A NOTE ON POSITIVE EIGENFUNCTIONS AND HIDDEN CONVEXITY

LORENZO BRASCO AND GIOVANNI FRANZINA

Abstract. We give a simple convexity-based proof of the following fact: the only eigenfunction of the $p-$Laplacian that does not change sign is the first one. The method of proof covers also more general nonlinear eigenvalue problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a connected open set and $p \in (1, \infty)$. A (positive) number λ is said to be a *Dirichlet eigenvalue* of the $p-$Laplace operator if

$$-\mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u) = \lambda\,|u|^{p-2}\,u, \qquad \text{in } \Omega, \tag{1.1}$$

holds for some nontrivial function $u \in W^{1,p}_0(\Omega)$. In this case the function u is called *eigenfunction*. Here solutions to (1.1) are always intended in a weak sense. Observe that eigenvalues can be characterized as critical values of the nonlinear Dirichlet integral $\int_\Omega |\nabla u|^p$, restricted to the manifold

$$S_p = \{u \in W^{1,p}_0(\Omega) \,:\, \|u\|_{L^p(\Omega)} = 1\}.$$

The corresponding critical points are the eigenfunctions, normalized by the constraint on the L^p norm. The first eigenvalue $\lambda_1(\Omega)$ plays a distinguished role, since it corresponds to the global minimum of the Dirichlet integral on S_p. Modulo the choice of its sign, the first (normalized) eigenfuction is unique (see [3]).

The aim of this short note is to show that a subtle form of *hidden convexity*[1] implies the well-known result, that the only eigenfunctions with constant sign are the ones associated with $\lambda_1(\Omega)$. This has been derived in various places, under different assumptions on the regularity of Ω (see [1, 4, 5] and [6] for example). We believe that the most simple and direct proof of this fact was given by Kawohl and Lindqvist ([4]), in turn inspired by [6].

The proof in [4] is based on a clever use of the equation, but it does not clearly display the reason behind such a remarkable result. As we will show, it is just a matter of convexity of the energy functional $\int_\Omega |\nabla u|^p$. More precisely, $\int_\Omega |\nabla u|^p$ enjoys a sort of geodesic convexity on the intersection between the cone of positive functions and the manifold S_p. This allows one to conclude that on this space the global analysis of $\int_\Omega |\nabla u|^p$ is trivial, because convexity implies that *there can not be any critical point, except for the global minimizer*, which as already said is unique (except for its sign). In the end, we believe this to be the deep reason why it is impossible to have positive eigenfunctions corresponding to a $\lambda > \lambda_1(\Omega)$.

1991 *Mathematics Subject Classification.* 35P30, 47A75.
Key words and phrases. Nonlinear eigenvalue problems, uniqueness of eigenfunctions.

[1] We owe this terminology to Bernd Kawohl.

The plan of this small note is the following. In the next section, we prove a convexity property of variational integrals, whose Lagrangians depend homogeneously on the gradient. This is a variation on the convexity principle used by Belloni and Kawohl in [3] (see also [2]), in order to prove the uniqueness of the first (normalized) eigenfunction of the $p-$Laplacian. Then, Section 3 shows how this convexity can be used to derive the above claimed result about positive eigenfunctions. For the sake of generality, we will give the result in a slightly more general version (see Theorem 3.1), that can be applied to more general nonlinear eigenvalue problems such as

$$-\mathrm{div}\nabla H(x,\nabla u)=b(x)\,\lambda\,|u|^{p-2}\,u, \tag{1.2}$$

where $H:\Omega\times\mathbb{R}^N\to\mathbb{R}$ is C^1 convex and $p-$positively homogeneous in the gradient variable and $b\in L^\infty(\Omega)$, with $b\geq 0$.

2. The Hidden Convexity Lemma

The main tool of our proof is the following convexity principle, used by Belloni and Kawohl in [3] for the functional $\int|\nabla u|^p$. In order to make the paper self-contained, we repeat here the proof. The statement is slightly more general than in [3] so as to include a wider list of functionals. Moreover, we relax the strict posivity requirement on the functions.

Lemma 2.1. *Given $\Omega\subset\mathbb{R}^N$ an open set, let $p\geq 1$ and let $H:\Omega\times\mathbb{R}^N\to\mathbb{R}_+$ be a measurable function such that*

$$\begin{aligned}&z\mapsto H(x,z)\quad \textit{is convex and positively homogeneous of degree } p, \textit{ i.e.}\\ &\qquad\qquad H(x,tz)=t^pH(x,z) \textit{ for every } t\geq 0,\ (x,z)\in\Omega\times\mathbb{R}^N.\end{aligned} \tag{2.1}$$

If for every $u_0,u_1\in W^{1,p}(\Omega)$ such that $u_0,u_1\geq 0$ on Ω and

$$\int_\Omega H(x,\nabla u_i(x))\,dx<+\infty,\qquad i=0,1,$$

$\sigma_t(x)$ is defined by

$$\sigma_t(x):=\Big((1-t)\,u_0(x)^p+t\,u_1(x)^p\Big)^{\frac{1}{p}},\qquad t\in[0,1],\ x\in\Omega,$$

then the mapping

$$t\mapsto\int_\Omega H(x,\nabla\sigma_t(x))\,dx\qquad \textit{is convex on } [0,1]. \tag{2.2}$$

Proof. We claim that

$$\begin{aligned}\int_\Omega H(x,\nabla\sigma_t(x))\,dx&\leq(1-t)\int_\Omega H(x,\nabla u_0(x))\,dx\\ &\qquad+t\int_\Omega H(x,\nabla u_1(x))\,dx,\qquad t\in[0,1].\end{aligned} \tag{2.3}$$

It is easily seen that for every $t\in[0,1]$, σ_t defines an element of $W^{1,p}(\Omega)$. Indeed, this is nothing but the composition of the vector-valued Sobolev map

$$\Big((1-t)^{\frac{1}{p}}\,u_0,t^{\frac{1}{p}}\,u_1\Big)\in W^{1,p}(\Omega;\mathbb{R}^2),$$

with the ℓ_p norm, i.e. $\|(x,y)\|_{\ell^p} = (|x|^p + |y|^p)^{1/p}$. Moreover, the latter is a C^1 function outside the origin and ∇u_i vanishes almost everywhere on the set $u_i^{-1}(\{0\})$, $i = 0, 1$. Thus the usual chain rule formula holds, i.e. we obtain

$$\begin{aligned}\nabla\sigma_t &= \sigma_t^{1-p}\,\Big[(1-t)\,\nabla u_0(x)\,u_0^{p-1} + t\,\nabla u_1(x)\,u_1^{p-1}\Big]\\ &= \sigma_t\,\left[\frac{(1-t)\,u_0^p}{\sigma_t^p}\,\frac{\nabla u_0}{u_0} + \frac{t\,u_1^p}{\sigma_t^p}\,\frac{\nabla u_1}{u_1}\right],\end{aligned}$$

almost everywhere in Ω. Observe that the latter is a convex combination of $\nabla u_0/u_0$ and $\nabla u_1/u_1$. Using the convexity and homogeneity of H in the z variable, we then get

$$\begin{aligned}H(x,\nabla\sigma_t) &\le \sigma_t^p\,\left[\frac{(1-t)\,u_0^p}{\sigma_t^p}\,H\left(x,\frac{\nabla u_0}{u_0}\right) + \frac{t\,u_1^p}{\sigma_t^p}\,H\left(x,\frac{\nabla u_1}{u_1}\right)\right]\\ &= (1-t)\,u_0^p\,H\left(x,\frac{\nabla u_0}{u_0}\right) + t\,u_1^p\,H\left(x,\frac{\nabla u_1}{u_1}\right)\\ &= (1-t)\,H(x,\nabla u_0) + t\,H(x,\nabla u_1).\end{aligned}$$

By integrating over Ω, the claim (2.3) is proved. Finally, for every $t_0, t_1 \in [0,1]$ we have

$$\sigma_{(1-\lambda)\,t_0+\lambda\,t_1} = \Big((1-\lambda)\,\sigma_{t_0}^p + \lambda\,\sigma_{t_1}^p\Big)^{\frac{1}{p}}, \qquad \lambda\in[0,1].$$

Thus (2.2) follows by inequality (2.3), replacing u_0 and u_1 by σ_{t_0} and σ_{t_1}, respectively. □

Remark 2.2. It is noteworthy that the curves of the form

$$\sigma_t(x) = \Big((1-t)\,u_0(x)^p + t\,u_1(x)^p\Big)^{\frac{1}{p}}, \qquad t\in[0,1],$$

are the constant speed geodesics of $C_p = \{u \in L^p(\Omega) : u \ge 0\}$, endowed with the metric

$$d_p(u_0,u_1) = \left(\int_\Omega |u_0(x)^p - u_1(x)^p|\,dx\right)^{\frac{1}{p}}.$$

Indeed, we have

$$\begin{aligned}d_p(\sigma_t,\sigma_s) &= \left(\int_\Omega |\sigma_t(x)^p - \sigma_s(x)^p|\,dx\right)^{\frac{1}{p}}\\ &= |t-s|\,\left(\int_\Omega |u_0(x)^p - u_1(x)^p|\,dx\right)^{\frac{1}{p}} = |t-s|\,d_p(u_0,u_1),\end{aligned}$$

for all $s, t \in [0,1]$.

3. The main result

We are going to prove the main result of this note. The argument is very simple and just based on the convexity principle of Lemma 2.1, but a mild approximation argument is needed. Since this is essentially a *uniqueness result*, we do not insist on the sharp hypotheses needed to obtain existence for the problem defining $\lambda_1(\Omega)$ below. Rather, we will directly assume that this is well-defined. In what follows, for simplicity we will denote by $\nabla H(x,z)$ the gradient of H with respect to the z variable.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, having finite measure and $p > 1$. Let $H : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ be a function such that*

$$
\begin{aligned}
z \mapsto H(x,z) \ \ &\text{is } C^1 \text{ convex and homogeneous of degree } p, \text{ i.e.} \\
&H(x,tz) = |t|^p H(x,z) \text{ for every } t \in \mathbb{R}, (x,z) \in \Omega \times \mathbb{R}^N.
\end{aligned}
\tag{3.1}
$$

Assume that the variational problem

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega H(x, \nabla u)\, dx \,:\, \|u\|_{L^p(\Omega)} = 1 \right\},$$

admits at least one solution. If there exist λ and a strictly positive $v \in W_0^{1,p}(\Omega)$ such that

$$\frac{1}{p} \int_\Omega \langle \nabla H(x, \nabla v(x)), \nabla \varphi(x) \rangle \, dx = \lambda \int_\Omega |v(x)|^{p-2}\, v(x)\, \varphi(x)\, dx, \tag{3.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$, then

$$\lambda = \lambda_1(\Omega). \tag{3.3}$$

Proof. First of all, we can assume that $\|v\|_{L^p(\Omega)} = 1$, since equation (3.2) is $(p-1)$-homogeneous and $v \not\equiv 0$. Moreover, by testing the equation with $\varphi = v$ and by homogeneity of H, we get

$$\int_\Omega H(x, \nabla v(x))\, dx = \frac{1}{p} \int_\Omega \langle \nabla H(x, \nabla v(x)), \nabla v(x) \rangle \, dx = \lambda \geq \lambda_1(\Omega),$$

since v is admissible for the problem defining $\lambda_1(\Omega)$. Let us assume by contradiction that (3.3) is not true. This means that we have

$$\lambda_1(\Omega) - \lambda < 0. \tag{3.4}$$

Then we take a minimizer $u \in W_0^{1,p}(\Omega)$ for (3.3). Thanks to the homogeneity of H, we can suppose that $u \geq 0$ without loss of generality. Indeed, the function $\widetilde{u} = |u|$ is nonnegative and still satisfies the constraint on the L^p norm. Since $H(x,z) = H(x,-z)$, we get $H(x, \nabla \widetilde{u}) = H(x, \nabla u)$ almost everywhere and

$$\int_\Omega H(x, \nabla \widetilde{u}(x))\, dx = \int_\Omega H(x, \nabla u(x))\, dx = \lambda_1(\Omega).$$

For every $\varepsilon \ll 1$, we set for simplicity

$$u_\varepsilon = u + \varepsilon \qquad \text{and} \qquad v_\varepsilon = v + \varepsilon.$$

Now we define as before the following curve of functions

$$\sigma_t(x) = \Big((1-t)\, v_\varepsilon(x)^p + t\, u_\varepsilon(x)^p \Big)^{\frac{1}{p}}, \qquad x \in \Omega,\ t \in [0,1],$$

connecting v_ε and u_ε. By Lemma 2.1 we can infer that

$$
\begin{aligned}
\int_\Omega H(x, \nabla \sigma_t(x))\, dx &\leq (1-t) \int_\Omega H(x, \nabla v_\varepsilon(x))\, dx + t \int_\Omega H(x, \nabla u_\varepsilon(x))\, dx \\
&= t \left[\int_\Omega H(x, \nabla u(x))\, dx - \int_\Omega H(x, \nabla v(x))\, dx \right] \\
&\quad + \int_\Omega H(x, \nabla v_\varepsilon(x))\, dx, \qquad t \in [0,1],
\end{aligned}
$$

where we used that $\nabla u_\varepsilon = \nabla u$ and $\nabla v_\varepsilon = \nabla v$. The last estimate implies that

$$\int_\Omega \frac{H(x,\nabla\sigma_t(x)) - H(x,\nabla\sigma_0(x))}{t}\,dx \le \int_\Omega \Big[H(x,\nabla u(x)) - H(x,\nabla v(x))\Big]\,dx$$
$$= \lambda_1(\Omega) - \lambda, \qquad t\in(0,1],$$

where we used that $\sigma_0 = v_\varepsilon$ by construction. By the convexity of H in the left-hand side we can infer

$$\int_\Omega \left\langle \nabla H(x,\nabla\sigma_0(x)), \frac{\nabla\sigma_t(x) - \nabla\sigma_0(x)}{t}\right\rangle dx \le \lambda_1(\Omega) - \lambda.$$

Since $\nabla\sigma_0 = \nabla v_\varepsilon = \nabla v$, we can use the equation (3.2) and test it with $\varphi = \sigma_t - \sigma_0 \in W^{1,p}_0(\Omega)$. This yields

$$p\,\lambda \int_\Omega v(x)^{p-1}\,\frac{\sigma_t(x)-\sigma_0(x)}{t}\,dx \le \lambda_1(\Omega)-\lambda, \qquad \text{for every } \varepsilon\ll 1,\ t\in(0,1].$$

We first pass to the limit as t goes to 0. Observe that by concavity of the function $s\mapsto s^{1/p}$, we have

$$v^{p-1}\,\frac{\sigma_t-\sigma_0}{t} \ge v^{p-1}\,(\sigma_1-\sigma_0) = v^{p-1}\,(u_\varepsilon - v_\varepsilon)\in L^1(\Omega), \quad t\in(0,1],$$

then by Fatou Lemma the limit as t goes to 0 gives

$$\lambda\int_\Omega \left(\frac{v(x)}{v_\varepsilon(x)}\right)^{p-1}\big[u_\varepsilon(x)^p - v_\varepsilon(x)^p\big]\,dx \le \lambda_1(\Omega)-\lambda, \qquad \text{for every } \varepsilon\ll 1.$$

We now send ε to 0. Since u and v have the same L^p norm, we finally end up with

$$0 = \lambda\left[\int_\Omega u(x)^p\,dx - \int_\Omega v(x)^p\,dx\right] \le \lambda_1(\Omega)-\lambda,$$

where we also used that $v>0$ on Ω. This contradicts assumption (3.4), hence the Theorem is proved. □

Remark 3.2. Observe that we required the solution v to (3.2) to be *strictly* positive on Ω. This is not a big deal, since in many situations of interest Harnack's inequality holds true and guarantees that nontrivial nonnegative solutions of (3.2) do not vanish at interior points of Ω. This is the case for example for $H:\Omega\times\mathbb{R}^N\to\mathbb{R}_+$ satisfying (3.1) and the growth conditions

$$c_1\,|z|^p \le H(x,z) \le c_2\,|z|^p, \qquad (x,z)\in\Omega\times\mathbb{R}^N,$$

with two positive constants $c_1 \ge c_2 > 0$.

The uniqueness of positive eigenfunctions of the $p-$Laplacian is now an easy consequence of Theorem 3.1.

Corollary 3.3. *Let $\Omega\subset\mathbb{R}^N$ be a connected open set having finite measure. Then the only Dirichlet eigenfunctions of the $p-$Laplacian having constant sign are those corresponding to the first eigenvalue, that is defined by*

$$\lambda_1(\Omega) = \min_{u\in W^{1,p}_0(\Omega)}\left\{\int_\Omega |\nabla u(x)|^p\,dx\ :\ \|u\|_{L^p(\Omega)} = 1\right\}.$$

Remark 3.4. The same conclusions can be drawn for the Dirichlet eigenfunctions of the so called *pseudo* $p-$*Laplacian* $\widetilde{\Delta}_p$, defined by

$$\widetilde{\Delta}_p u := \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p-2} \, \partial_{x_i} u \right).$$

Here the eigenvalue problem (introduced in [2]) consists in finding the positive numbers $\lambda > 0$, such that the equation

$$-\widetilde{\Delta}_p u = \lambda \, |u|^{p-2} \, u,$$

has nontrivial solutions in $W^{1,p}_0(\Omega)$. The proof amounts to applying again Theorem 3.1, now with the variational integral

$$\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u(x)|^p \, dx.$$

Remark 3.5. We observe that the statement of Theorem 3.1 holds true and the proof is exactly the same if we replace the L^p constraint $\|u\|_{L^p(\Omega)} = 1$ by the following weighted one

$$\int_\Omega b(x) \, |u(x)|^p \, dx = 1.$$

Here $b(x)$ is any nonnegative bounded measurable weight such that the nonlinear eigenvalue problem

$$-\mathrm{div} \, \nabla H(x, \nabla u(x)) = \lambda \, b(x) \, |u(x)|^{p-2} \, u(x),$$

admits a first eigenfunction.

Acknowledgement. The authors would like to thank Bernd Kawohl for his kind interest in this work, as well as the Centre International de Rencontres Mathématiques (Marseille) and its facilities, where this paper has been written.

References

[1] A. Anane, Simplicité et isolation de la première valeur propre du $p-$Laplacien avec poids, C. R. Acad. Sci. Paris, Sér. I Math., **305** (1987), 725–728.

[2] M. Belloni, B. Kawohl, The pseudo $p-$Laplace eigenvalue problem and viscosity solution as $p \to \infty$, ESAIM Control Optim. Calc. Var., **10** (2004), 28–52.

[3] M. Belloni, B. Kawohl, A direct uniqueness proof for equations involving the $p-$Laplace operator, Manuscripta Math., **109** (2002), 229–231.

[4] B. Kawohl, P. Lindqvist, Positive eigenfunctions for the $p-$Laplace operator revisited, Analysis, **26** (2006), 539–544.

[5] P. Lindqvist, On the equation $\mathrm{div}\,(|\nabla|^{p-2}\,\nabla u) + \lambda\,|u|^{p-2}\,u = 0$, Proc. AMS, **109** (1990), 157–164.

[6] M. Ôtani, T. Teshima, On the first eigenvalue of some quasilinear elliptic equations, Proc. Japan Acad. Ser. A Math. Sci., **64** (1988), 8–10.

Laboratoire d'Analyse, Topologie, Probabilités UMR6632,, Aix-Marseille Université,, 39 Rue Frédéric Joliot Curie,, 13453 Marseille Cedex 13,, France

E-mail address: `brasco@cmi.univ-mrs.fr`

Dipartimento di Matematica,, Università di Trento,, Via Sommarive 14,, 38100 Povo (TN),, Italy

E-mail address: `franzina@science.unitn.it`