# The degenerate two wells problem for piecewise affine maps

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**Abstract**

The *two wells problem* consists in finding maps $u$ which satisfy some boundary conditions and whose gradient $Du$ assumes values in the two wells $\mathbb{S}_A$, $\mathbb{S}_B$. Here $\mathbb{S}_A$ (similarly $\mathbb{S}_B$) is the *well* generated by a square matrix $A$, i.e., $\mathbb{S}_A$ is the set of matrices of the form $RA$, where $R$ is a rotation. We study specifically the case when at least one of the two matrices $A$, $B$ is singular and we characterize piecewise affine maps $u$ satisfying almost everywhere the differential inclusion $Du(x) \in \mathbb{S}_A \cup \mathbb{S}_B$. In particular we describe the *lamination* and *angle* properties, which turn out to be different from those of the nonsingular case described in detail in [15]. We also show that the two wells problem can be solved in some cases involving singular matrices, in strict contrast to the nonsingular (and not *orthogonal*) case.

## 1 Introduction

Arrigo Cellina, to whom this paper is dedicated, gave a relevant contribution to the field of *differential inclusions*, with application to several different types of problems, starting from the celebrated book [1] on set-valued maps, published in 1984 in collaboration with Jean-Pierre Aubin. In the context of *vector-valued maps*, which we consider in the research presented here, we mainly refer to the article [5], in collaboration with Stefania Perrotta, where a Dirichlet problem for differential inclusions involving *orthogonal matrices* is studied.

The differential inclusion problem is the following: let $A, B$ be given $2 \times 2$ matrices and let us denote respectively by $\mathbb{S}_A$ and $\mathbb{S}_B$ the *wells* generated by $A$ and $B$. That is, the set of matrices $\mathbb{S}_A$ (similarly $\mathbb{S}_B$) is the *well*, generated by $A$, of the form

$$\mathbb{S}_A = SO(2)\, A = \{RA : \text{ where } R \in SO(2), \text{ i.e. } R \text{ is a } \textit{rotation}\}.$$

For a given open set $\Omega \subset \mathbb{R}^2$ and a given boundary datum $\varphi$, the $2-$dimensional *Dirichlet problem* for two wells consists in finding a map $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies the conditions

$$\begin{cases} Du\,(x) \in \mathbb{S}_A \cup \mathbb{S}_B & \text{a.e. } x \in \Omega \\ u\,(x) = \varphi\,(x) & x \in \partial\Omega. \end{cases} \tag{1}$$

Above $Du$ represents the $2 \times 2$ gradient matrix of the map $u$. The *two wells problem* is relevant in nonlinear elasticity and is a model for vector-valued differential inclusions. The original mathematical formulation is due to Ball and James [2], [3] (see [9], [15] for details and references).

The case when $A, B$ are *orthogonal matrices* has been first considered in the quoted paper [5] by Cellina and Perrotta (for $3 \times 3$ matrices and zero boundary condition). Is this a special case? Why to consider only *orthogonal* matrices *$A, B$?*

The reason relies on the fact that it turns out to be impossible to solve the same two wells problem in the same context, i.e., by mean of *piecewise affine maps*, unless we start from orthogonal matrices $A, B$. Precisely, we recently proved in [15] that the two wells problem, for *nonsingular* matrices $A$ and $B$, can be solved by means of piecewise affine maps if and only if

$$BA^{-1} \in O\,(2) \quad \text{and} \quad \det\left(BA^{-1}\right) = -1. \tag{2}$$

To give a more complete picture we recall that a map $u$, solving (1) with $\varphi$ affine and satisfying some natural compatibility conditions, exists in the Sobolev class $W^{1,\infty}(\Omega;\mathbb{R}^2)$ of Lipschitz continuous maps (see [4], [6], [7], [8], [9], [16], [18]). However, for *orthogonal matrices* it was also proved (see [5], [10], [11], [12], [13], [14], [17]) that solutions exist in the class of piecewise *affine* maps. As we said, in the class of nonsingular matrices $A$, $B$ (i.e., with nonzero determinant) the Dirichlet problem (1) $-$ for instance with zero boundary datum $-$ can be solved, in the class of piecewise affine maps, (see [15]) essentially only if $A$ and $B$ are orthogonal matrices (namely, they satisfy (2)).

What about singular matrices $A$ and/or $B$? In fact this is a case which does not enter in the previous analysis. The aim of this paper is to show that if at least one of the two matrices has zero determinant then different properties happen to the geometry of the singularities of maps $u$ satisfying the differential inclusion

$$Du\,(x) \in \mathbb{S}_A \cup \mathbb{S}_B\,, \qquad \text{a.e. } x \in \Omega; \tag{3}$$

precisely, if $u$ is a piecewise affine map as in (3), then its gradient matrix $Du$ at every internal vertex of gradient-discontinuity satisfy new *lamination* and *angle conditions*, different from that one valid in the nonsingular case and described in detail in [15]. The new geometrical situation is proposed in Section 3 for one singular matrix and in Section 4 when both matrices $A$ and $B$ are singular.

Finally in Section 5 we show that the Dirichlet problem can be solved in some cases involving singular matrices, in strict contrast with respect to what happens for the nonsingular case, where the Dirichlet problem for the two well problem lacks a solution in the class of piecewise affine maps, unless $A$ and $B$ are orthogonal matrices.

## 2 General notations

The set of $2 \times 2$ *orthogonal* matrices is denoted by $O(2)$, this is the set of $R \in \mathbb{R}^{2\times 2}$ such that $R^t R = I$. The set of *special orthogonal* matrices, denoted by $SO(2)$, is the set of $R \in O(2)$ such that $\det R = 1$. We write a generic matrix in $SO(2)$ as

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

with $\varphi \in (-\pi, \pi]$.

We recall that the *singular values* of a $2 \times 2$ matrix $A$, denoted by

$$0 \le \lambda_1(A) \le \lambda_2(A),$$

are defined to be the square root of the eigenvalues of the symmetric and positive semidefinite matrix $A^t A \in \mathbb{R}^{2\times 2}$. As well known, the singular values decomposition theorem asserts that for any $A \in \mathbb{R}^{2\times 2}$, there exist $R, Q \in O(2)$ such that

$$RAQ = \operatorname{diag}(\lambda_1(A), \lambda_2(A)) = \begin{pmatrix} \lambda_1(A) & 0 \\ 0 & \lambda_2(A) \end{pmatrix}.$$

In the sequel we will adopt the following notation and definitions.

**Notation 1** *If $A$ is a $2 \times 2$ matrix, we denote with $\mathbb{S}_A$ the set of matrices*

$$\mathbb{S}_A = SO(2) \cdot A = \{RA \colon R \in SO(2)\}.$$

**Definition 2 (Piecewise affine maps)** *Let $\Omega \subset \mathbb{R}^2$ be an open set and let $u \colon \Omega \to \mathbb{R}^2$.*

*(i) We define the* singular set of $u$ *as the set*

$$\Sigma_u = \{x \in \Omega \colon u \textit{ is not differentiable in } x\}.$$

*(ii) We say that a map $u \colon \Omega \to \mathbb{R}^2$ is* piecewise affine *in $\Omega$ if*
*- $u$ is continuous on $\Omega$,*
*- $\Sigma_u$ is relatively closed in $\Omega$,*
*- $\Omega \setminus \Sigma_u$ has a finite number of connected components*
*- $Du$ is constant on each connected component of $\Omega \setminus \Sigma_u$.*

*(iii) We say that $u$ is* locally piecewise affine *if for every open set $\Omega'$ such that $\overline{\Omega'}$ is a compact subset of $\Omega$, we have that $u$ is piecewise affine in $\Omega'$.*

**Definition 3** *Let $\Sigma$ be a locally finite union of closed segments in an open set $\Omega \subset \mathbb{R}^2$. We say that a point of $\Omega$ is a* vertex *of $\Sigma$ if either it is an end point of a segment or a point where at least two segments meet.*

# 3 The semi-degenerate case

In the present section we consider the case where one of the matrix is invertible, say $A$, and the other is not. We first fix the notations.

**Notation 4** *Let $\mu = \lambda_2\left(BA^{-1}\right)$ be the largest singular value of $BA^{-1}$ (the smallest being, under our hypotheses, necessarily $0$). When $\mu \geq 1$ we let*

$$m = \sqrt{\mu^2 - 1} \quad \textit{and} \quad \nu_\pm = \left(\pm 1, \sqrt{\mu^2 - 1}\right) = (\pm 1, m)\,.$$

*Note that if $\mu = 1$, then $\nu_+ \| \nu_- \| e_1$. We also define $\theta \in [0, \pi/2]$ through*

$$\cos\theta = \frac{1}{\mu} \quad \textit{and} \quad \sin\theta = \frac{\sqrt{\mu^2 - 1}}{\mu} = \frac{m}{\mu}\,. \tag{4}$$

*As in [15], with a change of variables we can reduce all of our analysis to the case where*

$$A = I \quad \textit{and} \quad B = \Lambda = \operatorname{diag}(0, \mu)\,,$$

*with $\mu \geq 1$, to be consistent with (4). We will assume this special structure in all the following lemmas.*

We now analyze when edges and vertices can occur and we start with edges.

**Lemma 5 (Edge)** *Let $\nu = (\nu_1, \nu_2) \neq 0$ and $\varphi, \psi \in (-\pi, \pi]$. The map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$R_{-\varphi}\, u(x) = \begin{cases} x & \textit{if } \langle x; \nu \rangle = x_1\nu_1 + x_2\nu_2 > 0 \\ R_\psi \Lambda x & \textit{if } \langle x; \nu \rangle = x_1\nu_1 + x_2\nu_2 < 0 \end{cases}$$

*is continuous across the line $\langle x; \nu \rangle = 0$ if and only if $\mu \geq 1$, $\nu$ is parallel to $\nu_\pm$ and $\psi = \pm\theta$. More precisely (see Figure 1, middle case)*

$$R_{-\varphi}\, u(x) = \begin{cases} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \textit{if } x_1 + m x_2 > 0 \\ \begin{pmatrix} 0 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R_\theta \Lambda \begin{pmatrix} -m x_2 \\ x_2 \end{pmatrix} & \textit{if } x_1 + m x_2 < 0 \end{cases}$$

*or (see Figure 1, left hand case)*

$$R_{-\varphi}\, u(x) = \begin{cases} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \textit{if } x_1 - m x_2 < 0 \\ \begin{pmatrix} 0 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R_{-\theta} \Lambda \begin{pmatrix} m x_2 \\ x_2 \end{pmatrix} & \textit{if } x_1 - m x_2 > 0. \end{cases}$$

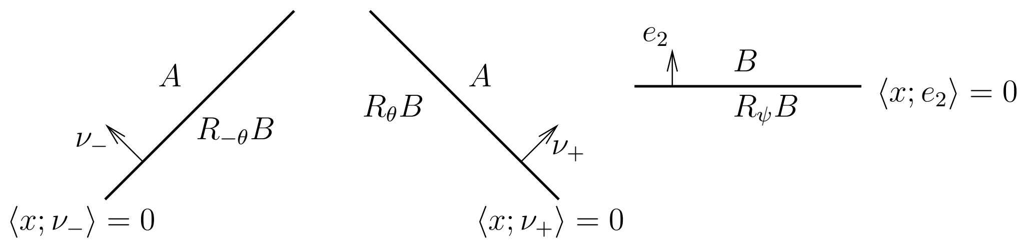


Figure 1: a scheme of 3 possible edges in the semi-degenerate case

**Remark 6** *(i) Note that within the well $\mathbb{S}_B$ there might be rank one connections, while this cannot happen within the well $\mathbb{S}_A$. More precisely if $\varphi, \psi \in (-\pi, \pi]$, then the map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$R_{-\varphi}\, u\,(x) = \begin{cases} \Lambda x & \text{if } \langle x;\nu\rangle = x_1\nu_1 + x_2\nu_2 > 0 \\ R_\psi \Lambda x & \text{if } \langle x;\nu\rangle = x_1\nu_1 + x_2\nu_2 < 0 \end{cases}$$

*is continuous across the line $\langle x;\nu\rangle = 0$ if and only if $\nu$ is parallel to $e_2 = (0,1)$. In other words, for every $\varphi, \psi \in (-\pi, \pi]$, the map $R_{-\varphi}\, u$ does not depend on $x_1$ more precisely (see Figure 1, right hand case)*

$$R_{-\varphi}\, u\,(x) = \begin{cases} x = (0, \mu x_2) & \text{if } x_2 > 0 \\ ((-\mu \sin\psi)\, x_2, (\mu\cos\psi)\, x_2) & \text{if } x_2 < 0. \end{cases}$$

*(ii) If $\mu = 1$, the lemma reads as*

$$R_{-\varphi}\, u\,(x) = \begin{cases} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{if } x_1 > 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} & \text{if } x_1 < 0 \end{cases}$$

**Proof** The map $u$ is continuous only if

$$\det(I - R_\psi \Lambda) = 1 - \mu\cos\psi = 0.$$

This can happen if and only if $\mu \geq 1$ and $\psi = \pm\theta$ where

$$\cos\theta = \frac{1}{\mu} \quad \text{and} \quad \sin\theta = \frac{\sqrt{\mu^2 - 1}}{\mu} = \frac{m}{\mu}\,.$$

It remains to show that $\nu$ is parallel to $\nu_\pm = (\pm 1, m)$. We discuss only the case where $\psi = \theta$, the case $\psi = -\theta$ being handled similarly. Since the map is continuous across the line $\langle x;\nu\rangle = 0$, we should have

$$\begin{pmatrix} 1 & \mu\sin\theta \\ 0 & 1 - \mu\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

This is equivalent to $x_1 + mx_2 = 0$ and thus $\nu$ is parallel to $\nu_+ = (1, m)$, as wished. ■

Recall that $\mu \geq 1$. If $\mu = 1$ then the directions $\nu_\pm = (\pm 1, 0)$ are parallel to $e_1$ while $\nu$ is parallel to $e_2$, hence all the edges of the singular set are parallel to $e_2$, respectively $e_1$, and a vertex can occur either with a *T-shaped configuration* (i.e., 3 angles of measure $\pi/2$, $\pi/2$, and $\pi$) or with 4 orthogonal angles; a direct verification excludes both cases. Thus a vertex can exist only if $\mu > 1$.

We will show that these vertices can only be of order 3, 4, 5 and 6. We now consider separately all these cases. In the following lemmas we assume that $\mu > 1$. The configurations considered in each lemma are schematized in Figure 2 below.

**Lemma 7 (Vertex of order 3)** *Let $\varphi, \chi, \psi \in (-\pi, \pi]$. The map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$R_{-\varphi}\, u(x) = \begin{cases} x & \text{if } \langle x; \nu_+ \rangle < 0 \text{ and } \langle x; \nu_- \rangle < 0 \\ R_\chi \Lambda x & \text{if } \langle x; \nu_+ \rangle > 0 \text{ and } \langle x; e_2 \rangle < 0 \\ R_\psi \Lambda x & \text{otherwise} \end{cases}$$

*is continuous, across the lines of discontinuities of the gradient, if and only if $\chi = -\psi = \theta$.*

**Remark 8** *(i) Therefore the only possibility of having, under the hypotheses of the lemma, a continuous $u$ is that (see Figure 2, the four cases in (3a))*

$$R_{-\varphi}\, u\,(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if } x_1 + mx_2 < 0 \text{ and } x_1 - mx_2 > 0 \\ R_\theta \Lambda x = (-mx_2, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_2 < 0 \\ R_{-\theta} \Lambda x = (mx_2, x_2) & \text{otherwise.} \end{cases}$$

*(ii) There can also be, for example, (but it is essentially the same, the only difference is the aperture of the angles) a vertex of order 3 of the form (see Figure 2, the four cases in (3b))*

$$R_{-\varphi}\, u\,(x_1, x_2) = \begin{cases} (mx_2, x_2) & \text{if } x_1 - mx_2 > 0 \text{ and } x_2 > 0 \\ (-mx_2, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_2 < 0 \\ (x_1, x_2) & \text{otherwise.} \end{cases}$$

**Proof** We can apply Lemma 5 to the two lines of discontinuity ($\langle x; \nu_+ \rangle = 0$ and $\langle x; \nu_- \rangle = 0$) to find that

$$\chi = -\psi = \theta.$$

Remark 6 ensures the continuity across $x_2 = 0$. ■

**Lemma 9 (Vertex of order 4)** *Let $\varphi, \chi_1, \chi_2, \psi \in (-\pi, \pi]$. The map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$R_{-\varphi}\, u(x) = \begin{cases} x & \text{if } \langle x;\nu_+\rangle < 0 \text{ and } \langle x;\nu_-\rangle < 0 \\ R_{\chi_1}\Lambda x & \text{if } \langle x;\nu_-\rangle > 0 \text{ and } \langle x;e_2\rangle < 0 \\ R_{\chi_2}\Lambda x & \text{if } \langle x;\nu_+\rangle > 0 \text{ and } \langle x;e_2\rangle < 0 \\ R_{\psi}\Lambda x & \text{if } \langle x;e_2\rangle > 0 \end{cases}$$

*is continuous, across the lines of discontinuities of the gradient, if and only if $\chi_1 = -\chi_2 = \theta$.*

**Remark 10** *(i) Therefore the only possibility of having, under the hypotheses of the lemma, a continuous $u$ is that (see Figure 2, the first case in (4))*

$$R_{-\varphi}\, u(x) = \begin{cases} (x_1, x_2) & \text{if } x_1 + mx_2 < 0 \text{ and } x_1 - mx_2 > 0 \\ R_{-\theta}\Lambda x = (mx_2, x_2) & \text{if } x_1 - mx_2 < 0 \text{ and } x_2 < 0 \\ R_{\theta}\Lambda x = (-mx_2, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_2 < 0 \\ R_{\psi}\Lambda x = ((-\mu\sin\psi)\, x_2, (\mu\cos\psi)\, x_2) & \text{if } x_2 > 0. \end{cases}$$

*(ii) There can also be (but it is essentially the same, the only difference is the aperture of the angles) one other vertex of order 4 (see Figure 2, the second case in (4)).*

*(iii) However there cannot be a vertex of order 4 of the form*

$$R_{-\varphi}\, u(x) = \begin{cases} x & \text{if } \langle x;\nu_+\rangle > 0 \text{ and } \langle x;\nu_-\rangle < 0 \\ R_{\chi_1}\Lambda x & \text{if } \langle x;\nu_+\rangle > 0 \text{ and } \langle x;\nu_-\rangle > 0 \\ R_{\psi} x & \text{if } \langle x;\nu_+\rangle < 0 \text{ and } \langle x;\nu_-\rangle > 0 \\ R_{\chi_2}\Lambda x & \text{if } \langle x;\nu_+\rangle < 0 \text{ and } \langle x;\nu_-\rangle < 0 \end{cases}$$

*i.e. necessarily $\chi_1 = -\chi_2 = \theta$ but no $\psi$ can verify the third part*

$$R_{-\varphi}\, u(x) = \begin{cases} (x_1, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_1 - mx_2 > 0 \\ (mx_2, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_1 - mx_2 < 0 \\ \begin{pmatrix} x_1\cos\psi - x_2\sin\psi \\ x_1\sin\psi + x_2\cos\psi \end{pmatrix} & \text{if } x_1 + mx_2 < 0 \text{ and } x_1 - mx_2 < 0 \\ (-mx_2, x_2) & \text{if } x_1 + mx_2 < 0 \text{ and } x_1 - mx_2 > 0. \end{cases}$$

**Proof** We can apply Lemma 5 to the two lines of discontinuity ($\langle x;\nu_+\rangle = 0$ and $\langle x;\nu_-\rangle = 0$) to find that

$$\chi = -\psi = \theta.$$

Remark 6 ensures the continuity across $x_2 = 0$. ■

The proofs of the next two lemmas is similar to the above ones and we do not discuss the details.

**Lemma 11 (Vertex of order 5)** *Let $\varphi, \chi_1, \chi_2, \chi_3, \psi \in (-\pi, \pi]$. The map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$R_{-\varphi}\, u(x) = \begin{cases} x & \text{if } \langle x; \nu_+ \rangle < 0 \text{ and } \langle x; \nu_- \rangle < 0 \\ R_{\chi_1} \Lambda x & \text{if } \langle x; \nu_+ \rangle < 0 \text{ and } \langle x; \nu_- \rangle > 0 \\ R_\psi x & \text{if } \langle x; \nu_+ \rangle > 0 \text{ and } \langle x; \nu_- \rangle > 0 \\ R_{\chi_2} \Lambda x & \text{if } \langle x; \nu_- \rangle < 0 \text{ and } \langle x; e_2 \rangle > 0 \\ R_{\chi_3} \Lambda x & \text{if } \langle x; \nu_+ \rangle > 0 \text{ and } \langle x; e_2 \rangle < 0 \end{cases}$$

*is continuous, across the lines of discontinuities of the gradient, if and only if*

$$\chi_1 = -\theta,\ \chi_2 = -3\theta,\ \chi_3 = \theta,\ \psi = -2\theta.$$

**Remark 12** *(i) Therefore the only possibility, under the hypotheses of the lemma, to have a continuous u is that (see Figure 2, the first case in (5))*

$$R_{-\varphi}\, u\,(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if } x_1 + mx_2 < 0 \text{ and } x_1 - mx_2 > 0 \\ R_{-\theta} \Lambda x = (mx_2, x_2) & \text{if } x_1 + mx_2 < 0 \text{ and } x_1 - mx_2 < 0 \\ R_{-2\theta} x & \text{if } x_1 + mx_2 > 0 \text{ and } x_1 - mx_2 < 0 \\ R_{-3\theta} \Lambda x = R_{-2\theta}\,(mx_2, x_2) & \text{if } x_1 - mx_2 > 0 \text{ and } x_2 > 0 \\ R_\theta \Lambda x = (-mx_2, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_2 < 0. \end{cases}$$

*(ii) There can also be another vertex of order 5 (essentially the symmetric one), see Figure 2, the second case in (5).*

**Lemma 13 (Vertex of order 6)** *Let $\varphi, \chi, \chi', \psi, \psi', \omega \in (-\pi, \pi]$. The map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$R_{-\varphi}\, u(x) = \begin{cases} x & \text{if } \langle x; \nu_+ \rangle < 0 \text{ and } \langle x; \nu_- \rangle < 0 \\ R_\chi \Lambda x & \text{if } \langle x; \nu_+ \rangle > 0 \text{ and } \langle x; e_2 \rangle < 0 \\ R_{\chi'} \Lambda x & \text{if } \langle x; \nu_- \rangle > 0 \text{ and } \langle x; e_2 \rangle < 0 \\ R_\omega x & \text{if } \langle x; \nu_+ \rangle > 0 \text{ and } \langle x; \nu_- \rangle > 0 \\ R_{\omega+\psi} \Lambda x & \text{if } \langle x; \nu_- \rangle < 0 \text{ and } \langle x; e_2 \rangle > 0 \\ R_{\omega+\psi'} \Lambda x & \text{if } \langle x; \nu_+ \rangle < 0 \text{ and } \langle x; e_2 \rangle > 0 \end{cases}$$

*is continuous, across the lines of discontinuities of the gradient, if and only if*

$$\chi = \psi' = -\chi' = -\psi = \theta.$$

**Remark 14** *Therefore the only possibility, under the hypotheses of the lemma, to have a continuous u is that (see Figure 2, (6))*

$$R_{-\varphi}\, u\,(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if } x_1 + mx_2 < 0 \text{ and } x_1 - mx_2 > 0 \\ R_\theta \Lambda x = (-mx_2, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_2 < 0 \\ R_{-\theta} \Lambda x = (mx_2, x_2) & \text{if } x_1 - mx_2 < 0 \text{ and } x_2 < 0 \\ R_\omega x = R_\omega\,(x_1, x_2) & \text{if } x_1 + mx_2 > 0 \text{ and } x_1 - mx_2 < 0 \\ R_{\omega-\theta} \Lambda x = R_\omega\,(mx_2, x_2) & \text{if } x_1 - mx_2 > 0 \text{ and } x_2 > 0 \\ R_{\omega+\theta} \Lambda x = R_\omega\,(-mx_2, x_2) & \text{if } x_1 + mx_2 < 0 \text{ and } x_2 > 0 \end{cases}$$

Finally we have the main result of the present section, which follows from the previous lemmas in a straightforward way. In the theorem we will assume that $A, B \in \mathbb{R}^{2\times 2}$ with $A$ invertible and $B \neq 0$ non-invertible (the case where $B = 0$ is elementary, see Remark 16). We recall that $\mu = \lambda_2(BA^{-1})$ (while, of course, $\lambda_1(BA^{-1}) = 0$) and, if $\mu \geq 1$,

$$\nu_\pm = \left(\pm 1, \sqrt{\mu^2 - 1}\right) = (\pm 1, m)\,.$$

We let $\nu \neq 0$ be a (uniquely defined up to rescaling) vector such that $B\nu = 0$.

**Theorem 15 (The semi-degenerate case)** *Let $\Omega$ be an open set of $\mathbb{R}^2$. Let $A, B \in \mathbb{R}^{2\times 2}$ with $A$ invertible and $B \neq 0$ non-invertible. Let $u\colon \Omega \to \mathbb{R}^2$ be a piecewise affine map satisfying*

$$Du \in \mathbb{S}_A \cup \mathbb{S}_B \quad a.e. \text{ in } \Omega.$$

Case 1: $\mu > 1$. *Then every edge of $\Sigma_u$ is perpendicular to one of the three vectors: $\nu_+, \nu_-, \nu$. Moreover if an edge perpendicular to $\nu_\pm$ is crossed, then the matrix $Du$ changes from a rotation of $A$ to a rotation of $B$ or vice-versa. While if an edge perpendicular to $\nu$ is crossed, then the matrix $Du$ changes between two different rotations of $B$. The segments can possibly meet in vertices of order 3, 4, 5 or 6. The possible local configurations around a vertex are the 13 cases depicted in Figure 2.*

Case 2: $\mu = 1$. *In this case $\nu_+$, $\nu_-$ are parallel each other and every edge of $\Sigma_u$ is perpendicular either to $\nu_+$ (and $\nu_-$) or $\nu$. Moreover if an edge perpendicular to $\nu_\pm$ is crossed, then the matrix $Du$ changes from a rotation of $A$ to a rotation of $B$ or vice-versa. While if an edge perpendicular to $\nu$ is crossed, then the matrix $Du$ changes between two different rotations of $B$. In this case no vertex is possible.*

Case 3: $\mu < 1$. *No interface between $A$ and $B$ can exist. Hence either $\Sigma_u = \emptyset$ or $\Sigma_u$ is composed by segments perpendicular to $\nu$ and $Du(x) \in \mathbb{S}_B$ for a.e. $x \in \Omega$.*

**Remark 16** *When $A$ is invertible and $B = 0$, we are in the case of incompatible wells with no rank one connection inside each of the wells. Therefore if $u$ is a piecewise $C^1$ solution of*

$$Du \in \mathbb{S}_A \cup \mathbb{S}_B \quad a.e. \text{ in } \Omega,$$

*then $u$ is affine and thus $\Sigma_u = \emptyset$.*

**Proof** If $\mu < 1$ then there is not a $\theta$ satisfying (4); hence no edge between $A$ and $B$ can exist, as stated in Lemma 5 and Remark 6. If $\mu = 1$ the directions $\nu_\pm = (\pm 1, 0)$ are parallel each other, hence all the edges of the singular set are parallel either to $\nu_\pm$ or to $\nu$. As we already said a vertex cannot occur. We therefore discuss only the last possibility $\mu > 1$.

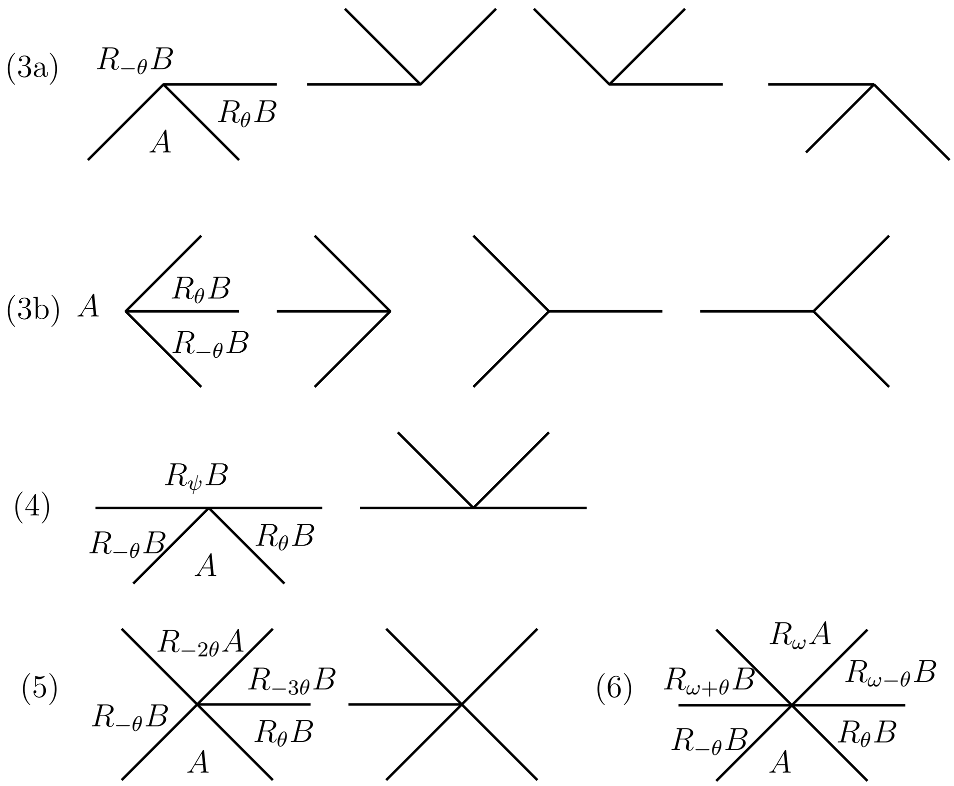


Figure 2: vertices of order 3, 4, 5 and 6

As in [15], by mean of a change of variables, we can reduce ourselves to the case when $A = I$ and $B = \mathrm{diag}(0, \mu)$. By Lemma 5 and Remark 6 we know that every segment of $\Sigma_u$ is perpendicular to one of the vectors: $\nu, \nu_\pm$ . Then around a vertex of $\Sigma_u$ we can have edges with 6 possible directions.

By deciding if a direction is or is not used, we obtain a total of $2^6 = 64$ possible configurations. However only 13 (cf. Figure 2) of them are possible. To exclude the others we notice that the map $u$ must satisfy the following conditions:

(i) the configuration must have at least a vertex where at least two edges with different directions meet; i.e., we exclude the empty configuration, configurations with only one edge and configurations with two aligned edges;

(ii) on both sides of the edges which are perpendicular to $\nu$ (the horizontal edges) the matrix $Du$ must be a rotation of $B$;

(iii) crossing a side which is perpendicular to $\nu_\pm$ the matrix $Du$ changes from a rotation of $A$ to a rotation of $B$ (or vice-versa);

(iv) the configuration with order 4 where the two directions perpendicular to $\nu$ are not used is not possible in view of Remark 10 (iii). In fact this configuration would be possible only if $\lambda\mu = -1$ (see [15], where the two matrices are invertible), but we have here $\lambda = 0$.

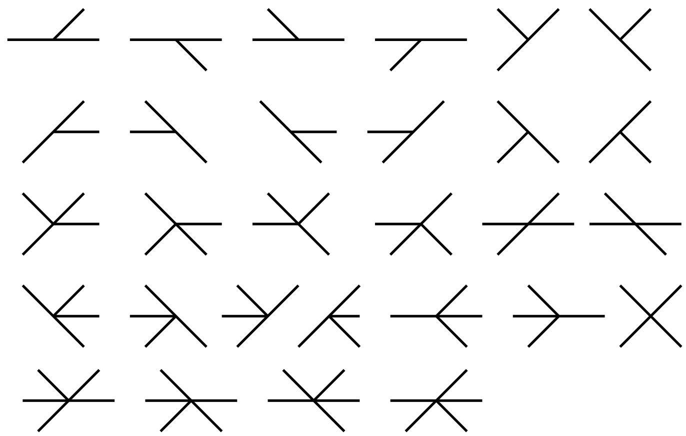

Figure 3: the represented vertices cannot be in the singular set of any map $u$

Conditions (ii)-(iv) allow us to exclude the vertices represented in Figure 3. Notice that in the configurations with 4 and 6 edges, the map is not uniquely determined even if we fix the value in one region. ∎

We end this section with some other considerations. An example of a complete singular set $\Sigma_u$ is depicted in Figure 4.

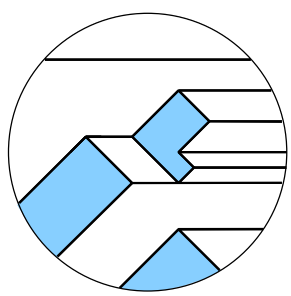

Figure 4: an example of the singular set: in the dark regions the map has gradient in the non-singular well $\mathbb{S}_A$, while in the complementary regions its gradient belong to the singular well $\mathbb{S}_B$

Interestingly enough we can show, with the example depicted in Figure 5, that a set $\Sigma$ satisfying the local properties considered in the previous proposition, is not always the singular set of some Lipschitz continuous map $u$.

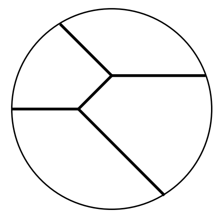

Figure 5: a set $\Sigma$ which locally satisfies the conditions stated in Lemmas 5, 7 at every vertex and every edge, but which does not correpond to the singular set of any map

Thus in this case we cannot have a *recovery* theorem (as in Theorem 4.9 of [11] or in the *sufficient condition* of Theorem 25 in [15]). Moreover there are also cases (when we use the configuration with vertices of order 4 or 6) where a map with the prescribed singular set can be constructed but is not uniquely determined even if we fix the value in one region and the set $\Omega$ is simply connected.

# 4 The fully-degenerate case

We will now discuss the case where both matrices $A$ and $B$ are not invertible. We start with a very elementary lemma whose proof is straightforward.

**Lemma 17** *Let $A \in \mathbb{R}^{2\times 2}$ with $\det A = 0$. Then there exists $\alpha \in \mathbb{R}^2$ such that*

$$\mathbb{S}_A = (SO(2)e_1) \otimes \alpha.$$

*Thus if $A, B \in \mathbb{R}^{2\times 2}$ with $\det A = \det B = 0$. Then there exist $\alpha, \beta \in \mathbb{R}^2$ such that*

$$\mathbb{S}_A \cup \mathbb{S}_B = (SO(2)e_1) \otimes \{\alpha, \beta\}.$$

We now have a lemma concerning edges between two different wells.

**Lemma 18** *Let $\alpha, \beta \in \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^2$ be an open set. Let $u$ be piecewise affine and such that*

$$Du \in (SO(2)e_1) \otimes \{\alpha, \beta\} \quad \text{a.e. in } \Omega.$$

*Then three cases can happen for the lamination between the two different wells (we do not consider here discontinuities in the same well; see the remark below).*

Case 1 (double laminations). *If the vectors $\alpha$ and $\beta$ are linearly independent, then only two lines of discontinuity are possible; one with normal $\alpha + \beta$ and the other one with normal $\alpha - \beta$.*

Case 2 (single lamination). *If the vectors $\alpha$ and $\beta$ are linearly dependent and $\alpha \neq 0$ (or $\beta \neq 0$), then only one line of discontinuity is possible, with normal $\alpha$ (respectively $\beta$).*

Case 3 (no lamination). *If $\alpha=\beta=0$, then $u$ is constant on every connected component.*

**Remark 19** *As in the semi-degenerate case, lamination can occur within the same well. The line of discontinuities are then orthogonal to $\alpha$ and $\beta$ respectively. For example in the first well we can have, for every $\varphi,\psi\in(-\pi,\pi]$,*

$$u(x)=\begin{cases}(R_\varphi e_1\otimes\alpha)\,x=\begin{pmatrix}(\alpha_1x_1+\alpha_2x_2)\cos\varphi\\(\alpha_1x_1+\alpha_2x_2)\sin\varphi\end{pmatrix} & \text{if } \langle\alpha;x\rangle=\alpha_1x_1+\alpha_2x_2>0\\ (R_\psi e_1\otimes\alpha)\,x=\begin{pmatrix}(\alpha_1x_1+\alpha_2x_2)\cos\psi\\(\alpha_1x_1+\alpha_2x_2)\sin\psi\end{pmatrix} & \text{if } \langle\alpha;x\rangle=\alpha_1x_1+\alpha_2x_2<0\end{cases}$$

**Proof** In Case 3 nothing is to be proved, so we discuss the two other cases. We start with the following observation. Along a line of discontinuity of the gradient we have that

$$Du=\begin{cases}R_\theta e_1\otimes\alpha=\begin{pmatrix}\alpha_1\cos\theta & \alpha_2\cos\theta\\ \alpha_1\sin\theta & \alpha_2\sin\theta\end{pmatrix}\\ R_\varphi e_1\otimes\beta=\begin{pmatrix}\beta_1\cos\varphi & \beta_2\cos\varphi\\ \beta_1\sin\varphi & \beta_2\sin\varphi\end{pmatrix}\end{cases}.$$

Since the map $u$ is continuous across the line of discontinuity of the gradient we should have

$$\det\left(R_\theta e_1\otimes\alpha-R_\varphi e_1\otimes\beta\right)=\det\left(R_{(\theta-\varphi)}e_1\otimes\alpha-e_1\otimes\beta\right)=0$$

which leads to

$$(\alpha_2\beta_1-\alpha_1\beta_2)\sin(\theta-\varphi)=0.$$

Therefore only three possibilities can happen

$$\alpha_2\beta_1-\alpha_1\beta_2=0,\quad \varphi=\theta\quad\text{or}\quad\varphi=\theta+\pi.$$

*Case 1.* We now consider the case where $\alpha$ and $\beta$ are linearly independent. According to the previous computations we must have $\varphi=\theta$ or $\varphi=\theta+\pi$. Therefore only two possibilities happen for the gradient

$$Du=\begin{cases}R_\theta e_1\otimes\alpha=\begin{pmatrix}\alpha_1\cos\theta & \alpha_2\cos\theta\\ \alpha_1\sin\theta & \alpha_2\sin\theta\end{pmatrix}\\ R_\theta e_1\otimes\beta=\begin{pmatrix}\beta_1\cos\theta & \beta_2\cos\theta\\ \beta_1\sin\theta & \beta_2\sin\theta\end{pmatrix}\end{cases}$$

or

$$Du=\begin{cases}R_\theta e_1\otimes\alpha=\begin{pmatrix}\alpha_1\cos\theta & \alpha_2\cos\theta\\ \alpha_1\sin\theta & \alpha_2\sin\theta\end{pmatrix}\\ R_{(\theta+\pi)}e_1\otimes\beta=R_\theta e_1\otimes(-\beta)=-\begin{pmatrix}\beta_1\cos\theta & \beta_2\cos\theta\\ \beta_1\sin\theta & \beta_2\sin\theta\end{pmatrix}\end{cases}.$$

The two lines of discontinuities of the gradient are therefore orthogonal in the first case to $\alpha - \beta$ and in the second case to $\alpha + \beta$. More precisely in the first case the line of discontinuity is parallel to

$$(\alpha_1 - \beta_1)\, x_1 + (\alpha_2 - \beta_2)\, x_2 = 0$$

and in the second case the line is parallel to

$$(\alpha_1 + \beta_1)\, x_1 + (\alpha_2 + \beta_2)\, x_2 = 0$$

The case 1 is therefore settled.

*Case 2.* We now consider the case where $\alpha$ and $\beta$ are linearly dependent. Assume, without loss of generality, that $\alpha \neq 0$, we therefore have $\beta = t\alpha$ and hence along a line of discontinuity of the gradient we have that

$$Du = \begin{cases} R_\theta e_1 \otimes \alpha \\ R_\varphi\,(te_1) \otimes \alpha. \end{cases}$$

Thus the only possible line of discontinuity has normal $\alpha$. More precisely the line of discontinuity is parallel to the line

$$\alpha_1 x_1 + \alpha_2 x_2 = 0.$$

Case 2 is established and hence the proof of the theorem is complete. ■

# 5 The Dirichlet problem in the degenerate case

We finally discuss an example of the Dirichlet problem in the degenerate case. This turns out to be elementary. It is in fact essentially a reduction to a scalar problem. We emphasize that this example strictly contrasts with the nonsingular case; in fact, if $A$ and $B$ are invertible but not orthogonal matrices (in the sense of (2)), then we proved in [15] that the corresponding Dirichlet problem lacks a solution in the class of piecewise affine maps.

**Example 20** *Let $a, b, c$ be positive real numbers. Let $\Omega$ be the rectangle in $\mathbb{R}^2$ with vertices in $(a,b), (-a,b), (a,-b), (-a,-b)$. Let us consider the degenerate matrices*

$$A = \begin{pmatrix} \frac{c}{a} & 0 \\ \frac{c}{a} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{c}{b} \\ 0 & \frac{c}{b} \end{pmatrix}.$$

*Then there exists a piecewise (in fact with only four pieces) affine solution of*

$$\begin{cases} Du\,(x) \in \mathbb{S}_A \cup \mathbb{S}_B & a.e.\ x \in \Omega \\ u\,(x) = 0 & x \in \partial\Omega. \end{cases}$$

**Proof** We will solve a scalar differential problem for each component $u^1$, $u^2$ of the map $u : \Omega \to \mathbb{R}^2$, as in the more general "pyramid" construction in Theorem 2.10 in [9]. In this specific case we can exhibit an explicit solution given by $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$ with

$$u^1(x_1, x_2) = u^2(x_1, x_2) = \begin{cases} c - \frac{c}{a}|x_1|, & \text{when } a|x_2| \le b|x_1| \\ c - \frac{c}{b}|x_2|, & \text{when } a|x_2| > b|x_1| \end{cases}$$

whose gradient is given by

$$Du = \begin{pmatrix} u^1_{x_1} & u^1_{x_2} \\ & \\ u^2_{x_1} & u^2_{x_2} \end{pmatrix} = \pm \begin{pmatrix} \frac{c}{a} & 0 \\ \frac{c}{a} & 0 \end{pmatrix} = \pm A \in \mathbb{S}_A$$

when $a|x_2| < b|x_1|$, and similarly $Du = \pm B \in \mathbb{S}_B$ in the other case. ■

**Acknowledgement 21** *As we said above, this paper is dedicated to Arrigo Cellina. We know Arrigo since long time and we appreciate his mathematical rigor associated to fantasy and originality, which made the mathematics that he touched in his studies interesting to us, often attracting our interest and mathematical curiosity.*

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