Existence and uniqueness for a p-Laplacian nonlinear eigenvalue problem

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Abstract

We consider the Dirichlet eigenvalue problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda \|u\|_q^{p-q}|u|^{q-2}u,$$

where the unknowns $u \in W_0^{1,p}(\Omega)$ (the eigenfunction) and $\lambda > 0$ (the eigenvalue), Ω is an arbitrary domain in $\mathbb{R}^N$ with finite measure, $1 < p < \infty$, $1 < q < p^*$, $p^* = Np/(N-p)$ if $1 < p < N$ and $p^* = \infty$ if $p \geq N$. We study several existence and uniqueness results as well as some properties of the solutions. Moreover, we indicate how to extend to the general case some proofs known in the classical case $p = q$.

1 Introduction

Let Ω be a domain (i.e., a connected open set) in $\mathbb{R}^N$ with finite measure, $1 < p < \infty$, $1 < q < p^*$ where $p^* = Np/(N-p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$. It is well-known that the Sobolev space $W_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ and that

$$\lambda_1(p,q) := \inf_{u \in W_0^{1,p}(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u|^p dx}{(\int_\Omega |u|^q dx)^{p/q}} > 0. \tag{1.1}$$

It is also well-known that the Rayleigh quotient in (1.1) admits a minimizer which does not change sign in Ω. The Euler-Lagrange equation associated with this minimization problem is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda \|u\|_q^{p-q}|u|^{q-2}u, \tag{1.2}$$

where $\|u\|_q$ denotes the norm of u in $L^q(\Omega)$. Usually in the literature, the function u is normalized in order to get rid of the apparently redundant factor $\|u\|_q^{p-q}$. However, we prefer to keep it since it allows to think of this problem as an eigenvalue problem. Indeed, (1.2) is homogeneous; i.e., if u is a solution then ku is also a solution for all $k \in \mathbb{R}$, as one would expect from an eigenvalue problem. It turns out that $\lambda_1(p,q)$ is the smallest eigenvalue of (1.2) and we refer to it as the first eigenvalue.

The case $p = q$ has been largely investigated by many authors and it has been often considered as a *typical eigenvalue problem* (cf. e.g., García

Azorero and Peral Alonso [1]); for extensive references on this subject we refer to Lindqvist [17]. For the case $p \neq q$ we refer again to [1], Ôtani [20, 21] and Drábek, Kufner and Nicolosi [5] who consider an even more general class of *nonhomogeneous eigenvalue problems*.

In this paper, we study several results and we indicate how to adapt to the general case some proofs known in the case $p = q$.

First of all, we discuss the simplicity of $\lambda_1(p,q)$. We recall that $\lambda_1(p,q)$ is simple if $q \leq p$, as it is proved in Idogawa and Ôtani [11]. If $q > p$ then $\lambda_1(p,q)$ is not necessarily simple: for example, simplicity does not hold if Ω is a sufficiently thin annulus, see Kawohl [12] and Nazarov [19]. However, if Ω is a ball then the simplicity of $\lambda_1(p,q)$ is guaranteed also in the case $q > p$: here we briefly describe the argument of Erbe and Tang [7].

By adapting the argument of Kawohl and Lindqvist [13], we prove that if $q \leq p$ then the only eigenvalue admitting a non-negative eigenfunction is the first one.

Moreover, by exploiting our point of view, we also give an alternative proof of a uniqueness result of Drábek [4, Thm. 1.1] for the equation $-\Delta_p u = |u|^{q-2}u$, see Theorem 4.4.

Finally, in the general case $1 < q < p^*$, we observe that the point spectrum $\sigma(p,q)$ is closed as in the case $p = q$ considered in [17] and we indicate how to apply the Ljusternik-Schnirelman min-max procedure in order to define a divergent sequence of eigenvalues $\lambda_n(p,q)$, $n \in \mathbb{N}$. Note that the existence of infinitely many solutions to equation (1.2) is also proved in [1] where the cases $q < p$ and $q > p$ are treated separately; instead, here we adopt a unified approach.

We point out that in this paper we do not assume that Ω is bounded as largely done in the literature, but only that its measure is finite.

2 The eigenvalue problem

Let Ω be a domain in $\mathbb{R}^N$ with finite measure and $1 < p < \infty$. By $W^{1,p}(\Omega)$ we denote the Sobolev space of those functions in $L^p(\Omega)$ with first order weak derivatives in $L^p(\Omega)$ endowed with its usual norm. By $W_0^{1,p}(\Omega)$ we denote the closure in $W^{1,p}(\Omega)$ of the C^∞-functions with compact support in Ω.

It is well-known that the Poincaré inequality holds. Namely, for every $1 < q < p^*$ there exists $C > 0$ depending only on N, p, q such that

$$\|u\|_{L^q(\Omega)} \leq C|\Omega|^{\frac{1}{q}-\frac{1}{p}+\frac{1}{N}}\|\nabla u\|_{L^p(\Omega)}, \tag{2.1}$$

for all $u \in W_0^{1,p}(\Omega)$. In particular it follows that $\lambda_1(p,q)$ defined in (1.1) is positive and satisfies the inequality

$$\lambda_1(p,q) > \frac{1}{C^p|\Omega|^{p(\frac{1}{q}-\frac{1}{p}+\frac{1}{N})}}\,. \tag{2.2}$$

Moreover, since the measure of Ω is finite, the embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ is compact: this combined with the reflexivity of $W_0^{1,p}(\Omega)$ guarantees the existence of a minimizer in (1.1). As we mentioned in the introduction, equation (1.2) is the Euler-Lagrange equation corresponding to the minimization problem (1.1). It is then natural to give the following definition where, as usual, equation (1.2) is interpreted in the weak sense.

Definition 2.1. Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1 < p < \infty$ and $1 < q < p^*$. We say that $\lambda > 0$ is an eigenvalue of equation (1.2) if there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx = \lambda \|u\|_q^{p-q} \int_\Omega |u|^{q-2} u \varphi dx, \tag{2.3}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. The eigenfunctions corresponding to λ are the solutions u to (2.3).

It is clear that all eigenvalues are positive and that $\lambda_1(p,q)$ is the least eigenvalue. Moreover, the eigenfunctions corresponding to $\lambda_1(p,q)$ are exactly the minimizers in (1.1). We recall the following known result.

Theorem 2.2. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1 < p < \infty$ and $1 < q < p^*$. Let $\lambda > 0$ be an eigenvalue of equation (2.3) and $u \in W_0^{1,p}(\Omega)$ be a corresponding eigenfunction. Then u is bounded and its first derivatives are locally Hölder continuous. Moreover, if $u \geq 0$ in Ω then $u > 0$ in Ω.*

As done in [17, Lemma 5.2] for the case $q = p$, the boundedness of u can be proved by using the method of [15, Lemma 5.1]. The Hölder regularity of the first order derivatives follows by Tolksdorf [24]. We note that the argument in [17] allows to give a quantitative bound for u. Namely, by a slight modification of [17, Lemma 5.2] one can prove that there exists a constant $M > 0$, depending only on p, q, N, such that

$$\|u\|_{L^\infty(\Omega)} \leq M \lambda^{\frac{1}{\delta p}} \|u\|_{L^1(\Omega)}, \tag{2.4}$$

where $\delta = 1/N$ if $q \leq p$ and $\delta = 1/q - 1/p + 1/N$ if $q > p$. We refer to Franzina [8] for details. Finally, the fact that a non-negative eigenfunction does not vanish in Ω can be deduced by the strong maximum principle in García-Meliàn and Sabina de Lis [9, Theorem 1].

Corollary 2.3. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1 < p < \infty$ and $1 < q < p^*$. Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ be an eigenfunction corresponding to $\lambda_1(p,q)$. Then either $u > 0$ or $u < 0$ in Ω.*

Proof. Clearly u is a minimizer in (1.1). Then also $|u|$ is a minimizer, hence a first eigenfunction. Thus by Theorem 2.2 $|u|$ cannot vanish in Ω. □

3 On the simplicity of $\lambda_1(p,q)$

It is known that if $q \leq p$ then $\lambda_1(p,q)$ is simple. In fact we have the following theorem by Idogawa and Ôtani [11, Theorem 4] the proof of which works word by word also when Ω is not bounded.

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure and $1 < q \leq p < \infty$. Then $\lambda_1(p,q)$ is simple; i.e., the eigenfunctions corresponding to $\lambda_1(p,q)$ define a linear space of dimension one.*

We refer to Kawohl, Lucia and Prashanth [14] for a recent generalization of the previous result to some indefinite quasilinear problems. For the case $p = q$ we refer to Lindqvist [18].

In general, Theorem 3.1 does not hold if $q > p$; see [12] and [19] where the case of a sufficiently thin annulus is considered. However, as one may expect, if Ω is a ball then $\lambda_1(p,q)$ is simple. Basically, this depends on the following theorem, cf. [12].

Theorem 3.2. *Let Ω be a ball in $\mathbb{R}^N$ centered at zero, $1 < p < \infty$ and $1 < q < p^*$. Then the eigenfunctions corresponding to $\lambda_1(p,q)$ are radial functions.*

Theorem 3.2 allows to pass to spherical coordinates and to reduce our problem to an ODE as follows. If Ω is a ball centered at zero and u is a radial function, $u(x) = \phi(|x|)$, then

$$-\Delta_p u = -(p-1)|\phi'(r)|^{p-2}\phi''(r) - \frac{N-1}{r}|\phi'(r)|^{p-2}\phi'(r),$$

which is well-defined for all $r > 0$ such that ϕ is twice differentiable in r. Recall that by standard regularity theory an eigenfunction u is twice differentiable on the set $\{x \in \Omega : \nabla u(x) \neq 0\}$. By writing (1.2) in spherical coordinates and multiplying both sides by r^{N-1} it follows that if $u = \phi(|x|)$ is a radial eigenfunction corresponding to the eigenvalue λ and $\|u\|_{L^q(\Omega)} = 1$ then

$$-(r^{N-1}|\phi'|^{p-2}\phi')' = \lambda r^{N-1}|\phi|^{q-2}\phi. \tag{3.1}$$

If in addition u is a first eigenfunction then u does not change sign in Ω; thus, by integrating equation (3.1), one can easily prove that ϕ' vanishes only at $r = 0$. Hence ϕ is twice differentiable for all $r > 0$ and (3.1) is satisfied in the classical sense for all $r > 0$.

To prove the simplicity of $\lambda_1(p,q)$ we use the following Lemma. The proof is more or less standard (further details can be found in Franzina [8]).

Lemma 3.3. *Let $1 < p < \infty$, $1 < q < p^*$ and $\lambda, c > 0$. Then the Cauchy problem*

$$\begin{gathered} -(r^{N-1}|\phi'|^{p-2}\phi')' = \lambda r^{N-1}|\phi|^{q-2}\phi, \quad r \in (0,R), \\ \phi(0) = c, \quad \phi'(0) = 0, \end{gathered} \tag{3.2}$$

has at most one positive solution ϕ in $C^1[0,R] \cap C^2(0,R)$.

Proof. We consider the operator T of $C[0,R]$ to $C[0,R]$ defined by

$$T(\phi)(r) = c - \int_0^r g^{-1}\Big(\frac{\lambda}{t^{N-1}}\int_0^t s^{N-1}|\phi|^{q-2}\phi\, ds\Big)\, dt, \quad r \in [0,R], \tag{3.3}$$

for all $\phi \in C[0,R]$, where $g(t) = |t|^{p-2}t$ if $t \neq 0$ and $g(0) = 0$ and g^{-1} denotes the inverse function of g. It's easily seen that every positive solution to the Cauchy problem (3.2) is a fixed point of the operator T of class $C^1[0,R] \cap C^2(0,R)$.

Now let $\phi_1, \phi_2 \in C^1[0,R] \cap C^2(0,R)$ be two positive solutions to problem (3.2). One can prove that there exists $\epsilon_1 > 0$ such that

$$\|T(\phi_1) - T(\phi_2)\|_{C[0,\varepsilon]} \leqslant C_1(\varepsilon)\|\phi_1 - \phi_2\|_{C[0,\varepsilon]},$$

for all $\varepsilon \in [0, \epsilon_1]$ where $C_1(\varepsilon) < 1$. It follows that $\phi_1 = \phi_2$ in a neighborhood of zero. Furthermore, let $R_0 = \sup\{\varepsilon > 0 : \phi_1 = \phi_2 \text{ on } [0,\varepsilon]\}$. Arguing by contradiction, assume that $R_0 < R$. Then one can prove that there exists $0 < \epsilon_2 < R - R_0$ such that

$$\|T(\phi_1) - T(\phi_2)\|_{C[R_0,R_0+\varepsilon]} \leqslant C_2(\varepsilon)\|\phi_1 - \phi_2\|_{C[R_0,R_0+\varepsilon]},$$

for all $\varepsilon \in [0, \epsilon_2]$, where $C_2(\varepsilon) < 1$. This implies that $\phi_1 = \phi_2$ in a neighborhood of R_0, a contradiction. □

We point out that Lemma 3.3 does not immediately imply that $\lambda_1(p,q)$ is simple in a ball; if $N > 1$ further technical work is required and the main step is the following Lemma for which we refer to Erbe and Tang [7, Lemma 3.1].

Lemma 3.4. *Let $N > 1$, $1 < p < \infty$, $1 < q < p^*$ and $c_1, c_2 > 0$. Let $\phi_1, \phi_2 \in C^1[0,R] \cap C^2(0,R)$ be two positive solutions to the Cauchy problem (3.2) with $c = c_1, c_2$ respectively. If $c_1 \leq c_2$ then $\phi_1 \leq \phi_2$.*

By using Lemmas 3.3 and 3.4 we can deduce the validity of the following result.

Theorem 3.5. *Let Ω be a ball in $\mathbb{R}^N$, $1 < p < \infty$ and $1 < q < p^*$. Then $\lambda_1(p,q)$ is simple.*

Proof. For the case $N = 1$ we refer to [21, Theorem I]. Assume now that $N > 1$ and that Ω is a ball of radius R centered at zero. Let u_1, u_2 be two nonzero eigenfunctions corresponding to the first eigenvalue $\lambda_1(p,q)$. We have to prove that u_1 and u_2 are proportional. To do so we can directly assume that $\|u_1\|_q = \|u_2\|_q = 1$. Moreover, by Corollary 2.3 we can assume without loss of generality that $u_1, u_2 > 0$ on Ω. By Theorem 3.2 u_1, u_2 are radial functions hence they can be written as $u_1 = \phi_1(|x|)$, $u_2 = \phi_2(|x|)$ for suitable positive functions $\phi_1, \phi_2 \in C^1[0,R] \cap C^2(0,R)$ satisfying condition $\phi_1'(0) = \phi_2'(0) = 0$ and equation (3.1) with $\lambda = \lambda_1(p,q)$. If $\phi_1(0) \neq \phi_2(0)$, say $\phi_1(0) < \phi_2(0)$, then by Lemma 3.4 $\phi_1 \leq \phi_2$ in Ω hence $\|u_1\|_q < \|u_2\|_q$, since by continuity $u_1 < u_2$ in a neighborhood of zero. A contradiction. Thus $\phi_1(0) = \phi_2(0)$, hence by Lemma 3.3 $\phi_1 = \phi_2$ or equivalently $u_1 = u_2$. □

4 Further uniqueness results

By Corollary 2.3 the first eigenvalue admits a non-negative eigenfunction. It is well-known that no other eigenvalues enjoy this property when $p = q$. This can be proved also in the case $q \le p$. The proof of the following theorem exploits an argument used by Ôtani and Teshima [22] in the case of bounded smooth open sets combined with an argument of Lindqvist and Kawhol [13] which allows to deal with rough boundaries.

Theorem 4.1. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure and $1 < q \le p$. If λ is an eigenvalue of* (1.2) *admitting a positive eigenfunction then $\lambda = \lambda_1(p,q)$.*

Proof. We argue by contradiction and assume that $\lambda_1(p,q) < \lambda$. Let $u_1 \in W_0^{1,p}(\Omega) \setminus \{0\}$ be a positive eigenfunction corresponding to $\lambda_1 = \lambda_1(p,q)$ and u be a positive eigenfunction corresponding to λ. We directly assume that $u_1 \le u$ in Ω, otherwise one can use the approximation argument of [13] (which works also in the case of unbounded domains). Since $q \le p$ it follows that for all nonnegative test functions φ,

$$
\begin{aligned}
\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\varphi dx &= \lambda \|(\lambda_1/\lambda)^{\frac{1}{p-1}} u_1\|_q^{p-q} \int_\Omega \left((\lambda_1/\lambda)^{\frac{1}{p-1}} u_1\right)^{q-1} \varphi dx \\
&\le \lambda \|\eta u\|_q^{p-q} \int_\Omega (\eta u)^{q-1} \varphi dx \\
&= \int_\Omega |\nabla \eta u|^{p-2} \nabla \eta u \cdot \nabla \varphi dx,
\end{aligned}
\tag{4.1}
$$

where $\eta = (\lambda_1/\lambda)^{\frac{1}{p-1}}$. By choosing $\varphi = \max\{u_1 - \eta u, 0\}$ in (4.1) and using the argument in the proof of [22, Lemma 3] (see also [13]) we deduce that $u_1 \le \eta u$ and by iteration $u_1 \le \eta^n u$ for all $n \in \mathbb{N}$. Since $0 < \eta < 1$, by passing to the limit as $n \to \infty$ we obtain $u_1 = 0$, a contradiction. □

By Theorem 4.1 we deduce the validity of the following corollary which is well-known in the case of bounded smooth domains (cf. e.g. Huang [10]).

Corollary 4.2. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure and $1 < q < p$. The equation*

$$
-\Delta_p v = |v|^{q-2} v \tag{4.2}
$$

has a unique positive solution in $W_0^{1,p}(\Omega) \setminus \{0\}$.

Proof. Existence follows immediately by observing that if u is a nonzero eigenfunction of (1.2) then

$$
v = \frac{u}{\lambda^{\frac{1}{p-q}} \|u\|_q}
$$

is a solution to (4.2), hence the first eigenfunction provides a positive solution to (4.2). We now prove uniqueness. Observe that if $v \ne 0$ is a solution to (4.2) then v is an eigenfunction corresponding to the eigenvalue

$$
\lambda = \frac{1}{\|v\|_q^{p-q}}. \tag{4.3}
$$

Accordingly, by Theorem 4.1 two positive solutions v_1, v_2 of (4.2) would be eigenfunctions corresponding to $\lambda_1(p,q)$. Thus such solutions would be proportional by Theorem 3.1. Since $p \neq q$ proportionality implies coincidence and then $v_1 = v_2$. □

We recall that the solutions to (4.2) are exactly the critical points of the functional

$$J(v) := \frac{1}{p}\int_\Omega |\nabla v|^p dx - \frac{1}{q}\int_\Omega |v|^q dx,$$

defined for all $v \in W_0^{1,p}(\Omega)$. The functional J can be used to give a condition equivalent to the simplicity of $\lambda_1(p,q)$. In fact, we have the following

Lemma 4.3. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1 < p < \infty$ and $1 < q < p^*$ with $q \neq p$. Let S_{pq} be the set of all nontrivial solutions to (4.2). If $w \in S_{pq}$ is a point of minimum for the restriction of J to S_{pq} then w is an eigenfunction corresponding to $\lambda_1(p,q)$. In particular, $\lambda_1(p,q)$ is simple if and only if the restriction of J to S_{pq} has a unique (up to the sign) point of minimum.*

Proof. Note that if $v \in S_{pq}$ then $\int_\Omega |\nabla v|^p dx = \int_\Omega |v|^q dx$. Thus

$$J(v) = \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega |v|^q dx,$$

for all $v \in S_{pq}$. Moreover, $v \in S_{pq}$ if and only if v is an eigenfunction of equation (1.2) corresponding to an eigenvalue λ satisfying (4.3). It follows that a function $w \in S_{pq}$ minimizes the restriction $J_{|S_{pq}}$ of J to S_{pq} if and only if w minimizes the functional defined on S_{pq} by (4.3). In particular, if $w \in S_{pq}$ minimizes $J_{|S_{pq}}$ and $\lambda_1(p,q)$ is simple then $w = ku$, $k \in \mathbb{R}$ where u is the eigenfunction corresponding to $\lambda_1(p,q)$ uniquely determined by the conditions $u > 0$ in Ω and $\|u\|_q = 1$; moreover, since ku satisfies equation (4.2) then $k = \pm\lambda_1(p,q)^{\frac{1}{q-p}}$, hence w is uniquely determined up to the sign. To conclude the proof it suffices to observe that if v is an eigenfunction corresponding to $\lambda_1(p,q)$ and satisfies (4.3) with $\lambda = \lambda_1(p,q)$ then v minimizes $J_{|S_{pq}}$. □

By Lemma 4.3 and Theorem 3.1 we deduce the following result of Drábek [4, Thm. 1.1] for the case $1 < q < p$.

Theorem 4.4. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure and $1 < q < p$. Equation (4.2) has a unique (up to the sign) nontrivial solution $w \in W_0^{1,p}(\Omega)$ with the following property: $J(w) \leq J(v)$ if $v \in W_0^{1,p}(\Omega)$ is a nontrivial solution to equation (4.2). Moreover, w is an eigenfunction corresponding to $\lambda_1(p,q)$.*

Proof. The proof follows immediately by Lemma 4.3 and by observing that by Theorem 3.1 $\lambda_1(p,q)$ is simple since $q < p$. □

5 On the spectrum $\sigma(p,q)$

We denote by $\sigma(p,q)$ the set of all the eigenvalues of (1.2). We refer to $\sigma(p,q)$ as the spectrum of the p-Laplacian.

The following result is well-known in the case $p=q$: the proof given in [17, Theorem 5.1] does not require any significant modification.

Theorem 5.1. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1<p<\infty$ and $1<q<p^*$. Then $\sigma(p,q)$ is a closed set.*

As in the case $p=q$, it is possible to produce an infinite sequence of eigenvalues by means of a min-max procedure which generalizes the well-known min-max Courant principle. Namely, for all $n\in\mathbb{N}$ we set

$$\lambda_n(p,q)=\inf_{\mathcal{M}\in\mathfrak{M}(p,q)}\sup_{u\in\mathcal{M}}\frac{\int_\Omega|\nabla u|^p dx}{\left(\int_\Omega|u|^q dx\right)^{p/q}}, \tag{5.1}$$

where $\mathfrak{M}(p,q)$ is the family of those conic subsets $\mathcal{M}$ of $W_0^{1,p}(\Omega)\setminus\{0\}$, whose intersection with the unit sphere of $L^q(\Omega)$ is compact in $W_0^{1,p}(\Omega)$ and whose Krasnoselskii's genus $\gamma(\mathcal{M})$ is greater than or equal to n. Recall that

$$\gamma(\mathcal{M})=\min\left\{k\in\mathbb{N}:\exists F\in C\left(\mathcal{M},\mathbb{R}^k\setminus\{0\}\right),F(f)=-F(-f)\,\forall f\in\mathcal{M}\right\}, \tag{5.2}$$

where $C(\mathcal{M},\mathbb{R}^k\setminus\{0\})$ denotes the space of all continuous functions of $\mathcal{M}$ to $\mathbb{R}^k\setminus\{0\}$. It is understood that $\gamma(\mathcal{M})=\infty$ if the set in the right-hand side of (5.2) is empty.

The following theorem is proved by applying the abstract result of Szulkin [23, Cor. 4.1, p. 132] as done in Cuesta [3, Prop. 4.5, p. 85] where one can find a detailed proof for the case $p=q$.

Theorem 5.2. *Let Ω be domain in $\mathbb{R}^N$ with finite measure, $1<p<\infty$ and $1<q<p^*$. Then $\lambda_n(p,q)\in\sigma(p,q)$ and $\lim_{n\to\infty}\lambda_n(p,q)=\infty$.*

Proof. Let I and E be the functions of $W_0^{1,p}(\Omega)$ to $\mathbb{R}$ defined by

$$I(u)=\Big(\int_\Omega|u|^q dx\Big)^{p/q},\quad E(u)=\int_\Omega|\nabla u|^p dx,$$

for all $u\in W_0^{1,p}(\Omega)$ and let $M=\{u\in W_0^{1,p}(\Omega):\ I(u)=1\}$. Note that I and E are of class C^1 and that M is a closed submanifold of $W_0^{1,p}(\Omega)$ of codimension one whose tangent space at a point u is given by $T_uM=\ker d_uI$. It is clear that the eigenvalues λ of (1.2) are exactly the critical levels of E restricted to M; i.e., are those real numbers λ for which there exists $u\in M$ such that $E(u)=\lambda$ and $T_uM\subset\ker d_uE$. It is not difficult to adapt the argument in Cuesta [3] to prove that E satisfies the well-known Palais-Smale condition on M. Thus, by applying [23, Cor. 4.1, p. 132] to the functions I, E it follows that the numbers $\lambda_n(p,q)$ are critical levels of E restricted to M.

It remains to prove that $\lim_{n\to\infty}\lambda_n(p,q)=\infty$. To do so we use an argument in Zeidler [25, Ch. 44]. First of all, we recall that by [25, Lemma 44.32] for every $n\in\mathbb{N}$ there exist a finite-dimensional subspace X_n of $W_0^{1,p}(\Omega)$ and an odd continuous operator P_n of $W_0^{1,p}(\Omega)$ to X_n such that for every $u\in W_0^{1,p}(\Omega)$ we have that $\|P_nu\|_{W_0^{1,p}(\Omega)}\le\|u\|_{W_0^{1,p}(\Omega)}$ and P_nu_n converges weakly in $W_0^{1,p}(\Omega)$ to u for all sequences u_n, $n\in\mathbb{N}$ weakly convergent in $W_0^{1,p}(\Omega)$ to u. Clearly, it suffices to prove that for any fixed $L>0$ there exists $n\in\mathbb{N}$ such that $\sup_{u\in\mathcal{A}}E(u)>L$ for all symmetric subsets $\mathcal{A}$ of M such that $\mathcal{A}$ is compact in $W_0^{1,p}(\Omega)$ and $\gamma(\mathcal{A})\ge n$. Assume to the contrary that there exists $L>0$ such that this is not the case and set $B_L=\{u\in M:\ E(u)\le L\}$. By means of a simple contradiction argument one can prove that there exists $n_L\in\mathbb{N}$ such that $\inf_{u\in B_L}\|P_{n_L}u\|_{W_0^{1,p}(\Omega)}>0$ hence $P_{n_L}u\neq 0$ for all $u\in B_L$. Let $k_L=\dim X_{n_L}+1$. By assumption there exists a symmetric subset $\mathcal{A}$ of M such that $\mathcal{A}$ is compact, $\gamma(\mathcal{A})\ge k_L$ and $\sup_{u\in\mathcal{A}}E(u)\le L$. Since $\mathcal{A}\subset B_L$ then $P_{n_L}u\neq 0$ for all $u\in\mathcal{A}$ hence $\gamma(P_{n_L}(\mathcal{A}))\ge k_L$; on the other hand $P_{n_L}(\mathcal{A})\subset X_{n_L}$ hence $\gamma(P_{n_L}(\mathcal{A}))\le\dim X_{n_L}=k_L-1$, a contradiction. □

We remark that, despite the results of Binding and Rynne [2] who have recently provided examples of nonlinear eigenvalue problems for which not all eigenvalues are variational, it is not clear yet whether for our problem the variational eigenvalues exhaust the spectrum if $N>1$, not even in the classical case $p=q$. However, a complete description of $\sigma(p,q)$ is available for $N=1$, see Ôtani [21] and Drábek and Manásevich [6].

The following theorem is a restatement of [6, Theorems 3.1, 4.1]. We include a detailed proof of (5.3) for the convenience of the reader. Recall that the function defined by

$$\mathrm{arcsin}_{pq}(t)=\frac{q}{2}\int_0^{\frac{2t}{q}}\frac{ds}{(1-s^q)^{\frac{1}{p}}}\,,$$

for all $t\in[0,q/2]$, is a strictly increasing function of $[0,q/2]$ onto $[0,\pi_{pq}/2]$ where $\pi_{pq}=2\,\mathrm{arcsin}_{pq}(q/2)=B(1/q,1-1/p)$ and B denotes the Euler Beta function. The inverse function of arcsin_{pq}, which is denoted by $\sin_{pq}$, is extended to $[-\pi_{pq},\pi_{pq}]$ by setting $\sin_{pq}(\theta)=\sin_{pq}(\pi_{pq}-\theta)$ for all $\theta\in]\pi_{pq}/2,\pi_{pq}]$, $\sin_{pq}(\theta)=-\sin_{pq}(-\theta)$ for all $\theta\in[-\pi_{pq},0[$, and then it is extended by periodicity to the whole of $\mathbb{R}$.

Theorem 5.3. *If $N=1$ and $\Omega=(0,a)$ with $a>0$ then*

$$\lambda_1(p,q)=q^{\frac{p(1-q)}{q}}\Big(\frac{2\pi_{pq}}{a^{\frac{1}{q}-\frac{1}{p}+1}}\Big)^p\Big(1-\frac{1}{p}\Big)\Big(\frac{1}{q}-\frac{1}{p}+1\Big)^{\frac{p-q}{q}}, \tag{5.3}$$

and $\lambda_n(p,q)=n^p\lambda_1(p,q)$ for all $n\in\mathbb{N}$. Moreover, $\sigma(p,q)=\{\lambda_n(p,q):\ n\in\mathbb{N}\}$, $\lambda_n(p,q)$ is simple for all $n\in\mathbb{N}$ and the corresponding eigenspace is spanned by the function $u_n(x)=\sin_{pq}\big(\frac{n\pi_{pq}}{a}x\big)$, $x\in(0,a)$.

Proof. Let u be an eigenfunction corresponding to $\lambda_1(p,q)$ with $u>0$ on $(0,a)$ and $\|u\|_{L^q(0,a)}=1$. By [21, Lemma 2.5] it follows that

$$\frac{p-1}{p}|u'(x)|^p+\frac{\lambda_1(p,q)}{q}|u(x)|^q=\frac{p-1}{p}|u'(0)|^p, \tag{5.4}$$

for all $x\in(0,a)$. Recall that $u'(x)>0$ for all $x\in(0,a/2)$ and $u'(a/2)=0$. Thus, by setting $y=u(x)/u(a/2)$ and by means of a change if variables in integrals it follows by (5.4) that

$$\begin{aligned} a&=2\int_0^{a/2}dx\\ &=2\Big(\frac{q(p-1)}{\lambda_1(p,q)p}\Big)^{1/q}|u'(0)|^{p/q-1}\int_0^1(1-y^q)^{-1/p}dy\\ &=\frac{2}{q}\Big(\frac{q(p-1)}{\lambda_1(p,q)p}\Big)^{1/q}|u'(0)|^{p/q-1}\pi_{pq}.\end{aligned} \tag{5.5}$$

By integrating (5.4) and recalling that $\lambda_1(p,q)=\|u'\|^p_{L^p(0,a)}$ it follows that

$$\Big(\frac{p-1}{p}+\frac{1}{q}\Big)\lambda_1(p,q)=\frac{p-1}{p}|u'(0)|^pa. \tag{5.6}$$

By combining (5.5) and (5.6) we deduce (5.3).

By [21, Proposition 4.2, Theorem II], one can easily deduce that $\sigma(p,q)=\{n^p\lambda_1(p,q):\ n\in\mathbb{N}\}$ which, combined with the argument used in [3, Proposition 4.6] for the case $p=q$, allows to conclude that $\lambda_n(p,q)=n^p\lambda_1(p,q)$. For the proof of the last part of the statement we refer to [6]. □

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