

CHARACTERIZATION OF ELLIPSES AS UNIFORMLY DENSE SETS WITH RESPECT TO A FAMILY OF CONVEX BODIES

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ABSTRACT. Let $K \subset \mathbb{R}^N$ be a convex body containing the origin. A measurable set $G \subset \mathbb{R}^N$ with positive Lebesgue measure is said to be uniformly K -dense if, for any fixed $r > 0$, the measure of $G \cap (x + rK)$ is constant when x varies on the boundary of G (here, $x + rK$ denotes a translation of a dilation of K). We first prove that G must always be strictly convex and at least $C^{1,1}$ -regular; also, if K is centrally symmetric, K must be strictly convex, $C^{1,1}$ -regular and such that $K = G - G$ up to homotheties; this implies in turn that G must be $C^{2,1}$ -regular. Then for $N = 2$, we prove that G is uniformly K -dense if and only if K and G are homothetic to the same ellipse. This result was already proven by Amar, Berrone and Gianni in [3]. However, our proof removes their regularity assumptions on K and G and, more importantly, it is susceptible to be generalized to higher dimension since, by the use of Minkowski's inequality and an affine inequality, avoids the delicate computations of the higher-order terms in the Taylor expansion near $r = 0$ for the measure of $G \cap (x + rK)$ (needed in [3]).

1. INTRODUCTION

Let K be a convex body containing the origin of \mathbb{R}^N and G be a measurable subset of \mathbb{R}^N with positive Lebesgue measure $V(G)$. For each fixed $r > 0$, we define a density function $\delta_K : \mathbb{R}^N \times (0, \infty) \rightarrow \mathbb{R}$ as follows:

$$(1.1) \quad \delta_K(x, r) = \frac{V(G \cap (x + rK))}{V(rK)}, \quad x \in \mathbb{R}^N.$$

Here, $x + rK$ denotes the translation by a vector x of a dilation of K by a factor $r > 0$.

We say that G is *uniformly K -dense*, or *K -dense* for short, if there is a function $c : (0, \infty) \rightarrow (0, \infty)$ such that

$$\delta_K(x, r) = c(r) \quad \text{for every } (x, r) \in \partial G \times (0, \infty).$$

When K is the unit ball B of \mathbb{R}^N , K -dense sets have been studied in [14] in connection with the so-called *stationary isothermic surfaces* — the time-invariant level surfaces of solutions of the heat equation. There, it is proved that a set G is uniformly dense (B -dense in our terminology) if and only if the solution $U = U(x, t)$ of the following Cauchy problem

$$(1.2) \quad U_t = \Delta U \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad U = \mathcal{X}_G \quad \text{on } \mathbb{R}^N \times \{0\},$$

2010 *Mathematics Subject Classification.* 52A10, 52A20, 52A39, 52A40.

Key words and phrases. Uniformly dense sets, convex bodies, affine inequalities.

is such that

$$U(x, t) = a(t) \quad \text{for } (x, t) \in \partial G \times (0, \infty),$$

for some function $a : (0, \infty) \rightarrow (0, 1)$ (here, \mathcal{X}_G denotes the characteristic function of the set G). The latter condition qualifies ∂G as a stationary isothermic surface for U . The aforementioned equivalence easily follows from the fact that the solution of (1.2) can be written as

$$U(x, t) = \frac{|B|}{\pi^{N/2} \sqrt{t}} \int_0^\infty \delta_B(x, \sqrt{4t} \sigma) \sigma^N e^{-\sigma^2} d\sigma.$$

The study of stationary isothermic surfaces was motivated by a problem posed by M.S. Klamkin in [8]. Contributions in that field can be found in [1]-[2], [15]-[16] for the case of the initial-Dirichlet boundary value problem, [22] for the initial-Neumann boundary value problem and in [17]-[19] for some generalizations to nonlinear problems.

Problem (1.2) is the simplest setting in which stationary isothermic surfaces have been considered and their equivalence with B -dense sets have been instrumental to obtain an almost complete characterization for them. In fact, in [14], it is shown that if u is the solution of (1.2) and ∂G is connected, bounded and stationary for u , then ∂G is a sphere; if ∂G is connected, unbounded and stationary, then it is a straight line, if $N = 2$; it is either a spherical cylinder or a minimal surface (which reduces to a plane, if its total curvature is finite), if $N = 3$; its principal curvatures must satisfy certain necessary constraints for $N \geq 4$; it is also shown that the right helicoid is a stationary isothermic surface with infinite total curvature. Finally, it is observed in [14] that, if E is an *ellipsoid*, then E -dense sets are obtained as affine images of B -dense ones; in particular, any bounded E -dense set must be homothetic to E , and hence an ellipsoid itself.

The case of general K -dense sets have been considered by Amar, Berrone and Gianni in [3], when $N = 2$. There, by calculating, for a fixed $x \in G$, the Taylor expansion of the function δ_K in (1.1) as $r \rightarrow 0^+$ up to the third order, it is proved that, if ∂G is C^4 -smooth, ∂K is C^2 -smooth and G is K -dense, then both G and K must be homothetic to an ellipse E . It is reasonable to conjecture that this conclusion still holds when $N \geq 3$, that is

G is K -dense if and only if K and G are homothetic ellipsoids.

However, as we explain below, it seems difficult to extend the analysis employed in [3] to the case $N \geq 3$: other means must be developed. The purpose of this paper is to investigate in this direction.

A geometrical analysis of the the computations made in [3] and [14] gives some useful information: (i) the first relevant coefficient in the Taylor expansion for δ_K is related to the volume of certain subsets of K and can be used to give information on its symmetry; (ii) the second one is somewhat related to a weighted curvature of ∂G at x ; (iii) in the third one, the derivatives (up to the order 2) of the curvature appear. It is reasonable to expect that the higher-order coefficients contain information about higher-order derivatives of the curvature of ∂G . However, in general dimension, it is relatively easy to compute the first and second coefficient while it is very difficult to compute the higher-order ones. In any case, higher-order terms can only

give local information about the surface ∂G ; thus, to have hope to prove the conjectured result, we must use some global information.

The main result of this paper is an improvement of Amar, Berrone and Gianni's result.

Theorem 1.1. *Let $K \subset \mathbb{R}^2$ be a convex body and let G be a bounded measurable set in \mathbb{R}^2 .*

If G is K -dense, then K and G are homothetic to the same ellipse.

The improvements we introduce are mainly two: we remove the regularity assumptions on K and G ; our proof only relies on items (i) and (ii), Minkowski's inequality for mixed volumes and a variant of the affine isoperimetric inequality: we thus avoid the use of the higher-order local information mentioned in (iii). In the remainder of this section, we explain in detail the main steps of our argument.

We begin by showing, in general dimension, that a K -dense set G is necessarily strictly convex and, *no matter how regular K is*, at least $C^{1,1}$ -smooth, that is its boundary is locally the graph of a differentiable function with Lipschitz continuous derivatives (see Theorems 2.3 and 2.5). These two properties are proved by showing that G is a level set for a regular value of a $C^{1,1}$ -smooth convex function.

Then, we continue our investigation and observe that the requirement that G is K -dense does not only imply the regularity of G but also that of K itself, and more: in fact, under the additional assumption that K is centrally symmetric, we show that K must be $C^{1,1}$ -smooth, strictly convex and that $K = G - G$ (i.e. K is *Minkowski sum* of G and $-G$) up to homotheties (Theorem 2.8).

A first by-product of this result is that the gain on the regularity of K implies a gain in that of G , that must be $C^{2,1}$ -smooth. However, the true importance of Theorem 2.8 is that it points towards the direction of the desired conjecture, in the sense that, with the additional assumption that also G be centrally symmetric, we obtain that G is a dilate of K — as predicted by the conjecture — and moreover, by a bootstrap argument, we find that both K and G must be C^∞ -smooth. Notice that this always holds in the case $N = 2$, since one is able to prove that the K -density of G implies the central symmetry of K and G .

The next step of our argument is the computation of the first and second coefficient in the Taylor expansion for $\delta_K(x, r)$ in general dimension. Differently from what was done in [3], we privilege a geometrical point of view; in fact, we obtain the following formula:

$$(1.3) \quad \delta_K(x, r) = \delta_0(x) - \delta_1(x)r + o(r) \quad \text{as } r \rightarrow 0^+,$$

where

$$(1.4) \quad \delta_0(x) = \frac{V(K \cap H_{\nu(x)}^+)}{V(K)}, \quad x \in \partial G,$$

and

$$(1.5) \quad \delta_1(x) = \frac{1}{2V(K)} \sum_{i=1}^{N-1} m_i(x) \kappa_i(x), \quad x \in \partial G.$$

Here,

$$(1.6) \quad m_i(x) = \int_{K \cap \pi_{\nu(x)}} \langle \xi, e_i(x) \rangle^2 d\mathcal{H}_\xi^{N-1}, \quad i = 1, \dots, N-1;$$

$\nu(x)$ denotes the inward normal unit normal to ∂G at x ; for any $u \in \mathbb{S}^{N-1}$, H_u^+ and π_u are respectively the half-space $\{y \in \mathbb{R}^N : \langle y \cdot u \rangle \geq 0\}$ and the hyperplane ∂H_u^+ ; $\kappa_1(x), \dots, \kappa_{N-1}(x)$ and $e_1(x), \dots, e_{N-1}(x)$ are respectively the principal curvatures and directions of ∂G at x ; \mathcal{H}_ξ^{N-1} is the $(N-1)$ -dimensional Hausdorff measure.

When G is K -dense, easy consequences of (1.3), (1.4) and (1.5) are:

$$(1.7) \quad V(K \cap H_{\nu(x)}^+) = \frac{1}{2} V(K) \quad x \in \partial G,$$

and

$$(1.8) \quad \sum_{i=1}^{N-1} m_i(x) \kappa_i(x) = c V(K), \quad x \in \partial G,$$

where c is a constant. Condition (1.7) gives some sort of symmetry for K (that, for $N = 2$, implies its central symmetry, as already observed in [3]). Condition (1.8) is a constraint between the curvatures of ∂G and certain *moments of inertia* of the central sections of K . When $N = 2$, it means that the curvature of ∂G and the radial function of K must be somewhat related. This last information is crucial since it implies that K and G must be homothetic and, with the help of Minkowski's inequality and an affine inequality, that both must be ellipses.

2. CONVEXITY AND REGULARITY OF K -DENSE SETS.

Let \mathcal{K}_0^N be the set of convex bodies of \mathbb{R}^N that contain the origin in their interior; for $K \in \mathcal{K}_0^N$ let $|\cdot|_K : \mathbb{R}^N \rightarrow \mathbb{R}^+$ denote the *gauge* of the set K , that is

$$|x|_K = \min\{r > 0 : x \in rK\}.$$

It is well-known that $x + rK = \{y \in \mathbb{R}^N : |y - x|_K \leq r\}$; since $|\cdot|_K$ is a seminorm, $B_K(x, r)$ is a convenient notation for the set $x + rK$. When $K = B$, $|\cdot|_B$ is the euclidean norm and we shall drop the subscript B .

Given a measure μ on \mathbb{R}^+ and set $\phi(t) = \mu([0, t])$, we define a function $f^\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ as follows:

$$(2.1) \quad f^\phi(x) = \int_G \phi(|y - x|_K) dy = \int_G \phi(|x - y|_{-K}) dy;$$

f^ϕ is thus the convolution of the characteristic function \mathcal{X}_G and the composition of ϕ with the gauge of $-K$.

If μ is a Borel and locally finite measure, we can use the *layer-cake* representation theorem (see [10] for instance) in order to write:

$$(2.2) \quad f^\phi(x) = \int_0^{+\infty} V(G \cap \{y : |y - x|_K > t\}) d\mu = \int_0^{+\infty} V(G \setminus B_K(x, t)) d\mu.$$

If G is K -dense, the last integral does not depend on x , for $x \in \partial G$. Conversely, if $f^\phi(x)$ is constant on ∂G for every choice of the measure μ ,

for each given $r > 0$ we can set $\mu = \delta_r$ (the Dirac's delta measure centered at r) in (2.2) and obtain that $f^\phi(x) = V(G \setminus B_K(x, r))$. When G has finite measure, the assumption on f^ϕ and the fact that r is arbitrary imply that G must be K -dense. Thus, we can state the following characterization.

Theorem 2.1. *Let G be a bounded¹, measurable subset of \mathbb{R}^N . Then the following conditions are equivalent:*

- (i) G is K -dense,
- (ii) for every Borel, locally finite measure μ on \mathbb{R}^+ , the function f^ϕ defined in (2.1) does not depend on x , for $x \in \partial G$.

The following lemma is instrumental to prove the convexity of G ; its proof is straightforward.

Lemma 2.2. *Let the function $\phi(t) = \mu([0, t])$ be convex, increasing and non-constant, and let f^ϕ be the function defined in (2.1). Then:*

- (i) f^ϕ is convex and hence, in particular, continuous;
- (ii) f^ϕ is coercive, that is $f^\phi \rightarrow +\infty$ as $|x| \rightarrow \infty$.

Theorem 2.3. *Let G be a bounded K -dense set; then G is strictly convex.*

Moreover, if the function $\phi(t) = \mu([0, t])$ is convex and strictly increasing, then G is a regular level set for f^ϕ .

Proof. First, we show that, if ϕ satisfies the assumptions, then f^ϕ cannot be constant on a segment whose middle point belongs to \overline{G} .

By contradiction, let x and y be the endpoints of a segment on which f^ϕ is constant and suppose the midpoint $\frac{1}{2}(x + y) \in \overline{G}$; then

$$\int_G \{\phi(|z - x|_K)/2 + \phi(|z - y|_K)/2 - \phi(|z - (x + y)/2|_K)\} dz = 0.$$

Since the integrand is always non-negative, we get that

$$2\phi(|z - (x + y)/2|_K) = \phi(|z - x|_K) + \phi(|z - y|_K)$$

for every $z \in \overline{G}$, since both ϕ and $|\cdot|_K$ are continuous². Thus, if we choose $z = \frac{1}{2}(x + y)$ we get a contradiction.

Therefore, we can claim that the function f^ϕ does not reach its minimum on the boundary of G , otherwise f^ϕ would be constant on the convex hull of ∂G which certainly contains a segment whose middle point belongs to \overline{G} . Hence, there exists a positive number s such that the set A where $f^\phi < s$ is open, bounded and convex; also, $\partial G \subseteq \partial A = \{x \in \mathbb{R}^N : f^\phi(x) = s\}$. It is now easy to check that this property implies that $A \subseteq G \subseteq \overline{A}$ and, in particular, that G is convex and hence strictly convex. \square

Corollary 2.4. *Let G be a K -dense body; then the function*

$$x \mapsto \max_{y \in G} |y - x|_K$$

is constant on ∂G .

¹It is possible to replace this assumption by asking that $V(G) < \infty$; however, it turns out that there not exists any unbounded K -dense set of finite measure.

²This is clear when G is connected. Otherwise, it is sufficient that, for each $x \in \overline{G}$, every neighborhood of x has intersection with G of positive measure. This is guaranteed by the fact that G is K -dense.

Proof. Let x and $z \in \partial G$ and suppose by contradiction that

$$d_1 = \max_{y \in G} |y - x|_K < \max_{y \in G} |y - z|_K = d_2.$$

Then $G \setminus B_K(z, d_1) \neq \emptyset$ and hence $V(G \setminus B_K(z, d_1)) > 0$, being G a body and $B_K(z, d_1)$ open; thus,

$$\begin{aligned} V(G \cap B_K(x, d_1)) &= V(G) = \\ V(G \setminus B_K(z, d_1)) + V(G \cap B_K(z, d_1)) &> V(G \cap B_K(z, d_1)). \end{aligned}$$

□

We now study the regularity of K -dense sets.

Theorem 2.5. *Let G be a K -dense body; then ∂G is of class $C^{1,1}$, that is ∂G is locally the graph of a $C^{1,1}$ -smooth function.*

Proof. Set $f = f^\phi$ with $\phi(t) = t$. By Theorem 2.3, it is sufficient to show that $f \in C^{1,1}$.

Consider the incremental ratio of f at x in a canonical direction e_i :

$$\frac{f(x + te_i) - f(x)}{t} = \int_G \frac{|x - z + te_i|_{-K} - |x - z|_{-K}}{t} dz.$$

Since $|\cdot|_{-K}$ is almost everywhere differentiable and its gradient is a bounded map over \mathbb{R}^N , by the dominated convergence theorem, we obtain that the partial derivative $\partial_{x_i} f(x)$ exists and equals

$$\int_G \frac{\partial}{\partial x_i} |x - z|_{-K} dz = \int_{\mathbb{R}^N} \mathcal{X}_G(x - z) \frac{\partial}{\partial z_i} |z|_{-K} dz,$$

and the second factor in the integrand is bounded almost everywhere by a constant, say, L . Thus, for $x, y \in \mathbb{R}^N$, we obtain the estimate:

$$\begin{aligned} |\partial_{x_i} f(x) - \partial_{x_i} f(y)| &\leq L \int_{\mathbb{R}^N} |\mathcal{X}_G(x - z) - \mathcal{X}_G(y - z)| dz \leq \\ &L P(\partial G) |x - y|, \end{aligned}$$

since G is convex and bounded (here, $P(\partial G)$ denotes the perimeter of G).

Therefore, f is differentiable and has Lipschitz continuous partial derivatives. □

Since the function $|\cdot|_K$ has the same regularity of ∂K at all points of \mathbb{R}^N except the origin, then if $\partial K \in C^{m,1}$ for some integer m , by the same arguments used in the proof of Theorem 2.5, we can easily prove the following result.

Theorem 2.6. *Let G be a bounded K -dense set, and let $\partial K \in C^{m,1}$ for some integer m . Then $\partial G \in C^{m+1,1}$.*

Corollary 2.7. *Let G be a bounded K -dense set. If the class of homothetical images of K contains G , then $\partial G \in C^\infty$.*

Proof. We show that $G \in C^{m,1}$ for every $m \in \mathbb{N}$ by induction on m . The base step is exhibited Theorem 2.5; the inductive step is the matter of Theorem 2.6. □

The following result shows that, surprisingly, at least when K is centrally symmetric, the existence of a K -dense set implies some regularity of K itself.

Theorem 2.8. *Let K be a convex body symmetric with respect to the origin of \mathbb{R}^N , and let G be a K -dense body. Then it holds that*

- (a) $K = G - G$, up to homotheties;
- (b) K is strictly convex;
- (c) ∂K and ∂G are respectively $C^{1,1}$ -smooth and $C^{2,1}$ -smooth.

Proof. Recall that, since K is convex, to each point $x \in \partial K$ we can associate its (non-empty) *normal cone* $N_K(x)$, which is the set of vectors w such that $\langle y - x, w \rangle \geq 0$ for every $y \in K$. Thus, in order to prove the differentiability of ∂K , we only need to prove that $N_K(x) \cap \mathbb{S}^{N-1}$ contains only one vector for every $x \in \partial K$.

- (a) Without loss of generality, let us suppose that

$$\max_{y \in G} |y - x|_K = 1 \quad \text{for every } x \in \partial G.$$

We have that

$$\max_{y \in G-x} |y|_K = 1,$$

and hence $G - x \subseteq K$ for every $x \in \partial G$. It follows that $G - G \subseteq K$. Indeed, if $z \in G - G$, then $z = x - y$ for some points $x, y \in G$; since G is convex, there are points x_1 e x_2 in ∂G and a number $0 \leq \lambda \leq 1$ such that $x = \lambda x_1 + (1 - \lambda)x_2$. Hence,

$$z = \lambda(y - x_1) + (1 - \lambda)(y - x_2).$$

Since K is convex and contains both $y - x_1$ and $y - x_2$, we get that $z \in K$.

Viceversa, let x be an exposed point of ∂K and let $u \in \mathbb{S}^{N-1}$ be such that H_u is the supporting hyperplane which intersects K only at the point x .

Next, choose $y \in \partial G$ such that the (inward) unit normal vector to ∂G at y , $\nu_G(y)$, coincides with $-u$ (it exists since we already know that G is smooth and strictly convex). Also, pick a point $z \in \partial G$ that maximizes the K -distance from y , that is, such that $|y - z|_K = 1$. Note that $y - z \in (G - z) \cap \partial K$ and, since $G - z \subseteq K$, we get the following reverse inclusion for the normal cones:

$$N_K(y - z) \cap \mathbb{S}^{N-1} \subseteq \{-\nu_{G-z}(y - z)\} = \{-\nu_G(y)\} = \{u\}.$$

Hence, our choice of x and ω allows us to write $x = y - z$; Thus, $G - G$ contains all the exposed points of ∂K and hence, being K a closed convex set, it must contain also K .

- (b) It easily follows from (a) and Theorem 2.3.

- (c) It follows from (a) and Theorem 2.5, since the Minkowski sum of $C^{1,1}$ sets is $C^{1,1}$ (see for instance [9]). \square

Corollary 2.9. *If, in addition to the assumptions of Theorem 2.8, G is centrally symmetric, then $G = K$ (up to homotheties) and ∂G (and ∂K) is C^∞ -smooth.*

3. ASYMPTOTICS AS $r \rightarrow 0^+$.

Consistently with what defined in Section 1, given a unit vector $u \in \mathbb{S}^{N-1}$, we write $H_u^+ = \{x \in \mathbb{R}^N : \langle x, u \rangle \geq 0\}$ and $H_u^- = H_{-u}^+$. Also, since our focus is on K -dense sets, without loss of generality, we can always suppose that G is convex.

Theorem 3.1. *Let G and K be convex bodies and suppose that ∂G is differentiable at x . Then*

$$\lim_{r \rightarrow 0^+} \delta_K(x, r) = \frac{V(K \cap H_{\nu(x)}^+)}{V(K)}.$$

In particular, if G is K -dense, then

$$(3.1) \quad V(K \cap H_u^+) = \frac{1}{2} V(K) \quad \text{for all } u \in \mathbb{S}^{N-1}.$$

Proof. For $r > 0$ we have:

$$(3.2) \quad r^{-N} V(G \cap (x + rK)) = V\left(\frac{G-x}{r} \cap K\right).$$

Since ∂G is differentiable at x , as r decreases to 0, $\frac{G-x}{r} \cap K$ increases to $H_{\nu(x)}^+ \cap K$. The first claim of the theorem then follows from the monotone convergence theorem.

Now, suppose that G is K -dense. Then, the Gauss map from ∂G to \mathbb{S}^{N-1} that takes any $x \in \partial G$ to the outward normal unit vector $\nu(x)$ is surjective. Hence, for every $u \in \mathbb{S}^{N-1}$, there exist $x, x' \in \partial G$ such that $u = \nu(x) = -\nu(x')$.

Since G is K -dense, then the quantity $V(K \cap H_{\nu(x)}^+)$ does not depend on x , for $x \in \partial G$. Thus, our choice of x and x' enables us to write that

$$V(K \cap H_{\nu(x)}^+) = V(K \cap H_{\nu(x')}^+) = V(K \cap H_{\nu(x)}^-).$$

Since $V(K \cap H_{\nu(x)}^-) + V(K \cap H_{\nu(x)}^+) = V(K)$, then we find that

$$V(K \cap H_{\nu(x)}^+) = \frac{1}{2} V(K).$$

□

Corollary 3.2. *If G is K -dense, then*

$$(3.3) \quad \int_{K \cap \pi_u} \langle y, w \rangle dy = 0 \quad \text{for every } u, w \in \mathbb{S}^{N-1} \quad \text{with } \langle u, w \rangle = 0.$$

In particular, when $N = 2$, K is centrally symmetric.

Proof. Let u and $v \in \mathbb{S}^{N-1}$, then:

$$\begin{aligned} V(K \cap H_v^+ \cap H_u^+) + V(K \cap H_v^+ \cap H_u^-) &= V(K \cap H_v^+) = \\ V(K \cap H_v^-) &= V(K \cap H_v^- \cap H_u^+) + V(K \cap H_v^- \cap H_u^-), \end{aligned}$$

and also

$$\begin{aligned} V(K \cap H_v^+ \cap H_u^+) + V(K \cap H_v^- \cap H_u^+) &= V(K \cap H_u^+) = \\ V(K \cap H_u^-) &= V(K \cap H_v^+ \cap H_u^-) + V(K \cap H_v^- \cap H_u^-). \end{aligned}$$

Thus,

$$(3.4) \quad V(K \cap H_v^+ \cap H_u^+) = V(K \cap H_v^- \cap H_u^-).$$

Now fix $\varepsilon > 0$, a unit vector u and choose $v = -u \cos \varepsilon + w \sin \varepsilon$, where w is a unit vector orthogonal to u ; we can write that

$$K \cap H_v^+ \cap H_u^+ = \{y + tu : \langle y, u \rangle = 0, |y + tu|_K \leq 1, 0 \leq t \leq \langle y, w \rangle \tan \varepsilon\}$$

and, by a re-scaling in the variable t , we get that

$$\frac{1}{\varepsilon} V(K \cap H_v^+ \cap H_u^+) = \frac{\tan \varepsilon}{\varepsilon} V(\{y + \tau u : \langle y, u \rangle = 0, |y + \tau u \tan \varepsilon|_K \leq 1, 0 \leq \tau \leq \langle y, w \rangle\}).$$

As $\varepsilon \rightarrow 0$, $v \rightarrow u$ and we can easily infer that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} V(H_v^+ \cap H_u^+ \cap K) &= \\ &= V(\{y + \tau u : \langle y, u \rangle = 0, |y|_K \leq 1, 0 \leq \tau \leq \langle y, w \rangle\}) = \\ &= \int_{K \cap \pi_u \cap H_w^+} \langle y, w \rangle dy. \end{aligned}$$

By the same argument, we obtain that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} V(H_v^- \cap H_u^- \cap K) = - \int_{K \cap \pi_u \cap H_w^-} \langle y, w \rangle dy$$

and hence (3.4) implies (3.3).

By a simple inspection of the proof, in the case $N = 2$ we easily obtain that $-K = K$. \square

Theorem 2.8 and Corollary 3.2 immediately imply the following result.

Corollary 3.3. *Let $G \subset \mathbb{R}^2$ be a K -dense body, then $G \in C^{2,1}$.*

We now compute the second term in the asymptotic expansion for δ_K .

Theorem 3.4. *Let $G \subset \mathbb{R}^N$ be a convex body with C^2 -smooth boundary, let $x \in \partial G$ and denote by $\kappa_1(x), \dots, \kappa_{N-1}(x)$ the principal curvatures of ∂G at x with respect to the inward normal unit vector.*

Then, we have the formula:

$$(3.5) \quad \lim_{r \rightarrow 0^+} \frac{\delta_K(x, r) - \delta_0(x)}{r} = - \frac{1}{2V(K)} \sum_{i=1}^{N-1} m_i(x) \kappa_i(x),$$

where the coefficients $m_i(x)$ are given by (1.6). Therefore, (1.3) holds.

Proof. We choose a coordinate system $\{e_1, \dots, e_{N-1}, \nu\}$ around the point $x \in \partial G$ such that e_i , for $i = 1, \dots, N-1$, is the i -th principal direction of ∂G at x and $\nu = \nu(x)$ is the normal.

In these coordinates $B_K(x, r)$ can be written as

$$B_K(x, r) = \left\{ x + \sum_{i=1}^{N-1} z_i e_i + z_N \nu : z \in \mathbb{R}^N, \left| \sum_{i=1}^{N-1} z_i e_i + z_N \nu \right|_K \leq r \right\}.$$

Also, in these same coordinates, ∂G can be locally parametrized by a convex function $\psi \in C^2$ and, clearly, $\psi(0) = 0$ and $\nabla \psi(0) = 0$. Furthermore, our choice of the axes e_1, \dots, e_N allow us to write that

$$\psi(z') = \frac{1}{2} \sum_{i=1}^{N-1} \kappa_i(x) z_i^2 + o(|z'|^2),$$

for $z' = (z_1, \dots, z_{N-1}) \in \mathbb{R}^{N-1}$ in a sufficiently small neighborhood of 0.

We need to estimate the measure of the remainder set

$$R(x, r) = B_K(x, r) \cap H_{\nu(x)}^+ \setminus G;$$

for sufficiently small $r > 0$, $R(x, r)$ can be written as

$$\left\{ x + \sum_{i=1}^{N-1} z_i e_i + z_N \nu : \left| \sum_{i=1}^{N-1} z_i e_i + z_N \nu \right|_K \leq r, 0 \leq z_N \leq \psi(z'), z' \in V \right\},$$

where V is some neighborhood of 0 in \mathbb{R}^{N-1} . Next, we make the following change of variables: $z_i = r\xi_i$, for $i = 1, \dots, N-1$ and $z_N = r^2\xi_N$; since $|\cdot|_K$ is positively homogeneous, we get that

$$V(R(x, r)) = r^{N+1}V(S_r),$$

where S_r is the set

$$\left\{ \xi \in \mathbb{R}^N : \xi' \in r^{-1}V, \left| \sum_{i=1}^{N-1} \xi_i e_i + r\xi_N \nu \right|_K \leq 1; 0 \leq \xi_N \leq \frac{\psi(r\xi_1, \dots, r\xi_{N-1})}{r^2} \right\}.$$

Now, if we define the set

$$S_0 = \left\{ \xi \in \mathbb{R}^N : \left| \sum_{i=1}^{N-1} \xi_i e_i(x) \right|_K < 1; 0 \leq \xi_N < \frac{1}{2} \sum_{i=1}^{N-1} \kappa_i(x) \xi_i^2 \right\},$$

we easily check that

$$S_0 \subseteq \bigcup_{r>0} \left(\bigcap_{0<t<r} S_t \right) \subseteq \bigcap_{r>0} \left(\bigcup_{0<t<r} S_t \right) \subseteq \bar{S}_0.$$

Since $V(S_0) = V(\bar{S}_0)$, the smoothness assumptions on ∂G give the sufficient uniform boundedness to infer that

$$\lim_{r \rightarrow 0^+} \frac{V(R(x, r))}{r^{N+1}} = V(S_0).$$

By the definition of S_0 , $V(S_0)$ is easily computed as

$$V(S_0) = \int_{K \cap \pi_{\nu(x)}} \frac{1}{2} \sum_{i=1}^{N-1} \kappa_i(x) \xi_i^2 d\xi = \frac{1}{2} \sum_{i=1}^{N-1} m_i(x) \kappa_i(x),$$

that implies the desired formula (3.5). \square

Corollary 3.5. *Let G be a C^2 -smooth K -dense body. Then there exists a positive constant α such that (1.8) holds.*

Proof. That the right-hand side of (1.8) does not depend on x for $x \in \partial G$ clearly follows from (1.1) and (1.3). Since K is a convex body, then the $m_i(x)$'s are all positive; if α were zero, then all the curvatures would be zero for every $x \in \partial G$ and this is impossible, since G is a convex body. \square

4. ALTERNATIVE PROOF OF THE CONJECTURE IN THE
TWO-DIMENSIONAL CASE

In this section, we present our new proof of the result of Amar, Berrone and Gianni [3]. We stress the fact that, besides dropping the smoothness assumptions needed in [3], our proof only needs the pointwise information given by (1.8), since it relies on some global information provided by the Minkowski and affine isoperimetric inequalities. So far, we were not able to reproduce this proof in general dimension.

We need to introduce some terms and notations that we borrow from the theory of convex bodies (see [24] and [12], for instance). We limit our presentation to the case $N = 2$.

Given a convex body K , we denote by ρ_K and h_K its *radial function* and *support function*, respectively; by our notations, we have that $\rho_K(u) = 1/|u|_K$ for $u \in \mathbb{S}^1$. The only moment of inertia $m = m_1$ in (1.6) can be easily computed and, by setting $u = \nu(x)$, re-defined as a function on \mathbb{S}^1 as

$$(4.1) \quad m(u) = \frac{2}{3} \rho_K(u^\perp)^3, \quad u \in \mathbb{S}^1,$$

where u^\perp is the unit vector obtained from u by a clockwise rotation of 90 degrees.

The *curvature function* f_K of K can be defined as a non-negative function on \mathbb{S}^1 such that the *mixed volume* $V(K, G)$ can be written as

$$(4.2) \quad V(K, G) = \frac{1}{2} \int_{\mathbb{S}^1} f_K(u) h_G(u) du,$$

for every compact convex set G . When K is smooth, $f_K(u)$ is the reciprocal of the curvature κ_K of ∂K at the point on ∂K at which the normal unit vector equals u . The Minkowski's first inequality for mixed volumes tells us that

$$(4.3) \quad V(K, G) \geq \sqrt{V(K)V(G)};$$

the sign of equality holds if and only if K and G are homothetic.

We recall that the *affine area* $\Omega(K)$ of K is defined by

$$(4.4) \quad \Omega(K) = \int_{\mathbb{S}^1} f_K(u)^{2/3} du;$$

we will make use of an inequality, that relates $\Omega(K)$, $V(K)$ and the volume of the polar set K^* (with respect to the origin) of K and can be found in [5] or [13]:

$$(4.5) \quad \Omega(K)^3 \leq 8 V(K)^2 V(K^*);$$

here, the sign of equality holds if and only if there exists a positive constant λ such that $f_K(u) = \lambda h_K(u)^{-3}$, for all $x \in \partial K$.

In [20] Petty proves that the latter condition holds if and only if K is an ellipse.

Theorem 4.1. *Let $K \subset \mathbb{R}^2$ be a convex body. If $G \subset \mathbb{R}^2$ is a K -dense body, then G and K are homothetic and both ∂K and ∂G are C^∞ -smooth.*

Proof. Since K is centrally symmetric by Corollary 3.2, then Corollary 3.3 implies that formula (1.8) holds and, by (4.1) and in view of the geometric meaning of the curvature function, can be written as

$$(4.6) \quad \rho_K(u^\perp)^3 = cV(K) f_G(u), \quad u \in \mathbb{S}^1,$$

(with a slight abuse of notation) where c is some positive constant. Also, being K centrally symmetric, $\rho_K(-u^\perp) = \rho_K(u^\perp)$ and hence $f_G(-u) = f_G(u)$ for every $u \in \mathbb{S}^1$; this means that also G is centrally symmetric.

Thus, by Corollary 2.9, K and G differ by a homothety and both ∂K and ∂G are C^∞ -smooth. \square

Proof of Theorem 1.1. In view of Theorem 4.1, we know that G and K have smooth boundaries and only differ by a homothety; without loss of generality, we shall assume that $G = K$. Thus, (4.6) reads:

$$(4.7) \quad \rho_K(u^\perp)^3 = cV(K) f_K(u), \quad u \in \mathbb{S}^1.$$

Our goal is to show that (4.7) leads inequality (4.5) into an equality; then we shall conclude that K is an ellipse.

By a well-known formula, we then compute:

$$\begin{aligned} 2V(K) &= \int_{\mathbb{S}^1} \rho_K(u)^2 du = \int_{\mathbb{S}^1} \rho_K(u^\perp)^2 du = \\ &= [cV(K)]^{2/3} \int_{\mathbb{S}^1} f_K(u)^{2/3} du = [cV(K)]^{2/3} \Omega(K), \end{aligned}$$

that gives:

$$c^{-2} = \frac{\Omega(K)^3}{8V(K)}.$$

On the other hand, by the definition (4.2), (4.7) also gives:

$$\begin{aligned} V(K, (K^*)^\perp) &= \frac{1}{2} \int_{\mathbb{S}^1} f_K(u) h_{K^*}(u^\perp) du = \frac{1}{2} \int_{\mathbb{S}^1} \frac{f_K(u)}{\rho_K(u^\perp)} du = \\ &= [cV(K)]^{-1} \frac{1}{2} \int_{\mathbb{S}^1} \rho_K(u^\perp)^2 du = c^{-1} \end{aligned}$$

where we have used the well-known fact that $h_{K^*} = 1/\rho_K$.

Therefore, by applying (4.3) and (4.5) successively, we obtain that

$$\begin{aligned} \frac{\Omega(K)^3}{8V(K)} &= c^{-2} = V(K, (K^*)^\perp)^2 \geq V(K) V((K^*)^\perp) = \\ &= V(K) V(K^*) \geq \frac{\Omega(K)^3}{8V(K)}, \end{aligned}$$

that is inequality (4.5) holds with the sign of equality. \square

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