ONE-DIMENSIONAL SYMMETRY FOR SEMILINEAR EQUATIONS WITH UNBOUNDED DRIFT

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Abstract. We consider semilinear equations with unbounded drift in the whole of $\mathbb{R}^n$ and we show that monotone solutions with finite energy are one-dimensional.

1. Introduction

In the paper [10] E. De Giorgi formulated the celebrated conjecture that bounded monotone solutions to the Allen-Cahn equation

$$\Delta u = u^3 - u \tag{1}$$

are necessarily one-dimensional (in the sense that the level sets are hyperplanes) at least if $n \leq 8$. This conjecture has been proved by Ghoussoub and Gui in [19](see also [3]) in dimension $n = 2$, and by Ambrosio and Cabré [2] in dimension $n = 3$ (see also [1]), and a counterexample has been given by del Pino, Kowalczyk and Wei in [11] for $n \geq 9$. Under the additional assumption that u connects -1 to 1, a proof has been presented by Savin [24] in dimension $n \leq 8$.

In this paper we consider the semilinear elliptic equation

$$\Delta u + c(z)u_z + \langle \nabla_y g(y), \nabla_y u \rangle + f(u) = 0, \tag{2}$$

where we write $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$. A solution u of (2) of the form

$$u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n, \tag{3}$$

where $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\omega \in \mathbb{R}^n$ with $|\omega| = 1$ will be called *one-dimensional.*

We are interested in symmetry results for solutions u which are monotone in the z-variable, i.e. satisfy

$$u_z(x) > 0 \qquad \forall x \in \mathbb{R}^n. \tag{4}$$

In particular, we will show that, under suitable assumptions, monotone solutions to (2) are necessarily one-dimensional (see Theorem 1.1).

Our methods rely on the geometric approach developed in [14] (see also [7, 8, 12, 13, 18, 25]), and our computations follow those in [15, 16], where the authors prove Liouville type results for stable solutions to elliptic equations in complete Riemmanian manifolds with nonnegative Ricci curvature.

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1.1. **Main result.** Let us state the main result of this paper:

Theorem 1.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $g \in C^2(\mathbb{R}^{n-1})$, $c \in C^1(\mathbb{R})$ and that*

$$c'(z)\,\mathrm{I}_{n-1} \geq \nabla^2_y g(y) \qquad \textit{for every } (y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}, \tag{5}$$

where I_{n-1} denotes the identity matrix on $\mathbb{R}^{n-1}$. Let $C \in C^2(\mathbb{R})$ be a primitive of c, and let u be a solution to (2) satisfying (4) and one of the following conditions:

a)

$$\int_{\mathbb{R}^n} |\nabla u|^2 e^{g(y)+C(z)} \mathrm{d}z\mathrm{d}y < +\infty\,; \tag{6}$$

b) *For all $z \in \mathbb{R}$*

$$\int_{\mathbb{R}^{n-1}} |\nabla u|^2 e^{g(y)+C(z)} \mathrm{d}y \leq K \qquad \text{for some } K > 0\,; \tag{7}$$

c) *$n = 2$ and for all $(y,z) \in \mathbb{R}^n$*

$$|\nabla u|^2 e^{g(y)+C(z)} \leq K \qquad \text{for some } K > 0\,; \tag{8}$$

then u is one-dimensional, and

$$\langle \left(c'(z)\,\mathrm{I}_{n-1} - \nabla^2_y g(y)\right) \nabla_y u, \nabla_y u \rangle = 0\,. \tag{9}$$

In particular, if the strict inequality holds in (5) for some (y,z) then u depends only on z.

From Theorem 1.1 we get the following corollaries which extend a result in [7], valid for the Ornstein-Uhlenbeck case $C(z) = -z^2/2$, $g(y) = -|y|^2/2$.

Corollary 1.2. *Let C, g bounded above and satisfying (5). Assume also that $n = 2$ or*

$$\int_{\mathbb{R}^{n-1}} e^{g(y)} \mathrm{d}y < +\infty\,. \tag{10}$$

Let $u \in W^{1,\infty}(\mathbb{R}^n)$ be a solution to (2) satisfying (4), then u is one-dimensional.

Proof. If C, g are bounded above and $u \in W^{1,\infty}(\mathbb{R}^n)$, then (8) holds, and moreover condition (10) implies (7). The thesis then follows directly from Theorem 1.1. □

Remark 1.3. From [21, Th. 2.4 and Rem. 2.5] it follows that, if $\nabla^2 g(y)$ and $c'(z)$ are uniformly bounded below, every bounded solution to (2) belongs to $W^{1,\infty}(\mathbb{R}^n)$.

Corollary 1.4. *Let C, g be concave, satisfying (5) and $-C, -g$ coercive. Let $u \in L^\infty(\mathbb{R}^n)$ be a solution to (2) satisfying (4), then u is one-dimensional.*

Proof. In [9, Thm 2.5, Cor. 4.3] it is proved that if $-C, -g$ are convex and coercive then any (weak) solution to (2) such that

$$\int_{\mathbb{R}^n} u^2 e^{g(y)+C(z)} \mathrm{d}z\mathrm{d}y < +\infty$$

also satisfies (6) (see Remark 2.3). In particular, any bounded solution to (2) satisfies (6), and we can conclude by Theorem 1.1. □

When $c(z) \equiv c \in \mathbb{R}$, solutions to (2) correspond to traveling (or standing if $c = 0$) wave solutions to the reaction-diffusion equation:

$$v_t = \Delta v + \langle \nabla_y g, \nabla_y v \rangle + f(v) \qquad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{11}$$

A traveling wave solution is a particular solution v to (11), uniformly translating in the z-direction at constant speed c, of the form

$$v(t, x) = u(y, z - ct).$$

We refer to [26, 29] and references therein for classical results about existence and uniqueness of traveling waves in infinite cylinders.

Corollary 1.5. *Let g be concave and let $v(t, x) = u(y, z - ct)$ be a traveling or a standing wave solution to* (11)*. If u satisfies one of the three conditions of Theorem 1.1, then u is one-dimensional. Moreover u depends only on z unless $n = 2$, g is constant and* (8) *holds.*

Conditions (6), (7), (8) are quite restrictive. However, traveling wave solutions satisfying these conditions are relevant to propagation and are sometimes called *variational traveling waves*. We refer to [22, 23] for a general analysis of such solutions, including necessary and sufficient conditions for existence.

If these conditions are not satisfied, equation (11) admits in general traveling waves which are not one-dimensional even in the case $n = 2$ and $g = 0$ (see [4, 5, 20]).

2. A Ornstein-Uhlenbeck type equations.

More generally, we shall consider the following equation of Ornstein-Uhlenbeck type:

$$\Delta u + \langle \nabla G(x), \nabla u \rangle + f(u) = 0 \qquad x \in \mathbb{R}^n, \tag{12}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and $G \in C^2(\mathbb{R}^n)$.

Notice that solutions to (12) are critical point of the functional

$$I(u) := \int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + F(u) \right) e^{G(x)} \mathrm{d}x, \tag{13}$$

where $F'(t) = -f(t)$. We define the function $\lambda_G \in C^0(\mathbb{R}^n)$ as

$$\lambda_G(x) := \text{ maximal eigenvalue of } \nabla^2 G(x). \tag{14}$$

Observe that, if $G(x) := g(y) + C(z)$, then (12) reduces to (2), and $\lambda_G(x) \geq C''(z)$ for every $x \in \mathbb{R}^n$.

2.1. h-stable solutions.

We denote by μ the measure on $\mathbb{R}^n$ with density $e^{G(x)}$ w.r.t. the Lebesgue measure, and we let $W^{k,p}_\mu(\mathbb{R}^n) \subset W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$, for $k, p \in \mathbb{N}$, be the corresponding Sobolev spaces. Notice that, if G is concave, then μ is a finite measure iff

$$\lim_{|x| \to +\infty} G(x) = -\infty\,.$$

We introduce now the notion of h-stability for solutions to (12).

Definition 2.1. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. A solution u to (12) is h-stable if*

$$\int_{\mathbb{R}^n} \left(|\nabla\varphi|^2 - f'(u)\varphi^2\right)\mathrm{d}\mu \geq \int_{\mathbb{R}^n} h(x)\varphi^2\mathrm{d}\mu \qquad \forall\, \varphi \in C^1_c(\mathbb{R}^n)\,. \tag{15}$$

If $h \equiv 0$, then u is said to be stable.

We recall that a function $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ is a weak solution to (12) if

$$\int_{\mathbb{R}^n} \big(\langle \nabla u, \nabla\varphi\rangle - f(u)\varphi\big)\mathrm{d}\mu = 0 \qquad \forall\, \varphi \in C^1_c(\mathbb{R}^n)\,. \tag{16}$$

Note that every critical point of the functional I in (13) is a weak solution to (12). By classical elliptic regularity theory, if u is a weak solution then $u \in C^{2,\alpha}(\mathbb{R}^n)$ for all $\alpha < 1$, in particular it is also a classical solution to (12).

Remark 2.2. The function $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ is a local minimizer of the functional I in (13) if I does not decrease under compactly supported perturbations, i.e.

$$I(u) \leq I(v) \qquad \text{whenever } v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } \{u \neq v\} \subset K \subset\subset \mathbb{R}^n.$$

Every local minimizer of I is a stable weak solution to (12).
In [7] the authors show that, when $G(x) = -|x|^2/2$, monotone solutions to (12) are -1-stable (i.e. stable with respect to the constant function $h \equiv -1$).

In the following we will consider h-stable solutions to (12) which have *finite energy*, in the sense that

$$|\nabla u| \in L^2_\mu(\mathbb{R}^n). \tag{17}$$

Note that if $G(y,z) = g(y) + C(z)$, this condition reduces to (6). When $n = 2$, we can substitute this condition with

$$|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^n). \tag{18}$$

Remark 2.3. If the measure μ is finite then $L^\infty(\mathbb{R}^n) \subset L^2_\mu(\mathbb{R}^n)$. If the function G is concave, by [9, Thm 2.5, Cor. 4.3] this implies that every bounded solution to (12) belongs to $W^{2,2}_\mu(\mathbb{R}^n)$ and hence satisfies (17).

On the other hand, assumption (17) can be satisfied also when μ is not finite: for instance, if $G(x) = g(y)$ is such that (10) holds and $f(s) = s - s^3$, the function

$$u(z) = \tanh\left(\frac{z}{\sqrt{2}}\right)$$

is a monotone stable solution to (12) with finite energy.

3. λ_G-stability and finite energy imply one-dimensional symmetry

We now show that λ_G-stable solutions to (12), where λ_G is defined in (14), which satisfy (17) or (18) are one-dimensional. Similar results for stable solutions have been obtained in the setting of Riemannian manifolds with nonnegative Ricci curvature in [15, 16].

Given a differentiable function $v : \mathbb{R}^n \to \mathbb{R}$, we set $v_i := \partial_i v$ for all $i = 1, \dots, n$.

Lemma 3.1. *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ be a weak solution to* (12)*. Then for any $i = 1, \dots, n$ and $\varphi \in C^1_c(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \big(\langle \nabla u_i, \nabla\varphi\rangle - \langle \nabla u, \nabla G_i\rangle \varphi - f'(u)u_i\varphi\big)\, \mathrm{d}\mu(x) = 0. \tag{19}$$

Proof. It suffices to prove (19) for $\varphi \in C^\infty_c(\mathbb{R}^n)$. From (16), applied with φ replaced by φ_i, we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \langle \nabla u, \nabla\varphi_i\rangle - f(u)\varphi_i \, \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^n} - \langle \nabla u_i, \nabla\varphi\rangle - \langle \nabla u, \nabla\varphi\rangle G_i + f'(u)u_i\varphi + f(u)\varphi G_i \, \mathrm{d}\mu(x) \qquad (20) \\ &= \int_{\mathbb{R}^n} - \langle \nabla u_i, \nabla\varphi\rangle - \langle \nabla u, \nabla(\varphi G_i)\rangle + \langle \nabla u, \nabla G_i\rangle \varphi + f'(u)u_i\varphi + f(u)\varphi G_i \, \mathrm{d}\mu(x). \end{aligned}$$

Recalling (16), applied with φ replaced by φG_i, we obtain the thesis. □

Proposition 3.2. *Let $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and u be a h-stable solution to* (12)*. Then for every $\varphi \in C^1_c(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \Big(|\nabla^2 u|^2 - |\nabla|\nabla u||^2 + \big\langle \big(h(x)\mathrm{I}_n - \nabla^2 G(x)\big)\ \nabla u, \nabla u\big\rangle\Big)\varphi^2 \mathrm{d}\mu(x) \le \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla\varphi|^2 \mathrm{d}\mu(x)\,. \tag{21}$$

Proof. Let $\varphi \in C^1_c(\mathbb{R}^n)$. Using (19) with test function $u_i\varphi^2$ we obtain

$$\int_{\mathbb{R}^n} \big\langle \nabla u_i, \nabla(u_i\varphi^2)\big\rangle - f'(u)u_i^2\varphi^2 \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla G_i\rangle u_i\varphi^2 \mathrm{d}\mu(x). \tag{22}$$

Summing over i, (22) gives

$$\int_{\mathbb{R}^n} |\nabla^2 u|^2\varphi^2 + \frac{1}{2}\big\langle \nabla|\nabla u|^2, \nabla\varphi^2\big\rangle - f'(u)|\nabla u|^2\varphi^2 \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \big\langle \nabla^2 G(x)\ \nabla u, \nabla u\big\rangle \varphi^2 \, \mathrm{d}\mu(x). \tag{23}$$

Using (15) with test function $|\nabla u|\varphi$ we then get

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)|\nabla u|^2\varphi^2 \mathrm{d}\mu(x) &\le \int_{\mathbb{R}^n} |\nabla(|\nabla u|\varphi)|^2 - f'(u)|\nabla u|^2\varphi^2 \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^n} \varphi^2|\nabla|\nabla u||^2 + |\nabla u|^2|\nabla\varphi|^2 + \frac{1}{2}\big\langle \nabla|\nabla u|^2, \nabla\varphi^2\big\rangle - f'(u)|\nabla u|^2\varphi^2 \mathrm{d}\mu(x). \qquad (24) \end{aligned}$$

Substituting (23) in (24) we get the result. □

Corollary 3.3. *Recalling that $|\nabla^2 u|^2 - |\nabla|\nabla u||^2 \ge 0$ (see Remark 3.4), from* (21) *it follows*

$$\int_{\mathbb{R}^n} \big\langle \big(h(x)\mathrm{I}_n - \nabla^2 G(x)\big)\ \nabla u, \nabla u\big\rangle \varphi^2 \mathrm{d}\mu(x) \le \int_{\mathbb{R}^n} |\nabla u|^2|\nabla\varphi|^2 \mathrm{d}\mu(x). \tag{25}$$

If $h \ge \lambda_G$, from (21) *and the definition of λ_G in* (14) *it follows*

$$\int_{\mathbb{R}^n} \Big(|\nabla^2 u|^2 - |\nabla|\nabla u||^2\Big)\varphi^2 \mathrm{d}\mu(x) \le \int_{\mathbb{R}^n} |\nabla u|^2|\nabla\varphi|^2 \mathrm{d}\mu(x). \tag{26}$$

Remark 3.4. The Poincaré type formula (26) was first obtained by Sternberg and Zumbrun [28]. Notice that the quantity $|\nabla^2 u|^2 - |\nabla|\nabla u||^2$ has a geometric interpretation, in the sense that it can be expressed in terms of the principal curvatures of level sets of u. More precisely, letting

$$L_{u,x} := \{y \in \mathbb{R}^n \mid u(y) = u(x)\},$$

we denote by $\nabla_T u$ the tangential gradient of u along $L_{u,x} \cap \{\nabla u \neq 0\}$, and by $k_1, \ldots, k_{n-1}$ the principal curvatures of $L_{u,x} \cap \{\nabla u \neq 0\}$. Then the following formula holds (as proved in Lemma 2.1 in [27])

$$|\nabla^2 u|^2 - |\nabla|\nabla u||^2 = |\nabla_T |\nabla u||^2 + |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \qquad \text{on } L_{u,x} \cap \{\nabla u \neq 0\}, \tag{27}$$

so that (26) becomes

$$\int_{\{\nabla u \neq 0\}} (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2)\varphi^2 \, \mathrm{d}\mu(x) + \int_{\{\nabla u = 0\}} (|\nabla^2 u|^2 - |\nabla|\nabla u||^2)\varphi^2 \, \mathrm{d}\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, \mathrm{d}\mu(x). \tag{28}$$

where $\mathcal{K} := \sum_{j=1}^{n-1} k_j^2$. By Stampacchia's Theorem, since $\mu << \mathcal{L}^n$, we get

$$\begin{aligned} &\nabla|\nabla u|(x) = 0 \quad \mu\text{-a.e } x \in \{|\nabla u| = 0\} \\ &\nabla u_j(x) = 0 \quad \mu\text{-a.e } x \in \{|\nabla u| = 0\} \subseteq \{u_j = 0\} \end{aligned}$$

for any $j = 1, \ldots, n$. Hence (28) gives

$$\int_{\{\nabla u \neq 0\}} (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2)\varphi^2 \, \mathrm{d}\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, \mathrm{d}\mu(x). \tag{29}$$

We refer to [28] and [14] for more details.

We now state the main result of this section.

Theorem 3.5. *Assume that $G \in C^2(\mathbb{R}^n)$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $h \geq \lambda_G$. Let u be a h-stable solution to* (12) *such that one of the following conditions hold:*

i) *u satisfies* (17)*;*
ii) *$n = 2$ and u satisfies* (18)*.*

Then u is one-dimensional, i.e. there exists $\omega \in \mathbb{S}^{n-1}$ and $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n. \tag{30}$$

Moreover,

$$\left\langle \left(h(x)\mathrm{I}_n - \nabla^2 G(x)\right) \nabla u, \nabla u \right\rangle = 0 \qquad \forall\, x \in \mathbb{R}^n. \tag{31}$$

In particular, if u_0 is not constant, there are C and g of class C^2 such that

$$G(x) = C(\langle x, \omega \rangle) + g(x'), \tag{32}$$

where $x' := x - \langle x, \omega \rangle\, \omega$, and $\lambda_G(x) = h(x) = C''(\langle x, \omega \rangle)$ for all $x \in \mathbb{R}^n$.

Proof. Let us fix $R > 1$ and let us define $\varphi(x) := \Phi(|x|)$ where $\Phi \in C^\infty(\mathbb{R})$, $|\Phi'(t)| \leq 3$ for any $t \in [R, R+1]$

$$\Phi(t) := \begin{cases} 1 & \text{if} \quad t \leq R \\ 0 & \text{if} \quad t \geq R+1. \end{cases} \tag{33}$$

Obviously $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $|\nabla\varphi(x)| \leq |\Phi'(|x|)| \leq 3$. Hence for every $R > 1$ (29) yields

$$\int_{\{\nabla u \neq 0\} \cap \bar{B}_R} \left(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2\right) \mathrm{d}\mu(x) \leq 9 \int_{\bar{B}_{R+1} \setminus B_R} |\nabla u|^2 \, \mathrm{d}\mu(x) \tag{34}$$

where $B_R := \{y \in \mathbb{R}^n \mid |y| < R\}$.

If $\nabla u \in L^2_\mu(\mathbb{R}^n)$, then

$$\lim_{R \to \infty} \int_{\bar{B}_{R+1} \setminus B_R} |\nabla u|^2 \, \mathrm{d}\mu(x) = 0. \tag{35}$$

Hence (34) and (35) yield

$$k_j(x) = 0 \quad \text{and} \quad |\nabla_T |\nabla u||(x) = 0 \tag{36}$$

for every $j = 1, \ldots, n-1$ and every $x \in \{\nabla u \neq 0\}$. From this and Lemma 2.11 in [14] we get the one-dimensional symmetry of u.

If $n = 2$ and $|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^n)$, we take in (29) the following test function

$$\varphi(x) = \max\left[0, \min\left(1, \frac{\ln R^2 - \ln|x|}{\ln R}\right)\right], \tag{37}$$

Reasoning as in [14, Cor. 2.6], we then obtain

$$\int_{\{\nabla u \neq 0\} \cap \bar{B}_R} \left(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2\right) \mathrm{d}\mu(x) \leq \int_{B_{R^2} \setminus B_R} \frac{1}{|x|^2 \, (\ln R)^2} |\nabla u|^2 e^{G(x)} \mathrm{d}x.$$

When $R \to +\infty$, since $|\nabla u|^2 e^{G(x)}$ is bounded, the r.h.s. term of the previous inequality vanishes, and we conclude agian that u is one-dimensional.

Assume now that u is not constant. If we take in (25) the same test functions as above, we get

$$\int_{\mathbb{R}^n} \left\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\nabla u, \nabla u \right\rangle \mathrm{d}\mu(x) = 0\,.$$

Using the fact that $u(x) = u_0(\langle \omega, x\rangle)$, we obtain that $\left\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\omega, \omega \right\rangle = 0$ for all x such that $u_0'(\langle \omega, x\rangle) \neq 0$. Since u is not constant and is a solution to the elliptic equation (12), the set of points such that $u_0'(\langle \omega, x\rangle) = 0$ has zero measure, so, by the regularity of G we conclude that

$$\left\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\omega, \omega \right\rangle = 0 \qquad \forall\, x \in \mathbb{R}^n\,,$$

which gives (31) and (32). □

Theorem 3.5 direclty implies the following Liouville type result (cfr. [15]).

Corollary 3.6. *Let $h \in C^0(\mathbb{R}^n)$ with $h \geq \lambda_G$, and u be a h-stable solution solution to* (12) *with finite energy. If $\lambda_G(x) < h(x)$ for some $x \in \mathbb{R}^n$, then u is constant. In particular, if u is a stable solution and $\lambda_G(x) < 0$ for some $x \in \mathbb{R}^n$, then u is constant.*

Remark 3.7. Recalling Remark 2.3, when the measure μ is finite and G is concave, Theorem 3.5 implies that bounded solutions to (12) which are λ_G-stable are one-dimensional.

4. Monotonicity implies λ_G-stability

In this section we assume that, for every $x \in \mathbb{R}^n$, e_n is the eigenvector associated to the maximal eigenvalue $\lambda_G(x)$ of $\nabla^2 G(x)$. This implies that there exist two functions g and C such that

$$G(x) = g(y) + C(z) \quad \text{and} \quad \lambda_G(x) = C''(z). \tag{38}$$

We prove that solutions to (12) which are monotone along the z-axis are stable.

Theorem 4.1. *Assume that G satisfies* (38) *and u is a solution to* (12) *satisfying* (4). *Then u is λ_G-stable.*

Proof. Equation (19) with $i = n$ reads

$$\int_{\mathbb{R}^n} \langle \nabla u_z, \nabla \varphi \rangle - C''(z) u_z \varphi - f'(u) u_z \varphi \, \mathrm{d}\mu(x) = 0. \tag{39}$$

Let $\varphi \in C^1_c(\mathbb{R}^n)$. Taking as test function $\frac{\varphi^2}{u_z}$ in (39), we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \left\langle \nabla u_z, \nabla \left(\frac{\varphi^2}{u_z} \right) \right\rangle - C''(z)\varphi^2 - f'(u)\varphi^2 \, \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^n} |\nabla \varphi|^2 - \left| \frac{\varphi}{u_z} \nabla u - \nabla \varphi \right|^2 - C''(z)\varphi^2 - f'(u)\varphi^2 \, \mathrm{d}\mu(x) \\ &\leq \int_{\mathbb{R}^n} |\nabla \varphi|^2 - C''(z)\varphi^2 - f'(u)\varphi^2 \, \mathrm{d}\mu(x), \end{aligned}$$

which is the stability condition (15). $\square$

5. Proof of Theorem 1.1

Observe that in (2), $G(x) = g(y) + C(z)$, and by assumption $C''(z) \geq \nabla^2 g(y)$. So (38) holds, and by Theorem 4.1 every solution to (2) satisfying (4) is λ_G-stable.

If either a) or c) holds, the thesis follows from Theorem 3.5.

Let us assume that u satisfies b). We define $\psi_R(y) := \Phi(|y|)$ where Φ is as in (33) and $\varphi_S(z)$ as follows. We fix $S > 1$ and let

$$\varphi_S(z) := \begin{cases} 3 & \text{if } |z| \leq S \\ 4 - \frac{z^2}{S^2} & \text{if } S \leq |z| \leq 2S \\ 0 & \text{if } |z| \geq 2S. \end{cases}$$

We compute (29) with test function $\psi_R(y)\varphi_S(z)$ and obtain, recalling (7),

$$\begin{aligned} \int_{\{\nabla u \neq 0\}} \left(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \right) \psi_R^2 \varphi_S^2 \, \mathrm{d}\mu(x) &\leq \int_{\mathbb{R}^n} |\nabla u|^2 {\varphi'}_S^2(z) \nabla^2 \psi_R(y) \mathrm{d}\mu(x) \\ \leq \frac{4}{S^2} \int_{\mathbb{R}^n} |\nabla u|^2 \nabla^2 \psi_R(y) \mathrm{d}\mu(x) &\leq \frac{36K}{S^2}. \end{aligned}$$

If we let $R \to +\infty$ we obtain

$$\int_{\{\nabla u \neq 0\} \cap \{|z| \leq S\}} \Big(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \Big) \mathrm{d}\mu(x) \leq \frac{4K}{S^2}.$$

Letting $S \to +\infty$ we then obtain (36) and we conclude as in the proof of Theorem 3.5. $\square$

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