# Relaxation of elastic energies with free discontinuities and constraint on the strain

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## 1 Introduction

Many variational problems in mechanics involve energies with bulk and interfacial contributions. Such energies can be analytically represented as follows. In the context of fracture mechanics, let $\Omega$ denote a reference configuration of a (possibly brittle) elastic body in $\mathbb{R}^n$ ($n = 1, 2$ or $3$) and let $u$ parametrize its displacement, which we regard as smooth outside a subset $K \subseteq \Omega$. If we assume that $K$ itself is smooth enough so that a normal $\nu$ exists together with the traces $u^{\pm}$ of $u$ on both sides of $K$ $\mathcal{H}^{n-1}$ almost everywhere on $K$ ($\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure), then the elastic energy of this deformation may be written in the form

$$\int_{\Omega\setminus K} f(\nabla u)\,dx + \int_K g(u^+ - u^-, \nu)\,d\mathcal{H}^{n-1}. \tag{1.1}$$

In this framework we may include Griffith's theory of fracture, in which case $K$ is interpreted as the crack site and $g$ is a constant, and the surface integral is simply proportional to the area of the fracture. In Barenblatt's model $g = g(|u^+ - u^-|)$ is a function of the opening of the crack.

Equilibrium configurations can be obtained by solving minimum problems related to the energies (1.1) above. In order to attack such problems via the so-called direct methods of the Calculus of Variations, suitable weak formulations of the energies in (1.1) have been proposed in the spaces $SBV(\Omega;\mathbb{R}^n)$ of special functions of bounded variation (in the nonlinear setting) or $SBD(\Omega)$ of special functions of bounded deformation (in the linear framework) (see e.g. Ambrosio-Braides [1], Ambrosio-Fusco-Pallara [3], Ambrosio-Coscia-Dal Maso [2], Francfort-Marigo [16], Buliga [11]). In both cases $u$ is interpreted as a (possibly discontinuous) function defined in the whole $\Omega$ and the role of $K$ is played by the set $J_u$ of essential discontinuity points for $u$. In those spaces $J_u$ turns out to be smooth enough as to define a normal $\nu_u$ $\mathcal{H}^{n-1}$-almost everywhere on $J_u$ and $u$ is smooth enough so that an approximate gradient $\nabla u$ is defined almost everywhere in $\Omega$ as well as the traces $u^{\pm}$ of $u$ on both sides of $J_u$. A weak formulation of the energy in (1.1) can then be written as

$$\int_\Omega f(\nabla u)\,dx + \int_{J_u} g(u^+ - u^-, \nu_u) dH^{n-1}. \tag{1.2}$$

Notice that, since $\nabla u$ is defined almost everywhere on $\Omega$, the first integral can be directly computed on $\Omega$.

In this paper we study the behaviour of the energies (1.2) under the addition of some constraints on the singular part of the strain $(u^+ - u^-) \odot \nu_u$ on $J_u$. More precisely, we consider a fixed closed cone $K_0$ of matrices of the form $a \odot b$, and we require that the condition

$$(u^+ - u^-) \odot \nu_u \in K_0$$

holds a.e. on $J_u$. Various interesting situations fall within this formulation, among which we recall:

(i) $K_0 = \{a \odot b : \ \langle a, b\rangle = 0\}$ (no-slip condition). In this case the opening of the crack is perpendicular to the crack surface. This is a zero-divergence condition on $J_u$;

(ii) $K_0 = \{a \odot b : \ b = \lambda a \text{ with } \lambda \geq 0\}$. This condition can be interpreted as an infinitesimal non-interpenetration condition;

(iii) $K_0 = \{a \odot b : \langle a, b\rangle \geq 0\}$. This is a detachment condition on the opening of a crack.

We focus on the case of linearly elastic bulk energy densities, namely of the form

$$\frac{1}{2}\int_\Omega (A\mathcal{E}u, \mathcal{E}u)\,dx + \int_{J_u} g((u^+ - u^-) \odot \nu_u) dH^{n-1}, \tag{1.3}$$

where $\mathcal{E}u$ denotes the symmetrized gradient and $A$ is a fixed fourth-order tensor.

If the energy in (1.3) satisfies suitable structure and growth conditions (which are satisfied, e.g., if $g$ is constant) and $K_0$ is the restriction of a convex cone of symmetric matrices, then the addition of this constraint still gives a lower semicontinuous energy by the closure properties of measures which take values in a convex cone (see Anzellotti [4]). In general (for example for surface energy densities of Barenblatt's type) the constraint is not closed on sets of functions with equibounded energy, and these energies are not lower semicontinuous. The study of their relaxation; i.e., the computation of their lower semicontinuous envelope, allows to describe the macroscopic behaviour of their minimizers. The main goal of this paper is to describe this relaxation when $g$ is positively homogeneous of degree one. This model case is of particular importance, since in general a fundamental role is played by the tangent cone of $g$ at the origin, which defines a positively homogeneous function of degree one.

The relaxation of the energies in (1.3) gives functionals defined on the space of functions of bounded deformation; i.e., on those functions $u \in L^1(\Omega; R^n)$ whose strain $Eu = \big(Du + (Du)^T\big)/2$, defined in the sense of distributions, is a measure. For the measure $Eu$ the Radon-Nikodým decomposition $Eu = \mathcal{E}dx + E^s u$ holds. If $g$ is positively homogeneous of degree one and non degenerate (i.e., $g(\xi) = 0$ only if $\xi = 0$) then we show that the lower semicontinuous envelope of the energy in (1.3) is finite only on the set $\mathcal{U}(\Omega)$ of those functions $u \in BD(\Omega)$ such that the projection $P_{K^\perp} Eu$ of the strain measure $Eu$ on the cone orthogonal to $K$ (which denotes the convex hull of $K_0$) is absolutely continuous with respect to the Lebesgue measure and belongs to $L^2(\Omega)$. On $\mathcal{U}(\Omega)$ the relaxed energy can be represented as

$$\int_\Omega \omega(\mathcal{E}u)dx + \int_\Omega \omega^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big) d|E^s u|. \tag{1.4}$$

In Theorem 5.1 we give an explicit formula for the function $\omega$ in terms of $g$, $A$ and $K$. The complex form of $\omega$ is due to the interplay of the two energy densities $(A\xi, \xi)$ and $g$, which has a different effect on $K$ and $K^\perp$. This relaxation theorem provides a microscopic interpretation of energies with constraint on the strain studied by Anzellotti in [4]. As already observed in Braides-Defranceschi-Vitali [10], energies obtained by relaxation are a strict subclass of all energies of the form (1.4). Note that the energy density $\omega$ satisfies a non-standard growth condition

$$c_1(|\xi| - 1) \le \omega(\xi) \le c_2(|\xi|^2 + 1),$$

so that this relaxation theorem does not fit in the framework of any of the general integral representation results as in Buttazzo [12], Braides-Chiado' Piat [8], Bouchitté-Fonseca-Mascarenas [7], Barroso-Fonseca-Toader [5].

An interesting limit case is when $g = 0$; i.e., when fracture is allowed only if $(u^+ - u^-) \odot \nu_u \in K_0$ and in that case is not penalized at all. In this case the relaxed energy is simply

$$\frac{1}{2}\int_\Omega \big(AP_{K^\perp}\mathcal{E}u, P_{K^\perp}\mathcal{E}u\big)\, dx;$$

i.e., macroscopical deformations with $Eu$ taking values in $K$ have energy zero. By taking $K_0$ as in (ii) or (iii) above we recover the masonry-like functionals introduced by Giaquinta and Giusti in [17].

## 2 Notation and preliminaries

In order to fix up the notation, let us briefly review some well-known definitions and results.

We shall denote by $(\cdot,\cdot)$ and $|\cdot|$ the scalar product and the corresponding norm in $\mathbb{R}^n$ ($n \ge 1$). The same notation will be used in the vector space $M^{m\times n}$ of $m \times n$ matrices with real entries, identified with the space $\mathbb{R}^{mn}$. $M^{sym}$ will stand for the subspace of $M^{n\times n}$ consisting of the symmetric matrices, while $M_0^{sym}$ will denote the subspace of $M^{sym}$ of the matrices with null trace. It is easy to see that $\{tI :\ t \in \mathbb{R}\}$ ($I$ identity matrix) is the orthogonal space to $M_0^{sym}$ in $M^{sym}$; for every $\xi \in M^{sym}$ the corresponding decomposition is given by $\xi = \xi^D + \frac{1}{3}(\operatorname{tr}\xi)I$, where $\xi^D = \xi - \frac{1}{3}(\operatorname{tr}\xi)I$ is the *deviator* of $\xi$.

If $a, b \in \mathbb{R}^n$ the *tensor product* $a \otimes b$ is the $n \times n$ matrix whose entries are $a_i b_j$ with $i, j = 1, \dots, n$. The *symmetric tensor product* is defined by $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. It turns out that $|a \odot b|^2 = \frac{1}{2}\big(|a|^2|b|^2 + (a,b)^2\big)$.

The Lebesgue measure on $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$; if $E$ is a Lebesgue measurable set we shall also use $|E|$ in place of $\mathcal{L}^n(E)$. The open ball in $\mathbb{R}^n$ with centre $x$ and radius $\rho$ will be denoted by $B_\rho(x)$.

In the sequel $\Omega$ will be an open subset of $\mathbb{R}^n$ $(n \geq 1)$.

$L^p(\Omega;\mathbb{R}^m)$ and $W^{1,p}(\Omega;\mathbb{R}^m)$ will stand for the usual Lebesgue and Sobolev spaces of $\mathbb{R}^m$-valued functions ($m \in \mathbb{N}$); if $m = 1$ we shall simply write $L^p(\Omega)$ and $W^{1,p}(\Omega)$. For the spaces of compactly supported smooth functions we shall use the notation $C^k_c(\Omega;\mathbb{R}^m)$ ($k \in \mathbb{N}$ or $k = 0, \infty$).

*Convex functions*

Let $\phi: \mathbb{R}^m \to [0, +\infty]$ be convex (and proper, i.e., different from the constant $+\infty$); then the limit

$$\phi^\infty(\xi) = \lim_{t \to +\infty} \frac{\phi(t\xi)}{t}$$

exists for every $\xi \in \mathbb{R}^m$, and $\phi^\infty$ is called *recession function* of $\phi$. The same definition can be given if $\phi$ is defined only on a cone of $\mathbb{R}^m$. Clearly, $\phi^\infty$ is positively homogeneous of degree 1; moreover, if $\phi$ is lower semicontinuous, so is $\phi^\infty$.

If $f_1, f_2: \mathbb{R}^m \to [0, +\infty]$ are proper convex functions, then the *infimal convolution* of $f_1$ and $f_2$ is defined as

$$(f_1 \,\square\, f_2)(\xi) = \inf\{f_1(\xi - \eta) + f_2(\eta) : \ \eta \in \mathbb{R}^m\},$$

and it turns out to be a convex function (see [20] Th. 5.4). Moreover, it is not difficult to check (see [20] p. 38) that if $\varphi$ and $\psi$ are non-negative convex functions on $\mathbb{R}^m$, with $\varphi(0) = 0$ and $\psi$ positively homogeneous of degree 1, then the convex hull of $\varphi \wedge \psi = \min(\varphi, \psi)$ is given by $\varphi \square \psi$.

*Measures*

We shall denote by $\mathcal{M}^+(\Omega)$ the set of positive Radon measures on $\Omega$, i.e., the positive Borel measures on $\Omega$ which are finite on compact subsets. $\mathcal{M}(\Omega;\mathbb{R}^m)$ $(m \geq 1)$ will stand for the space of $\mathbb{R}^m$-valued Borel measures. Notice, (see, e.g., [3] Th. 1.6) that if $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ then $|\mu|(\Omega) < +\infty$, where the *total variation* measure $|\mu|$ is defined by

$$|\mu|(E) = \sup \sum_{h=1}^{\infty} |\mu(E_h)|, \qquad \text{for every Borel subset } E \text{ of } \Omega,$$

with $(E_h)$ ranging over all sequences of pairwise disjoint Borel sets such that $E = \bigcup_h E_h$.

Let $\nu \in \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$. The Radon-Nikodým Theorem and the Besicovitch Derivation Theorem (see, e.g., [3] Th. 1.28 and Th. 2.22) yield the representation

$$\mu = h\nu + \mu^s, \qquad \text{with} \quad \mu^s \perp \nu, \quad h(x) = \lim_{\rho \to 0} \frac{\mu\big(B_\rho(x)\big)}{\nu\big(B_\rho(x)\big)} \quad \text{for } \nu\text{-a.e. } x \in \Omega.$$

The function $h$ is called the *Radon-Nikodým derivative* of $\mu$ with respect to $\nu$, and denoted by $d\mu/d\nu$ or $\mu/\nu$.

Let $\mu$ and $(\mu_h)$ be a measure and a sequence of measures in $\mathcal{M}(\Omega;\mathbb{R}^m)$, respectively. We say that $(\mu_h)$ *locally weakly** converges to $\mu$ if

$$\lim_{h \to +\infty} \int_\Omega \varphi \, d\mu_h = \int_\Omega \varphi \, d\mu$$

for every $\varphi \in C^0_c(\Omega)$. If this equality holds for every $\varphi$ which is the uniform limit of a sequence in $C^0_c(\Omega)$ then we say that $(\mu_h)$ converges *weakly** to $\mu$. It is easy to see that the weak* convergence of a sequence $(\mu_h)$ is equivalent to the local weak* convergence together with the condition $\sup_h |\mu_h|(\Omega)| < +\infty$.

Let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ and let $f\colon \mathbb{R}^m \to [0,+\infty[$ be a convex function. Then we set

$$\int_\Omega f(\mu) = \int_\Omega f\Big(\frac{d\mu^a}{d\mathcal{L}^n}\Big)\, d\mathcal{L}^n + \int_\Omega f^\infty\Big(\frac{d\mu^s}{d|\mu^s|}\Big)\, d|\mu^s|, \tag{2.1}$$

where $\mu = \mu^a + \mu^s$ is the Lebesgue decomposition of $\mu$ ($\mu^a \ll \mathcal{L}^n$, $\mu^s \perp \mathcal{L}^n$).

The $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ will be denoted by $\mathcal{H}^{n-1}$. A subset $E$ of $\mathbb{R}^n$ is said to be *countably* $(n-1)$*-rectifiable* if it is contained, up to a $\mathcal{H}^{n-1}$-negligible set, in the union of a countable family $(S_j)$ of $(n-1)$-dimensional $C^1$ submanifolds of $\mathbb{R}^n$. The definition of an *approximate tangent space* (hence, of an approximate normal direction) at $\mathcal{H}^{n-1}$-a.e. point of $E$ can be given through the tangent spaces to the manifolds $S_j$.

*Approximate discontinuity points and approximate jump points*

In the following definition $B^+_\rho(x,\nu) = \{y \in B_\rho(x) :\ (y-x,\nu) > 0\}$ ($\nu$ unit vector in $\mathbb{R}^n$) and $B^-_\rho(x,\nu) = \{y \in B_\rho(x) :\ (y-x,\nu) < 0\}$.

**Definition 2.1** *Let $u \in L^1_{loc}(\Omega;\mathbb{R}^m)$ and $x \in \Omega$.*

*We say that $u$ has* approximate limit *at $x$ if there exists $z \in \mathbb{R}^m$ such that:*

$$\lim_{\rho\to 0} \fint_{B_\rho(x)} |u(y) - z|\, dy = 0.$$

*The set $S_u$ where this property fails is called* approximate discontinuity set *of $u$.*

*We say that $x$ is an* approximate jump point *of $u$ if there exists $a, b, \nu \in \mathbb{R}^m$, with $|\nu| = 1$, such that $a \neq b$ and*

$$\lim_{\rho\to 0} \fint_{B^+_\rho(x,\nu)} |u(y) - a|\, dy = 0, \qquad \lim_{\rho\to 0} \fint_{B^-_\rho(x,\nu)} |u(y) - b|\, dy = 0.$$

*The set of approximate jump points of $u$ is denoted by $J_u$.*

The vector $z$ is uniquely determined for any point $x \in \Omega \setminus S_u$ and is called the *approximate limit* of $u$ at $x$ and denoted by $\tilde{u}(x)$. The triplet $(a, b, \nu)$, which turns out to be uniquely determined up to a permutation of $a$ and $b$ and a change of sign of $\nu$, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. On $\Omega \setminus S_u$ we set $u^+ = u^- = \tilde{u}$.

*The space $BV$*

We recall that the space $BV(\Omega)$ of real *functions of bounded variation* is the space of the functions $u \in L^1(\Omega)$ whose distributional derivative is representable by a measure in $\Omega$, i.e.

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi\, dD_i u, \qquad \text{for every } \varphi \in C^\infty_c(\Omega) \text{ and } i = 1, \dots, n$$

for some $Du = (D_1 u, \dots, D_n u) \in \mathcal{M}(\Omega;\mathbb{R}^n)$. The space of $\mathbb{R}^m$-valued functions whose components are in $BV(\Omega)$ will be denoted by $BV(\Omega;\mathbb{R}^m)$. We again write $Du = (D_i u_j) \in \mathcal{M}(\Omega; M^{n\times n})$.

For a thorough treatment of $BV$ functions we refer to [3].

**Theorem 2.2 (Federer-Vol'pert)** *For any $u \in BV(\Omega;\mathbb{R}^m)$ the set $S_u$ is countably $(n-1)$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. Moreover, $Du \llcorner J_u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u$, and $\nu_u(x)$ gives the approximate normal direction to $J_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$.*

The space $SBV(\Omega;\mathbb{R}^m)$ ($SBV(\Omega)$ if $m=1$) of *special functions of bounded variation* can be defined as the space of the functions $u\in BV(\Omega;\mathbb{R}^m)$ such that the singular part of the derivative with respect to the Lebesgue measure $\mathcal{L}^n$ is given by $(u^+-u^-)\otimes\nu_u\mathcal{H}^{n-1}\llcorner J_u$. For such $u$, denoting by $\nabla u$ the density of the absolutely continuous part of $Du$, we have:

$$Du=\nabla u\mathcal{L}^n+(u^+-u^-)\otimes\nu_u\mathcal{H}^{n-1}\llcorner J_u. \tag{2.2}$$

It turns out (see [3] Proposition 4.4 or [15] Lemma 2.3) that $SBV(\Omega)$ contains the bounded "piecewise Sobolev" functions. More precisely, if $\Omega$ is bounded, $K$ is a closed subset of $\mathbb{R}^n$ with $\mathcal{H}^{n-1}(K\cap\Omega)<+\infty$ and $u\in L^\infty(\Omega)$ with $u\in W^{1,1}(\Omega\backslash K)$ then $u\in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_u\backslash K)=0$.

In particular, $SBV(\Omega;\mathbb{R}^n)$ contains the following space $S(\Omega)$ of "piecewise $C^1$-functions" (here $m=n$ since $u$ will be interpreted as a mechanical deformation):

**Definition 2.3** *If $\Omega$ is a bounded open subset of $\mathbb{R}^n$ ($n\ge 1$) we define $S(\Omega)$ as the set of all functions $u\in L^\infty(\Omega;\mathbb{R}^n)$ satisfying the following conditions:*

i) *there exists a closed set $K$ in $\mathbb{R}^n$ with $\mathcal{H}^{n-1}(K\cap\Omega)<+\infty$ and $K=\bigcup_j\overline{S}_j$, where $(S_j)_{j\in J}$ is a finite family of $(n-1)$-dimensional $C^1$ submanifolds of $\mathbb{R}^n$ such that $\mathcal{H}^{n-1}(\overline{S}_j\setminus S_j)=0$, and $u\in C^1(\Omega\setminus K;\mathbb{R}^n)$;*

ii) $\int_{\Omega\setminus K}|Du|\,dx<+\infty$.

Since $S(\Omega)\subseteq SBV(\Omega;\mathbb{R}^n)$, it is convenient to rely on the previous theory (though we could proceed more directly) to obtain the decomposition (2.2), with $J_u\subseteq K$, for every $u\in S(\Omega)$.

*The space BD*

The space $BD(\Omega)$ of *functions of bounded deformation* is the space of the functions $u\in L^1(\Omega;\mathbb{R}^n)$ whose symmetric distributional derivative

$$Eu=\frac{1}{2}\big(Du+(Du)^T\big)$$

is a measure in $\mathcal{M}(\Omega;M^{sym})$. Clearly, $BV(\Omega;\mathbb{R}^n)$ is a subspace of $BD(\Omega)$. If $u\in BD(\Omega)$ the density of the absolutely continuous part of $Eu$ with respect to the Lebesgue measure is denoted by $\mathcal{E}u$. Thus

$$Eu=\mathcal{E}u\,\mathcal{L}^n+E^su,\qquad\text{with}\quad E^su\perp\mathcal{L}^n.$$

We can further decompose $E^su$ as $E^su=E^ju+E^cu$, where $E^ju=E^su\llcorner J_u$ and $E^cu=E^su\llcorner(\Omega\setminus J_u)$. It is proved in [2] that

$$E^ju=(u^+-u^-)\odot\nu_u\mathcal{H}^{n-1}\llcorner J_u.$$

Moreover, $|E^cu|(B)=0$ whenever $B$ is a Borel subset of $\Omega$ and $\mathcal{H}^{n-1}(B)<+\infty$.

The space $SBD(\Omega)$ of *special functions with bounded deformation* in $\Omega$ is defined as the set of all $u\in BD(\Omega)$ with $E^cu=0$ or, equivalently, with

$$Eu=\mathcal{E}u\,\mathcal{L}^n+(u^+-u^-)\odot\nu_u\mathcal{H}^{n-1}\llcorner J_u.$$

We say that a sequence $(u_h)$ in $BD(\Omega)$ weakly converges to a function $u\in BD(\Omega)$ if

$$u_h\ \to\ u\quad\text{in}\quad L^1(\Omega;\mathbb{R}^n)$$

$$\big(|Eu_h|(\Omega)\big)_h\quad\text{is bounded.}$$

**Remark 2.4** Let $(u_h)$ be a sequence in $BD(\Omega)$ converging in $L^1(\Omega;\mathbb{R}^n)$ to a function $u$ and such that $\big(|Eu_h|(\Omega)\big)_h$ is bounded. This implies the convergence of $(Eu_h)$ to $Eu$ in $\mathcal{D}'(\Omega;M^{sym})$; hence $Eu$ is a measure, and

$$\int_\Omega \psi dEu_h \to \int_\Omega \psi dEu$$

for every $\psi \in C^0_c(\Omega;\mathbb{R}^n)$. Therefore, $u \in BD(\Omega)$, and $(Eu_h)$ weakly* converges to $Eu$.

If $u \in S(\Omega)$ then

$$Eu = \mathcal{E}u\,\mathcal{L}^n + (u^+ - u^-)\odot \nu_u \mathcal{H}^{n-1} \,\llcorner\, J_u.$$

In particular $u \in SBD(\Omega)$.

We refer to [22], [21] and [2] for a detailed study of the properties of $BD$ functions.

*Relaxation*

Let $F\colon X \to \overline{\mathbb{R}}$ be a functional on a topological space $(X,\tau)$. The *relaxed functional* $\overline{F}$ of $F$, or *lower semicontinuous envelope* of $F$, with respect to the $\tau$-topology, is the greatest $\tau$-lower semicontinuous functional majorized by $F$. If $(X,\tau)$ satisfies the first countability axiom then:

$$\overline{F}(x) = \inf\big\{\liminf_{h\to+\infty} F(x_h) : \quad x_h \to x \text{ in } X\big\},$$

and the infimum is attained. In the general case this formula *defines* the so-called *sequential lower semicontinuous envelope* of $F$. For a general treatment of this subject we refer to [12] and [14].

## 3 Convex cones

Let $m \geq 1$ and let $K$ be a cone in $\mathbb{R}^m$, i.e., a subset which is closed under positive scalar multiplication. Let us assume that a scalar product $\langle\cdot,\cdot\rangle$ (with the corresponding norm $\|\cdot\|$) is defined in $\mathbb{R}^m$. The *orthogonal cone* to $K$ (with respect to the given scalar product) is defined as:

$$K^\perp = \{\eta \in \mathbb{R}^m : \langle \xi,\eta\rangle \leq 0 \text{ for every } \xi \in K\}.$$

Assume that $K$ is closed and convex. Then we can consider the *orthogonal projection* $P_K$ (with respect to $\langle\cdot,\cdot\rangle$) onto $K$. Notice that $K^\perp$ turns out to be closed and convex too, and for every $\xi \in \mathbb{R}^m$ :

$$\text{(3.1)}\qquad \begin{aligned} &\text{a)}\quad \xi = P_K\xi + P_{K^\perp}\xi, \qquad \langle P_K\xi, P_{K^\perp}\xi\rangle = 0;\\ &\text{b)}\quad \xi = \xi_1+\xi_2,\ \ \xi_1\in K,\ \xi_2 \in K^\perp \quad \Rightarrow \quad \|P_K\xi\| \leq \|\xi_1\|, \quad \|P_{K^\perp}\xi\| \leq \|\xi_2\|. \end{aligned}$$

If $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ we set:

$$P_K\mu = \Big(P_K \frac{d\mu}{d|\mu|}\Big)|\mu|.$$

This defines a measure in $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$, called the *(orthogonal) projection* of $\mu$ onto $K$. Clearly, if $\nu \in \mathcal{M}^+(\Omega)$ and $|\mu| \ll \nu$ then

$$P_K\mu = \Big(P_K\frac{d\mu}{d\nu}\Big)\nu.$$

Property (a) above immediately yields $\mu = P_K\mu + P_{K^\perp}\mu$. Moreover, it is easy to see that if $\mu_1 \perp \mu_2$ then $P_K(\mu_1+\mu_2) = P_K\mu_1 + P_K\mu_2$.

For reference convenience we state the following result ([4], Theorem 3.3):

**Theorem 3.1** *Let $\mu$ and $(\mu_h)$ be a measure and a sequence of measures in $\mathcal{M}(\Omega;\mathbb{R}^m)$, respectively. Let $K$ be a closed convex cone in $\mathbb{R}^m$. Assume that $P_K\mu_h \ll \mathcal{L}^n$ for every $h \in \mathbb{N}$, and $(dP_K\mu_h/d\mathcal{L}^n)_h$ is bounded in $L^2(\Omega;\mathbb{R}^m)$. If $(\mu_h)$ weakly* converges to $\mu$, then $P_K\mu \ll \mathcal{L}^n$, $dP_K\mu/d\mathcal{L}^n \in L^2(\Omega;\mathbb{R}^m)$, and*

$$\int_\Omega \Big|\frac{dP_K\mu}{d\mathcal{L}^n}\Big|^2\,dx \le \liminf_{h\to+\infty}\int_\Omega \Big|\frac{dP_K\mu_h}{d\mathcal{L}^n}\Big|^2\,dx.$$

In the sequel we shall deal with closed convex cones in the vector space $M^{sym}$, which can be identified with a space $\mathbb{R}^m$. In §9 we shall consider, in $M^{sym}$ with $n=3$, the scalar product determined by the operator $A$ of the stress-strain relation in the classic theory of linearized elasticity:

$$A\colon \xi \mapsto 2\mu\xi + \lambda(\operatorname{tr}\xi)I\colon M^{sym} \to M^{sym}$$

($\lambda$ and $\mu$ are the Lamé coefficients, $I$ is the identity matrix). $A$ is a positive definite symmetric linear operator; hence $\langle \xi,\eta\rangle = (A\xi,\eta)$ defines a scalar product in $M^{sym}$.

For each of the following choices of $K_0$ we shall be concerned with the convex hull $K$ of $K_0$, and with the orthogonal cone $K^\perp$ (with respect to the above defined scalar product):

I) $K_0 = \{a\odot b:\ a,b\in\mathbb{R}^3,\ (a,b)=0\}$;

II) $K_0 = \{a\odot a:\ a\in\mathbb{R}^3\} = \{a\odot b:\ a,b\in\mathbb{R}^3 \text{ with } b=\lambda a\ (\lambda\ge 0)\}$;

III) $K_0 = \{a\odot b:\ a,b\in\mathbb{R}^3,\ (a,b)\ge 0\}$.

**Proposition 3.2** *Let $K_0$ be as above, and let $K$ be the convex hull of $K_0$.*
*In case* (I)

$$K = M_0^{sym} = \{\xi\in M^{sym}:\ \operatorname{tr}\xi = 0\},\qquad K^\perp = \{tI:\ t\in\mathbb{R}\}.$$

*In case* (II) *$K$ is the set $M^+$ of the positive semi-definite symmetric matrices.*
*In case* (III)

$$K = \{\xi\in M^{sym}:\ \operatorname{tr}\xi\ge 0\},\qquad K^\perp = \{tI:\ t\le 0\}.$$

*In any case $K$ is closed.*

**Proof** The equality $K = M_0^{sym}$ in case (I) follows immediately from the fact that $M_0^{sym}$ is a vector space spanned by the elements $e_i\odot e_j$ for $i,j\in\{1,2,3\}$ with $i\ne j$, $(e_1+e_2)\odot(e_1-e_2)$ and $(e_2+e_3)\odot(e_2-e_3)$. Moreover the decomposition $\xi = \xi^D + \frac{1}{3}(\operatorname{tr}\xi)I$ easily yields that $K^\perp = \{tI:\ t\in\mathbb{R}\}$.

Let us consider case (II). The set $K_0$ is contained in $M^+$, hence the convex hull of $K_0$ is contained in $M^+$ too. On the other hand, notice that for every positive semi-definite symmetric matrix $\xi$ there exists an orthogonal matrix $Q$ with the property that $Q^T\xi Q$ is diagonal, say $\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)$, with $\lambda_i\ge 0$. Then

$$Q^T\xi Q = \lambda_1 e_1\odot e_1 + \ldots + \lambda_n e_n\odot e_n,$$

and

$$\xi = \lambda_1 Q(e_1 \odot e_1)Q^T + \ldots + \lambda_n Q(e_n \odot e_n)Q^T.$$

Since $[Q(a \odot a)Q^T] = (Qa) \odot (Qa)$, we conclude that $\xi$ belongs to the convex hull of $K_0$.

As to case (III), notice that $K \subseteq \{\xi \in M^{sym} : \ \operatorname{tr}\xi \geq 0\}$ since the right-hand side is a convex set. Moreover, $K \supseteq M_0^{sym}$ since $M_0^{sym}$ is the convex hull of $\{a \odot b : \ (a, b) = 0\}$. The result now follows from the decomposition $\xi = \xi^D + \frac{1}{3}(\operatorname{tr}\xi)I$. □

## 4 Semicontinuous functionals with constraint on the jump

In the following we shall deal with the relaxation of energies which involve a constraint on the jump of the form $(u^+ - u^-) \odot \nu_u \in K_0$, with $K_0$ a fixed cone. Before facing this problem we describe a class of energies which take such a constraint into account and are lower semicontinuous.

Let $K$ be a closed convex cone in $\mathbb{R}^m$. We say that a measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ is a $K$-valued measure if

$$\frac{d\mu}{d|\mu|} \in K \qquad |\mu|\text{-a.e. on } \Omega.$$

In the lemma below (see [4], Lemma 2.5) the closedness of the set of $K$-valued measures with respect to weak* convergence is proved. It is worth pointing out that this easily implies a lower semicontinuous result for functionals on $SBD$ or $SBV$.

**Lemma 4.1** *Let $\mu$ and $(\mu_h)$ be a measure and a sequence of measures in $\mathcal{M}(\Omega; \mathbb{R}^m)$, respectively. Let $\mu_h$ be $K$-valued for every $h \in \mathbb{N}$. If $(\mu_h)$ weakly* converges to $\mu$ then $\mu$ is $K$-valued, too.*

Let $\mathcal{F}: SBD(\Omega) \to [0, +\infty]$ be sequential lower semicontinuous with respect to the following convergence:

$$\begin{array}{ll} u_h \to u & \text{in } L^1(\Omega; \mathbb{R}^n) \\ \mathcal{E}u_h \rightharpoonup \mathcal{E}u & \text{weakly in } L^1(\Omega; M^{sym}) \\ (u_h^+ - u_h^-) \odot \nu_{u_h} \mathcal{H}^{n-1} \llcorner J_{u_h} \rightharpoonup (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u & \text{weakly}^* \text{ in } \mathcal{M}(\Omega; M^{sym}). \end{array} \tag{4.1}$$

Let $K$ be a closed convex cone in $M^{sym}$ and let $\mathcal{F}_K: SBD(\Omega) \to [0, +\infty]$ be defined by:

$$\mathcal{F}_K(u) = \begin{cases} F(u) & \text{if } \ (u^+ - u^-) \odot \nu_u \in K \ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.2}$$

**Theorem 4.2** *$\mathcal{F}_K$ is sequential lower semicontinuous with respect to convergence* (4.1).

**Proof** Let $(u_h)$ be a sequence in $SBD(\Omega)$ converging to a function $u \in SBD(\Omega)$ in the sense (4.1). We have to prove that

$$\mathcal{F}_K(u) \leq \liminf_{h \to +\infty} \mathcal{F}_K(u_h).$$

We can assume that $\mathcal{F}_K(u_h) < +\infty$ for every $h \in \mathbb{N}$. Therefore, $(u_h^+ - u_h^-) \odot \nu_{u_h} \in K \ \mathcal{H}^{n-1}$-a.e. on $J_{u_h}$. By Lemma 4.1 we deduce that $(u^+ - u^-) \odot \nu_u \in K \ \mathcal{H}^{n-1}$-a.e. on $J_u$. The lower semicontinuity of $\mathcal{F}$ now concludes the proof. □

**Remark 4.3** A perfectly analogous result holds for functionals on $SBV$, provided $\mathcal{E}u$ and $(u^+ - u^-) \odot \nu_u$ are replaced by $\nabla u$ and $(u^+ - u^-) \otimes \nu_u$, respectively, in (4.1) and (4.2).

As an example we can consider the functional

$$\mathcal{F}(u) = \int_\Omega f(\mathcal{E}u)\, dx + \int_{J_u} g\big((u^+ - u^-) \odot \nu_u\big)\, \mathcal{H}^{n-1} + \alpha \mathcal{H}^{n-1}(J_u)$$

($u \in SBD(\Omega)$), where $\alpha > 0$,

$$\begin{aligned} f&: M^{sym} \to [0, +\infty[ \qquad \text{is convex} \\ g&: M^{sym} \to [0, +\infty[ \qquad \text{is convex and positively 1-homogeneous,} \end{aligned}$$

and there exists $\phi: [0, +\infty[ \to [0, +\infty[$, with $\lim_{t\to+\infty} \phi(t)/t = +\infty$, and $\beta > 0$ such that

$$f(\xi) \geq \phi(|\xi|) \quad \text{and} \quad g(\xi) \geq \beta|\xi| \qquad \text{for every } \xi \in M^{sym}.$$

The lower semicontinuity of $\mathcal{F}$ with respect to the $L^1_{\mathrm{loc}}(\Omega)$-convergence follows from Corollary 1.2 in [6] (actually, there is lower semicontinuity of each of the three terms separately). This paper also presents a compactness criterion for sequences in $SBD$ and some examples of existence theorems which they give rise to. In view of the previous theorem the same applications can be rephrased in terms of $\mathcal{F}_K$.

We can apply the results outlined above with $f(\mathcal{E}u) = (A\mathcal{E}u, \mathcal{E}u)$, $g = 0$ and $K$ the convex hull of any of the three cones $K_0$ considered in §3. For instance, choose $n = 3$ and $K_0$ as in (II) (notice that, by Proposition 3.2, $\{a \odot b : \ a, b \in \mathbb{R}^3\} \cap K = K_0$). In this way we obtain the existence of solutions to the minimum problem

$$\min\Big\{ \int_\Omega (A\mathcal{E}u, \mathcal{E}u)\, dx + \alpha \mathcal{H}^2(J_u) - \int_\Omega (h, u)\, dx : \\ u \in C \text{ a.e. },\ u^+(x) - u^-(x) = t(x)\nu_{u(x)} \text{ with } t(x) \geq 0 \ \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u \Big\},$$

where $C$ is a fixed compact set of $\mathbb{R}^3$ and $h \in L^1(\Omega; \mathbb{R}^3)$ reresents an external force. Similarly, boundary conditions may be added.

On the line of Remark 4.3, analogous considerations can be made for functionals on $SBV$.

In the rest of the paper we shall study the case of the constraint $K$ when the surface energy density is a positively 1-homogeneous convex function (clearly, the case without constraint falls in this setting too, taking $K = M^{sym}$). Hence we drop the term $\mathcal{H}^{n-1}(J_u)$ in the functional $\mathcal{F}$ (or, equivalently, we choose $\alpha = 0$), thus losing the lower semicontinuity.

## 5 Setting of the problem and main result

The main object of this paper is the computation of the lower semicontinuous envelope of the functional $F$ defined below. In §9 we shall show some results which can be obtained by a suitable choice of the function $g$ and the set $K_0$.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ $(n \geq 1)$ and let $F: L^1(\Omega;\mathbb{R}^n) \to [0,+\infty]$ be given by

$$(5.1)\qquad F(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \|\mathcal{E}u\|^2\,dx + \int_{J_u} g\big((u^+ - u^-)\odot \nu_u\big)d\mathcal{H}^{n-1}, & \text{if } u \in S(\Omega) \text{ and} \\ & (u^+ - u^-)\odot \nu_u \in K_0 \\ & \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $S(\Omega)$ is the space of "piecewise $C^1$-functions" introduced in Definition 2.3, and:

- the norm $\|\cdot\|$ in the term $\frac{1}{2}\int_\Omega \|\mathcal{E}u\|^2\,dx$ (the *elastic part* of the energy) is defined as follows. Let $A: M^{sym} \to M^{sym}$ be a positive definite symmetric linear operator, i.e., $A$ is linear and

  a) $(A\xi,\eta) = (\xi, A\eta)$ for every $\xi,\eta \in M^{sym}$;

  b) there exists $\alpha > 0$ such that $(A\xi,\xi) \geq \alpha|\xi|^2$ for every $\xi \in M^{sym}$.

  Then we can consider on $M^{sym}$ the scalar product $\langle \xi,\eta \rangle = (A\xi,\eta)$ and the corresponding norm $\|\xi\| = \langle \xi,\xi \rangle^{1/2}$. Clearly, $\|\cdot\|$ is equivalent to the usual norm $|\cdot|$ in $M^{sym}$, i.e.:

  $$(5.2)\qquad \sqrt{\alpha}|\xi| \leq \|\xi\| \leq M|\xi|$$

  for a suitable $M > 0$.

- $g: M^{sym} \to [0,+\infty[$ (the *surface energy density*) is a convex function positively homogeneous of degree 1.

- $K_0$ describes the admissible singular part of the strain. We assume that $K_0$ is a closed cone in $M^{sym}$ consisting of matrices of the form $a \odot b$ $(a, b \in \mathbb{R}^n)$.

We shall assume that $\Omega$ is a *strictly star shaped* Lipschitz bounded open subset of $\mathbb{R}^n$, i.e. (see [4]) $\overline{\Omega} \subseteq \rho\Omega$ for every $\rho > 1$. This regularity condition will be used in Step 9 of the proof of Theorem 5.1 and in the proof of Theorem 9.2 to apply an approximation result for $BD$ functions proved in [4].

In the sequel, if $H$ is a closed convex cone in $M^{sym}$, we shall denote by $P_H$ the orthogonal projection onto $H$ with respect to the scalar product $\langle\cdot,\cdot\rangle$ introduced above (see §2 for the definition of projection of a measure).

**Theorem 5.1** *Let $\Omega$ be a strictly star shaped Lipschitz bounded open subset of $\mathbb{R}^n (n \geq 1)$. Let $F: L^1(\Omega;\mathbb{R}^n) \to [0,+\infty]$ be the functional defined in* (5.1) *and let $\overline{F}$ be its lower semicontinuous envelope with respect to the $L^1(\Omega;\mathbb{R}^n)$ topology.*

*Assume that $g$ and $K_0$ satisfy the following conditions:*

a) $\beta = \min_{|\xi|=1} g(\xi) > 0$;

b) *the convex hull $K$ of $K_0$ is closed, and*

$$(5.3)\qquad a \odot (b + c) \in K_0 \qquad \textit{whenever } a \odot b,\ a \odot c \in K_0;$$

c) *the convex hull* $f_1$ *of the function*

$$f_0(\xi) = \begin{cases} g(\xi), & \text{if } \xi \in K_0 \\ +\infty, & \text{if } \xi \in M^{sym} \setminus K_0 \end{cases}$$

*is Lipschitz on* $K$.

*Then*

i) $\overline{F}(u) < +\infty$ *if and only if* $u$ *belongs to the following space:*

$$\begin{aligned} \mathcal{U}(\Omega) &= \{u \in BD(\Omega) : P_{K^\perp} Eu \ll \mathcal{L}^n,\ P_{K^\perp}\mathcal{E}u \in L^2(\Omega; M^{sym})\} \\ &= \{u \in BD(\Omega) : P_{K^\perp} E^s u = 0,\ P_{K^\perp}\mathcal{E}u \in L^2(\Omega; M^{sym})\}. \end{aligned}$$

ii)
$$\overline{F}(u) = \int_\Omega \omega(\mathcal{E}u)\,dx + \int_\Omega \omega^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)\,d|E^s u|,$$

*for every* $u \in \mathcal{U}(\Omega)$, *with* $\omega$ *defined as follows: let* $f$ *be the convex hull of* $f_1 \wedge f_2 = \min(f_1, f_2)$, *where*

$$f_2(\xi) = \begin{cases} \frac{1}{2}\|\xi\|^2, & \text{if } \xi \in K \\ +\infty, & \text{if } \xi \in M^{sym} \setminus K; \end{cases}$$

*then* $\omega$ *is the convex hull of the function*

$$\xi \mapsto f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2 \colon M^{sym} \to [0, +\infty[.$$

We defer the proof of Theorem 5.1 to the next sections. Here we gather a few comments on the result.

**Remark 5.2** $F$ is a convex functional. Indeed, let $u, v \in L^1(\Omega; \mathbb{R}^n)$ and $0 < \lambda < 1$. We may assume that $F(u) < +\infty$ and $F(v) < +\infty$. In particular, $u, v \in S(\Omega)$, so that $w_\lambda = \lambda u + (1-\lambda)v \in S(\Omega)$, too. Moreover, $J_{w_\lambda} \subseteq J_u \cup J_v$, up to a set of $\mathcal{H}^{n-1}$-measure zero, and

$$(w_\lambda^+ - w_\lambda^-) \odot \nu_{w_\lambda} = \lambda(u^+ - u^-) \odot \nu_u + (1-\lambda)(v^+ - v^-) \odot \nu_v$$

$\mathcal{H}^{n-1}$-a.e. on $J_{w_\lambda}$. Since $(u^+ - u^-) \odot \nu_u, (v^+ - v^-) \odot \nu_v \in K_0$ $\mathcal{H}^{n-1}$-a.e. on $J_u$ and $J_v$ respectively, and $\nu_u = \nu_v$ $\mathcal{H}^{n-1}$-a.e. on $J_u \cap J_v$, we conclude that $(w_\lambda^+ - w_\lambda^-) \odot \nu_{w_\lambda} \in K_0$ $\mathcal{H}^{n-1}$-a.e. on $J_{w_\lambda}$ by the convexity assumption (5.3) on $K_0$. It is now immediate to get the inequality $F(w_\lambda) \leq \lambda F(u) + (1-\lambda)F(v)$.

**Remark 5.3** Condition (b) of Theorem 5.1 is not automatically satisfied; indeed, the convex hull of a closed cone in $\mathbb{R}^m$ need not be closed (consider, e.g., the set $\{(x,y,z) \in \mathbb{R}^3 : z \geq \sqrt{x^2+y^2}\} \cup \{(0,t,-t) : t \in \mathbb{R}\}$). A sufficient condition for the convex hull to be closed is contained in the following lemma.

**Lemma 5.4** *Let $K_0$ be a closed cone in $\mathbb{R}^m$, and $K$ the convex hull of $K_0$. If $K$ contains no lines then $K$ is closed.*

**Proof** Let $C$ be the convex hull of $K_0 \cap S^{m-1}$. We show that $0 \notin C$.

First notice that 0 cannot be an extreme point of $C$, otherwise 0 should belong to $K_0 \cap S^{m-1}$ ([20], Corollary 18.3.1). Then, if $0 \in C$ we could find two distinct points $x_1, x_2 \in C$ and $0 < \lambda < 1$ with $\lambda x_1 + (1-\lambda)x_2 = 0$. This would imply that the line through $x_1$ and $x_2$ is contained in $K$, since $K$ is easily seen to be a cone. Thus $0 \notin C$.

$K_0 \cap S^{m-1}$ is compact, therefore $C$ is compact, too ([20], Theorem 17.2); therefore, we deduce the existence of an hyperplane separating 0 from $C$, i.e., the existence of $\nu \in S^{m-1}$ and $\gamma > 0$ such that $(\nu, \xi - \gamma) > 0$ for every $\xi \in C$. In particular

$$(\nu, \xi) \geq \gamma|\xi| \qquad \text{for every } \xi \in K_0. \tag{5.4}$$

Let now $(\eta_h)$ be a sequence in $K$ converging to a point $\xi$. Each $\eta_h$ can be written as a sum $\xi_1^h + \ldots + \xi_m^h$ of points $\xi_i^h$ in $K_0$. In view of (5.4) for every $h$ and $i$ we have

$$\gamma|\xi_i^h| \leq \sum_j \gamma|\xi_j^h| \leq \sum_j (\nu, \xi_j^h) = (\nu, \eta_h).$$

The sequence $(\xi_i^h)_h$ is therefore bounded for every $i$. As a consequence, we can find an increasing sequence $(h_k)$ in $\mathbb{N}$ such that $(\xi_i^{h_k})_k$ converges for every $i$. Since $K_0$ is closed, each limit is in $K_0$, which implies that $\xi \in K$. $\square$

The following proposition describes some properties of the function $f_1$.

**Proposition 5.5** *Let $g$ and $f_1$ be as in Theorem 5.1. Let $C$ be the convex envelope of the set $\{\xi \in K_0 : g(\xi) \leq 1\}$. Then:*

a) *$C$ is closed and bounded, and $f_1$ is the gauge function of the set $C$, i.e.,*

$$f_1(\xi) = \inf\{\lambda \geq 0 : \xi \in \lambda C\}$$

*for every $\xi \in M^{sym}$. Moreover, $C = \{\xi \in M^{sym} : f_1(\xi) \leq 1\}$ and $\{\xi \in M^{sym} : f_1(\xi) < +\infty\} = K$;*

b) *if $K$ does not contains lines, then there exists $M_0 > 0$ such that $f_1(\xi) \leq M_0|\xi|$ for every $\xi \in K$.*

**Proof** a) The set $\{\xi \in K_0 : g(\xi) \leq 1\}$ is closed and bounded; by Theorem 17.2 in [20] the same property is shared by its convex hull, i.e., $C$.

Define $h(\xi) = \inf\{\lambda \geq 0 : \xi \in \lambda C\}$ for every $\xi \in M^{sym}$. We claim that $g(\xi) = h(\xi)$ for every $\xi \in K_0$.

If $\xi \in K_0 \setminus \{0\}$ then $\xi/g(\xi) \in C$ so that $h(\xi/g(\xi)) \leq 1$ and, by the positive homogeneity of $h$, we have $h(\xi) \leq g(\xi)$. On the other hand, if $\xi \in K_0 \setminus \{0\}$ then $h(\xi) > 0$, and $\xi/h(\xi) \in C$; it follows that $g(\xi/h(\xi)) \leq 1$, i.e., $g(\xi) \leq h(\xi)$.

Since $h$ is a convex function not greater than $f_0$, then $h \leq f_1$. Let us turn to the reverse inequality. It is enough to show that $f_1(\xi) \leq h(\xi)$ when $\xi \neq 0$ and $h(\xi) < +\infty$; this implies that

$0 < h(\xi) < +\infty$. Since $\xi/h(\xi) \in C$, we can express $\xi/h(\xi)$ as a convex combination $\sum_i \alpha_i \xi_i$ of a finite number of elements of the set $\{\zeta \in K_0 : g(\zeta) \leq 1\}$. Thus

$$f_1(\xi/h(\xi)) \leq \sum_i \alpha_i f_1(\xi_i) \leq \sum_i \alpha_i g(\xi_i) \leq \sum_i \alpha_i = 1.$$

The positive homogeneity of $f_1$ now yields $f_1(\xi) \leq h(\xi)$.

The property $C = \{\xi \in M^{sym} : f_1(\xi) \leq 1\}$ can be deduced from Corollary 9.7.1 in [20]. This yields, in particular, that $\{\xi \in M^{sym} : f_1(\xi) < +\infty\} \subseteq K$. The reverse inclusion is obvious.

b) Let $\xi \in K\setminus\{0\}$ and $\lambda = f_1(\xi)$; then $\lambda > 0$. Since $\xi/\lambda \in C$ we can write $\xi/\lambda = \xi_1 + \xi_2 + \ldots + \xi_N$ for suitable $\xi_i \in K_0$ with $g(\xi_i) \leq 1$. If $g(\xi_i) < 1$ for every $i$, then we could determine $0 < \delta < 1$ such that $\xi/(\delta\lambda) = \sum_i (\xi_i/\delta) \in C$ : this contradicts the minimality of $\lambda$. Therefore $g(\xi_{i_0}) = 1$ for some $i_0$. Letting $M = \max\{g(\zeta) : |\zeta| = 1\}$ we have $|\xi_{i_0}| \geq 1/M$.

Let us now use the assumption that $K$ does not contains lines. As shown in the proof of Lemma 5.4, formula (5.4) holds for suitable $\nu$ and $\gamma$. Therefore

$$\frac{|\xi|}{\lambda} = \sum_i (\xi_i, \nu) \geq \sum_i \gamma|\xi_i| \geq \gamma|\xi_{i_0}| \geq \frac{\gamma}{M}.$$

We conclude that

$$f_1(\xi) = \lambda \leq \frac{M}{\gamma}|\xi|.$$

$\square$

**Corollary 5.6** *The functions $f_1$ and $f$ in Theorem 5.1 are lower semicontinuous.*

**Proof** The semicontinuity of $f_1$ follows from the proposition above (since the sublevel sets of $f_1$ are closed). Moreover, $f = f_1 \Box f_2$, (*infimal convolution* of $f_1$ and $f_2$ : see §2). Therefore, the lower semicontinuity of $f_1$ (and $f_2$) implies the lower semicontinuity of $f$ by Corollary 9.2.2 in [20]. $\square$

**Proposition 5.7** *Assume that $K$ contains no lines. Let $C$ be as in Proposition 5.5. Then $f_1$ is Lipschitz on $K$ if there exists $\gamma_0 > 0$ such that every $\xi \in C$ with $f_1(\xi) = 1$ admits a unit vector $\nu$ in the normal cone to $C$ at $\xi$ satisfying the condition $(\xi, \nu) \geq \gamma_0|\xi|$.*

**Proof** Let $\xi, \eta \in K$ be fixed. We have to prove that

$$|f_1(\xi) - f_1(\eta)| \leq L|\xi - \eta|$$

for a suitable $L$ independent of $\xi$ and $\eta$. By the homogeneity of $f_1$ and by (b) of Proposition 5.5, it is enough to suppose $f_1(\eta) = 1 < f_1(\xi)$. Let $\xi' = \xi/f_1(\xi)$; then $f_1(\xi') = 1$, and, by assumption, we can find a unit vector $\nu$ in the normal cone to $C$ at $\xi'$ such that $(\xi', \nu) \geq \gamma_0|\xi'|$. Moreover, if $H$ denotes the supporting hyperplane to $C$ at $\xi'$ determined by $\nu$, then

$$\begin{aligned} |\xi - \eta| &\geq d(\xi, C) \geq d(\xi, H) = (\xi - \xi', \nu) = \\ &= \Big(1 - \frac{1}{f_1(\xi)}\Big)(\xi, \nu) = \big(f_1(\xi) - 1\big)(\xi', \nu) \geq \gamma_0 \frac{|\xi|}{f_1(\xi)}\big(f_1(\xi) - 1\big). \end{aligned}$$

By (b) of Proposition 5.5 we conclude that

$$f_1(\xi) - 1 \leq \frac{M_0}{\gamma_0}|\xi - \eta|.$$

□

**Remark 5.8** If $\varphi: \xi \mapsto f(P_K\xi): M^{sym} \to [0, +\infty[$ is convex, then the function $\omega$ is given by

$$\omega(\xi) = f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2, \qquad \text{for every } \xi \in M^{sym}.$$

Indeed, $\xi \mapsto \frac{1}{2}\|P_{K^\perp}\xi\|^2$ is convex, as can be easily verified by means of property (3.1)(b); it follows that $\xi \mapsto f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2$ is convex, too.

An important simple case in which $\varphi$ is convex is when $K$ is a vector subspace of $M^{sym}$ (see, e.g., the application to Hencky's plasticity in §9); indeed, in this case, the projection $P_K$ is linear.

As to the general case we point out the following results.

**Proposition 5.9** *Let $K$ be a closed convex cone in $M^{sym}$, and let $\phi: K \to [0, +\infty[$ be a lower semicontinuous convex function. Then the following properties are equivalent:*

a) $\xi \mapsto \phi(P_K\xi): M^{sym} \to [0, +\infty[$ *is convex;*

b) $\phi(P_K(\xi + \eta)) \leq \phi(\xi)$ *for every* $\xi \in K$ *and for every* $\eta \in K^\perp$.

**Proof** (b)⇒(a) Let $\xi_0, \xi_1 \in M^{sym}$, $\lambda \in [0, 1]$ and $\xi_\lambda = \xi_0 + \lambda(\xi_1 - \xi_0) = \lambda\xi_1 + (1-\lambda)\xi_0$. Since $\xi_\lambda = [\lambda P_K\xi_1 + (1-\lambda)P_K\xi_0] + [\lambda P_{K^\perp}\xi_1 + (1-\lambda)P_{K^\perp}\xi_0]$, property (b) yields

$$\phi(P_K\xi_\lambda) \leq \phi\big(\lambda P_K\xi_1 + (1-\lambda)P_K\xi_0\big) \leq \lambda\phi(P_K\xi_1) + (1-\lambda)\phi(P_K\xi_0),$$

from which (a) follows.

(a)⇒(b) Let $\xi$ and $\eta$ be as in (b), and let $\lambda \in ]0, 1[$. By (a)

$$\begin{aligned}\phi(P_K(\xi+\eta)) &= \phi\Big(P_K\big(\lambda\frac{\xi}{\lambda} + (1-\lambda)\frac{\eta}{1-\lambda}\big)\Big) \leq \lambda\phi\big(P_K(\frac{\xi}{\lambda})\big) + (1-\lambda)\phi\big(P_K(\frac{\eta}{1-\lambda})\big)\\ &= \lambda\phi\big(\frac{\xi}{\lambda}\big) \to \phi(\xi), \text{ for } \lambda \to 1^-.\end{aligned}$$

(Here we have used the continuity of lower semicontinuous convex functions along line segments: see Corollary 7.5.1 in [20]). We therefore get the inequality contained in (b). □

## 6 A convergence lemma

In this section we prove a convergence lemma which will be crucial in the estimate of the lower semicontinuous envelope of $F$ from above. It generalizes Reshetnyak's Theorem (see below) to a case where the growth of the integrand function is quadratic in some directions.

Let $n \geq 1$ and let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. We shall adopt the definition (2.1) for $\int f(\mu)$ when $f$ is convex and $\mu$ is a measure.

**Theorem 6.1 (Reshetnyak)** *Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be a convex function such that $0 \leq f(\xi) \leq c(1+|\xi|)$ for every $\xi \in \mathbb{R}^m$ and a suitable constant $c \geq 0$. Let $(\mu_h)$ be a sequence in $\mathcal{M}(\Omega;\mathbb{R}^m)$ which weakly* converges to a measure $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$. Then*

$$\int_\Omega f(\mu) \leq \liminf_{h\to+\infty} \int_\Omega f(\mu_h).$$

*If, in addition, $\lim_h |\mu_h|(\Omega) = |\mu|(\Omega)$ then*

$$\int_\Omega f(\mu) = \lim_{h\to+\infty} \int_\Omega f(\mu_h).$$

**Proof** In the case of $f$ positively homogeneous of degree 1, see [3] Theorems 2.38, 2.39 (the former result was originally proved in [18] Th. 3 or [19] Th. 2, while the latter was proved in [19] Th. 3). In the general case it is possible to reduce to $f$ positively homogeneous of degree 1 by means of the auxiliary function $\overline{f}\colon \mathbb{R}^m \times [0,+\infty[ \to [0,+\infty[$ defined as

$$\overline{f}(\xi,t) = \begin{cases} tf(\xi/t) & \text{if } t > 0 \\ f^\infty(\xi) & \text{if } t = 0 \end{cases}$$

(see, e.g., [18] Th. 2′). $\square$

In this section we shall only assume that the norm $\|\cdot\|$, the set $K$ and the function $f$ satisfy the following conditions.

A) Let $m \geq 1$. Let $\langle\cdot,\cdot\rangle$ be a scalar product in $\mathbb{R}^m$ with the corresponding norm $\|\cdot\|$.

B) Let $K$ be a closed convex cone in $\mathbb{R}^m$.

C) Let $f\colon K \to [0,+\infty[$ be a lower semicontinuous convex function whose recession function $f^\infty$ is Lipschitz on $K$ and such that there exist $c_0, c_1 > 0$ satisfying:

$$f^\infty(\xi) - c_0 \leq f(\xi) \leq c_1\|\xi\|, \quad \text{for every } \xi \in K. \tag{6.1}$$

Moreover, we assume $f(0) = 0$.

Let us define $\omega$ as the convex envelope of the function

$$\xi \mapsto f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2\colon \mathbb{R}^m \to [0,+\infty[.$$

We point out that $\omega$ is finite on the whole $\mathbb{R}^m$.

Under these assumptions the following result holds.

**Lemma 6.2** *Let $\mu$ and $(\mu_h)$ be a measure and a sequence of measures in*

$$\{\nu \in \mathcal{M}(\Omega;\mathbb{R}^m):\quad P_{K^\perp}\nu \ll \mathcal{L}^n,\ \frac{dP_{K^\perp}\nu}{d\mathcal{L}^n} \in L^2(\Omega;\mathbb{R}^m)\},$$

*respectively. Assume that:*

a) $(\mu_h)_h$ *weakly* converges to* $\mu$;

b) $\left(\frac{dP_{K^\perp}\mu_h}{d\mathcal{L}^n}\right)_h$ *converges to* $\frac{dP_{K^\perp}\mu}{d\mathcal{L}^n}$ *in* $L^2(\Omega;\mathbb{R}^m)$.

*Then*

$$\liminf_{h\to+\infty}\int_\Omega \omega(\mu_h) \geq \int_\Omega \omega(\mu).$$

*Moreover, if in addition* $\lim\limits_{h\to+\infty}|\mu_h|(\Omega) = |\mu|(\Omega)$, *then*

$$\lim_{h\to+\infty}\int_\Omega \omega(\mu_h) = \int_\Omega \omega(\mu).$$

For the proof we need the following result.

**Proposition 6.3** a) *There exist $L, \gamma > 0$ such that*

$$\omega(\xi) - \omega(\xi') \leq L\|\xi - \xi'\| + \|P_{K^\perp}\xi\|^2 + \gamma$$

*for all* $\xi, \xi' \in \mathbb{R}^m$.

b) *The subdifferential $\partial\omega$ is bounded on any set of the form $W_r = \{\xi \in \mathbb{R}^m : \|P_{K^\perp}\xi\| \leq r\}$, with $r \geq 0$. In particular, for every $r \geq 0$ there exists $L_r > 0$ such that*

$$\omega(\xi) - \omega(\xi') \leq L_r\|\xi - \xi'\|$$

*whenever $\xi \in W_r$ and $\xi' \in \mathbb{R}^m$. Moreover, $\omega$ is Lipschitz on $W_r$.*

**Proof** <u>*Step*</u> 1 Assume, in addition, that the function $f$ is Lipschitz. Let us introduce the auxiliary function $\Phi = f \,\square\, f_2^\perp$ (*infimal convolution* of $f$ and $f_2^\perp$ : see §2), where $f$ is considered extended to the whole $\mathbb{R}^m$ with value $+\infty$ on $\mathbb{R}^m \setminus K$, and

$$f_2^\perp(\xi) = \begin{cases} \frac{1}{2}\|\xi\|^2, & \text{if } \xi \in K^\perp, \\ +\infty, & \text{if } \xi \in \mathbb{R}^m \setminus K^\perp. \end{cases}$$

Then

$$\Phi(\xi) = \inf\{f(\xi_1) + \frac{1}{2}\|\xi_2\|^2 : \xi_1 \in K,\ \xi_2 \in K^\perp,\ \xi_1 + \xi_2 = \xi\} \tag{6.2}$$

for every $\xi \in \mathbb{R}^m$. It turns out that $\Phi$ is convex and the infimum in (6.2) is attained ([20] Corollary 9.2.2).

We shall prove the following properties, where $L_f$ is a Lipschitz constant for $f$ on $K$:

i) for every $\xi \in \mathbb{R}^m$, if $(\xi_1, \xi_2) \in K \times K^\perp$ is an optimal pair for $\xi$ in (6.2) then:

$$\langle \xi_2, \xi_2 - P_{K^\perp}\xi \rangle \leq L_f \|\xi_2 - P_{K^\perp}\xi\|, \tag{6.3}$$

ii) $\Phi(\xi) \geq \Phi(P_K\xi) - L_f^2$, for every $\xi \in \mathbb{R}^m$;

iii) $\Phi \leq \omega \leq \Phi + c_0$ on $\mathbb{R}^m$, where $c_0$ is the constant in (6.1);

iv) there exists a constant $c > 0$ such that

$$\Phi^\infty - c \leq \Phi \qquad \text{on } K, \tag{6.4}$$

$$\omega^\infty - c \leq \omega \leq \omega^\infty \qquad \text{on } K. \tag{6.5}$$

Let $\xi \in \mathbb{R}^m$ and let $(\xi_1, \xi_2)$ be as in (i). The pair $(\xi_1 + \delta(P_K\xi - \xi_1), \xi_2 + \delta(P_{K^\perp}\xi - \xi_2)$, with $0 < \delta < 1$, is admissible in (6.2); hence:

$$\begin{aligned} f(\xi_1) + \frac{1}{2}\|\xi_2\|^2 &\leq f(\xi_1 + \delta(P_K\xi - \xi_1)) + \frac{1}{2}\|\xi_2 + \delta(P_{K^\perp}\xi - \xi_2)\|^2 \\ &\leq f(\xi_1) + L_f\delta\|P_K\xi - \xi_1\| + \frac{1}{2}\|\xi_2\|^2 + \delta\langle \xi_2, P_{K^\perp}\xi - \xi_2 \rangle + \frac{1}{2}\delta^2\|P_{K^\perp}\xi - \xi_2\|^2. \end{aligned}$$

It follows that

$$0 \leq L_f\|P_K\xi - \xi_1\| + \langle \xi_2, P_{K^\perp}\xi - \xi_2 \rangle + \frac{1}{2}\delta\|P_{K^\perp}\xi - \xi_2\|^2.$$

Letting $\delta$ tend to 0, and taking into account that $P_K\xi + P_{K^\perp}\xi = \xi_1 + \xi_2$ we conclude that

$$\langle \xi_2, \xi_2 - P_{K^\perp}\xi \rangle \leq L_f\|P_K\xi - \xi_1\| = L_f\|\xi_2 - P_{K^\perp}\xi\|,$$

hence condition (6.3).

Let us consider (ii). Let $\xi \in \mathbb{R}^m$ and let $\xi_1, \xi_2$ be an optimal pair for $\xi$ in (6.2). Then, in view of (i),

$$\begin{aligned} \|\xi_2 - P_{K^\perp}\xi\|^2 &= \langle \xi_2, \xi_2 - P_{K^\perp}\xi \rangle - \langle P_{K^\perp}\xi, \xi_2 - P_{K^\perp}\xi \rangle \\ &\leq L_f\|\xi_2 - P_{K^\perp}\xi\| - \langle P_{K^\perp}\xi, P_K\xi - \xi_1 \rangle \leq L_f\|\xi_2 - P_{K^\perp}\xi\|, \end{aligned}$$

from which $\|\xi_2 - P_{K^\perp}\xi\| \leq L_f$. Then,

$$\begin{aligned} \Phi(\xi) = f(\xi_1) + \frac{1}{2}\|\xi_2\|^2 &\geq f(\xi_1) \geq f(P_K\xi) - L_f\|\xi_1 - P_K\xi\| \\ &= f(P_K\xi) - L_f\|\xi_2 - P_{K^\perp}\xi\| \geq \Phi(P_K\xi) - L_f^2, \end{aligned}$$

i.e., property (ii).

Since $\Phi$ is convex and $\Phi(\xi) \leq f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2$ for every $\xi \in \mathbb{R}^m$, then $\Phi \leq \omega$ : the first part of (iii).

Let now $\xi_1 \in K$ and $\xi_2 \in K^\perp$ be optimal for $\xi$ in (6.2). Then, for every $0 < \lambda < 1$,

$$\begin{aligned} \omega(\xi) &= \omega\big(\lambda\frac{\xi_1}{\lambda} + (1-\lambda)\frac{\xi_2}{1-\lambda}\big) \leq \lambda\omega\big(\frac{\xi_1}{\lambda}\big) + (1-\lambda)\omega\big(\frac{\xi_2}{1-\lambda}\big) \\ &\leq \lambda f\big(\frac{\xi_1}{\lambda}\big) + (1-\lambda)\frac{1}{2}\Big\|\frac{\xi_2}{1-\lambda}\Big\|^2 \\ &= \lambda f\big(\frac{\xi_1}{\lambda}\big) + \frac{1}{1-\lambda}\frac{1}{2}\|\xi_2\|^2 \to f^\infty(\xi_1) + \frac{1}{2}\|\xi_2\|^2, \qquad \text{as } \lambda \to 0^+. \end{aligned}$$

Thus, by (6.1),

$$\omega(\xi) \leq f(\xi_1) + \frac{1}{2}\|\xi_2\|^2 + c_0 = \Phi(\xi) + c_0.$$

Therefore, (iii) is proved.

Let $\xi \in K$ be fixed, and let $\xi_1, \xi_2$ be an optimal pair for $\xi$ in (6.2). We have

$$\begin{aligned}\Phi^\infty(\xi) - \Phi(\xi) &= \Phi^\infty(\xi) - f(\xi_1) - \frac{1}{2}\|\xi_2\|^2 \\ &\leq f^\infty(\xi) - f(\xi_1) \leq f(\xi) - f(\xi_1) + c_0 \\ &\leq L_f\|\xi - \xi_1\| + c_0 = L_f\|\xi_2\| + c_0.\end{aligned}$$

Moreover, since $\xi \in K$, (6.3) implies that $\|\xi_2\|^2 \leq L_f\|\xi_2\|$, hence $\|\xi_2\| \leq L_f$. We conclude that

$$\Phi^\infty(\xi) - \Phi(\xi) \leq L_f^2 + c_0,$$

i.e., property (6.4).

Finally, let us consider (6.5). The second inequality is immediate taking into account the convexity of $\omega$ and the fact that $\omega(0) = 0$; indeed, for every $\xi \in \mathbb{R}^m$ and $t > 1$,

$$\omega(\xi) = \omega(\frac{1}{t}t\xi) \leq \frac{1}{t}\omega(t\xi) \to \omega^\infty(\xi) \qquad \text{as } t \to +\infty.$$

As to the first inequality, from (iii) and (6.4) it follows that for every $\xi \in K$

$$\omega^\infty \leq \Phi^\infty \leq \Phi + c \leq \omega + c,$$

i.e., the first inequality in (6.5).

<u>*Step*</u> 2 Since $\omega$ is convex and finite on the whole $\mathbb{R}^m$, for every $\xi_0 \in \mathbb{R}^m$ there exists $\xi_0^* \in \mathbb{R}^m$ such that

(6.6) $$\omega(\xi) \geq \omega(\xi_0) + \langle\, \xi_0^*, \xi - \xi_0 \,\rangle \qquad \text{for every } \xi \in \mathbb{R}^m.$$

It turns out that

(6.7) $$\|P_K\xi_0^*\| \leq c_1.$$

where $c_1$ is the constant in (6.1).

Indeed, taking $\xi = t\eta$ in (6.6), with $\eta \in \mathbb{R}^m$ and $t > 0$, we have

$$\frac{\omega(t\eta)}{t} \geq \frac{\omega(\xi_0)}{t} + \langle\, \xi_0^*, \eta \,\rangle - \frac{\langle\, \xi_0^*, \xi_0 \,\rangle}{t};$$

and, as $t \to +\infty$,

(6.8) $$\omega^\infty(\eta) \geq \langle\, \xi_0^*, \eta \,\rangle, \qquad \text{for every } \eta \in \mathbb{R}^m.$$

In particular, choosing $\eta = P_K\xi_0^*$, we obtain

$$\|P_K\xi_0^*\|^2 \leq \omega^\infty(P_K\xi_0^*) \leq f^\infty(P_K\xi_0^*) \leq c_1\|P_K\xi_0^*\|,$$

which implies (6.7).

*Step* 3 Under the additional assumption that $f$ is Lipschitz we prove that the subdifferential $\partial\omega$ is bounded on $K$; in particular, $\omega$ is Lipschitz on $K$.

Let $\xi_0 \in K$ be fixed, and $\xi_0^* \in \mathbb{R}^m$ satisfy the subdifferential inequality (6.6). By (6.7) it remains to estimate $P_{K^\perp}\xi_0^*$.

Take $\eta = \xi_0$ in (6.8). By (6.5)

$$\langle\, \xi_0^*, \xi_0 \,\rangle \leq \omega^\infty(\xi_0) \leq \omega(\xi_0) + c;$$

therefore, for every $\xi \in \mathbb{R}^m$,

$$\omega(\xi) \geq \langle\, \xi_0^*, \xi_0 \,\rangle - c + \langle\, \xi_0^*, \xi - \xi_0 \,\rangle = \langle\, \xi_0^*, \xi \,\rangle - c.$$

Let $\xi = \frac{P_{K^\perp}\xi_0^*}{\|P_{K^\perp}\xi_0^*\|}$; then

$$\omega\Big(\frac{P_{K^\perp}\xi_0^*}{\|P_{K^\perp}\xi_0^*\|}\Big) \geq \langle\, \xi_0^*, \frac{P_{K^\perp}\xi_0^*}{\|P_{K^\perp}\xi_0^*\|} \,\rangle - c = \|P_{K^\perp}\xi_0^*\| - c.$$

Since $\omega$ is continuous, this yields

$$\|P_{K^\perp}\xi_0^*\| \leq c + \sup_{\|\eta\|=1} \omega(\eta) < +\infty.$$

*Step* 4 We prove (a) of the proposition under the additional assumption that the function $f$ is Lipschitz.

By (6.5) we have that for every $\xi \in K$

$$\omega(2\xi) \leq \omega^\infty(2\xi) = 2\omega^\infty(\xi) \leq 2\big(\omega(\xi) + c\big),$$

therefore

$$\frac{1}{2}\omega(2\xi) \leq \omega(\xi) + c.$$

It follows, taking estimate (ii) and (iii) above, and the Lipschitz property of $\omega$ on $K$ (proved in Step 3) into account, that for every $\xi, \xi' \in \mathbb{R}^m$

$$\begin{aligned}
\omega(\xi) - \omega(\xi') &= \omega\big(\frac{1}{2}(2P_K\xi) + \frac{1}{2}(2P_{K^\perp}\xi)\big) - \omega(\xi') \\
&\leq \frac{1}{2}\omega(2P_K\xi) + \frac{1}{2}\omega(2P_{K^\perp}\xi) - \Phi(\xi') \\
&\leq \omega(P_K\xi) + c + \frac{1}{2}\cdot\frac{1}{2}\|2P_{K^\perp}\xi\|^2 - \big(\Phi(P_K\xi') - L_f^2\big) \\
&\leq \omega(P_K\xi) - \omega(P_K\xi') + \|P_{K^\perp}\xi\|^2 + L_f^2 + c + c_0 \\
&\leq L_\omega\|P_K\xi - P_K\xi'\| + \|P_{K^\perp}\xi\|^2 + L_f^2 + c + c_0 \\
&\leq L_\omega\|\xi - \xi'\| + \|P_{K^\perp}\xi\|^2 + L_f^2 + c + c_0.
\end{aligned}$$

where $L_\omega$ is a Lipschitz constant for $\omega$ on $K$.

*Step* 5 We prove (b) of the proposition under the additional assumption that the function $f$ is Lipschitz.

Let $r > 0$, and $\xi_0 \in W_r = \{\xi \in \mathbb{R}^m : \|P_{K^\perp}\xi\| \le r\}$. Let $\xi_0^* \in \partial\omega(\xi_0)$. By (6.7) we only have to estimate $\|P_{K^\perp}\xi_0^*\|$. Let $\zeta_0^* \in \partial\omega(P_K\xi_0)$; by Step 3 we may assume that $\|\zeta_0^*\| \le M$, with $M$ independent of $\xi_0$. Then, if we set $\widehat{P_{K^\perp}\xi_0^*} = \frac{P_{K^\perp}\xi_0^*}{\|P_{K^\perp}\xi_0^*\|}$,

$$\begin{aligned}\omega(\xi_0 + \widehat{P_{K^\perp}\xi_0^*}) &\ge \omega(\xi_0) + \langle\, \xi_0^*,\, \widehat{P_{K^\perp}\xi_0^*}\,\rangle \\ &\ge \omega(P_K\xi_0) + \langle\, \zeta_0^*,\, P_{K^\perp}\xi_0\,\rangle + \|P_{K^\perp}\xi_0^*\| \\ &\ge \omega(P_K\xi_0) - M\|P_{K^\perp}\xi_0\| + \|P_{K^\perp}\xi_0^*\|;\end{aligned}$$

therefore

$$\|P_{K^\perp}\xi_0^*\| \le \omega(\xi_0 + \widehat{P_{K^\perp}\xi_0^*}) - \omega(P_K\xi_0) + Mr,$$

and, by (a) (proved in Step 4 if $f$ is Lipschitz)

$$\begin{aligned}\|P_{K^\perp}\xi_0^*\| &\le L\|\xi_0 + \widehat{P_{K^\perp}\xi_0^*} - P_K\xi_0\| + \|P_{K^\perp}(\xi_0 + \widehat{P_{K^\perp}\xi_0^*})\|^2 + \gamma + Mr \\ &\le L(\|P_{K^\perp}\xi_0\| + 1) + (\|P_{K^\perp}\xi_0\| + \|\widehat{P_{K^\perp}\xi_0^*}\|)^2 + \gamma + Mr \\ &\le L(r+1) + (r+1)^2 + \gamma + Mr.\end{aligned}$$

*Step* 6 Let us show that (a) and (b) hold without the additional assumption that $f$ is Lipschitz on $K$.

Let $\psi$ be the convex envelope of

$$\xi \mapsto f^\infty(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2 \colon \mathbb{R}^m \to [0, +\infty[.$$

Since $f^\infty$ is Lipschitz by assumption, we can apply Step 4 and Step 5 with $f$ replaced by $f^\infty$, and, consequentely, $\omega$ replaced by $\psi$. In particular, there exist $L, \gamma > 0$ such that

$$\psi(\xi) - \psi(\xi') \le L\|\xi - \xi'\| + \|P_{K^\perp}\xi\|^2 + \gamma$$

for all $\xi, \xi' \in \mathbb{R}^m$. On the other hand, by assumption (C) on $f$, we have

$$\omega \le \psi \le \omega + c_0 \qquad \text{on } \mathbb{R}^m; \tag{6.9}$$

we conclude that

$$\begin{aligned}\omega(\xi) - \omega(\xi') &\le \psi(\xi) - \psi(\xi') + c_0 \\ &\le L\|\xi - \xi'\| + \|P_{K^\perp}\xi\|^2 + \gamma + c_0,\end{aligned}$$

for all $\xi, \xi' \in \mathbb{R}^m$. Thus, property (a) is proved.

As for (b), let $r \ge 0$, $\xi_0 \in W_r$ and $\xi_0^* \in \partial\omega(\xi_0)$. Define $\widehat{\xi}_0^* = \frac{\xi_0^*}{\|\xi_0^*\|}$; it turns out that

$$\omega(\xi_0 + \widehat{\xi}_0^*) \ge \omega(\xi_0) + \langle\, \xi_0^*, \widehat{\xi}_0^*\,\rangle = \omega(\xi_0) + \|\xi_0^*\|,$$

hence, by (6.9),

$$\|\xi_0^*\| \le \psi(\xi_0 + \widehat{\xi}_0^*) - \psi(\xi_0) + c_0.$$

Since $\xi_0 + \widehat{\xi}_0^* \in W_{r+1}$ and, as pointed out above, property (b) holds with $\omega$ replaced by $\psi$, we conclude that $\|\xi_0^*\| \le M + c_0$, where $M$ is a bound for $\partial\psi$ on $W_{r+1}$. $\square$

Before addressing the proof of Lemma 6.2 we state, for reference convenience, the following result.

**Lemma 6.4** *For every $\lambda > 0$ let $\varphi_\lambda$ be the convex envelope of the function*

$$\xi \mapsto \omega(\xi) \wedge (\lambda\|\xi\|) \colon \mathbb{R}^m \to [0, +\infty[.$$

*Let $L, \gamma$ and $L_r$ be as in Proposition 6.3. Then*

a) *$\varphi_\lambda = \omega$ on $W_r = \{\xi \in \mathbb{R}^m : \|P_{K^\perp}\xi\| \le r\}$ whenever $\lambda \ge L_r$;*

b) *$(\varphi_\lambda)_{\lambda>0}$ is non-decreasing and converges pointwise to $\omega$ on $\mathbb{R}^m$ as $\lambda \to +\infty$;*

c) *if $\lambda \ge L$ then*

$$\omega(\xi) \le \varphi_\lambda(\xi) + \|P_{K^\perp}\xi\|^2 + \gamma$$

*for every $\xi \in \mathbb{R}^m$.*

**Proof** Recall (see §2) that $\varphi_\lambda = \omega \,\square\, (\lambda\|\cdot\|)$. Then, for every $\xi \in \mathbb{R}^m$ we get

$$\varphi_\lambda(\xi) = \inf\{\omega(\xi_1) + \lambda\|\xi_2\| : \xi_1 + \xi_2 = \xi\}. \tag{6.10}$$

Let $\xi \in W_r$ and $\xi_1, \xi_2 \in \mathbb{R}^m$ with $\xi_1 + \xi_2 = \xi$. By (b) of Proposition 6.3 we have

$$\omega(\xi_1) \ge \omega(\xi) - L_r\|\xi_1 - \xi\| = \omega(\xi) - L_r\|\xi_2\|;$$

hence, if $\lambda \ge L_r$,

$$\omega(\xi) \le \omega(\xi_1) + \lambda\|\xi_2\|.$$

By the arbitrariness of $\xi_1$ and $\xi_2$ it follows that $\omega(\xi) \le \varphi_\lambda(\xi)$; since the opposite inequality is obvious, the proof of (a) is complete. Clearly, $(\varphi_\lambda)_{\lambda>0}$ is non-decreasing as $\lambda$ increases, therefore (b) follows immediately from (a) taking into account that $\bigcup_{r\ge 0} W_r = \mathbb{R}^m$.

Let $\lambda > 0$ and $\xi, \xi_1, \xi_2 \in \mathbb{R}^m$, with $\xi_1 + \xi_2 = \xi$. By (a) of Proposition 6.3 we have

$$\omega(\xi) \le \omega(\xi_1) + L\|\xi_2\| + \|P_{K^\perp}\xi\|^2 + \gamma.$$

The inequality in (c) follows from (6.10) by the arbitrariness of $\xi_1$ and $\xi_2$. $\square$

**Proof** (of Lemma 6.2) Notice that if $\nu \in \mathcal{M}(\Omega; \mathbb{R}^m)$, from the equality $P_{K^\perp}\nu = P_{K^\perp}\nu^a + P_{K^\perp}\nu^s$ we deduce that $P_{K^\perp}\nu \ll \mathcal{L}^n$ if and only if $P_{K^\perp}\nu^s$ is the zero measure; in this case

$$\frac{dP_{K^\perp}\nu}{dx} = \frac{dP_{K^\perp}\nu^a}{dx} = P_{K^\perp}\frac{d\nu^a}{dx},$$

and, since $\nu^s = P_K\nu^s$,

$$\frac{d\nu^s}{d|\nu^s|} = \frac{dP_K\nu^s}{d|\nu^s|} = P_K\frac{d\nu^s}{d|\nu^s|} \in K \qquad |\nu^s|\text{-a.e. in } \Omega. \tag{6.11}$$

If $\mu^a$ and $\mu_h^a$ ($h \in \mathbb{N}$) denote the absolutely continuous part of $\mu$ and $\mu_h$, respectively, we set $d\mu^a/dx = g$ and $d\mu_h^a/dx = g_h$.

Let $\lambda > 0$ and let $\varphi_\lambda$ be as in the preceding lemma. Since $\varphi_\lambda \leq \omega$, by Theorem 6.1 we have

$$\liminf_{h\to+\infty} \int_\Omega \omega(\mu_h) \geq \liminf_{h\to+\infty} \int_\Omega \varphi_\lambda(\mu_h) \geq \int_\Omega \varphi_\lambda(\mu).$$

By property (a) of Lemma 6.4, $\varphi_\lambda^\infty = \omega^\infty$ on $K$ if $\lambda \geq L_0$ (the constant $L_r$ in (b) of Proposition 6.3 with $r = 0$). Therefore, by (6.11) applied with $\nu = \mu$, we have

$$\int_\Omega \varphi_\lambda(\mu) = \int_\Omega \varphi_\lambda(g)\, dx + \int_\Omega \omega^\infty\Big(\frac{d\mu^s}{d|\mu^s|}\Big) d|\mu^s|$$

if $\lambda \geq L_0$. An application of the monotone convergence theorem yields the convergence to $\int_\Omega \omega(\mu)$ as $\lambda \to +\infty$. The first part of Lemma 6.2 is thus proved.

Assume now that $\lim\limits_{h\to+\infty} |\mu_h|(\Omega) = |\mu|(\Omega)$. It remains to prove that

$$\limsup_{h\to+\infty} \int_\Omega \omega(\mu_h) \leq \int_\Omega \omega(\mu).$$

Let $\lambda \geq L_0$, so that $\varphi_\lambda^\infty = \omega^\infty$ on $K$. Since, by (6.11), $\frac{d\mu_h^s}{d|\mu_h^s|} \in K$ $|\mu_h^s|$-a.e., we get

$$\int_\Omega \omega(\mu_h) = \int_\Omega \omega(g_h)\, dx + \int_\Omega \varphi_\lambda^\infty\Big(\frac{d\mu_h^s}{d|\mu_h^s|}\Big) d|\mu_h^s| = \int_\Omega \varphi_\lambda(\mu_h) + \int_\Omega \big(\omega(g_h) - \varphi_\lambda(g_h)\big)\, dx.$$

Therefore, by Theorem 6.1 and the inequality $\varphi_\lambda \leq \omega$,

$$\begin{aligned}\limsup_{h\to+\infty} \int_\Omega \omega(\mu_h) &= \int_\Omega \varphi_\lambda(\mu) + \limsup_{h\to+\infty} \int_\Omega \big(\omega(g_h) - \varphi_\lambda(g_h)\big)\, dx \\ &\leq \int_\Omega \omega(\mu) + \limsup_{h\to+\infty} \int_\Omega \big(\omega(g_h) - \varphi_\lambda(g_h)\big)\, dx.\end{aligned}$$

Now it is enough to show that

$$\lim_{\lambda\to+\infty}\ \limsup_{h\to+\infty} \Big(\int_\Omega \omega(g_h) - \int_\Omega \varphi_\lambda(g_h)\Big) = 0. \tag{6.12}$$

Let $r > 0$ and $\lambda \geq L_r \vee L$ (we may assume $\lambda \geq L_0$, too). By (a) and (c) of Lemma 6.4

$$\begin{aligned}0 \leq \int_\Omega \big(\omega(g_h) - \varphi_\lambda(g_h)\big)\, dx &= \int_{\{x\in\Omega:\|P_{K^\perp} g_h(x)\|>r\}} \big(\omega(g_h) - \varphi_\lambda(g_h)\big)\, dx \\ &\leq \int_{\{x\in\Omega:\|P_{K^\perp} g_h(x)\|>r\}} \big(\|P_{K^\perp} g_h(x)\|^2 + \gamma\big)\, dx.\end{aligned}$$

Notice that

$$\{x \in \Omega : \|P_{K^\perp} g_h(x)\| > r\} \subseteq$$
$$\{x \in \Omega : \|P_{K^\perp} g_h(x)\| > r, \|P_{K^\perp} g\| \leq r - 1\} \cup \{x \in \Omega : \|P_{K^\perp} g\| > r - 1\}.$$

Since $(P_{K^\perp} g_h)_h$ converges to $P_{K^\perp} g$ in $L^2(\Omega;\mathbb{R}^m)$, it is easy to see that

$$\lim_{h\to+\infty} \int_{\{x\in\Omega:\|P_{K^\perp} g_h(x)\|>r, \|P_{K^\perp} g\|\le r-1\}} \big(\|P_{K^\perp} g_h(x)\|^2+\gamma\big)\,dx = 0;$$

hence

$$0 \le \limsup_{h\to+\infty} \int_\Omega \big(\omega(g_h)-\varphi_\lambda(g_h)\big)\,dx \le \int_{\{x\in\Omega:\|P_{K^\perp} g\|>r-1\}} \big(\|P_{K^\perp} g(x)\|^2+\gamma\big)\,dx.$$

The last term tends to zero as $r\to+\infty$, thus yielding 6.12. □

## 7 Proof of the main theorem

The proof of Theorem 5.1 will be carried out through several steps.

*Step* 1 If $\overline{F}(u) < +\infty$ then $u \in \mathcal{U}(\Omega)$.

By assumption there exists a sequence $(u_h)$ in $S(\Omega)$ converging to $u$ in $L^1(\Omega;\mathbb{R}^n)$ and such that $F(u_h) \le M$ for every $h\in\mathbb{N}$ and for a suitable constant $M$. Then

$$\frac{1}{2}\Big(\int_\Omega \|\mathcal{E}u_h\|^2 + \beta\int_{J_u} |u_h^+ - u_h^-|d\mathcal{H}^{n-1}\Big) \le F(u_h) \le M.$$

In particular, $\big(|Eu_h|(\Omega)\big)$ is bounded; by Remark 2.4 we have that $u\in BD(\Omega)$ and $(Eu_h)$ weakly* converges to $Eu$. Let us now consider that $P_{K^\perp}Eu_h \ll \mathcal{L}^n$ for every $h\in\mathbb{N}$ and $(\|P_{K^\perp}\mathcal{E}u_h\|_{L^2})$ is bounded. Then we can apply Theorem 3.1 and conclude that $u\in\mathcal{U}(\Omega)$.

*Step* 2 $f^\infty = f_1$ on $K$, and there exist $c_0, c_1$ such that $f^\infty(\xi) - c_0 \le f(\xi) \le c_1\|\xi\|$ for every $\xi\in K$.

We have $f^\infty(\xi) \le f_1^\infty(\xi) = f_1(\xi)$ for every $\xi\in K$. By the Lipschitz continuity of $f_1$ on $K$ there exists $c_1$ such that $f_1(\xi)\le c_1\|\xi\|$ for $\xi\in K$. Since $f\le f_1$ on $K$, this immediately yields that $f(\xi)\le c_1\|\xi\|$ for $\xi\in K$. Moreover, for a suitable constant $c_0>0$

$$\frac{1}{2}\|\xi\|^2 \ge f_1(\xi) - c_0$$

for every $\xi\in K$; it follows that $f_1\wedge f_2 \ge f_1 - c_0$, from which $f\ge f_1-c_0$ on $K$. In particular, $f^\infty \ge f_1^\infty = f_1$, so that $f^\infty = f_1$ on $K$ (since $f\le f_1$ on $K$). Now, the inequality $f\ge f_1 - c_0$ can be rewritten as $f \ge f^\infty - c_0$ on $K$.

*Step* 3 For every $u\in\mathcal{U}(\Omega)$ let

$$G(u) = \int_\Omega \omega(\mathcal{E}u)dx + \int_\Omega \omega^\infty\Big(\frac{E^s u}{|E^s u|}\Big)|E^s u|.$$

Here we prove that $G$ is lower semicontinuous on $\mathcal{U}(\Omega)$ with respect to the $L^1(\Omega;\mathbb{R}^n)$ topology.

Let $(u_h)$ be a sequence in $\mathcal{U}(\Omega)$ converging to a function $u \in \mathcal{U}(\Omega)$ in $L^1(\Omega;\mathbb{R}^n)$. We can assume that $\big(G(u_h)\big)_h$ has a finite limit.

For every $\lambda > 0$ let $\varphi_\lambda$ be the convex envelope of the function

$$\xi \mapsto \omega(\xi) \wedge (\lambda\|\xi\|) \colon M^{sym} \to [0,+\infty[\,.$$

Since $f_1(\xi) \geq \beta|\xi|$ if $\xi \in K$, by Step 2 we deduce that

$$\omega(\xi) \geq \beta|P_K\xi| + \frac{1}{2}\|P_{K^\perp}\xi\|^2 \qquad \text{for every } \xi \in K$$

(the right-hand side is convex (see Remark 5.8)). The boundedness of $\big(G(u_h)\big)$ implies that $(Eu_h)$ is bounded in $\mathcal{M}(\Omega;M^{sym})$. As we remarked in Step 1, the weak* convergence of $(Eu_h)$ to $Eu$ follows.

Since $\varphi_\lambda$ is convex and grows at most linearly, by Reshetnyak's Theorem 6.1 we have

$$\int_\Omega \varphi_\lambda(Eu) \leq \liminf_{h\to+\infty} \int_\Omega \varphi_\lambda(Eu_h) \leq \liminf_{h\to+\infty} G(u_h).$$

By Step 2 we are in a position to apply Lemma 6.4. Therefore, $(\varphi_\lambda)_\lambda$ is non-decreasing and converges to $\omega$ on $M^{sym}$ as $\lambda \to +\infty$, and $\varphi_\lambda = \omega$ on $K$. By the monotone convergence theorem we get

$$\int_\Omega \omega(\mathcal{E}u)dx + \int_\Omega \omega^\infty(E^s u) \leq \liminf_{h\to+\infty} G(u_h),$$

i.e., $G(u) \leq \liminf_{h\to+\infty} G(u_h)$.

<u>*Step*</u> 4 $\quad G \leq \overline{F}$ on $\mathcal{U}(\Omega)$.

Let $u \in \mathcal{U}(\Omega)$ with $\overline{F}(u) < +\infty$, and let $(u_h)$ be a sequence in $L^1(\Omega;\mathbb{R}^n)$ with $\liminf_{h\to+\infty} F(u_h) = \overline{F}(u)$. We can assume $F(u_h) < +\infty$ for every $h \in \mathbb{N}$; then $u_h \in S(\Omega)$, $(u_h^+ - u_h^-) \odot \nu_{u_h} \in K_0$ on $J_{u_h}$ up to a set of $\mathcal{H}^{n-1}$-measure zero, and $\frac{1}{2}\int_\Omega \|P_{K^\perp}\mathcal{E}u_h\|^2 dx \leq F(u_h) < +\infty$. In particular, $u_h \in \mathcal{U}(\Omega)$ for every $h$. Notice now that

$$\omega(\xi) \leq f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2 \leq \frac{1}{2}\|P_K\xi\|^2 + \frac{1}{2}\|P_{K^\perp}\xi\|^2 = \frac{1}{2}\|\xi\|^2 \qquad \text{for every } \xi \in M^{sym},$$

and

$$\omega^\infty(\xi) \leq f_1^\infty(\xi) = f_1(\xi) \leq g(\xi) \qquad \text{for every } \xi \in K_0.$$

Therefore, $G(u_h) \leq F(u_h)$ for every $h \in \mathbb{N}$. Taking the lower semicontinuity of $G$ (established in Step 3) into account, we conclude that $G(u) \leq \liminf_{h\to+\infty} G(u_h) \leq \liminf_{h\to+\infty} F(u_h) = \overline{F}(u)$.

It is now useful to "localize" the functional $F$, by defining for every open subset $A$ of $\Omega$

$$F(u,A) = \begin{cases} \dfrac{1}{2}\displaystyle\int_A \|\mathcal{E}u\|^2 dx + \int_{J_u\cap A} g\big((u^+ - u^-)\odot\nu_u\big)d\mathcal{H}^{n-1}, & \text{if } u \in S(\Omega) \text{ and} \\ & (u^+ - u^-)\odot\nu_u \in K_0 \\ & \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty, & \text{otherwise.} \end{cases}$$

Moreover, for every $u \in L^2(\Omega;\mathbb{R}^n) \cap \mathcal{U}(\Omega)$ and $A \in \mathcal{A}(\Omega)$ (the family of all open subsets of $\Omega$) we set

$$(7.1) \quad F_2(u, A) = \inf\big\{\liminf_{h\to+\infty} F(u_h, A) : (u_h) \text{ in } L^2(\Omega;\mathbb{R}^n) \cap \mathcal{U}(\Omega), \quad u_h \to u \text{ in } L^2(\Omega;\mathbb{R}^n)\big\}.$$

In other words, $F_2(\cdot, A)$ is the lower semicontinuous envelope of $F(\cdot, A)$ on $L^2(\Omega;\mathbb{R}^n) \cap \mathcal{U}(\Omega)$ with respect to the $L^2(\Omega;\mathbb{R}^n)$ topology.

We would like to represent $F_2$ in integral form on $W^{1,2}(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega)$. To this end we first need the following technical step:

*Step* 5 $F_2(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a non-negative Borel measure on $\Omega$ for every function $u \in W^{1,2}(\Omega;\mathbb{R}^n)$.

Let $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ be fixed. By Theorem 14.23 in [14] the following properties guarantee that $F_2(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a non-negative Borel measure on $\Omega$: for any $A, A_1, A_2 \in \mathcal{A}(\Omega)$

$$(7.2) \qquad F_2(u, A_1) \le F_2(u, A_2) \qquad \text{if } A_1 \subseteq A_2,$$

$$(7.3) \qquad F_2(u, A_1 \cup A_2) \ge F_2(u, A_1) + F_2(u, A_2) \qquad \text{if } A_1 \cap A_2 = \emptyset,$$

$$(7.4) \qquad F_2(u, A) = \sup\{F_2(u, A') : A' \in \mathcal{A}(\Omega),\ A' \subset\subset A\},$$

$$(7.5) \qquad F_2(u, A_1 \cup A_2) \le F_2(u, A_1) + F_2(u, A_2).$$

Properties (7.2) and (7.3) can be easily proved. As for the others, we need the following fact:

$$(7.6)\ F_2(u, A' \cup B) \le F_2(u, A'') + F_2(u, B), \qquad \text{whenever } A', A'', B \in \mathcal{A}(\Omega) \text{ with } A' \subset\subset A''.$$

Let $(u_h)$ and $(v_h)$ be two arbitrary sequences in $L^2(\Omega;\mathbb{R}^n) \cap \mathcal{U}(\Omega)$ converging to $u$ in $L^2(\Omega;\mathbb{R}^n)$. Suppose that the limits

$$\lim_{h\to+\infty} F(u_h, A''), \quad \lim_{h\to+\infty} F(v_h, B)$$

exist and are finite; in particular we can assume that $(u_h)$ and $(v_h)$ are sequences in $L^2(\Omega;\mathbb{R}^n) \cap S(\Omega)$, and that $\int_{A''} \|\mathcal{E}u_h\|^2 dx, \int_B \|\mathcal{E}v_h\|^2 dx \le M$ for every $h \in \mathbb{N}$ and a suitable constant $M > 0$.

By the arbitrariness of $(u_h)$ and $(v_h)$ the claim will be proved if we show that

$$F_2(u, A' \cup B) \le \lim_{h\to+\infty} F(u_h, A'') + \lim_{h\to+\infty} F(v_h, B).$$

Fix $k \in \mathbb{N}$. Let $A_0 = A'$ and let $A_1, \dots, A_k$ be open subsets of $\Omega$ with boundary of measure zero and satisfying the property

$$A' = A_0 \subset\subset A_1 \subset\subset \dots \subset\subset A_k \subset\subset A''.$$

For every $i \in \{1, \dots, k\}$ let $\varphi_i \in C_c^\infty(A_i)$ with $\varphi_i = 1$ on $A_{i-1}$ and $0 \le \varphi_i \le 1$. Define

$$w_{h,i} = \varphi_i u_h + (1 - \varphi_i) v_h \qquad (h \in \mathbb{N},\ 1 \le i \le k).$$

Then $w_{h,i} \in S(\Omega)$. Moreover, up to a set of $\mathcal{H}^{n-1}$-measure zero, $J_{w_{h,i}} \subseteq J_{u_h} \cup J_{v_h}$, and

$$(w_{h,i}^+ - w_{h,i}^-) \odot \nu_{w_{h,i}} = \varphi_i (u_h^+ - u_h^-) \odot \nu_{u_h} + (1 - \varphi_i)(v_h^+ - v_h^-) \odot \nu_{v_h}$$

$\mathcal{H}^{n-1}$-a.e. on $J_{w_{h,i}}$. Since $(u_h^+ - u_h^-) \odot \nu_{u_h} \in K_0$ and $(v_h^+ - v_h^-) \odot \nu_{v_h} \in K_0$ $\mathcal{H}^{n-1}$-a.e. on $J_{u_h}$ and $J_{v_h}$ respectively, and $\nu_{u_h} = \nu_{v_h}$ $\mathcal{H}^{n-1}$-a.e. on $J_{u_h} \cap J_{v_h}$, we conclude that $(w_{h,i}^+ - w_{h,i}^-) \odot \nu_{w_{h,i}} \in K_0$ $\mathcal{H}^{n-1}$-a.e. on $J_{w_{h,i}}$ by the convexity assumption on $K_0$.

Let $C_i = \overline{A}_i \setminus A_{i-1}$ $(1 \le i \le k)$; then

$$
\begin{aligned}
F(w_{h,i}, A' \cup B) = &\frac{1}{2} \int_{(A'\cup B)\cap A_{i-1}} \|\mathcal{E} u_h\|^2 dx + \int_{J_{u_h} \cap (A'\cup B)\cap A_{i-1})} g\big((u_h^+ - u_h^-) \odot \nu_{u_h}\big) d\mathcal{H}^{n-1} \\
&+ \frac{1}{2} \int_{B\setminus \overline{A}_i} \|\mathcal{E} v_h\|^2 dx + \int_{J_{v_h} \cap (B\setminus \overline{A}_i)} g\big((v_h^+ - v_h^-) \odot \nu_{v_h}\big) d\mathcal{H}^{n-1} \\
&+ \frac{1}{2} \int_{B\cap C_i} \|\mathcal{E} w_{h,i}\|^2 dx + \int_{J_{w_{h,i}} \cap C_i} g\big((w_{h,i}^+ - w_{h,i}^-) \odot \nu_{w_{h,i}}\big) d\mathcal{H}^{n-1}.
\end{aligned} \tag{7.7}
$$

As for the last integral in (7.7), by the convexity of $g$ we have:

$$
\begin{aligned}
\int_{J_{w_{h,i}} \cap C_i} g\big((w_{h,i}^+ - w_{h,i}^-) \odot \nu_{w_{h,i}}\big) d\mathcal{H}^{n-1} \le &\int_{J_{u_h} \cap C_i} g\big((u_h^+ - u_h^-) \odot \nu_{u_h}\big) d\mathcal{H}^{n-1} \\
&+ \int_{J_{v_h} \cap C_i} g\big((v_h^+ - v_h^-) \odot \nu_{v_h}\big) d\mathcal{H}^{n-1}.
\end{aligned} \tag{7.8}
$$

Consider now the volume integral over $B \cap C_i$ in (7.7). The measure strain $Ew_{h,i}$ is given by

$$
Ew_{h,i} = \varphi_i Eu_h + (1 - \varphi_i) Ev_h + (D\varphi_i) \odot (u_h - v_h).
$$

Hence

$$
\frac{1}{2} \int_{B\cap C_i} \|\mathcal{E} w_{h,i}\|^2 dx \le c\Big( \int_{B\cap C_i} \big(\|\mathcal{E} u_h\|^2 + \|\mathcal{E} v_h\|^2\big) dx + N_k^2 \int_{B\cap C_i} |u_h - v_h|^2 dx \Big),
$$

where $c > 0$ is a constant independent of $h$ and $k$, and $N_k = \sup\{|D\varphi_i(x)| : 1 \le i \le k,\ x \in \Omega\}$. Notice that for every $h \in \mathbb{N}$

$$
\sum_{i=1}^{k} \int_{B\cap C_i} \big(\|\mathcal{E} u_h\|^2 + \|\mathcal{E} v_h\|^2\big) dx \le \int_{A''\cap B} \big(\|\mathcal{E} u_h\|^2 + \|\mathcal{E} v_h\|^2\big) dx \le 2M;
$$

it follows that for a suitable $i_h \in \{1, \dots, k\}$

$$
\int_{B\cap C_{i_h}} \big(\|\mathcal{E} u_h\|^2 + \|\mathcal{E} v_h\|^2\big) dx \le 2M/k.
$$

Therefore,

$$
\frac{1}{2} \int_{B\cap C_{i_h}} \|\mathcal{E} w_{h,i_h}\|^2 dx \le c\Big( 2M/k + N_k^2 \int_{\Omega} |u_h - v_h|^2 dx \Big).
$$

This, together with (7.7) and (7.8), yields

$$
F(w_{h,i_h}, A' \cup B) \le F(u_h, A'') + F(v_h, B) + c\Big( 2M/k + N_k^2 \int_{\Omega} |u_h - v_h|^2 dx \Big).
$$

Since the sequence $(w_{h,i_h})$ converges to $u$ in $L^2(\Omega;\mathbb{R}^n)$, we have

$$F_2(u, A' \cup B) \leq \lim_{h\to+\infty} F(u_h, A'') + \lim_{h\to+\infty} F(v_h, B) + 2cM/k.$$

The claim is now immediate by taking the limit as $k$ tends to infinity.

We can now turn to (7.4) and (7.5).

An approximation of $u$ in $W^{1,2}(\Omega;\mathbb{R}^n)$ by means of a sequence of $C^1(\Omega;\mathbb{R}^n)$ functions gives that for every compact subset $K$ of $A$

$$F_2(u, A\setminus K) \leq \frac{1}{2}\int_{A\setminus K} \|\mathcal{E}u\|^2\,dx.$$

Hence, for every $\varepsilon > 0$ we can choose $K$ such that $F_2(u, A\setminus K) < \varepsilon$. Let $A'$ and $A''$ be open sets such that $K \subseteq A' \subset\subset A'' \subset\subset A$. By (7.6) with $B = A\setminus K$ it turns out that

$$F_2(u,A) \leq F_2(u,A'') + F_2(u, A\setminus K) \leq F_2(u,A'') + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we get (7.4). Let us prove (7.5). Given $\varepsilon > 0$, by (7.4) there exists $G \subset\subset A_1 \cup A_2$ such that $F_2(u, A_1\cup A_2) - \varepsilon \leq F_2(u,G)$. Let $A' \in \mathcal{A}(\Omega)$ with $A' \subset\subset A_1$ and $G \subseteq A' \cup A_2$. Then (7.6) ensures that

$$F_2(u, A_1\cup A_2) - \varepsilon \leq F_2(u, A'\cup A_2) \leq F_2(u,A_1) + F_2(u,A_2).$$

Since $\varepsilon > 0$ is arbitrary, (7.5) holds.

*Step* 6 There exists a convex function $\psi\colon M^{n\times n} \to \mathbb{R}$ such that

$$F_2(u,A) = \int_A \psi(Du)dx$$

for every $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ and for every open subset $A$ of $\Omega$.

Since $F_2(\cdot, A)$ is easily verified to be convex (recall Remark 5.2), the integrand function $\psi$ in the representation of $F_2$ has to be convex (use linear functions).

By Theorem 1.1 in [13] there exists a Carathéodory function $\psi\colon \Omega\times M^{n\times n} \to \mathbb{R}$ such that the integral representation

$$F_2(u,A) = \int_A \psi(x, Du)dx \tag{7.9}$$

holds for every $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ and for every $A \in \mathcal{A}(\Omega)$, provided the following five properties are satisfied:

i) (measure) $F_2(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure;

ii) (locality property) $F_2(u,A) = F_2(v,A)$ whenever $u = v$ a.e. on $A$;

iii) (growth condition) there exist $a \in L^1(\Omega)$ and $b \in \mathbb{R}$ such that

$$F_2(u,A) \leq \int_A \big(a(x) + b|Du|^2\big)dx;$$

iv) (translation invariance) $F_2(u+c,A)=F_2(u,A)$ for every $c\in\mathbb{R}^n$;

v) (semicontinuity) $F_2(\cdot,A)$ is sequentially lower semicontinuous with respect to the weak topology of $W^{1,2}(\Omega;\mathbb{R}^n)$.

Step 5 guarantees that condition (i) is satisfied. Let us now verify (ii) through (v).

(ii) Fix $A\in\mathcal{A}(\Omega)$ and $u,v\in W^{1,2}(\Omega;\mathbb{R}^n)$ with $u=v$ a.e. on $A$. Let $(u_h)$ be a sequence in $\mathcal{U}(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$ converging in $L^2(\Omega;\mathbb{R}^n)$ to $u$ and such that $\lim_{h\to+\infty}F(u_h,A)=F_2(u,A)$. Given $A'\subset\subset A$, consider a function $\varphi\in C^1_c(A)$ with $\varphi=1$ on $A'$ and $0\le\varphi\le 1$. Define $w_h=\varphi u_h+(1-\varphi)v$. Then $(w_h)$ converges to $v$ in $L^2(\Omega;\mathbb{R}^n)$ and

$$F_2(v,A')\le\liminf_{h\to+\infty}F(w_h,A')=\liminf_{h\to+\infty}F(u_h,A')\le\lim_{h\to+\infty}F(u_h,A)=F_2(u,A).$$

Taking the supremum for $A'\subset\subset A$ (recall that $F_2(v,\cdot)$ is a measure) we conclude that $F_2(v,A)\le F_2(u,A)$. The opposite inequality follows by exchanging the roles of $u$ and $v$.

(iii) Clearly there exists a constant $c$ such that $F(u,A)\le c\int_A|Du|^2dx$ for every $u\in C^1(\Omega;\mathbb{R}^n)\cap W^{1,2}(\Omega;\mathbb{R}^n)$ and $A\in\mathcal{A}(\Omega)$. Hence, the same inequality holds for $F_2$ on $W^{1,2}(\Omega;\mathbb{R}^n)$.

(iv) The translation invariance of $F_2$ comes easily from the corresponding property for $F$.

(v) For any $A\in\mathcal{A}(\Omega)$ the function $F_2(\cdot,A)$ is, by definition, lower semicontinuous on $W^{1,2}(\Omega;\mathbb{R}^n)$ with respect to the $L^2(\Omega;\mathbb{R}^n)$ topology. Since $F_2(\cdot,A)$ is convex we obtain the sequential lower semicontinuity with respect to the weak topology of $W^{1,2}(\Omega;\mathbb{R}^n)$, too.

Finally, let us show that we can assume $\psi$ in (7.9) to be independent of $x$. Let $B(x_0,r)$ and $B(y_0,r)$ be any pair of congruent balls contained in $\Omega$. Since the integrand functions defining $F$ are independent of $x$, the evaluation of $F_2$ by means of its definition gives $F_2(u_1,B(x_0,r))=F_2(u_2,B(y_0,r))$, where $u_1\colon x\mapsto(\xi,x)$ and $u_2\colon x\mapsto(\xi,x+x_0-y_0)$. Thus, by the integral representation,

$$\int_{B(x_0,r)}\psi(x,\xi)\,dx=\int_{B(y_0,r)}\psi(x,\xi)\,dx$$

for every $\xi\in M^{n\times n}$. This equality implies that $\psi(x_0,\xi)=\psi(y_0,\xi)$ at every pair of Lebesgue points of the function $\psi(\cdot,\xi)$. Letting $\xi$ vary in a countable dense subset of $M^{n\times n}$, and using the continuity of $\psi(x,\cdot)$ we get the existence of a set $N\subseteq\Omega$ with $|N|=0$ such that $\psi(x,\xi)=\psi(y,\xi)$ for every $\xi\in M^{n\times n}$ and for every $x,\ y\in\Omega\setminus N$. Therefore, we can assume that $\psi$ is independent of $x$.

<u>*Step*</u> 7 Let $\psi$ be the function given by Step 6. Then

$$\psi(\xi)\le f(P_K\xi^s)+\frac{1}{2}\|P_{K^\perp}\xi^s\|^2$$

for every $\xi\in M^{n\times n}$, where $\xi^s=\frac{1}{2}(\xi+\xi^T)$ is the symmetric part of $\xi$.

Let $\xi\in M^{n\times n}$ be fixed. We would like to construct a suitable approximating sequence for the function $u\colon x\mapsto(\xi,x)$. Notice that (see §2) $f=f_1\,\square\,f_2$; hence

$$f(P_K\xi^s)=\inf\{f_1(\eta)+\frac{1}{2}\|P_K\xi^s-\eta\|^2:\eta\in K,\ P_K\xi^s-\eta\in K\},$$

and, by [20], Corollary 17.1.6,

$$f_1(\eta) = \inf\Big\{\sum_{i=1}^n \lambda_i g(a_i \odot b_i) : \lambda_i \geq 0,\ a_i \odot b_i \in K_0,\ |a_i| = |b_i| = 1,\ \sum_{i=1}^n \lambda_i a_i \odot b_i = \eta\Big\}$$

for every $\eta \in K$.

Let $\eta \in K$ be fixed, and $\lambda_i \geq 0$, $a_i \odot b_i \in K_0$ $(1 \leq i \leq n)$, with $|a_i| = |b_i| = 1$, and $\sum_{i=1}^n \lambda_i a_i \odot b_i = \eta$. For every $h \in \mathbb{N}$ let $u_h : \mathbb{R}^n \to \mathbb{R}^n$ be defined as follows:

$$u_h(x) = \frac{1}{2} \sum_{i=1}^n \frac{\lambda_i}{h} \big([h(b_i, x)]a_i + [h(a_i, x)]b_i\big)$$

(here $[s]$ denotes the integer part of $s$). Then $u_h$ is piecewise constant, $J_{u_h} \subseteq \bigcup_{i=1}^n S'_{i,h} \cup S''_{i,h}$, where

$$S'_{i,h} = \{x \in \mathbb{R}^n : h(b_i, x) \in \mathbb{Z}\}, \qquad S''_{i,h} = \{x \in \mathbb{R}^n : h(a_i, x) \in \mathbb{Z}\};$$

moreover, taking $\nu_{u_h} = b_i$ on $S'_{i,h}$ and $\nu_{u_h} = a_i$ on $S''_{i,h}$,

$$(u_h^+ - u_h^-) \odot \nu_{u_h} = \frac{1}{2} \sum_{i=1}^n \big(1_{S'_{i,h}} + 1_{S''_{i,h}}\big) \frac{\lambda_i}{h} a_i \odot b_i, \qquad \text{on } J_{u_h}.$$

In particular, $(u_h^+ - u_h^-) \odot \nu_{u_h} \in K_0$ on $J_{u_h}$.

Let $u_0(x) = (\xi - \eta, x)$ $(x \in \mathbb{R}^n)$, and let $B$ be an open ball contained in $\Omega$. In the sequel it will not be restrictive to assume $|B| = 1$. The sequence $(u_0 + u_h)_h$ converges uniformly to $u : x \mapsto (\xi, x)$ on $\mathbb{R}^n$; therefore,

$$F_2(u, B) \leq \liminf_{h \to +\infty} F(u_0 + u_h, B).$$

Since $g$ (1-homogeneous convex function) is subadditive, we have

$$F(u_0 + u_h, B) \leq \frac{1}{2}\|\xi^s - \eta\|^2 + \sum_{i=1}^n \lambda_i g(a_i \odot b_i) \frac{\mathcal{H}^{n-1}(S'_{i,h} \cap B) + \mathcal{H}^{n-1}(S''_{i,h} \cap B)}{2h}$$

for every $h \in \mathbb{N}$. Notice now that

$$\lim_{h \to +\infty} \frac{\mathcal{H}^{n-1}(S'_{i,h} \cap B)}{h} = \lim_{h \to +\infty} \frac{\mathcal{H}^{n-1}(S''_{i,h} \cap B)}{h} = |B| = 1.$$

By the integral representation of the previous step, for any $\eta \in K$ such that $P_K \xi^s - \eta \in K$ we get that

$$\psi(\xi) = F_2(u, B) \leq \frac{1}{2}\|\xi^s - \eta\|^2 + \sum_{i=1}^n \lambda_i g(a_i \odot b_i) \leq \frac{1}{2}\|P_K \xi^s - \eta\|^2 + \frac{1}{2}\|P_{K^\perp} \xi^s\|^2 + \sum_{i=1}^n \lambda_i g(a_i \odot b_i).$$

By the arbitrariness of $\lambda_i$ and $a_i \odot b_i$ $(1 \leq i \leq n)$ we conclude that

$$\psi(\xi) \leq f_1(\eta) + \frac{1}{2}\|P_K \xi^s - \eta\|^2 + \frac{1}{2}\|P_{K^\perp} \xi^s\|^2.$$

By the arbitrariness of $\eta$ we finally obtain that $\psi(\xi) \leq f(P_K\xi^s) + \frac{1}{2}\|P_{K^\perp}\xi^s\|^2$.

*Step* 8 Let $\psi$ be the function given by Step 6, and let $\omega$ be as in Theorem 5.1 . Then $\psi(\xi) \leq \omega(\xi^s)$ for every $\xi \in M^{n\times n}$, where $\xi^s = \frac{1}{2}(\xi + \xi^T)$ is the symmetric part of $\xi$.

Let us consider the following functions:

$$\varphi: \xi \mapsto f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2 : M^{sym} \to [0, +\infty[$$
$$\tilde{\varphi}: \xi \mapsto f(P_K\xi^s) + \frac{1}{2}\|P_{K^\perp}\xi^s\|^2 : M^{n\times n} \to [0, +\infty[.$$

By the previous step $\psi \leq \tilde{\omega}$ on $M^{n\times n}$, where $\tilde{\omega}$ is the convex hull of $\tilde{\varphi}$. Thus, it is enough to show that $\tilde{\omega}(\xi) \leq \omega(\xi^s)$ for every $\xi \in M^{n\times n}$. Let $\sum \alpha_i\eta_i$ be a convex combination of elements of $M^{sym}$, with $\sum \alpha_i\eta_i = \xi^s$. Then $\xi = \sum \alpha_i(\eta_i + \xi^a)$ ($\xi^a$ denotes the antisymmetric part of $\xi$), and

$$\tilde{\omega}(\xi) \leq \sum \alpha_i\tilde{\varphi}(\eta_i + \xi^a) = \sum \alpha_i\varphi(\eta_i).$$

The arbitrariness of the convex combination yields the desired inequality.

*Step* 9 $\overline{F} \leq G$ on $\mathcal{U}(\Omega)$.

Let $u \in \mathcal{U}(\Omega)$. By Theorem 10.2 in [4] there exists a sequence $(u_h)$ in $C^\infty(\overline{\Omega}; \mathbb{R}^n)$ such that

$$\begin{aligned} u_h \to u \qquad & \text{in } L^1(\Omega; \mathbb{R}^n) \\ |Eu_h|(\Omega) \to |Eu|(\Omega) & \\ P_{K^\perp}\mathcal{E}u_h \to P_{K^\perp}\mathcal{E}u \qquad & \text{in } L^2(\Omega; M^{sym}). \end{aligned}$$

The first two conditions also imply the weak* convergence of $(Eu_h)$ to $Eu$ (see Remark 2.4). By the definition of $F_2$, and Steps 6 and 8 we have

$$\overline{F}(u) \leq \liminf_{h\to+\infty} \overline{F}(u_h) \leq \liminf_{h\to+\infty} F_2(u_h, \Omega) = \liminf_{h\to+\infty} \int_\Omega \psi(Du_h)dx \leq \liminf_{h\to+\infty} \int_\Omega \omega(\mathcal{E}u_h)dx.$$

We are in a position to apply Lemma 6.2 with $\mu_h = \mathcal{E}u_h\,\mathcal{L}^n$, and $\mu = Eu$. Therefore

$$\overline{F}(u) \leq \int_\Omega \omega(\mathcal{E}u)dx + \int_\Omega \omega^\infty\Big(\frac{dE^su}{d|E^su|}\Big)d|E^su| = G(u).$$

The proof of Theorem 5.1 is thus complete. □

## 8 The degenerate case

In view of the applications to masonry-like materials we now study the relaxation of the functional $F$ in (5.1) when $g$ is the null function. We shall rely on Theorem 5.1 through a simple perturbation argument.

**Theorem 8.1** *Let $\Omega$ be a strictly star shaped Lipschitz bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$. Let $K_0$ be a closed cone in $M^{sym}$ consisting of matrices of the form $a \odot b$ $(a, b \in \mathbb{R}^n)$ and such that:*

$$a \odot (b + c) \in K_0 \qquad \textit{whenever } a \odot b,\ a \odot c \in K_0.$$

*Moreover, we assume that the convex hull $K$ of $K_0$ does not contain lines.*

*Let $F\colon BD(\Omega) \to [0, +\infty]$ be defined by*

$$F(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \|\mathcal{E}u\|^2 dx, & \textit{if } u \in S(\Omega) \textit{ and} \\ & (u^+ - u^-) \odot \nu_u \in K_0 \ \mathcal{H}^{n-1}\textit{-a.e. on } J_u, \\ +\infty, & \textit{otherwise.} \end{cases}$$

*Denote by $\overline{F}$ its sequential lower semicontinuous envelope with respect to the weak convergence in $BD(\Omega)$.*

*Then $K$ is closed and $\{u \in BD(\Omega) : \overline{F}(u) < +\infty\} = \mathcal{U}(\Omega)$, where:*

$$\mathcal{U}(\Omega) = \big\{u \in BD(\Omega) : P_{K^\perp} E^s u = 0,\ P_{K^\perp}\mathcal{E}u \in L^2(\Omega; M^{sym})\big\}.$$

*Moreover,*

$$\overline{F}(u) = \frac{1}{2}\int_\Omega \|P_{K^\perp}\mathcal{E}u\|^2\, dx$$

*for every $u \in \mathcal{U}(\Omega)$.*

**Proof** The set $K$ is closed by Lemma 5.4. From the proof of the same lemma (see (5.4)), we obtain the existence of $\nu \in M^{sym}$ and $\gamma > 0$ such that

$$(\nu, \xi) \geq \gamma|\xi| \qquad \text{for every } \xi \in K_0. \tag{8.1}$$

Let $g(\xi) = (\nu, \xi)$, and, for every $\beta > 0$ and $u \in L^1(\Omega; \mathbb{R}^n)$, let

$$F_\beta(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \|\mathcal{E}u\|^2 dx + \beta\int_{J_u} g\big((u^+ - u^-) \odot \nu_u\big) d\mathcal{H}^2, & \text{if } u \in S(\Omega) \text{ and} \\ & (u^+ - u^-) \odot \nu_u \in K_0\ \mathcal{H}^2\text{-a.e. on } J_u, \\ +\infty, & \text{otherwise in } L^1(\Omega; \mathbb{R}^n). \end{cases}$$

Moreover, we define

$$G(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \|P_{K^\perp}\mathcal{E}u\|^2 dx, & \text{if } u \in \mathcal{U}(\Omega) \\ +\infty, & \text{otherwise in } L^1(\Omega; \mathbb{R}^n). \end{cases}$$

Extend the definition of $F$ to the whole $L^1(\Omega; \mathbb{R}^n)$ with value $+\infty$. Clearly, $G \leq F \leq F_\beta$ on $L^1(\Omega; \mathbb{R}^n)$. For every $u \in L^1(\Omega; \mathbb{R}^n)$ set

$$\overline{F}(u) = \inf\big\{\liminf_{h\to+\infty} F(u_h) :\ (u_h) \text{ in } BD(\Omega),\ u_h \to u \text{ in } L^1(\Omega; \mathbb{R}^n) \text{ and } \big(|Eu_h|(\Omega)\big)_h \text{ bounded } \big\}$$

(on $BD(\Omega)$ this is the sequential lower semicontinuous envelope of $F$ with respect to the weak convergence). Analogous definitions are given for $\overline{G}$ and $\overline{F}_\beta$. By Theorem 3.1 (and Remark 2.4) $G = \overline{G}$; hence

$$G \le \overline{F} \le \overline{F}_\beta.$$

By the coerciveness of $g$, the functional $\overline{F}_\beta$ coincides with the lower semicontinuous envelope of $F_\beta$ with respect to the $L^1(\Omega;\mathbb{R}^n)$ topology, too.

We would like to apply Theorem 5.1 to $F_\beta$. In view of (8.1) and the closedness of $K$ the only condition to be verified is that the convex hull $f_1$ of

$$f_0(\xi) = \begin{cases} \beta g(\xi), & \text{if } \xi \in K_0, \\ +\infty, & \text{if } \xi \in M^{sym} \setminus K_0 \end{cases} \tag{8.2}$$

is Lipschitz on $K$. We notice that $\beta g \le f_0$, hence $\beta g \le f_1$; moreover, every $\xi \in K$ can be obtained as a convex combination $\sum_i \alpha_i \xi_i$, with $\xi_i \in K_0$, so that

$$f_1(\xi) \le \sum_i \alpha_i f_1(\xi_i) \le \sum_i \alpha_i \beta g(\xi_i) = \beta g(\sum_i \alpha_i \xi_i) = \beta g(\xi),$$

by the linearity of $g$. We conclude that $f_1 = \beta g$ on $K$; in particular $f_1$ is Lipschitz on $K$.

We are now in a position to apply Theorem 5.1 to $F_\beta$; then $\overline{F}_\beta$ is finite on the set $\mathcal{U}(\Omega)$, and

$$\overline{F}_\beta(u) = \int_\Omega \omega_\beta(\mathcal{E}u)dx + \int_\Omega \omega_\beta^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u|,$$

for every $u \in \mathcal{U}(\Omega)$. Here $\omega_\beta$ is computed as prescribed in Theorem 5.1, with $f_0$ as above. It follows that

$$\lim_{\beta\to 0} \omega_\beta(\xi) = \frac{1}{2}\|P_{K^\perp}\xi\|^2, \qquad \text{for every } \xi \in M^{sym}.$$

An easy application of the dominated convergence theorem yields now

$$\lim_{\beta\to 0} \overline{F}_\beta(u) = G(u),$$

for every $u \in \mathcal{U}(\Omega)$. We conclude that $\overline{F} = G$. □

## 9 Applications

In this section we apply Theorems 5.1 and 8.1 assuming that $n = 3$ and $A$ is the operator expressing the stress-strain relation in the classic theory of linear elasticity. Let us recall the equation

$$\sigma = 2\mu\mathcal{E}u + \lambda(\operatorname{tr}\mathcal{E}u)I, \tag{9.1}$$

where $u\colon \Omega \to \mathbb{R}^3$ is the displacement field, $\sigma, \mathcal{E}$ are the stress and the strain tensors, respectively, and $\lambda, \mu$ are the Lamé coefficients, satisfying the relations $3\kappa = 3\lambda + 2\mu > 0$ and $\mu > 0$. Therefore, we shall consider the operator

$$A\colon \xi \mapsto 2\mu\xi + \lambda(\operatorname{tr}\xi)I\colon M^{sym} \to M^{sym}. \tag{9.2}$$

Clearly, $A$ is a positive definite symmetric linear operator, thus satisfying the requirement of Theorem 5.1. Moreover, we note that $Q^T(A\xi)Q = A(Q^T\xi Q)$ for any $\xi \in M^{sym}$ and $Q$ orthogonal matrix.

The strain energy associated to the constitutive relation (9.1) is given by

$$W(u) = \int_\Omega \left(\mu|\mathcal{E}u|^2 + \frac{\lambda}{2}(\operatorname{tr}\mathcal{E}u)^2\right) dx \;=\; \frac{1}{2}\int_\Omega \left(A\mathcal{E}u, \mathcal{E}u\right) dx \;=\; \frac{1}{2}\int_\Omega \|\mathcal{E}u\|^2\, dx, \tag{9.3}$$

where $\|\cdot\|$ is the norm induced by the scalar product $\langle\,\xi,\eta\,\rangle = (A\xi,\eta)$ in $M^{sym}$, according to §5.

Let us consider the functional $F$ introduced in (5.1), with the operator $A$ given by (9.2). Thus, the volume integral represents the strain energy relative to the elastic part of the body, while $K_0$ is the constraint prescribed for the fracture. In the sequel we shall consider different constraints $K_0$ (see (I), (II) and (III) below and in §3).

*Hencky's plasticity*

Let us define

$$K_0 = \{a \odot b:\; a, b \in \mathbb{R}^3,\; (a,b) = 0\} = \{a \odot b:\; \operatorname{tr} a \odot b = 0\}. \tag{I}$$

It turns out that the convex hull $K$ of $K_0$ is the subspace $M_0^{sym}$ consisting of the matrices with null trace and $K^\perp = \{tI:\; t \in \mathbb{R}\}$ (see Proposition 3.2). Hence

$$P_K\xi = \xi^D = \xi - \frac{1}{3}(\operatorname{tr}\xi)I, \qquad P_{K^\perp}\xi = \frac{1}{3}(\operatorname{tr}\xi)I$$

for every $\xi \in M^{sym}$.

Let $g\colon M^{sym} \to [0,+\infty[$ be given by $g(\xi) = \sqrt{2}c|\xi|$, where $c$ is a fixed constant. Then

$$g(a \odot b) = c|a||b| \quad \text{if } a \odot b \in K_0.$$

It is now easy to see that all the assumptions of Theorem 5.1 are satisfied ($K = M_0^{sym}$ is a vector space, hence $f_1$ is Lipschitz on $M_0^{sym}$ since it is a finite convex function positively homogeneous of degree 1). Notice that:

$$\mathcal{U}(\Omega) = \{u \in BD(\Omega) : \operatorname{div} u \ll \mathcal{L}^3,\; \operatorname{div} u \in L^2(\Omega)\}.$$

Moreover, since $K = M_0^{sym}$ is a vector space, we have (recall Remark 5.8)

$$\omega(\xi) = f(P_K\xi) + \frac{1}{2}\|P_{K^\perp}\xi\|^2 = f(\xi^D) + \frac{\kappa}{2}(\operatorname{tr}\xi)^2.$$

Therefore, we get the first part of the following theorem; for the exact computation of the function $f$ we refer to [10] Corollary 3.2, where this result was originally proved (for any bounded open Lipschitz set $\Omega$). It is worth noticing that

$$\mu|\xi|^2 + \frac{\lambda}{2}(\operatorname{tr}\xi)^2 = \mu|\xi^D|^2 + \frac{\kappa}{2}(\operatorname{tr}\xi)^2.$$

**Theorem 9.1** *Let* $F\colon L^1(\Omega;\mathbb{R}^3)\to[0,+\infty]$ *be defined as follows:*

$$F(u)=\begin{cases}\displaystyle\int_\Omega\big(\mu|\mathcal{E}u|^2+\frac{\lambda}{2}(\operatorname{tr}\mathcal{E}u)^2\big)\,dx+c\int_{J_u}|u^+-u^-|\,d\mathcal{H}^2 & \text{if } u\in S(\Omega) \text{ and } \\ & (u^+-u^-)\perp\nu_u\ \mathcal{H}^2\text{-a.e. on } J_u,\\ +\infty, & \text{otherwise.}\end{cases}$$

*Then the lower semicontinuous envelope* $\overline{F}$ *of* $F$ *with respect to the* $L^1(\Omega;\mathbb{R}^3)$ *topology is finite on* $\mathcal{U}(\Omega)=\{u\in BD(\Omega):\operatorname{div}u\ll\mathcal{L}^3,\ \operatorname{div}u\in L^2(\Omega)\}$ *and for every* $u\in\mathcal{U}(\Omega)$ :

$$\overline{F}(u)=\int_\Omega\Big(f\big((\mathcal{E}u)^D\big)+\frac{\kappa}{2}(\operatorname{div}u)^2\Big)\,dx+\int_\Omega f^\infty\big((E^su)^D\big),$$

*where* $f\colon M_0^{sym}\to[0,+\infty[$ *is the convex function whose conjugate is:*

$$f^*(\sigma)=\begin{cases}(1/4\mu)|\sigma|^2 & \text{if } \sigma\in C\cap M_0^{sym},\\ +\infty & \text{otherwise,}\end{cases}$$

*and* $C$ *is Tresca's convex set:*

$$C=\{\sigma\in M^{sym}:\ \lambda_M(\sigma)-\lambda_m(\sigma)\le 2c\}$$

*(*$\lambda_M(\sigma)$ *and* $\lambda_m(\sigma)$ *denote the maximum and minimum eigenvalues of* $\sigma$ *respectively).*

*Masonry-like materials*

In ideal masonry-like materials, which are incapable of sustaining tensile stress, the amount of energy to create an admissible fracture is zero. Thus, we now assume that the surface energy density $g$ in functional (5.1) is the null function, i.e., $F\colon L^1(\Omega;\mathbb{R}^3)\to[0,+\infty]$ is given by

$$(9.4)\quad F(u)=\begin{cases}\displaystyle\int_\Omega\big(\mu|\mathcal{E}u|^2+\frac{\lambda}{2}(\operatorname{tr}\mathcal{E}u)^2\big)\,dx & \text{if } u\in S(\Omega)\text{ and}\\ & (u^+-u^-)\odot\nu_u\in K_0\ \mathcal{H}^2\text{-a.e. on } J_u,\\ +\infty, & \text{otherwise.}\end{cases}$$

We choose the admissible set $K_0$ for the discontinuities according to one of the following models:

- the relative displacement along a fracture is normal to the fracture surface itself; therefore:

$$\text{(II)}\qquad K_0=\{a\odot a:\ a\in\mathbb{R}^3\}.$$

- the angle between the relative displacement along a fracture and the normal to the fracture surface is less or equal to $\pi/2$; therefore:

$$\text{(III)}\qquad K_0=\{a\odot b:\ a,b\in\mathbb{R}^3,\ (a,b)\ge 0\}.$$

When $K_0$ is chosen as in (II), the computation of the relaxed functional (9.4) is a simple corollary of Theorem 8.1. We explicitely note that the functional $\overline{F}(u)=\frac{1}{2}\int_\Omega\|P_{K^\perp}\mathcal{E}u\|^2\,dx$ (with the norm $\|\cdot\|$ as in (9.3)) is just the same as that proposed in [17] and [4] in modelling masonry structures. Indeed, in [4] the projection is onto the cone $A^{-1}M^-$; but, by Proposition 3.2, $K=M^+$, thus $K^\perp=\{\eta\in M^{sym}:\ \forall\xi\in M^+\ (\xi,A\eta)\le 0\}$, i.e. $K^\perp$ is the image through $A^{-1}$ of the orthogonal cone to $M^+$ with respect to the standard scalar product in $M^{sym}$; hence we recover $A^{-1}M^-$.

Finally, let us consider case (III), where $K^\perp=\{tI:\ t\le 0\}$. Then, for any $\xi\in M^{sym}$ it turns out that $P_{K^\perp}\xi=\frac{1}{3\kappa}(\operatorname{tr}\xi)^-I$, where $\alpha^-$ is defined as $\min(\alpha,0)$.

**Theorem 9.2** *Let $F\colon BD(\Omega) \to [0,+\infty]$ be defined by* (9.4)*, with $K_0$ as in* (II)*. Then the sequential lower semicontinuous envelope $\overline{F}$ with respect to the weak convergence in $BD(\Omega)$ is finite on the set*

$$\mathcal{U}(\Omega) = \big\{u \in BD(\Omega) : \ (\operatorname{tr} E^s u/|E^s u|)^- = 0 \quad |E^s u| - a.e., \ (\operatorname{tr} \mathcal{E}u)^- \in L^2(\Omega; M^{sym})\big\}.$$

*Furthermore, for every $u \in \mathcal{U}(\Omega)$,*

$$\overline{F}(u) = \frac{1}{6\kappa} \int_\Omega \big((\operatorname{tr} \mathcal{E}u)^-\big)^2 \, dx.$$

**Proof** For every $h \in \mathbb{N}$ let $K_0^h = \{a \odot b : \ (a,b) \geq \frac{1}{h}|a||b|\}$. It is easy to see that

$$K = \overline{\bigcup_h K_h},$$

where $K_h$ is the convex hull of $K_0^h$. Let $F_h\colon BD(\Omega) \to [0,+\infty]$ be defined by

$$F_h(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \|\mathcal{E}u\|^2 dx, & \text{if } u \in S(\Omega) \text{ and} \\ & (u^+ - u^-) \odot \nu_u \in K_0^h \ \mathcal{H}^2\text{-a.e. on } J_u, \\ +\infty, & \text{otherwise,} \end{cases}$$

Since $K_0^h$ satisfies the requirements for $K_0$ in Theorem 8.1, if we set

$$\mathcal{U}_h(\Omega) = \{u \in BD(\Omega) : P_{K_h^\perp} E^s u = 0, \ P_{K_h^\perp} \mathcal{E}u \in L^2(\Omega; M^{sym})\},$$

we obtain that

$$\overline{F}_h(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \|P_{K_h^\perp} \mathcal{E}u\|^2 \, dx, & \text{if } u \in \mathcal{U}_h(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} \mathcal{U}(\Omega) =& \{u \in BD(\Omega) : P_{K^\perp} E^s u = 0, \ P_{K^\perp} \mathcal{E}u \in L^2(\Omega; M^{sym})\} \\ =& \big\{u \in BD(\Omega) : \ (\operatorname{tr} E^s u/|E^s u|)^- = 0 \quad |E^s u| - a.e., \ (\operatorname{tr} \mathcal{E}u)^- \in L^2(\Omega; M^{sym})\big\}. \end{aligned}$$

Fix $u \in \mathcal{U}(\Omega)$. By Theorem 10.2 in [4] there exists a sequence $(u_j)$ in $C^\infty(\overline{\Omega}; \mathbb{R}^n)$ such that

$$\begin{gathered} u_j \to u \qquad \text{in } L^1(\Omega; \mathbb{R}^n) \\ |Eu_j|(\Omega) \to |Eu|(\Omega) \\ P_{K^\perp} \mathcal{E}u_j \to P_{K^\perp} \mathcal{E}u \qquad \text{in } L^2(\Omega; M^{sym}). \end{gathered}$$

Fix $j \in \mathbb{N}$. Since $u_j \in \mathcal{U}_h(\Omega)$ for every $h$, we have

$$\overline{F}(u_j) \leq \overline{F}_h(u_j) = \frac{1}{2} \int_\Omega \|P_{K_h^\perp} \mathcal{E}u_j\|^2 \, dx.$$

It is easy to see that

$$P_{K_h^\perp} \xi \to P_{K^\perp} \xi \qquad \text{for every } \xi \in M^{sym}.$$

By the dominated convergence theorem it turns out that

$$\overline{F}(u_j) \leq \frac{1}{2} \int_\Omega \|P_{K^\perp} \mathcal{E} u_j\|^2 \, dx.$$

Therefore, taking the convergence of $(u_j)$ into account,

$$\overline{F}(u) \leq \liminf_{j\to\infty} \overline{F}(u_j) = \frac{1}{2} \int_\Omega \|P_{K^\perp} \mathcal{E} u\|^2 \, dx.$$

Let $G$ be as in the proof of the previous theorem. Clearly, the inequality $G \leq \overline{F}$ still holds on $BD(\Omega)$. This concludes the proof. $\square$

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