Asymptotics of an optimal compliance-network problem

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Abstract

We consider the problem of the optimal location of a Dirichlet region in a d-dimensional domain Ω subjected to a given force f in order to minimize the p-compliance of the configuration. We look for the optimal region among the class of all closed connected sets of assigned length l. Then we let l tends to infinity and we look at the Γ-limit of a suitable rescaled functional, in order to get information of the asymptotic distribution of the optimal region. We highlight as well the case where the Dirichlet region is searched among discrete sets of finite cardinality.

Introduction

We consider the problem of finding the best location of the Dirichlet region Σ in a d-dimensional domain Ω associated to an elliptic equation in divergence form, namely

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in} \Sigma \cup \partial\Omega, \end{cases}$$

where f is a nonnegative function belonging in $L^q(\Omega)$, q being the conjugate exponent of p and $\Delta_p u$ stands for $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$.We are interested in the minimization of the p-compliance functional defined by

$$C(\Sigma) = \int_\Omega f u_{f,\Sigma,\Omega} dx,$$

where $u_{f,\Sigma,\Omega}$ stands for the unique solution of the above equation. The admissible class of control variables Σ we consider here is the class of all closed connected sets with given one dimensional Hausdorff. It is easy to obtain the optimal configuration Σ_l of the above optimization problem (see Theorem 1) as a consequence of Šverák result (see e.g. [4], [11]). we are interested in the asymptotic behavior of Σ_l as $l \to +\infty$; more precisely we want to obtain the limit distribution of Σ_l as a limit probability measure that minimizes the Γ-limit functional of the suitable rescaled p-compliance functional. In the literature, they are similar results among them we may cite location problems studied in [2], irrigation problems in [10] and compliance in [6]. This problem has been considered in the case of dimension 2 in [5] and the result is exactly as the one in Theorem 2 provided $d = 2$. This is an extension of this result in higher dimension i.e. $d \geq 3$, assuming that $p > d - 1$. The proofs follow the guidelines of [5] and difficulties are mainly of technical nature. In [5], the two dimensional setting has been used in the proofs of Γ-$\liminf$ and Γ-$\limsup$ inequalities. More precisely, the Lemma 1 in [5] (analogous of Lemma 3 here) which is crucial for the Γ-$\liminf$ inequality, follows from the classical Poincaré's inequality wile here we consider the Poincaré 's inequality using the notion of p-capacity. for the construction of the recovering sequence of the Γ-$\limsup$ inequality, the analogous of the Lemma 7 is enough in the case where the dimension is two. The case of higher dimension require more since the boundary of the unit cube is not an one dimensional set. To overcome this difficulty, we prove an other result (Lemma 8) which study the difference between two solutions of the p-Laplacian equation with two different Dirichlet conditions. This result together with Lemma 7 are sufficient for the construction of the recovering sequence. In the last section, we deal with the case where the Dirichlet region is searched among the class of discrete sets of a finite numbers of elements under the assumption that $p > d$. This problem is in connection with the location problem studied in [2].

1 The p-compliance under length constraint

Let $p > d - 1$ be fixed and $q = p/(p-1)$ the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}^d$ and l a positive given real number, we define

$$\mathcal{A}_l(\Omega) = \{\Sigma \subset \overline{\Omega}, \quad \text{closed and connected}, \quad 0 < \mathcal{H}^1(\Sigma) \leq l\}.$$

For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity, we denote by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases}$$

that is $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega \setminus \Sigma). \tag{1}$$

By the maximum principle, the nonnegativity of the function f implies that of u. For $f \geq 0$, we define the p-compliance functional as follows:

$$\begin{aligned} C(\Sigma) = F_p(\Sigma, f, \Omega) &= \int_\Omega f u_{f,\Sigma,\Omega} dx = \int_\Omega |\nabla u_{f,\Sigma,\Omega}|^p dx \\ &= q \max \left\{ \int_\Omega (v - \frac{1}{p} |\nabla v|^p) dx : v \in W_0^{1,p}(\Omega \setminus \Sigma) \right\}, \end{aligned}$$

where q stands for the conjugate exponent of p. The existence of the minimal p-compliance configuration is just a consequence of a generalized Šverák compactness-continuity result (see [4]).

Theorem 1 *For any real number $l > 0$, Ω bounded open subset of $\mathbb{R}^d$, $d \geq 2$ and f a nonnegative function belonging to $L^q(\Omega)$, the problem*

$$\min\{C_p(\Sigma) : \Sigma \in \mathcal{A}_l(\Omega)\} \tag{2}$$

admits at least one solution.

Here we are interested to the asymptotic behavior of the optimal set Σ_l of the problem (2) as $l \to +\infty$. Let us associate to every $\Sigma \in \mathcal{A}_l(\Omega)$ a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and define a functional $F_l : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$F_l(\mu) = \begin{cases} l^{\frac{q}{d-1}} C_p(\Sigma) & \text{if } \mu = \mu_\Sigma, \Sigma \in \mathcal{A}_l(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

The scaling factor $l^{\frac{q}{d-1}}$ is needed in order to avoid the functional to degenerate to the trivial limit functional which vanishes everywhere. Our main result deals with the behavior as $l \to +\infty$ of the functional F_l, and we state it in terms of Γ-convergence.

Theorem 2 *The functional F_l defined in* (3) *Γ-converges, with respect to the weak* topology on the class $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$, to the functional F defined on $\mathcal{P}(\overline{\Omega})$ by*

$$F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx, \tag{4}$$

where μ_a stands for the density of the absolutely continuous part of μ with respect to the Lebesgue measure, and θ is a positive constant depending only on d and p and is defined by

$$\theta = \inf\{\liminf_{l\to+\infty} l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) : \Sigma_l \in \mathcal{A}_l(I^d)\} \tag{5}$$

$I^d = (0,1)^d$ being the unit cube in $\mathbb{R}^d$.

According to the general theory of Γ-convergence (see [7]), we deduce the following consequence of Theorem 2:

- if Σ_l is a solution of the minimization problem (2), then up to a subsequence $\mu_{\Sigma_l} \rightharpoonup \mu$ as $l \to +\infty$, where μ is a minimizer of F;
- since F has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, the whole sequence μ_{Σ_l} converges to the unique minimizer μ of F given by $\mu = cf^{\frac{q(d-1)}{q+d-1}}\mathcal{L}^d$ where c is such that μ is a probability measure, that is $c = 1/\left(\int_\Omega f^{\frac{q(d-1)}{q+d-1}} dx\right)$
- the minimal value of F is equal to $\theta c^{\frac{q+d-1}{d-1}}$, and the sequence of the values $\inf\{F_p(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}_l(\Omega)\}$ is asymptotically equivalent to $l^{\frac{q}{d-1}}\theta c^{\frac{q+d-1}{d-1}}$.

2 Γ-convergence result

We will prove the Γ-convergence result in two steps corresponding to Γ-lim inf and Γ-lim sup.

2.1 Γ-lim inf inequality

Before proving the Γ-lim inf inequality, we need some results and constructions. We start by a construction of a set $G_{\varepsilon,l}$ which will be useful later. Let Ω be a domain, I^d be a unit cube in $\mathbb{R}^d$ and a be a positive number such that the cube $(-a,a)^d$, that we will denote by I_a^d contains Ω. Let M be a union of d segments of length 1 joining at the center of the unit cube I^d and connecting two parallel faces of the unit cube in the given direction. The segments are made in such a way that their endpoints coincide with the middle points of the faces of I^d. We consider the set $G_{\varepsilon,l}$ to be the homogenization of the set M of order $\lfloor(\frac{\varepsilon l}{2ad})^{1/(d-1)}\rfloor$ into I^d_a. It is clear that due to the particularity of the set M, the set $G_{\varepsilon,l}$ is connected and $\mathcal{H}^1(G_{\varepsilon,l}) \approx \varepsilon l$.

Lemma 3 *1. Let $Q_R \subset \mathbb{R}^d$ be a cube of side R and $A \subset \overline{Q}_R$ a closed subset of $\overline{Q}_R$ of positive p-capacity, then there exists a constant $C = C(d,p)$ such that, for all functions $v \in C^\infty(\overline{Q}_R)$ with nonnegative mean value and vanishing on A, we have*

$$\int_{Q_R} |v|^p dx \leq \frac{CR^d}{cap_p(A, Q_{2R})} \int_{Q_R} |\nabla v|^p dx,$$

where $cap_p(A, Q_{2R})$ stands for the relative p-capacity of the set A inside Q_{2R}.

2. For any $\varepsilon > 0$, any $0 < l < +\infty$, any domain Ω and any function with non zero mean value $v \in W^{1,p}_0(\Omega \setminus G_{\varepsilon,l}) \subset W^{1,p}_0(\Omega)$ ($G_{\varepsilon,l}$ is the network constructed above) it holds $||v||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0) l^{\frac{1}{1-d}} ||v||_{W^{1,p}_0(\Omega)}$, where $\varepsilon_0 = cap_p(M, 2I^d)$

3. As a consequence, if we have a nonnegative function $f \in L^q(\Omega)$, then the function $u_{f,G_{\varepsilon,l},\Omega}$ satisfies $||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0) l^{\frac{q}{1-d}} ||f||^{q/(d-1)}_{L^q(\Omega)}$

Proof: The first assertion is a variant of the well-known Poincaré inequality. See [9] for more comment. For proving the second one, we first choose the function v to be a nonnegative smooth function on a large cube I^d_a which vanish outside $\Omega \setminus G_{\varepsilon,l}$. We consider the subdivision of cube I^d_a into subcubes which are coming from the homogenization of order $\lfloor(\frac{\varepsilon l}{2ad})^{1/(d-1)}\rfloor$ of the unit cube into I^d_a and consider the associated network $G_{\varepsilon,l}$. The side of subcubes is of order $l^{1/(1-d)}$. Let us denote the subcubes by Q_j. The set $I^d_a \setminus G_{\varepsilon,l}$ can be seen as the homogenized of order $k = \lfloor(\frac{\varepsilon l}{ad})^{1/(d-1)}\rfloor$ of $I^d \setminus M$ into I^d_a (M is the set constructed above). Let us set $\varepsilon_0 = cap_p(M, 2I^d)$ and notice that v vanishes on $G_{\varepsilon,l}$. By applying the first statement of this Lemma, it follows that

$$\int_{Q_j} |v|^p dx \leq \frac{Ck^{-d}}{cap_p(k^{-1}M, 2Q_j)} \int_{Q_j} |\nabla v|^p dx \leq \frac{Cl^{p/(1-d)}}{cap_p(M, 2I^d)} \int_{Q_j} |\nabla v|^p dx$$

and by summing up over j we get

$$\int_{I_a^d} |v|^p dx \leq \frac{C}{\varepsilon_0} l^{p/(1-d)} \int_{I_a^d} |\nabla v|^p dx.$$

Using the fact that v vanishes outside Ω, we may restrict the integrand to Ω, raise each term of the inequality to the power $1/p$ and thus getting the result by noticing that the L^p norm of the gradient $||\nabla v||_{L^p(\Omega)}$ stands for the norm $||v||_{W_0^{1,p}(\Omega)}$. The general case follows by density. For the last inequality, we use the weak version of the PDE which gives

$$\int_\Omega |\nabla u_{f,G_{\varepsilon,l},\Omega}|^p dx = \int_\Omega f u_{f,G_{\varepsilon,l},\Omega} dx \leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)}.$$

Since $u_{f,G_{\varepsilon,l},\Omega} \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l})$ we get

$$\begin{aligned} ||u_{f,G_{\varepsilon,l},\Omega}||^p_{W_0^{1,p}(\Omega)} &\leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)} \\ &\leq C(d,\varepsilon_0,\varepsilon) l^{1/(1-d)} ||u_{f,G_{\varepsilon,l},\Omega}||_{W_0^{1,p}(\Omega)} ||f||_{L^q(\Omega)}, \end{aligned},$$

and the desired result follows. □

Before proving the Γ-lim inf inequality, we need the following estimate which will be helpful.

Lemma 4 *Let $f, g \in L^q(\Omega)$ be given and u_f and u_g denote the solution of p-Laplacian equation with respective right hand side f, g and with Dirichlet boundary condition on $\Sigma_l' = \Sigma_l \cup G_{\varepsilon,l}$ (where Σ_l is an element of $\mathcal{A}_l(\Omega)$ and $G_{\varepsilon,l}$ the above constructed network), then*

$$l^{q/(d-1)} ||u_f - u_g||_{L^1(\Omega)} \leq C|\Omega|^{1/q} ||f - g||^{1/(d-1)}_{L^q(\Omega)},$$

where $C = C(d, p, \varepsilon_0, \varepsilon)$. In particular, if $\Omega = Q$ a cube centered at x_0, $g = f(x_0)$ and x_0 is a Lebesgue point for f, then

$$l^{q/(d-1)} ||u_f - u_g||_{L^1(Q)} \leq C|Q| \left(\frac{\int_Q |f(x) - f(x_0)|^q dx}{|Q|} \right)^{1/p} = |Q| r(Q).$$

Proof: For any $p \geq 2$, and any pair of vectors (z, w) we have the following monotonicity formulas (see [8])

$$|z - w|^p \leq C(|z|^{p-2} z - |w|^{p-2} w) \cdot (z - w).$$

Here $p > d-1 \geq 2$ and from monotonicity formulas, it follows that

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \leq C||u_f - u_g||_{L^p(\Omega)}||f-g||_{L^q(\Omega)},$$

where we have used $z = \nabla u_f$ and $w = \nabla u_g$. From Lemma 3, we have the inequality $||v||_{L^p(\Omega)} \leq Cl^{1/(1-d)}||v||_{W_0^{1,p}(\Omega)}$ which holds for every function v vanishing on Σ_l'. Since the function $u_f - u_g$ vanishes on Σ_l', we have

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \leq Cl^{1/(1-d)}||u_f - u_g||_{W_0^{1,p}(\Omega)}||f-g||_{L^q(\Omega)},$$

which gives

$$||u_f - u_g||_{W_0^{1,p}(\Omega)} \leq Cl^{1/(1-d)(p-1)}||f-g||^{1/(p-1)}_{L^q(\Omega)},$$

and using Hölder inequality, we get

$$\begin{aligned} ||u_f - u_g||_{L^1(\Omega)} &\leq |\Omega|^{1/q}||u_f - u_g||_{L^p(\Omega)} \\ &\leq C|\Omega|^{1/q}l^{1/(1-d)}||u_f - u_g||_{W_0^{1,p}(\Omega)} \\ &\leq C|\Omega|^{1/q}l^{q/(1-d)}||f-g||^{1/(p-1)}_{L^q(\Omega)}, \end{aligned}$$

and the first part of the statement follows. The second part is an obvious consequence of the first part. □

In the following proposition, we prove that the Γ-lim inf functional is bounded below by the candidate limit functional F in (4).

Proposition 5 *Under the same hypotheses of Theorem 2, denoting by F^- the functional Γ-$\liminf_l F_l$, it holds $F^-(\mu) \geq F(\mu)$ for any $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any sequence $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that μ_{Σ_l} weakly* converges to μ, we have*

$$\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \geq F(\mu).$$

Proof: Let $\Sigma_l' = \Sigma_l \cup G_{\varepsilon,l}$ and set $u_l' = u_{f,\Sigma_l',\Omega}$. Since $u_l \geq u_l'$, it is enough to estimate the integral $l^{\frac{q}{d-1}}\int_\Omega f u_l' dx$. It is obvious that $0 \leq u_l' \leq u_{f,G_{\varepsilon,l},\Omega}$ and Lemma 3 gives

$$||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d,\varepsilon_0,\varepsilon,f)l^{\frac{q}{1-d}}.$$

It follows that $l^{\frac{q}{d-1}}u_l'$ is L^p bounded, so up to a subsequence $l^{\frac{q}{d-1}}u_l' \rightharpoonup w$ weakly in $L^p(\Omega)$. Thus

$$\lim_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega g u_l' dx = \int_\Omega g w dx, \quad \forall g \in L^q(\Omega).$$

So it is enough to estimate w from below. We will show that, for almost any $x_0 \in \Omega$, it holds

$$w(x_0) \geq \theta \frac{f(x_0)^{1/(p-1)}}{(\mu_a + \varepsilon)^{\frac{q}{d-1}}}. \tag{6}$$

To this aim, we first estimate w on a cube Q centered at the point $x_0 \in \Omega$. We assume that x_0 is a Lebesgue point for f and $|Q|^{-1}\mu(Q) \to \mu_a(x_0)$ as Q shrinks around x_0. Assume also $f(x_0) > 0$ otherwise (6) would be trivial. We have

$$\lim_{l \to +\infty} l^{\frac{q}{d-1}} \int_Q u'_l dx = \int_Q w dx,$$

we use

$$u'_l \geq u_{f,\Sigma'_l,Q} \geq u_{f(x_0),\Sigma'_l,Q} - |u_{f,\Sigma'_l,Q} - u_{f(x_0),\Sigma'_l,Q}| \text{ in } Q,$$

where the first inequality comes from the fact that we add Dirichlet boundary condition on Q. The second part of Lemma 4 gives

$$\int_Q |u_{f,\Sigma'_l,Q} - u_{f(x_0),\Sigma'_l,Q}| dx \leq l^{\frac{q}{1-d}} |Q| r(Q).$$

It remains to estimate the second term. First of all let us define the number $L(l,Q) = \mathcal{H}^1(\Sigma'_l \cap Q)$ and observe that

$$u_{f(x_0),\Sigma'_l,Q} = f(x_0)^{1/(p-1)} u_{1,\Sigma'_l,Q}.$$

For simplicity of the notation, we denote $u_{1,\Sigma'_l,Q}$ by v_l. By a change of variables, if we assume the side of cube Q to be λ and we define $v_{l,\lambda} = \lambda^{-q} v_l(\lambda x)$ (thinking for instance that both cubes are centered at the origin), we get $v_{l,\lambda} = u_{1,\lambda^{-1}\Sigma'_l,I^d}$. It is easy to see that

$$\lambda^{-1}\Sigma'_l \in \mathcal{A}_{L(l,Q)/\lambda}(I^d);$$

moreover, it holds $L(l,Q) \to +\infty$ as $l \to +\infty$, since

$$L(l,Q) \geq \mathcal{H}^1(G_{\varepsilon,l} \cap Q) \approx \varepsilon l |Q|. \tag{7}$$

Using (7) and the fact that $\mu_l = l^{-1}\mathcal{H}^1(\Sigma_l)$, we may estimate the ratio between $L(l,Q)$ and l. It follows from the weak* convergence of μ_l to μ that $\limsup_{l \to +\infty} \mu_l(Q) \leq \mu(\overline{Q})$. So we have

$$\limsup_{l \to +\infty} \frac{L(l,Q)}{l} \leq \mu(\overline{Q}) + \varepsilon |Q|. \tag{8}$$

Using the definition of θ and the change of variables $y = \lambda x$ we have,

$$\begin{aligned}
\liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \int_Q v_l(y)dy &= \liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \lambda^{d+q} \int_{I^d} v_{l,\lambda}(x)dx \\
&= \liminf_{l\to+\infty} \left(\lambda^{-1}L(l,Q)\right)^{\frac{q}{d-1}} \lambda^{d+q+\frac{q}{d-1}} \int_{I^d} v_{l,\lambda}(x)dx^{\cdot} \\
&\geq \lambda^{d+q+\frac{q}{d-1}}\theta
\end{aligned}$$

hence using the fact that $\lambda^d = |Q|$ we get

$$\begin{aligned}
\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_Q v_l(y)dy &\geq \liminf_{l\to+\infty} \left(\frac{l}{L(l,Q)}\right)^{\frac{q}{d-1}} \liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \int_Q v_l(y)dy \\
&\geq \lambda^{d+q+\frac{q}{d-1}}\theta \left(\frac{1}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}} \\
&= \left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}} |Q|\theta.
\end{aligned}$$

This implies that

$$|Q|^{-1}\int_Q w dx \geq -r(Q) + \left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}} \theta f(x_0)^{1/(p-1)}.$$

We know that $r(Q)$ tends to 0 when the cube Q shrinks to x_0, whenever x_0 is a Lebesgue point for f. Now we let the cube Q shrinks toward x_0 with x_0 satisfying the previous assumption, then we get

$$w(x_0) \geq \frac{\theta f(x_0)^{1/(p-1)}}{(\mu_a(x_0)+\varepsilon)^{\frac{q}{d-1}}}.$$

It follows that

$$\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_l dx \geq \int_\Omega f w dx \geq \theta \int_\Omega \frac{f^q}{(\mu_a+\varepsilon)^{\frac{q}{d-1}}} dx,$$

and the desired inequality holds by letting ε tend to 0 that is

$$\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_l dx \geq \theta \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx.$$

$\square$

2.2 Γ-lim sup inequality

Before proving the Γ-lim sup inequality we introduce a definition and prove some preliminaries results. We start by the definition of tiling set.

Definition 6 *A set $\Sigma \in \mathcal{A}_l(I^d)$ is called tiling set if $\Sigma \cap \partial I^d$ coincides with the 2^d vertices of I^d.*

If $\Sigma \in \mathcal{A}_l(I^d)$ is tiling set and Σ_k is the homogenization of order k of Σ into I^d, then Σ_k remains connected and

$$\mathcal{H}^1(\Sigma_k) = k^{d-1}\mathcal{H}^1(\Sigma).$$

Lemma 7 *Given $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ a tiling set, a domain $\Omega \subset \mathbb{R}^d$ and $f \in L^q(\Omega)$, we consider the sequence of sets*

$$\Sigma^k = \bigcup_{y \in k^{-1}\mathbb{Z}^d} (y + k^{-1}\Sigma_0 \cup \partial I^d) \cap \overline{\Omega}$$

and consider the sequence of functions $(u_k)_k$ given by

$$u_k = k^q u_{f,\Sigma^k,\Omega},$$

then $u_k \rightharpoonup c(\Sigma_0) f^{1/(p-1)}$ in $L^p(\Omega)$ as $k \to +\infty$, where $c(\Sigma_0)$ is a constant given by $\int_\Omega u_{1,\Sigma_0,I^d} dx$.

Proof: Let us set $\varepsilon_0 = cap_p(\Sigma_0) > 0$, then thanks to Lemma 3 the sequence $(u_k)_k$ is bounded in $L^p(\Omega)$. So up to a subsequence it converges weakly in $L^p(\Omega)$ to some function. Let us consider the subsequence (denoted by the same indices) $(u_k)_k$ and its weak limit $w_{f,\Sigma_0,\Omega}$. It is obvious that the pointwise value of this limit function depends only on the local behavior of f. In fact, we may produce small cubes around each point $x \in \Omega$ which do not affect each other and if $f = \sum_j f_j 1_{A_j}$ is piecewise constant (the pieces A_j being disjoint open sets, for instance), then for k large enough the value of u_k at $x \in A_j$ depends only of f_j (u_k vanishes on $k^{-1}\partial I^d$). From the rescaling property of the p-Laplacian operator Δ_p, if f is a piecewise constant function, it holds $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} w_{1,\Sigma_0,\Omega}$. It is clear that in the case $f = 1$, since we are simply homogenizing the function u_{1,Σ_0,I^d}, the limit of the whole sequence $(u_k)_k$ exists and does not depend on the global geometry of Ω, but it is a constant and it is the same constant if we have I^d instead of Ω. An easy computation shows that the constant is $c(\Sigma_0)$. It remains to extend the equality for non piecewise constant function belonging to $L^q(\Omega)$. Let $f \in L^q(\Omega)$ be a generic function and $(f_n)_n$ a sequence of piecewise constant functions approaching f in $L^q(\Omega)$. Up to a subsequence it holds $k^q u_{f,\Sigma_k,\Omega} \rightharpoonup w_{f,\Sigma_0,\Omega}$ and $k^q u_{f_n,\Sigma^k,\Omega} \rightharpoonup f_n^{1/(p-1)} c(\Sigma_0)$ as $k \to +\infty$. By Lemma 4 it holds also

$$||k^q u_{f,\Sigma^k,\Omega} - k^q u_{f_n,\Sigma^k,\Omega}||_{L^1(\Omega)} \leq C||f - f_n||_{L^q(\Omega)}^{1/(p-1)}.$$

taking into account the lower semicontinuity of the $L^1(\Omega)$-norm with respect to the $L^p(\Omega)$-weak topology, we get, passing to the limit as $k \to +\infty$,

$$||w_{f,\Sigma_0,\Omega} - f_n^{1/(p-1)} c(\Sigma_0)||_{L^1(\Omega)} \leq C||f - f_n||_{L^q(\Omega)}^{1/(p-1)}.$$

We now pass to the limit as $n \to +\infty$ and using Fatou's Lemma (up to a subsequence f_n converges pointwise a.e. to f), we get $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} c(\Sigma_0)$ and the proof is over. □

This result remains true even if Σ_0 is not tiling. In fact we have never used the fact that Σ_0 is tiling in the proof. We keep it for the up coming construction. One problem in the previous Lemma is that we have used the whole boundary of the unit cube which is not an one dimensional set (if $d \geq 3$) and consequently the set Σ^k is not an one dimensional set. In the following Lemma, we prove an estimate on an unit cube which will be useful for proving that $u_{f,\Sigma^k,\Omega}$ may be approximated by $u_{f,\Sigma_l^k,\Omega}$ where Σ_l^k is an one dimensional closed and connected set.

Lemma 8 *Let $\Sigma \in \mathcal{A}_l(I^d)$ be a tiling set such that the corresponding rescaled state functions $l^{\frac{q}{1-d}} u_{f,\Sigma,I^d}$ are uniformly L^p bounded, then there exists $T_l \in \mathcal{A}_l(I^d)$ such that $\mathcal{H}^1(T_l) \ll l$ and if we denote by $u_l = u_{f,\Sigma \cup T_l, I^d}$ and v_l the solution of the equation*

$$\begin{cases} -\Delta_p u = f \ \ in \ \ I^d \setminus \Sigma \cup T_l^\alpha \\ u = 0 \ \ in \ \ \Sigma \cup T_l, \end{cases}$$

then $v_l \leq u_l + c_l l^{\frac{q}{1-d}}$ on I^d where c_l is a constant dependent of l and goes to zero as l goes to infinity.

Proof: Let $\Sigma \in \mathcal{A}_l(I^d)$ be a tiling set such that the sequence $(\tilde{u}_l)_l = (l^{\frac{q}{d-1}} u_{f,\Sigma,I^d})_l$ is L^p bounded and denote by u_l the solution of the equation

$$\begin{cases} -\Delta_p u = f \ \text{ in } \ I^d \setminus \Sigma \\ u = 0 \ \text{ on } \ \Sigma \cup \partial I^d, \end{cases}$$

and by v_l^k the solution of the equation

$$\begin{cases} -\Delta_p u = f \ \text{ in } \ I^d \setminus \Sigma \cup \Sigma_k \\ u = 0 \ \text{ on } \ \Sigma \cup \Sigma_k, \end{cases}$$

where Σ_k is grid of length k contained in the boundary of I^d and converges to it in Hausdorff distance. Since Σ is tiling, we may choose Σ_k such that $\Sigma \cup \Sigma_k$ is connected for all k. For l fixed, $(\Sigma \cup \Sigma_k)_k$ is a sequence of connected sets which converges to the connected set $\Sigma \cup \partial I^d$ then by generalized Šverak continuity result (see [4]) the sequence $(v_l^k)_k$ converges strongly to u_l in $W^{1,p}(I^q)$ as $k \to +\infty$. As consequence $(l^{\frac{q}{d-1}} v_l^k)_l$ (as well as $l^{\frac{q}{d-1}}(v_l^k - u_l)$) is L^p bounded more precisely there exists a constant c_k such that

$$\|l^{\frac{q}{d-1}}(v_l^k - u_l)\|_{L^p(I^d)} \leq c_k. \tag{9}$$

Moreover c_k may be as small as we want for k large enough. Now let k depends on l say $k = k(l)$ and consider the set $\Sigma_l = \Sigma \cup \Sigma_{k(l)}$. We may choose $k(l)$ such that $k(l) \lll l$ and $k(l) \to +\infty$ as $l \to +\infty$. This make the length of Σ_l to be asymptotically equivalent to l. $(\Sigma_l)_l$ is a sequence of connected sets converging to the connected set $\overline{I}^d$ the closure of the unit cube then the associated sequence of solutions converges strongly to zero in $W^{1,p}(I^d)$ and $(l^{\frac{q}{d-1}} v_l^{k(l)})_l$ are L^p bounded. Moreover $l^{\frac{q}{d-1}} v_l^{k(l)}$ satisfies the inequality (9). From the maximum principle we get $v_l - u_l \geq 0$ (setting $v_l = v_l^{k(l)}$) and from the above boundedness and Hölder inequality it holds

$$0 \leq \int_{I^d} (v_l - u_l) dx \leq c_l l^{\frac{q}{1-d}}.$$

We obtain easily the existence of some constant c_l (it may be different from the above constant c_l but it goes to zero as $l \to +\infty$) such that the inequality

$$v_l - u_l \leq c_l l^{\frac{q}{1-d}}$$

holds in I^d and the proof is over. □

Due to the terminology suggested in [5], the sets satisfying the hypothesis of the Lemma 8 will be called almost boundary-covering sets. We have proved the Lemma for the unit cube but the result remains true for a cube of any side as well as an open domain with Lipschitz boundary. Now we built an almost boundary-covering set that will be used for the construction of the recovering sequence for the Γ-lim sup inequality.

Lemma 9 *For any $\varepsilon > 0$, there exists $l_0 > 0$ such that for all $l > l_0$ we find a set $\Sigma \in \mathcal{A}_l(I^d)$ which is almost boundary-covering, with*

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx < (1+\varepsilon)\theta$$

and consequently if we denote by $u_{1,\Sigma}$ the solution of the same equation which vanish only on Σ and not on whole the boundary of I^d we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma} dx < (1+\varepsilon)\theta + c_l.$$

Proof: Given a small positive number δ $(0 < \delta \ll 1)$, by definition of θ, we may find a set $\Sigma_1 \in \mathcal{A}_{l_1}(I^d)$ such that

$$l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx < (1+\delta)\theta$$

and moreover the number l_1 may be chosen as large as we want. Now, we want to enlarge the set Σ_1 to get a set Σ_2 which is almost boundary-covering. Let $\gamma = \cup_{j=1}^{2^d} S_j$ where S_j is the shortest segment joining Σ_1 to the j^{th} vertice of I^d cube. We set $\Sigma_2 = \Sigma_1 \cup T_{l_1} \cup \gamma$ where T_{l_1} is the grid T_l of the previous Lemma with l replaced by l_1. Up to adding one segment, we may assume Σ_2 connected. The length $l_2 = \mathcal{H}^1(\Sigma_2)$ does not exceed the number $l_1 + \mathcal{H}^1(T_{l_1}) + (2^d+1)\sqrt{d}$. It is possible to chose l_1 so that

$$\left(\frac{l_1 + \mathcal{H}^1(T_{l_1}) + (2^d+1)\sqrt{d}}{l_1}\right)^{\frac{q}{d-1}} \leq 1 + \delta.$$

This implies

$$l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d} dx \leq \left(\frac{l_2}{l_1}\right)^{\frac{q}{d-1}} l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx \leq (1+\delta)^2\theta.$$

Now if we are given a large number l, we homogenize the set Σ_2 of order $k = \lfloor \left(\frac{l}{l_2}\right)^{\frac{1}{d-1}} \rfloor$ into I^d and the homogenized set Σ belongs to $\mathcal{A}_{k^{d-1}l_2}(I^d)$ and is still almost boundary-covering. For this set Σ it holds (using the rescaling property of the p-Laplacian operator)

$$(k^{d-1}l_2)^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx = l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d} dx.$$

Noticing that $l^{\frac{q}{d-1}} \leq \left(\frac{k+1}{k}\right)^q \left(k^{d-1}l_2\right)^{\frac{q}{d-1}}$, we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx \leq \left(\frac{k+1}{k}\right)^q (1+\delta)^2\theta.$$

If $l > l_2\delta^{-1}$, using the fact that $\delta \ll 1$, an easy computation shows that $1 + 1/k < 1 + \delta$ so that we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx \leq (1+\delta)^{2+q}\theta.$$

Now it is sufficient to choose δ so small that $(1+\delta)^{2+q} < 1+\varepsilon$, choose $l_0 = l_2\delta^{-1}$ and the result follows. □

We have all the ingredients for proving the Γ-lim sup inequality. We will start from a particular class of measures. Let us call piecewise constant probability measures those probability measures $\mu \in \mathcal{P}(\overline{\Omega})$ which are of the form

$$\mu = \rho dx, \text{ with, } \rho \in L^1(\Omega), \int_\Omega \rho dx = 1, \ \rho > 0,$$

for a piecewise constant function $\rho = \sum_{j=1}^m \rho_j I_{\Omega_j}$, the pieces Ω_j being disjoint Lipschitz open subsets with the possible exception of $\Omega_0 = \Omega \setminus \cup_{j=1}^m \Omega_j$.

Proposition 10 *Under the same hypotheses of Theorem 2, we have*

$$F^+(\mu) \leq F(\mu), \quad \textit{where} \quad F^+ = \Gamma - \limsup_{l\to+\infty} F_l,$$

for any piecewise constant measure $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any such a measure μ and $\varepsilon > 0$, there exists a family of sets $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that the measure μ_{Σ_l} weakly converges to the measure μ and moreover*

$$\limsup_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \leq (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^{\frac{q}{d-1}}} dx.$$

Proof: Apply Lemma 9 and take an almost boundary-covering set $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ such that

$$l_0^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_0,I^d} dx < (1+\varepsilon)\theta.$$

Now, we define the set Σ_l^j by homogenizing into Ω_j the set Σ_0 of order $k(l,j)$ that is

$$\Sigma_l^j = \overline{\Omega_j} \cap k(l,j)^{-1}(\mathbb{Z}^d + \Sigma_0).$$

Since Σ_0 is tiling , for $k(l,j)$ large enough Σ_l^j remains connected and

$$\mathcal{H}^1(\Sigma_l^j) = |\Omega_j| K(l,j)^{d-1} \mathcal{H}^1(\Sigma_0) \leq |\Omega_j| K(l,j)^{d-1} l_0.$$

Let $\Sigma_{l_1} \in \mathcal{A}_{l_1}(\Omega)$ be a set contained in the internal boundary of the union of Ω_j and converges to it in the Hausdorff topology as $l_1 \to +\infty$ (Σ_{l_1} may obtained by homogenizing some kind of grid contained in ∂I^d of some order into $\cup_{j=0}^m \partial\Omega_j$). Due to the connectedness of Σ_{l_1}, the corresponding solution converges to the solution associated to the internal boundary of $\cup_{j=0}^m \Omega_j$ as well. Then we choose $\Sigma_l = \cup_{j=0}^m \Sigma_l^j \cup \Sigma_{l_1}$. We may assume Σ_l connected otherwise we add some segments to connect all the pieces. The family of sets Σ_l is admissible (i.e. $\Sigma_l \in \mathcal{A}_l(\Omega)$ and $\mu_{\Sigma_l} \rightharpoonup \mu$) if we have, as $l \to +\infty$,

$$\sum_{j=0}^m |\Omega_j| k(l,j)^{d-1} l_0 + l_1 \leq l \quad \text{and is asymptotic to} \quad l;$$

$$\frac{k(l,j)^{d-1} l_0}{l} \to \rho_j \quad \text{for} \quad j = 0, \cdots, m.$$

It is easy to see that all theses conditions are satisfied if we set

$$k(l,j) = \left\lfloor \left(\frac{l - l_1}{l_0} \rho_j \right)^{\frac{1}{d-1}} \right\rfloor.$$

Let us introduce the following sets

$$\Gamma_l^j = \overline{\Omega}_j \cap k(l,j)^{-1}(\mathbb{Z}^d + \partial I^d), \quad \Gamma_l = \bigcup_j \Gamma_l^j.$$

Thanks to Lemma 8 we have

$$\int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l^j,\Omega_j} dx \leq \int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l \cup \Gamma_l^j,\Omega_j} dx + c_l l_0^{\frac{q}{1-d}}.$$

In fact, we consider subcubes $Q_{k(l,j)}$ which are obtained by the partition of Ω_j made by Γ_l^j, then in each subcube $Q_{k(l,j)}$, the Lemma 8 gives

$$u_{f,\Sigma_l^j} \leq u_{f,\Sigma_l^j,Q_{k(l,j)}} + c_l (k(l,j) l_0^{\frac{1}{d-1}})^{-q}.$$

By multiplying this inequality by f (notice that $f \geq 0$), Integrating over $Q_{k(l,j)}$ and summing up, we get

$$\int_{\Omega_j} f u_{f,\Sigma_l^j,\Omega_j} dx \leq \int_{\Omega_j} f u_{f,\Sigma_l^j} dx \leq \int_{\Omega_j} f u_{f,\Sigma_l^j \cup \Gamma_l^j,\Omega_j} dx + c_l (k(l,j) l_0^{\frac{1}{d-1}})^{-q}$$

where the first inequality comes from the maximum principle and the second is obtained by observing that on each cube $Q_{k(l,j)}$ it holds $u_{f,\Sigma_l^j \cup \Gamma_l^j,\Omega_j} = u_{f,\Sigma_l^j,Q_{k(l,j)}}$

We choose l_1 to be a function of l (for example $l_1 = l^{\frac{d-1}{d}}$) in such a way that l_1 goes to $+\infty$ whenever l goes to $+\infty$. We are interested in the estimate of the value of $F_l(\Sigma_l)$

$$\begin{aligned} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx &= \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}} \right)^{\frac{q}{d-1}} \int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l,\Omega} dx \\ &\leq \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}} \right)^{\frac{q}{d-1}} \left(\int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l,\Omega_j} dx + c_{l_1} \right) \\ &\leq \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}} \right)^{\frac{q}{d-1}} \left(\int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l^j \cup \Gamma_l^j,\Omega_j} dx + c_{l_1} + c_l l_0^{\frac{q}{1-d}} \right) \end{aligned}$$

where c_{l_1} goes to zero as l_1 tend to infinity. By applying Lemma 7 to each Ω_j we get the following weak convergence in L^p.

$$k(l,j)^q u_{f,\Sigma_l^j \cup \Gamma_l^j,\Omega_j} \rightharpoonup c(\Sigma_0) f^{1/(p-1)} \quad \text{as} \quad l \to +\infty$$

and the term $\left(\frac{l}{k(l,j)^{d-1}} \right)^{\frac{q}{d-1}}$ converges to $\left(\frac{l_0}{\rho_j} \right)^{\frac{q}{d-1}}$ as $l \to +\infty$ for $j = 0, \cdots, m$. The choice of the set Σ_0 implies that $l_0^{\frac{q}{d-1}} c(\Sigma_0) < (1+\varepsilon)\theta$, so we have

$$\limsup_{l \to +\infty} l^{\frac{q}{d-1}} \int_{\Omega_j} f u_{f,\Sigma_l,\Omega} dx \leq (1+\varepsilon)\theta \rho_j^{\frac{q}{d-1}} \int_{\Omega_j} f^q dx, \ \text{ for } \ j = 0, \cdots, m$$

and summing up we get

$$\limsup_{l \to +\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \leq (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^{\frac{q}{d-1}}} dx.$$

□

We have to extend the result to non piecewise constant measures. By the general theory of Γ-convergence, we know that it is enough to prove the Γ-lim sup inequality on a class which is dense in energy. Hence, due to the lower semicontinuity of the functional F, it is sufficient to prove the following

Proposition 11 *For any measure $\mu \in \mathcal{P}(\overline{\Omega})$ there exists a sequence $(\mu_n)_n$ of piecewise constant measures such that $\mu_n \rightharpoonup \mu$ and*

$$\limsup_n F(\mu_n) \leq F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx.$$

Proof: First observe that the inequality is trivial whenever $F(\mu) = +\infty$. Assume now that $F(\mu) < +\infty$ and start proving the inequality for measures which are absolutely continuous with respect to the Lebesgue measure and have positive densities bounded away from zero. Given a measure $\mu = \rho dx$, with $\rho \geq c > 0$, it is possible to find a sequence of measures $\mu_n = \rho_n dx$ such $\rho_n \to \rho$ strongly in L^1 and μ_n are piecewise constant with $\rho_n \geq c$. The pointwise $a.e$ convergence of ρ_n to ρ may be assumed and the inequality $F(\mu) \geq \limsup_n F(\mu_n)$ follows easily (we have even an equality). So we have extended the result to any absolutely continuous measure with density bounded below away from zero. To get the result for any measure $\mu \in \mathcal{P}(\overline{\Omega})$, it is sufficient to prove that any measure μ may be approximated weakly* by absolutely continuous measure μ_n with densities bounded below away from zero and $\limsup_n F(\mu_n) \leq F(\mu)$. Let us take $\mu = \rho dx + \mu^s$, where μ^s is the singular part of the measure μ with respect to the Lebesgue measure and ρ the density of the absolutely continuous part. We construct the sequence of absolutely continuous measure μ_n by setting $\mu_n = ((1-1/n)\rho + a_n + \phi_n)dx$, where $a_n = n^{-1} \int_\Omega \rho dx$ and $\phi_n dx \rightharpoonup \mu^s$ with $\int_\Omega \phi_n dx = \int_{\overline{\Omega}} d\mu^s$. The fact that $F(\mu) < +\infty$ implies that ρ cannot vanish, hence $a_n > 0$ and $\rho_n = (1-1/n)\rho + a_n + \phi_n$ is bounded below by the positive constant a_n. We have as well that μ_n weakly* converges to μ and

$$\begin{aligned} F(\mu_n) = \theta \int_\Omega \frac{f^q}{((1-1/n)\rho + a_n + \phi_n)^{\frac{q}{d-1}}} &\leq \theta \int_\Omega \frac{f^q}{((1-1/n)\rho)^{\frac{q}{d-1}}} dx \\ = (1 - \frac{1}{n})^{-q/(d-1)} F(\mu) \end{aligned}.$$

Passing to the lim sup on the inequality, we get the desired result. ☐

3 Some estimate on θ

In this section we will prove some estimate on the constant θ and in particular we will show that θ is neither 0 nor $+\infty$ so that our limit functional is not trivial.

Proposition 12 *We have*

$$\theta < +\infty.$$

Proof: Let $\Sigma_l \in \mathcal{A}_l(I^d)$ be a tiling set. For any positive integer number n, let us denote by Σ_l^n the homogenization of the set Σ_l of order n into I^d. Clearly, Σ_l^n is connected and $\mathcal{H}^1(\Sigma_l^n) \leq n^{d-1}l$. Using the rescaling property of the p-Laplacian operator, it follows that

$$\theta \leq \liminf_n (n^{d-1}l)^{\frac{q}{d-1}} F_p(\Sigma_l^n, 1, I^d) = l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) < +\infty$$

which concludes the proof. $\square$

Proposition 13

$$\theta \geq \frac{(d-1)q^{-q}}{(q+d-1)w_{d-1}^{\frac{q}{d-1}}},$$

where w_r stands for the volume of unit ball in $\mathbb{R}^r$.

Proof: First, we prove that

$$F_p(\Sigma_l, 1, I^d) \geq q^{-q} D_q(\Sigma_l \cup \partial I^d),$$

where $D_r(\Sigma) = \int_{I^d} h_\Sigma(x)^r dx$ and $h_\Sigma(x) = d(x, \Sigma)$ is the distance from x to Σ. For every real number A and for every real number $r > 1$, we have

$$\begin{aligned} F_p(\Sigma_l, 1, I^d) &= q \max \left\{ \int_{I^d} (v - \frac{1}{p}|\nabla v|^p) dx : v \in W_0^{1,p}(I^d \setminus \Sigma_l) \right\} \\ &\geq q \int_{I^d} (A h_{\Sigma_l \cup \partial I^d}(x)^r - \frac{1}{p}|\nabla (A h_{\Sigma_l \cup \partial I^d}(x)^r)|^p) dx. \end{aligned}$$

It is well known that the distance function is 1-Lipschitz and satisfies $|\nabla h_{\Sigma_l \cup \partial I^d}| = 1$ (and consequently $|\nabla (h_{\Sigma_l \cup \partial I^d})^r| = r(h_{\Sigma_l \cup \partial I^d})^{r-1}$). Choosing $r = q$ the conjugate exponent of p, we get

$$F_p(\Sigma_l, 1, I^d) \geq q(A - A^q \left(\frac{q^p}{p}\right)) \int_{I^d} h_{\Sigma_l \cup \partial I^d}(x)^q dx.$$

The result follows by optimizing on A (the optimal choice is $A = q^{-q}$). In [10] it has been proved that for any set $\Sigma_l \in \mathcal{A}_l(I^d)$ it holds

$$\liminf_l l^{\frac{q}{d-1}} \int_{I^d} h_{\Sigma_l}(x)^q dx \geq \frac{d-1}{(q+d-1)w_{d-1}^{\frac{q}{d-1}}}.$$

Here the same proof may be adapted by doing some modification and getting the same result even if $\Sigma_l \cup \partial I^d$ is not an one dimensional set i.e.

$$\liminf_l l^{\frac{q}{d-1}} \int_{I^d} h_{\Sigma_l \cup \partial I^d}(x)^q dx \geq \frac{d-1}{(q+d-1)w_{d-1}^{\frac{q}{d-1}}},$$

and the desired result holds. □

4 asymptotic of p-compliance-location problem

In this section we consider the case where the control variable is look for among discrete sets of finite elements. Let $p > d$ be fixed and $q = p/(p-1)$ the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}^d$ and n a positive given integer number, we define

$$\mathcal{A}_n(\Omega) = \{\Sigma \subset \overline{\Omega} : 0 < \mathcal{H}^0(\Sigma) \leq n\}.$$

For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity(since $p > d$, every point has positive p-capacity), we denote as before by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u = f \ \text{ in } \ \Omega \setminus \Sigma \\ \quad u = 0 \ \text{ in } \ \Sigma \cup \partial\Omega, \end{cases}$$

that is $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega \setminus \Sigma). \tag{10}$$

For $f \geq 0$, we define the p-compliance functional as before and the existence of the minimal p-compliance configuration is a consequence of the continuity of Sobolev functions when $p > d$.

Theorem 14 *For any integer number $n > 0$, Ω bounded open subset of $\mathbb{R}^d$, $d \geq 2$ and f a nonnegative function belonging to $L^q(\Omega)$, the problem*

$$\min\{C_p(\Sigma) : \Sigma \in \mathcal{A}_n(\Omega)\} \tag{11}$$

admits at least one solution.

As before, we are interested to the asymptotic behavior of the optimal set Σ_n of the problem (11) as $n \to +\infty$. Let us associate to every $\Sigma \in \mathcal{A}_n(\Omega)$ a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = n^{-1}\delta_\Sigma$$

and define a functional $G_n : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$G_n(\mu) = \begin{cases} n^{\frac{q}{d}} C_p(\Sigma) & \text{if } \ \mu = \mu_\Sigma, \Sigma \in \mathcal{A}_n(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{12}$$

The scaling factor $n^{\frac{q}{d}}$ is needed in order to avoid the functional to degenerate to the trivial limit functional which vanishes everywhere. Again the main result deals with the behavior as $n \to +\infty$ of the functional G_n, and is stated in terms of Γ-convergence.

Theorem 15 *The functional G_n defined in (12) Γ-converges, with respect to the weak* topology on the class $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$, to the functional G defined on $\mathcal{P}(\overline{\Omega})$ by*

$$G(\mu) = \theta_1 \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d}}} dx, \tag{13}$$

where μ_a stands for the density of the absolutely continuous part of μ with respect to the Lebesgue measure, and θ_1 is a positive constant depending only on d and p and is defined by

$$\theta_1 = \inf\{\liminf_{n \to +\infty} n^{\frac{q}{d}} F_p(\Sigma_n, 1, I^d) : \Sigma_n \in \mathcal{A}_n(I^d)\} \tag{14}$$

$I^d = (0,1)^d$ being the unit cube in $\mathbb{R}^d$.

We deduce the following consequence of Theorem 15:

- if Σ_n is a solution of the minimization problem (11), then up to a subsequence $\mu_{\Sigma_n} \rightharpoonup \mu$ as $n \to +\infty$, where μ is a minimizer of G;

- since G has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, the whole sequence μ_{Σ_n} converges to the unique minimizer μ of G given by $\mu = cf^{\frac{qd}{q+d}}\mathcal{L}^d$ where c is such that μ is a probability measure that is $c = 1/\left(\int_\Omega f^{\frac{qd}{q+d}}dx\right)$

- the minimal value of G is equal to $\theta_1 c^{\frac{q+d}{d}}$, and the sequence of the values $\inf\{F_p(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}_n(\Omega)\}$ is asymptotically equivalent to $n^{\frac{q}{d}} \inf\left\{G(\mu) : \mu \in \mathcal{P}(\overline{\Omega})\right\}$.

This problem is in connection with the location problem, that is the minimization of the functional $\int_\Omega f(x)d_\Sigma(x)dx$ where $d_\Sigma(x)$ stands for the distance from x to Σ and $\Sigma \in \mathcal{A}_n(\Omega)$. For more details, the reader may consult [2]. We will not prove Theorem 15 since the proof follows the same line as the proof of Theorem 2 but we will point out some necessaries modifications. The Lemma 3 is crucial for the proof of the Γ-lim inf inequality. This Lemma remains valid in the case of discrete set provided that the power $d-1$ is replaced by d. In this case it suffices that v vanishes on one point since point has positive p-capacity (remember that $p > d$). An other important element in the proof of the Γ-lim inf inequality is the set $G_{\varepsilon,l}$. Here, we will call it $G_{\varepsilon,n}$ and its construction is obtained by the homogenization of order $\lfloor\left(\frac{\varepsilon n}{2ad}\right)^{1/d}\rfloor$ of the center of the unit cube into the cube $I_a^d = (-a, a)^d$ which contains Ω. For the Γ-lim sup inequality, proofs are essentially the same except the fact that we do not need tiling set and replace l by n. We conclude this section with the estimate of the constant θ_1. To prove the finiteness it suffice to use the set Σ_n which is the homogenization of order n of the center of the unit cube into the unit cube. For the lower bound, the proof follows that of Proposition 13 and gives

$$\theta_1 \geq \frac{d}{(q+d)w_d^{\frac{q}{d}}}.$$

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