

A shape optimization problem for the p -Laplacian

Anisa Chorwadwala · Rajesh Mahadevan

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Abstract It is known that the torsional rigidity for a punctured ball, with the puncture having the shape of a ball, is minimum when the balls are concentric and the first eigenvalue for the Dirichlet Laplacian for such domains is also a maximum in this case. These results have been obtained by Ashbaugh and Chatelain (private communication), Harrell et. al. [12], Kesavan [13] and Ramm and Shivakumar [18]. In this paper we extend these results to the case of p -Laplacian for $1 < p < \infty$. For proving these results, we follow the same line of ideas as in the aforementioned articles, namely, study the sign of the shape derivative using the moving plane method and comparison principles. In the process, we obtain some interesting new side results such as the Hadamard perturbation formula for the torsional rigidity functional for the Dirichlet p -Laplacian, the existence and uniqueness result for a nonlinear pde and some extensions of known comparison results for nonlinear pdes.

Keywords shape optimization · Dirichlet p -Laplacian · shape derivative analysis · moving plane method · comparison principles

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Anisa Chorwadwala,
Indian Institute of Science Education and Research (IISER),
Central Tower, Sai Trinity Building,
Sutarwadi Road, Pashan, Pune 411021, INDIA.
E-mail: anisa@iiserpune.ac.in

Rajesh Mahadevan,
Departamento de Matemática,
Facultad de Cs. Físicas y Matemáticas,
Av. Esteban Iturra s/n, Barrio Universitario,
Casilla 160C, Concepción, CHILÉ.
E-mail: rmahadevan@udec.cl

1 Introduction

The p -Laplacian Δ_p is the non-linear operator defined as $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2} \nabla f)$. Let B_1 be an open ball in \mathbb{R}^N . Let B_0 be another open ball whose closure is contained in B_1 , and is free to move inside B_1 . Let $\Omega = B_1 \setminus \overline{B_0}$. We consider the following domain optimization problems:

- i. Given $y \in W_0^{1,p}(\Omega)$, the unique solution of the equation

$$\left. \begin{aligned} -\Delta_p u &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \right\} \quad (1.1)$$

we are interested in minimizing the p -torsional rigidity

$$E(\Omega) := \int_{\Omega} |\nabla y|^p dx = \int_{\Omega} y dx \quad (1.2)$$

with respect to the position of the hole B_0 .

- ii. Given the eigenvalue problem

$$\left. \begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (1.3)$$

whose principal eigenvalue is

$$\lambda_1(\Omega) := \inf \left\{ \frac{\|\nabla \varphi\|_{L^p(\Omega)}^p}{\|\varphi\|_{L^p(\Omega)}^p} \mid \varphi \in W_0^{1,p}(\Omega) \right\}, \quad (1.4)$$

we are interested in maximizing $\lambda_1(\Omega)$ with respect to the position of B_0 .

The following results were obtained, in the linear case, i.e., for $p = 2$, by Ashbaugh and Chatelain (private communication), Harrell et. al. [12], Kesavan [13], Ramm and Shivakumar [18]: the torsional rigidity is minimum if and only if B_0 and B_1 are concentric. Also, the first eigenvalue λ_1 of problem (1.4) attains its maximum if and only if the balls are concentric.

The analogues of these results for manifolds were obtained in Anisa and Aithal [2] in the setting of space-forms (complete simply connected Riemannian manifolds of constant sectional curvature) and in Anisa and Vemuri (On two functionals connected to the Laplacian in a class of doubly connected domains in rank-one symmetric spaces of non-compact type, preprint) in the setting of rank-one symmetric spaces of non-compact type. We extend these results, in a different direction, to the non-linear setting. Our main results are Theorem 6.1 and Theorem 6.2.

The proofs in [13, 18] rely on *shape differentiation* [20], the *moving plane method* [4, 11] and various maximum principles. In the non-linear case, carrying out this program involves several technical difficulties. We develop the shape calculus for the torsional rigidity function for p -Laplacian. A formula for the Hadamard perturbation of the first Dirichlet eigenvalue for the p -Laplacian is given. This, however, is not new and may also be seen in the works of García Melián and Sabina de Lis [10], Lamberti [14] and Ly [16]. For the Steklov eigenvalue this is done in Del Pezzo and Fernández

Bonder [9]. Subsequently, we analyze the sign of the shape derivative. We do this by proving a suitable strong comparison result. In the case of the eigenvalue problem, before this, we also need to prove a general weak comparison principle for the p -Laplacian with non-vanishing boundary condition (cf. Theorem 3.1). This result is new and can be of independent interest in itself. An existence and uniqueness result for a nonlinear pde is required for applying this comparison principle and this result is also proved (cf. Proposition 4.1).

The Section 2 establishes notations, contains some definitions and technical preliminaries. In Section 3, we recall some existing weak and strong comparison principles for the p -Laplacian and prove an extension of a weak comparison principle. In Section 4, we prove the existence and uniqueness of non-negative solution for a nonlinear pde needed for an application of the comparison principle. In Section 5, following [20] we obtain the Hadamard perturbation formula for the torsional rigidity functional (1.2) and for the first eigenvalue of the Dirichlet p -Laplacian (1.4). Finally, in Section 6 we prove the main results by analyzing the sign of the shape derivatives.

2 Preliminaries

In this section we introduce some definitions and recall some results which will be used later on.

SHAPE DERIVATIVE: Given a functional J which depends on the domain Ω (usually, a smooth open set in \mathbb{R}^N) and given, a variation of the domain Ω by a fairly smooth perturbative vector field V which has its support in a neighborhood of $\partial\Omega$, the infinitesimal variation of J in the direction V is defined as

$$J'(\Omega; V) = \lim_{t \rightarrow 0} \frac{J(\Omega_t) - J(\Omega)}{t} \quad (2.1)$$

where Ω_t is the diffeomorphic image $\Phi_t(\Omega)$ of Ω under the smooth perturbation of identity $\Phi_t(x) = (I + tV)(x)$.

The shape derivative is a tool widely used in problems of optimization with respect to the domain as it permits to understand the variations of shape functionals (cf. Simon [19], [20]).

We define $B(t) := (D\Phi_t)^{-1}$, $\gamma(t) := |\det D\Phi_t|$ and $A(t) := \gamma(t) B(t) B(t)^*$ where $B(t)^*$ shall denote the transpose of $B(t)$. It will be convenient to denote $\gamma(t)$, $B(t)$, $B(t)^*$ and $A(t)$ respectively, by γ_t , B_t , B_t^* and A_t . We observe that

$$D\Phi_t = I + tDV \quad (D\Phi_t)^* = I + t(DV)^* \quad (2.2)$$

and so, B_t , B_t^* , A_t , γ_t and F_t are analytic functions of t near $t = 0$. We record that

$$\gamma'(0) = \operatorname{div} V \quad (2.3)$$

$$(B_t^*)'(0) = -(DV)^*. \quad (2.4)$$

So, for small t , we have

$$\gamma_t \approx \gamma(0) + t\gamma'(0) = 1 + t \operatorname{div} V \quad (2.5)$$

Also, for t sufficiently small say $|t| < t_0$, there exists a constant $C > 0$ such that

$$|(D\Phi_t)^*\xi| \leq C|\xi| \text{ for all } \xi \in \mathbb{R}^n. \quad (2.6)$$

Consequently, by substituting $B_t^*\eta$ for ξ , η arbitrary in \mathbb{R}^n , we have

$$|B_t^*\eta| \geq C^{-1}|\eta| \text{ for all } \eta \in \mathbb{R}^n. \quad (2.7)$$

PUCCI-SERRIN IDENTITY: We shall find it very useful to employ the extended version of the Pucci-Serrin identity proved by Degiovanni et. al. [8] which gives the following identity for the p -Laplacian. Assume that $u \in C^1(\overline{\Omega})$ is a solution of the equation

$$\left. \begin{array}{l} -\Delta_p u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{array} \right\} \quad (2.8)$$

Then, for all $V \in C^1(\overline{\Omega})$ the following identity holds

$$-\frac{(p-1)}{p} \int_{\Omega} \left| \frac{\partial u}{\partial v} \right|^p V \cdot n \, dS = \int_{\Omega} \operatorname{div} V \frac{|\nabla u|^p}{p} \, dx - \int_{\Omega} \langle (DV)^*\nabla u, |\nabla u|^{p-2}\nabla u \rangle \, dx + \int_{\Omega} V \cdot \nabla u f \, dx \quad (2.9)$$

The Pucci-Serrin identity may be obtained by using $V \cdot \nabla u$ as a test function in (2.8) and after several integration by parts whenever $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. However, by standard regularity results for solutions of the p -Laplacian equation, they are known to belong to only $C^{1,\alpha}(\overline{\Omega})$ (cf. Tolksdorff [22]) as the coefficients $|\nabla u|^{p-2}$ degenerates near the critical points of u . This formula can be justified by regularizing the coefficient first and then passing to the limit cf. [8] (see also García-Melían and Sabina de Lis [10] and the work of Del Pezzo and Fernández Bonder [9] for such arguments).

A POSITIVE DEFINITE MATRIX: Define a strictly convex function $\Gamma : \mathbb{R}^N \rightarrow \mathbb{R}$ by $\Gamma(x) = \frac{|x|^p}{p}$. Let $A = D\Gamma$. Then $A = (A_1, A_2, \dots, A_N) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and is given by

$$A(x) = |x|^{p-2}x. \quad (2.10)$$

Clearly, $A \in \mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$. The matrix $\mathcal{A} := \left[\frac{\partial A_i}{\partial x_j}(x) \right]_{i,j=1}^N$ corresponds to the symmetric matrix

$|x|^{p-2}\operatorname{Id} + (p-2)|x|^{p-4}x \otimes x$, which is the Hessian of the convex function Γ . It can be seen that $(p-1)|x|^{p-2}$ and $|x|^{p-2}$ are eigenvalues of \mathcal{A} with multiplicity one and $(n-1)$ respectively. Therefore, for any $\xi \in \mathbb{R}^n$, we have

$$\langle \mathcal{A}\xi, \xi \rangle \geq \min\{1, p-1\}|x|^{p-2}|\xi|^2. \quad (2.11)$$

3 Comparison Theorems for the p -Laplacian

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Let $\beta : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous function and $u \mapsto \beta(x, u)$ is locally Lipschitz on $\mathbb{R} \setminus \{0\}$ uniformly for $x \in \Omega$ and assume that $\frac{\partial \beta}{\partial u}$ is of constant sign for all $(x, u) \in \Omega \times (\mathbb{R} \setminus \{0\})$. Let $f, g \in W^{-1, \frac{p}{p-1}}(\Omega)$, $f', g' \in W^{1-\frac{1}{p}, p}(\partial\Omega)$ with $f \geq g$ in Ω (in the sense of distributions), $f' \geq g'$ on $\partial\Omega$. Let $u, v \in W^{1,p}(\Omega)$ solve (in the weak sense)

$$\begin{aligned} -\Delta_p u &= \beta(x, u) + f(x), & -\Delta_p v &= \beta(x, v) + g(x) & \text{in } \Omega, \\ u &= f', & v &= g' & \text{on } \partial\Omega. \end{aligned} \quad (3.1)$$

Then one is interested in the following comparison results:

- (WCP) Weak Comparison Principle: Is it true that $u \geq v$ in Ω ?
 (SCP) Strong Comparison Principle: If $u, v \in \mathcal{C}^1(\bar{\Omega})$, $u \not\equiv v$, $u \geq v$ in Ω , is it true that $u > v$ in Ω and $\frac{\partial u}{\partial n}(x_0) < \frac{\partial v}{\partial n}(x_0)$ for any $x_0 \in \partial\Omega$? Here, n is the unit outward normal to Ω on $\partial\Omega$.

The Weak Comparison Principle (WCP) holds when $\frac{\partial \beta}{\partial u} \leq 0$ for which we refer to Tolksdorff [21].

The Weak Comparison Principle also holds when $\frac{\partial \beta}{\partial u} \geq 0$ under the following assumptions and for Dirichlet boundary data:

(A-1) $\frac{\partial \beta}{\partial u} \geq 0 \forall (x, u) \in \Omega \times (\mathbb{R} \setminus \{0\})$, $\beta(x, 0) \geq 0 \forall x \in \Omega$.

(A-2) The problem

$$\begin{aligned} -\Delta_p u &= \beta(x, u) + f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

(where $f \in L^\infty(\Omega)$, $f \geq 0$ in Ω) admits a unique non-negative solution $u \in W_0^{1,p}(\Omega)$.

(A-3) $f, g \in L^\infty(\Omega)$, $0 \leq g \leq f$ on Ω and $0 = g' = f'$ on $\partial\Omega$.

This result is proved in [6]. However, for our purposes the zero Dirichlet data assumption in (A-2) and (A-3) is too restrictive. We show that this result also holds for inhomogeneous Dirichlet boundary data, that is, by relaxing the condition (A-2) and (A-3) to (A-2') and (A-3') respectively:

(A-2') The problem

$$\begin{aligned} -\Delta_p u &= \beta(x, u) + f & \text{in } \Omega, \\ u &= f' & \text{on } \partial\Omega. \end{aligned}$$

(where $f \in L^\infty(\Omega)$, $f \geq 0$ in Ω and $f' \geq 0$ on $\partial\Omega$) admits a unique non-negative solution $u \in W^{1,p}(\Omega)$.

(A-3') $f, g \in L^\infty(\Omega)$, $0 \leq g \leq f$ on Ω and $0 \leq g' \leq f'$ on $\partial\Omega$.

We prove the following results along the same lines as in [6].

Theorem 3.1 *Let the assumptions (A-1) (A-2') and (A-3') hold then the WCP holds for bounded solutions.*

Proof. Let us denote $L_+^\infty(\Omega) = \{h \in L^\infty(\Omega) \mid h \geq 0 \text{ in } \Omega\}$. Given $f \in L_+^\infty(\Omega)$ and $f' \in W^{1-\frac{1}{p}, p}(\partial\Omega)$ with $f' \geq 0$ on $\partial\Omega$, define the nonlinear operator $T_{f, f'}$ on $L_+^\infty(\Omega)$ by letting $T_{f, f'}(u) = v$, where v is the weak solution of

$$\begin{aligned} -\Delta_p v &= \beta(x, u) + f && \text{in } \Omega, \\ v &= f' && \text{on } \partial\Omega \end{aligned} \quad (3.2)$$

Since $\frac{\partial\beta}{\partial u} \geq 0$ and $u \geq 0$, it follows that $\beta(x, u) \geq \beta(x, 0) \geq 0$. So, the right hand side in (3.2) is non-negative as also the boundary data. By appealing to the WCP proved by Tolksdorff [21] we conclude that indeed $T_{f, f'}(u) = v \geq 0$ and $T_{f, f'}$ maps $L_+^\infty(\Omega)$ into itself.

Claim. Let $f_1, f_2, u_1, u_2 \in L_+^\infty(\Omega)$. If $f_1 \leq f_2$, $u_1 \leq u_2$ and $f'_1 \leq f'_2$ then $T_{f_1, f'_1}(u_1) \leq T_{f_2, f'_2}(u_2)$

Indeed, following the condition $\frac{\partial\beta}{\partial u} \geq 0$ we conclude that $f_1^* := \beta(x, u_1) + f_1 \leq \beta(x, u_2) + f_2 =: f_2^*$. Let $v_1 = T_{f_1, f'_1}(u_1)$ and $v_2 = T_{f_2, f'_2}(u_2)$. Then

$$\begin{aligned} -\Delta_p v_1 &= f_1^*, & -\Delta_p v_2 &= f_2^* && \text{in } \Omega, \\ v_1 &= f'_1, & v_2 &= f'_2 && \text{on } \partial\Omega. \end{aligned}$$

So, again by the weak comparison result proved in [21] we obtain $v_1 \leq v_2$ in Ω . This proves the claim.

Now, let u, v be bounded solutions of the non-linear pdes in (3.1). To begin with, $T_{f, f'}(u) = u$ and $T_{g, g'}(v) = v$. Now, using the claim we obtain the inequalities,

$$0 \leq T_{f, f'}(0) \leq T_{f, f'}(u) = u, \quad 0 \leq T_{g, g'}(0) \leq T_{g, g'}(v) = v$$

We can then show by an inductive application of the claim that following chains of inequalities hold

$$0 \leq T_{f, f'}(0) \leq T_{f, f'}^2(0) \leq \dots \leq T_{f, f'}^n(0) \leq \dots \leq u = T_{f, f'}(u) \quad (3.3)$$

$$0 \leq T_{g, g'}(0) \leq T_{g, g'}^2(0) \leq \dots \leq T_{g, g'}^n(0) \leq \dots \leq v = T_{g, g'}(v) \quad (3.4)$$

The pointwise limits $u^*(x) = \lim_{n \rightarrow \infty} [T_{f, f'}^n(0)](x)$ and $v^*(x) = \lim_{n \rightarrow \infty} [T_{g, g'}^n(0)](x)$ exist and must clearly satisfy $T_{f, f'}(u^*) = u^*$ and $T_{g, g'}(v^*) = v^*$ respectively. So, by the uniqueness assumption in (A-2'), it follows that $u^* = u$ and $v^* = v$.

Again, by applying the claim above, for any $n \geq 0$, we obtain $T_{g, g'}^n(0) \leq T_{f, f'}^n(0)$. Therefore, upon taking the limit as n goes to infinity we obtain $v \leq u$. This proves the theorem. \square

4 Existence and uniqueness for a nonlinear Dirichlet problem

Let λ_1 be the first eigenvalue of the Dirichlet p -Laplacian as in (1.4) on a bounded domain Ω . Let \mathcal{O} be an open proper subset of Ω . We prove the existence and uniqueness result for a nonlinear partial differential equation on \mathcal{O} given Dirichlet data $f' \geq 0$ on $\partial\Omega$. This shall be needed for applying the comparison principle of the previous section, later in Section 6.

Proposition 4.1 *Given $f' \in W^{1-\frac{1}{p},p}(\partial\mathcal{O})$ and $f' \geq 0$ on $\partial\mathcal{O}$, the problem*

$$\left. \begin{aligned} -\Delta_p w &= \lambda_1 |w|^{p-2} w && \text{in } \mathcal{O}, \\ w &= f' && \text{on } \partial\mathcal{O}. \end{aligned} \right\} \quad (4.1)$$

admits a unique non-negative solution.

Proof. Let us first prove that if a solution exists then it is non-negative. Let u be a solution of the above problem. As $u \geq 0$ on $\partial\mathcal{O}$, we obtain that $u^- \in W_0^{1,p}(\mathcal{O})$. Therefore, taking u^- as a test function, we have

$$\int_{\mathcal{O}} |\nabla u|^{p-2} \langle \nabla u, \nabla u^- \rangle dx = \lambda_1 \int_{\mathcal{O}} |u|^{p-2} u u^- dx.$$

From this we obtain

$$\int_{\mathcal{O}} |\nabla u^-|^p dx = \lambda_1 \int_{\mathcal{O}} |u^-|^p dx.$$

We cannot have $u^- \neq 0$, for otherwise, from the variational characterization of the first eigenvalue we can conclude that $\lambda_1(\mathcal{O}) \leq \lambda_1 = \lambda_1(\Omega)$. However, this cannot happen, \mathcal{O} being a proper open subset of Ω we must have $\lambda_1(\Omega) < \lambda_1(\mathcal{O})$.

Existence. We denote by f' again a $W^{1,p}(\mathcal{O})$ function whose trace on $\partial\mathcal{O}$ is f' . We can then obtain a weak solution of (4.1) by minimizing the functional $J(w) = \int_{\mathcal{O}} |\nabla w|^p dx - \lambda_1(\Omega) \int_{\mathcal{O}} |w|^p dx$ on the affine space $A := W_0^{1,p}(\mathcal{O}) + f'$. Indeed, if w is a minimizer of J then we shall have

$$0 = \left. \frac{d}{dt} \right|_{t=0} J(w + t\phi) = \int_{\mathcal{O}} |\nabla w|^{p-2} \langle \nabla w, \nabla \phi \rangle dx - \lambda_1 \int_{\mathcal{O}} |w|^{p-2} w \phi dx \quad \forall \phi \in \mathcal{C}_0^1(\mathcal{O}). \quad (4.2)$$

which is just the weak formulation of (4.1). As A is a closed convex subset of the reflexive Banach space $W^{1,p}(\mathcal{O})$, for showing the existence of a minimizer of J on A , it is enough to prove that J is coercive and weakly sequentially lower semi-continuous on A .

J is weakly sequentially lower semi-continuous on A : This is true since $\int_{\Omega} |\nabla w|^p dx$ is lower semi-continuous for the weak topology on $W^{1,p}(\mathcal{O})$ and $\int_{\Omega} |w|^p dx$ is continuous for the weak topology on $W^{1,p}(\mathcal{O})$ due to the compact inclusion of $W^{1,p}(\mathcal{O})$ in $L^p(\mathcal{O})$.

J is coercive on A : Let $w_n := f' + \phi_n \in A$ be a sequence such that $\|w_n\|_{W^{1,p}(\mathcal{O})} \rightarrow \infty$ as $n \rightarrow \infty$. If $\int_{\mathcal{O}} |w_n|^p dx$ is a bounded sequence, then the coercivity is immediate.

So, let us assume that $\int_{\mathcal{O}} |w_n|^p dx \rightarrow \infty$ as $n \rightarrow \infty$. We may write $w_n := f' + \varphi_n$ with $\varphi \in W_0^{1,p}(\mathcal{O})$. Let $B_n := \frac{\int_{\mathcal{O}} |w_n|^p dx}{\int_{\mathcal{O}} |\varphi_n|^p dx}$. It can be argued, using the triangle inequality, that $\int_{\mathcal{O}} |\varphi_n|^p dx \rightarrow \infty$ and $B_n \rightarrow 1$ as $n \rightarrow \infty$. From the Poincaré inequality on \mathcal{O} , we conclude that $\int_{\mathcal{O}} |\nabla \varphi_n|^p dx \rightarrow \infty$ as $n \rightarrow \infty$. Setting $A_n := \frac{\int_{\mathcal{O}} |\nabla w_n|^p dx}{\int_{\mathcal{O}} |\nabla \varphi_n|^p dx}$, we obtain using the triangle inequality, that $\int_{\mathcal{O}} |\nabla w_n|^p dx \rightarrow \infty$ and $A_n \rightarrow 1$ as $n \rightarrow \infty$. Now,

$$\begin{aligned} J(w_n) &= A_n \left(\int_{\mathcal{O}} |\nabla \varphi_n|^p dx - \lambda_1(\Omega) \frac{B_n}{A_n} \int_{\mathcal{O}} |\varphi_n|^p dx \right) \\ &\geq A_n \left(1 - \frac{B_n \lambda_1(\Omega)}{A_n \lambda_1(\mathcal{O})} \right) \int_{\mathcal{O}} |\nabla \varphi_n|^p dx \end{aligned} \quad (4.3)$$

where the last inequality has been obtained by applying Poincaré inequality in the domain \mathcal{O} . Since we have $0 < \lambda_1(\Omega) < \lambda_1(\mathcal{O})$, since A_n and B_n converge to 1 as $n \rightarrow \infty$, it follows that $A_n \left(1 - \frac{B_n \lambda_1(\Omega)}{A_n \lambda_1(\mathcal{O})} \right)$ is bounded below by a positive constant $C > 0$. Once again, we have the coercivity of J .

Uniqueness. Suppose u, v are two different solutions of (4.2) in A . Let $w_1 := \nabla \log u$ and $w_2 := \nabla \log v$. As $f(x) = |x|^p$ is a strictly convex function we have

$$|w_1|^p \geq |w_2|^p + p |w_2|^{p-2} \langle w_2, w_2 - w_1 \rangle \quad (4.4)$$

and equality holds if and only if $w_1 = w_2$. If we prove that $w_1 = w_2$ then we are done because in that case we will have $0 = \nabla \log u - \nabla \log v = \nabla \log \left(\frac{u}{v} \right)$. That is, $\log \left(\frac{u}{v} \right) = k$ for some constant k . As a result we get $u = e^k v$. But as $u \equiv v = f' \neq 0$ on $\partial \mathcal{O}$ we get $u \equiv v$ in \mathcal{O} . Therefore, it suffices to prove that

$$|w_1|^p = |w_2|^p + p |w_2|^{p-2} \langle w_2, w_2 - w_1 \rangle. \quad (4.5)$$

The proof of (4.5) is the same as the proof of Lemma 3.1 in Lindqvist [15]. We include the proof here for completeness. The function u solves (4.1). We use $u - v^p u^{1-p}$ as a test function in the equation for u . Similarly, we use $v - u^p v^{1-p}$ as a test function in (4.1) with v as a solution. Then we integrate by parts and sum the two identities. This new identity can be reduced to

$$\begin{aligned} 0 &= \int_{\mathcal{O}} [u^p \{ |w_1|^p - |w_2|^p - p |w_2|^{p-2} \langle w_2, w_2 - w_1 \rangle \} \\ &\quad + v^p \{ |w_2|^p - |w_1|^p - p |w_1|^{p-2} \langle w_1, w_1 - w_2 \rangle \}] dx. \end{aligned} \quad (4.6)$$

by using the following:

$$\nabla(u - v^p u^{1-p}) = \left\{ 1 + (p-1) \left(\frac{v}{u} \right)^p \right\} \nabla u - p \left(\frac{v}{u} \right)^{p-1} \nabla v,$$

and,

$$\nabla(v - u^p v^{1-p}) = \left\{ 1 + (p-1) \left(\frac{u}{v} \right)^p \right\} \nabla v - p \left(\frac{u}{v} \right)^{p-1} \nabla u.$$

But by (4.4) the integrand in (4.6) is non-negative (being the sum of two non-negative terms) and so, it follows from (4.6) that this integrand is equal to zero almost everywhere in \mathcal{O} . Therefore, each of the terms in the integrand must be zero. This proves (4.5). \square

5 Shape derivatives of torsional rigidity and eigenvalue functionals

Let Ω be a smooth domain in \mathbb{R}^N and let \mathcal{D} be a domain such that $\Omega_t \subset \mathcal{D}$, for t sufficiently small, for the smooth perturbations Φ_t associated to a smooth vector field V . Consider the Dirichlet boundary value problem on Ω_t :

$$\left. \begin{aligned} -\Delta_p u &= 1 && \text{in } \Omega_t, \\ u &= 0 && \text{on } \partial\Omega_t. \end{aligned} \right\} \quad (5.1)$$

Let $y_t \in \mathcal{C}^{1,\alpha}(\overline{\Omega_t})$ be the unique solution of problem (5.1). Throughout this section $y = y(\Omega)$ denotes the unique solution of (5.1) for $t = 0$. Denote $(y_t \circ \Phi_t)|_{\Omega}$ by y^t ($t \in \mathbb{R}$). We also denote the torsional rigidity $E(\Omega_t)$ by $E(t)$.

Proposition 5.1 *The shape derivative of the torsional rigidity functional $E(\Omega_t)$ exists at $t = 0$ and*

$$\left. \frac{d}{dt} \right|_{t=0} E(\Omega_t) = \int_{\partial\Omega} \left| \frac{\partial y}{\partial n} \right|^p \langle V, n \rangle dS. \quad (5.2)$$

(Here, n denotes unit outward normal on $\partial\Omega$.)

Proof. Let y be the unique solution of (5.1) on Ω corresponding to $t = 0$.

STEP 1: We first show that $y^t \rightarrow y$ strongly in $W_0^{1,p}(\Omega)$.

This can be obtained using the Γ -convergence (cf. Attouch [3], Braides [5], Dal Maso [7]) of a suitable family of functionals. Consider the following family of functionals defined over $W_0^{1,p}(\Omega)$:

$$F(t, y) := \frac{1}{p} \int_{\Omega} |B_t^*(\nabla y)|^p \gamma_t dx - \int_{\Omega} y \gamma_t dx \quad (5.3)$$

Since B_t^* converges uniformly to I and γ_t converges uniformly to the constant 1, it is classical to show the Γ -convergence of the family of convex integral functionals $F(t, \cdot)$, as $t \rightarrow 0$, to the following functional

$$F(y) := F(0, y) = \frac{1}{p} \int_{\Omega} |\nabla y|^p dx - \int_{\Omega} y dx \quad (5.4)$$

See Theorem 5.14 in Dal Maso [7] for instance. Furthermore, the family $F(t, \cdot)$ is equicoercive following the inequalities (2.7) and (2.5). By standard results on Γ -convergence (cf. Theorems 7.8 and 7.12 Dal Maso [7]), the minimizer of $F(t, \cdot)$ converges weakly in $W_0^{1,p}(\Omega)$ to the minimizer of $F(\cdot)$ and the minima converge. Now, for each $t \in \mathbb{R}$, y_t satisfies the equation

$$\int_{\Omega_t} |\nabla y_t|^{p-2} \langle \nabla y_t, \nabla \psi \rangle dx = \int_{\Omega_t} \psi dx \quad \forall \psi \in \mathcal{C}_0^\infty(\Omega_t). \quad (5.5)$$

By the change of variable $\Phi_t : \Omega \longrightarrow \Omega_t$, the equation (5.5) can be re-written as

$$\int_{\Omega} |B_t^*(\nabla y^t)|^{p-2} \langle A_t(\nabla y^t), \nabla \varphi \rangle dx = \int_{\Omega} \gamma_t \phi dx \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega). \quad (5.6)$$

Therefore, y^t satisfies :

$$\left. \begin{aligned} -\operatorname{div}(|B_t^*(\nabla y^t)|^{p-2} A_t(\nabla y^t)) - \gamma_t &= 0 && \text{in } \Omega, \\ y^t &= 0 && \text{on } \partial\Omega. \end{aligned} \right\} \quad (5.7)$$

which is the Euler equation for the minimization of the convex functional $F(t, \cdot)$ and therefore, y^t is the minimizer of $F(t, \cdot)$. Whereas, y , being the solution of problem (5.1) for $t = 0$, is the minimizer of F . So, by the Γ -convergence result, we have the convergence of the minimum values

$$\lim_{t \rightarrow 0} \left(\frac{1}{p} - 1 \right) \int_{\Omega} |B_t^*(\nabla y^t)|^p \gamma_t dx = \left(\frac{1}{p} - 1 \right) \int_{\Omega} |\nabla y|^p dx. \quad (5.8)$$

and the weak convergence in $W_0^{1,p}(\Omega)$, as $t \rightarrow 0$ of y_t to y . It remains to show the strong convergence.

Since B_t^* and γ_t converge uniformly to I and 1 respectively, and y_t remains bounded in $W_0^{1,p}(\Omega)$, we can conclude from (5.8) that

$$\lim_{t \rightarrow 0} \int_{\Omega} |\nabla y^t|^p dx = \int_{\Omega} |\nabla y|^p dx. \quad (5.9)$$

Therefore, since the L^p norm is uniformly convex, we can conclude from the weak convergence of ∇y^t to ∇y in $L^p(\Omega)$ and the convergence of their norms (5.9) that the convergence of ∇y^t to ∇y is strong in $L^p(\Omega)$. By Poincaré inequality, as the L^p norm of the gradients is an equivalent norm on $W_0^{1,p}(\Omega)$, we obtain the desired conclusion.

STEP 2: We observe that the torsional rigidity $E(t)$ of the domain Ω_t is given by

$$E(t) = \frac{p}{p-1} \sup_{\phi \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} \phi \gamma_t dx - \frac{1}{p} \int_{\Omega} |B_t^*(\nabla \phi)|^p \gamma_t dx \right\} = \frac{p}{p-1} \sup_{\phi \in W_0^{1,p}(\Omega)} (-F(t, \phi)) \quad (5.10)$$

and the supremum is attained at $\phi = y^t$ for $y^t = y_t \circ \Phi(t)$ and y_t is the solution of (5.1) on Ω_t .

Indeed, the supremum in the above corresponds to the negative of the infimum in the following

$$\begin{aligned} & \inf_{\phi \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} |B_t^*(\nabla \phi)|^p \gamma_t dx - \int_{\Omega} \phi \gamma_t dx \right\} \\ &= \inf_{\phi \in W_0^{1,p}(\Omega_t)} \left\{ \frac{1}{p} \int_{\Omega_t} |\nabla \phi|^p dx - \int_{\Omega_t} \phi dx \right\} \end{aligned}$$

and this is attained by y_t which is the solution of the Euler-Lagrange equation (5.1) on Ω_t . We can calculate this value which turns out to be

$$F(t, y^t) = -\frac{p-1}{p} \int_{\Omega_t} y_t dx = -\frac{p-1}{p} E(t).$$

This proves our affirmation.

STEP 3: We now show that the shape derivative exists, that is the limit, $\lim_{t \rightarrow 0} \frac{E(t) - E(0)}{t}$, exists and

$$\lim_{t \rightarrow 0} \frac{E(t) - E(0)}{t} = -\frac{p}{p-1} \frac{\partial F}{\partial t}(0, y) \quad (5.11)$$

We obtain from the variational characterization (5.10) of $E(t)$ that

$$E(t) - E(0) \geq \frac{p}{p-1} (F(0, y) - F(t, y)). \quad (5.12)$$

Thus,

$$\liminf_{t \downarrow 0} \frac{E(t) - E(0)}{t} \geq \frac{p}{p-1} \lim_{t \downarrow 0} \frac{F(0, y) - F(t, y)}{t} = -\frac{p}{p-1} \frac{\partial F}{\partial t}(0, y). \quad (5.13)$$

Once again by applying the variational characterization of $E(t)$ we have

$$E(t) - E(0) \leq \frac{p}{p-1} (F(0, y^t) - F(t, y^t)).$$

Therefore, by applying the integral form of the mean value theorem in the above in the first variable

$$\begin{aligned} \frac{E(t) - E(0)}{t} &\leq \frac{p}{p-1} \frac{F(0, y^t) - F(t, y^t)}{t} \\ &= -\frac{p}{p-1} \int_0^1 \frac{\partial F(st, y^t)}{\partial s} ds \end{aligned}$$

In order to conclude the reverse inequality

$$\limsup_{t \downarrow 0} \frac{E(t) - E(0)}{t} \leq -\frac{p}{p-1} \frac{\partial F}{\partial t}(0, y) \quad (5.14)$$

it is enough to show that

$$\liminf_{t \downarrow 0} \frac{\partial F}{\partial t}(st, y^t) = \frac{\partial F}{\partial t}(0, y) \text{ for every } s \in [0, 1]. \quad (5.15)$$

By a straightforward computation it is seen that, for any $\psi \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \frac{\partial F}{\partial s}(s, \psi) &= \frac{\partial}{\partial s} \left(\frac{1}{p} \int_{\Omega} |B_s^*(\nabla \psi)|^p \gamma_s dx - \int_{\Omega} \psi \gamma_s dx \right) \\ &= \left(\int_{\Omega} \frac{1}{p} \gamma_s |B_s^* \nabla \psi|^p dx + \int_{\Omega} |\nabla \psi|^{p-2} \langle (B_s^*)' \nabla \psi, B_s^* \nabla \psi \rangle dx - \int_{\Omega} \gamma_s' \psi dx \right). \end{aligned}$$

So, in particular, by taking $\psi = y^t$ we get

$$\frac{\partial F}{\partial s}(st, y^t) = \left(\int_{\Omega} \frac{1}{p} \gamma_{st} |B_{st}^* \nabla y^t|^p dx + \int_{\Omega} |\nabla y^t|^{p-2} \langle (B_{st}^*)' \nabla y^t, B_{st}^* \nabla y^t \rangle dx - \int_{\Omega} \gamma_{st}' y^t dx \right).$$

Due to the strong convergence of ∇y^t to ∇y in $L^p(\Omega)$ and the analyticity of B_s , γ_s in s , it is now straightforward to pass to the limit as $t \rightarrow 0$ and we obtain easily, using (2.3) and (2.4), that

$$\begin{aligned} \liminf_{t \downarrow 0} \frac{\partial F}{\partial s}(st, y^t) &= \left(\int_{\Omega} \frac{1}{p} \operatorname{div} V |\nabla y|^p dx + \int_{\Omega} |\nabla y|^{p-2} \langle -(DV)^* \nabla y, \nabla y \rangle dx - \int_{\Omega} \operatorname{div} V y dx \right) \\ &= \frac{\partial F}{\partial t}(0, y) \end{aligned}$$

for every $s \in [0, 1]$, proving the claim (5.15).

STEP 4: To obtain the expression for the shape derivative (5.2), it is enough to integrate by parts in the term $-\int_{\Omega} \operatorname{div} V y dx$ which appears in the expression for $\frac{\partial F}{\partial t}(0, y) = -\frac{p-1}{p} E'(0)$ and apply the Pucci-Serrin identity (2.9). \square

We now recall the shape derivative for the eigenvalue functional. Consider the eigenvalue problem:

$$\left. \begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u && \text{in } \Omega_t, \\ u &= 0 && \text{on } \partial\Omega_t. \end{aligned} \right\} \quad (5.16)$$

The first eigenvalue $\lambda_1(t) := \lambda_1(\Omega_t)$ is simple and is characterized as the minimum of the problem

$$\lambda_1(\Omega_t) := \inf \left\{ \frac{\|\nabla \phi\|_{L^p(\Omega_t)}^p}{\|\phi\|_{L^p(\Omega_t)}^p} \mid \phi \in W_0^{1,p}(\Omega_t) \right\}. \quad (5.17)$$

We fix $y_{1,t} := y_1(\Omega_t)$ to be a corresponding eigenfunction which is positive (using the Krein-Rutman theorem) and normalize it to satisfy

$$\int_{\Omega_t} |y_{1,t}|^p dx = 1. \quad (5.18)$$

For $t = 0$, we denote the corresponding eigenvalue and eigenfunction by λ_1 and y_1 respectively.

Proposition 5.2 *The map $t \mapsto \lambda_1(t)$ is differentiable at 0 and*

$$\lambda_1'(0) = -(p-1) \int_{\partial\Omega} \left| \frac{\partial y_1}{\partial n} \right|^p \langle V, n \rangle dS \quad (5.19)$$

Proof. As we have mentioned before, this result has been shown previously by de Lis and García-Melián [10] and by Lamberti [14] (also see Ly [16]). This can also be proved along the same lines as in Proposition 5.1. \square

6 Main Results

Let $0 < r_0 < r_1$, B_1 be the ball $B(0, r_1)$ and let B_0 be any open ball of radius r_0 such that $\overline{B_0} \subset B_1$. Consider the family $\mathcal{F} = \{B_1 \setminus \overline{B_0}\}$ of domains in \mathbb{R}^N . We study the extrema of the functionals $E(\Omega)$ and $\lambda_1(\Omega)$ over \mathcal{F} , associated to the problems (1.1), (1.3) respectively.

We state our main results :

Put $\Omega_0 = B(0, r_1) \setminus \overline{B(0, r_0)}$.

Theorem 6.1 *The minimum value of the torsional rigidity functional $E(\Omega)$ on \mathcal{F} is attained only when $\Omega = \Omega_0$, i.e., when the balls are concentric.*

Theorem 6.2 *The first Dirichlet eigenvalue $\lambda_1(\Omega)$ is maximum on \mathcal{F} only when $\Omega = \Omega_0$, i.e., when the balls are concentric.*

Before proceeding to the proof we make the following observation and reduction. The functionals to be optimized are invariant under the isometries of \mathbb{R}^N . Therefore, it is enough to study these optimization problems for the class of domains $\Omega(s) := B_1 \setminus \overline{B(se_1, r_0)}$, $0 \leq s < r_1 - r_0$ where e_1 is the unit vector in the direction of the first coordinate axis. In order to study the optimality of the domain $\Omega(s)$ in the class \mathcal{F} we need to study perturbations of the domain which correspond to translations of the inner ball along the direction of the first coordinate axis. For this purpose we consider a smooth vector field $V(x) = \rho(x)e_1 \forall x \in B_1$ where $\rho : \mathbb{R}^N \rightarrow [0, 1]$ is a smooth function with compact support in B_1 such that $\rho \equiv 1$ on a neighborhood of $B(se_1, r_0)$. Let $\{\Phi_t\}_{t \in \mathbb{R}}$ be the one-parameter family of diffeomorphisms of B_1 associated with V . We see that, for t sufficiently close to 0, $\Phi_t(\Omega(s)) = \Omega(s+t)$. So, if we define $j, j_1 : (r_0 - r_1, r_1 - r_0) \rightarrow \mathbb{R}$ as follows:

$$j(s) = E(\Omega(s)) \quad \text{and} \quad j_1(s) = \lambda_1(\Omega(s))$$

we see that the minimization of E in the class \mathcal{F} corresponds to studying the minimum of j on the interval $(r_0 - r_1, r_1 - r_0)$ and that the problem of maximization of λ_1 in the class \mathcal{F} corresponds to studying the maximum of j_1 on the interval $(r_0 - r_1, r_1 - r_0)$. Also, the shape derivative of E and λ_1 at $\Omega(s)$ for the vector field V are the ordinary derivatives at s of j and j_1 respectively. We have seen in Proposition 5.1 and 5.2 that these shape derivatives exist and so the derivative of both j and j_1 exist. The optimization problems can be studied by analyzing the sign of the derivatives of j and j_1 .

First, we note that both j and j_1 are even functions and since they are differentiable, we have $j'(0) = 0 = j_1'(0)$.

We shall adopt the following notations. Given s in $(0, r_1 - r_0)$ we simply denote $\Omega(s)$ as Ω and $B(se_1, r_0)$ as B_0 and n shall denote the unit outward normal to Ω on $\partial\Omega$. Let H denote the hyperplane $H := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \mid x_1 = s\}$. Let r_H be reflection function about H . We define \mathcal{O} to be the subdomain $\mathcal{O} := \{x \in \Omega \mid x_1 > s\}$ in Ω . Then we see that the reflection of \mathcal{O} about H , namely $\mathcal{O}' = r_H(\mathcal{O})$ is contained in B_1 , whereas B_0 is symmetric with respect to H . Thus $\mathcal{O}' \subset \Omega$. For $x \in \mathcal{O}$, let x' denote the reflection of x about H , namely, the point $r_H(x)$. With these notations, if y be the solution of the equation (1.1) in Ω then, from the expression of the shape derivative (5.2) for E we obtain that

$$j'(s) = \int_{x \in \partial B_0} \left| \frac{\partial y}{\partial n}(x) \right|^p n_1(x) dS \tag{6.1}$$

since V is zero on ∂B_1 and since, for all $x \in \partial B_0$, $\langle V, n \rangle(x) = \rho e_1 \cdot n(x) = n_1(x)$, the first component of the normal vector. Similarly, if y_1 be the solution of (1.3) in Ω then, from the expression of the shape derivative of λ_1 , viz. (5.19), we obtain that

$$j'_1(s) = -(p-1) \int_{x \in \partial B_0} \left| \frac{\partial y_1}{\partial n}(x) \right|^p n_1(x) dS \quad (6.2)$$

Proof of Theorem 6.1 Let y be the solution of the boundary value problem (1.1) in Ω . We recall that $y \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ by regularity results in Tolksdorff [22], and by the strong maximum principle (cf. Theorem 5, Vazquez [23]) we have $y > 0$ in Ω . We now consider the subdomain \mathcal{O} and let us define \tilde{y} on \mathcal{O} by $\tilde{y}(x) := y(x')$ the value of y at the reflection x' of x about H . Let us note that $\frac{\partial \tilde{y}}{\partial n}(x) = \frac{\partial y}{\partial n}(x')$ and $n_1(x') = -n_1(x)$ for all $x \in \partial B_0$. Now, we may rewrite the expression (6.1) as follows:

$$\begin{aligned} j'(s) &= \int_{x \in \partial B_0 \cap \partial \mathcal{O}} \left\{ \left| \frac{\partial y}{\partial n}(x) \right|^p n_1(x) dS + \int_{x' \in \partial B_0 \cap \partial \mathcal{O}'} \left| \frac{\partial y}{\partial n}(x') \right|^p \right\} n_1(x') dS \\ &= \int_{x \in \partial B_0 \cap \partial \mathcal{O}} \left\{ \left| \frac{\partial y}{\partial n}(x) \right|^p - \left| \frac{\partial \tilde{y}}{\partial n}(x) \right|^p \right\} n_1(x) dS. \end{aligned} \quad (6.3)$$

We shall show that $j'(s) \geq 0 \forall s \in [0, r_1 - r_0)$ and is zero only if $s = 0$. We have already observed that $j'(0) = 0$ by symmetry considerations. It is clear that $n_1(x) < 0 \forall x \in \partial \mathcal{O} \cap \partial B_0 \cap H^c$. So when $s \neq 0$, we shall prove that $j'(s) > 0$ by showing that

$$\frac{\partial \tilde{y}}{\partial n}(x) < \frac{\partial y}{\partial n}(x) < 0 \quad \forall x \in \partial \mathcal{O} \cap \partial B_0 \cap H^c. \quad (6.4)$$

We shall prove inequality (6.4) in a few steps.

STEP 1: First we prove that $\frac{\partial y}{\partial n} < 0$ on ∂B_0 .

We begin by noticing that at every point x_0 on ∂B_0 , the interior sphere property holds, that is, there exists an open ball $B = B_R(z_0) \subset \Omega$ such that $\partial B \cap \partial B_0 = \{x_0\}$ and the unit outward normal n to Ω and to B coincide at x_0 . For $K > 0$ and $\alpha > 0$ we define a function $b : B \rightarrow \mathbb{R}$ as $b(x) = K \left(e^{-\alpha|x-z_0|^2} - e^{-\alpha R^2} \right)$. We have $b > 0$ in $B_R(z_0) \setminus B_{\frac{R}{2}}(z_0)$, $b(x_0) = 0$, in fact, $b \equiv 0$ on ∂B and that $\frac{\partial b}{\partial n}(x_0) < 0$. Moreover, it can be shown that

$$-\Delta_p b(x) = -2K e^{-\alpha|x-z_0|^2} |\nabla b|^{p-2} (2\alpha^2|x-z_0|^2 - N\alpha).$$

We may therefore, choose α large enough (independent of K) so that

$$-\Delta_p b \leq 0 \text{ in } B_R(z_0) \setminus B_{\frac{R}{2}}(z_0). \quad (6.5)$$

We know that y satisfies (1.1). Since y is bounded below by a positive constant on $\partial B_{\frac{R}{2}}(z_0)$, we may choose K small enough so that $b \leq y$ on $\partial B_{\frac{R}{2}}(z_0)$. Thus we have

$$b \leq y \text{ on } \partial \left(B_R(z_0) \setminus B_{\frac{R}{2}}(z_0) \right), \quad -\Delta_p b \leq 0 \text{ and } -\Delta_p y > 0 \text{ in } B_R(z_0) \setminus B_{\frac{R}{2}}(z_0). \quad (6.6)$$

Then by the WCP of Tolksdorff [21] we have $b \leq y$ in $B_R(z_0) \setminus B_{\frac{R}{2}}(z_0)$. This, along with $b(x_0) = 0 = y(x_0)$, implies that $\frac{\partial b}{\partial n}(x_0) \geq \frac{\partial y}{\partial n}(x_0)$. Since $\frac{\partial b}{\partial n}(x_0) < 0$ we get $\frac{\partial y}{\partial n}(x_0) < 0$. We have thus proved that $\frac{\partial y}{\partial n} < 0$ on $\partial B_0 \cap \partial \mathcal{O} \cap H^c$.

STEP 2: Now we prove the first inequality in (6.4).

On \mathcal{O} , the function y satisfies

$$\begin{aligned} -\Delta_p y &= 1 && \text{in } \mathcal{O}, \\ y &= 0 && \text{on } \partial \mathcal{O} \cap \partial B_0, \\ y &= y^* && \text{on } \partial \mathcal{O} \cap H, \\ y &= 0 && \text{on } \partial \mathcal{O} \cap \partial B_1; \end{aligned}$$

while \tilde{y} satisfies

$$\begin{aligned} -\Delta_p \tilde{y} &= 1 && \text{in } \mathcal{O}, \\ \tilde{y} &= 0 && \text{on } \partial \mathcal{O} \cap \partial B_0, \\ \tilde{y} &= y^* && \text{on } \partial \mathcal{O} \cap H, \\ \tilde{y} &> 0 && \text{on } \partial \mathcal{O} \cap \partial B_1; \end{aligned}$$

where y^* denote the common value of y and \tilde{y} on H . Therefore, we have

$$\begin{aligned} -\Delta_p \tilde{y} &= -\Delta_p y && \text{in } \mathcal{O}, \\ \tilde{y} &\geq y && \text{on } \partial \mathcal{O}. \end{aligned}$$

So, by the WCP of Tolksdorff [21] we get, $\tilde{y} \geq y$ in \mathcal{O} . Since, $\tilde{y} \equiv 0 \equiv y$ on $\partial B_0 \cap \partial \mathcal{O}$ we conclude that

$$\frac{\partial \tilde{y}}{\partial n}(x) \leq \frac{\partial y}{\partial n}(x) \text{ for all } x \in \partial B_0 \cap \partial \mathcal{O}. \quad (6.7)$$

By the result of Step 1 and (6.7), we can obtain a neighborhood \mathcal{N} of $\partial B_0 \cap \partial \mathcal{O}$ and positive numbers η, ε_0 small enough so that

$$|t\nabla \tilde{y}(x) + (1-t)\nabla y(x)| \geq \eta, \quad t\tilde{y}(x) + (1-t)y(x) \leq \varepsilon_0 \quad \forall t \in [0, 1], \quad \forall x \in \overline{\mathcal{N}}. \quad (6.8)$$

Let $w := \tilde{y} - y$, then $w \geq 0$ on $\partial \mathcal{O}$ with $w = 0$ on $\partial \mathcal{O} \cap (H \cup \partial B_0)$ and $w > 0$ on $\partial \mathcal{O} \cap \partial B_1$. Let $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be the map as defined in (2.10). Then, we have

$$\operatorname{div}(A(\nabla \tilde{y}) - A(\nabla y)) = \Delta_p \tilde{y} - \Delta_p y = 0 \text{ in } \mathcal{N}.$$

By Mean Value Theorem we get,

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} \left\langle \int_0^1 (\nabla A_i)(t\nabla \tilde{y} + (1-t)\nabla y) dt, \nabla w \right\rangle = 0.$$

Let $a_{ij}(x) = \int_0^1 \frac{\partial A_i}{\partial x_j}(t\nabla\tilde{y}(x) + (1-t)\nabla y(x)) dt$. Then, w satisfies

$$\begin{aligned} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial w}{\partial x_j} \right) &= 0 \quad \text{in } \mathcal{N}, \\ w &\geq 0 \quad \text{in } \mathcal{N}, \\ w &= 0 \quad \text{on } \partial\mathcal{N} \cap \partial B_0. \end{aligned} \tag{6.9}$$

By (2.11), $[a_{ij}(x)]_{i,j=1}^N \geq \min\{1, p-1\} \int_0^1 |t\nabla\tilde{y}(x) + (1-t)\nabla y(x)|^{p-2} dt$ and so, by (6.8) we conclude that $[a_{ij}(x)]_{i,j=1}^N$ is a uniformly positive definite matrix when $x \in \overline{\mathcal{N}}$. Therefore, by the maximum principle for uniformly elliptic operators (cf. Theorem 5, Ch. 2, Protter and Weinberger [17]), since w is a non-constant function, it follows that the minimum of w will be attained on $\partial\mathcal{N}$. Since $\inf_{\partial\mathcal{N}} w = 0$ it follows that $w > 0$ in \mathcal{N} . Further, by the same argument as in the Hopf's Lemma for uniformly elliptic operators (cf. Theorem 7, Ch. 2, Protter and Weinberger [17]) we have $\frac{\partial w}{\partial n} < 0$ on $\partial\mathcal{N} \cap \partial B_0$. That is, $\forall x \in \partial\mathcal{O} \cap \partial B_0$ we have the following:

$$\frac{\partial \tilde{y}}{\partial n}(x) < \frac{\partial y}{\partial n}(x) < 0.$$

□

Proof of Theorem 6.2 Recall from Section 5 that y_1 is the principal eigenfunction of (1.3) (that is, the unique solution of (1.3) for $\lambda = \lambda_1(\Omega)$) characterized by $y_1 > 0$ in Ω and $\int_{\Omega} y_1^p dx = 1$. We now consider the subdomain \mathcal{O} and let us define \tilde{y}_1 on \mathcal{O} by $\tilde{y}_1(x) := y_1(x')$ the value of y_1 at the reflection x' of x about H . Let us note that $\frac{\partial \tilde{y}_1}{\partial n}(x) = \frac{\partial y_1}{\partial n}(x')$ and $n_1(x') = -n_1(x)$ for all $x \in \partial B_0$. Now, we may rewrite the expression (6.2) as follows:

$$\begin{aligned} j'_1(s) &= -(p-1) \int_{x \in \partial B_0 \cap \partial \mathcal{O}} \left\{ \left| \frac{\partial y_1}{\partial n}(x) \right|^p n_1(x) dS - (p-1) \int_{x' \in \partial B_0 \cap \partial \mathcal{O}'} \left| \frac{\partial y_1}{\partial n}(x') \right|^p \right\} n_1(x') dS \\ &= -(p-1) \int_{x \in \partial B_0 \cap \partial \mathcal{O}} \left\{ \left| \frac{\partial y_1}{\partial n}(x) \right|^p - \left| \frac{\partial \tilde{y}_1}{\partial n}(x) \right|^p \right\} n_1(x) dS. \end{aligned} \tag{6.10}$$

We shall show that $j'_1(s) \leq 0 \forall s \in [0, r_1 - r_0]$ and is zero only if $s = 0$. We have already observed that $j'_1(0) = 0$ by symmetry considerations. It is clear that $n_1(x) < 0 \forall x \in \partial\mathcal{O} \cap \partial B_0 \cap H^c$. So when $s \neq 0$, we shall prove that $j'_1(s) < 0$ by showing that

$$\frac{\partial \tilde{y}_1}{\partial n}(x) < \frac{\partial y_1}{\partial n}(x) < 0 \quad \forall x \in \partial\mathcal{O} \cap \partial B_0 \cap H^c. \tag{6.11}$$

We shall prove inequality (6.11) in a few steps.

STEP 1: First we prove that $\frac{\partial y_1}{\partial n} < 0$ on ∂B_0 .

As in the proof of Theorem 6.1 we begin by noticing that at every point x_0 on ∂B_0 , the interior sphere

property holds, that is, there exists an open ball $B = B_R(z_0) \subset \Omega$ such that $\partial B \cap \partial B_0 = \{x_0\}$ and the unit outward normal n to Ω and to B coincide at x_0 . We recall that y_1 satisfies (1.3) and $y_1 > 0$ in Ω . We construct an auxiliary function b , as in the proof of Theorem 6.1, with α sufficiently large so that $-\Delta_p b \leq 0$ on $B_R(z_0) \setminus B_{\frac{R}{2}}(z_0)$ and K small so that $b \leq y_1$ in $\partial(B_R(z_0) \setminus B_{\frac{R}{2}}(z_0))$, to obtain $b \leq y_1$ in $B_R(z_0) \setminus B_{\frac{R}{2}}(z_0)$ and consequently to obtain $\frac{\partial y_1}{\partial n} < 0$ on ∂B_0 .

STEP 2: Now we prove the first inequality in (6.11).

On \mathcal{O} , the function y_1 satisfies

$$\begin{aligned} -\Delta_p y_1 &= \lambda_1 y_1^{p-1} && \text{in } \mathcal{O}, \\ y_1 &= 0 && \text{on } \partial \mathcal{O} \cap \partial B_0, \\ y_1 &= y_1^* && \text{on } \partial \mathcal{O} \cap H, \\ y_1 &= 0 && \text{on } \partial \mathcal{O} \cap \partial B_1; \end{aligned}$$

whereas \tilde{y}_1 satisfies

$$\begin{aligned} -\Delta_p \tilde{y}_1 &= \lambda_1 \tilde{y}_1^{p-1} && \text{in } \mathcal{O}, \\ \tilde{y}_1 &= 0 && \text{on } \partial \mathcal{O} \cap \partial B_0, \\ \tilde{y}_1 &= y_1^* && \text{on } \partial \mathcal{O} \cap H, \\ \tilde{y}_1 &> 0 && \text{on } \partial \mathcal{O} \cap \partial B_1; \end{aligned}$$

where y_1^* denotes the common value of y_1 and \tilde{y}_1 on H . Therefore, we have

$$\begin{aligned} -\Delta_p \tilde{y}_1 &= \lambda_1 \tilde{y}_1^{p-1} && \text{in } \mathcal{O}, \\ \tilde{y}_1 &\geq y_1 && \text{on } \partial \mathcal{O}. \end{aligned}$$

Therefore, by the WCP (cf. Theorem 3.1 proved in Section 3) for $\frac{\partial \beta}{\partial u} \geq 0$ we get, $\tilde{y}_1 \geq y_1$ in \mathcal{O} . Since, $\tilde{y}_1 \equiv 0 \equiv y_1$ on $\partial B_0 \cap \partial \mathcal{O}$ we conclude that

$$\frac{\partial \tilde{y}_1}{\partial n}(x) \leq \frac{\partial y_1}{\partial n}(x) \text{ for all } x \in \partial B_0 \cap \partial \mathcal{O}. \quad (6.12)$$

By the result of Step 1 and (6.12), we can obtain a neighborhood \mathcal{N} of $\partial B_0 \cap \partial \mathcal{O}$ and positive numbers η, ε_0 small enough so that

$$|t \nabla \tilde{y}_1(x) + (1-t) \nabla y_1(x)| \geq \eta, \quad t \tilde{y}_1(x) + (1-t) y_1(x) \leq \varepsilon_0 \quad \forall t \in [0, 1], \quad \forall x \in \overline{\mathcal{N}}.$$

Let $w := \tilde{y}_1 - y_1$, then $w \geq 0$ on $\partial \mathcal{O}$ with $w = 0$ on $\partial \mathcal{O} \cap (H \cup \partial B_0)$ and $w > 0$ on $\partial \mathcal{O} \cap \partial B_1$. We have

$$-\operatorname{div}(A(\nabla \tilde{y}_1) - A(\nabla y_1)) = -\Delta_p \tilde{y}_1 + \Delta_p y_1 = \lambda_1 (\tilde{y}_1^{p-1} - y_1^{p-1}) \geq 0 \text{ in } \mathcal{N},$$

where the map A is as defined in (2.10). By Mean Value Theorem we get,

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} \left\langle \int_0^1 (\nabla A_i)(t \nabla \tilde{y}_1 + (1-t) \nabla y_1) dt, \nabla w \right\rangle \geq 0.$$

Let $a_{ij}(x) = \int_0^1 \frac{\partial A_i}{\partial x_j}(t\nabla\tilde{y}_1(x) + (1-t)\nabla y_1(x)) dt$, then w satisfies

$$\begin{aligned} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial w}{\partial x_j} \right) &\geq 0 && \text{in } \mathcal{N}, \\ w &\geq 0 && \text{in } \mathcal{N}, \\ w &= 0 && \text{on } \partial\mathcal{N} \cap \partial B_0. \end{aligned} \tag{6.13}$$

As in the proof of Proposition 6.1, we observe that the matrix $[a_{ij}(x)]_{i,j=1}^N$ is a uniformly positive definite matrix when $x \in \overline{\mathcal{N}}$. Then by the maximum principle for uniformly elliptic operators (cf. Theorem 5, Ch. 2, Protter and Weinberger [17]), since w is a non-constant function, it follows that the minimum of w will be attained on $\partial\mathcal{N}$. Since $\inf_{\partial\mathcal{N}} w = 0$ it follows that $w > 0$ in \mathcal{N} . Further, by the same argument as in the Hopf's Lemma for uniformly elliptic operators (cf. Theorem 7, Ch. 2, Protter and Weinberger [17]), we have $\frac{\partial w}{\partial n} < 0$ on $\partial\mathcal{N} \cap \partial B_0$. That is, $\forall x \in \partial\mathcal{N} \cap \partial B_0$ we have the following:

$$\frac{\partial \tilde{y}_1}{\partial n}(x) < \frac{\partial y_1}{\partial n}(x) < 0.$$

□

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