FRACTIONAL REGULARITY FOR NONLINEAR ELLIPTIC PROBLEMS WITH MEASURE DATA

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Abstract. We consider nonlinear elliptic equations of the type

$$-\operatorname{div} a(x, Du) = \mu$$

having a Radon measure on the right-hand side and prove fractional differentiability results of Calderón-Zygmund type for very weak solutions. We extend some of the results achieved by G. Mingione (2007) [29], in turn improving a regularity result by Cirmi & Leonardi (2010) [8].

Contents

1. Introduction

We study nonlinear elliptic equations with a right-hand side being merely a measure. Our aim is to establish quantified higher differentiability properties of Calderón-Zygmund type for the gradient of the weak solutions to such equations. Precisely, we deal with the following Dirichlet problems

$$\begin{cases} -\operatorname{div} a(x, Du) = \mu & \text{in } \Omega \subset \mathbb{R}^n, n \geq 2 \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, we assume that μ, in the most general case, is a signed Radon measure with finite total mass $|\mu|(\Omega) < \infty$, and $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory regular vector field

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satisfying standard monotonicity and p-growth conditions, i. e., for every z_1, $z_2 \in \mathbb{R}^n$, $x \in \Omega$,

$$\nu(s^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1|^2 \;\leq\; \langle a(x,z_2)-a(x,z_1), z_2-z_1\rangle, \tag{1.2}$$

$$|a(x,z_2)-a(x,z_1)| \leq L(s^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1|, \tag{1.3}$$

$$|a(x,0)| \leq Ls^{p-1}, \tag{1.4}$$

where $0<\nu\leq L$ and $s\geq 0$.

Also, we will deal with a fractional Sobolev type differentiability assumption on the coefficients of the map $x \mapsto a(x,z)$:

$$|a(x_2,z)-a(x_1,z)| \;\leq\; \big(g(x_1)+g(x_2)\big)|x_2-x_1|^\alpha(s^2+|z|^2)^{\frac{p-1}{2}}, \quad \forall\, x_1,x_2\in\Omega, \quad \forall\, z\in\mathbb{R}^n, \tag{1.5}$$

for every $0<\alpha\leq 1$, where $0\leq g\in L^r(\Omega)$, $r\geq 2\chi/(\chi-1)$, with χ being the higher integrability exponent of Gehring's theory (see forthcoming Lemma 3.2). For an exhaustive discussion about such assumption, introduced as a notion of *fractional differentiability* by DeVore & Sharpley ([10]), we refer to Section 7 in [33] and Section 1.2 in [19] (see, also, [17, 18, 21]). Roughly speaking, the function g plays the role of an α-derivative of the function $x\to a(x,z)$ and so (1.5) serves to describe the α-Hölder continuity in a weak way, since the pointwise Hölder semi-norm g may blow-up at some points.

We will focus mainly on the case when μ belongs to the Lebesgue space $L^q(\Omega)$ in a range of q that does not necessarily permit to obtain the existence of finite energy solutions $u\in W^{1,p}_0(\Omega)$ to problem (1.1). However, we can deal with the (very) weak solutions $u\in W^{1,1}_0(\Omega)$ obtained via the Boccardo & Gallouët standard approximation procedure; that is, a function $u\in W^{1,1}_0(\Omega)$ such that $a(x,Du)\in L^1(\Omega\times\mathbb{R}^n)$ and

$$\int_\Omega a(x,Du)D\phi\,\mathrm{d}x \;=\; \int_\Omega \phi\,\mathrm{d}\mu \quad \forall\,\phi\in C^\infty_0(\Omega).$$

The approach to show existence of weak solutions developed by the aforementioned authors in [4, 5] is the following: one considers regular right-hand sides f_k which converge in the weak sense of measure to μ, and the weak solutions u_k to the regularized problems (1.1) with μ replaced by f_k. Then, exploiting the classic theory of elliptic equations with regular data allows to establish a priori estimates for the solutions u_k, being stable when passing to the limit on k. Finally, when $2-1/n<p\leq n$, one can deduce the existence of a solution u to (1.1) such that[1]

$$Du\in L^q, \quad \text{for every } 1\leq q<\frac{n(p-1)}{n-1}=:m. \tag{1.6}$$

On the other hand, when $p>n$, μ belongs to $W^{-1,p'}(\Omega)$, and the existence of a unique solution in the natural space $W^{1,p}_0(\Omega)$ follows by standard methods (see [27]). For other related regularity results for the weak solutions to (1.1) in which similar techniques

[1] We recall that the lower bound $p>2-1/n$ serves to ensure the existence of the very weak solutions u, otherwise it is not guaranteed that the gradient of u belongs to L^1.

are involved, we refer for instance to [11, 6, 1, 7, 20] and the references therein; see, also, [30, 31, 15, 32, 21, 22, 12, 13] for explicit local and potential estimates, both in the Lebesgue scale and in the Lorentz-Morrey one.

It is worth observing that the integrability of the gradient of the solutions u stated in (1.6) is optimal, in the sense of the inclusion $q < m$, since in general $Du \notin L^m(\Omega)$. In the present paper, we are interested in estimating the oscillations of the gradient rather than its size; that is, we want to investigate the higher regularity of Du. In this respect, let us focus for a while on the basic case $-\Delta u = \mu$; i.e., we are considering $p \equiv 2$, $a(x, Du) \equiv Du$ and the assumption in (1.5) does not really take part. In this case, the standard Calderón-Zygmund theory [23] asserts

$$f \in L^{1+\varepsilon} \implies Du \in W^{1,1+\varepsilon} \quad \text{for every } \varepsilon > 0\,.$$

This does not hold when $\varepsilon = 0$, since the inclusion $Du \in W^{1,1}$ may fail. For this, it suffices to consider the classic example given by the following Dirichlet problem in the unit ball

$$\begin{cases} -\Delta u = \delta_0 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

with δ_0 being the Dirac measure centered in the origin. The unique solution is now given by the Green's function

$$u(x) := c(n) \begin{cases} |x|^{2-n} - 1 & \text{if } n > 2 \\ \log|x| & \text{if } n = 2. \end{cases} \tag{1.7}$$

Assume by contradiction that $Du \in W^{1,1}(B_1)$. Then, by the critical Sobolev embedding, it follows $Du \in L^m(B_1)$, where we recall that $m = n/(n-1)$ since here $p = 2$, and this limit case does not hold true, as one can see by computing the summability of the function u in (1.7).

Therefore, in general we can not have a second derivative of u in some Sobolev space. On the other hand, it is still possible to establish an optimal Calderón-Zygmund theory for nonlinear elliptic problems of type (1.1), provided that the right Sobolev spaces are considered. This has been firstly analyzed by G. Mingione in [29], in which it has been proved that the solutions to (1.1), under (1.2)–(1.4) and a Lipschitz assumption on the coefficients, satisfy the following fractional Sobolev inclusion

$$Du \in W^{1-\varepsilon,1}_{\mathrm{loc}}(\Omega), \quad \forall\, 0 < \varepsilon < 1. \tag{1.8}$$

More in general, for $p \geq 2$, it holds

$$Du \in W^{\frac{2}{p}\delta - \frac{\varepsilon}{q}, q}_{\mathrm{loc}}(\Omega), \quad \forall\, 0 < \varepsilon < 2q\delta/p, \quad \forall\, p-1 \leq q < m, \tag{1.9}$$

with

$$\delta = \delta(p,q) := \frac{p}{2q}\left(n - \frac{q(n-1)}{p-1}\right) \tag{1.10}$$

giving the optimal exponent depending on the couple (p, q). We notice that, in accordance with (1.8), $p = 2$ and $q = 1$ yield $\delta = 1$, but in general $\delta \leq 1$.

Precisely, in order to prove the sharp inclusion (1.8)-(1.9) for the solutions to general nonlinear elliptic problems as in (1.1), condition (1.5) in the purely Lipschitz case $\alpha = 1$ and $g \equiv const.$ is required.

This result has been extended to the purely Hölder case, that is condition (1.5) with $\alpha \leq 1$ and $g \equiv const.$, by Cirmi & Leonardi in [8], in which, among other results (see, also, [9]), it has been proved that, when $p = 2$, the following inclusion holds

$$Du \in W^{\alpha\delta - \frac{\varepsilon}{q}, q}_{\mathrm{loc}}(\Omega), \quad \forall\, 0 < \varepsilon < q\alpha\delta, \quad \forall\, 1 \leq q < n/(n-1). \tag{1.11}$$

In the present paper, we will show that, under the assumptions (1.2)–(1.5), the solutions u to (1.1) when $p = 2$ satisfy

$$Du \in W^{\min\{\alpha,\delta\} - \frac{\varepsilon}{q}, q}_{\mathrm{loc}}(\Omega), \quad \forall\, 0 < \varepsilon < q\min\{\alpha,\delta\}, \quad \forall\, 1 \leq q < n/(n-1). \tag{1.12}$$

We immediately notice that in the Lipschitz case when $\alpha = 1$ all the mentioned results collapse in the same higher differentiability of the solutions. On the other hand, in the Hölder case when $\alpha < 1$ the regularity stated in (1.12) provides an improvement with respect to that in (1.11), since $\alpha\delta \leq \min\{\alpha, \delta\}$ plainly holds.

Precisely, we will prove the following theorems.

Theorem 1.1. *Under the assumptions* (1.2)–(1.5)*, with* $2 - 1/n < p \leq n$*, let* $u \in W^{1,1}_0(\Omega)$ *be a solution to problem* (1.1)*. Then*

$$Du \in W^{\frac{2}{p}\min\{\alpha,\delta\} - \frac{\varepsilon}{q}, q}_{\mathrm{loc}}(\Omega), \quad \forall\, 0 < \varepsilon < \frac{2q}{p}\min\{\alpha,\delta\}, \quad \forall\, 1 \leq q < m,$$

where m *and* δ *are given by* (1.6) *and* (1.10)*, respectively.*

Theorem 1.2. *Let the assumptions and notation of* Theorem 1.1 *hold, and let* $0 < \sigma < 2q\min\{\alpha,\delta\}/p$*. Then there exists a constant* $c \equiv c(n, p, q, \alpha, L/\nu, \|g\|_{L^r})$ *such that*

$$[Du]^q_{W^{\sigma/q,q}(B_{R/2})} \leq cR^{-\sigma}\int_{B_R}(|Du|^q + s^q)\,\mathrm{d}x + cR^{\frac{2q}{p}\min\{\alpha,\delta\}-\sigma}|\mu|(\overline{B_R})^{\frac{q}{p-1}}, \tag{1.13}$$

for every ball $B_R \subset\subset \Omega$ *of radius* $R > 0$*; where* $[\,\cdot\,]_{W^{\sigma/q,q}}$ *denotes the fractional Sobolev seminorm* (*see* Section 2.2 *below*)*. Moreover, for every open subset* $\Omega' \subset\subset \Omega$ *there exists a constant* $c \equiv c(n, p, q, \alpha, L/\nu, \mathrm{dist}(\Omega', \partial\Omega), \Omega, \|g\|_{L^r})$ *such that the following local estimate holds*

$$\int_{\Omega'} |Du|^q\,\mathrm{d}x + [Du]^q_{W^{\sigma/q,q}(\Omega')} \leq c|\mu|(\Omega)^{\frac{q}{p-1}} + cs^q|\Omega|\,. \tag{1.14}$$

Therefore, in the present paper we will extend the higher differentiability results in [29] to the more general case involving the weak Hölder continuity assumption (1.5), as well as providing an improvement to the results in [8] when $p = 2$ and no functions g are considered. Moreover, exploiting some techniques from [33], here we will be able to deal with the case $2 - 1/n < p < 2$, too.

The strategy of the proofs follows the one developed in [29] (see, also, [3] for extended results in the parabolic framework). In a first step, we will prove some comparison estimates for the solutions u to (1.1) and the solutions to the corresponding homogeneous problem (see Section 3), by also applying the Hölder continuity theory by De Giorgi-Nash-Moser and the higher integrability theory by Gehring. We will work locally by combining these estimates together with some properties of the fractional Sobolev spaces, in order to show that the initial fractional differentiability of Du can be improved in a precise range depending on α and δ (see Lemma 4.1). Then, we will show that, thanks to some precise properties of the involved quantities, this procedure can be iterated to finally obtaining the desired result (see Lemma 4.2). Clearly, we have to produce some modifications in view of the novel assumptions we are considering, that is, (1.5), and to handle the case $2 - 1/n < p < 2$.

The paper is organized as follows. In Section 2, we fix notation; we give details on the structure of the problem and we briefly recall the definition and a few basic properties of the fractional Sobolev spaces. In Section 3, we state and prove comparison regularity estimates and other needed results. Section 4 is devoted to the proof of the main results, and to further extensions not covered by Theorem 1.1 and 1.2.

2. Preliminaries

As usual, we denote by

$$B_R(x_0) = B(x_0; R) := \{x \in \mathbb{R}^n \,:\, |x - x_0| < R\}$$

the open ball centered in $x_0 \in \mathbb{R}^n$ with radius $R > 0$. When not important and clear from the context we shall use the shorter notation $B_R = B(x_0; R)$. We denote by λB the concentric ball scaled by a factor $\lambda > 0$, that is $\lambda B := B(x_0; \lambda R)$.

Moreover, if $f \in L^1(A)$ and the n-dimensional Lebesgue measure $|A|$ of the set A is finite and strictly positive, we write

$$(f)_A := \fint_A f(x)\,\mathrm{d}x = \frac{1}{|A|}\int_A f(x)\,\mathrm{d}x.$$

2.1. The map $V(z)$ and the monotonicity of $a(x,z)$.

For any given $s \geq 0$, and $p > 1$, consider the locally bi-Lipschitz bijection V of $\mathbb{R}^n$ defined as follows

$$V(z) = V_s(z) := (s^2 + |z|^2)^{\frac{p-2}{4}} z, \quad \forall\, z \in \mathbb{R}^n.$$

For any z_1, $z_2 \in \mathbb{R}^n$ and any $s \geq 0$, we have

$$c^{-1}(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} \;\leq\; \frac{|V(z_2) - V(z_1)|^2}{|z_2 - z_1|^2} \;\leq\; c(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}},$$

where $c \equiv c(n,p)$ is independent of s. Also,

$$\begin{cases} |V(z)|^2 = |z|^2 & \text{if } p = 2, \\ |z|^p \leq |V(z)|^2 \leq 2(s^p + |z|^p) & \text{if } p > 2, \\ |V(z)|^2 \leq |z|^p & \text{if } p \in [1,2). \end{cases} \tag{2.1}$$

The strict monotonicity properties of the vector field a, see (1.2), can be reformulated by means of the map V. Indeed, combining (1.2) and (2.1), it follows

$$c^{-1}|V(z_2) - V(z_1)|^2 \;\leq\; \langle a(x,z_2) - a(x,z_1), z_2 - z_1\rangle, \quad \forall\, z_1, z_2 \in \mathbb{R}^n, \tag{2.2}$$

where $c \equiv c(n,p,\nu) > 0$. Moreover, when $p \geq 2$, assumptions (1.2) also implies

$$c^{-1}|z_2 - z_1|^p \leq \langle a(x,z_2) - a(x,z_1), z_2 - z_1 \rangle.$$

Finally inequality (1.2) together with (1.4) and a standard use of Young's inequality yields

$$c^{-1}(s^2 + |z|^2)^{\frac{p-2}{2}}|z|^2 - c\, s^p \;\leq\; \langle a(x,z), z \rangle, \quad \forall\, z \in \mathbb{R}^n,$$

where $c \equiv c(n,p,L/\nu)$; while (1.3) with (1.4), and again Young's inequality, gives

$$|a(x,z)| \leq c(s^2 + |z|^2)^{\frac{p-1}{2}}.$$

2.2. **Fractional Sobolev spaces.** We briefly recall the definition of Sobolev spaces of fractional order $W^{t,q}$, as well as some related basic facts. We refer to the classics [25, 2, 34], and to the recent books [35, 26]; see, also, [14].

Let $A \subset \mathbb{R}^n$ and $k \in \mathbb{N}$. We start by fixing the fractional exponent t in $(0,1)$. For any $q \in [1,+\infty)$, we define $W^{t,q}(A,\mathbb{R}^k) \equiv W^{t,q}(A)$ as follows

$$W^{t,q}(A) := \left\{ |w| \in L^q(A) \;:\; \frac{|w(x) - w(y)|}{|x-y|^{\frac{n}{q}+t}} \in L^q(A \times A) \right\};$$

i. e., an intermediary Banach space between $L^q(A)$ and $W^{1,q}(A)$, endowed with the natural norm

$$\|w\|_{W^{t,q}(A)} := \left(\int_A |w|^q \,\mathrm{d}x \,+\, \int_A \int_A \frac{|w(x) - w(y)|^q}{|x-y|^{n+tq}} \mathrm{d}x\, \mathrm{d}y \right)^{\frac{1}{q}},$$

where the term

$$[w]_{W^{t,q}(A)} := \left(\int_A \int_A \frac{|w(x) - w(y)|^q}{|x-y|^{n+tq}} \mathrm{d}x\, \mathrm{d}y \right)^{\frac{1}{q}}$$

is the so-called *Gagliardo* (*semi*)*norm* of u.

For a vector valued function $w : A \to \mathbb{R}^k$ and $h \in \mathbb{R}$, we define the *finite difference operator* $\tau_{i,h}$, for $i \in \{1,...,n\}$ as

$$\tau_{i,h} w(x) = \tau_{i,h}(w)(x) := w(x + he_i) - w(x), \tag{2.3}$$

with $\{e_i\}_{i=1,\dots,n}$ being the standard basis of $\mathbb{R}^n$. In the following we always assume that x, $x + he_i \in A$ in order to make (2.3) worth.

Now, we recall the critical Sobolev embedding of $W^{t,q}(A)$ in $L^{nq/(n-tq)}(A)$, whose proof can be found in [14, Theorem 6.7]; see, also, [16, Lemma 3].

Proposition 2.1. *Let $w \in W^{t,q}(A)$, with $q \geq 1$ and $t \in (0,1]$, such that $tq < n$, and let $A \subset \mathbb{R}^n$ be an extension domain for $W^{t,q}$. Then $w \in L^{nq/(n-tq)}(A)$, and there exists a constant $c \equiv c(n,q,t,A)$ such that*

$$\|w\|_{L^{\frac{nq}{n-tq}}(A)} \;\leq\; c\|w\|_{W^{t,q}(A)}.$$

Next result is nothing but a fractional Poincaré inequality; see, for instance, [28] and related references.

Proposition 2.2. *If B_R is a ball and $w \in W^{t,q}(B_R)$, then*

$$\int_{B_R} |w-(w)_{B_R}|^q \,\mathrm{d}x \,\le\, c(n)R^{tq}[w]^q_{W^{t,q}(B_R)}.$$

3. Regularity for the homogeneous problem and comparison results

In this section we show some comparison estimates between the solution u_k to the following regularized problem with the regularizing datum $f_k \in L^\infty(\Omega)$,

$$\begin{cases} -\mathrm{div}\, a(x,Du_k) = f_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and the solutions to analogous homogeneous problems. Note that in the remaining of the paper we will always write u instead of u_k. We will show how to recover regularity and estimates for the original solutions only in the conclusion of the proofs of Theorem 1.1 and 1.2.

Lemma 3.1. ([33, Theorem 9.1]). *Let $a_0 : \mathbb{R}^n \to \mathbb{R}^n$ satisfy (1.2)–(1.4), for any $1 < p \le n$, and let $v_0 \in W^{1,p}(A)$ be a weak solution to*

$$-\mathrm{div}\, a_0(Dv_0) = 0 \quad \textit{in } A.$$

Then there exists $c \equiv c(n,p,L/\nu,q)$ such that, for $q \in (0,2]$, we have

$$\int_{B_{R/2}} |D(V(Dv_0))|^q \,\mathrm{d}x \,\le\, \frac{c}{R^q}\int_{B_R} |V(Dv_0)-V(z)|^q \,\mathrm{d}x, \quad \forall\, z \in \mathbb{R}^n, \tag{3.2}$$

for every ball $B_R \subseteq A$.

It is important to remark that the constant c in (3.2) is independent of the choice of $z \in \mathbb{R}^n$.

In the following lemma, we collected a few basic consequences of De Giorgi's regularity theory and Gehring's lemma for elliptic problems; see [24, Chapter 6-7], and, also, [29, Lemma 3.3] for a sketch of the proof.

Lemma 3.2. *Let $v \in W^{1,p}(A)$, $1 < p \le n$ be a weak solution to*

$$-\mathrm{div}\, a(x,Dv) = 0 \quad \textit{in } A,$$

under the assumptions

$$|a(x,z)| \le c(s^2+|z|^2)^{\frac{p-1}{2}}, \quad c^{-1}|z|^p - c\,s^p \le \langle a(x,z),z\rangle$$

for every $x \in A$ and $z \in \mathbb{R}^n$, where $c \equiv c(L/\nu)$ and ν, L are the numbers given in (1.2)–(1.4). There exists $\varpi \equiv \varpi(n,p,L/\nu) \in (0,1]$, such that for every $q \in (0,p]$ there exists $c \equiv c(n,p,L/\nu,q)$ such that, whenever $B_R \subseteq A$ and $0 < \rho \le R$, it holds

$$\int_{B_\rho} (|Dv|+s)^q \,\mathrm{d}x \,\le\, c\left(\frac{\rho}{R}\right)^{n-q+\varpi q}\int_{B_R} (|Dv|+s)^q \,\mathrm{d}x.$$

Moreover, there exists $\chi \equiv \chi(n,p,L/\nu) > 1$*, such that* $Dv \in L^{p\chi}_{\rm loc}(A,\mathbb{R}^n)$ *and*

$$\left(\fint_{B_{R/2}} (|Dv|+s)^{p\chi}\,{\rm d}x\right)^{\frac{1}{p\chi}} \le c\left(\fint_{B_R} (|Dv|+s)^{q}\,{\rm d}x\right)^{\frac{1}{q}}, \tag{3.3}$$

where again $c \equiv c(n,p,L/\nu,q)$.

The next density estimate is of crucial importance in order to get the gradient differentiability and integrability results in the case $p<2$.

Proposition 3.3. ([33, Proposition 9.4]) *Under the assumptions* (1.2)–(1.4)*, with* $2-1/n < p \le n$*, let* $u \in W^{1,p}_0(\Omega)$ *be the unique solution to* (3.1) *and let* $B_\rho \subset\subset \Omega$ *be a ball with radius* $\rho > 0$*, then for every* $1 \le q < m$*, with* m *defined in* (1.6)*, the following inequality holds*

$$\rho^{\frac{q(n-1)}{p-1}}\fint_{B_\rho} (|Du|+s)^{q}\,{\rm d}x \le c\left(\int_{\Omega} |f|\,{\rm d}x\right)^{\frac{q}{p-1}},$$

where $c \equiv c(n,p,L/\nu,q,{\rm dist}(B_\rho,\partial\Omega))$.

Finally, we are in position to establish some comparison lemmata. For almost every $x_0 \in \Omega$, consider a ball $B_R = B(x_0,R) \subset\subset \Omega$, with $R \le 1$, and let $v \in u + W^{1,p}_0(B_R)$ be the unique weak solution to

$$\begin{cases} -{\rm div}\, a(x,Dv) = 0 & \text{in } B_R \\ v = u & \text{on } \partial B_R. \end{cases} \tag{3.4}$$

Also, take a ball $B_{\bar R} \equiv B(x_0,\bar R) \subseteq B(x_0,R)$ and let $v_0 \in v + W^{1,p}_0(B_{\bar R})$ be the unique weak solution to

$$\begin{cases} -{\rm div}\, a(x_0,Dv_0) = 0 & \text{in } B_{\bar R} \\ v_0 = v & \text{on } \partial B_{\bar R}. \end{cases} \tag{3.5}$$

Lemma 3.4. ([29, Lemma 4.1]) and ([33, Lemma 9.5]). *Under the assumptions* (1.2)–(1.4)*, with* $2-1/n < p \le n$*, let* $u \in W^{1,p}_0(\Omega)$ *be the unique solution to* (3.1) *and* $v \in u + W^{1,p}_0(B_R)$ *to* (3.4)*. Then, for any* $1 \le q < m$*, with* m *defined in* (1.6)*, the following inequality holds*

$$\begin{aligned} &\int_{B_R} \left(|V(Du)-V(Dv)|^{\frac{2q}{p}} + |Du-Dv|^q\right){\rm d}x \\ &\qquad \le c\,R^{\sigma(q)}\left(\int_{B_R}|f|\,{\rm d}x\right)^{\frac{q}{p-1}} \\ &\qquad\quad + c\,\mathbb{1}_{\{p<2\}}\,R^{\sigma(q)(p-1)}\left(\int_{B_R}|f|\,{\rm d}x\right)^{q}\left(\int_{B_R}(|Du|+s)^q\,{\rm d}x\right)^{2-p}, \end{aligned} \tag{3.6}$$

where $\sigma(q) := n - q(n-1)/(p-1)$*,* $c \equiv c(n,p,\nu,q)$ *and* $\mathbb{1}_{\{p<2\}}$ *denotes the usual characteristic function of the set* $\{p<2\}$*, that is* $\mathbb{1}_{\{p<2\}} = 1$ *if* $p<2$ *and* $\mathbb{1}_{\{p<2\}} = 0$ *if* $p \ge 2$.

Lemma 3.5. *Under the assumptions* (1.2)–(1.5)*, let v and v_0 be the unique weak solutions to* (3.4) *and* (3.5)*, respectively. Then the following inequality holds whenever $q \in (0,p]$*

$$\fint_{B_{\bar R}} |V(Dv_0)-V(Dv)|^{\frac{2q}{p}}\,\mathrm{d}x \,\le\, c\,\bar R^{\frac{2q\alpha}{p}} \fint_{B_{2\bar R}} (|Dv|^q+s^q)\,\mathrm{d}x,$$

where $c \equiv c\big(n,p,L/\nu,\|g\|_{L^r}\big)$.

Proof. The proof relies on that of [33, Lemma 9.6] and extends that of [8, Lemma 4.3], in which the case $g \equiv const.$ has been considered.

It is known (see Theorem 6.1 of [24]) that v_0 is a Q-minimum of the functional $w \mapsto \fint_{B_R} (|Dw|^p+s^p)\,\mathrm{d}x$ with $Q \equiv Q(n,p,L/\nu)$. This fact implies that

$$\fint_{B_{\bar R}} |Dv_0|^p\,\mathrm{d}x \,\le\, c(n,p,L/\nu) \fint_{B_{\bar R}} (|Dv|^p+s^p)\,\mathrm{d}x. \tag{3.7}$$

Using Lemma 3.2 and some standard calculations, we get

$$\left(\fint_{B_{\bar R}} (|Dv|^p+s^p)\,\mathrm{d}x\right)^{\frac{q}{p}} \,\le\, c \fint_{B_{2\bar R}} (|Dv|+s)^q\,\mathrm{d}x \tag{3.8}$$

Now using (2.1), (2.2), the facts that v and v_0 are solutions to (3.4) and (3.5) respectively, the assumption (1.5) and Young's inequality, we get

$$\begin{aligned}
\fint_{B_{\bar R}} (s^2+|Dv_0|^2+|Dv|^2)^{\frac{p-2}{2}}|Dv-Dv_0|^2\,\mathrm{d}x \,&\le\, \fint_{B_{\bar R}} |V(Dv_0)-V(Dv)|^2\,\mathrm{d}x \\
&\le\, \fint_{B_{\bar R}} \langle a(x_0,Dv)-a(x_0,Dv_0), Dv-Dv_0\rangle\,\mathrm{d}x \\
&=\, \fint_{B_{\bar R}} \langle a(x_0,Dv)-a(x,Dv), Dv-Dv_0\rangle\,\mathrm{d}x \\
&\le\, c\,\bar R^{\alpha} \fint_{B_{\bar R}} (g(x_0)+g(x))(s^2+|Dv_0|^2+|Dv|^2)^{\frac{p-1}{2}}|Dv-Dv_0|\,\mathrm{d}x \\
&\le\, \frac{1}{2} \fint_{B_{\bar R}} (s^2+|Dv_0|^2+|Dv|^2)^{\frac{p-2}{2}}|Dv-Dv_0|^2\,\mathrm{d}x \\
&\quad + c\,\bar R^{2\alpha} \fint_{B_{\bar R}} (g(x_0)+g(x))^2(s^2+|Dv_0|^2+|Dv|^2)^{\frac{p}{2}}\,\mathrm{d}x.
\end{aligned} \tag{3.9}$$

Absorbing the first term in the right-hand side of (3.9), using again (2.1) and Hölder's inequality with exponents $r/2>1$ and $r/(r-2)$ to the second term in the right-hand side, we arrive at

$$
\begin{aligned}
&\fint_{B_{\bar R}} |V(Dv)-V(Dv_0)|^2\,\mathrm{d}x \\
&\qquad\qquad \le\; c\,\bar R^{2\alpha}\,\|g\|^2_{L^r(\Omega)}\left(\fint_{B_{\bar R}} (s^2+|Dv_0|^2+|Dv|^2)^{\frac{pr}{2(r-2)}}\,\mathrm{d}x\right)^{\frac{r-2}{r}}.
\end{aligned}
$$

Now, in view on the integrability assumptions on g (recall (1.5)), one can choose r such that $pr/(2(r-2)) = p\chi/2$; and thus, by applying (3.3) to Dv and Dv_0 with $q=p$, it follows

$$
\fint_{B_{\bar R}} |V(Dv)-V(Dv_0)|^2\,\mathrm{d}x \;\le\; c\,\bar R^{2\alpha}\,\|g\|^2_{L^r(\Omega)}\fint_{B_{\bar R}} (s^p+|Dv_0|^p+|Dv|^p)\,\mathrm{d}x.
$$

Finally, using (3.7), (3.8) and Hölder's inequality (with exponents $p/q>1$ and $p/(p-q)$), we arrive at the desired result.

□

Combining Lemma 3.4 with Lemma 3.5, and in particular using (3.6) twice, we obtain the following

Lemma 3.6. *Let u, v and v_0 as in* Lemma 3.4 *and* Lemma 3.5*. Then the following inequality holds*

$$
\begin{aligned}
&\int_{B_R} |V(Du)-V(Dv_0)|^{\frac{2q}{p}}\,\mathrm{d}x \\
&\qquad \le c\,R^{\sigma(q)}\left(\int_{B_{2R}} |f|\,\mathrm{d}x\right)^{\frac{q}{p-1}} \\
&\qquad\quad + c\,\mathbb{1}_{\{p<2\}}\,R^{\sigma(q)(p-1)}\left(\int_{B_{2R}} |f|\,\mathrm{d}x\right)^{q}\left(\int_{B_{2R}} (|Du|+s)^q\,\mathrm{d}x\right)^{2-p} \\
&\qquad\quad + c\,R^{\frac{2q\alpha}{p}}\int_{B_{2R}} (|Du|+s)^q\,\mathrm{d}x.
\end{aligned}
$$

4. Proofs of the main results

In this section we will prove Theorem 1.1 and 1.2. First, we recall the definition of δ in (1.10),

$$
\delta = \frac{p}{2q}\left(n-\frac{q(n-1)}{p-1}\right) \;\le\; 1, \tag{4.1}
$$

and we define

$$
\gamma(t,\tau) := \frac{\tau}{\tau+1-t}, \quad \forall\, t,\tau \ge 0. \tag{4.2}
$$

Lemma 4.1. *Let $u\in W^{1,p}_0(\Omega)$ be the unique solution to* (3.1)*, under the assumptions* (1.2)–(1.5) *with $2-1/n<p\le n$, and let q be such that $1\le q<m$, with m defined in* (1.6)*. Assume that there exists $\bar t\in[0,\min\{\alpha,\delta\})$ such that*

$$
V(Du)\in W^{\bar t,2q/p}_{\mathrm{loc}}(\Omega) \tag{4.3}
$$

and that, for every couple of open subsets $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, there exists $c_1 = c_1(\mathrm{dist}(\Omega', \partial\Omega''))$ such that

$$[V(Du)]^{2q/p}_{W^{\bar{t},2q/p}(\Omega')} \leq c_1 \int_{\Omega''} (|Du|^q + s^q)\,\mathrm{d}x + c_1 \left(\int_{\Omega''} |f|\,\mathrm{d}x\right)^{\frac{q}{p-1}}.$$

Then

$$V(Du) \in W^{t,2q/p}_{\mathrm{loc}}(\Omega), \quad \textit{for every } t \in [0, \bar{\gamma}), \tag{4.4}$$

where

$$\bar{\gamma} := \begin{cases} \gamma(\bar{t}, \delta) & \textit{if } \alpha > \delta \\ \gamma(\bar{t}, \alpha) & \textit{if } \alpha \leq \delta\,, \end{cases}$$

with δ and γ defined by (4.1) and (4.2), respectively. Also,

$$[V(Du)]^{2q/p}_{W^{t,2q/p}(\Omega')} \leq c \int_{\Omega''} (|Du|^q + s^q)\,\mathrm{d}x + c \left(\int_{\Omega''} |f|\,\mathrm{d}x\right)^{\frac{q}{p-1}}$$

for every open subsets $\Omega' \subset\subset \Omega'' \subset\subset \Omega$; c being a constant depending only on $n, p, L/\nu$, q, $\|g\|_{L^r}$, $\mathrm{dist}(\Omega', \partial\Omega''), t$. Moreover, the following estimate holds, for any $1 \leq i \leq n$,

$$\sup_{0<|h|<\mathrm{dist}(\Omega',\partial\Omega'')} \int_{\Omega'} \frac{|\tau_{i,h} V(Du(x))|^{2q/p}}{|h|^{\bar{\gamma}2q/p}}\,\mathrm{d}x \leq c \int_{\Omega''} (|Du|^q + s^q)\,\mathrm{d}x + c \left(\int_{\Omega''} |f|\,\mathrm{d}x\right)^{\frac{q}{p-1}}.$$

Proof. The proof extends that of Lemma 6.2 in [29] and Lemma 11.1 in [33]. Nevertheless, we have to make some modifications due to the different regularity assumptions on the maps $x \mapsto a(x, z)$ that we are handling. For this, we will focus on the steps where the Hölder regularity does arise. Also, we would note that here we deal with the super and the sub-quadratic case in a unified way.

Fix a couple of arbitrary open subsets Ω' and Ω'' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, and consider a real number h such that

$$0 < |h| << \min\left\{1, \mathrm{dist}(\Omega', \Omega'')\right\}.$$

Take $x_0 \in \Omega'$ and denote by

$$B := B(x_0, |h|^\beta)$$

the ball centered in x_0 with radius $|h|^\beta$. We need to make use of the auxiliary homogeneous problems in the enlarged balls $8B$ and $4B$. Thus, we define $v \in u + W^{1,p}_0(8B)$ as the unique solutions to problem (3.4), with B_R replaced by $8B$ and $v_0 \in v + W^{1,p}_0(4B)$ the one to problem (3.5) with $B_{\bar{R}}$ replaced by $4B$. Now, for any fixed $1 \leq i \leq n$, we have the following estimate

$$\begin{aligned}
\int_B |\tau_{i,h} V(Du)|^{\frac{2q}{p}}\,\mathrm{d}x \quad &\leq \quad c \int_B |\tau_{i,h} V(Dv_0)|^{\frac{2q}{p}}\,\mathrm{d}x + c \int_B |V(Du) - V(Dv_0)|^{\frac{2q}{p}}\,\mathrm{d}x \\
&\qquad + c \int_B |V(Du(x + he_i)) - V(Dv_0(x + he_i))|^{\frac{2q}{p}}\,\mathrm{d}x \\
&\leq \quad c \int_{4B} |V(Du) - V(Dv_0)|^{\frac{2q}{p}}\,\mathrm{d}x + c \int_B |\tau_{i,h} V(Dv_0)|^{\frac{2q}{p}}\,\mathrm{d}x \\
&:= \quad I_1 + I_2,
\end{aligned}$$

where we have also used that, since $|h|<1$, $B(x_0,|h|^\beta)+B(0,|h|)\subset B(x_0,4|h|^\beta)=4B$. In order to estimate I_1 we start using Lemma 3.6, which gives

$$\begin{aligned}
&\int_{4B}|V(Du)-V(Dv_0)|^{\frac{2q}{p}}\,\mathrm{d}x\\
&\qquad\le c\,|h|^{\beta\left(n-\frac{q(n-1)}{p-1}\right)}\left(\int_{8B}|f|\,\mathrm{d}x\right)^{\frac{q}{p-1}}\\
&\qquad\quad+c\,\mathbb{1}_{\{p<2\}}\,|h|^{\beta[n-q(n-1)]}\left(\int_{8B}|f|\,\mathrm{d}x\right)^{q}\left(\fint_{8B}(|Du|+s)^q\,\mathrm{d}x\right)^{2-p}\\
&\qquad\quad+c\,|h|^{\frac{2\beta\alpha q}{p}}\int_{8B}(|Du|+s)^q\,\mathrm{d}x.
\end{aligned}$$

When $2-1/n<p<2$ we can estimate the second term in the right-hand side of the previous inequality using Proposition 3.3. We have

$$\begin{aligned}
&|h|^{\beta[n-q(n-1)]}\left(\int_{8B}|f|\,\mathrm{d}x\right)^{q}\left(\fint_{8B}(|Du|+s)^q\,\mathrm{d}x\right)^{2-p}\\
&\qquad\le c\,|h|^{\beta[n-q(n-1)]-\beta\frac{q(n-1)(2-p)}{p-1}}\left(\int_{8B}|f|\,\mathrm{d}x\right)^{q}\left(\int_{8B}|f|\,\mathrm{d}x\right)^{\frac{q(2-p)}{p-1}}\\
&\qquad= c\,|h|^{\beta\left(n-\frac{q(n-1)}{p-1}\right)}\left(\int_{8B}|f|\,\mathrm{d}x\right)^{\frac{q}{p-1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_1 &\le c\,|h|^{\beta\left(n-\frac{q(n-1)}{p-1}\right)}\left(\int_{8B}|f|\,\mathrm{d}x\right)^{\frac{q}{p-1}}+c\,|h|^{\frac{2\beta\alpha q}{p}}\int_{8B}(|Du|+s)^q\,\mathrm{d}x\\
&\le c\left(|h|^{\frac{2\beta\delta q}{p}}+|h|^{\frac{2\beta\alpha q}{p}}\right)\Lambda(8B)\\
&\le c\,|h|^{\frac{2\beta q}{p}\min\{\alpha,\delta\}}\Lambda(8B)\,, \qquad (4.5)
\end{aligned}$$

where we have set

$$\Lambda(A):=\int_A(|Du|+s)^q\,\mathrm{d}x+\left(\int_A|f|\,\mathrm{d}x\right)^{\frac{q}{p-1}}\quad\text{for every measurable set } A\subset\mathbb{R}^n. \qquad (4.6)$$

In order to estimate I_2, firstly we apply (3.2) in Lemma 3.1, with $a_0(\cdot)\equiv a(x_0,\cdot)$ and q replaced by $2q/p\le 2$, there. We get

$$\int_B|D(V(Dv_0))|^{\frac{2q}{p}}\,\mathrm{d}x\le c\,|h|^{-\frac{2\beta q}{p}}\int_{2B}|V(Dv_0)-V(z)|^{\frac{2q}{p}}\,\mathrm{d}x,\quad\forall\, z\in\mathbb{R}^n.$$

Therefore, using the last estimate with the definition of the operator $\tau_{i,h}$ and the fact that $|h|<1$ and $0<\beta<1$, it follows

$$\begin{aligned} I_2 &\le c\,|h|^{\frac{2q}{p}}\int_{2B}|D(V(Dv_0))|^{\frac{2q}{p}}\,\mathrm{d}x \\ &\le c\,|h|^{(1-\beta)\frac{2q}{p}}\int_{4B}|V(Dv_0)-V(z)|^{\frac{2q}{p}}\,\mathrm{d}x \\ &\le c\,|h|^{(1-\beta)\frac{2q}{p}}\int_{4B}|V(Du)-V(z)|^{\frac{2q}{p}}\,\mathrm{d}x + c\int_{4B}|V(Du)-V(Dv_0)|^{\frac{2q}{p}}\,\mathrm{d}x, \quad \forall\, z\in\mathbb{R}^n. \end{aligned}$$

Combining the last inequality with (4.5),

$$\begin{aligned} &\int_B|\tau_{i,h}V(Du)|^{\frac{2q}{p}}\,\mathrm{d}x \\ &\qquad \le c\,|h|^{\frac{2\beta q}{p}\min\{\alpha,\delta\}}\,\Lambda(8B)+c\,|h|^{(1-\beta)\frac{2q}{p}}\int_{4B}|V(Du)-V(z)|^{\frac{2q}{p}}\,\mathrm{d}x. \end{aligned} \tag{4.7}$$

Now we have to choose $z\in\mathbb{R}^n$ in the latter inequality. First, suppose $\bar t=0$. In this case, we take $z=0$ and thus, (2.1) gives

$$\int_B|\tau_{i,h}V(Du)|^{\frac{2q}{p}}\,\mathrm{d}x \le c\left(|h|^{\frac{2\beta q}{p}\min\{\alpha,\delta\}}+|h|^{(1-\beta)\frac{2q}{p}}\right)\Lambda(8B).$$

In the case $\bar t>0$, we take $z:=V^{-1}((V(Du))_{8B})$. Keeping in mind this choice, in view of the assumption (4.3) we can use fractional Poincaré inequality given in Proposition 2.2 to get

$$\int_{8B}|V(Du)-V((Du)_{8B})|^{\frac{2q}{p}}\,\mathrm{d}x \le c\,|h|^{\frac{2\beta q\bar t}{p}}[V(Du)]^{\frac{2q}{p}}_{W^{\bar t,2q/p}(8B)}. \tag{4.8}$$

Summing up (4.7) with (4.8), we obtain

$$\int_B|\tau_{i,h}V(Du)|^{\frac{2q}{p}}\,\mathrm{d}x \le c\,|h|^{\frac{2\beta q}{p}\min\{\alpha,\delta\}}\,\Lambda(8B)+c\,|h|^{\frac{2\beta q\bar t}{p}+(1-\beta)\frac{2q}{p}}[V(Du)]^{\frac{2q}{p}}_{W^{\bar t,2q/p}(8B)}.$$

Now, consider the characteristic function $\mathbb{1}_{\{\bar t>0\}}$ such that $\mathbb{1}_{\{\bar t>0\}}=0$ if $\bar t=0$ and $\mathbb{1}_{\{\bar t>0\}}=1$ if $\bar t>0$, and the set function $\bar\Lambda$ defined by

$$\bar\Lambda(A):=\Lambda(A)+\mathbb{1}_{\{\bar t>0\}}[V(Du)]^{\frac{2q}{p}}_{W^{\bar t,2q/p}(A)}, \quad \forall\, A\subset\mathbb{R}^n,$$

where Λ is defined by (4.6). Thus, for any $0\le\bar t<\min\{\alpha,\delta\}$, we get

$$\int_{\Omega'}|\tau_{i,h}V(Du)|^{\frac{2q}{p}}\,\mathrm{d}x \le c\left(|h|^{\frac{2\beta q}{p}\min\{\alpha,\delta\}}+|h|^{\frac{2\beta q\bar t}{p}+(1-\beta)\frac{2q}{p}}\right)\bar\Lambda(\Omega''),$$

where a covering argument has been also used.

Finally, we take $\beta=1/(\min\{\alpha,\delta\}+1-\bar t)$, and this is admissible, since $\bar t<\min\{\alpha,\delta\}$ yields $\beta<1$. We obtain

$$\int_{\Omega'}|\tau_{i,h}V(Du)|^{\frac{2q}{p}}\,\mathrm{d}x \le c\,\bar\Lambda(\Omega'')\begin{cases}|h|^{\frac{2\gamma(\bar t,\delta)q}{p}} & \text{if } \alpha>\delta\\ |h|^{\frac{2\gamma(\bar t,\alpha)q}{p}} & \text{if } \alpha\le\delta.\end{cases}$$

From now on, the proof can be completed arguing exactly as in the proof of [29, Lemma 6.2], by taking into account our different function γ. □

Lemma 4.2. *Let $u \in W_0^{1,p}(\Omega)$ be the unique solution to (3.1), under the assumptions (1.2)–(1.5), with $2 - 1/n < p \leq n$, and let q be such that $1 \leq q < m$, with m defined in (1.6). Then*

$$V(Du) \in W^{t,2q/p}_{\rm loc}(\Omega), \quad Du \in W^{2t/p,q}_{\rm loc}(\Omega), \quad \text{for every } 0 \leq t < \min\{\alpha, \delta\}$$

and

$$[V(Du)]^{2q/p}_{W^{t,2q/p}(\Omega')} + [Du]^q_{W^{2t/p,q}(\Omega')} \leq c \int_{\Omega''} (|Du|^q + s^q)\,{\rm d}x + c\left(\int_{\Omega''} |f|\,{\rm d}x\right)^{\frac{q}{p-1}}$$

for every open subsets $\Omega' \subset\subset \Omega'' \subset\subset \Omega$; c being a constant depending only on $n, p, L/\nu$, q, t, ${\rm dist}(\Omega', \partial\Omega'')$, $\|g\|_{L^r}$. Moreover, the following estimate holds, for any $1 \leq i \leq n$,

$$\sup_{0<|h|<{\rm dist}(\Omega',\partial\Omega'')} \int_{\Omega'} \frac{|\tau_{i,h}Du(x)|^q}{|h|^{t2q/p}}\,{\rm d}x \;\leq\; c\int_{\Omega''} (|Du|^q + s^q)\,{\rm d}x + c\left(\int_{\Omega''} |f|\,{\rm d}x\right)^{\frac{q}{p-1}}.$$

Proof. The proof will follow by using the iterative argument in the proof of Lemma 6.3 in [29]. We just have to take into account the different range of validity of (4.4) depending of α. The key-point is in the structure of the function $t \mapsto \gamma(t,\tau)$ defined by (4.2). For any fixed $0 < \tau \leq 1$, we have

$$\gamma(\cdot,\tau) \text{ is increasing}, \quad \gamma(\tau,\tau) = \tau \quad \text{and} \quad t \in (0,\tau) \longrightarrow \gamma(t,\tau) \in (t,\tau).$$

For this, if we consider the sequence $\{s_k\}_{k\geq1}$ and $\{t_k\}_{k\geq1}$ defined inductively by

$$s_1 := \frac{\tau}{4(1+\tau)}, \quad s_{k+1} := \gamma(s_k,\tau), \quad \text{and} \quad t_1 = 2s_1, \quad t_{k+1} := \frac{\gamma(s_k,\tau)+\gamma(t_k,\tau)}{2},$$

it follows that

$$s_k \nearrow \tau, \quad s_k < t_k < \tau \text{ for any } k \geq 1 \quad \text{and} \quad t_k \nearrow \tau.$$

At this time, we have to distinguish the case $\alpha > \delta$ versus that $\alpha \leq \delta$. In the first case, we fix $\tau = \delta$ and start the iteration with $\bar{t} = 0$ by proceeding as in the proof of [29, Lemma 6.3]. In the second case, we can argue in the same way, but we can simply remember to fix $\tau = \alpha$. □

Proof of Theorem 1.1 *and* 1.2.

Once we established the previous lemmata in which the regularity exponents derived by the weak Hölder assumption in (1.5) have arisen, the proofs of Theorem 1.1 and 1.2 closely follow those of Theorem 1.2 and Theorem 1.3 in [29] (and in turn that of [8, Theorem 2.8]). For the reader's convenience, we sketch the main steps.

Firstly, we recall that all the regularity results and the estimates established in the previous section are valid for the weak solutions $u_k \in W_0^{1,p}(\Omega)$ to problem (1.1) with μ replaced by a regular function f_k (see the beginning of Section 3). As stated in the Introduction, it is known that, up to subsequences,

$$\begin{aligned} &u_k \to u \;\text{ weakly in } W_0^{1,q}(\Omega), \\ &u_k \to u \;\text{ strong in } L^q(\Omega) \text{ and a.e. in } \Omega, \end{aligned} \tag{4.9}$$

for all $1 \leq q < n(p-1)/(n-1)$; with u being a solution to problem (1.1). Now, applying Lemma 4.2 to each u_k, we deduce, for any $k \in \mathbb{N}$,

$$Du_k \in W^{\frac{\sigma}{q},q}(\Omega), \quad \forall\ 0 \leq \sigma < \min\left\{\frac{2q\alpha}{p}, \frac{2q\delta}{p}\right\}, \quad \forall\, 1 \leq q < m;$$

and, for every $\Omega' \subset\subset \Omega$,

$$\|Du_k\|^q_{W^{\frac{\sigma}{q},q}(\Omega')} \leq c\int_\Omega (s^q + |Du_k|^q)\,\mathrm{d}x + \left(\int_\Omega |f_k|\,\mathrm{d}x\right)^{\frac{q}{p-1}} \leq c\,[|\mu|(\Omega)]^{\frac{q}{p-1}} + c\,s^q|\Omega|, \tag{4.10}$$

where we used some uniform estimates established in [29] (see, in particular, (5.3)–(5.5) there). Therefore (1.14) plainly follows from (4.9), (4.10) and a standard lower semicontinuity argument. Finally, in order to obtain estimate (1.13), it suffices to use standard scaling arguments. The proof is complete. □

4.1. **Further extensions.** We conclude this section by stating the natural extensions of [29, Theorem 1.4] and [29, Corollary 1.5], in which a gain in differentiability is shown passing to $V(Du)$ and considering non-degerate problems, respectively. The proofs in the weak Hölder case do not present any relevant variations with respect to those in the Lipschitz case.

Theorem 4.3. *Let the assumptions in* Theorem 1.1 *hold, and let $u \in W^{1,1}_0(\Omega)$ be a solution to problem* (1.1). *Then*

- *If $p \geq 2$, then*

$$V(Du) \in W^{\min\left\{\alpha, \frac{p}{2(p-1)}\right\}-\varepsilon,\, \frac{2(p-1)}{p}}(\Omega), \quad \forall\,\varepsilon \in \left(0, \min\left\{\alpha, \frac{p}{2(p-1)}\right\}\right).$$

  Morevorer, for any open subset $\Omega' \subset\subset \Omega$,

$$[V(Du)]^{2(p-1)/p}_{W^{\min\left\{\alpha, \frac{p}{2(p-1)}\right\},\, \frac{2(p-1)}{p}}(\Omega')} \leq \bar{c}\,|\mu|(\Omega) + \bar{c}\,s^{p-1}|\Omega|, \quad \forall\,\Omega' \subset\subset \Omega.$$

- *If $2 - 1/n < p < 2$, then*

$$V(Du) \in W^{\min\left\{\alpha, \frac{(np-2n+1)p}{2(p-1)}\right\}-\varepsilon,\, \frac{2}{p}}(\Omega), \quad \forall\,\varepsilon \in \left(0, \min\left\{\alpha, \frac{(np-2n+1)p}{2(p-1)}\right\}\right).$$

  Morevorer, for any open subset $\Omega' \subset\subset \Omega$,

$$[V(Du)]^{2/p}_{W^{\min\left\{\alpha, \frac{(np-2n+1)p}{2(p-1)}\right\},\, \frac{2}{p}}(\Omega')} \leq \bar{c}\,|\mu|(\Omega)^{\frac{1}{p-1}} + \bar{c}\,s\,|\Omega|, \tag{4.11}$$

where $\bar{c} \equiv \bar{c}(n, p, L/\nu, \varepsilon, \mathrm{dist}(\Omega', \partial\Omega), \Omega, \|g\|_{L^r})$.

Corollary 4.4. *Let the assumptions in* Theorem 1.1 *hold. Let $u \in W^{1,1}_0(\Omega)$ be a solution to problem* (1.1). *In addition, assume that $s > 0$. Then*

$$Du \in W^{\min\left\{\alpha, \frac{p}{2(p-1)}\right\}-\varepsilon,\, \frac{2(p-1)}{p}}(\Omega), \quad \forall\,\varepsilon \in \left(0, \min\left\{\alpha, \frac{p}{2(p-1)}\right\}\right).$$

Also, estimate (4.11) *holds with $V(Du)$ replaced by Du, provided that the constant $\bar{c}$ is substituted with $s^{(2-p)(p-1)/p}c(n,p)\bar{c}$.*

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