Remarks on the Total Variation of the Jacobian

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The total variation $TV(u)$ of the Jacobian determinant of non-smooth vector fields u has recently been studied in [2]. We focus on the subclass $u(x) = \varphi(x/|x|)$ of homogeneous extensions of smooth functions $\varphi : \partial B^n \to \mathbf{R}^n$. In the case $n = 2$, we explicitely compute $TV(u)$ for some relevant examples exhibiting a gap with respect to the total variation $|\mathrm{Det} Du|$ of the distributional determinant. We then provide examples of functions with $|\mathrm{Det} Du| = 0$ and $TV(u) = +\infty$. We finally show that this gap phenomenon doesn't occur if $n \geq 3$.

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1 Introduction and statements

In a recent paper [2], I. Fonseca, N. Fusco and P. Marcellini address the study of the *Jacobian determinant* $\det Du$ of fields $u : \Omega \to \mathbf{R}^n$ outside the traditional regularity Sobolev space $W^{1,n}(\Omega; \mathbf{R}^n)$, where $\Omega \subset \mathbf{R}^n$ is a fixed open set and $n \geq 2$. We refer to [2] for motivations, applications and related references. More precisely, denote

$$\det Du(x) := \frac{\partial(u^1, u^2, \dots, u^n)}{\partial(x_1, x_2, \dots, x_n)}$$

the *Jacobian determinant*, i.e., the determinant of the $n \times n$ Jacobian matrix of the gradient $Du = Du(x)$ of a smooth vector-valued map $u : \Omega \to \mathbf{R}^n$, where $u = (u^1, u^2, \dots, u^n)$ and $x = (x_1, \dots, x_n)$.

If $u \in W^{1,n}(\Omega; \mathbf{R}^n)$, since

$$|\det Du(x)| \leq n^{-n/2} \, |Du(x)|^n, \tag{1.1}$$

the Jacobian determinant is a function of class $L^1(\Omega)$. In this case the set function

$$\Omega \supset A \mapsto m(A) := \int_A \det Du(x) \, dx$$

is a measure in Ω, with *total variation* $|m|$ in Ω given by

$$|m|(\Omega) := \int_\Omega |\det Du(x)|\, dx\,.$$

Under weaker assumptions on u, taking account of the integration by part formula after multiplication by a test function, it is possible to consider the *distributional determinant*

$$\mathrm{Det}Du := \sum_{i=1}^n \frac{\partial}{\partial x_i}\,(u^1\,(\mathrm{adj}Du)_i^1)\,, \tag{1.2}$$

where $\mathrm{adj}Du$ is the matrix of the adjoints of Du, so that

$$(\mathrm{adj}Du)_i^1 := (-1)^{i+1}\frac{\partial(u^2,\ldots,u^n)}{\partial(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}\,.$$

Now, if $u:\Omega\to\mathbf{R}^n$ is a smooth map, by Laplace formula

$$\begin{aligned}&\sum_{i=1}^n \frac{\partial}{\partial x_i}\,(u^1\,(\mathrm{adj}Du)_i^1)\\ &= \sum_{i=1}^n \frac{\partial u^1}{\partial x_i}\,(\mathrm{adj}Du)_i^1 + u^1\,\sum_{i=1}^n \frac{\partial}{\partial x_i}\,(\mathrm{adj}Du)_i^1 = \det Du + 0\,.\end{aligned} \tag{1.3}$$

Then, if $u\in W^{1,n}(\Omega;\mathbf{R}^n)$, by (1.1) and by $W^{1,n}$-density of smooth maps, (1.2) coincides a.e. with the pointwise Jacobian determinant $\det Du$. Anyway, (1.2) is well defined e.g. if u is a bounded function in $L^\infty(\Omega;\mathbf{R}^n)$ and the distributional gradient Du is a summable function in the class $L^{n-1}(\Omega;\mathbf{R}^{n^2})$. Another possibility to make (1.2) be mathematically precise is to require $u\in W^{1,p}(\Omega;\mathbf{R}^n)$ for some $p\geq n^2/(n+1)$. In this case, in fact, by the Sobolev's embedding theorem $u\in L^{n^2}(\Omega;\mathbf{R}^n)$ whereas $(\mathrm{adj}Du)_i^1\in L^{n^2/(n^2-1)}(\Omega)$. Then, since $L^{n^2/(n^2-1)}$ is the dual space to L^{n^2}, we have

$$u^1\,(\mathrm{adj}Du)_i^1\in L^1(\Omega) \qquad \forall\, i=1,\ldots,n\,.$$

Motivated by the study of the relations between the distribution $\mathrm{Det}Du$ and the "*total variation*" of the Jacobian determinant, given $u\in L^\infty_{\mathrm{loc}}(\Omega;\mathbf{R}^n)\cap W^{1,p}(\Omega;\mathbf{R}^n)$, the authors in [2] consider the following limit formula

$$\begin{aligned}TV_p(u,A) := \inf\Big\{\liminf_{j\to+\infty}\int_A |\det Du_k(x)|\,dx \mid u_k\in W^{1,n}(A;\mathbf{R}^n)\,,\\ u_k \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(A;\mathbf{R}^n)\Big\}\end{aligned} \tag{1.4}$$

for every open set $A \subset \Omega$. It is obtained by a *relaxation* procedure in the weak $W^{1,p}$ topology, where $n-1<p<n$, i.e., below the natural growth exponent n.

Note that even if a priori definition (1.4) may depend on the choice of p, the representation formulas given in [2] turn out to be independent of p. Also, for certain classes of functions weak convergence in $W^{1,p}$ may be equivalently replaced by strong convergence in $W^{1,p}$. Moreover, it has been first noted by Malý [7] and by Giaquinta, Modica and Souček [4], see also Jerrard and Soner [6], that for some maps $u \in L^\infty(\Omega;\mathbf{R}^n)\cap W^{1,p}(\Omega;\mathbf{R}^n)$, with $n-1<p<n$, it may happen that the distribution $\mathrm{Det}Du$ is identically equal to zero whereas the total variation of the Jacobian determinant is different from zero. Finally, when $\mathrm{Det}Du$ is a measure, it turns out that, in general, the total variation of the Jacobian determinant (1.4) is not the total variation of the measure $\mathrm{Det}Du$.

In this paper we focus on a particular subclass of Sobolev functions $u \in L^\infty_{\mathrm{loc}}(\Omega;\mathbf{R}^n)\cap W^{1,p}(\Omega;\mathbf{R}^n)$, where $n^2/(n+1)<p<n$. To this aim, denote by B^n_r the n-ball of radius r centered at the origin, $B^n := B^n_1$, the unit ball, and by ∂B^n_r its boundary, so that $S^{n-1} := \partial B^n$ is the unit $(n-1)$-sphere in $\mathbf{R}^n$. We will consider the homogeneous extension $u : B^n \to \mathbf{R}^n$ of Lipschitz-continuous functions $\varphi : S^{n-1} \to \mathbf{R}^n$, given by

$$u(x) := \varphi\left(\frac{x}{|x|}\right). \tag{1.5}$$

Of course $u \in W^{1,p}(B^n;\mathbf{R}^n)$ for every $p<n$ whereas, since the image of B^n by u is at most $(n-1)$-dimensional, by the *area formula* [3, 3.2.3] we have $\det Du(x) = 0$ a.e. in B^n. We stress now that condition $|\mathrm{Det}Du| = 0$ is related to a *homological* property of such maps u. In fact, in [5, Vol. I], Sec. 3.2.4, it is given the following result, the proof of which is brought back for the sake of clearness.

Proposition 1.1 *If* $\varphi_\# [\![S^{n-1}]\!] = 0$, *then* $|\mathrm{Det}Du| = 0$.

Proof: following the notation from Geometric measure theory [3], $[\![S^{n-1}]\!]$ is the $(n-1)$-dimensional *current* in $\mathbf{R}^n$ given by integration of $(n-1)$-forms on S^{n-1}, equipped with the natural orientation. Also, $\varphi_\# [\![S^{n-1}]\!]$ is the current image via φ in the target space $\mathbf{R}^n$, or push forward, defined by duality as $\varphi_\# [\![S^{n-1}]\!](\omega) := [\![S^{n-1}]\!](\varphi^\# \omega)$ for every smooth $(n-1)$-form ω in the target $\mathbf{R}^n$. We recall by [5] that the *boundary* in $B^n \times \mathbf{R}^n$ of the n-current G_u carried by the graph of u is given by

$$\partial G_u \llcorner B^n \times \mathbf{R}^n = -\delta_0 \times \varphi_\# [\![S^{n-1}]\!], \tag{1.6}$$

where δ_x is the Dirac unit mass centered at $x \in B^n$. Hence, if $\varphi_\# [\![S^{n-1}]\!] = 0$ we have that the graph G_u has no boundary in $B^n \times \mathbf{R}^n$, i.e., u is a *Cartesian map*. This corresponds to a series of integration by part formulas which yield, in particular,

$$\int_{B^n} \sum_{i=1}^{n} D_i(\phi\, u^1)\,(\mathrm{adj}Du)_i^1\, dx = 0 \qquad \forall\, \phi \in C_c^1(B^n)\,.$$

By (1.2) and (1.3) this gives

$$-<\mathrm{Det}Du, \phi> + \int_{B^n} \phi\, \det Du\, dx = \int_{B^n} \sum_{i=1}^{n} D_i(\phi\, u^1)\,(\mathrm{adj}Du)_i^1\, dx$$

and hence the assertion $|\mathrm{Det}Du| = 0$, since $\det Du(x) = 0$ a.e. in B^n. □

For example, following [7] and [4], if $n = 2$ and $\varphi : S^1 \to \mathbf{R}^2$ is defined by

$$\varphi(\theta) := \begin{cases} (-1+\cos 4\theta, \sin 4\theta) & \text{if} \quad 0 \le \theta < \pi/2 \\ (1-\cos 4\theta, \sin 4\theta) & \text{if} \quad \pi/2 \le \theta < \pi \\ (-1+\cos 4\theta, -\sin 4\theta) & \text{if} \quad \pi \le \theta < 3\pi/2 \\ (1-\cos 4\theta, -\sin 4\theta) & \text{if} \quad 3\pi/2 \le \theta < 2\pi \end{cases} \tag{1.7}$$

where we identify $[0, 2\pi]$ with S^1 via $\theta \mapsto (\cos\theta, \sin\theta)$, since the image of S^1 through φ is the boundary of the union of the two unit disks of the target space $\mathbf{R}^2$ centered at $(-1, 0)$ and $(1, 0)$, and $\varphi(S^1)$ is covered twice with opposite orientation, one has $\varphi_{\#}[\![\, S^1\,]\!] = 0$ and hence $|\mathrm{Det}Du| = 0$, by Proposition 1.1.

Most importantly, due to the *non-trivial homotopy type* of the mapping φ, it is proved that u cannot be approximated by smooth maps $\{u_k\}$ neither in the strong $W^{1,p}$ topology with $\det Du_k \to \det Du$ in L^p, if $p < 2$, see [7], nor weakly with the mass in the sense of currents, i.e., $G_{u_k} \rightharpoonup G_u$ and $\mathbf{M}(G_{u_k}) \to \mathbf{M}(G_u)$, see [4].

In [2], an upper and a lower bound for the total variation of the Jacobian determinant are provided. This formulas allow to compute exactly (1.4) for a large class of functions, but do not comprehend the examples of the type (1.7) or similar ones, in which the geometry of the target space comes into play.

In this note we give a contribution in this direction by proving the following results. In the first one we explicitly compute the gap between $|\mathrm{Det}Du|$ and $TV_p(u, B^2)$ in the relevant example (1.7). In the second one we give an example of Sobolev function for which the distributional determinant is zero, whereas the total variation of the Jacobian determinant is $+\infty$. Finally, in the third one we show that this gap phenomenon does not occur in case of dimension $n \ge 3$. This is due to the Hurewicz homomorphism theorem, compare [8], and to the commutativity of the higher order *homotopy groups*, see Remark 2.2.

Theorem 1.2 *If $n = 2$, $4/3 < p < 2$, $u : B^2 \to \mathbf{R}^2$ is given by (1.5) and $\varphi : S^1 \to \mathbf{R}^2$ is defined by (1.7), then for every radius $0 < r < 1$*

$$TV_p(u, B_r^2) = 2\,\omega_2\,, \tag{1.8}$$

ω_2 being the measure of the unit disk in $\mathbf{R}^2$.

Theorem 1.3 *If $n = 2$ and $4/3 < p < 2$, there exist functions $u \in W^{1,p}(B^2; \mathbf{R}^2)$ such that $|\mathrm{Det}Du| = 0$ in B^2 and $TV_p(u, B^2) = +\infty$.*

Theorem 1.4 *Let $n \geq 3$, $n^2/(n+1) < p < n$ and $u : B^n \to \mathbf{R}^n$ be given by (1.5), where $\varphi : S^{n-1} \to \mathbf{R}^n$ is Lipschitz-continuous. Then, if $\varphi_\#[\![S^{n-1}]\!] = 0$, we have that $TV_p(u, B^n) = 0$.*

Remark 1.5 The result of Theorem 1.2 has been independently obtained by E. Paolini [10].

2 Proofs

We first show that in (1.4) we can actually impose a Dirichlet type condition.

Proposition 2.1 *Let $n^2/(n+1) < p < n$ and $u : B^n \to \mathbf{R}^n$ be given by (1.5), where $\varphi : S^{n-1} \to \mathbf{R}^n$ is Lipschitz-continuous. Then for every $0 < r < 1$*

$$\begin{aligned} TV_p(u, B_r^n) = \inf \Big\{ &\liminf_{j \to +\infty} \int_{B_r^n} |\det Du_k(x)|\, dx \mid u_k \in \mathrm{Lip}(B_r^n; \mathbf{R}^n)\,, \\ &u_k \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(B_r^n; \mathbf{R}^n)\,, \quad u_{k|\partial B_r^n} = u_{|\partial B_r^n} \Big\}. \end{aligned} \tag{2.1}$$

Proof: due to the invariance of $\int |\det Dv|\, dx$ under reparametrization, and to the homogeneity of u, it suffices to prove the claim for $r = 1$. By (1.1), for every $v \in W^{1,n}(B^n; \mathbf{R}^n)$ we can find in a standard way a sequence $\{v_k\} \subset \mathrm{Lip}(B^n; \mathbf{R}^n)$ converging to u in strong $W^{1,n}$-sense, hence in weak $W^{1,p}$-sense and with $\int_{B^n} |\det Dv_k|\, dx \to \int_{B^n} |\det Dv|\, dx$. Then by a diagonal procedure we can suppose $u_k \in \mathrm{Lip}(B_r^n; \mathbf{R}^n)$ in (1.4). To prove the claim, it then suffices to find, for every given sequence $\{u_k\} \subset \mathrm{Lip}(B^n; \mathbf{R}^n)$ weakly converging to u in $W^{1,p}(B^n; \mathbf{R}^n)$, a sequence $\{v_k\} \subset \mathrm{Lip}(B^n; \mathbf{R}^n)$ weakly converging to u in $W^{1,p}(B^n; \mathbf{R}^n)$, with $v_{k|\partial B^n} = u_{|\partial B^n}$, such that

$$\liminf_{k \to +\infty} \int_{B^n} |\det Dv_k(x)|\, dx \leq \liminf_{k \to +\infty} \int_{B^n} |\det Du_k(x)|\, dx\,. \tag{2.2}$$

To this aim, denote by ν and $\tau := (\tau_1, \ldots, \tau_{n-1})$ the outward unit normal and an orthonormal basis to the tangent $(n-1)$-space to ∂B_ρ^n, respectively. Then, setting

$D_\tau u := (D_{\tau_1} u, \dots, D_{\tau_{n-1}} u)$, we have that $|Du|^2 = |D_\nu u|^2 + |D_\tau u|^2$. Moreover, if $G \in \mathbf{R}^{nN}$ is an $N \times n$ matrix, we denote by $|M_{(j)}(G)|$ the square root of the sum of the squares of the determinants of all minors of order j of G, for $j = 1, \dots, \underline{n} := \min(n, N)$, and set $|M_{(0)}(G)| := 1$. If $u_k \rightharpoonup u$ weakly in $W^{1,p}(B^n; \mathbf{R}^n)$, and the right-hand side of (2.2) is finite, otherwise there is nothing to prove, possibly passing to a subsequence we can suppose that it is a finite limit. Then $|M_{(j)}(Du_k)|$ is equibounded in $L^{p/j}(B^n)$ for every $j = 1, \dots, n-1$, whereas $u_k \to u$ strongly in $L^{n^2}(B^n)$, by Rellich's theorem. As a consequence, since

$$\frac{j}{p} + \frac{n-1-j}{p} = \frac{n-1}{p} < \frac{n^2-1}{n^2} \qquad \forall\, j = 0, \dots, n-1\,,$$

by duality we obtain

$$\lim_{k \to +\infty} \int_{B^n} |u_k - u| \left(\sum_{j=0}^{n-1} |M_{(j)}(Du_k)|\, |M_{(n-1-j)}(Du)| \right) dx = 0\,.$$

Setting then

$$\begin{aligned} f_k(\rho) &:= \int_{\partial B^n_\rho} |u_k - u| \left(\sum_{j=0}^{n-1} |M_{(j)}(D_\tau u_k)|\, |M_{(n-1-j)}(D_\tau u)| \right) d\mathcal{H}^{n-1}\,, \\ g_k(\rho) &:= \int_{\partial B^n_\rho} |u_k - u|^p \, d\mathcal{H}^{n-1}\,, \\ h_k(\rho) &:= \int_{\partial B^n_\rho} |D_\tau u_k|^p \, d\mathcal{H}^{n-1}\,, \end{aligned}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$, possibly passing to a subsequence, by the *coarea formula* [3, 3.2.12] we have

$$\int_0^1 f_k(\rho)\, d\rho \le \varepsilon_k\,, \qquad \int_0^1 g_k(\rho)\, d\rho \le \varepsilon_k \quad \text{and} \quad \int_0^1 h_k(\rho)\, d\rho \le C\,,$$

where $C > 0$ is an absolute constant and $\{\varepsilon_k\} \subset (0,1)$ is a decreasing sequence with $\varepsilon_k \searrow 0$. Then

$$\begin{aligned} &\mathrm{meas}(\{\rho \in (0,1) \mid f_k(\rho) \le \sqrt{\varepsilon_k}\}) \ge (1 - \sqrt{\varepsilon_k})\,, \\ &\mathrm{meas}(\{\rho \in (0,1) \mid g_k(\rho) \le \sqrt{\varepsilon_k}\}) \ge (1 - \sqrt{\varepsilon_k})\,, \\ &\mathrm{meas}(\{\rho \in (0,1) \mid h_k(\rho) \le C/\sqrt{\varepsilon_k}\}) \ge (1 - \sqrt{\varepsilon_k})\,. \end{aligned}$$

As a consequence, for k large enough we can find a sequence $\{r_k\} \subset (0,1)$ of radii with $r_k \to 1$ and

$$0 < 1 - 4\sqrt{\varepsilon_k} \le r_k \le 1 - \sqrt{\varepsilon_k} \qquad \forall\, k \tag{2.3}$$

for which

$$f_k(r_k) \le \sqrt{\varepsilon_k}\,, \qquad g_k(r_k) \le \sqrt{\varepsilon_k} \quad \text{and} \quad h_k(r_k) \le \frac{C}{\sqrt{\varepsilon_k}}\,. \tag{2.4}$$

Define now $v_k : B^n \to \mathbf{R}^n$ by

$$v_k(x) := \begin{cases} u_k(x) & \text{if} \quad |x| \le r_k \\ \dfrac{|x| - r_k}{1 - r_k}\, u(x) + \dfrac{1 - |x|}{1 - r_k}\, u_k\left(r_k\, \dfrac{x}{|x|}\right) & \text{if} \quad r_k < |x| < 1\,. \end{cases}$$

For a.e. $r_k < |x| < 1$ we have

$$D_\nu v_k(x) = \frac{1}{1 - r_k} \left(u(x) - u_k\left(r_k\, \frac{x}{|x|}\right)\right)$$

and for every $i = 1, \ldots, n-1$

$$D_{\tau_i} v_k(x) = t\, D_{\tau_i} u(x) + (1-t)\, D_{\tau_i} u_k\left(r_k\, \frac{x}{|x|}\right)$$

for some $0 \le t \le 1$. Moreover if $u(x) = v(y)$, $y := R\, \dfrac{x}{|x|}$, we have $D_{\tau_i} u(x) = \dfrac{R}{|x|}\, D_{\tau_i} v(y)$. Hence for a.e. $r_k < |x| < 1$, writing $\det Dv_k$ in coordinates (ν, τ), and using the Laplace formula, we estimate

$$\begin{aligned} |\det Dv_k(x)| \le \frac{c(n)}{1 - r_k} \left(\frac{r_k}{|x|}\right)^{n-1} \left| u(x) - u_k\left(r_k\, \frac{x}{|x|}\right)\right| \times \\ \left(\sum_{j=0}^{n-1} |M_{(j)}(D_\tau u_k(y_k))|\, |M_{(n-1-j)}(D_\tau u(y_k))| \right), \end{aligned}$$

where $y_k(x) := r_k\, \dfrac{x}{|x|}$. Then by the coarea formula and by changing variables

$$\int_{B^n \setminus B^n_{r_k}} |\det Dv_k(x)|\, dx \le c(n)\, f_k(r_k)$$

which goes to zero by (2.4), so that (2.2) holds. Moreover $\{v_k\} \subset \mathrm{Lip}(B^n; \mathbf{R}^n)$ and $v_{k|\partial B^n} = u_{|\partial B^n}$. Finally, to show that $v_k \rightharpoonup u$ weakly in $W^{1,p}(B^n; \mathbf{R}^n)$, and conclude with the assertion, since $r_k < \rho < 1$ yields $1 < \rho/r_k < 2$ for k large, we readily estimate

$$\int_{B^n\setminus B^n_{r_k}} |Du_k|^p\,dx \le c\,(p)\int_{B^n\setminus B^n_{r_k}} (|D_\nu u_k|^p + |D_\tau u_k|^p)\,dx$$
$$\le c\,(n,p)\left((1-r_k)^{1-p}\,g_k(r_k) + (1-r_k)\,h_k(r_k) + (1-r_k)\int_{\partial B^n} |D_\tau u|^p\,d\mathcal{H}^{n-1}\right)$$

which is equibounded since $r_k \to 1$ and by (2.3) and (2.4)

$$(1-r_k)^{1-p}\,g_k(r_k) \le \sqrt{\varepsilon_k}^{\,2-p} \qquad \text{and} \qquad (1-r_k)\,h_k(r_k) \le 4C\,.$$

□

Proof of Theorem 1.2: first note that if $v : B^2_r \to \mathbf{R}^2$ is a Lipschitz-continuous function such that $v(x) = u(x)$ for each point $x \in \partial B^2_r$, then

$$\int_{B^2_r} |\det Dv(x)|\,dx \ge 2\,\omega_2. \tag{2.5}$$

In fact, $v_{|\partial B^2_\rho} : \partial B^2_\rho \to \mathbf{R}^2$, $\rho \in [0,r]$, defines a 1-parameter continuous family of loops in $\mathbf{R}^2$, moving $u_{|\partial B^2_r}$ to one point. Then, by definition of u and by the non-trivial homotopy type of the loop $\varphi_{|S^1}$, see [8], $v_{|B^2_r}$ must completely cover at least one of the two disks $B^2(p_i,1) \subset \mathbf{R}^2$, $p_i := ((-1)^i, 0)$, $i = 1,2$. Also, since $v_{|\partial B^2_r} = u_{|\partial B^2_r}$ has *index* zero with respect to any point of $\mathbf{R}^2 \setminus \varphi(S^1)$, the multiplicity function of v is at least 2 in each point contained in such disk, see [1]. By the area formula we finally obtain (2.5). As a consequence, by Proposition 2.1 we obtain the lower bound "$\ge$" in (1.8). To obtain the upper bound "$\le$", it suffices to define a weakly approximating smooth sequence $u_k : B^2_r \to \mathbf{R}^2$ such that

$$\lim_{k\to+\infty}\int_{B^2_r} |\det Du_k|\,dx = 2\,\omega_2\,. \tag{2.6}$$

To this aim, taking polar coordinates $x := \rho\,(\cos\theta, \sin\theta)$, we define for each $k \in \mathbf{N}$, $k > 3/r$,

$$u_k(\rho,\theta) := \begin{cases} \varphi(\theta) & \text{if} \quad 3/k \le \rho < r \\ \widetilde{u}_k(\rho,\theta) & \text{if} \quad 2/k \le \rho \le 3/k \\ \widehat{u}_k(\rho,\theta) & \text{if} \quad 1/k \le \rho \le 2/k \\ (0,0) & \text{if} \quad 0 \le \rho \le 1/k \end{cases}$$

where φ is given by (1.7), for $2/k \le \rho \le 3/k$

$$\widetilde{u}_k(\rho,\theta) := \begin{cases} \varphi(\theta) & \text{if} \quad \theta \in [0,\pi/2] \cup [\pi, 3\pi/2] \\ (k\rho - 2)\,\varphi(\theta) & \text{if} \quad \theta \in [\pi/2,\pi] \cup [3\pi/2, 2\pi] \end{cases}$$

and finally for $1/k \le \rho \le 2/k$

$$\widehat{u}_k(\rho,\theta) := \begin{cases} \varphi(\theta) & \text{if} \quad \theta \in [0,(k\rho-1)\,\pi/2] \\ \varphi((k\rho-1)\,\pi/2) & \text{if} \quad \theta \in [(k\rho-1)\,\pi/2,(4-k\rho)\,\pi/2] \\ \varphi(\theta) & \text{if} \quad \theta \in [(4-k\rho)\,\pi/2,3\pi/2] \\ (0,0) & \text{if} \quad \theta \in [3\pi/2,2\pi]\,. \end{cases}$$

Clearly $u_k : B^2_r \to \mathbf{R}^2$ is a sequence of Lipschitz continuous functions with $\mathrm{Lip}(u_k, B^2_r) \le c/k$ and $u_k \equiv u$ in $B^2_r \setminus B^2_{3/k}$; moreover, since $p < 2$, it is not difficult to show that $u_k \rightharpoonup u$ weakly in $W^{1,p}(B^2_r;\mathbf{R}^2)$. Finally, by the area formula

$$\int_{B^2_{3/k}} |\det Du_k(x)|\,dx = \int_{u_k(B^2_{3/k})} \mathcal{H}^0(B_{3/k} \cap u_k^{-1}(y))\,d\mathcal{H}^2(y) = 2\,\omega_2$$

for each k, so that (2.6) holds. $\square$

Proof of Theorem 1.3: it suffices to define a function $u \in W^{1,p}(B^2;\mathbf{R}^2)$, with 1-dimensional image, such that the current G_u carried by its graph has no boundary in $B^2 \times \mathbf{R}^2$ and for which there is a sequence $\{B_j\}$ of pairwise disjoint balls contained in B^2 such that the restriction $u_{|B_j}$ behaves like the function of Theorem 1.2, so that $TV_p(u,B_j) = 2\,\omega_2$ for every j. In fact, arguing as in Proposition 1.1, conditions $\partial G_u \llcorner B^2 \times \mathbf{R}^2 = 0$ and $\det Du = 0$ will give $|\mathrm{Det}Du| = 0$ whereas, by superadditivity of the set function $A \mapsto TV_p(u,A)$, we will obtain

$$TV_p(u,B^2) \ge \sum_{j=1}^{+\infty} TV_p(u,B_j) = \sum_{j=1}^{+\infty} 2\,\omega_2 = +\infty\,.$$

To this aim, following an example by [9], we set $B_j := B^2(c_j, 2^{-(j+1)})$, where

$$c_j = \left(-1 + \sum_{k=0}^{j-1} 2^{-k}, 0\right), \qquad j = 1,2,\ldots$$

Moreover we define $u_{|B_j} := u^{(j)} : B_j \to \mathbf{R}^2$ by

$$u^{(j)}(x) := \begin{cases} \varphi\left(\dfrac{x-c_j}{|x-c_j|}\right) & \text{if} \quad j = 1,3,5,\ldots \\ \psi\left(\dfrac{x-c_j}{|x-c_j|}\right) & \text{if} \quad j = 2,4,6,\ldots \end{cases}$$

where φ is given by (1.7) and $\psi : S^1 \to \mathbf{R}^2$ is defined by

$$\psi(\theta) := \varphi(-\theta + \pi) .$$

If $Q_j := c_j + [-2^{-(j+1)}, 2^{-(j+1)}]^{\,2}$ denotes the square circumscribing B_j, we extend $u_{|B_j}$ to Q_j as the continuous map which is constant in the x_1-variable (note that $Q_j \subset B^2$ for every $j \geq 1$). Then $u \equiv 0$ over all the sides of the boundary of the Q_j's which are parallel to the x_1-axis, whereas on the sides parallel to the x_2-axis,

$$L_j^k := c_j + \{((-1)^k \, 2^{-(j+1)}, x_2) \mid -2^{-(j+1)} \leq x_2 \leq 2^{-(j+1)}\} , \qquad k = 1, 2 ,$$

$u_{|L_j^2}$ and $u_{|L_{j+1}^1}$ parametrize the circles $\partial B^2(p_i, 1) \subset \mathbf{R}^2$, $p_i = ((-1)^i, 0)$, $i = 1, 2$, with the same order and orientation. We can thus define u over the convex hull of L_j^2 and L_{j+1}^1, the right-hand side of ∂Q_j and the left-hand side of ∂Q_{j+1}, as the continuous map which is constant along the straight lines connecting the corresponding points in L_j^2 and L_{j+1}^1 (points on which u takes the same value). We finally define u in the strip connecting L_1^1 to the boundary of B^2 as the continuous map constant in the x_1-variable, and set $u \equiv 0$ in the rest of B^2. Then, it is not difficult to show that $u \in W^{1,p}(B^2; \mathbf{R}^2)$ and that $\det Du = 0$ in B^2, whereas due to the construction

$$\partial G_u \llcorner B_j \times \mathbf{R}^2 = -\delta_{c_j} \times f_\# [\![\, S^1 \,]\!],$$

where $f = \varphi$ if j is odd, $f = \psi$ if j is even, so that since $f_\# [\![\, S^1 \,]\!] = 0$ we have $\partial G_u \llcorner B^2 \times \mathbf{R}^2 = 0$, as required. $\square$

Proof of Theorem 1.4: let $N \geq n$, $f : S^{n-1} \to \mathbf{R}^N$ be a Lipschitz-continuous function and $T \in \mathcal{R}_n(\mathbf{R}^N)$ be an n-dimensional *integer multiplicity rectifiable current* with boundary $\partial T = f_\# [\![\, S^{n-1} \,]\!]$. Then, by Thm. 1 and Prop. 1 in [11], for every $\varepsilon > 0$ it can be performed a Lipschitz-continuous function $v_\varepsilon : B^n \to \mathbf{R}^N$ with boundary values f, $v_{\varepsilon | \partial B^n} = f$, and with *n-dimensional mapping area* comparable to the mass of T. In particular, if $N = n$ and $f = \varphi$, this means

$$\int_{B^n} |\det D v_\varepsilon| \, dx \leq \mathbf{M}(T) + \varepsilon . \tag{2.7}$$

Remark 2.2 This result does hold under the condition $n \geq 3$, and it is based on the Hurewicz theorem. Roughly speaking, if $n \geq 3$, a homologically trivial $(n-1)$-dimensional cycle of a 1-connected $(n-1)$-skeleton K is homotopically trivial, hence contractible in K. Note that Theorem 1.2 actually shows that it is false if $n = 2$, simply by taking $T = 0$.

Now, since $\varphi_{\#}[\![S^{n-1}]\!] = 0$, we may apply this result, with $T = 0$ and $f = \varphi$, and define $u_k^{(\varepsilon)} : B^n \to \mathbf{R}^n$ by

$$u_k^{(\varepsilon)}(x) := \begin{cases} v_\varepsilon(kx) & \text{if} \quad |x| < 1/k \\ u(x) & \text{if} \quad 1/k \leq |x| < 1\,. \end{cases}$$

Then by the area formula and (2.7) we have

$$\int_{B^n} |\det Du_k^{(\varepsilon)}(x)|\, dx = \int_{B^n_{1/k}} |\det Dv_\varepsilon(kx)|\, dx = \int_{B^n} |\det Dv_\varepsilon(x)|\, dx \leq \varepsilon\,.$$

Moreover $u_k^{(\varepsilon)} \in \mathrm{Lip}(B^n; \mathbf{R}^n)$ and since

$$\int_{B^n_{1/k}} |Du_k^{(\varepsilon)}|^p\, dx = k^{p-n} \int_{B^n} |Dv_\varepsilon|^p\, dx$$

and $p < n$, we infer that $u_k^{(\varepsilon)} \rightharpoonup u$ weakly in $W^{1,p}(B^n; \mathbf{R}^n)$ as $k \to +\infty$. We then obtain $TV_p(u, B^n) \leq \varepsilon$ and hence the assertion, as $\varepsilon \to 0^+$. $\square$

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