On some asymptotical shape problems

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Abstract

We consider the problem of the optimal location of a Dirichlet region in a d-dimensional domain Ω subjected to a given force f in order to minimize a given functional. We look for the optimal region among the class of all closed connected sets of assigned total length l. Then we let l tends to infinity and we look at the Γ-limit of a suitable rescaled functional, in order to get information of the asymptotic distribution of the optimal region.

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Introduction

We consider the problem of finding the best location of the Dirichlet region Σ in a d-dimensional domain Ω associated to an elliptic equation in divergence form, namely

$$\begin{cases} -\Delta_p u = f \text{ in } \Omega \setminus \Sigma \\ u = 0 \text{ in} \Sigma \cup \partial\Omega, \end{cases}$$

where f is a nonnegative function belonging in $L^q(\Omega)$, q being the conjugate exponent of p. We are interested in the minimization of the functional defined by

$$\mathcal{F}(\Sigma) = \int_\Omega F(x, u, \nabla u) dx,$$

where $u = u_{f,\Sigma,\Omega}$ stands for the unique solution of the above equation. The admissible class of control variables Σ we consider here is the class of all closed connected sets with given one dimensional Hausdorff measure. It is easy to obtain the optimal configuration Σ_l of the above optimization problem (see Theorem 1.1) as a consequence of Šverák result (see e.g. [3], [10]) and some assumption on F. we are interested in the asymptotic behavior of Σ_l as $l \to +\infty$; more precisely we want to obtain the limit distribution of Σ_l as a limit probability measure that minimizes the Γ-limit functional of the suitable rescaled functional. This problem has been considered in the case where $F = f(x)u$ in [5] for dimension 2 and [4] for higher dimension. This paper treats the general case. While in [5] and [4] the weak L^p convergence of the rescaled solution of the p-Laplacian equation is enough here we go further and prove strong L^p convergence of the rescaled solution and its gradient.

1 Setting of the problems and existence of optimal shape

Let $p > d-1$ be fixed and $q = p/(p-1)$ the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}^d$ and l a positive given real number, we define

$$\mathcal{A}_l(\Omega) = \{\Sigma \subset \overline{\Omega}, \quad \text{closed and connected}, \quad 0 < \mathcal{H}^1(\Sigma) \leq l\}.$$

The constraint $p > d-1$ allows each set in $\mathcal{A}_l(\Omega)$ to be of positive p-capacity. For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity, we denote by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \setminus \Sigma \\ \quad u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases}$$

that is $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega \setminus \Sigma). \tag{1}$$

By the maximum principle, the nonnegativity of the function f implies that of u. For $f \geq 0$, we define the p-compliance functional as follows:

$$\begin{aligned} C(\Sigma) = F_p(\Sigma, f, \Omega) &= \int_\Omega f u_{f,\Sigma,\Omega} dx = \int_\Omega |\nabla u_{f,\Sigma,\Omega}|^p dx \\ &= q \max \left\{ \int_\Omega (v - \frac{1}{p}|\nabla v|^p) dx : v \in W_0^{1,p}(\Omega \setminus \Sigma) \right\}, \end{aligned}$$

where q stands for the conjugate exponent of p. The existence of the minimal p-compliance configuration is just a consequence of Šverák compactness-continuity result (see [10] for $p = 2$ and [3] for general p). In this paper we consider a general functional namely

$$\mathcal{F}(\Sigma) = \int_{\Omega} F(x, u, \nabla u) dx,$$

where $u = u_{f,\Sigma,\Omega}$ stands for the unique solution of the equation (1) and $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a function satisfying the following hypothesis:

1. for a.e. $x \in \Omega$ the function $F(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R} \times \mathbb{R}^d$;
2. for all $(u, z) \in \mathbb{R} \times \mathbb{R}^d$ the function $F(\cdot, u, z)$ is Lebesgue measurable on Ω;
3. $F(\cdot, u, z) \geq c(|u|^p + |z|^p) \quad \forall (u, z) \in \mathbb{R} \times \mathbb{R}^d$ for some constant $c > 0$
4. for the technical computational reason, we assume the following "nondecreasing" condition on F : for a given unit vector n,

$$F(\cdot, \alpha, \beta n) \geq F(\cdot, \alpha', \beta' n) \;\; \forall \alpha \geq \alpha' \;\; , \beta \geq \beta' \tag{2}$$

Up to identification of $\mathbb{R}^{d+1}$ and $\mathbb{R} \times \mathbb{R}^d$, we will consider sometime F as a function from $\Omega \times \mathbb{R}^{d+1}$ to $\mathbb{R}$. Due to the hypothesis made on F and the Šverák compactness-continuity result of [10] and [3], we have easily the following existence result.

Theorem 1.1 *For any real number $l > 0$, Ω bounded open subset of $\mathbb{R}^d$, $d \geq 2$ and f a nonnegative function belonging to $L^q(\Omega)$, the problem*

$$\min\{\mathcal{F}(\Sigma) : \Sigma \in \mathcal{A}_l(\Omega)\} \tag{3}$$

admits at least one solution.

For the convenience of the reader let us describe briefly the existence of an optimal shape. Let $\{\Sigma_n\}_n \subset \mathcal{A}_l(\Omega)$ be a minimizing sequence in the optimization problem (3), since $\{\Sigma_n\}_n$ is a sequence a closed connected subsets of Ω such that $\sup_n \mathcal{H}^1(\Sigma_n) \leq l$, by Blaschke theorem (compactness of the sequence $\{\Sigma_n\}_n$ in the Hausdorff topology) and by Gołab theorem (lower semicontinuity of the $\mathcal{H}^1$ with respect to the Hausdorff topology), up to extracting a subsequence, $\{\Sigma_n\}_n$ converges in Hausdorff distance to some $\Sigma \in \mathcal{A}_l(\Omega)$ and $\mathcal{H}^1(\Sigma) \leq \liminf_{n\to\infty} \mathcal{H}^1(\Sigma_n)$. For the lower semicontinuity of the functional we need the Šverák continuity-compactness result which is stated in this terms: let $\{\Omega_n\}_n$ be a sequence of open and bounded sets contained in a fix bounded set D. If we assume that the number of the connected components of the complements of Ω_n in D is uniformly bounded by some number t then $\{\Omega_n\}_n$ converges in the Hausdorff topology to some open and bounded set $\Omega \subset D$ and the number of the connected components of the complement of Ω is less or equal t. Moreover, if we denote by $u_n \in W_0^{1,p}(\Omega_n)$ the distributional solution of the p-Laplacian equation $-\Delta_p u_n = f$ in Ω_n for some f in $W^{-1,q}(D)$, then up to subsequence, u_n converges strongly in $W^{1,p}(D)$ (u_n are extended by zero outside Ω_n) to the function u which is the distributional solution of the equation $-\Delta_p u = f$ in Ω. This result is interesting only in the case where p satisfies $d-1 < p \leq d$ because the case where $p > d$ is trivial due to the fact that functions in $W^{1,p}(D)$ are continuous and the convergence of solutions follows easily. To apply this result to our problem, we choose $\Omega_n = \Omega \setminus \Sigma_n$ and notice that $\{\Omega_n\}_n$ converges to $\Omega \setminus \Sigma$ in the Hausdorff topology where Σ is the limit of Σ_n (note that to not overburden terminologies, we do not make difference between the Hausdorff convergence of compact sets and of open sets). From the lower semicontinuity of the function F and the continuity with respect to the domains variation of solutions, the lower semicontinuity of the functional $\mathcal{F}$ follows easily and also the existence of an optimal shape.

Since problem (3) has solution, we are interested in the asymptotic behavior of the optimal set Σ_l as $l \to +\infty$. As it is usual in this kind of problems, let us associate to every $\Sigma \in \mathcal{A}_l(\Omega)$ a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and define functionals $F_l : \mathcal{A}_l(\Omega) \to [0, +\infty)$ and $G_l : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$F_l(\Sigma) := \int_\Omega F(x, v_l)dx,$$

where $v_l(x) = (l^{\frac{q}{d-1}} u, l^{\frac{q}{p(d-1)}} \nabla u)$ and u the unique solution of equation (1)

$$G_l(\mu) = \begin{cases} F_l(\Sigma) & \text{if } \mu = \mu_\Sigma, \quad \Sigma \in \mathcal{A}_l(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

The scaling factors $l^{\frac{q}{d-1}}$ and $l^{\frac{q}{p(d-1)}}$ are needed in order to avoid the functional to degenerate to the trivial limit functional which is constant. Our main result deals with the behavior as $l \to +\infty$ of the functional G_l, and we state it in terms of Γ-convergence.

Theorem 1.2 *The functional G_l defined in* (4) Γ*-converges, with respect to the weak* topology on the class $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$, to the functional G defined on $\mathcal{P}(\overline{\Omega})$ by*

$$G(\mu) = \int_\Omega F\left(x, \theta\frac{f^{q-1}}{\mu_a^{\frac{q}{d-1}}}, \left(\theta\frac{f^q}{\mu_a^{\frac{q}{d-1}}}\right)^{1/p} n\right) dx, \tag{5}$$

where n is a unit given vector, μ_a stands for the density of the absolutely continuous part of μ with respect to the Lebesgue measure and θ is a positive constant depending only on d and p and is defined by

$$\theta = \inf\{\liminf_{l\to+\infty} l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) : \Sigma_l \in \mathcal{A}_l(I^d)\} \tag{6}$$

$I^d = (0,1)^d$ *being the unit cube in $\mathbb{R}^d$.*

According to the general theory of Γ-convergence (see [6]), we deduce the following consequence of Theorem 1.2:

Corollary 1.3
- *if Σ_l is a solution of the minimization problem* (3)*, then up to a subsequence $\mu_{\Sigma_l} \rightharpoonup \mu$ as $l \to +\infty$, where μ is a minimizer of G;*
- *If G has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, then the whole sequence μ_{Σ_l} converges to the unique minimizer μ of G*

We will prove the Γ-convergence result in two steps corresponding to Γ-$\liminf$ and Γ-$\limsup$ inequality.

2 Γ-$\liminf$ inequality

This section is devoted to the proof of the Γ-$\liminf$ inequality of the Theorem 1.2 Before proving the Γ-$\liminf$ inequality, we need some results and constructions. We start by a construction of a set $G_{\varepsilon,l}$ which will be useful later. Let Ω be a domain, I^d be a unit cube in $\mathbb{R}^d$ and a be a positive number such that the cube $(-a,a)^d$, that we will denote by I_a^d contains Ω. Let M be a union of d segments of length 1 joining at the center of the unit cube I^d and connecting two parallel faces of the unit cube in the given direction. The segments are made in such a way that their endpoints coincide with the middle points of the faces of I^d. We consider the set $G_{\varepsilon,l}$ to be the homogenization of the set M of order $\lfloor(\frac{\varepsilon l}{2ad})^{1/(d-1)}\rfloor$ into I_a^d. It is clear that due to the particularity of the set M, the set $G_{\varepsilon,l}$ is connected and $\mathcal{H}^1(G_{\varepsilon,l}) \approx \varepsilon l$.

Lemma 2.1 *1. Let $Q_R \subset \mathbb{R}^d$ be a cube of side R and $A \subset \overline{Q}_R$ a closed subset of $\overline{Q}_R$ of positive p-capacity, then there exists a constant $C = C(d,p)$ such that, for all functions $v \in C^\infty(\overline{Q}_R)$ with nonnegative mean value and vanishing on A, we have*

$$\int_{Q_R} |v|^p dx \leq \frac{CR^d}{cap_p(A, Q_{2R})} \int_{Q_R} |\nabla v|^p dx,$$

where $cap_p(A, Q_{2R})$ stands for the relative p-capacity of the set A inside Q_{2R}.

2. *For any $\varepsilon > 0$, any $0 < l < +\infty$, any domain Ω and any function with non zero mean value $v \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l}) \subset W_0^{1,p}(\Omega)$ ($G_{\varepsilon,l}$ is the network constructed above) it holds $||v||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0) l^{\frac{1}{1-d}} ||v||_{W_0^{1,p}(\Omega)}$, where $\varepsilon_0 = cap_p(M, 2I^d)$*

3. *As a consequence, if we have a nonnegative function $f \in L^q(\Omega)$, then the function $u_{f,G_{\varepsilon,l},\Omega}$ satisfies $||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0) l^{\frac{q}{1-d}} ||f||_{L^q(\Omega)}^{q/(d-1)}$*

Proof: The first assertion is a variant of the well-known Poincaré inequality. See [8] for more comment. For proving the second one, we first choose the function v to be a nonnegative smooth function on a large cube I_a^d which vanish outside $\Omega \setminus G_{\varepsilon,l}$. We consider the subdivision of cube I_a^d into subcubes which are coming from the homogenization of order $\lfloor (\frac{\varepsilon l}{2ad})^{1/(d-1)} \rfloor$ of the unit cube into I_a^d and consider the associated network $G_{\varepsilon,l}$. The side of subcubes is of order $l^{1/(1-d)}$. Let us denote the subcubes by Q_j. The set $I_a^d \setminus G_{\varepsilon,l}$ can be seen as the homogenized of order $k = \lfloor (\frac{\varepsilon l}{ad})^{1/(d-1)} \rfloor$ of $I^d \setminus M$ into I_a^d (M is the set constructed above). Let us set $\varepsilon_0 = cap_p(M, 2I^d)$ and notice that v vanishes on $G_{\varepsilon,l}$. By applying the first statement of this Lemma, it follows that

$$\int_{Q_j} |v|^p dx \leq \frac{Ck^{-d}}{cap_p(k^{-1}M, 2Q_j)} \int_{Q_j} |\nabla v|^p dx \leq \frac{Cl^{p/(1-d)}}{cap_p(M, 2I^d)} \int_{Q_j} |\nabla v|^p dx$$

and by summing up over j we get

$$\int_{I_a^d} |v|^p dx \leq \frac{C}{\varepsilon_0} l^{p/(1-d)} \int_{I_a^d} |\nabla v|^p dx.$$

Using the fact that v vanishes outside Ω, we may restrict the integrand to Ω, raise each term of the inequality to the power $1/p$ and thus getting the result by noticing that the L^p norm of the gradient $||\nabla v||_{L^p(\Omega)}$ stands for the norm $||v||_{W_0^{1,p}(\Omega)}$. The general case follows by density. For the last inequality, we use the weak version of the PDE which gives

$$\int_\Omega |\nabla u_{f,G_{\varepsilon,l},\Omega}|^p dx = \int_\Omega f u_{f,G_{\varepsilon,l},\Omega} dx \leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)}.$$

Since $u_{f,G_{\varepsilon,l},\Omega} \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l})$ we get

$$\begin{aligned} ||u_{f,G_{\varepsilon,l},\Omega}||^p_{W_0^{1,p}(\Omega)} &\leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)} \\ &\leq C(d,\varepsilon_0,\varepsilon) l^{1/(1-d)} ||u_{f,G_{\varepsilon,l},\Omega}||_{W_0^{1,p}(\Omega)} ||f||_{L^q(\Omega)}, \end{aligned},$$

and the desired result follows. □

Before proving the Γ-lim inf inequality, we recall the following estimate which will be helpful. For the proof see [5] for Lemma 2.2 and Lemma 2.3 and [4] for Lemma 2.4 (Lemma 2.4 is proved in the same way as the Lemma 2.2)

Lemma 2.2 *Assume $d = 2$ and $p \geq 2$. If $f, g \in L^q(\Omega)$ and u_f and u_g denote the respective solution of the p-Laplacian Equation with Dirichlet boundary conditions on Σ'_l, then*

$$l^q ||u_f - u_g||_{L^1(\Omega)} \leq C ||f - g||^{1/(p-1)}_{L^q(\Omega)} |\Omega|^{1/q},$$

where the constant C depends only on p. If $\Omega = Q$ (a square centered at x_0), $g = f(x_0)$ and x_0 is a Lebesgue point for f, we have

$$l^q ||u_f - u_g||_{L^1(Q)} \leq C|Q| \left(\frac{\int_Q |f(x) - f(x_0)|^q dx}{|Q|} \right)^{1/p} = |Q| r(Q).$$

Lemma 2.3 *Assume $d = 2$ and $p \leq 2$. If $f, g \in L^q(\Omega)$ and u_f and u_g denote the respective solution of the p-Laplacian Equation with Dirichlet boundary conditions on Σ'_l, then*

$$l^q ||u_f - u_g||_{L^1(\Omega)} \leq C ||f - g||_{L^q(\Omega)} |\Omega|^{1/q} \left(||f||^q_{L^q(\Omega)} + ||||^q_{L^q(\Omega)} \right)^{(2-p)/p},$$

where the constant C depends only on p. If $\Omega = Q$ (a square centered at x_0), $g = f(x_0)$ and x_0 is a Lebesgue point for f, with $f(x_0) \neq 0$, we have

$$l^q ||u_f - u_g||_{L^1(Q)} \leq C|Q| |f(x_0)|^{(2-p)/(p-1)} \left(\frac{\int_Q |f(x) - f(x_0)|^q dx}{|Q|} \right)^{1/q} = |Q| r(Q).$$

Lemma 2.4 *Assume $d \geq 3$ and $p > d-1$. If $f, g \in L^q(\Omega)$ and u_f and u_g denote the respective solution of p-Laplacian equation with Dirichlet boundary condition on $\Sigma_l' = \Sigma_l \cup G_{\varepsilon,l}$, then*

$$l^{q/(d-1)}||u_f - u_g||_{L^1(\Omega)} \leq C|\Omega|^{1/q}||f-g||_{L^q(\Omega)}^{1/(d-1)},$$

where $C = C(d, p, \varepsilon_0, \varepsilon)$. In particular, if $\Omega = Q$ a cube centered at x_0, $g = f(x_0)$ and x_0 is a Lebesgue point for f, then

$$l^{q/(d-1)}||u_f - u_g||_{L^1(Q)} \leq |Q|r(Q) = C|Q|\left(\frac{\int_Q |f(x) - f(x_0)|^q dx}{|Q|}\right)^{1/p}.$$

In the following proposition, we prove that the Γ-lim inf functional is bounded below by the candidate limit functional G in (5).

Proposition 2.5 *Under the same hypotheses of Theorem* 1.2, *denoting by G^- the functional Γ-$\liminf_l G_l$, it holds $G^-(\mu) \geq G(\mu)$ for any $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any sequence $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that μ_{Σ_l} weakly* converges to μ, we have*

$$\liminf_{l \to +\infty} \int_\Omega F(x, l^{\frac{q}{d-1}} u_{f,\Sigma_l,\Omega}, l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l,\Omega}) dx \geq G(\mu).$$

Proof: Let $\Sigma_l' = \Sigma_l \cup G_{\varepsilon,l}$ and set $u_l' = u_{f,\Sigma_l',\Omega}$. It is obvious that $0 \leq u_l' \leq u_{f,G_{\varepsilon,l},\Omega}$ and Lemma 2.1 gives

$$||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d, \varepsilon_0, \varepsilon, f) l^{\frac{q}{1-d}}.$$

It follows that $l^{\frac{q}{d-1}} u_l'$ is L^p bounded, so up to a subsequence $l^{\frac{q}{d-1}} u_l'$ converges weakly in $L^p(\Omega)$. For the sequence $(u_l)_l$, we have two situations which correspond to the L^p boundedness or not of the sequence $(l^{\frac{q}{d-1}} u_l)_l$. For the case where the sequence $(l^{\frac{q}{d-1}} u_l)_l$ is not L^p bounded the inequality in the proposition is trivial since the lim inf of the functional is $+\infty$ by the coerciveness of the function F. Now we assume that $(l^{\frac{q}{d-1}} u_l)_l$ is L^p bounded and this also implies the L^p boundedness of $(l^{\frac{q}{p(d-1)}} \nabla u_l)_l$. In fact we have

$$\int_\Omega |\nabla u_l|^p dx = \int_\Omega f u_l dx$$

since u_l is the solution of the equation (1) with Σ replaced by Σ_l and multiplying this equality by $l^{\frac{q}{d-1}}$ we get the required L^p boundedness. From this L^p

boundedness, we have that the sequence of sets $(\Sigma_l)_l$ spread over all the domain Ω and the corresponding solution u_l converges strongly to zero in $W^{1,p}(\Omega)$. From the L^p boundedness of $(l^{\frac{q}{d-1}}u_l)_l$, up to extracting a subsequence, there exists a function w in $L^p(\Omega)$ such that

$$\lim_{l\to+\infty} l^{\frac{q}{d-1}}\int_\Omega gu_l dx = \int_\Omega gwdx, \quad \forall g \in L^q(\Omega).$$

Moreover this convergence is strong in $L^p(\Omega)$. In fact from the strong convergence of u_l to zero in $L^p(\Omega)$ we may extract a subsequence (denoted by the same indices) which converges pointwise to zero and from the inequalities

$$0 \le \int_\Omega l^{\frac{q}{d-1}}u_l dx \le C(\Omega)\|l^{\frac{q}{d-1}}u_l\|_{L^p(\Omega)} \le C(\Omega)$$

and the nonnegativity of functions $l^{\frac{q}{d-1}}u_l$ we may have the existence of some constant C (which may be different from the above $C(\Omega)$) such that $0 \le l^{\frac{q}{d-1}}u_l \le C$. This gives (up to extraction of further subsequence) the pointwise of $l^{\frac{q}{d-1}}u_l$ to some function u which turns out to be equal to w. By the dominated convergence theorem we have

$$\lim_{l\to+\infty}\int_\Omega |l^{\frac{q}{d-1}}u_l|^p dx = \int_\Omega u^p dx = \int_\Omega w^p dx.$$

This equality and the weak convergence give the strong L^p convergence of $l^{\frac{q}{d-1}}u_l$ to w. We will estimate w from below more precisely we will show that, for almost any $x_0 \in \Omega$, it holds

$$w(x_0) \ge \theta \frac{f(x_0)^{1/(p-1)}}{(\mu_a(x_0)+\varepsilon)^{\frac{q}{d-1}}}. \tag{7}$$

We begin by estimating first w on a cube Q centered at the point $x_0 \in \Omega$. We assume that x_0 is a Lebesgue point for f and $|Q|^{-1}\mu(Q) \to \mu_a(x_0)$ as Q shrinks around x_0. Assume also $f(x_0) > 0$ otherwise (7) would be trivial. We have

$$\lim_{l\to+\infty} l^{\frac{q}{d-1}}\int_Q u_l dx = \int_Q wdx,$$

we use

$$u_l \ge u_l' \ge u_{f,\Sigma_l',Q} \ge u_{f(x_0),\Sigma_l',Q} - |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}| \ \text{ in } \ Q,$$

where the first inequality comes from the maximum principle and the second from the fact that we add Dirichlet boundary condition on Q. The second part of Lemma 2.2,Lemma 2.3 or Lemma 2.4 (depending on the dimension d and p) gives

$$\int_Q |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}| dx \le l^{\frac{q}{1-d}} |Q| r(Q).$$

It remains to estimate the first term. First of all let us define the number $L(l,Q) = \mathcal{H}^1(\Sigma_l' \cap Q)$ and observe that

$$u_{f(x_0),\Sigma_l',Q} = f(x_0)^{1/(p-1)} u_{1,\Sigma_l',Q}.$$

For simplicity of the notation, we denote $u_{1,\Sigma_l',Q}$ by v_l. By a change of variables, if we assume the side of cube Q to be λ and we define $v_{l,\lambda} = \lambda^{-q} v_l(\lambda x)$ (thinking for instance that both cubes are centered at the origin), we get $v_{l,\lambda} = u_{1,\lambda^{-1}\Sigma_l',I^d}$. It is easy to see that

$$\lambda^{-1}\Sigma_l' \in \mathcal{A}_{L(l,Q)/\lambda}(I^d);$$

moreover, it holds $L(l,Q) \to +\infty$ as $l \to +\infty$, since

$$L(l,Q) \ge \mathcal{H}^1(G_{\varepsilon,l} \cap Q) \approx \varepsilon l |Q|. \tag{8}$$

Using (8) and the fact that $\mu_l = l^{-1}\mathcal{H}^1 \llcorner \Sigma_l$, we may estimate the ratio between $L(l,Q)$ and l. It follows from the *weak** convergence of μ_l to μ that $\limsup_{l\to+\infty} \mu_l(Q) \le \mu(\overline{Q})$. So we have

$$\limsup_{l\to+\infty} \frac{L(l,Q)}{l} \le \mu(\overline{Q}) + \varepsilon|Q|. \tag{9}$$

Using the definition of θ and the change of variables $y = \lambda x$ we have,

$$\begin{aligned}
\liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \int_Q v_l(y)dy &= \liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \lambda^{d+q} \int_{I^d} v_{l,\lambda}(x)dx \\
&= \liminf_{l\to+\infty} \left(\lambda^{-1}L(l,Q)\right)^{\frac{q}{d-1}} \lambda^{d+q+\frac{q}{d-1}} \int_{I^d} v_{l,\lambda}(x)dx \\
&\ge \lambda^{d+q+\frac{q}{d-1}}\theta
\end{aligned}$$

hence using the fact that $\lambda^d = |Q|$ we get

$$\begin{aligned}
\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_Q v_l(y)dy &\ge \liminf_{l\to+\infty} \left(\frac{l}{L(l,Q)}\right)^{\frac{q}{d-1}} \liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \int_Q v_l(y)dy \\
&\ge \lambda^{d+q+\frac{q}{d-1}}\theta \left(\frac{1}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^{\frac{q}{d-1}} \\
&= \left(\frac{|Q|}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^{\frac{q}{d-1}} |Q|\theta.
\end{aligned}$$

This implies that

$$|Q|^{-1}\int_Q w dx \geq -r(Q) + \left(\frac{|Q|}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^{\frac{q}{d-1}} \theta f(x_0)^{1/(p-1)}.$$

We know that $r(Q)$ tends to 0 when the cube Q shrinks to x_0, whenever x_0 is a Lebesgue point for f. Now we let the cube Q shrinks toward x_0 with x_0 satisfying the previous assumption, then we get

$$w(x_0) \geq \frac{\theta f(x_0)^{1/(p-1)}}{(\mu_a(x_0) + \varepsilon)^{\frac{q}{d-1}}}.$$

It remains to do some estimate on the gradient. Since the sequence functions $(l^{\frac{q}{p(d-1)}}\nabla u_l)_l$ is L^p bounded, up to subsequence it converges weakly to some function $z \in L^p(\Omega, \mathbb{R}^d)$. Using the fact that

$$\lim_{l\to+\infty}\int_\Omega |l^{\frac{q}{p(d-1)}}\nabla u_l|^p dx = \lim_{l\to+\infty} l^{\frac{q}{d-1}}\int_\Omega f u_l dx = \int_\Omega f w dx,$$

and arguing as in the case of the functions $l^{\frac{q}{d-1}}u_l$ we get the strong L^p convergence of $(l^{\frac{q}{p(d-1)}}\nabla u_l)_l$ to the function z and $|z| = (fw)^{1/p}$ therefore we may write $z = (fw)^{1/p}n$ where n is the vector defined by

$$n = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

The unit vector n may be assumed to be the pointwise limit as $l \to +\infty$ of $\frac{\nabla u_l}{|\nabla u_l|}$. Let us denote by v_l the vector $(l^{\frac{q}{d-1}}u_l, l^{\frac{q}{p(d-1)}}\nabla u_l) \in L^p(\Omega, \mathbb{R}^{d+1})$. Assume that

$$\sup_l \|F(\cdot, v_l(\cdot))\|_{L^1(\Omega)} < +\infty,$$

then we have the L^p boundedness of v_l thanks to the coerciveness of F (the third condition on the function F). Let $(v_l)_l$ be a subsequence not relabeled which converges strongly to some $v = (w, (fw)^{1/p}n) \in L^p(\Omega, \mathbb{R}^{d+1})$ (this happens because of the fact that u_l is solution and the above part of this proof). Then by the lower semicontinuity of F and Fatou's Lemma it holds

$$\liminf_{l\to+\infty}\int_\Omega F(x, v_l(x))dx \geq \int_\Omega \liminf_{l\to+\infty} F(x, v_l(x))dx \geq \int_\Omega F(x, v(x))dx.$$

Using the condition (2) we get

$$\liminf_{l\to+\infty}\int_\Omega F(x, v_l(x))dx \geq \int_\Omega F\left(x, \frac{\theta f^{q-1}}{(\mu_a(x)+\varepsilon)^{\frac{q}{d-1}}}, \left(\frac{\theta f^q}{(\mu_a(x)+\varepsilon)^{\frac{q}{d-1}}}\right)^{1/p} n\right) dx.$$

passing to the limit as $\varepsilon \to 0$ and again using the lower semicontinuity of F and Fatou's Lemma it holds

$$\liminf_{l\to+\infty} \int_\Omega F(x, v_l(x))dx \geq \int_\Omega F\left(x, \frac{\theta f^{q-1}}{(\mu_a(x))^{\frac{q}{d-1}}}, \left(\frac{\theta f^q}{(\mu_a(x))^{\frac{q}{d-1}}}\right)^{1/p} n\right) dx.$$

□

3 Γ-lim sup inequality

The goal of this section is to prove the Γ-lim sup inequality of the theorem 1.2. Before proving the Γ-lim sup inequality we introduce a definition and prove some preliminaries results. We start by the definition of tiling set.

Definition 3.1 *A set $\Sigma \in \mathcal{A}_l(I^d)$ is called tiling set if $\Sigma \cap \partial I^d$ coincides with the 2^d vertices of I^d.*

If $\Sigma \in \mathcal{A}_l(I^d)$ is tiling set and Σ_k is the homogenization of order k of Σ into I^d, then Σ_k remains connected and

$$\mathcal{H}^1(\Sigma_k) = k^{d-1}\mathcal{H}^1(\Sigma).$$

Lemma 3.2 *Given $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ a tiling set, a domain $\Omega \subset \mathbb{R}^d$ and $f \in L^q(\Omega)$, we consider the sequence of sets*

$$\Sigma^k = \bigcup_{y\in k^{-1}\mathbb{Z}^d} (y + k^{-1}\Sigma_0 \cup \partial I^d) \cap \overline{\Omega}$$

and consider the sequence of functions $(u_k)_k$ and $(v_k)_k$ given by

$$u_k = k^q u_{f,\Sigma^k,\Omega}, \quad v_k = k^{q/p}\nabla u_{f,\Sigma^k,\Omega}$$

then

$$u_k \to c(\Sigma_0)f^{1/(p-1)} \text{ in } L^p(\Omega), \quad v_k \to \left(c(\Sigma_0)f^{q)}\right)^{1/p} n \text{ in } L^p(\Omega, \mathbb{R}^d)$$

as $k \to +\infty$, where $c(\Sigma_0)$ is a constant given by $\int_\Omega u_{1,\Sigma_0,I^d}dx$ and n is some given unit vector.

Proof:

We will prove weak L^p convergence and the use of the procedure as the one in the proof of the Γ-$\liminf$ inequality will gives the desired result. Let us set $\varepsilon_0 = cap_p(\Sigma_0) > 0$, then thanks to Lemma 2.1 the sequence $(u_k)_k$ is bounded in $L^p(\Omega)$. So up to a subsequence it converges weakly in $L^p(\Omega)$ to some function. Let us consider the subsequence (denoted by the same indices) $(u_k)_k$ and its weak limit $w_{f,\Sigma_0,\Omega}$. It is obvious that the pointwise value of this limit function depends only on the local behavior of f. In fact, we may produce small cubes around each point $x \in \Omega$ which do not affect each other and if $f = \sum_j f_j 1_{A_j}$ is piecewise constant (the pieces A_j being disjoint open sets, for instance), then for k large enough the value of u_k at $x \in A_j$ depends only of f_j (u_k vanishes on $k^{-1}\partial I^d$). From the rescaling property of the p-Laplacian operator Δ_p, if f is a piecewise constant function, it holds $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} w_{1,\Sigma_0,\Omega}$. It is clear that in the case $f = 1$, since we are simply homogenizing the function u_{1,Σ_0,I^d}, the limit of the whole sequence $(u_k)_k$ exists and does not depend on the global geometry of Ω, but it is a constant. An easy computation shows that the constant is $c(\Sigma_0)$. It remains to extend the equality for non piecewise constant function belonging to $L^q(\Omega)$. Let $f \in L^q(\Omega)$ be a generic function and $(f_n)_n$ a sequence of piecewise constant functions approaching f in $L^q(\Omega)$. Up to a subsequence it holds $k^q u_{f,\Sigma_k,\Omega} \rightharpoonup w_{f,\Sigma_0,\Omega}$ and $k^q u_{f_n,\Sigma^k,\Omega} \rightharpoonup f_n^{1/(p-1)} c(\Sigma_0)$ as $k \to +\infty$. By Lemma 2.2, Lemma 2.3 or Lemma 2.4 (depending on d and p) it holds also

$$||k^q u_{f,\Sigma^k,\Omega} - k^q u_{f_n,\Sigma^k,\Omega}||_{L^1(\Omega)} \leq C||f - f_n||_{L^q(\Omega)}^{1/(p-1)}.$$

taking into account the lower semicontinuity of the $L^1(\Omega)$-norm with respect to the $L^p(\Omega)$-weak topology, we get, passing to the limit as $k \to +\infty$,

$$||w_{f,\Sigma_0,\Omega} - f_n^{1/(p-1)} c(\Sigma_0)||_{L^1(\Omega)} \leq C||f - f_n||_{L^q(\Omega)}^{1/(p-1)}.$$

We now pass to the limit as $n \to +\infty$ and using Fatou's Lemma (up to a subsequence f_n converges pointwise a.e. to f), we get $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} c(\Sigma_0)$ and the proof is over. □

This result remains true even if Σ_0 is not tiling. In fact we have never used the fact that Σ_0 is tiling in the proof. We keep it for the up coming construction. One problem in the previous Lemma is that we have used the whole boundary of the unit cube which is not an one dimensional set (if $d \geq 3$) and consequently the set Σ^k is not an one dimensional set. In the following

Lemma, we prove an estimate on an unit cube which will be useful for proving that, in the case where $d \geq 3$, $u_{f,\Sigma^k,\Omega}$ may be approximated by $u_{f,\Sigma_l^k,\Omega}$ where Σ_l^k is an one dimensional closed and connected set. It is obvious that in the case $d = 2$, ∂I^2 is an one dimensional set then there is no need of approximations.

Lemma 3.3 *Let $\Sigma \in \mathcal{A}_l(I^d)$ be a tiling set such that the corresponding rescaled state functions $l^{\frac{q}{1-d}} u_{f,\Sigma,I^d}$ are uniformly L^p bounded, then there exists $T_l \in \mathcal{A}_l(I^d)$ such that $\mathcal{H}^1(T_l) \ll l$ and if we denote by $u_l = u_{f,\Sigma,I^d}$ and v_l the solution of the equation*

$$\begin{cases} -\Delta_p u = f \ \ in \ \ I^d \setminus \Sigma \cup T_l^\alpha \\ u = 0 \ \ in \ \ \Sigma \cup T_l, \end{cases}$$

then $v_l \leq u_l + c_l l^{\frac{q}{1-d}}$ on I^d where c_l is a constant dependent of l and goes to zero as l goes to infinity.

Proof: Let $\Sigma \in \mathcal{A}_l(I^d)$ be a tiling set such that the sequence $(\tilde{u}_l)_l = (l^{\frac{q}{d-1}} u_{f,\Sigma,I^d})_l$ is L^p bounded and denote by u_l the solution of the equation

$$\begin{cases} -\Delta_p u = f \ \text{ in } \ I^d \setminus \Sigma \\ u = 0 \ \text{ on } \ \Sigma, \end{cases}$$

and by v_l^k the solution of the equation

$$\begin{cases} -\Delta_p u = f \ \text{ in } \ I^d \setminus \Sigma \cup \Sigma_k \\ u = 0 \ \text{ on } \ \Sigma \cup \Sigma_k, \end{cases}$$

where Σ_k is grid of length k contained in the boundary of I^d and converges to it in Hausdorff distance. Since Σ is tiling, we may choose Σ_k such that $\Sigma \cup \Sigma_k$ is connected for all k. For l fixed, $(\Sigma \cup \Sigma_k)_k$ is a sequence of connected sets which converges to the connected set $\Sigma \cup \partial I^d$ then by Šverák continuity result (see [3], [10] depending on p) the sequence $(v_l^k)_l$ converges strongly to u_l in $W^{1,p}(I^q)$ as $k \to +\infty$. As consequence $(l^{\frac{q}{d-1}} v_l^k)_l$ (as well as $l^{\frac{q}{d-1}}(v_l^k - u_l)$) is L^p bounded more precisely there exists a constant C_k such that

$$\|l^{\frac{q}{d-1}}(v_l^k - u_l)\|_{L^p(\Omega)} \leq C_k. \tag{10}$$

Moreover C_k may be as small as we want for k large enough. Now let k depends on l say $k = k(l)$ and consider the set $\Sigma_l = \Sigma \cup \Sigma_{k(l)}$. We may choose $k(l)$ such that $k(l) \ll l$ and $k(l) \to +\infty$ as $l \to +\infty$. This make the length of Σ_l to be asymptotically equivalent to l. $(\Sigma_l)_l$ is a sequence of connected sets converging to the connected set $\overline{I}^d$ the closure of the unit cube then the associated sequence of solutions converges strongly to zero in $W^{1,p}(I^d)$ and $(l^{\frac{q}{d-1}} v_l^{k(l)})_l$ are L^p bounded. Moreover $l^{\frac{q}{d-1}} v_l^{k(l)}$ satisfies the inequality (10). From the maximum principle we get $v_l - u_l \geq 0$ (setting $v_l = v_l^{k(l)}$) and from the above boundedness and Hölder inequality it holds

$$0 \leq \int_{I^d} (v_l - u_l)dx \leq c_l l^{\frac{q}{1-d}}.$$

We obtain easily the existence of some constant c_l (it may be different from the above constant c_l but it goes to zero as $l \to +\infty$) such that the inequality

$$v_l - u_l \leq c_l l^{\frac{q}{1-d}}$$

holds in I^d and taking $T_l = \Sigma_{k(l)}$ we are done. As consequent, if we pass to the limit in (10) (k replaced by $k(l)$) as $l \to +\infty$ we get

$$\lim_{l\to+\infty} l^{\frac{q}{d-1}} v_l^{k(l)} = \lim_{l\to+\infty} l^{\frac{q}{d-1}} u_l \quad \text{in} \quad L^p(I^d).$$

Using the same techniques as in the proof of Γ-$\liminf$ we get also

$$\lim_{l\to+\infty} l^{\frac{q}{p(d-1)}} \nabla v_l^{k(l)} = \lim_{l\to+\infty} l^{\frac{q}{p(d-1)}} \nabla u_l \quad \text{in} \quad L^p(I^d, \mathbb{R}^d).$$

□

Due to the terminology suggested in [5], the sets satisfying the hypothesis of the Lemma 3.3 will be called almost boundary-covering sets if $d \geq 3$ and boundary-covering sets if $d = 2$. We have proved the Lemma for the unit cube but the result remains true for a cube of any side as well as an open domain with Lipschitz boundary. Now we built an almost boundary-covering set that will be used for the construction of the recovering sequence for the Γ-$\limsup$ inequality. This result is stated in the different way than its analogous in [5] and [4] but the idea is the same.

Lemma 3.4 *There exists a minimizer set $\Sigma \in \mathcal{A}_l(I^d)$ for the function θ which is almost boundary-covering, with*

$$\lim_{l\to+\infty} l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx = \theta$$

and consequently if we denote by $u_{1,\Sigma}$ the solution of the same equation which vanish only on Σ and not on whole the boundary of I^d we get

$$\lim_{l \to +\infty} l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma} dx = \theta.$$

Proof: By definition of θ, we may find a set $\Sigma_1 \in \mathcal{A}_{l_1}(I^d)$ such that

$$\lim_{l_1 \to +\infty} l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx = \theta.$$

Now, we want to enlarge the set Σ_1 to get a set Σ_2 which is almost boundary-covering. Let $\gamma = \cup_{j=1}^{2^d} S_j$ where S_j is the shortest segment joining Σ_1 to the j^{th} vertice of I^d cube. We set $\Sigma_2 = \Sigma_1 \cup T_{l_1} \cup \gamma$ where T_{l_1} is a uniform grid contained in the boundary of the unit cube I^d of length $l_1' \ll l_1$ and $l_1' \to +\infty$ whenever $l_1 \to +\infty$. Up to adding one segment, we may assume Σ_2 connected. The length $l_2 = \mathcal{H}^1(\Sigma_2)$ does not exceed the number $l_1 + (l_1' + (2^d + 1)\sqrt{d})$ and $\frac{l_2}{l_1} \to 1$ as $l_1 \to +\infty$. This implies

$$\lim_{l_2 \to +\infty} l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d} dx \le \lim_{l_1 \to +\infty} \left(\frac{l_2}{l_1}\right)^{\frac{q}{d-1}} \lim_{l_1 \to +\infty} l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx = \theta.$$

For the opposite inequality, using the fact that the length of the set γ is asymptotically irrelevant, we may have the inequality $u_{1,\Sigma_1,I^d} \le u_{1,\Sigma_2,I^d} + c_{l_1} l_1^{\frac{q}{1-d}}$ thank to a procedure similar to the one of Lemma 3.3. It holds

$$\theta = \lim_{l_1 \to +\infty} l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx \le \lim_{l_1 \to +\infty} \left(\frac{l_1}{l_2}\right)^{\frac{q}{d-1}} \lim_{l_1 \to +\infty} l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d} dx,$$

and the result follows by observing that $\frac{l_1}{l_2} \to 1$ as $l_1 \to +\infty$.

□

We have all the ingredients for proving the Γ-lim sup inequality. We will start from a particular class of measures. Let us call piecewise constant probability measures those probability measures $\mu \in \mathcal{P}(\overline{\Omega})$ which are of the form

$$\mu = \rho dx, \text{ with, } \rho \in L^1(\Omega), \int_\Omega \rho dx = 1, \ \rho > 0,$$

for a piecewise constant function $\rho = \sum_{j=1}^m \rho_j I_{\Omega_j}$, the pieces Ω_j being disjoint Lipschitz open subsets with the possible exception of $\Omega_0 = \Omega \setminus \cup_{j=1}^m \Omega_j$.

Proposition 3.5 *Under the same hypotheses of Theorem* 1.2*, we have*

$$F^+(\mu) \leq G(\mu), \quad \textit{where} \quad F^+ = \Gamma - \limsup_{l \to +\infty} G_l,$$

for any piecewise constant measure $\mu \in \mathcal{P}(\overline{\Omega})$*. This means that for any such a measure* μ *and* $\varepsilon > 0$*, there exists a family of sets* $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ *such that the measure* μ_{Σ_l} *weakly** *converges to the measure* μ *and moreover*

$$\limsup_{l \to +\infty} \int_\Omega F(x, l^{\frac{q}{d-1}} u_{f,\Sigma_l,\Omega}, l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l,\Omega}) dx \leq G(\mu).$$

Proof: Apply Lemma 3.4 and take an almost boundary-covering and minimizing set $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ such that

$$\lim_{l_0 \to +\infty} l_0^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_0,I^d} dx = \theta.$$

Now, we define the set Σ_l^j by homogenizing into Ω_j the set Σ_0 of order $k(l,j)$ that is

$$\Sigma_l^j = \overline{\Omega_j} \cap k(l,j)^{-1}(\mathbb{Z}^d + \Sigma_0).$$

Since Σ_0 is tiling , for $k(l,j)$ large enough Σ_l^j remains connected and

$$\mathcal{H}^1(\Sigma_l^j) = |\Omega_j| K(l,j)^{d-1} \mathcal{H}^1(\Sigma_0) \leq |\Omega_j| K(l,j)^{d-1} l_0.$$

Let $\Sigma_{l_1} \in \mathcal{A}_{l_1}(\Omega)$ be a set contained in the internal boundary of the union of Ω_j and converges to it in the Hausdorff topology as $l_1 \to +\infty$ (in the case of dimension 2 we take Σ_{l_1} to be $\cup_{j=0}^m \partial\Omega_j$ which an one dimensional set). Due to the connectedness of Σ_{l_1}, the corresponding solution converges to the solution associated to the internal boundary of $\cup_{j=0}^m \Omega_j$ as well. Then we choose $\Sigma_l = \cup_{j=0}^m \Sigma_l^j \cup \Sigma_{l_1}$. We may assume Σ_l connected otherwise we add some segments to connect all the pieces. The family of sets Σ_l is admissible (i.e. $\Sigma_l \in \mathcal{A}_l(\Omega)$ and $\mu_{\Sigma_l} \rightharpoonup \mu$) if we have, as $l \to +\infty$,

$$\sum_{j=0}^m |\Omega_j| k(l,j)^{d-1} l_0 + l_1 \leq l \quad \text{and is asymptotic to} \quad l;$$

$$\frac{k(l,j)^{d-1} l_0}{l} \to \rho_j \quad \text{for} \quad j = 0, \cdots, m.$$

It is easy to see that all theses conditions are satisfied if we set

$$k(l,j) = \left\lfloor \left(\frac{l - l_1}{l_0} \rho_j \right)^{\frac{1}{d-1}} \right\rfloor.$$

Let us introduce the following sets

$$\Gamma_l^j = \overline{\Omega}_j \cap k(l,j)^{-1}(\mathbb{Z}^d + \partial I^d), \quad \Gamma_l = \bigcup_j \Gamma_l^j.$$

We choose l_1 to be a function of l (for example $l_1 = l^{\frac{d-1}{d}}$) in such a way that l_1 goes to $+\infty$ whenever l goes to $+\infty$. This allows us to manage the difference between the solutions of equation with the Dirichlet boundary condition on the whole boundary of Ω_j and with the Dirichlet boundary condition on a one dimensional subset of the boundary of Ω_j of length l_1. We are interested in the estimate of the value of $F_l(\Sigma_l)$

$$\begin{aligned} F_l(\Sigma_l) &= \int_\Omega F(x, l^{\frac{q}{d-1}} u_{f,\Sigma_l,\Omega}, l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l,\Omega}) dx \\ &= \sum_{j=0}^{m} \int_{\Omega_j} F(x, l^{\frac{q}{d-1}} u_{f,\Sigma_l,\Omega}, l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l,\Omega}) dx. \end{aligned}$$

By applying Lemma 3.2 to each Ω_j we get the following strong convergence in L^p.

$$k(l,j)^q u_{f,\Sigma_l^j \cup \Gamma_l^j, \Omega_j} \to c(\Sigma_0) f^{q-1}, \quad k(l,j)^{q/p} \nabla u_{f,\Sigma_l^j \cup \Gamma_l^j, \Omega_j} \to (c(\Sigma_0) f^q)^{1/p} n_j$$

$l \to +\infty$ where n_j may be assumed to be the pointwise limit of the normalized gradient. The term $\left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}}$ converges to $\left(\frac{l_0}{\rho_j}\right)^{\frac{q}{d-1}}$ as $l \to +\infty$ for $j = 0, \cdots, m$. Therefore the following convergence hold in the strong L^p sense

$$l^{\frac{q}{d-1}} u_{f,\Sigma_l^j \cup \Gamma_l^j, \Omega_j} \to \left(\frac{l_0}{\rho_j}\right)^{\frac{q}{d-1}} c(\Sigma_0) f^{q-1}, \quad \text{as} \quad l \to +\infty$$

$$l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l^j \cup \Gamma_l^j, \Omega_j} \to \left(\frac{l_0}{\rho_j}\right)^{\frac{q}{p(d-1)}} (c(\Sigma_0) f^q)^{1/p} n_j, \quad \text{as} \quad l \to +\infty.$$

Thanks to the Lemma 3.3 the following also hold in each Ω_j

$$l^{\frac{q}{d-1}} u_{f,\Sigma_l,\Omega} \to \left(\frac{l_0}{\rho_j}\right)^{\frac{q}{d-1}} c(\Sigma_0) f^{q-1}, \quad \text{as} \quad l \to +\infty$$

$$l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l,\Omega} \to \left(\frac{l_0}{\rho_j}\right)^{\frac{q}{p(d-1)}} (c(\Sigma_0) f^q)^{1/p} n_j, \quad \text{as} \quad l \to +\infty$$

where $u_{f,\Sigma_l,\Omega}$ is seen as its restriction to Ω_j. Putting all those results together we have

$$\begin{aligned}\limsup_{l\to+\infty} F_l(\Sigma_l) &= \limsup_{l\to+\infty} \int_\Omega F(x, l^{\frac{q}{d-1}} u_{f,\Sigma_l,\Omega}, l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l,\Omega}) dx \\ &\le \sum_{j=0}^{m} \limsup_{l\to+\infty} \int_{\Omega_j} F(x, l^{\frac{q}{d-1}} u_{f,\Sigma_l,\Omega}, l^{\frac{q}{p(d-1)}} \nabla u_{f,\Sigma_l,\Omega}) dx \\ &\le \sum_{j=0}^{m} \int_{\Omega_j} F\left(x, \left(\frac{l_0}{\rho_j}\right)^{\frac{q}{d-1}} c(\Sigma_0) f^{q-1}, \left(\frac{l_0}{\rho_j}\right)^{\frac{q}{p(d-1)}} (c(\Sigma_0) f^q)^{1/p} n_j\right).\end{aligned}$$

The choice of the set Σ_0 implies that $\lim_{l_0\to+\infty} l_0^{\frac{q}{d-1}} c(\Sigma_0) = \theta$, so using the lower semicontinuity of the function F we have

$$\begin{aligned}\limsup_{l\to+\infty} F_l(\Sigma_l) &\le \sum_{j=0}^{m} \int_{\Omega_j} F\left(x, (\rho_j)^{\frac{q}{1-d}} \theta f^{q-1}, (\rho_j)^{\frac{q}{p(1-d)}} (\theta f^q)^{1/p} n_j\right) \\ &= \int_\Omega F\left(x, \theta \frac{f^{q-1}}{\rho^{\frac{q}{d-1}}}, \left(\theta \frac{f^q}{\rho^{\frac{q}{d-1}}}\right)^{1/p} n\right) dx,\end{aligned}$$

where $n = n_j$ on Ω_j and the proof is over. □

We have to extend the result to non piecewise constant measures. By the general theory of Γ-convergence (see [6]), we know that it is enough to prove the Γ-lim sup inequality on a class which is dense in energy. Hence, due to the lower semicontinuity of the functional F, it is sufficient to prove the following

Proposition 3.6 *For any measure $\mu \in \mathcal{P}(\overline{\Omega})$ there exists a sequence $(\mu_k)_k$ of piecewise constant measures such that $\mu_k \rightharpoonup \mu$ and*

$$\limsup_k G(\mu_k) \le G(\mu) = \int_\Omega F\left(x, \frac{\theta f^{q-1}}{\mu_a^{\frac{q}{d-1}}}, \left(\frac{\theta f^q}{\mu_a^{\frac{q}{d-1}}}\right)^{1/p} n\right) dx.$$

Proof: First observe that the inequality is trivial whenever $G(\mu) = +\infty$. Assume now that $F(\mu) < +\infty$ and start proving the inequality for measures which are absolutely continuous with respect to the Lebesgue measure and have positive densities bounded away from zero. Given a measure $\mu = \rho dx$, with $\rho \ge c > 0$, it is possible to find a sequence of measures $\mu_k = \rho_k dx$

such $\rho_k \to \rho$ strongly in L^1 and μ_k are piecewise constant with $\rho_k \geq c$. The pointwise *a.e* convergence of ρ_k to ρ may be assumed and the inequality $G(\mu) \geq \limsup_k G(\mu_k)$ follows easily. So we have extended the result to any absolutely continuous measure with density bounded below away from zero. To get the result for any measure $\mu \in \mathcal{P}(\overline{\Omega})$, it is sufficient to prove that any measure μ may be approximated *weakly** by absolutely continuous measure μ_k with densities bounded below away from zero and $\limsup_k G(\mu_k) \leq G(\mu)$. Let us take $\mu = \rho dx + \mu^s$, where μ^s is the singular part of the measure μ with respect to the Lebesgue measure and ρ the density of the absolutely continuous part. We construct the sequence of absolutely continuous measure μ_k by setting $\mu_k = ((1-1/k)\rho + a_k + \phi_k)dx$, where $a_k = k^{-1}\int_\Omega \rho dx$ and $\phi_k dx \rightharpoonup \mu^s$ with $\int_\Omega \phi_k dx = \int_{\overline{\Omega}} d\mu^s$. The fact that $G(\mu) < +\infty$ implies that ρ cannot vanish thanks to the third condition on F, hence $a_k > 0$ and $\rho_k = (1-1/k)\rho + a_k + \phi_k$ is bounded below by the positive constant a_k. We have as well that μ_k *weakly** converges to μ and

$$\begin{aligned} G(\mu_k) &= \int_\Omega F\left(x, \frac{\theta f^{q-1}}{\rho_k^{\frac{q}{d-1}}}, \left(\frac{\theta f^q}{\rho_k^{\frac{q}{d-1}}}\right)^{1/p} n_k\right) dx \\ &\leq \int_\Omega F\left(x, \frac{\theta f^{q-1}}{t_k^{\frac{q}{d-1}}}, \left(\frac{\theta f^q}{t_k^{\frac{q}{d-1}}}\right)^{1/p} n_k\right) dx, \end{aligned}$$

where we have used (2) and set $t_k = (1-1/k)\rho$. Passing to the lim sup on the inequality and using Fatou's Lemma (assuming that n_k converges pointwise to n), we get the desired result. □

4 Some estimate on θ

In this section we will prove some estimate on the constant θ and in particular we will show that θ is neither 0 nor $+\infty$ so that our limit functional is not trivial.

Proposition 4.1 *We have*

$$\theta < +\infty.$$

Proof: Let $\Sigma_l \in \mathcal{A}_l(I^d)$ be a tiling set. For any positive integer number n, let us denote by Σ_l^n the homogenization of the set Σ_l of order n into I^d. Clearly,

Σ_l^n is connected and $\mathcal{H}^1(\Sigma_l^n) \leq n^{d-1}l$. Using the rescaling property of the p-Laplacian operator, it follows that

$$\theta \leq \liminf_n (n^{d-1}l)^{\frac{q}{d-1}} F_p(\Sigma_l^n, 1, I^d) = l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) < +\infty$$

which concludes the proof. □

Proposition 4.2

$$\theta \geq \frac{(d-1)q^{-q}}{(q+d-1)w_{d-1}^{\frac{q}{d-1}}},$$

where w_r stands for the volume of unit ball in $\mathbb{R}^r$.

Proof: First, we prove that

$$F_p(\Sigma_l, 1, I^d) \geq q^{-q} D_q(\Sigma_l \cup \partial I^d),$$

where $D_r(\Sigma) = \int_{I^d} h_\Sigma(x)^r dx$ and $h_\Sigma(x) = d(x, \Sigma)$ is the distance from x to Σ. For every real number A and for every real number $r > 1$, we have

$$\begin{aligned} F_p(\Sigma_l, 1, I^d) &= q \max \left\{ \int_{I^d} (v - \frac{1}{p} |\nabla v|^p) dx : v \in W_0^{1,p}(I^d \setminus \Sigma_l) \right\} \\ &\geq q \int_{I^d} (A h_{\Sigma_l \cup \partial I^d}(x))^r - \frac{1}{p} |\nabla (A h_{\Sigma_l \cup \partial I^d}(x)^r)|^p) dx. \end{aligned}$$

It is well known that the distance function is 1-Lipschitz and satisfies $|\nabla h_{\Sigma_l \cup \partial I^d}| = 1$ (and consequently $|\nabla (h_{\Sigma_l \cup \partial I^d})^r| = r (h_{\Sigma_l \cup \partial I^d})^{r-1}$). Choosing $r = q$ the conjugate exponent of p, we get

$$F_p(\Sigma_l, 1, I^d) \geq q(A - A^q \left(\frac{q^p}{p} \right)) \int_{I^d} h_{\Sigma_l \cup \partial I^d}(x)^q dx.$$

The result follows by optimizing on A (the optimal choice is $A = q^{-q}$). In [9] it has been proved that for any set $\Sigma_l \in \mathcal{A}_l(I^d)$ it holds

$$\liminf_l l^{\frac{q}{d-1}} \int_{I^d} h_{\Sigma_l}(x)^q dx \geq \frac{d-1}{(q+d-1)w_{d-1}^{\frac{q}{d-1}}}.$$

Here the same proof may be adapted by doing some modification and getting the same result even if $\Sigma_l \cup \partial I^d$ is not an one dimensional set i.e.

$$\liminf_l l^{\frac{q}{d-1}} \int_{I^d} h_{\Sigma_l \cup \partial I^d}(x)^q dx \geq \frac{d-1}{(q+d-1)w_{d-1}^{\frac{q}{d-1}}},$$

and the desired result holds. □

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