \int_\Omega \nabla v_\epsilon\, d\mathcal{X}. \tag{3.25}$$

Now we are going to show that, as $\epsilon \searrow 0$,

$$\nabla v_\epsilon \quad \text{converges in } L^2(|\mathcal{X}|) \ \text{ to the Radon-Nikodym derivative } \frac{\mathcal{X}}{|\mathcal{X}|}. \tag{3.26}$$

To do that we use the technique used in [2, Theorem 5.2]. We first notice that the functional $\Psi : [C(\overline{\Omega}, \mathbb{R}^N)]^* \to \mathbb{R}$ defined by

$$\Psi(\nu) := \int_\Omega \left| \frac{\nu}{|\nu|} - w \right|^2 d|\nu|$$

is lower semicontinuous with respect to the the weak convergence of measures for any $w \in C(\overline{\Omega}, \mathbb{R}^N)$. Next, we observe that

$$\lim_{\epsilon \to 0^+} \limsup_{p \to \infty} \int_\Omega \left| \frac{\mathcal{X}_p}{|\mathcal{X}_p|} - \nabla v_\epsilon \right|^2 d|\mathcal{X}_p| = 0, \tag{3.27}$$

where $v_\epsilon$ are smooth functions uniformly converging to $u_\infty$ with $\|\nabla v_\epsilon\|_\infty \le 1$. Indeed,

$$\begin{aligned} \int_\Omega \left| \frac{\mathcal{X}_p}{|\mathcal{X}_p|} - \nabla v_\epsilon \right|^2 d|\mathcal{X}_p| &\le 2 \int_\Omega |\nabla u_p|^{p-1} \left( 1 - \frac{\nabla v_\epsilon \cdot \nabla u_p}{|\nabla u_p|} \right) dx \\ &\le 2 \int_\Omega |\nabla u_p|^{p-2} \left( |\nabla u_p|^2 - \nabla v_\epsilon \cdot \nabla u_p \right) dx + \omega_p \\ &= 2 \int_\Omega f(u_p - v_\epsilon)\, dx + \int_{\partial\Omega} (u_p - v_\epsilon) d\, \mathcal{X}_p \cdot \eta + \omega_p, \end{aligned}$$

where $\omega_p := \sup_{t \ge 0} t^{p-1} - t^p$ tends to 0 as $p \to \infty$. Then, having in mind (3.22) and the uniform convergence of $u_p$ and $v_\epsilon$ to $u_\infty$, we obtain (3.27). Now, from (3.27), taking into account the lower semicontinuity of $\Psi$, setting $p = p_i$ and passing to the limit as $i \to \infty$, we obtain

$$\lim_{\epsilon \to 0^+} \int_\Omega \left| \frac{\mathcal{X}}{|\mathcal{X}|} - \nabla v_\epsilon \right|^2 d|\mathcal{X}| = 0.$$

Consequently, (3.26) holds true.

Now, having in mind (3.26), if we take the limit in (3.25) as $\epsilon \searrow 0$, we get

$$\int_\Omega f u_\infty + \int_{\partial\Omega} u_\infty \, d\mathcal{V} = \lim_{\epsilon \downarrow 0} \int_\Omega \nabla v_\epsilon \cdot \frac{\mathcal{X}}{|\mathcal{X}|} \, d|\mathcal{X}| = \int_\Omega d|\mathcal{X}|. \tag{3.28}$$

Giving a function $\varphi \in C^1(\overline{\Omega})$ with $\|\nabla \varphi\|_\infty \le 1$, by (3.23) and (3.28), we have

$$\begin{aligned} &\int_\Omega u_\infty f \, dx + \int_{\partial\Omega} u_\infty d\mathcal{V} = \int_\Omega d|\mathcal{X}| \ge \int_\Omega \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \nabla \varphi \, d|\mathcal{X}| \\ &= \int_\Omega \nabla \varphi \, d\mathcal{X} = \int_\Omega \varphi f \, dx + \int_{\partial\Omega} \varphi d\mathcal{V}. \end{aligned}$$

Then, by approximation, given a Lipschitz continuous function $w$ with $\|\nabla w\|_\infty \le 1$, we obtain

$$\int_\Omega u_\infty f\,dx + \int_{\partial\Omega} u_\infty d\mathcal{V} \ge \int_\Omega wf\,dx + \int_{\partial\Omega} w d\mathcal{V}.$$

Therefore $u_\infty$ is a Kantorovich potential for the classical transport problem associated to the measures $f_+\,d\mathcal{L}^N \llcorner \Omega + \mathcal{V}^+$ and $f_-\,d\mathcal{L}^N \llcorner \Omega + \mathcal{V}^-$. Observe that the total masses for both measures are the same. □

This provides a proof for Theorem 3.2, and moreover we have that $\mathcal{V}^+$ *and* $\mathcal{V}^-$ *are import and export masses in our original problem.*

*Proof of Theorem 3.2.* It is enough to show that

$$\begin{aligned}&\int_\Omega u_\infty(x)(f_+(x) - f_-(x))\,dx \\ &\qquad = \min\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu - \int_{\partial\Omega} g_1 d\pi_1\#\mu + \int_{\partial\Omega} g_2 d\pi_2\#\mu \ :\ \mu \in \mathcal{A}(f_+, f_-)\right\}.\end{aligned}$$

Let us first assume that

$$g_1(x) - g_2(y) < |x-y| \quad \forall\, x, y \in \partial\Omega.$$

And take $\tilde{f}_1 := f_+\mathcal{L}^N \llcorner \Omega + \mathcal{V}^+$ and $\tilde{f}_2 := f_-\mathcal{L}^N \llcorner \Omega + \mathcal{V}^-$, being $\mathcal{V}$ the measure obtained in Theorem 3.4. We have that

$$\int_{\overline{\Omega}} u_\infty d(\tilde{f}_1 - \tilde{f}_2) = \min_{\nu\in\Pi(\tilde{f}_1,\tilde{f}_2)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_0,$$

for some $\nu_0 \in \Pi(\tilde{f}_1, \tilde{f}_2)$. Then, since $\pi_1\#\nu_0 \llcorner \partial\Omega = \mathcal{V}^+$ and $\pi_2\#\nu_0 \llcorner \partial\Omega = \mathcal{V}^-$,

$$\int_\Omega u_\infty(f_+ - f_-) = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_0 - \int_{\partial\Omega} g_1\,d\pi_1\#\nu_0 + \int_{\partial\Omega} g_2\,d\pi_2\#\nu_0,$$

and consequently, since $\nu_0 \in \mathcal{A}(f_+, f_-)$, the above equality jointly with (2.7) gives

$$\begin{aligned}\inf&\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu - \int_{\partial\Omega} g_1\,d\pi_1\#\mu + \int_{\partial\Omega} g_2\,d\pi_2\#\mu \ :\ \mu \in \mathcal{A}(f_+, f_-)\right\} \\ &\qquad = \int_\Omega u_\infty(f_+ - f_-)\,dx,\end{aligned}$$

and the above infimum is in fact a minimum attained at $\nu_0$.

The result under condition (1.3) now follows by approximation. Indeed, let $g_{1,n}, g_{2,n}$ be Lipschitz continuous functions on the boundary $\partial\Omega$ satisfying

$$g_{1,n}(x) - g_{2,n}(y) < |x-y| \quad \forall\, x, y \in \partial\Omega,$$

and

$$g_{i,n} \rightrightarrows g_i \quad \text{uniformly on } \ \partial\Omega, \quad i = 1, 2.$$

By the previous argument, there exist $u_{\infty,n} \in W^{1,\infty}(\Omega)$, with $\|\nabla u_{\infty,n}\|_\infty \leq 1$ and $g_{1,n} \leq u_{\infty,n} \leq g_{2,n}$ on $\partial\Omega$, and there exist measures $\mu_n \in \mathcal{A}(f_+, f_-)$ satisfying

$$\begin{aligned}
&\int_\Omega u_{\infty,n}(x)(f_+(x) - f_-(x))\,dx \\
&= \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu_n - \int_{\partial\Omega} g_{1,n} d\pi_1\#\mu_n + \int_{\partial\Omega} g_{2,n} d\pi_2\#\mu_n \\
&= \min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu - \int_{\partial\Omega} g_{1,n} d\pi_1\#\mu + \int_{\partial\Omega} g_{2,n} d\pi_2\#\mu : \ \mu \in \mathcal{A}(f_+, f_-) \right\}.
\end{aligned} \tag{3.29}$$

By the Morrey-Sobolev's embedding and Arzela-Ascoli compactness criterion we can suppose that, for a subsequence if necessary,

$$u_{\infty,n} \rightrightarrows u_\infty \quad \text{uniformly in } \overline{\Omega}.$$

Moreover,

$$\|\nabla u_\infty\|_\infty \leq 1, \quad \text{and} \quad g_1 \leq u_\infty \leq g_2 \ \text{ on } \partial\Omega.$$

On the other hand, let us see that

$$\mu_n(\overline{\Omega}\times\overline{\Omega}) \leq \int_\Omega (f_+(x) + f_-(x))\,dx \qquad \forall\, n \in \mathbb{N}. \tag{3.30}$$

Indeed,

$$\mu_n(\overline{\Omega}\times\overline{\Omega}) = \int_\Omega f_-(x)\,dx + \pi_2\#\mu_n(\partial\Omega). \tag{3.31}$$

Now, if we define

$$\tilde{\mu_n} := \mu_n - \mu_n \llcorner (\partial\Omega \times \partial\Omega),$$

we have $\tilde{\mu_n} \in \mathcal{A}(f_+, f_-)$, and

$$\begin{aligned}
&\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\tilde{\mu_n} - \int_{\partial\Omega} g_1\,d\pi_1\#\tilde{\mu_n} + \int_{\partial\Omega} g_2\,d\pi_2\#\tilde{\mu_n} \\
&\quad = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu_n - \int_{\partial\Omega} g_1\,d\pi_1\#\mu_n + \int_{\partial\Omega} g_2\,d\pi_2\#\mu_n \\
&\qquad - \int_{\partial\Omega\times\partial\Omega} \left(|x-y| - g_{1,n}(x) + g_{2,n}(y)\right)\,d\mu_n.
\end{aligned}$$

Hence, since

$$|x-y| - g_{1,n}(x) + g_{2,n}(y) > 0 \quad \forall x, y \in \partial\Omega,$$

from (3.29), we deduce that

$$\mu_n(\partial\Omega \times \partial\Omega) = 0.$$

That is, there is not transportation of mass directly between the boundary, and hence all the mass exported will come from $f_+$ (so, $\pi_2\#\mu_n(\partial\Omega) \le \int_\Omega f_+$), and all the mass imported will go to cover $f_-$ (so, $\pi_1\#\mu_n(\partial\Omega) \le \int_\Omega f_-$):

$$\begin{aligned}\int_\Omega f_+(x)\,dx + \pi_1\#\mu_n(\partial\Omega) &= \mu_n(\overline{\Omega}\times\overline{\Omega}) \\ &\ge \mu_n(\overline{\Omega}\times\partial\Omega) + \mu_n(\partial\Omega\times\overline{\Omega}) - \mu_n(\partial\Omega\times\partial\Omega) \\ &= \mu_n(\overline{\Omega}\times\partial\Omega) + \mu_n(\partial\Omega\times\overline{\Omega}) \\ &= \pi_2\#\mu_n(\partial\Omega) + \pi_1\#\mu_n(\partial\Omega)\,,\end{aligned}$$

and we get

$$\pi_2\#\mu_n(\partial\Omega) \le \int_\Omega f_+(x)\,dx,$$

(similarly, we get $\pi_1\#\mu_n(\partial\Omega) \le \displaystyle\int_\Omega f_-(x)\,dx$). Consequently, by (3.31), we obtain (3.30).

Now, we can assume that $\mu_n \rightharpoonup \mu_0$ weakly* as measures in $\overline{\Omega}\times\overline{\Omega}$, with $\mu_0 \in \mathcal{A}(f_+, f_-)$. Then, passing to the limit in (3.29) we end the proof. □

3.1. **Construction of transport plans via transport maps.**

Once export/import masses on the boundary are fixed, let us call them $\mathcal{V}^+$ and $\mathcal{V}^-$, and an optimal transport plan $\mu$ is taken for $f_+\mathcal{L}^N\llcorner\Omega + \mathcal{V}^+$ and $f_-\mathcal{L}^N\llcorner\Omega + \mathcal{V}^-$, with $\mu(\partial\Omega\times\partial\Omega) = 0$, we know which part of $f_+$, we call $\tilde{f}_+$ such part, is going to be exported and which part of $f_-$, we call it $\tilde{f}_-$, is covered by imported material:

$$\tilde{f}_+ = \pi_1\#(\mu\llcorner\overline{\Omega}\times\partial\Omega)$$

and

$$\tilde{f}_- = \pi_2\#(\mu\llcorner\partial\Omega\times\overline{\Omega}).$$

Now, let us state two facts.

1. **Existence of optimal maps.** Since $\tilde{f}_+$ and $\tilde{f}_-$ are absolutely continuous with respect to the Lebesgue measure, by the Sudakov Theorem ([2, Theorem 6.2]), there exists $t_2 : \operatorname{supp}(\tilde{f}_+) \to \partial\Omega$ an optimal map pushing $\tilde{f}_+$ forward $\mathcal{V}^+$, and there exists $t_1 : \operatorname{supp}(\tilde{f}_-) \to \partial\Omega$ an optimal map pushing $\tilde{f}_-$ forward $\mathcal{V}^-$. These maps are described by

$$\begin{aligned}g_2(t_2(x)) + |x - t_2(x)| &= \min_{y\in\partial\Omega}(g_2(y) + |x-y|) \quad \text{for } a.e.\ x \in \operatorname{supp}(\tilde{f}_+), \\ g_1(t_1(x)) - |x - t_1(x)| &= \max_{y\in\partial\Omega}(g_1(y) - |x-y|) \quad \text{for } a.e.\ x \in \operatorname{supp}(\tilde{f}_-).\end{aligned}$$

Moreover, there exists $t_0 : \operatorname{supp}(f_+) \to \Omega$ an optimal map pushing $f_+ - \tilde{f}_+$ forward to $f_- - \tilde{f}_-$.

All these maps are such that

$$\mu^*(x,y) = \tilde{f}_+(x)\delta_{y=t_2(x)} + (f_+(x) - \tilde{f}_+(x))\delta_{y=t_0(x)} + \tilde{f}_-(y)\delta_{x=t_1(y)}$$

is an optimal transport plan for our problem, which is given in terms of transport maps.

2. **Kantorovich potentials.** The limit $u_\infty$ is a Kantorovich potential, in the classical sense, for each of the three transports problems that appears in the above description, the transport of $\tilde{f}_+$ to $\mathcal{V}^+$ on $\partial\Omega$, of $\mathcal{V}^-$ on $\partial\Omega$ to $\tilde{f}_-$ and of $f_+ - \tilde{f}_+$ to $f_- - \tilde{f}_-$ inside $\Omega$. In fact, it holds that

$$\begin{aligned} u_\infty(x) &= g_2(t_2(x)) + |x - t_2(x)| \quad \text{for } a.e.\ x \in \operatorname{supp}(\tilde{f}_+), \\ u_\infty(x) &= g_1(t_1(x)) - |x - t_1(x)| \quad \text{for } a.e.\ x \in \operatorname{supp}(\tilde{f}_-), \\ u_\infty(x) &= u_\infty(t_0(x)) + |x - t_0(x)| \quad \text{for } a.e.\ x \in \operatorname{supp}(f_+ - \tilde{f}_+). \end{aligned}$$

So, in the support of the mass that is exported from $\Omega$ and in the support of the mass that is covered by mass imported from $\partial\Omega$, the potential is given only in terms of $g_2$ and $g_1$ respectively:

$$\begin{aligned} u_\infty(x) &= \min_{y \in \partial\Omega} (g_2(y) + |x - y|) \quad \text{for } a.e.\ x \in \operatorname{supp}(\tilde{f}_+), \\ u_\infty(x) &= \max_{y \in \partial\Omega} (g_1(y) - |x - y|) \quad \text{for } a.e.\ x \in \operatorname{supp}(\tilde{f}_-). \end{aligned}$$

To show that these two points hold we argue as follows: let $t_0,\ t_1,\ t_2$ be any optimal transport plan given as above, then it follows that

$$\mu^*(x,y) = \tilde{f}_+(x)\delta_{y=t_2(x)} + (f_+(x) - \tilde{f}_+(x))\delta_{y=t_0(x)} + \tilde{f}_-(y)\delta_{x=t_1(y)}$$

is an optimal transport plan for our problem. Now, we can take, for each $x \in \operatorname{supp}(\tilde{f}_+)$ a point $y_x \in \partial\Omega$ where $\min_{y\in\partial\Omega}(g_2(y) + |x-y|)$ is attained in such a way that $\tilde{t}_2(x) = y_x$ is Borel measurable (see for example [6]), and for each $x \in \operatorname{supp}(\tilde{f}_-)$ a point $z_x \in \partial\Omega$ where $\max_{y\in\partial\Omega}(g_1(y) - |x-y|)$ is attained in such a way that $\tilde{t}_1(x) = z_x$ is Borel measurable, then it holds that

$$g_2(t_2(x)) + |x - t_2(x)| \geq g_2(\tilde{t}_2(x)) + |x - \tilde{t}_2(x)|,$$

$$g_1(t_1(x)) - |x - t_1(x)| \leq g_1(\tilde{t}_1(x)) - |x - \tilde{t}_1(x)|,$$

and the cost of the transport for the plan

$$\tilde{f}_+(x)\delta_{y=\tilde{t}_2(x)} + (f_+(x) - \tilde{f}_+(x))\delta_{y=t_0(x)} + \tilde{f}_-(y)\delta_{x=\tilde{t}_1(y)}$$

is, in fact, equal to the one for $\mu^*$. Hence, the above inequalities are equalities.

On the other hand, by putting this particular $\mu^*$ in (3.6), a careful computation using that $|\nabla u_\infty| \leq 1$ and that $u_\infty = g_1$ in $\operatorname{supp}(\mathcal{V}^-)$ and $u_\infty = g_2$ in $\operatorname{supp}(\mathcal{V}^+)$ gives 2. Alternatively, one can use the Dual Criteria for Optimality and Remark 3.3.

Then, from these facts, we conclude that all the relevant information to build transport maps (transport rays and sets, see [14, 15]) for this problem is encoded by the limit function $u_\infty$.

## 4. Examples

In this section we provide simple examples in which the solution to the mass transport problem described in §2.2 can be explicitly computed.

**Example 4.1.** Let $\Omega := (0,1)$, $f_+ := \chi_{(0,\frac{1}{2})}$, $f_- := \chi_{(\frac{3}{4},1)}$, and $g_1 \equiv 0$, $g_2 \equiv \frac{1}{2}$. For these data the equality (3.6) is written as

$$\begin{aligned}
&\max_{\substack{u \in C([0,1]) \\ u(x)-u(y) \leq |x-y| \\ 0 \leq u(0),\, u(1) \leq \frac{1}{2}}} \int_0^{\frac{1}{2}} u(x)dx - \int_{\frac{3}{4}}^1 u(x)dx \\
&= \min \left\{ \int_{[0,1]\times[0,1]} |x-y|\, d\mu + \frac{1}{2}\big(\pi_2\#\mu(0) + \pi_2\#\mu(1)\big) \ : \ \mu \in \mathcal{A}(f_+, f_-) \right\}.
\end{aligned} \tag{4.1}$$

It is easy to see that the maximum in (4.1) is taken in the function $u_\infty$ defined by

$$u_\infty(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{4}], \\ 1 - x & \text{if } x \in [\frac{1}{4}, 1], \end{cases}$$

for which

$$\int_0^{\frac{1}{2}} u_\infty(x)dx - \int_{\frac{3}{4}}^1 u_\infty(x)dx = \frac{9}{32}.$$

Moreover, for $\mu(x,y) = \chi_{(0,\frac{1}{4})}(x)\delta_{y=0} + \chi_{(\frac{1}{4},\frac{1}{2})}(x)\delta_{y=x+\frac{1}{2}}$, a simple calculation shows that $\pi_1\#\mu = f_+$ and $\pi_2\#\mu = f_- + \frac{1}{4}\delta_0$, and

$$\int_{[0,1]\times[0,1]} |x-y|\, d\mu + \frac{1}{2}\big(\pi_2\#\mu(0) + \pi_2\#\mu(1)\big) = \frac{9}{32},$$

therefore $\mu$ is minimizer in (4.1). This *optimal plan* exports the mass in $(0, \frac{1}{4})$ to the point 0 on the boundary and transports the mass from $(\frac{1}{4}, \frac{1}{2})$ to $(\frac{1}{2}, 1)$.

Observe that importing mass from the point 1 in the boundary is free of taxes, nevertheless, to import a little quantity of mass from that point would imply to export more mass to the point 0 where the taxes are sufficiently large to increase enough the total price of the operation. Nevertheless if we decrease a little bit the taxes on 0 the situation changes: consider $g_1 \equiv 0$, $g_2 \equiv \frac{1}{2} - b$ $(0 < b \leq \frac{1}{2})$; for these data

$$u_\infty(x) = \begin{cases} x + \frac{1}{2} - b & \text{if } x \in [0, \frac{b+1}{4}], \\ 1 - x - \frac{b}{2} & \text{if } x \in [\frac{b+1}{4}, 1 - \frac{b}{4}], \\ x - 1 & \text{if } x \in [1 - \frac{b}{4}, 1], \end{cases}$$

and $\mu(x,y) = \chi_{(0,\frac{b+1}{4})}(x)\delta_{y=0} + \chi_{(\frac{b+1}{4},\frac{1}{2})}(x)\delta_{y=x+\frac{1}{2}-\frac{b}{4}} + \chi_{(1-\frac{b}{4},1)}(y)\delta_{x=1}$ realize the maximum and the minimum in (3.6) with cost $\frac{9}{32} - \frac{b(b+2)}{10}$. Now this *optimal plan* exports the mass in $(0, \frac{b+1}{4})$ to the point 0 on the boundary, transports the mass from $(\frac{b+1}{4}, \frac{1}{2})$ to $(\frac{3}{4}, 1 - \frac{b}{4})$, and imports the mass to cover $(1 - \frac{b}{4}, 1)$ from the point 1 on the boundary.

On the contrary, let us now increase the taxes in 0: take $g_1 \equiv 0$, $g_2(0) = \frac{1}{2} + b$ $(0 < b \leq \frac{1}{2})$ and $g_2(1) = a$ $(0 \leq a \leq 1)$. Then, in this case

$$u_\infty(x) = \begin{cases} x + \frac{1}{2} + b & \text{if } x \in [0, \frac{1}{4} - \frac{(b-a)^+}{2}], \\ a \wedge b - (x - 1) & \text{if } x \in [\frac{1}{4} - \frac{(b-a)^+}{2}, 1], \end{cases}$$

and

$$\mu(x,y) = \chi_{(0,\frac{1}{4}-\frac{(b-a)^+}{2})}(x)\delta_{y=0} + \chi_{(\frac{1}{4}-\frac{(b-a)^+}{2},\frac{1}{2}-\frac{(b-a)^+}{2})}(x)\delta_{y=x+\frac{1}{2}-\frac{(b-a)^+}{2}} + \chi_{(\frac{1}{2}-\frac{(b-a)^+}{2},\frac{1}{2})}(x)\delta_{y=1}$$

realize the maximum and the minimum in (3.6) with total cost $\frac{9}{32} + \frac{b}{4} - \frac{((b-a)^+)^2}{4}$. In this case we are exporting the mass in $(0, \frac{1}{4} - \frac{(b-a)^+}{2})$ to 0, transporting the mass in $(\frac{1}{4} - \frac{(b-a)^+}{2}, \frac{1}{2} - \frac{(b-a)^+}{2})$ to $(\frac{3}{4}, 1)$ and exporting the mass in $(\frac{1}{2} - \frac{(b-a)^+}{2}, \frac{1}{2})$ to 1.

In the above example the masses $f_+$ and $f_-$ did not satisfy the mass balance condition. Let us see now another example in which the mass balance condition between both masses is satisfied, in order to show the difference between this transport problem and the classical one.

**Example 4.2.** Let $\Omega := (0,1)$, $f_+ := \chi_{(0,\frac{1}{2})}$, $f_- := \chi_{(\frac{1}{2},1)}$, and $g_1 \equiv 0$, $g_2 \equiv \frac{1}{2}$. For these data, the maximum and the minimum in (3.6) are taken in

$$u_\infty(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{8}], \\ -x + \frac{3}{4} & \text{if } x \in [\frac{1}{8}, \frac{7}{8}], \\ x - 1 & \text{if } x \in [\frac{7}{8}, 1], \end{cases}$$

and $\mu(x,y) = \chi_{(0,\frac{1}{8})}(x)\delta_{y=0} + \chi_{(\frac{1}{8},\frac{1}{2})}(x)\delta_{y=x+\frac{3}{8}} + \chi_{(\frac{7}{8},1)}(y)\delta_{x=1}$, and the cost of this transport problem is $\frac{7}{32}$. This *optimal plan* exports the mass in $(0, \frac{1}{8})$ to the point 0 on the boundary, transports the mass from $(\frac{1}{8}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{7}{8})$, and imports the mass to cover $(\frac{7}{8}, 1)$ from the point 1 on the boundary.

This transport problem would have coincided with the classical one if we had put $g_2 \equiv 1$.

Finally, let us show an example in which the import/export taxes coincide.

**Example 4.3.** Let $\Omega := (0,1)$, $f_+ := \chi_{(0,\frac{1}{2})}$, $f_- := \chi_{(\frac{1}{2},1)}$, and $g_1 = g_2 = g$, $g(0) := 0$, $g(1) := \frac{1}{2}$. Now,the maximum and the minimum in (3.6) are taken in

$$u_\infty(x) = \begin{cases} x & \text{if } x \in [0, \frac{3}{8}], \\ -x + \frac{3}{4} & \text{if } x \in [\frac{3}{8}, \frac{5}{8}], \\ x - \frac{1}{2} & \text{if } x \in [\frac{5}{8}, 1], \end{cases}$$

and $\mu(x,y) = \chi_{(0,\frac{3}{8})}(x)\delta_{y=0} + \chi_{(\frac{3}{8},\frac{1}{2})}(x)\delta_{y=x+\frac{1}{8}} + \chi_{(\frac{5}{8},1)}(y)\delta_{x=1}$, and the total cost of the transport process is $\frac{-1}{32}$. This *optimal plan* exports the mass in $(0,\frac{3}{8})$ to the point 0 on the boundary, transports the mass from $(\frac{3}{8},\frac{1}{2})$ to $(\frac{1}{2},\frac{5}{8})$, and imports the mass to cover $(\frac{5}{8},1)$ from the point 1 on the boundary. Observe that in this case the taxes are good enough to get benefits from export/import some mass.

## 5. Limited importation/exportation

In this section we show how to adapt the previous ideas to handle the case in which on top of the previous setting we add punctual restrictions on the amount of mass that can be exported/imported.

Let us now consider on the taxes $T_i$, $T_e$ the only restriction $T_i + T_e \geq 0$ on $\partial\Omega$ and let us take two functions $M_i, M_e \in L^\infty(\partial\Omega)$ with $M_i, M_e \geq 0$ on $\partial\Omega$ that are going to represent the limitations for import/export mass through the boundary. Accordingly, consider that the business man has the restriction of limiting the amount of mass by the punctual quantities of $M_e(x)$ for export at $x \in \partial\Omega$ and $M_i(x)$ for import at $x$ on the boundary. We then need to impose

$$-\int_{\partial\Omega} M_e \leq -\int_\Omega (f_+ - f_-) \leq \int_{\partial\Omega} M_i,$$

that says that the interplay with the boundary (that is, importation/exportation) is possible. Now, the main goal is, given the new set

$$\mathcal{A}_\ell(f_+,f_-) := \left\{ \begin{array}{ll} \mu \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}) : & \pi_1\#\mu \llcorner \Omega = f_+\mathcal{L}^N \llcorner \Omega,\ \pi_2\#\mu \llcorner \Omega = f_-\mathcal{L}^N \llcorner \Omega, \\ & \pi_1\#\mu \llcorner \partial\Omega \leq M_i,\ \pi_2\#\mu \llcorner \Omega \leq M_e \end{array} \right\},$$

to obtain

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu + \int_{\partial\Omega} T_i d(\pi_1\#\mu) + \int_{\partial\Omega} T_e d(\pi_2\#\mu) \ : \ \mu \in \mathcal{A}_\ell(f^+,f^-) \right\}.$$

In this situation, the dual problem proposed by the clever fellow is similar to the previous one but with the following new conditions on the payments $\varphi$ and $\psi$. The fellow proposes that he will do all the operation in such a way that (2.4) is satisfied, but now his charges will not necessarily satisfy (here is the main difference with the previous case) (2.5), nevertheless he will pay a compensation, when $\varphi < -T_i$ or $\psi < -T_e$ on the boundary, of amount

$$\int_{\partial\Omega} M_i(-\varphi - T_i)^+ + \int_{\partial\Omega} M_e(-\psi - T_e)^+.$$

We introduce the new functional $J_\ell : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}$, defined by

$$J_\ell(\varphi, \psi) := \int_\Omega \varphi f_+ + \int_\Omega \psi f_- - \int_{\partial\Omega} M_i(-\varphi - T_i)^+ - \int_{\partial\Omega} M_e(-\psi - T_e)^+,$$

and the set

$$\mathcal{B}_\ell := \left\{(\varphi, \psi) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \varphi(x) + \psi(y) \leq |x - y|\right\}.$$

Then the aim of the fellow is to obtain

$$\sup_{(\varphi,\psi)\in\mathcal{B}_\ell} J_\ell(\varphi, \psi).$$

In this situation we also get

$$\sup_{(\varphi,\psi)\in\mathcal{B}_\ell} J_\ell(\varphi, \psi) \leq \inf_{\mu\in\mathcal{A}_\ell(f^+,f^-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x - y| \, d\mu + \int_{\partial\Omega} T_i d(\pi_1\#\mu) + \int_{\partial\Omega} T_e d(\pi_2\#\mu),$$

and again we can show that there is no gap between both costs. In fact, if we now consider the energy functional

$$\Psi_\ell(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p} \, dx - \int_\Omega f(x)u(x) \, dx + \int_{\partial\Omega} j(x, u(x)),$$

where

$$j(x, r) = \begin{cases} M_i(x)(-T_i(x) - r) & \text{if } r < -T_i(x), \\ 0 & \text{if } -T_i(x) \leq r \leq T_e(x), \\ M_e(x)(r - T_e(x)) & \text{if } r > T_e(x), \end{cases}$$

the variational problem

$$\min_{u\in W^{1,p}(\Omega)} \Psi_\ell(u) \tag{5.1}$$

has a minimizer $u_p$ in $W^{1,p}(\Omega)$, which is a least energy solution (in an adequate sense, see (5.7)) of the nonlinear boundary problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta + \partial j(\cdot, u(\cdot)) \ni 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Then, we have the following result:

**Theorem 5.1.** *Given $u_p$ the solutions to (5.1), up to subsequence, $u_p \to u_\infty$ uniformly as $p \to \infty$. Moreover, $u_\infty$ is a maximizer of the variational problem*

$$\max\left\{\int_\Omega w(x)f(x) \, dx - \int_{\partial\Omega} j(x, w(x)) \colon w \in W^{1,\infty}(\Omega), \, \|\nabla w\|_{L^\infty(\Omega)} \leq 1\right\}. \tag{5.3}$$

*Proof.* The proof is analogous to the one of Theorem 3.1. Note that here we do not need to impose a condition like (1.3) since to obtain $u_p$ we are minimizing in the whole $W^{1,p}(\Omega)$ without any pointwise constraint, therefore we can use any $w$ with $\|\nabla w\|_{L^\infty(\Omega)} \leq 1$ as a test to obtain the required uniform bounds for $\|\nabla u_p\|_{L^p(\Omega)}$.

Finally, let us point out that to show (5.3) we use the fact that $j(x,\cdot)$ is lower semi-continuous. □

**Remark 5.2.** Fix $\mu \in \mathcal{A}_\ell(f_+, f_-)$ a measure where the minimum in (5.5) is taken. If $\mu_i := \pi_i \# \mu$, $i = 1, 2$, by the Kantorovich-Rubinstein Theorem, we have

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\nu : \ \nu \in \Pi(\mu_1,\mu_2)\right\} = \max\left\{\int_{\overline{\Omega}} u d(\mu_1-\mu_2) : \ u \in K_1(\overline{\Omega})\right\}. \tag{5.4}$$

Following the same lines of Remark 3.3 we have that $\mu$ is an optimal transport plan for (5.4). And $u_\infty$ is a Kantorovich potential for (5.4). Moreover, $u_\infty \leq -T_i$ on $\mathrm{supp}(\mu_1 \llcorner \partial\Omega)$ and $u_\infty \geq T_e$ on $\mathrm{supp}(\mu_2 \llcorner \partial\Omega)$.

**Theorem 5.3.** *Assume that $T_i$ and $T_e$ satisfy $T_i(x)+T_e(x) \geq 0$ on $\partial\Omega$. Then, the duality result holds,*

$$\int_\Omega u_\infty(x)(f^+(x)-f^-(x))dx - \int_{\partial\Omega} j(x,u_\infty(x))$$

$$= \sup\{J_\ell(\varphi,\psi) \ : \ (\varphi,\psi) \in \mathcal{B}_\ell\} \tag{5.5}$$

$$= \min\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu + \int_{\partial\Omega} T_i d\pi_1\#\mu + \int_{\partial\Omega} T_e d\pi_2\#\mu \ : \ \mu \in \mathcal{A}_\ell(f^+,f^-)\right\}.$$

The proof of this theorem uses the following result and an approximation argument for the case $T_i(x) + T_e(x) \geq 0$ similar to the given for Theorem 3.1.

**Theorem 5.4.** *Assume that $T_i, T_e$ verifies*

$$T_i(x) + T_e(x) > 0 \qquad \forall\, x \in \partial\Omega. \tag{5.6}$$

*Let $u_p$ be a sequence of minimizers to Problem* (5.1). *Then:*

*1. There exist $\mathcal{X}_p \cdot \eta \in L^\infty(\partial\Omega)$, $-\mathcal{X}_p \cdot \eta \in \partial j(x,u_p)$ a.e. $x \in \partial\Omega$ such that*

$$\int_{\partial\Omega} \mathcal{X}_p \cdot \eta\, \varphi = \int_\Omega |Du_p|^{p-2} Du_p \cdot \nabla\varphi - \int_\Omega f\varphi \quad \textit{for all } \varphi \in W^{1,p}(\Omega). \tag{5.7}$$

*Up to a subsequence, $\mathcal{X}_p \cdot \eta \to \mathcal{V}$ weakly$*$ in $L^\infty(\partial\Omega)$ with*

$$\mathcal{V}^+ \leq M_i, \ \mathcal{V}^- \leq M_e.$$

*2. $u_\infty$ is a Kantorovich potential for the classical transport problem for the measures*

$$f^+\mathcal{L}^N \llcorner \Omega + \mathcal{V}^+\, d\mathcal{H}^{N-1} \llcorner \partial\Omega \quad \textit{and} \quad f^-\mathcal{L}^N \llcorner \Omega + \mathcal{V}^-\, d\mathcal{H}^{N-1} \llcorner \partial\Omega.$$

*Proof.* The proof is similar to that of Theorem 3.4. Again, $u_\infty$ is a Kantorovich potential for the new transport problem, and moreover, the variational approach provides the required import/export masses on the boundary; $\mathcal{V}^+$ and $\mathcal{V}^-$.

The only difference occurs when performing the following computations (we use here the same notations as in the proof of Theorem 3.4):

We have that $\mathcal{X}_p \cdot \eta$ is a Radon measure with

$$\operatorname{supp}((\mathcal{X}_p \cdot \eta)^+) \subset \{x \in \partial\Omega : u_p(x) \leq -T_i(x)\}, \tag{5.8}$$

and

$$\operatorname{supp}((\mathcal{X}_p \cdot \eta)^-) \subset \{x \in \partial\Omega : u_p(x) \geq T_e(x)\}. \tag{5.9}$$

Let us see that

$$(\mathcal{X}_p \cdot \eta)^+ \leq M_i, \tag{5.10}$$

and

$$(\mathcal{X}_p \cdot \eta)^- \leq M_e. \tag{5.11}$$

For $\varphi$ a positive smooth function we have

$$\langle \mathcal{X}_p \cdot \eta, \varphi \rangle \geq \limsup_{t \to 0^+} -\int_{\partial\Omega} \frac{j(x, u_p + t\varphi) - j(x, u_p)}{t}.$$

Now, we observe that

$$\begin{aligned} &-\int_{\partial\Omega} \frac{j(x, u_p + t\varphi) - j(x, u_p)}{t} \\ &\geq -\int_{\{x\in\partial\Omega : u_p(x) \geq T_e(x)\}} M_e\varphi - \frac{1}{t}\int_{\{x\in\partial\Omega : u_p(x)+t\varphi(x) > T_e(x) > u_p(x)\}} M_e(u_p + t\varphi - T_e), \\ &\geq -\int_{\{x\in\partial\Omega : u_p(x) \geq T_e(x)\}} M_e\varphi - \int_{\{x\in\partial\Omega : u_p(x)+t\varphi(x) > T_e(x) > u_p(x)\}} M_e\varphi \to -\int_{\{x\in\partial\Omega : u_p(x) \geq T_e(x)\}} M_e\varphi. \end{aligned}$$

Hence,

$$\mathcal{X}_p \cdot \eta \geq -M_e \chi_{\{u_p \geq T_e\}},$$

so,

$$(\mathcal{X}_p \cdot \eta)^- \leq M_e \chi_{\{u_p \geq T_e\}}.$$

Since we have (5.9), we get (5.11). Similarly, we obtain (5.10).

Consequently, we have that (5.7) is satisfied for test functions $\varphi \in W^{1,p}(\Omega)$ and we can rewrite it as

$$\int_\Omega \mathcal{X}_p \cdot \nabla\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} \mathcal{X}_p \cdot \eta\, \varphi\,. \tag{5.12}$$

Taking $\varphi = u_p$ in (5.12) we get, taking into account the $L^\infty$-boundedness of $u_p$, (5.10) and (5.11), that there exists a constant $C$ such that

$$\int_\Omega |\nabla u_p|^p = \int_\Omega f u_p + \int_{\partial\Omega} \mathcal{X}_p \cdot \eta \, u_p \leq C, \tag{5.13}$$

that is

$$\text{the measures } \; \mathcal{X}_p \mathcal{L}^N \llcorner \Omega \quad \text{are equi-bounded in } \Omega. \tag{5.14}$$

Therefore, there exists a sequence $p_i \to \infty$ such that

$$u_{p_i} \rightrightarrows u_\infty \quad \text{uniformly in } \overline{\Omega}, \quad \text{with } \|\nabla u_\infty\|_\infty \leq 1,$$

$$\mathcal{X}_{p_i} \rightharpoonup \mathcal{X} \quad \text{weakly}^* \text{ as measures in } \Omega,$$

and

$$\mathcal{X}_{p_i} \cdot \eta \rightharpoonup \mathcal{V} \quad \text{weakly}^* \text{ in } L^\infty(\partial\Omega). \tag{5.15}$$

Moreover, we have that

$$\int_\Omega \nabla\varphi \, d\mathcal{X} = \int_\Omega f \varphi \, dx + \int_{\partial\Omega} \mathcal{V}\varphi \quad \forall\, \varphi \in C^1(\overline{\Omega}). \tag{5.16}$$

That is, formally,

$$\begin{cases} -\mathrm{div}(\mathcal{X}) = f & \text{in } \Omega \\ \mathcal{X} \cdot \eta = \mathcal{V} & \text{on } \partial\Omega. \end{cases}$$

From this point the proof runs as before.

Note that the proof of the approximation argument in this case is even simpler since the measures $\mathcal{V}^+$ and $\mathcal{V}^-$ are uniformly bounded by $M_i$ and $M_e$ on $\partial\Omega$. $\square$

Finally, let us present a simple example in which the limitation of export/import mass increases the total cost and modifies the Kantorovich potential and the optimal transport plan.

**Example 5.5.** Let $\Omega := (0,1)$, $f^+ := \chi_{(0,\frac{1}{2})}$, $f^- := \chi_{(\frac{1}{2},1)}$, and $g_1 \equiv 0$, $g_2 \equiv \frac{1}{2}$. For the first case studied, in which we do not limit the amount of mass that enters or leaves the domain (say $M_i = M_e = +\infty$), in Example 4.2 we explicitly compute that the cost of the transport problem is $\frac{56}{16^2}$ and that it is obtained with the Kantorovich potential

$$u_\infty(x) = \begin{cases} x + \frac{1}{2} & \text{if } \; x \in [0, \frac{1}{8}], \\ -x + \frac{3}{4} & \text{if } \; x \in [\frac{1}{8}, \frac{7}{8}], \\ x - 1 & \text{if } \; x \in [\frac{7}{8}, 1]; \end{cases}$$

being an optimal transport plan:

$$\mu(x,y) = \chi_{(0,\frac{1}{8})}(x)\delta_{y=0} + \chi_{(\frac{1}{8},\frac{1}{2})}(x)\delta_{y=x+\frac{3}{8}} + \chi_{(\frac{7}{8},1)}(y)\delta_{x=1}.$$

Now, we consider the limiting functions $M_i = M_e = \frac{1}{16}$. In this case the cost of the transport is $\frac{58}{16^2}$, it is attained at

$$u_\infty(x) = \begin{cases} x + \frac{5}{8} & \text{if } x \in [0, \frac{1}{16}], \\ -x + \frac{3}{4} & \text{if } x \in [\frac{1}{16}, \frac{15}{16}], \\ x - \frac{9}{8} & \text{if } x \in [\frac{15}{16}, 1], \end{cases}$$

and an optimal transport plan is given by

$$\mu(x,y) = \chi_{(0,\frac{1}{16})}(x)\delta_{y=0} + \chi_{(\frac{1}{16},\frac{1}{2})}(x)\delta_{y=x+\frac{7}{16}} + \chi_{(\frac{15}{16},1)}(y)\delta_{x=1}.$$

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