The monopolist's problem: existence, relaxation, and approximation

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Abstract

We study a variational problem arising from a generalization of an economic model introduced by Rochet and Choné in [5]. In this model a monopolist proposes a set Y of products with price list $p : Y \to \mathbb{R}$. Each rational consumer chooses which product to buy by solving a personal minimum problem, taking into account his/her tastes and economic possibilities. The monopolist looks for the optimal price list which minimizes costs, hence maximizes the profit. This leads to a minimum problem for functionals $\mathcal{F}(p)$ (the "pessimistic cost expectation") and $\mathcal{G}(p)$ (the "optimistic cost expectation"), which are in turn defined through two nested variational problems.

We prove that the minimum of $\mathcal{G}$ exists and coincides with the infimum of $\mathcal{F}$.

We also provide a variational approximation of $\mathcal{G}$ by smooth functionals defined in finite dimensional Euclidean spaces.

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1 Introduction

In this paper we consider variational problems coming from a generalization of an economic model proposed by Rochet and Choné in [5].

Before we enter in the details, just a few words about notations. Looking at the literature on these subjects, we realized that unfortunately conventions in the calculus of variations and in economic theories are opposite. In this paper we use standard conventions of the calculus of variations: this means that our consumers aim to minimize their stress (instead of maximizing their utility) and the goal of our monopolist is to minimize costs (instead of maximizing the profit).

It is obvious that the two approaches are completely equivalent, up to a sign change. The reason for our choice (beyond our mathematical background) is that the main tool in this analysis is Γ-convergence, which has been developed and presented in mathematical literature especially in the context of minimum problems.

This paper is organized as follows. In this section we first describe the model and the previous work, and then we introduce our assumptions and sketch the main results. In section 2 we recall for the convenience of the reader all what is needed about Γ-convergence in this paper. In section 3 we formally state our results, which we prove in section 4. Finally, in section 5 we present some counterexamples to show that some results are sharp.

1.1 The model

In this subsection we first introduce the mathematical setting of the problem, and then we describe the economic model leading to this problem.

From the mathematical point of view, the main ingredients are:

- a set X and a finite measure μ on X;
- a set Y;
- a subset $\mathcal{P} \subseteq \{\text{functions } p : Y \to \mathbb{R}\}$;
- a function $s : X \times Y \times \mathbb{R} \to \mathbb{R}$;
- a function $c : Y \times \mathbb{R} \to \mathbb{R}$.

For every $(x, p) \in X \times \mathcal{P}$, one defines

$$\begin{aligned} A_p(x) &:= \arg\min\{s(x, y, p(y)) : \ y \in Y\} \\ &= \{y \in Y : \ s(x, y, p(y)) \leq s(x, z, p(z)) \ \forall z \in Y\}, \end{aligned}$$

and then

$$M_p(x) := \max\{c(y, p(y)) : \ y \in A_p(x)\}, \qquad m_p(x) := \min\{c(y, p(y)) : \ y \in A_p(x)\}.$$

Finally, for every $p \in \mathcal{P}$ one defines

$$\mathcal{F}(p) := \int_X M_p(x)\, \mathrm{d}\mu, \qquad \mathcal{G}(p) := \int_X m_p(x)\, \mathrm{d}\mu.$$

Of course, at this level of generality, there is no reason for $A_p(x)$, $M_p(x)$, $m_p(x)$, $\mathcal{F}(p)$, $\mathcal{G}(p)$ to be well defined.

Now we describe what all these quantities represent in the model.

The set X and the measure μ represent consumers: for example X can be a finite set with the measure equally distributed, but X can also be a subset of $\mathbb{R}^n$ describing "consumer types", with μ representing the density of consumers with given characteristics. The set Y represents all the products the monopolist can produce: as in the previous case, it can be a finite set, but also an infinite set corresponding to some "qualities" of the possible products.

The subset $\mathcal{P}$ represents the admissible "price functions".

The real number $s(x, y, \pi)$ is the stress of consumer x when he buies product y at price π.

The real number $c(y, \pi)$ is the cost for the monopolist when he produces product y and sells it at price π.

Given an admissible price list $p \in \mathcal{P}$, each consumer x buies one of the products which minimize his stress, hence an element of $A_p(x)$. If $A_p(x)$ has more than one element, this model cannot predict which product the consumer will buy, therefore the exact cost given by x to the monopolist is unknown. However, this cost lies in any case between $m_p(x)$ and $M_p(x)$. Integrating over all consumers, it turns out that $\mathcal{G}(p)$ and $\mathcal{F}(p)$ represent the "cost forecast" for the monopolist in the optimistic and pessimistic situation, respectively. We remark that by definition

$$\mathcal{G}(p) \leq \mathcal{F}(p), \qquad \forall p \in \mathcal{P}. \tag{1.1}$$

We point out that in this model X, μ, Y, $\mathcal{P}$, s and c are given, and the monopolist looks for the element(s) in $\mathcal{P}$ which minimizes his expected costs. This leads to the minimum problems

$$\min\{\mathcal{G}(p) : \ p \in \mathcal{P}\}, \qquad \min\{\mathcal{F}(p) : \ p \in \mathcal{P}\}.$$

From the monopolist's point of view, it is also interesting to consider the subset of Y defined by

$$\mathcal{A}_p := \bigcup_{x \in X} A_p(x), \tag{1.2}$$

where p is an optimal price list for $\mathcal{F}$ or $\mathcal{G}$. This set contains all the products that can be bought by someone, so that the monopolist has only to produce the elements of $\mathcal{A}_p$. In some sense, this set could be an unknown of the problem, but as we have seen, it is completely determined by the optimal price list on Y.

1.2 Previous work

This model has been studied by Rochet and Choné [5], Monteiro and Page [7], Carlier [1], to which we refer for a brief survey of the theoretical literature on similar economic problems.

Notations and assumptions are very different in all these papers. For the convenience of the reader, we outline their results using our conventions.

Roughly speaking, the problem has been approached in at least two different ways.

The first one has been introduced in [5]. In this paper it is assumed that X and Y are convex subsets of $\mathbb{R}^n$, μ has a strictly positive density with respect to the Lebesgue measure on X, prices are strictly convex, $c(y,\pi) = C(y) - \pi$, where C is a twice differentiable convex function such that D^2C has strictly positive and bounded eigenvalues. The main assumption is however that $s(x,y,\pi) = \pi - x \cdot y$, where $x \cdot y$ denotes the scalar product. In this case $A_p(x)$ is always a singleton, and if consumer x buies product $y(x)$ at price $\pi(y(x))$, then the function $U(x) = s(x, y(x), \pi(y(x)))$ is concave and nonpositive. Moreover, the minimum problem for $\mathcal{G}(p)$ (or for $\mathcal{F}(p)$ which is the same in this case) becomes a minimum problem for an integral functional $\mathcal{I}(U)$, subject to a concavity and nonpositivity constraint on the dual variable U. Under suitable assumptions, this problem has a unique minimizer U, whose regularity has been studied by Carlier and Lachand-Robert [2]. The Euler-Lagrange equation satisfied by U was derived by Lions [6], and a numerical algorithm to approximate U has been proposed by Carlier and Lachand-Robert [3] in the case $n = 2$. From U, one can finally determine p and Y (in this approach Y is itself an unknown).

This dual approach has been generalized in [1] to stress functions such as $s(x,y,\pi) = \pi - h(x,y)$, where h satisfies suitable monotonicity, regularity and growth assumptions. The main idea is to replace concavity with a weaker notion, called h-concavity, in such a way that the minimum problem for $\mathcal{G}(p)$ (but not for $\mathcal{F}(p)$, which now can be different) can be rephrased in terms of a h-concave dual variable.

A completely different path was pursued in [7]. Their model in slightly different, because the equivalent of $s(x,y,\pi)$ is defined keeping into account external goods and budget constraints which force prices to be bounded; however the functional they consider is similar to our $\mathcal{G}(p)$. Under very general assumptions, they prove existence of minimizers using the direct method of the calculus of variations in the original variables. The existence result in [7] is therefore in the same spirit of the first part of our Theorem 1.1 (cf. Theorem 3.2).

1.3 Assumptions

In the following we list all the assumptions we make on X, μ, Y, $\mathcal{P}$, s and c.

(Hp 1) X is a compact metric space with distance d_X;

(Hp 2) μ is a Radon measure on X, normalized so that $\mu(X) = 1$;

(Hp 3) Y is a compact metric space with distance d_Y;

(Hp 4) s is continuous on $X \times Y \times \mathbb{R}$;

(Hp 5) c is continuous on $Y \times \mathbb{R}$;

(Hp 6) there exists $y_\star \in Y$ such that $c(y_\star, 0) = 0$, and $s(x, y_\star, 0) = 0$ for every $x \in X$;

(Hp 7) $\mathcal{P}$ is the set of all functions $p : Y \to \mathbb{R}$ which are lower semicontinuous and satisfy $p(y_\star) \leq 0$;

(Hp 8) $c(y, \pi) \geq 0$ for every $(y, \pi) \in Y \times (-\infty, 0]$;

(Hp 9) there exists two constants $c_2 \geq c_1 > 0$ such that

$$c_1(\pi_2 - \pi_1) \leq s(x, y, \pi_2) - s(x, y, \pi_1) \leq c_2(\pi_2 - \pi_1)$$

for every $(x, y, \pi_1, \pi_2) \in X \times Y \times \mathbb{R}^2$ such that $\pi_1 \leq \pi_2$.

Let us comment these assumptions.

Hypotheses (Hp 1) through (Hp 5) are technical, and are trivially satisfied *e.g.* if X and Y are finite sets.

With (Hp 6) we assume the existence of a special product $y_\star$, which can be thought as the "nothing". In this way $c(y_\star, 0) = 0$ means that the monopolist has cost zero when producing "nothing" and selling it at price zero; similarly, $s(x, y_\star, 0) = 0$ means that any consumer has stress zero when he buies "nothing" paying zero. In a certain sense, $y_\star$ is the only weapon a consumer has against a monopolist, *i.e.* the possibility of buying nothing. As already pointed out in [5], from the mathematical point of view, it is always possible to fulfill (Hp 6) by adding $y_\star$ as an isolated point to any set Y of products.

With (Hp 7) we restrict to lower semicontinuous price functions for technical reasons, and we force the monopolist to sell $y_\star$ (*i.e.* nothing) at price less than or equal to zero. Note that we do admit negative prices: this is not only for the sake of generality, or because negative prices are welcome by consumers, but mainly because this simplifies the proof of our relaxation result.

Assumption (Hp 8) says that selling a product y at a price ≤ 0 has a cost ≥ 0 (as every monopolist knows!).

Finally, assumption (Hp 9) says that the stress of a consumer is an increasing function of price (as every consumer knows!), and gives a lower and an upper bound on the growth.

1.4 Results

In this subsection we sketch our results, which will be restated in section 3 with more mathematical details.

The first result concerns well posedness and existence of minimizers (cf. Theorems 3.1, 3.2, and 3.3).

Theorem 1.1 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*

Then $A_p(x)$, $M_p(x)$, $m_p(x)$, $\mathcal{F}(p)$, $\mathcal{G}(p)$ are well defined.

Moreover, $\mathcal{G}$ has a minimum on $\mathcal{P}$ and

$$\inf\{\mathcal{F}(p):\ p\in\mathcal{P}\}=\min\{\mathcal{G}(p):\ p\in\mathcal{P}\}. \tag{1.3}$$

There are very simple examples where $\mathcal{F}(p)$ doesn't have minimum in $\mathcal{P}$ (cf. Example 5.1). Therefore in general the monopolist cannot reach the best in the pessimistic situation described by $\mathcal{F}$. However, by (1.3) he can arrange prices so that also in the pessimistic situation his costs are as close as he wants to the lowest costs in the optimistic situation! This shows also that allowing nonuniqueness in the choices of consumers has a negligible effect on the monopolist's profit. On the other hand, the only way to be sure that $A_p(x)$ is always a singleton is to impose severe restrictions on s and $\mathcal{P}$. We also remark that (Hp 9) is crucial in the proof of (1.3), as shown by Example 5.2.

The next problem we consider is how to compute the minimum of $\mathcal{G}$, since the expression of $\mathcal{G}(p)$ involves two nested minimum problems.

In order to overcome this difficulty, we approximate $\mathcal{G}$ by a sequence $\{\mathcal{G}_n\}$ of *smooth* functionals defined in *finite dimensional* Euclidean spaces. Roughly speaking, the domain of $\mathcal{G}_n$, denoted by $\mathcal{P}_n$, is the set of price functions defined only on a finite subset $Y_n \subseteq Y$. Moreover, the explicit expression of $\mathcal{G}_n$ involves only a finite number of algebraic operations (for the details see section 3.2).

The minimum problem for $\mathcal{G}$ in $\mathcal{P}$ is thus approximated as follows (cf. Theorem 3.4).

Theorem 1.2 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*

Then there exists a sequence of functionals $\mathcal{G}_n : \mathcal{P}_n \to \mathbb{R}$, where $\mathcal{P}_n$ is a subset of a finite dimensional vector space, such that

(1) for every $n \in \mathbb{N}$ we have that $\mathcal{G}_n \in C^\infty(\mathcal{P}_n)$;

(2) for every $n \in \mathbb{N}$ there exists

$$g_n := \min\{\mathcal{G}_n(p) : \ p \in \mathcal{P}_n\};$$

(3) as $n \to \infty$ we have that

$$g_n \to \min\{\mathcal{G}(p) : \ p \in \mathcal{P}\};$$

(4) the minimizers of $\mathcal{G}_n$ converge (up to subsequences) to minimizers of $\mathcal{G}$, in a suitable sense.

The main tool in the proof of Theorem 1.1 and 1.2 is De Giorgi's Γ-convergence. This notion allows to define a metric, hence a topology, on $\mathcal{P}$, with respect to which we apply to $\mathcal{G}$ the direct method of the calculus of variations, and we show that $\mathcal{G}$ is the relaxation of $\mathcal{F}$.

Finally, the convergence of minima and minimizers claimed in Theorem 1.2 follows from the Γ-convergence of the sequence $\{\mathcal{G}_n\}$ to $\mathcal{G}$ in $\mathcal{P}$. Due to the topology we consider in $\mathcal{P}$, this last one is a result of Γ-convergence with respect to Γ-convergence!

2 Gamma convergence: an overview

In this section we recall the main definitions and results from the theory of Γ-convergence. For a comprehensive introduction we refer to [4], where this subject is developed in full generality. For the sake of simplicity, here we outline the theory in a simpler setting, which is however enough for our purposes. This introduction is self contained, so the interested reader can prove by himself the results stated below, or simply deduce them from more general statements proved in [4].

Throughout this section, we assume that Z is a *compact metric* space, with distance d_Z.

Definition 2.1 A sequence of functions $F_n : Z \to \mathbb{R}$ is said to Γ-converge to $F_\infty : Z \to \mathbb{R}$ as $n \to +\infty$ if the following two properties are satisfied.

- *Liminf inequality*: for every sequence $\{z_n\}$ converging to $z_\infty \in Z$ we have that

$$\liminf_{n\to+\infty} F_n(z_n) \geq F_\infty(z_\infty).$$

- *Existence of a recovery sequence*: for every $z \in Z$ there exists a sequence $\{z_n\}$ converging to z in Z such that

$$\lim_{n \to +\infty} F_n(z_n) = F_\infty(z),$$

or equivalently by the first point

$$\limsup_{n \to +\infty} F_n(z_n) \leq F_\infty(z).$$

The Γ-limit, if it exists, is unique, and lower semicontinuous. The following result clarifies the variational character of Γ-convergence.

Theorem 2.2 *Let $\{F_n\}$ be a sequence of lower semicontinuous functions in Z which Γ-converges to F_∞ as $n \to +\infty$.*

Then

$$\lim_{n \to +\infty} \min\{F_n(z) : \ z \in Z\} = \min\{F_\infty(z) : \ z \in Z\}.$$

Moreover, if z_n is a minimizer of F_n for every $n \in \mathbb{N}$, then the sequence $\{z_n\}$ converges, up to subsequences, to a minimizer of F_∞.

A particular case of Γ-convergence is when F_n is equal to a given F for every $n \in \mathbb{N}$. In this case the Γ-limit of $\{F_n\}$ is called the *relaxation* of F, and can be characterized as the supremum of all lower semicontinuous functions which are less or equal than F.

The proof of a Γ-convergence (or a relaxation) result consists in two parts: the "liminf inequality", and the existence of a recovery sequence ("limsup inequality"). This second part of the proof can sometimes be simplified in different ways. A first possibility is described with the following result.

Proposition 2.3 *Let $F_n : Z \to \mathbb{R}$ ($n \in \mathbb{N}$), and let $H : Z \to \mathbb{R}$.*

Let us assume that for every $z \in Z$ there exists a sequence $\{z_n\} \subseteq Z$ such that

$$z_n \to z, \qquad \limsup_{n \to +\infty} F_n(z_n) \leq H(z).$$

Then for every $z \in Z$ there exists a sequence $\{z_n\} \subseteq Z$ such that

$$z_n \to z, \qquad \limsup_{n \to +\infty} F_n(z_n) \leq \overline{H}(z),$$

where $\overline{H}$ denotes the relaxation of H.

This result can be used as follows: "assume one needs to prove the limsup inequality with a candidate Γ-limit $F_\infty(z)$; then one can limit himself to prove the same inequality with some $H(z) \geq F_\infty(z)$ such that $\overline{H}(z) = F_\infty(z)$".

Another possible approach to the "limsup inequality" is based on the following notion.

Definition 2.4 A subset $D \subseteq Z$ is said to be *dense for the energy* $F : Z \to \mathbb{R}$ if, for every $z \in Z$, there exists a sequence $\{z_n\} \subseteq D$ such that

$$z_n \to z, \qquad F(z_n) \to F(z).$$

Proposition 2.5 *Let $F_n : Z \to \mathbb{R}$ ($n \in \mathbb{N}$), $F_\infty : Z \to \mathbb{R}$, and let $D \subseteq Z$ be a subset which is dense with respect to the energy F_∞.*

Assume that for every $z \in D$, and every $\eta > 0$ there exists $\{z_n\} \subseteq Z$ such that

$$z_n \to z, \qquad \limsup_{n \to +\infty} F_n(z_n) \leq F_\infty(z) + \eta.$$

Then for every $z \in Z$ there exists $\{z_n\} \subseteq Z$ such that

$$z_n \to z, \qquad \limsup_{n \to +\infty} F_n(z_n) \leq F_\infty(z).$$

Thanks to this result one can limit himself to prove the existence of a "recovery sequence up to η", and only for $z \in D$.

In this paper we use the topology induced by Γ-convergence on spaces of lower semicontinuous functions. In order to describe the general setting, let $\{U_i\}$ be a countable basis for the topology of Z. Let us denote by $\mathcal{LSC}(Z)$ the set of all lower semicontinuous real valued functions on Z. For every $i \in \mathbb{N}$, and every $(F, G) \in [\mathcal{LSC}(Z)]^2$, we set

$$\delta_i(F, G) := \left| \min\{F(z) : \ z \in \overline{U_i}\} - \min\{G(z) : \ z \in \overline{U_i}\} \right|,$$

where $\overline{U_i}$ denotes the closure of U_i, and finally we define

$$d_\Gamma(F, G) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\delta_i(F, G)}{\delta_i(F, G) + 1}.$$

Then we have the following result.

Theorem 2.6 *Let $\mathcal{LSC}(Z)$ and d_Γ be as above. Then*

- *d_Γ is a distance on $\mathcal{LSC}(Z)$;*

- *for every real number $K \geq 0$, the set*

$$\mathcal{LSC}_K(Z) := \{F \in \mathcal{LSC}(Z) : \ |F(z)| \leq K \ \forall z \in Z\}$$

is compact with respect to the topology induced by d_Γ;

- *a sequence $\{F_n\} \subseteq \mathcal{LSC}(Z)$ converges to $F_\infty \in \mathcal{LSC}(Z)$ with respect to d_Γ if and only if it Γ-converges to F_∞.*

In the last part of this section, we describe a generalization of Γ-convergence, which is often used in approximation results, including our Theorem 1.2.

When looking for a discrete approximation of a function $F : Z \to \mathbb{R}$, a standard approach is to choose a sequence $\{Z_n\}$ of closed (or better finite) subsets of Z such that

$$r(Z_n, Z) \to 0,$$

where for every closed subset $Z' \subseteq Z$ we set

$$r(Z', Z) := \max\{d_Z(w, Z') : \ w \in Z\}.$$

We remark that $r(Z', Z)$ is the smallest r such that Z is covered by the union of all the balls with centers in Z' and radius r.

In order to construct such a sequence $\{Z_n\}$, one can for example take a sequence $\{z_n\}$ dense in Z, and then set $Z_n := \{z_1, \ldots, z_n\}$.

Now let $F_n : Z_n \to \mathbb{R}$ be given for every $n \in \mathbb{N}$. From a formal point of view, one cannot say that $F_n \to F$ in the sense of Γ-convergence, since F_n and F are defined in different spaces. However, the definition of Γ-convergence can be generalized as follows.

Definition 2.7 Let $\{Z_n\}$ be a sequence of closed subsets of Z such that $r(Z_n, Z) \to 0$ as $n \to +\infty$. For every $n \in \mathbb{N}$, let $F_n : Z_n \to \mathbb{R}$, and let $F_\infty : Z \to \mathbb{R}$. We say that $\{F_n\}$ Γ-converges to F_∞ if the following two properties are satisfied.

- *Liminf inequality*: for every sequence $\{z_n\}$ such that $z_n \in Z_n$ for every $n \in \mathbb{N}$, and $z_n \to z_\infty$ in Z, we have that

$$\liminf_{n \to +\infty} F_n(z_n) \geq F_\infty(z_\infty).$$

- *Existence of a recovery sequence*: for every $z \in Z$ there exists a sequence $\{z_n\}$ such that $z_n \in Z_n$ for every $n \in \mathbb{N}$, and

$$z_n \to z, \qquad F_n(z_n) \to F_\infty(z).$$

This generalization (very useful in the applications) is actually apparent, at least when there exists $K \geq 0$ such that $F_n \in \mathcal{LSC}_K(Z_n)$ for every $n \in \mathbb{N}$. Indeed, defining $\widehat{F}_n : Z \to \mathbb{R}$ by

$$\widehat{F}_n(z) := \begin{cases} F_n(z) & \text{if } z \in Z_n \\ K & \text{if } z \in Z \setminus Z_n, \end{cases} \tag{2.1}$$

it turns out that $\{F_n\}$ Γ-converges to F_∞ in the sense of Definition 2.7 if and only if $\{\widehat{F}_n\}$ Γ-converges to F_∞ in the sense of Definition 2.1.

Moreover, the extension defined in (2.1) allows to consider $\mathcal{LSC}_K(Z_n)$ as a closed subset of the compact metric space $\mathcal{LSC}_K(Z)$ for every $K \geq 0$.

The following Proposition shows that $\mathcal{LSC}(Z_n)$ approximates $\mathcal{LSC}(Z)$ as $n \to +\infty$.

Proposition 2.8 *Let $\{Z_n\}$ be a sequence of closed subsets of Z such that $r(Z_n, Z) \to 0$ as $n \to +\infty$, and let $F \in \mathcal{LSC}(Z)$.*

(1) If we define $F_n : Z_n \to \mathbb{R}$ by

$$F_n(z) := \min\{F(w) : \ w \in Z, \ d_Z(w, z) \leq r(Z_n, Z)\} \qquad \forall z \in Z_n,$$

then the sequence $\{F_n\}$ Γ-converges to F in the sense of Definition 2.7.

(2) Considering $\mathcal{LSC}_K(Z_n)$ as a closed subset of $\mathcal{LSC}_K(Z)$, we have that

$$\lim_{n \to +\infty} r(\mathcal{LSC}_K(Z_n), \mathcal{LSC}_K(Z)) = 0.$$

3 Statements

This section is divided in two parts: in the first one we formally state our results concerning well posedness, lower semicontinuity and relaxation for $\mathcal{F}(p)$ and $\mathcal{G}(p)$; in the second one we approximate $\mathcal{G}$ by a sequence of smooth functionals $\{\mathcal{G}_n\}$.

In the following, we set

$$K := \frac{2k_1 + 1}{c_1}, \tag{3.1}$$

where

$$k_1 := \max\{|s(x, y, 0)| : \ (x, y) \in X \times Y\},$$

and c_1 is the constant of (Hp 9).

We consider $\mathcal{P}$ endowed with the topology coming from Γ-convergence, inherited as a subset of $\mathcal{LSC}(Y)$. We also remark that condition $p(y_\star) \leq 0$ is stable by Γ-convergence, hence $\mathcal{P}$ is closed in $\mathcal{LSC}(Y)$. For this reason, from Theorem 2.6 it follows that

$$\mathcal{P}_K := \{p \in \mathcal{P} : \ |p(y)| \leq K \ \forall y \in Y\}$$

is a compact metric space.

3.1 Well posedness, semicontinuity, relaxation

Theorem 3.1 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*
Then for every $p \in \mathcal{P}$ we have that

(1) for every $x \in X$, the function $s(x, y, p(y))$ attains its minimum on Y, hence $A_p(x)$ is well defined and compact;

(2) the set $\mathcal{A}_p$ defined in (1.2) is a compact subset of Y; moreover, the restriction of p to $\mathcal{A}_p$ is continuous, and $p(y) < K$ for every $y \in \mathcal{A}_p$, where K is the constant defined in (3.1);

(3) for every $x \in X$, the function $c(y, p(y))$ attains its maximum and minimum on $A_p(x)$, hence $M_p(x)$ and $m_p(x)$ are well defined;

(4) $m_p(x)$ is a bounded lower semicontinuous function of x, while $M_p(x)$ is a bounded upper semicontinuous function of x; as a consequence these functions are integrable with respect to the measure μ and therefore $\mathcal{F}(p)$ and $\mathcal{G}(p)$ are well defined.

Theorem 3.2 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*
Then

(1) $\mathcal{G}(p)$ is lower semicontinuous on $\mathcal{P}$;

(2) there exists

$$\min\{\mathcal{G}(p) : \ p \in \mathcal{P}\};$$

(3) if K is the constant defined in (3.1), then

$$\min\{\mathcal{G}(p) : \ p \in \mathcal{P}\} = \min\{\mathcal{G}(p) : \ p \in \mathcal{P}_K\}. \tag{3.2}$$

Theorem 3.3 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*
Then $\mathcal{G}(p)$ is the relaxation of $\mathcal{F}(p)$.
As a consequence

$$\inf\{\mathcal{F}(p) : \ p \in \mathcal{P}\} = \min\{\mathcal{G}(p) : \ p \in \mathcal{P}\}.$$

In order to contain this paper in a reasonable length, we don't study here regularity and qualitative properties of minimizers. We just remark that statement (2) of Theorem 3.1 implies in particular that any minimizer of $\mathcal{G}$ is continuous in $\mathcal{A}_p$, while (3.2) gives an *a priori* estimate on prices.

3.2 Approximation

The construction of the sequence $\{\mathcal{G}_n\}$ which approximates $\mathcal{G}$ is achieved in several steps, where we approximate all the ingredients of the model.

In the sequel, we assume that K is the real number defined in (3.1), and we set

$$M := \max\{|c(y,\pi)| : (y,\pi) \in Y \times [-K,K]\}. \tag{3.3}$$

Approximation of X and μ Let $\{X_n\}$ be a sequence of *finite* subsets of X such that $r(X_n, X) \to 0$ as $n \to +\infty$.

Now we approximate μ with a sequence $\{\mu_n\}$ of measures such that X_n is the support of μ_n. To this end, let $X_n = \{x_{n,1}, \ldots, x_{n,k_n}\}$ where $k_n = |X_n|$ is the number of elements of X_n. Let us set

$$X_{n,1} := \{x \in X : d_X(x, x_{n,1}) \leq d_X(x, x_{n,j}),\ j = 1, \ldots, k_n\},$$

and then, for $2 \leq i \leq k_n$,

$$X_{n,i} := \{x \in X \setminus (X_{n,1} \cup \ldots \cup X_{n,i-1}) : d_X(x, x_{n,i}) \leq d_X(x, x_{n,j}),\ j = i, \ldots, k_n\}.$$

In this way we have that each $X_{n,i}$ is a Borel subset of X; moreover

$$X = \bigcup_{i=1}^{k_n} X_{n,i}$$

for every $n \in \mathbb{N}$, and this union is disjoint. Now for every $x \in X$, we denote by δ_x the Dirac measure with support in x, and finally we set

$$\mu_n := \sum_{i=1}^{k_n} \mu(X_{n,i}) \delta_{x_{n,i}}.$$

It is easy to show that, for every $f : X \to \mathbb{R}$ we have that

$$\int_X f(x)\,\mathrm{d}\mu_n = \int_X f_n(x)\,\mathrm{d}\mu,$$

where $f_n(x)$ is the piecewise constant function which is equal to $f(x_{n,i})$ in $X_{n,i}$.

Approximation of Y and $\mathcal{P}$ Let $\{Y_n\}$ be a sequence of subsets of Y such that

(Y_n-1) $|Y_n| = n$ for every $n \in \mathbb{N}$;

(Y_n-2) $y_\star \in Y_n$ for every $n \geq 1$;

(Y_n-3) $r(Y_n, Y) \to 0$ as $n \to +\infty$.

Then for every $n \geq 1$ we set

$$P_{n,K} := \{p : Y_n \to \mathbb{R} \text{ such that } p(y_\star) \leq 0,\ |p(y)| \leq K\ \forall y \in Y_n\}.$$

By (Y_n-1) the set $\mathcal{P}_{n,K}$ can be identified with one half of an hypercube in $\mathbb{R}^n$. Moreover, from (Y_n-3) and Proposition 2.8, it follows that $\mathcal{P}_{n,K}$ can also be identified with a subset of the compact metric space $\mathcal{P}_K$ in such a way that $r(\mathcal{P}_{n,K}, \mathcal{P}_K) \to 0$ as $n \to +\infty$.

Approximation of s and c For every $n \in \mathbb{N}$, let $s_n(x, y, \pi)$ and $c_n(y, \pi)$ be functions which are C^∞ in π for every value of the other variable(s), and such that $s_n \to s$, and $c_n \to c$ uniformly on compact subsets of their domains. Sequences $\{s_n\}$ and $\{c_n\}$ with such properties can be easily obtained by convolution of s and c in the variable π.

Approximation of a Heaviside function For every $n \in \mathbb{N}$, let $\psi_n : \mathbb{R} \to \mathbb{R}$ be a function such that

(ψ1) $\psi_n \in C^\infty(\mathbb{R})$;

(ψ2) $0 \leq \psi_n(r) \leq 1$ for every $r \in \mathbb{R}$;

(ψ3) $\psi_n(r) = 1$ for every $r \leq 0$;

(ψ4) $\psi_n(r) = 0$ for every $r \geq 1/n$.

Approximation of the minimum of a set Let m be a positive integer, and let $r_1, \ldots, r_m$ be m real numbers. For every real number $q \neq 0$ we define

$$\mathrm{Mean}_q\{r_1, \ldots, r_m\} := \frac{1}{q} \log \left(\frac{1}{m} \sum_{i=1}^{m} e^{q r_i} \right).$$

It is easy to verify that

$$\min\{r_1, \ldots, r_m\} \leq \mathrm{Mean}_q\{r_1, \ldots, r_m\} \leq \max\{r_1, \ldots, r_m\}, \qquad \forall q \neq 0,$$

and that $\mathrm{Mean}_q\{r_1, \ldots, r_m\}$ tends to $\min\{r_1, \ldots, r_m\}$ as $q \to -\infty$. It is also clear that Mean_q doesn't depend on the order of the m arguments $r_1, \ldots, r_m$.

Approximation of m_p Let M be the constant defined in (3.3). For every $n \in \mathbb{N}$, and for every $(x,y,p) \in X \times Y_n \times \mathcal{P}_{n,K}$, we set

$$\begin{aligned}
\Psi_{n,p}(x,y) &:= 1 - \prod_{z \in Y_n} \psi_n \left[s_n(x,y,p(y)) - s_n(x,z,p(z)) \right]; \\
f_{n,p}(x,y) &:= c_n(y,p(y)) + 5M\Psi_{n,p}(x,y); \\
m_{n,p}(x) &:= \mathrm{Mean}_{-n}\{f_{n,p}(x,w) : \ w \in Y_n\},
\end{aligned}$$

where, with a little abuse of notation, $\{f_{n,p}(x,w) : \ w \in Y_n\}$ denotes the n real numbers (maybe with repetitions) obtained as w varies among the n elements of Y_n.

Approximation of $\mathcal{G}$ For every $n \in \mathbb{N}$, we define the functional $\mathcal{G}_n : \mathcal{P}_{n,K} \to \mathbb{R}$ by

$$\mathcal{G}_n(p) := \int_X m_{n,p}(x)\, \mathrm{d}\mu_n \qquad \forall p \in \mathcal{P}_{n,K}.$$

We point out that the integral in this definition is actually a finite sum.

We are now ready to state our approximation result.

Theorem 3.4 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*

Then the sequence $\{\mathcal{G}_n\}$ defined above Γ-converges to $\mathcal{G}$ in the sense of Definition 2.7 (with $Z_n = \mathcal{P}_{n,K}$ and $Z = \mathcal{P}_K$).

As a consequence

- *we have that*

$$\lim_{n \to +\infty} \min\{\mathcal{G}_n(p) : \ p \in \mathcal{P}_{n,K}\} = \min\{\mathcal{G}(p) : \ p \in \mathcal{P}\};$$

- *if p_n is a minimizer of $\mathcal{G}_n$ for every $n \in \mathbb{N}$, then the sequence $\{p_n\}$ converges, up to subsequences, to a minimizer of $\mathcal{G}$. Since p_n is defined in Y_n, and the limit is defined in Y, this convergence has to be intended in the sense of Definition 2.7 (with $Z_n = Y_n$ and $Z = Y$).*

We point out once more that the use of Γ-convergence in Theorem 3.4 is twofold, because it has been used also in the definition of the topology on $\mathcal{P}_K$.

Before we enter in the detail of the proofs, we outline the heuristic idea which inspires our definition of $\mathcal{G}_n$.

- The set
$$A_{n,p}(x) := \arg\min\{s_n(x,z,p(z)) : \ z \in Y_n\}$$
is the discrete counterpart of $A_p(x)$.

If $y \in A_{n,p}(x)$, then $s_n(x, y, p(y)) - s_n(x, z, p(z)) \leq 0$ for every $z \in Y_n$, hence by (ψ3) we have that $\Psi_{n,p}(x, y) = 0$; if $y \notin A_{n,p}(x)$, then $s_n(x, y, p(y)) - s_n(x, z, p(z)) > 0$ for some $z \in Y_n$, hence by (ψ4) $\Psi_{n,p}(x, y)$ should be approximately equal to 1. Therefore $\Psi_{n,p}(x, y)$ should be an approximation of the characteristic function of $Y \setminus A_p(x)$.

- The function $f_{n,p}(x, y)$ coincides with $c_n(y, p(y))$ if $y \in A_{n,p}(x)$, and it is very large otherwise. For this reason we expect that

$$\begin{aligned}\min\{f_{n,p}(x, y) : \ y \in Y_n\} \ &\approx \ \min\{c_n(y, p(y)) : \ y \in A_{n,p}(x)\} \\ &\approx \ \min\{c(y, p(y)) : \ y \in A_p(x)\} \ = \ m_p(x).\end{aligned}$$

- Since Mean_q of a set A is an approximation of the minimum of A as $q \to -\infty$, we expect $m_{n,p}(x)$ to be an approximation of $m_p(x)$ for large n.

- Finally, since integration with respect to μ_n is an approximation of integration with respect to μ, we expect $\mathcal{G}_n$ to be an approximation of $\mathcal{G}$. This is exactly the content of Theorem 3.4 above.

4 Proofs

In all our proofs, we repeatedly take subsequences, but we never relabel indices in order to avoid cumbersome notations.

4.1 Proof of Theorem 3.1

Step 1. We prove statement (1).

Let $(x, p) \in X \times \mathcal{P}$. Since p is lower semicontinuous, and the function $s(x, y, \pi)$ is continuous in (y, π) and increasing in π (by (Hp 4) and (Hp 9)), it follows that $s(x, y, p(y))$ is a lower semicontinuous function of y, and therefore it attains the minimum on the compact space Y. Moreover, the set $A_p(x)$ of minimum points is a nonempty and closed subset of Y, hence it is compact.

Step 2. We prove the following: "if $x_n \to x_\infty$ in X, $y_n \to y_\infty$ in Y, and $y_n \in A_p(x_n)$ for every $n \in \mathbb{N}$, then $y_\infty \in A_p(x_\infty)$ and $p(y_n) \to p(y_\infty)$".

To this end, let us set

$$\liminf_{n \to +\infty} p(y_n) =: \pi_\infty \geq p(y_\infty).$$

Up to subsequences we can assume that the lim inf is actually a limit. Now let $w \in Y$. Since $y_n \in A_p(x_n)$ we have that

$$s(x_n, y_n, p(y_n)) \leq s(x_n, w, p(w)),$$

so that, passing to the limit and using that s is continuous and increasing in the last variable, we have that

$$\begin{aligned} s(x_\infty, y_\infty, p(y_\infty)) &\leq s(x_\infty, y_\infty, \pi_\infty) \\ &= \lim_{n\to+\infty} s(x_n, y_n, p(y_n)) \\ &\leq \lim_{n\to+\infty} s(x_n, w, p(w)) \;=\; s(x_\infty, w, p(w)). \end{aligned}$$

Since w is arbitrary, this proves that $y_\infty \in A_p(x_\infty)$.

Now let us assume that $\pi_\infty > p(y_\infty)$. Then setting $w = y_\infty$ in the above inequalities, and remarking that s is strictly increasing in the third variable, we get immediately a contradiction. Therefore the lim inf of $p(y_n)$ is $p(y_\infty)$ for every sequence $y_n \to y_\infty$. This proves that $p(y_n) \to p(y_\infty)$.

Step 3. We prove that $\mathcal{A}_p$ is closed, hence compact.

To this end, let $\{y_n\} \subseteq \mathcal{A}_p$ be a sequence such that $y_n \to y_\infty$ in Y. Let x_n be such that $y_n \in A_p(x_n)$ for every $n \in \mathbb{N}$. Since X is compact, we can assume that x_n converges to some x_∞, up to subsequences.

By the result of the previous step, we know that $y_\infty \in A_p(x_\infty)$, hence in particular $y_\infty \in \mathcal{A}_p$.

Step 4. We prove that the restriction of p to $\mathcal{A}_p$ is continuous.

Assume that this is not the case. Then there exists a sequence $y_n \to y_\infty$ in $\mathcal{A}_p$ such that

$$\lim_{n\to+\infty} p(y_n) > p(y_\infty).$$

Let x_n be such that $y_n \in A_p(x_n)$. Since X is compact, we can assume that x_n converges to some x_∞, up to subsequences.

By the result of the step 2, we know that $y_\infty \in A_p(x_\infty)$, and $p(y_n) \to p(y_\infty)$, which contradicts our assumption.

Step 5. We prove that if $p \in \mathcal{P}$, and $y \in \mathcal{A}_p$, then $p(y) < K$.

Indeed, let us assume that $p(y) \geq K$. By (Hp 6), (Hp 9) and (3.1), for every $x \in X$ we have that

$$s(x, y, p(y)) \geq s(x, y, K) = s(x, y, K) - s(x, y, 0) + s(x, y, 0) \geq$$

$$\geq Kc_1 - k_1 = k_1 + 1 > 0 = s(x, y_\star, 0) \geq s(x, y_\star, p(y_\star)).$$

This proves that $s(x, y, p(y)) > s(x, y_\star, p(y_\star))$, and therefore $y \notin A_p(x)$ for every $x \in X$. This completes the proof of statement (2).

Step 6. We prove statement (3).

From step 4 we know that p is continuous if restricted to $\mathcal{A}_p$, and in particular if restricted to $A_p(x)$. Since c is continuous (by (Hp 5)), it follows that $c(y,p(y))$ is a continuous function of y on $A_p(x)$, which has been proved to be compact in step 1. This proves that $m_p(x)$ and $M_p(x)$ are well defined.

Step 7. We prove the semicontinuity of $m_p(x)$ and $M_p(x)$.

Let us assume that $x_n \to x_\infty$ in X, and let $y_n \in A_p(x_n)$ be such that $m_p(x_n) = c(y_n, p(y_n))$. Up to subsequences, we can assume that the lim inf of $m_p(x_n)$ is actually a limit, and that y_n converges to some y_∞ in Y.

From step 2 we know that $y_\infty \in A_p(x_\infty)$ and $p(y_n) \to p(y_\infty)$, hence by the continuity of c,

$$\lim_{n\to+\infty} m_p(x_n) = \lim_{n\to+\infty} c(y_n, p(y_n)) = c(y_\infty, p(y_\infty)) \geq m_p(x_\infty),$$

where the last inequality follows from the fact that y_∞ is a competitor in the definition of $m_p(x_\infty)$. This proves that $m_p(x)$ is a lower semicontinuous function of x.

The proof of the upper semicontinuity of $M_p(x)$ is completely analogous.

Step 8. We prove the boundedness of $m_p(x)$ and $M_p(x)$.

To this end, using the result of step 7 and the compactness of X, we have that $m_p(x)$ has a minimum on X, and $M_p(x)$ has a maximum on X. Therefore

$$\min\{m_p(x) : \ x \in X\} \leq m_p(x) \leq M_p(x) \leq \max\{M_p(x) : \ x \in X\}$$

for every $x \in X$. This proves statement (4).

4.2 Proof of Theorem 3.2

Compactness We prove that

$$\inf\{\mathcal{G}(p) : \ p \in \mathcal{P}\} = \inf\{\mathcal{G}(p) : \ p \in \mathcal{P}_K\}, \tag{4.1}$$

where K is the constant defined in (3.1). This proves that when looking for the minimum of $\mathcal{G}$, we can restrict ourselves to the compact set $\mathcal{P}_K$.

Step 1. We prove that the infimum of $\mathcal{G}$ over $\mathcal{P}$ is less or equal than zero.

Indeed among the competitors there is the function $\widehat{p}$ such that $\widehat{p}(y_\star) = 0$ and $\widehat{p}(y) = K$ for every $y \neq y_\star$. By statement (2) of Theorem 3.1, for every $x \in X$ we have that $A_{\widehat{p}}(x) = \{y_\star\}$, hence $m_{\widehat{p}}(x) = c(y_\star, 0) = 0$ (by (Hp 6)), and therefore $\mathcal{G}(\widehat{p}) = 0$.

Step 2. We prove that if $p \in \mathcal{P}$, and $\mathcal{G}(p) < 0$, then $p(y) > -K$ for every $y \in Y$.

Indeed let us assume by contradiction that $\mathcal{G}(p) < 0$, but $p(\overline{y}) \leq -K$ for some $\overline{y} \in Y$. Since $\mathcal{G}(p) < 0$, there exists $x \in X$ such that $m_p(x) < 0$. Let $y \in A_p(x)$ be such that $m_p(x) = c(y, p(y))$. By (Hp 8) we have that $p(y) > 0$. But in this case

$$s(x,y,p(y)) \geq s(x,y,0) \geq -k_1 > -k_1 - 1 = k_1 - c_1 K \geq$$

$$\geq s(x,\overline{y},0) - s(x,\overline{y},0) + s(x,\overline{y},-K) \geq s(x,\overline{y},p(\overline{y})).$$

We have thus proved that $s(x,y,p(y)) > s(x,\overline{y},p(\overline{y}))$ and this contradicts the assumption $y \in A_p(x)$.

Step 3. If $p \in \mathcal{P}$, and $\overline{p}(y) = \min\{p(y), K\}$, then $\mathcal{G}(\overline{p}) = \mathcal{G}(p)$.

Indeed, since products with price $\geq K$ cannot in any case be bought (by statement (2) of Theorem 3.1), then it is easy to verify that in this case $A_p(x) = A_{\overline{p}}(x)$ for every $x \in X$.

Step 4. If the infimum of $\mathcal{G}(p)$ in $\mathcal{P}$ is zero, then the price function $\widehat{p}$ of step 1 is a minimizer. Otherwise, we can restrict to price functions p such that $\mathcal{G}(p) < 0$. By step 2 such functions satisfy $p(y) > -K$ for every $y \in Y$. Moreover, by the truncation argument of step 3, we can replace any competitor with another one satisfying $p(y) \leq K$ for every $y \in Y$. This completes the proof of (4.1).

Lower semicontinuity We have to prove that, if $p_n \to p_\infty$ in $\mathcal{P}$, then

$$\liminf_{n\to+\infty} \mathcal{G}(p_n) \geq \mathcal{G}(p_\infty). \tag{4.2}$$

We recall that the convergence in $\mathcal{P}$ is equivalent to the following two requirements

- for every converging sequence $y_n \to y_\infty$ we have that
$$\liminf_{n\to+\infty} p_n(y_n) \geq p_\infty(y_\infty);$$
- for every $z \in Y$, there exists a recovery sequence $z_n \to z$ such that $p_n(z_n) \to p_\infty(z)$.

Step 1. Let $x \in X$ be fixed. We prove the following: "if $y_n \to y_\infty$ and $y_n \in A_{p_n}(x)$ for every $n \in \mathbb{N}$, then $y_\infty \in A_{p_\infty}(x)$ and $p_n(y_n) \to p_\infty(y_\infty)$".

The argument is similar to step 2 of the proof of Theorem 3.1. Let us set

$$\liminf_{n\to+\infty} p_n(y_n) =: \pi_\infty \geq p_\infty(y_\infty).$$

Up to subsequences we can assume that the lim inf is actually a limit. Now let $w \in Y$, and let $\{w_n\} \subseteq Y$ be a recovery sequence for w. Since $y_n \in A_{p_n}(x)$ we have that

$$s(x, y_n, p_n(y_n)) \leq s(x, w_n, p_n(w_n)),$$

so that, passing to the limit and using that s is continuous and increasing in the last variable, we have that

$$\begin{aligned} s(x, y_\infty, p_\infty(y_\infty)) &\leq s(x, y_\infty, \pi_\infty) \\ &= \lim_{n\to+\infty} s(x, y_n, p_n(y_n)) \\ &\leq \lim_{n\to+\infty} s(x, w_n, p_n(w_n)) = s(x, w, p_\infty(w)). \end{aligned}$$

Since w is arbitrary, this proves that $y_\infty \in A_{p_\infty}(x)$.

Now let us assume that $\pi_\infty > p_\infty(y_\infty)$. Then setting $w = y_\infty$ in the above inequalities, and remarking that s is strictly increasing in the third variable, we immediately get a contradiction. Therefore the lim inf of $p_n(y_n)$ is $p_\infty(y_\infty)$ for every sequence $y_n \to y_\infty$. This proves that $p_n(y_n) \to p_\infty(y_\infty)$.

Step 2. We prove that

$$\liminf_{n\to+\infty} m_{p_n}(x) \geq m_{p_\infty}(x) \qquad \forall x \in X. \tag{4.3}$$

To this end, let us assume that $m_{p_n}(x) = c(y_n, p_n(y_n))$ for every $n \in \mathbb{N}$. Up to subsequences, we can assume that the lim inf of $m_{p_n}(x)$ is actually a limit, and that y_n converges to some y_∞ in Y. By step 1, we know that $y_\infty \in A_{p_\infty}(x)$ and $p_n(y_n) \to p_\infty(y_\infty)$, hence by continuity of c,

$$\lim_{n\to+\infty} m_{p_n}(x) = \lim_{n\to+\infty} c(y_n, p_n(y_n)) = c(y_\infty, p_\infty(y_\infty)) \geq m_{p_\infty}(x),$$

where the last inequality follows from the fact that y_∞ is a competitor in the definition of $m_{p_\infty}(x)$.

Step 3. We prove that there exists a constant C such that $m_{p_n}(x) \geq C$ for every $n \in \mathbb{N}$.

Since $p_n \to p_\infty$ in the sense of Γ-convergence, then by Theorem 2.2

$$\lim_{n\to+\infty} \min\{p_n(y) : \ y \in Y\} = \min\{p_\infty(y) : \ y \in Y\}.$$

It follows that there exists a constant γ such that $p_n(y) \geq \gamma$ for every $n \in \mathbb{N}$ and every $y \in Y$.

Moreover, from statement (2) of Theorem 3.1, we know that $p(y) < K$ for every $y \in \mathcal{A}_p$. It follows that, for every $x \in X$,

$$m_{p_n}(x) \geq \min\{c(y,\pi) : \ (y,\pi) \in Y \times [\gamma, K]\} =: C.$$

Step 4. We prove (4.2).

By step 3 and (Hp 2), we can apply Fatou's Lemma, so that by (4.3)

$$\begin{aligned}
\liminf_{n\to+\infty} \mathcal{G}(p_n) &= \liminf_{n\to+\infty} \int_X m_{p_n}(x)\,\mathrm{d}\mu \\
&\geq \int_X \liminf_{n\to+\infty} m_{p_n}(x)\,\mathrm{d}\mu \\
&\geq \int_X m_{p_\infty}(x)\,\mathrm{d}\mu \ = \ \mathcal{G}(p_\infty).
\end{aligned}$$

Conclusion Since $\mathcal{P}_K$ is compact and $\mathcal{G}$ is lower semicontinuous, it follows that $\mathcal{G}$ has a minimum on $\mathcal{P}_K$. By (4.1) this is also a minimum on $\mathcal{P}$. This completes the proof of Theorem 3.2.

4.3 Proof of Theorem 3.3

The liminf inequality Let us assume that $p_n \to p_\infty$ in $\mathcal{P}$. By (1.1) and the lower semicontinuity of $\mathcal{G}$ we have that

$$\liminf_{n\to+\infty} \mathcal{F}(p_n) \geq \liminf_{n\to+\infty} \mathcal{G}(p_n) \geq \mathcal{G}(p_\infty).$$

This proves the liminf inequality of the relaxation result.

The limsup inequality We have to prove that for every $p \in \mathcal{P}$ there exists a recovery sequence $p_n \to p$ in $\mathcal{P}$ such that

$$\limsup_{n\to+\infty} \mathcal{F}(p_n) \leq \mathcal{G}(p).$$

To this end, we consider the subset $\mathcal{D} \subseteq \mathcal{P}$ defined as follows.

Definition 4.1 Let $p \in \mathcal{P}$. We say that $p \in \mathcal{D}$ if there exists a real number $\alpha > 0$ such that the following implication holds true for every converging sequence $y_n \to y_\infty$:

$$\liminf_{n\to+\infty} p(y_n) > p(y_\infty) \implies \liminf_{n\to+\infty} p(y_n) \geq p(y_\infty) + \alpha.$$

Thanks to Proposition 2.5, the limsup inequality is proved if we show that

- $\mathcal{D}$ is dense in $\mathcal{P}$ for the energy $\mathcal{G}$;
- there exists a "recovery sequence up to 2η" for every $p \in \mathcal{D}$, *i.e.* there exists a sequence $p_n \to p$ such that

$$\limsup_{n\to+\infty} \mathcal{F}(p_n) \leq \mathcal{G}(p) + 2\eta.$$

This is exactly the content of the next two lemmata, which are the technical core of this proof.

Lemma 4.2 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied, and let $\mathcal{D}$ be as in Definition 4.1.*

Then for every $p \in \mathcal{P}$ there exists a sequence $\{p_n\} \subseteq \mathcal{D}$ such that

$$p_n \to p, \qquad \mathcal{G}(p_n) \to \mathcal{G}(p).$$

Proof. For every $\varepsilon > 0$, let $\overline{p}_\varepsilon : Y \to \mathbb{R}$ be the function defined by

$$\overline{p}_\varepsilon(y) := \varepsilon \cdot \max\{z \in \mathbb{Z} : \; p(y) > \varepsilon z\}.$$

It is easy to prove that, for every $\varepsilon > 0$, we have that

(i) $\overline{p}_\varepsilon(y) \in \varepsilon\mathbb{Z}$ for every $y \in Y$;

(ii) $\overline{p}_\varepsilon$ is lower semicontinuous on Y;

(iii) $p(y) - \varepsilon \leq \overline{p}_\varepsilon(y) \leq p(y)$ for every $y \in Y$.

Now let us set $\mathcal{A}_p^\star = \mathcal{A}_p \cup \{y_\star\}$, where $\mathcal{A}_p$ is the subset of Y defined in (1.2), and let

$$p_\varepsilon(y) := \begin{cases} p(y) & \text{if } y \in \mathcal{A}_p^\star, \\ \overline{p}_\varepsilon(y) + 2\varepsilon & \text{if } y \in Y \setminus \mathcal{A}_p^\star. \end{cases}$$

If we show that

(1) $p_\varepsilon \in \mathcal{D}$ for every $\varepsilon > 0$,

(2) $p_\varepsilon \to p$ uniformly, hence also in $\mathcal{P}$,

(3) $\mathcal{G}(p_\varepsilon) = \mathcal{G}(p)$ for every $\varepsilon > 0$,

then any sequence $\{p_{\varepsilon_n}\}$ with $\varepsilon_n \to 0^+$ satisfies the thesis of the Lemma.

In order to prove claim (1), let $y_n \to y_\infty$ be any converging sequence, and let

$$\pi_\infty := \liminf_{n\to+\infty} p_\varepsilon(y_n).$$

Up to subsequences we can assume that the lim inf is actually a limit, and that we are in one of the following three cases.

- *Case 1*: $y_n \in \mathcal{A}_p^\star$ for every $n \in \mathbb{N}$. By statement (2) of Theorem 3.1, we have that $y_\infty \in \mathcal{A}_p^\star$, and $p(y_n) \to p(y_\infty)$, hence in this case

$$\pi_\infty = \lim_{n\to+\infty} p_\varepsilon(y_n) = \lim_{n\to+\infty} p(y_n) = p(y_\infty) = p_\varepsilon(y_\infty).$$

- *Case 2*: $y_n \notin \mathcal{A}_p^\star$ for every $n \in \mathbb{N}$, and $y_\infty \in \mathcal{A}_p^\star$. In this case by (iii) we have that $p_\varepsilon(y_n) = \overline{p}_\varepsilon(y_n) + 2\varepsilon \geq p(y_n) + \varepsilon$ for every $n \in \mathbb{N}$, and $p_\varepsilon(y_\infty) = p(y_\infty)$, hence

$$\pi_\infty = \lim_{n\to+\infty} p_\varepsilon(y_n) \geq \varepsilon + \liminf_{n\to+\infty} p(y_n) \geq \varepsilon + p(y_\infty) = \varepsilon + p_\varepsilon(y_\infty).$$

- *Case 3*: $y_n \notin \mathcal{A}_p^\star$ for every $n \in \mathbb{N}$, and $y_\infty \notin \mathcal{A}_p^\star$. In this case p_ε coincides with $2\varepsilon + \overline{p}_\varepsilon$ on every y_n and on y_∞, hence by (i) and (ii) it follows that $\pi_\infty = p_\varepsilon(y_\infty) + k\varepsilon$ for some $k \in \mathbb{N}$.

In all the three cases we have proved that either $\pi_\infty = p_\varepsilon(y_\infty)$ or $\pi_\infty \geq p_\varepsilon(y_\infty) + \varepsilon$, and therefore p_ε satisfies the condition of Definition 4.1 with $\alpha = \varepsilon$. This proves claim (1).

In order to prove claim (2), it is enough to remark that, for every $y \in Y$,

$$|p_\varepsilon(y) - p(y)| \leq |p_\varepsilon(y) - \overline{p}_\varepsilon(y)| + |\overline{p}_\varepsilon(y) - p(y)| \leq 3\varepsilon.$$

Finally, we prove that $A_{p_\varepsilon}(x) = A_p(x)$ for every $x \in X$ and $\varepsilon > 0$, hence $\mathcal{G}(p_\varepsilon) = \mathcal{G}(p)$ for every $\varepsilon > 0$.

Indeed, if $y \in A_p(x)$, then by definition of p_ε and the monotonicity of $s(x, y, \pi)$ in π, we have that

$$s(x, y, p_\varepsilon(y)) = s(x, y, p(y)) \leq s(x, w, p(w)) \leq s(x, w, p_\varepsilon(w))$$

for every $w \in Y$, hence $y \in A_{p_\varepsilon}(x)$. Conversely, if $y \notin A_p(x)$, and z is any element of $A_p(x)$, then

$$s(x, y, p_\varepsilon(y)) \geq s(x, y, p(y)) > s(x, z, p(z)) = s(x, z, p_\varepsilon(z)),$$

hence $y \notin A_{p_\varepsilon}(x)$. $\square$

Lemma 4.3 *Assume that assumptions (Hp 1) through (Hp 9) are satisfied, and let $\mathcal{D}$ be as in Definition 4.1.*

Then for every $p \in \mathcal{D}$, and every $\eta > 0$, there exists a sequence $\{p_n\} \subseteq \mathcal{P}$ such that

$$p_n \to p, \qquad \limsup_{n \to +\infty} \mathcal{F}(p_n) \leq \mathcal{G}(p) + 2\eta.$$

PROOF. Since $p \in \mathcal{D}$, then by definition there exists a constant $\alpha > 0$ such that

$$\liminf_{n \to +\infty} p(y_n) > p(y_\infty) \implies \liminf_{n \to +\infty} p(y_n) \geq p(y_\infty) + \alpha$$

for every converging sequence $y_n \to y_\infty$.

To begin with, let us assume that $p(y) \leq K$ for every $y \in Y$, where K is the constant defined in (3.1), and let us set

$$\begin{aligned} \gamma &:= \min\{p(y) : \ y \in Y\}, \\ \lambda &:= \min\{c(y, \pi) : \ (y, \pi) \in Y \times [\gamma, K]\}, \\ \Lambda &:= \max\{c(y, \pi) : \ (y, \pi) \in Y \times [\gamma, K]\}. \end{aligned}$$

For every $i \in \mathbb{N}$, we set

$$Y_i := \{y \in Y : \ \lambda + \eta(i-1) < c(y, p(y)) \leq \lambda + \eta i\},$$

and then we define $\Phi : Y \to \mathbb{R}$ as

$$\Phi(y) := \left(\frac{c_1}{c_2}\right)^i, \qquad \forall y \in Y_i$$

where c_1 and c_2 are the constants in (Hp 9).

Finally, for every $\varepsilon > 0$ we set

$$p_\varepsilon(y) := p(y) - \varepsilon\Phi(y)\,[\Lambda - c(y, p(y))] \qquad \forall y \in Y.$$

If we prove that

(1) $p_\varepsilon \in \mathcal{P}$ provided that $\varepsilon(\Lambda - \lambda) \leq \alpha$,

(2) $p_\varepsilon \to p$ uniformly in Y, hence in $\mathcal{P}$,

(3) $\mathcal{F}(p_\varepsilon) \leq \mathcal{G}(p) + 2\eta$ for ε small enough,

then any sequence $\{p_{\varepsilon_n}\}$ with $\varepsilon_n \to 0^+$ satisfies the thesis of the Lemma for n large enough.

Step 1. Since $0 \leq \Phi(y) \leq 1$ for every $y \in Y$, then

$$p(y) - \varepsilon(\Lambda - \lambda) \leq p_\varepsilon(y) \leq p(y), \qquad \forall \varepsilon > 0, \ \forall y \in Y. \tag{4.4}$$

This proves claim (2) provided that $p_\varepsilon \in \mathcal{P}$.

Step 2. We prove that if $y_n \to y_\infty$ in Y, and $p(y_n) \to p(y_\infty)$, then

$$\limsup_{n \to +\infty} \Phi(y_n) \leq \Phi(y_\infty). \tag{4.5}$$

Indeed, let us choose a subsequence such that the lim sup is actually a limit. Since there is only a finite number of nonempty Y_i's, we can assume (up to subsequences) that every y_n belongs to a fixed Y_{i_0}. This means that

$$\lambda + \eta(i_0 - 1) < c(y_n, p(y_n)) \leq \lambda + \eta i_0, \qquad \Phi(y_n) = \left(\frac{c_1}{c_2}\right)^{i_0}, \qquad \forall n \in \mathbb{N}.$$

Since $p(y_n) \to p(y_\infty)$, and c is continuous, there are only two possibilities:

- $y_\infty \in Y_{i_0}$, and therefore

$$\lim_{n\to+\infty} \Phi(y_n) = \left(\frac{c_1}{c_2}\right)^{i_0} = \Phi(y_\infty);$$

- $y_\infty \in Y_{i_0-1}$, and therefore

$$\lim_{n\to+\infty} \Phi(y_n) = \left(\frac{c_1}{c_2}\right)^{i_0} \leq \left(\frac{c_1}{c_2}\right)^{i_0-1} = \Phi(y_\infty).$$

In both cases, inequality (4.5) is proved.

Step 3. We prove claim (1). Since clearly $p_\varepsilon(y_\star) \leq p(y_\star) \leq 0$, we have only to prove the lower semicontinuity of p_ε, *i.e.* that

$$\liminf_{n\to+\infty} p_\varepsilon(y_n) \geq p_\varepsilon(y_\infty) \tag{4.6}$$

for every converging sequence $y_n \to y_\infty$ in Y.

Up to subsequences, we can assume that the lim inf is actually a limit, and that there exists

$$\pi_\infty := \lim_{n\to+\infty} p(y_n) \geq p(y_\infty).$$

Now we have two cases.

- *Case 1*: $\pi_\infty = p(y_\infty)$, *i.e.* $p(y_n) \to p(y_\infty)$. In this case by step 2 we have that

$$\begin{aligned}
\lim_{n\to+\infty} p_\varepsilon(y_n) &= \lim_{n\to+\infty} \{p(y_n) - \varepsilon\,[\Lambda - c(y_n, p(y_n))]\,\Phi(y_n)\} \\
&\geq p(y_\infty) - \varepsilon\,[\Lambda - c(y_\infty, p(y_\infty))] \limsup_{n\to+\infty} \Phi(y_n) \\
&\geq p(y_\infty) - \varepsilon\,[\Lambda - c(y_\infty, p(y_\infty))]\,\Phi(y_\infty) \;=\; p_\varepsilon(y_\infty),
\end{aligned}$$

  which proves (4.6) in this case.

- *Case 2*: $\pi_\infty > p(y_\infty)$. Since $p \in \mathcal{D}$, this means that $\pi_\infty \geq p(y_\infty) + \alpha$. Then by (4.4) it follows that

$$\liminf_{n\to+\infty} p_\varepsilon(y_n) \geq \liminf_{n\to+\infty} p(y_n) - \varepsilon(\Lambda - \lambda) \geq p(y_\infty) + \alpha - \varepsilon(\Lambda - \lambda).$$

  If $\varepsilon(\Lambda - \lambda) \leq \alpha$, then inequality (4.6) is proved also in this second case.

Step 4. We prove that if w and y are elements of Y such that

$$c(w, p(w)) > c(y, p(y)) + \eta, \tag{4.7}$$

then for every $\varepsilon > 0$

$$s(x, w, p_\varepsilon(w)) - s(x, w, p(w)) > s(x, y, p_\varepsilon(y)) - s(x, y, p(y)). \tag{4.8}$$

Let us assume that $y \in Y_i$ and $w \in Y_j$. By (4.7) it follows that $j > i$, hence $\Phi(y) \geq (c_2/c_1)\Phi(w)$. Therefore by (Hp 9)

$$\begin{aligned} s(x, y, p(y)) - s(x, y, p_\varepsilon(y)) &\geq c_1(p(y) - p_\varepsilon(y)) \\ &= \varepsilon c_1 \Phi(y) \left[\Lambda - c(y, p(y))\right] \\ &\geq \varepsilon c_2 \Phi(w) \left[\Lambda - c(y, p(y))\right] \\ &> \varepsilon c_2 \Phi(w) \left[\Lambda - c(w, p(w))\right] \\ &= c_2(p(w) - p_\varepsilon(w)) \\ &\geq s(x, w, p(w)) - s(x, w, p_\varepsilon(w)). \end{aligned}$$

This is equivalent to (4.8).

Step 5. We prove the following: "if $x \in X$, $y \in A_p(x)$, and $w \in A_{p_\varepsilon}(x)$ for some $\varepsilon > 0$, then $c(w, p(w)) \leq c(y, p(y)) + \eta$".

Let indeed $w \in Y$ be such that $c(w, p(w)) > c(y, p(y)) + \eta$. Then, using that $y \in A_p(x)$ and (4.8), we have that

$$\begin{aligned} s(x, w, p_\varepsilon(w)) &= s(x, w, p_\varepsilon(w)) - s(x, w, p(w)) + s(x, w, p(w)) \\ &> s(x, y, p_\varepsilon(y)) - s(x, y, p(y)) + s(x, y, p(y)) \\ &= s(x, y, p_\varepsilon(y)) \end{aligned}$$

which proves that $w \notin A_{p_\varepsilon}(x)$.

Step 6. We prove that for every $x \in X$, and every ε small enough, we have that

$$M_{p_\varepsilon}(x) \leq m_p(x) + 2\eta \tag{4.9}$$

To this end, let us set

$$\omega(\varepsilon) := \sup\{|c(y, \pi_1) - c(y, \pi_2)| : \ (y, \pi_1, \pi_2) \in Y \times [\gamma, K]^2, \ |\pi_2 - \pi_1| \leq \varepsilon(\Lambda - \lambda)\}.$$

Since c is uniformly continuous in $Y \times [\gamma, K]$, it follows that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, hence $\omega(\varepsilon) < \eta$ if ε is small enough.

Now let $y \in A_p(x)$ and $w \in A_{p_\varepsilon}(x)$ be such that

$$M_{p_\varepsilon}(x) = c(w, p_\varepsilon(w)), \qquad m_p(x) = c(y, p(y)).$$

By step 5, (4.4), and definition of ω, we have that

$$\begin{aligned} M_{p_\varepsilon}(x) \;=\; c(w,p_\varepsilon(w)) &= c(w,p(w))+c(w,p_\varepsilon(w))-c(w,p(w)) \\ &\leq c(y,p(y))+\eta+\omega(\varepsilon) \;<\; m_p(x)+2\eta \end{aligned}$$

for ε small enough.

Step 7. Integrating (4.9) with respect to μ, by (Hp 2) we obtain that

$$\mathcal{F}(p_\varepsilon) \leq \mathcal{G}(p) + 2\eta$$

for ε small enough. This completes the proof under the assumption that $p(y) \leq K$ for every $y \in Y$.

Step 8. If p is not bounded from above by K, we know from statement (2) of Theorem 3.1 that in any case $p(y) \leq K$ for every $y \in \mathcal{A}_p$. Now it is enough to define $p_\varepsilon(y)$ as in the previous case if $p(y) < K$, and to set $p_\varepsilon(y) = p(y)$ if $p(y) \geq K$. Since any $y \in Y$ with $p_\varepsilon(y) \geq K$ cannot belong to $\mathcal{A}_{p_\varepsilon}$, then everything works exactly as before. $\square$

4.4 Proof of Theorem 3.4

Technical Lemmata In order to simplify the proof, we first state and prove four lemmata.

Lemma 4.4 *Let X, X_n, μ, μ_n be as in section 3.2. Let $\gamma_n : X \to \mathbb{R}$ for every $n \in \mathbb{N}$, and let $\gamma_\infty : X \to \mathbb{R}$.*

Let us assume that there exists a constant C such that $\gamma_n(x) \geq C$ for every $(n,x) \in \mathbb{N} \times X$, and that

$$\liminf_{n\to+\infty} \gamma_n(x_n) \geq \gamma_\infty(x_\infty)$$

for every converging sequence $x_n \to x_\infty$ in X.

Then

$$\liminf_{n\to+\infty} \int_X \gamma_n(x)\,\mathrm{d}\mu_n \geq \int_X \gamma_\infty(x)\,\mathrm{d}\mu.$$

Lemma 4.5 *Let X, X_n, μ, μ_n be as in section 3.2. Let $\gamma_n : X \to \mathbb{R}$ for every $n \in \mathbb{N}$, and let $\gamma_\infty : X \to \mathbb{R}$.*

Let us assume that there exists a constant C such that $\gamma_n(x) \leq C$ for every $(n,x) \in \mathbb{N} \times X$, and that

$$\limsup_{n\to+\infty} \gamma_n(x_n) \leq \gamma_\infty(x_\infty)$$

for every converging sequence $x_n \to x_\infty$ in X.

Then

$$\limsup_{n\to+\infty} \int_X \gamma_n(x)\,\mathrm{d}\mu_n \le \int_X \gamma_\infty(x)\,\mathrm{d}\mu.$$

Lemma 4.6 *Let Y_n and Y be as in section 3.2. Let $\phi_n : Y_n \to \mathbb{R}$ for every $n \in \mathbb{N}$, and let $\phi_\infty : Y \to \mathbb{R}$.*

Let us assume that for every sequence $\{y_n\} \subseteq Y$ such that $y_n \in Y_n$, and $y_n \to y_\infty$ in Y we have that

$$\liminf_{n\to+\infty} \phi_n(y_n) \ge \phi_\infty(y_\infty).$$

Then

$$\liminf_{n\to+\infty} \mathrm{Mean}_{-n}\{\phi_n(y) :\ y \in Y_n\} \ge \inf\{\phi_\infty(y) :\ y \in Y\}.$$

Lemma 4.7 *Let Y_n and Y be as in section 3.2. Let $\phi_n : Y_n \to \mathbb{R}$ for every $n \in \mathbb{N}$.*

Let $\{y_n\} \subseteq Y$ be any sequence such that $y_n \in Y_n$ for every $n \in \mathbb{N}$.

Then

$$\limsup_{n\to+\infty} \mathrm{Mean}_{-n}\{\phi_n(y) :\ y \in Y_n\} \le \limsup_{n\to+\infty} \phi_n(y_n).$$

Proof of Lemma 4.4. Let $X_{n,i}$ and $x_{n,i}$ be as in section 3.2. For every $(n,x) \in \mathbb{N}\times X$, we denote by $T_n(x)$ the element $x_{n,i}$ such that $x \in X_{n,i}$. Since $r(X_n, X) \to 0$, it follows that $T_n(x) \to x$ as $n \to +\infty$.

Now let us define $\overline{\gamma}_n(x) := \gamma_n(T_n(x))$ so that by definition of μ_n,

$$\int_X \gamma_n(x)\,\mathrm{d}\mu_n = \int_X \overline{\gamma}_n(x)\,\mathrm{d}\mu.$$

By assumption we have that $\overline{\gamma}_n(x) \ge C$ for every $(n,x) \in \mathbb{N}\times X$, and moreover

$$\liminf_{n\to+\infty} \overline{\gamma}_n(x) = \liminf_{n\to+\infty} \gamma_n(T_n(x)) \ge \gamma_\infty(x) \qquad \forall x \in X.$$

Therefore thesis follows applying Fatou's Lemma to the sequence $\{\overline{\gamma}_n\}$ with respect to the fixed measure μ. $\square$

Proof of Lemma 4.5. Analogous to the proof of Lemma 4.4. $\square$

Proof of Lemma 4.6. Since the Mean_q of an n-uple is always greater or equal than the minimum, it is enough to show that

$$\liminf_{n\to+\infty} \min\{\phi_n(y) :\ y \in Y_n\} \ge \inf\{\phi_\infty(y) :\ y \in Y\}.$$

To this end, let us choose a subsequence such that the lim inf is actually a limit, and then let $y_n \in \arg\min\{\phi_n(y) : \ y \in Y_n\}$. Passing if necessary to a subsequence, we can assume that $\{y_n\}$ converges to some y_∞ in Y.

By assumption we have that

$$\lim_{n\to+\infty} \min\{\phi_n(y) : \ y \in Y_n\} = \lim_{n\to+\infty} \phi_n(y_n) \geq \phi_\infty(y_\infty) \geq \inf\{\phi_\infty(y) : \ y \in Y\},$$

which completes the proof. $\square$

Proof of Lemma 4.7. Let $\overline{y}_n \in \arg\min\{\phi_n(y) : \ y \in Y_n\}$ for every $n \in \mathbb{N}$. Since $|Y_n| = n$, we have that

$$\begin{aligned}
\mathrm{Mean}_{-n}\{\phi_n(y) : \ y \in Y_n\} &= -\frac{1}{n}\log\left(\frac{1}{n}\sum_{y\in Y_n} e^{-n\phi_n(y)}\right) \\
&= \phi_n(\overline{y}_n) - \frac{1}{n}\log\left(\frac{1}{n}\sum_{y\in Y_n} e^{-n(\phi_n(y)-\phi_n(\overline{y}_n))}\right) \\
&\leq \phi_n(y_n) - \frac{1}{n}\log\left[\frac{1}{n}\left(1 + \sum_{y\in Y_n\setminus\{\overline{y}_n\}} e^{-n(\phi_n(y)-\phi_n(\overline{y}_n))}\right)\right] \\
&\leq \phi_n(y_n) - \frac{1}{n}\log\left(\frac{1}{n}\right) \ = \ \phi_n(y_n) + \log\sqrt[n]{n}.
\end{aligned}$$

Letting $n \to +\infty$, thesis is proved. $\square$

We point out that our proof of Lemma 4.7 requires that $|Y_n| = n$, or at least that $\sqrt[n]{|Y_n|} \to 1$ as $n \to +\infty$.

The liminf inequality We have to prove that

$$\liminf_{n\to+\infty} \mathcal{G}_n(p_n) \geq \mathcal{G}(p_\infty) \tag{4.10}$$

for every sequence $\{p_n\}$ such that $p_n : Y_n \to [-K, K]$ for every $n \in \mathbb{N}$, and $p_n \to p_\infty$ in the sense of Definition 2.7 (with $Z_n = Y_n$ and $Z = Y$).

Applying Lemma 4.4 with

$$\gamma_n(x) := m_{n,p_n}(x), \qquad \gamma_\infty(x) := m_{p_\infty}(x),$$

we have that inequality (4.10) is proved if we show that

$$\liminf_{n\to+\infty} m_{n,p_n}(x_n) \geq m_{p_\infty}(x_\infty) \tag{4.11}$$

for every converging sequence $x_n \to x_\infty$ in X (the existence of C such that $\gamma_n(x) \geq C$ follows from the equi-boundedness of c_n in $Y \times [-K, K]$).

Now let us define $\phi_n : Y_n \to \mathbb{R}$ and $\phi_\infty : Y \to \mathbb{R}$ by

$$\phi_n(y) := f_{n,p_n}(x_n, y) = c_n(y, p_n(y)) + 5M\Psi_{n,p_n}(x_n, y),$$

$$\phi_\infty(y) := \begin{cases} c(y, p_\infty(y)) & \text{if } y \in A_{p_\infty}(x_\infty), \\ 2M & \text{otherwise.} \end{cases}$$

With these notations we have that

$$m_{n,p_n}(x_n) = \mathrm{Mean}_{-n}\{\phi_n(y) : \ y \in Y_n\}.$$

Moreover, since $\phi_\infty(y) = c(y, p_\infty(y)) \leq M$ for $y \in A_{p_\infty}(x_\infty)$, and $\phi_\infty(y) = 2M$ otherwise, it follows that

$$m_{p_\infty}(x_\infty) = \min\{\phi_\infty(y) : \ y \in Y\}.$$

Applying Lemma 4.6 we have that inequality (4.11) is proved if we show that

$$\liminf_{n\to+\infty} \phi_n(y_n) \geq \phi_\infty(y_\infty) \tag{4.12}$$

for every converging sequence $y_n \to y_\infty$ in Y.

In order to prove (4.12), we first choose a subsequence so that the lim inf is actually a limit, and then for every $n \in \mathbb{N}$ we choose $\overline{y}_n \in A_{n,p_n}(x_n)$. Passing if necessary to a subsequence, we can assume to be in one of the following two cases.

Case 1: $s_n(x_n, y_n, p_n(y_n)) \geq s_n(x_n, \overline{y}_n, p_n(\overline{y}_n)) + 1/n$ for every $n \in \mathbb{N}$. In this case by (ψ4) we have that $\Psi_{n,p_n}(x_n, y_n) = 1$, hence

$$\phi_n(y_n) = c_n(y_n, p_n(y_n)) + 5M \geq 2M \geq \phi_\infty(y_\infty),$$

from which (4.12) trivially follows.

Case 2: $s_n(x_n, y_n, p_n(y_n)) < s_n(x_n, \overline{y}_n, p_n(\overline{y}_n)) + 1/n$ for every $n \in \mathbb{N}$. Arguing as in step 1 of the "lower semicontinuity" part of the proof of Theorem 3.2, we can prove that in this case we have that $y_\infty \in A_{p_\infty}(x_\infty)$, and $p_n(y_n) \to p_\infty(y_\infty)$.

Now since $\phi_n(y_n) \geq c_n(y_n, p_n(y_n))$ for every $n \in \mathbb{N}$, passing to the limit we obtain that

$$\liminf_{n\to+\infty} \phi_n(y_n) \geq \lim_{n\to+\infty} c_n(y_n, p_n(y_n)) = c(y_\infty, p_\infty(y_\infty)) = \phi_\infty(y_\infty).$$

This proves (4.12) also in the second case, and completes the proof of the liminf inequality.

The limsup inequality We have to prove that, for every $p \in \mathcal{P}_K$, there exists a sequence $\{p_n\}$ such that $p_n \in \mathcal{P}_{n,K}$ for every $n \in \mathbb{N}$, $p_n \to p$ in the sense of Definition 2.7 (with $Z_n = Y_n$ and $Z = Y$), and

$$\limsup_{n\to+\infty} \mathcal{G}_n(p_n) \leq \mathcal{G}(p).$$

Thanks to Proposition 2.3 and Theorem 3.3, we can limit ourselves to show that

$$\limsup_{n\to+\infty} \mathcal{G}_n(p_n) \leq \mathcal{F}(p) \tag{4.13}$$

for a suitable sequence $p_n \to p$. To this end, we set

$$p_n(y) := \min\{p(w) : \ w \in Y, \ d_Y(w,y) \leq r(Y_n, Y)\} \qquad \forall y \in Y_n.$$

By statement (1) of Proposition 2.8, we have that $p_n \to p$ in the sense we need. Moreover, applying Lemma 4.5 with

$$\gamma_n(x) := m_{n,p_n}(x), \qquad \gamma_\infty(x) := M_{p_\infty}(x),$$

we have that inequality (4.13) is proved if we show that

$$\limsup_{n\to+\infty} m_{n,p_n}(x_n) \leq M_{p_\infty}(x_\infty) \tag{4.14}$$

for every converging sequence $x_n \to x_\infty$ in X (the existence of C such that $\gamma_n(x) \leq C$ follows from the equi-boundedness of c_n in $Y \times [-K, K]$).

Now let us define $\phi_n : Y_n \to \mathbb{R}$ by

$$\phi_n(y) := f_{n,p_n}(x_n, y) = c_n(y, p_n(y)) + 5M\Psi_{n,p_n}(x_n, y).$$

With these notations we have that

$$m_{n,p_n}(x_n) = \mathrm{Mean}_{-n}\{\phi_n(y) : \ y \in Y_n\}.$$

Applying Lemma 4.7 we have that inequality (4.14) is proved if we show that there exists a sequence $\{y_n\}$ such that $y_n \in Y_n$ for every $n \in \mathbb{N}$, and

$$\limsup_{n\to+\infty} \phi_n(y_n) \leq M_{p_\infty}(x_\infty). \tag{4.15}$$

For every $n \in \mathbb{N}$, let us choose $y_n \in A_{n,p_n}(x_n)$ (this implies in particular that $\Psi_{n,p_n}(x_n, y_n) = 0$). In order to prove (4.15), we choose a subsequence so that the lim sup is actually a limit. Passing if necessary to a subsequence, we can assume that $\{y_n\}$ converges to some y_∞ in Y. Arguing once more as in step 1 of the "lower semicontinuity" part of the proof of Theorem 3.2, we can prove that $y_\infty \in A_{p_\infty}(x_\infty)$, and $p_n(y_n) \to p_\infty(y_\infty)$, and therefore

$$\lim_{n\to+\infty} \phi_n(y_n) = \lim_{n\to+\infty} c_n(y_n, p_n(y_n)) = c(y_\infty, p_\infty(y_\infty)) \leq M_{p_\infty}(x_\infty),$$

where the last inequality follows from the fact that y_∞ is a competitor in the definition of $M_{p_\infty}(x_\infty)$.

This proves (4.15), and completes the proof of the limsup inequality.

5 Counterexamples

In the first example we show a very simple situation (one consumer, one product beyond $y_\star$) where the minimum of $\mathcal{F}$ doesn't exist.

Example 5.1 Let $X = \{x\}$, $Y = \{y_\star, y_1\}$, and for every $\pi \in \mathbb{R}$, let $s(x, y_\star, \pi) = \pi$, $s(x, y_1, \pi) = \pi - 2$, $c(y_\star, \pi) = -\pi$, $c(y_1, \pi) = 1 - \pi$.

All our assumptions (Hp 1) through (Hp 9) are satisfied.

In this case it is easy to see that, when looking for the infimum of $\mathcal{F}$ and $\mathcal{G}$, one can assume that $p(y_\star) = 0$. Under this condition, it turns out that

$$A_p(x) = \begin{cases} \{y_1\} & \text{if } p(y_1) < 2, \\ \{y_\star, y_1\} & \text{if } p(y_1) = 2, \\ \{y_\star\} & \text{if } p(y_1) > 2, \end{cases}$$

hence

$$\mathcal{F}(p) = M_p(x) = \begin{cases} 1 - p(y_1) & \text{if } p(y_1) < 2, \\ 0 & \text{if } p(y_1) \geq 2. \end{cases}$$

Therefore the infimum of $\mathcal{F}$ is -1, but it is not a minimum.

In the second example we show that, if the stress function $s(x, y, \pi)$ is not strictly increasing in π, then the infimum of $\mathcal{F}$ can be strictly greater than the minimum of $\mathcal{G}$.

Example 5.2 Let $X = \{x\}$, $Y = \{y_\star, y_1\}$, and for every $\pi \in \mathbb{R}$, let $c(y_\star, \pi) = -\pi$, $c(y_1, \pi) = 1 - 2\pi$, $s(x, y_\star, \pi) = \pi$, and

$$s(x, y_1, \pi) = \begin{cases} \pi & \text{if } \pi \leq 0, \\ 0 & \text{if } 0 \leq \pi \leq 1, \\ \pi - 1 & \text{if } \pi \geq 1. \end{cases}$$

In this case assumptions (Hp 1) through (Hp 8) are satisfied, but $s(x, y_1, \pi)$ is only nondecreasing in π.

Let us consider the price function $\overline{p}$ with $\overline{p}(y_\star) = 0$ and $\overline{p}(y_1) = 1$. Then $A_{\overline{p}}(x) = \{y_\star, y_1\}$, hence

$$\min\{\mathcal{G}(p) : \; p \in \mathcal{P}\} \leq \mathcal{G}(\overline{p}) = m_{\overline{p}}(x) = -1.$$

Now let us consider $\mathcal{F}(p)$. If $p(y_1) \leq 0$, then

$$\mathcal{F}(p) = M_p(x) \geq \min\{-p(y_\star), 1 - 2p(y_1)\} \geq 0.$$

If $p(y_1) > 0$, then $s(x, y_\star, p(y_\star)) \leq 0 \leq s(x, y_1, p(y_1))$, and therefore $y_\star \in A_p(x)$, hence $\mathcal{F}(p) = M_p(x) \geq -p(y_\star) \geq 0$.

In any case we have that $\mathcal{F}(p) \geq 0$, hence

$$\inf\{\mathcal{F}(p) : \; p \in \mathcal{P}\} \geq 0.$$

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