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Functionals defined on partitions of sets of finite perimeter, II: semicontinuity, relaxation and homogenization

Introduction.

In many problems of Mathematical Physics we find partitions of sets minimizing some kind of interface energy. For example, dealing with liquid crystals, we sometimes encounter interfacing small drops with different orientations ([E]), while, in the theory of Cahn- Hilliard fluids, we find regions corresponding to different densities of the fluid ([B]).
In a recent paper of ours ([AB]) we have studied a class of integral functionals describing this kind of phenomena in the framework of the Calculus of Variations, and in particular of Γ-convergence theory, proving integral representation and compactness results. In this paper we carry on this study, dealing with the problems of lower semicontinuity, relaxation and homogenization.
The functionals we are interested in are defined on partitions $\{E_1,\ldots,E_k\}$ of an open set Ω, and their values depend on an integral on the interfaces between the sets of the partition. Their integrands will depend on the interfacing sets, their orientation and possibly on a space variable; that is, our functional will be of the form

$$\sum_{i,j=1}^{k} \int_{\partial^* E_i \cap \partial^* E_j} \varphi_{ij}(x,\nu_i(x))\, d\mathcal{H}_{n-1}(x) \tag{0.1}$$

where $\partial^* E_i$, ν_i are the boundaries and the inner normals of the sets E_i, in a measure theoretic sense.
In Chapter 1 we recall the main definitions about sets of finite perimeter, which are the natural domain of functionals (0.1), and introduce a class of step functions of bounded variation which enables us to rewrite integral (0.1) in the more handy form

$$\int_{S_u} \varphi(x,u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x) \tag{0.2}$$

where S_u is the set of jumps of the function u, ν_u the normal to S_u and u^+, u^- are the (approximate) values of u on the two sides of S_u. We also recall the main notions of Γ-convergence and some results of [AB].
Chapter 2 is devoted to the study of the lower semicontinuity of the functional (0.2) with respect to convergence in measure. We prove necessity and sufficiency of an integral condition on the integrand f, BV-ellipticity, and discuss the conjecture of its equivalence with another algebraic one, bi-convexity. Other necessary and sufficient conditions, and their relations with the previous two are studied.
In chapter 3 we deal with the problem of relaxation: we give an integral representation of the greatest lower semicontinuous functional less or equal a given functional (0.2), and an integral formula for its integrand. Moreover we prove that the Γ-limit of Dirichlet type problems is a problem with a penalization on the boundary. This phenomenon is well known in non parametric area problems ([GI]).
In the last chapter we study the problem of homogenization: the "macroscopical" properties of a functional whose microscopical behaviour is described by (0.2), with φ periodic in the first variable; that is, the characterization of the Γ-limit of functionals

$$\int_{S_u} \varphi\big(\frac{x}{\epsilon},u^+(x),u^-(x),\nu_u(x)\big)\, d\mathcal{H}_{n-1}(x)$$

as $\epsilon \to 0^+$. We prove that these functionals Γ-converge to a functional

$$\int_{S_u} g(u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x)$$

and an integral formula for the integrand g of the Γ-limit is given.

1. Notation and preliminary results.

Let $\Omega \subset \mathbf{R}^n$ be an open set. We denote by $\mathbf{B}(\Omega)$ the class of Borel subsets of Ω and by $\mathbf{A}(\Omega)$ the class of open subsets of Ω. We denote by $|E|$, $\mathcal{H}_{n-1}(E)$ the Lebesgue n- dimensional measure and the Hausdorff $(n-1)$-dimensional measure of a Borel set $E \subset \mathbf{R}^n$ respectively, and we set

$$\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}.$$

Let $E \subset \Omega$ be a Borel set. We say that E is a set of finite perimeter in Ω if

$$P(E,\Omega) = \sup\left\{ \int_E \mathrm{div} g \, dx : g \in C_0^1(\Omega;\mathbf{R}^n),\ 0 \le g \le 1 \right\} < +\infty. \tag{1.1}$$

The real number defined in (1.1) is called perimeter of E in Ω. To every Borel set E we can associate its essential boundary $\partial^* E$, defined by

$$\partial^* E = \{x \in \mathbf{R}^n : \limsup_{\rho\to 0^+} \rho^{-n}|B_\rho(x) \cap E| > 0 \quad \text{and} \quad \limsup_{\rho\to 0^+} \rho^{-n}|B_\rho(x) \setminus E| > 0\}.$$

The essential boundary of sets of finite perimeter is closely related to the perimeter by the relation ([DG2], [V1], [FE1], [FE2])

$$\mathcal{H}_{n-1}(A \cap \partial^* E) = P(E,A) = \sup\left\{ \int_E \mathrm{div} g \, dx : g \in C_0^1(A;\mathbf{R}^n),\ 0 \le g \le 1 \right\}$$

for every open set $A \subset \Omega$. Moreover,

$$\mathcal{H}_{n-1}(\partial^* E \setminus E_{1/2}) = 0, \tag{1.2}$$

where $E_{1/2}$ is the set of points $x \in \Omega$ where E has density $1/2$.
We shall deal with partitions of the set Ω in a fixed number -say k- of sets of finite perimeter. We can index such a partition with the elements of a fixed finite set $T = \{z_1, \ldots, z_k\}$, whose nature will depend on the applications. For example, in the theory of the Cahn- Hilliard fluids T can be chosen as the set of minimal densities (see [B]), while dealing with small drops of liquid crystals, it is reasonable to take it as the set of the orientations of the crystal. To each partition $\{E_{z_1}, \ldots, E_{z_k}\}$ we can associate a function $u : \Omega \to T$ setting $u(x) = z_i$ on E_{z_i}.
The class of Borel functions $u : \Omega \to T$ whose level sets $\{u = z_i\}$ are sets of finite perimeter will be denoted by $BV(\Omega, T)$. In some applications the set T is chosen as a subspace of $\mathbf{R}^m$. In this case, $BV(\Omega, T)$ is a subset of the space $BV(\Omega;\mathbf{R}^m)$ of functions $u : \Omega \to \mathbf{R}^m$ with bounded variation.
We denote by S_u the Borel set

$$S_u = \bigcup_{i\in T} \partial^*\{u = i\} = \bigcup_{i,\, j\in T,\, i\ne j} \partial^*\{u = i\} \cap \partial^*\{u = j\}. \tag{1.3}$$

The most natural topology in $BV(\Omega, T)$ is given by (local) convergence in measure, which is induced by the distance

$$d_\Omega(u, v) = \sum_{k=1}^{\infty} |\{x \in \Omega_k : u(x) \ne v(x)\}| \tag{1.4}$$

($\Omega_k = \{x \in \Omega : |x| < k,\ \mathrm{dist}(x, \partial\Omega) > 2^{-k}\}$). Throughout this paper we shall use the following properties of $BV(\Omega, T)$, which can be desumed by the corresponding properties of sets of finite perimeter:
(1.5) The set

$$\{u \in BV(\Omega, T) : \mathcal{H}_{n-1}(S_u) \le C\}$$

is compact with respect to convergence in measure for every constant $C > 0$ ([GI], Theorem 1.19).

(1.6) If u, $v \in BV(\Omega, T)$ and $E \subset \Omega$ is a set of finite perimeter in Ω, then the function

$$w(x) = \begin{cases} u(x) & \text{if } x \in E; \\ \\ v(x) & \text{if } x \in \Omega \setminus E, \end{cases}$$

belongs to $BV(\Omega, T)$ ([V1], Theorem 14.5).

(1.7) Each point $x \in \Omega \setminus S_u$ is by definition a point of density 1 for a unique level set $\{u = i\}$. By (1.2) it is easily seen that in $\mathcal{H}_{n-1}$-almost every $x \in S_u$ there exist two different level sets $\{u = i\}$, $\{u = j\}$ which have density $1/2$ at x. We can say more ([V1], [V2]): in $\mathcal{H}_{n-1}$-almost every $x \in S_u$ there exist a unitary vector $\nu_u(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{\rho \to 0^+} \rho^{-n} |\{y \in B_\rho(x) : \langle y - x, \nu_u(x) \rangle > 0,\, u(y) \neq i\}| = 0,$$

and

$$\lim_{\rho \to 0^+} \rho^{-n} |\{y \in B_\rho(x) : \langle y - x, \nu_u(x) \rangle < 0,\, u(y) \neq j\}| = 0.$$

We set $i = u^+(x)$, $j = u^-(x)$. The triplet (u^+, u^-, ν_u) is defined $\mathcal{H}_{n-1}$-almost everywhere on S_u, and it is uniquely determined, up to a change of sign of ν_u and an interchange of u^+, u^-. The functional (0.1) can thus be represented in the form

$$\int_{S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \qquad u \in BV(\Omega, T)$$

with $\varphi : \Omega \times T \times T \times \mathbf{S}^{n-1} \to [0, +\infty]$ Borel function. We shall always tacitly assume that all the integrands φ satisfy the conditions

$$\varphi(x, u, v, \nu) = \varphi(x, v, u, -\nu), \qquad \varphi(x, u, u, \nu) = 0.$$

Moreover, φ will be prolonged by homogeneity whenever necessary:

$$\varphi(x, u, v, p) = \varphi\big(x, u, \frac{p}{|p|}\big)|p|$$

for all $p \in \mathbf{R}^n \setminus \{0\}$.

(1.8) If $\Omega \subset \mathbf{R}^n$ is an open set with Lipschitz continuous boundary $\partial\Omega$, every function $u \in BV(\Omega, T)$ has a trace u^* on $BV(\Omega, T)$, i.e., a Borel function $u^* : \partial\Omega \to T$ such that

$$\lim_{\rho \to 0^+} \frac{|\{x \in B_\rho(x) \cap \Omega : u(y) \neq u^*(x)\}}{\rho^n} = 0$$

for $\mathcal{H}_{n-1}$-almost every $x \in \partial\Omega$ ([GI], Theorem 2.10).

(1.9) Let $\Omega \subset \mathbf{R}^n$ as above, and assume that $\mathcal{H}_{n-1}(\partial\Omega) < +\infty$. If $u \in BV(\Omega, T)$, $v \in BV(\mathbf{R}^n \setminus \overline{\Omega}; T)$, then the function

$$w(x) = \begin{cases} u(x) & \text{if } x \in \Omega; \\ \\ v(x) & \text{if } x \in \mathbf{R}^n \setminus \overline{\Omega}, \end{cases}$$

belongs to $BV(\mathbf{R}^n; T)$, and

$$\begin{cases} S_w \subset \big(\Omega \cap S_u\big) \cup \big(S_v \cap \mathbf{R}^n \setminus \overline{\Omega}\big) \cup \big\{x \in \partial\Omega : u^*(x) \neq v^*(x)\big\} \cup N \\ \\ \nu_w = \pm\, \nu_\Omega \qquad \forall x \in \partial\Omega \setminus N, \end{cases}$$

for a suitable $\mathcal{H}_{n-1}$-negligible Borel set $N \subset \partial\Omega$ ([GI], Remark 2.3, Remark 2.14).

Let us recall some basic definitions and results about Γ- convergence (we refer to [DF1], [DF2], [DMO] for the bibliography on the subject). Let (X,d) be a separable metric space, and let (F_h) be a sequence of real extended valued functions defined in X. We set

$$\Gamma(d^-) - \limsup_{h\to+\infty} F_h(u) = \inf\big\{\limsup_{h\to+\infty} F_h(u_h) : (u_h) \subset X,\ u_h \to u\big\}, \tag{1.10}$$

$$\Gamma(d^-) - \liminf_{h\to+\infty} F_h(u) = \inf\big\{\liminf_{h\to+\infty} F_h(u_h) : (u_h) \subset X,\ u_h \to u\big\}, \tag{1.11}$$

for every $u \in X$. The functions in (1.10), (1.11) are both d-lower semicontinuous.

We say that the sequence (F_h) Γ-converges to F_∞ if

$$\Gamma(d^-) - \liminf_{h\to+\infty} F_h(u) = F_\infty(u) = \Gamma(d^-) - \limsup_{h\to+\infty} F_h(u) \qquad \forall u \in X.$$

The Γ-limit if exists is unique; moreover, every sequence (F_h) admits a Γ-converging subsequence.

The property which motivates the introduction of Γ-convergence in Calculus of Variations is the following: assume that (F_h) Γ- converges to F_∞ and

$$\inf_X F_h = \inf_K F_h \qquad \forall h \in \mathbf{N}$$

for a suitable compact set $K \subset X$. Then

$$\lim_{h\to+\infty} \inf_X F_h = \min\{F_\infty(x) : x \in X\} \tag{1.12}$$

and every sequence $(x_h) \subset K$ such that

$$\lim_{h\to+\infty} F_h(x_h) = \lim_{h\to+\infty} \inf_X F_h$$

admits a subsequence converging to a minimizer of F_∞.

If $F_h = F$ for every $h \in \mathbf{N}$, then the Γ-limit exists and is equal to

$$u \quad \longrightarrow \quad \min\big\{\liminf_{h\to+\infty} F(u_h) : (u_h) \subset X,\, u_h \to u\big\}. \tag{1.13}$$

In this case, the Γ-limit is called also relaxed functional.

In the following, we are interested in studying Γ-convergence of functionals defined on $BV(\Omega, T)$. To deal with this kind of problems it is convenient to introduce localized functionals $\mathcal{F}(u, A)$ depending on $u \in BV(\Omega, T)$ and $A \in \mathbf{A}(\Omega)$. Hence, we set

$$\Gamma(d_A^-) - \limsup_{h\to+\infty} \mathcal{F}_h(u, A) = \inf\big\{\limsup_{h\to+\infty} \mathcal{F}_h(u_h, A) : (u_h) \subset X,\ d_A(u_h, u) \to 0\big\},$$

$$\Gamma(d_A^-) - \liminf_{h\to+\infty} \mathcal{F}_h(u, A) = \inf\big\{\liminf_{h\to+\infty} \mathcal{F}_h(u_h, A) : (u_h) \subset X,\ d_A(u_h, u) \to 0\big\},$$

for every $u \in BV(\Omega, T)$, $A \in \mathbf{A}(\Omega)$, where $d_A(u, v)$ is defined as in (1.4).

We say that $\mathcal{F} : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0, +\infty[$ is a variational functional if the following three conditions are satisfied:

(1.14) $\mathcal{F}(u, A) = \mathcal{F}(v, A)$ whenever u, $v \in BV(\Omega, T)$, $A \in \mathbf{A}(\Omega)$ and $u = v$ almost everywhere in A;

(1.15) $\mathcal{F}(u, \cdot)$ is the restriction to $\mathbf{A}(\Omega)$ of a regular Borel measure in $BV(\Omega, T)$ for every $u \in BV(\Omega, T)$;

(1.16) $\mathcal{F}(\cdot, A)$ is d_A -lower semicontinuous in $BV(\Omega, T)$ for every open set $A \in \mathbf{A}(\Omega)$;

The following theorem, which has been proved in [AB], shows that the Γ-limit of a sequence of variational functionals is in many cases a variational functional which admits integral representation.

Theorem 1.1. *Let $\mathcal{F}_h : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0, +\infty[$ be a sequence of functionals satisfying (1.14), (1.15) and*

$$0 \le \mathcal{F}_h(u, A) \le \Lambda \mathcal{H}_{n-1}(A \cap S_u) \qquad \forall u \in BV(\Omega, T),\ A \in \mathbf{A}(\Omega), \tag{1.17}$$

for a suitable constant $\Lambda > 0$ independent of h. Then, there exists a subsequence $(\mathcal{F}_{h_k})$ and a variational functional $\mathcal{F}$ such that

$$\mathcal{F}(\cdot, A) = \Gamma(d_A^-) \lim_{k \to +\infty} \mathcal{F}_{h_k}(\cdot, A)$$

for every open set $A \subset \Omega$. Moreover, assume that for every open set $A \subset\subset \Omega$ there exists a continuous function $\omega_A : [0, +\infty[\to [0, +\infty[$ such that $\omega_A(0) = 0$ and

$$|\mathcal{F}(u, B) - \mathcal{F}(v, B + z)| < \omega_A(|z|) \mathcal{H}_{n-1}(B \cap S_u)$$

whenever $z \in \mathbf{R}^n$, B, $B + z \subset\subset A$, and $u(x) = v(x + z)$ in B . Then, there exists a unique continuous function $\varphi : \Omega \times T \times T \times \mathbf{S}^{n-1} \to [0, \Lambda]$ such that

$$\mathcal{F}(u, A) = \int_{A \cap S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \qquad \forall u \in BV(\Omega, T),\ A \in \mathbf{A}(\Omega).$$

Let $\Omega \subset \mathbf{R}^n$ be an open set, and let

$$\Omega_t = \big\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > t \big\} \quad (t > 0).$$

Arguing as in [AB], Lemma 4.4, it is possible to prove the following joint lemma.

Lemma 1.2. *There exists a constant $c = c(\Omega, t)$ such that, for every pair of functions u, $v \in BV(\Omega, T)$ we can find $s \in]0, t[$ such that*

$$w(x) = \begin{cases} u(x) & \text{if } x \in \Omega_s \\ v(x) & \text{if } x \in \Omega \setminus \overline{\Omega}_s \end{cases}$$

belongs to $BV(\Omega, T)$ and

$$\mathcal{F}(w, \Omega) \le \mathcal{F}(u, \Omega) + \mathcal{F}(v, \Omega \setminus \overline{\Omega}_t) + c\Lambda |\{x \in \Omega \setminus \Omega_t : u(x) \ne v(x)\}|$$

for every functional $\mathcal{F} : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0, +\infty[$ satisfying (1.14), (1.15), (1.17).

2. Semicontinuity.

In order to apply the tools and the theorems of the “direct” methods of the Calculus of Variations, we have first to investigate necessary and sufficient conditions for the lower semicontinuity of the functionals

$$u \quad \longrightarrow \quad \int_{S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \tag{2.1}$$

with respect to convergence in measure.

2.1. Necessary and sufficient condition: BV-ellipticity.

In what follows we shall bear in mind the semicontinuity theory for functionals defined in Sobolev spaces. Lower semicontinuity theorems for functionals of the Calculus of Variations related to the theory of non-linear elasticity have been proven in a number of papers, among which we recall [MR], [AF]. In these works it is proved that under suitable growth conditions on the integrand, the lower semicontinuity in the weak* topology of $W^{1,\infty}$ of functionals of the form

$$\int_\Omega f(x,u,\nabla u)\,dx$$

(u vector valued function, Ω bounded open set of some $\mathbf{R}^n$) is equivalent to the quasi-convexity of the function f (see [DA]), i.e., the condition

$$f(x,u,z)|\Omega| \le \int_\Omega f(x,u,z+\nabla w(y))\,dy$$

must hold for all $w \in C_0^\infty(\Omega)$.

Dealing with the functionals (2.1), we have been led to considering a condition on the integrand, similar to quasi-convexity, and closely related to the ellipticity conditions of Geometric Measure Theory (see [FE], Chapter 5). In the quasi convexity, one requires that the minimum is taken on the affine functions (among all C_0^∞ perturbations of a fixed affine function); here, we require that our functional must take its minimum on plane surfaces (among all perturbations with the same " boundary" values).

Let $x_0 \in \mathbf{R}^n$, $(i,j,\nu) \in T\times T\times \mathbf{S}^{n-1}$ with $i\neq j$, and let $u_0 : \mathbf{R}^n \to T$ be the function defined by

$$u_0(x) = \begin{cases} i & \text{if } \langle x-x_0,\nu\rangle > 0; \\ \\ j & \text{if } \langle x-x_0,\nu\rangle \le 0. \end{cases} \tag{2.2}$$

Let Ω be a set with Lipschitz continuous boundary containing x_0. We say that a function $\varphi : T\times T\times\mathbf{S}^{n-1} \to [0,+\infty]$ is BV-elliptic if for every triplet (i,j,ν) we have

$$\int_{\Omega\cap S_u} \varphi(u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x) \ge \int_{S_{u_0}} \varphi(u_0^+,u_0^-,\nu_{u_0})\,d\mathcal{H}_{n-1}(x)$$

whenever $u \in BV(\Omega,T)$ is a function with the same trace as u_0 on the boundary $\partial\Omega$. It is easy to see that the BV-ellipticity condition does not depend on the choice of Ω and x_0. A function $\varphi : \Omega\times T\times T\times\mathbf{S}^{n-1} \to [0,+\infty[$ is said to be BV-elliptic if the function $\varphi(x,\cdot,\cdot,\cdot)$ is BV-elliptic for every $x\in\Omega$.

The following theorem gives a characterization in terms of BV-ellipticity of a class of lower semicontinuous integral functionals of the type (2.1).

Theorem 2.1. *Let $c>0$, and let $\varphi : \Omega\times T\times T\times\mathbf{S}^{n-1} \to [c,+\infty[$ be a continuous function. Then, the functional (2.1) is lower semicontinuous with respect to convergence in measure if and only if the function $\varphi(x,\cdot,\cdot,\cdot)$ is BV-elliptic for every $x\in\Omega$.*

Proof. Without loss of generality, we can assume that Ω is a bounded open set.

Necessity. Let $x_0\in\Omega$, i, $j\in T$, $i\neq j$, and let $\nu\in\mathbf{S}^{n-1}$. We denote by Q_1 an open cube centered at x_0 with sides of length 1, either orthogonal or parallel to ν, and we set $Q_a = aQ_1$. Let u_0 be as in (2.2), and let $u\in BV(Q_1;T)$ be a function with the same trace as u_0 on the boundary ∂Q_1. We set

$$u_a(x) = u\big(\frac{x-x_0}{a}+x_0\big).$$

Let $(\nu_1,\ldots,\nu_{n-1})$ be the $(n-1)$ linearly independent edges unitary vectors normal to ν and let $\mathcal{S}$ be the set of mappings $\sigma:\{1,\ldots,n-1\}\to\{1,\ldots,h\}$; we consider h^{n-1} open cubes Q_σ centered in

$$x_\sigma = x_0 + a\sum_{i=1}^{n-1}\nu_i\big(\frac{2\sigma(i)-h-1}{2h}\big) \qquad \sigma\in\mathcal{S}$$

with sides of length a/h. Then, we set

$$u_h(x) = \begin{cases} i & \text{if } \langle x - x_0, \nu \rangle > \frac{1}{h}; \\ j & \text{if } \langle x - x_0, \nu \rangle < -\frac{1}{h}; \\ u_a(h(x - x_\sigma)) & \text{if } x \in Q_\sigma; \\ u_0 & \text{otherwise.} \end{cases}$$

By the boundary condition on u, we easily get that

$$\bigcup_{\sigma \in \mathcal{S}} [Q_\sigma \cap (\frac{S_{u_a}}{h} + x_\sigma)] \subset S_{u_h} \cap \overline{Q}_a \subset \bigcup_{\sigma \in \mathcal{S}} [Q_\sigma \cap (\frac{S_{u_a}}{h} + x_\sigma)] \cup N \qquad \forall h \in \mathbf{N},$$

with $\mathcal{H}_{n-1}(N) = 0$, hence

$$\int\limits_{Q_a \cap S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x) = \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{h}\right)^{n-1} \int\limits_{Q_\sigma \cap S_{u_a}} \varphi(x_\sigma + \frac{x}{h}, u_a^+, u_a^-, \nu_{u_a})\, d\mathcal{H}_{n-1}(x) \qquad \forall h \in \mathbf{N}.$$

Since we assume that the functional is lower semicontinuous, and since $u_h = u_0$ outside $\overline{Q}_a$, we obtain

$$a^{n-1} \int\limits_{Q_1 \cap S_u} \sup_{y \in Q_a} \varphi(y, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) = \int\limits_{Q_a \cap S_{u_a}} \sup_{y \in Q_a} \varphi(y, u_a^+, u_a^-, \nu_{u_a})\, d\mathcal{H}_{n-1}(x) \geq$$

$$\geq \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{h}\right)^{n-1} \int\limits_{Q_\sigma \cap S_{u_a}} \varphi(x_\sigma + \frac{x}{h}, u_a^+, u_a^-, \nu_{u_a})\, d\mathcal{H}_{n-1}(x) \geq \int\limits_{Q_a \cap S_{u_0}} \varphi(x, i, j, \nu)\, d\mathcal{H}_{n-1}(x).$$

Dividing both sides by a^{n-1} and letting $a \downarrow 0$ we get

$$\int\limits_{Q_1 \cap S_u} \varphi(x_0, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \geq \varphi(x_0, i, j, \nu),$$

and this implies the BV-ellipticity of $\varphi(x_0, \cdot, \cdot, \cdot)$.

Sufficiency. By Theorem 1.1 and (1.13), there exists a continuous function $\psi : \Omega \times T \times \mathbf{S}^{n-1} \to [c, +\infty[$ such that

$$\int\limits_{A \cap S_u} \psi(x, u^+, u^-, \nu_u) d\mathcal{H}_{n-1}(x) = \inf\Bigl\{\liminf_{h \to +\infty} \int\limits_{A \cap S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h}) d\mathcal{H}_{n-1}(x) : (u_h) \subset BV(\Omega, T),\ d_A(u_h, u) \to 0\Bigr\}$$

for every open set $A \subset \Omega$ and every $u \in BV(\Omega, T)$. The statement will follow by equality $\psi = \varphi$. The inequality $\psi \leq \varphi$ is trivial. To prove the opposite inequality, let $i,\, j \in T$ with $i \neq j$ and let $\nu \in \mathbf{S}^{n-1}$. Let $Q \subset\subset \Omega$ be an open cube with sides parallel or orthogonal to ν centered in $x_0 \in \Omega$, and let $(u_h) \subset BV(\Omega, T)$ be a sequence converging in measure in Q to the function $u_0 \in BV(\Omega, T)$ defined by (2.2), such that

$$\int\limits_{Q \cap S_{u_0}} \psi(x, i, j, \nu)\, d\mathcal{H}_{n-1}(x) = \lim_{h \to +\infty} \int\limits_{Q \cap S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x).$$

Let

$$Q_t = \bigl\{x \in Q : \operatorname{dist}(x, \partial Q) > t\bigr\} \qquad t > 0,$$

and let

$$s_k = 2^{-k}, \qquad t_k = 2^{-k-1} \qquad k \in \mathbf{N}.$$

Let $c_k = c(Q_{t_k}, 2^{-k+1})$ be given by joint Lemma 1.2. We can find a subsequence u_{h_k} such that

$$|\{x \in Q_{t_k} \setminus \overline{Q}_{s_k} : u_{h_k}(x) \neq u_0(x)\}| < \frac{2^{-k}}{c_k}.$$

By Lemma 1.2, we can find $w_k \in]s_k, t_k[$ such that the functions

$$v_k(x) = \begin{cases} u_{h_k}(x) & \text{if } x \in Q_{w_k}; \\ u_0(x) & \text{if } x \in Q \setminus \overline{Q}_{w_k} \end{cases}$$

belong to $BV(\Omega, T)$ and

$$\int_{Q\cap S_{v_k}} \varphi(x, v_k^+, v_k^-, \nu_{v_k})\, d\mathcal{H}_{n-1}(x) \le \int_{Q\cap S_{u_{h_k}}} \varphi(x, u_{h_k}^+, u_{h_k}^-, \nu_{u_{h_k}})\, d\mathcal{H}_{n-1}(x) + 2^{1-k}\|\varphi\|_\infty.$$

Since the functions v_k have the same trace as u_0 on the boundary, we get

$$\int_{Q\cap S_{u_0}} \varphi(x_0, i, j, \nu)\, d\mathcal{H}_{n-1}(x) \le \int_{Q\cap S_{v_k}} \varphi(x_0, v_k^+, v_k^-, \nu_{v_k})\, d\mathcal{H}_{n-1}(x) \qquad \forall k \in \mathbf{N}.$$

By letting $k \to +\infty$ we obtain

$$\int_{Q\cap S_{u_0}} \varphi(x_0, i, j, \nu)\, d\mathcal{H}_{n-1}(x) \le \int_{Q\cap S_{u_0}} \psi(x_0, i, j, \nu)\, d\mathcal{H}_{n-1}(x) \Bigl(1 + \frac{\sup_{x\in Q} |\varphi(x, i, j, \nu) - \varphi(x_0, i, j, \nu)|}{c}\Bigr).$$

Dividing both sides by $\mathcal{H}_{n-1}(Q \cap S_{u_0})$ and letting the diameter of Q go to 0 we get $\varphi(x_0, i, j, \nu) \ge \psi(x_0, i, j, \nu)$ and the statement is proved. **q.e.d.**

2.2. Algebraic conditions: bi-convexity.

Theorem 2.1 gives a complete characterization of lower semicontinuous functionals (2.1). Unfortunately, as for the case of quasi-convexity, the BV-ellipticity of the integrands, being an integral condition, is hardly ever easy to check. We are thus led to considering other kinds of conditions, of algebraic type, on the integrands. Let $\mathcal{L}_{n,k}$ be the space of linear mappings between $\mathbf{R}^n$ and $\mathbf{R}^k$, and let $T = \{z_1, \ldots, z_k\}$. We say that $\psi : T \times T \times \mathbf{S}^{n-1} \to [0, +\infty[$ is a bi-convex function if there exists a convex and positively 1-homogeneous function $\theta : \mathcal{L}_{n,k} \to \mathbf{R}$ such that

$$\varphi(z_i, z_j, \nu) = \theta\bigl((e_i - e_j) \otimes \nu\bigr) \qquad \forall i, j \in \{1, \ldots, k\},\ \nu \in \mathbf{S}^{n-1} \setminus \{0\}, \tag{2.3}$$

where $a \otimes b \in \mathcal{L}_{n,k}$ is the rank 1 mapping

$$a \otimes b(p) = \langle b, p\rangle a, \qquad p \in \mathbf{R}^n$$

for all $a \in \mathbf{R}^k$, $b \in \mathbf{R}^n$. We want to emphasize that (2.3) is an algebraic condition. In fact, since (2.3) determines θ only on rank 1 mappings, it exists if and only if

$$\varphi(z_{i_0}, z_{j_0}, p_0) \le \sum_{\lambda=1}^N \varphi(z_{i_\lambda}, z_{j_\lambda}, p_\lambda) \tag{2.4}$$

whenever

$$(e_{i_0} - e_{j_0}) \otimes p_0 = \sum_{\lambda=1}^N (e_{i_\lambda} - e_{j_\lambda}) \otimes p_\lambda \qquad \text{in } \mathcal{L}_{n,k}. \tag{2.5}$$

This condition resembles the convexity properties of elasticity theory (see [BA], [CA], [DA]). We conjecture that BV-ellipticity and bi-convexity are equivalent properties.
The "easy" implication is bi-convex functions being BV-elliptic, as the following result shows.

Proposition 2.2. *Every biconvex integrand is BV-elliptic.*

Proof. Since T is only an index set, we can assume without loss of generality that T is the canonical basis $\{e_1,\dots,e_k\}$ of $\mathbf{R}^k$. Let $\Omega = B_1$ be the unit ball in $\mathbf{R}^n$, let $(e_i,e_j,\nu)\in T\times T\times \mathbf{S}^{n-1}$, and let u_0 as in (2.2) with $x_0=0$. Since $BV(\Omega,T)\subset BV(\Omega;\mathbf{R}^k)$ the distributional derivative Du is representable as

$$Du(B)=\int_{B\cap S_u}(u^+-u^-)\otimes\nu_u\,d\mathcal{H}_{n-1}(x)\qquad \forall B\in BV(\Omega,T).$$

In particular, for every function $u\in BV(\Omega,T)$ with the same boundary values as u_0 we get

$$\int_{S_u}\big(u^+-u^-\big)\otimes\nu_u\,d\mathcal{H}_{n-1}(x)=\int_{S_{u_0}}\big(u_0^+-u_0^-\big)\otimes\nu_{u_0}\,d\mathcal{H}_{n-1}(x)=(e_j-e_i)\otimes\nu\mathcal{H}_{n-1}(\Omega\cap H_\nu),$$

where H_ν is the hyperplane normal to ν passing by 0. By the Jensen's inequality we obtain

$$\int_{S_u}\varphi(u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x)=\int_{S_u}\theta\big((u^+-u^-)\otimes\nu_u\big)\,d\mathcal{H}_{n-1}(x)\geq\theta\bigg(\int_{S_u}(u^+-u^-)\otimes\nu_u\,d\mathcal{H}_{n-1}\bigg)=$$

$$=\mathcal{H}_{n-1}(\Omega\cap H_\nu)\theta\big((e_j-e_i)\otimes\nu\big)=\int_{S_{u_0}}\theta(u_0^+,u_0^-,\nu_{u_0}),\,d\mathcal{H}_{n-1}(x),$$

and the statement follows. **q.e.d.**

The inverse problem -whether all BV-elliptic functions are bi-convex or not- is still an open problem. As for quasi-convexity, it is important to understand for which sets of triplets (i_0,j_0,ν_0), $(i_\lambda,j_\lambda,p_\lambda)$, $(\lambda=1,\dots,N)$ verifying (2.5) the relation (2.4) holds for all BV-elliptic functions. If (2.4) were proved for all triplets verifying (2.5), then the equivalence between BV- ellipticity and bi-convexity would be verified.
In some cases (2.4) can be checked, were it is possible to reduce to some geometrical construction of the triplets above. Let us show now that BV-ellipticity inequality holds for a wider class of functions than those verifying boundary conditions.
Let $\nu\in\mathbf{S}^{n-1}$, i, $j\in T$, Q_ν be a unit cube in $\mathbf{R}^n$ with center in 0 and one edge parallel to ν. We set

$$u_{ij}^\nu(x)=\begin{cases} i & \text{if } \langle x,\nu\rangle>0;\\ \\ j & \text{if } \langle x,\nu\rangle\leq 0.\end{cases}\tag{2.6}$$

We define P_{ij}^ν as the subset of $BV(Q_\nu;T)$ of all functions u such that their trace on the two sides orthogonal to ν is equal to the trace as u_ν^{ij}, and have equal traces on other opposite sides. By the same argument of the proof of the necessity part of theorem 2.1 it is possible to get the following result.

Proposition 2.3. *For every function $u\in P_{ij}^\nu$ and all BV-elliptic function φ, we have*

$$\varphi(i,j,\nu)\leq\int_{S_u}\varphi(u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x).\tag{2.7}$$

If the function u in Proposition 2.3 has polyhedral level sets, every set $\partial^*\{u=l\}\cap\partial^*\{u=k\}$ can be divided into plane surfaces $\Gamma_1^{lk},\dots,\Gamma_{m(lk)}^{lk}$. Let us set

$$P_m^{lk}=\mathcal{H}_{n-1}(\Gamma_m^{lk})\cdot\nu_m^{lk}\tag{2.8}$$

where ν_m^{lk} is the inner normal to $\{u=l\}$, orthogonal to the hyperplane containing Γ_m^{lk}. From (2.7) we obtain

$$(2.9).\qquad \varphi(i,j,\nu)\le\sum_{l,k\in T}\sum_{m=1}^{m(lk)}\varphi(l,k,\nu_m^{lk})\mathcal{H}_{n-1}(\Gamma_m^{lk})=\sum_{l,k\in T}\sum_{m=1}^{m(lk)}\varphi(l,k,p_m^{lk})$$

One possible way of dealing with the equivalence problem could be to investigate whether or not for all sets of triplets as in (2.5) there exists a polyhedral function $u\in P_{ij}^{\nu}$ such that every triplet (l,k,p_m^{lk}) constructed as in (2.8) is an element of the considered set (here we take $i_0=i$, $j_0=j$, $p_0=\nu$).
From (2.9) it is easy to get some algebraic conditions that each BV- elliptic function must verify.
Example 2.4. Let us take $u\in P_{ij}^{\nu}$ defined by

$$u(x)=\begin{cases} i & \text{if } \langle x,\nu\rangle>\frac14;\\ k & \text{if } |\langle x,\nu\rangle|\le\frac14;\\ j & \text{if } \langle x,\nu\rangle<\frac14.\end{cases}$$

Then we have

$$(2.10)\qquad \varphi(i,j,\nu)\le\varphi(i,k,\nu)+\varphi(k,j,\nu).$$

Let us remark that $(e_i-e_j)\otimes\nu=(e_i-e_k)\otimes\nu+(e_k-e_j)\otimes\nu$.
Example 2.5. Assume that $\nu\in\mathbf{S}^{n-1}$ is equal to p_1+p_2. It is possible to find a polyhedral function $u\in P_{ij}^{\nu}$, which takes only the values i, j, such that each face of the polyhedron $\{u=i\}$ is normal either to p_1 or to p_2, obtaining by (2.9) the convexity inequality

$$(2.11)\qquad \varphi(i,j,\nu)\le\varphi(i,j,p_1)+\varphi(i,j,p_2).$$

Example 2.6. Let us suppose card$(T)\ge6$, and $n=2$ for the sake of simplicity. Define u as in figure 1 (triangles ABC and CDE are equilateral).

Then we have

$$\varphi(i,j,\nu)\le\varphi(m,j,\frac{\nu}{2})+\varphi(k,j,\frac{\nu}{2})+\varphi(i,h,\frac{\nu}{2})+\varphi(i,l,\frac{\nu}{2})+\varphi(l,m,\frac{\nu_1}{2})+\varphi(h,m,\frac{\nu_2}{2})+\varphi(h,k,\frac{\nu_1}{2})+\varphi(l,k,\frac{\nu_2}{2}),$$

where ν_1, ν_2 are the vectors normal to AB and DE respectively, pointing towards the "l" set. Let us remark that again we have

$$(2.12)\qquad (e_i-e_j)\otimes\nu=(e_m-e_j)\otimes\frac{\nu}{2}+(e_k-e_j)\otimes\frac{\nu}{2}+(e_i-e_h)\otimes\frac{\nu}{2}+(e_i-e_l)\otimes\frac{\nu}{2}+$$
$$+(e_l-e_m)\otimes\frac{\nu_1}{2}+(e_h-e_m)\otimes\frac{\nu_1}{2}+(e_h-e_k)\otimes\frac{\nu_2}{2}+(e_l-e_k)\otimes\frac{\nu_2}{2},$$

so that (2.12) is a biconvexity-type inequality.
It is possible to find a function φ satisfying (2.10), (2.11), which does not satisfy the inequality corresponding to (2.12). Hence, the only conditions (2.10), (2.11) are not sufficient for bi-convexity.

Example 2.7. Let us consider the equality (here we take $T \supset \{1,2,3,4\}$ and $n=2$)

(2.13) $(e_4-e_1)\otimes(2,2) = (e_2-e_1)\otimes(2,1)+(e_3-e_1)\otimes(0,1)+(e_4-e_3)\otimes(1,2)+(e_4-e_2)\otimes(1,0)+(e_3-e_2)\otimes(1,1).$

Set

$$\nu = \nu_{4,1} = \frac{(2,2)}{2\sqrt{2}}, \quad \nu_{2,1} = \frac{(2,1)}{2\sqrt{2}}, \quad \nu_{3,1} = \frac{(0,1)}{2\sqrt{2}}, \ldots\ldots.$$

It does not seem possible to find a polyhedral function u in P^ν_{ij} such that for all $i,\, j = 1,\ldots,4$ the set $E_{ij} = \partial^*\{u=i\} \cap \partial^*\{u=j\}$ is a segment orthogonal to the vector ν_{ij} with the exact length $|\nu_{ij}|$ in order to obtain the bi-convexity inequality related to (2.13). It is not clear if u can be chosen so that E_{ij} can be divided into segments $(\Gamma^{ij}_m)_m$ such that each Γ^{ij}_m is orthogonal to ν_{ij} and $\sum_m \mathcal{H}_1(\Gamma^{ij}_m) \le |\nu_{ij}|$.
Similar reasonings in the quasi-convex case can be found in [CA], to whom we refer for further analysis.

Example 2.8. The BV-elliptic integrands of the form

$$\varphi(i,j,\nu) = \Theta(i,j)\psi(\nu)$$

with $\Theta(i,j) = \Theta(j,i)$ and $\psi(\nu) = \psi(-\nu)$ can be completely characterized. By (2.10), (2.11) we obtain that the conditions

$$\Theta(i,j) \le \Theta(i,k) + \Theta(k,j), \qquad \psi(p_1+p_2) \le \psi(p_1) + \psi(p_2)$$

are necessary for semicontinuity; i.e., Θ must be a pseudo-distance in T and ψ must be a convex and positively 1-homogeneous function. Coversely, the above conditions imply (we assume $\Theta(i,i)=0$)

$$\Theta(i,j)\psi(\nu) = \sup_{k\in T}\sup_{\xi\in K}\, [\Theta(i,k) - \Theta(j,k)]\langle \xi,\nu\rangle$$

where $K \subset \mathbf{R}^n$ is the subdifferentail of ψ at 0. It is easily seen that each function

$$[\Theta(i,k) - \Theta(j,k)]\langle \xi,\nu\rangle$$

is bi-convex (equality holds in (2.4)), hence the above conditions imply bi-convexity of φ and, a fortiori, its BV-ellipticity. For this class of integrands BV-ellipticity and bi-convexity are equivalent conditions.

2.3. Physical conditions.

In many situations the semicontinuity of the functional (2.1) assures that its values decrease whenever a partition $u \in BV(\Omega,T)$ is replaced by some other one $u_0 \in BV(\Omega; T\setminus\{i\})$ which coincides with u in $\Omega\setminus\{u=i\}$. Thinking in terms of liquid crystals, we can imagine that if a liquid crystal is taken away, the other ones will "flow" into the region left empty, reshaping to a configuration with less energy. In [AL] similar considerations have been made in the case where only one of the remaining regions is asked to fill the one which has been removed; that is, we pass from a partition $E_1,\ldots,E_k$ to another one $F_1,\ldots,F_k$ with $F_i=\emptyset$, $F_j = E_i\cup E_j$ for some j and all other F_k equal to E_k.

We say that a functional $F : BV(\Omega,T) \to [0,+\infty[$ is (B)-convex if for every function $u \in BV(\Omega,T)$ and for every $i\in T$ there exists a function $v \in BV(\Omega, T\setminus\{i\})$ such that $F(v)\le F(u)$ and

(2.14). $$\{u=j\} \subset \{v=j\} \qquad \forall j \in T\setminus\{i\}$$

A straightforward consequence of this definition is the following proposition.

Proposition 2.9. *If F is (B)-convex, then for every $u\in BV(\Omega,T)$ and every $S\subset T$ there exists a function $v\in BV(\Omega,S)$ such that $F(v)\le F(u)$ and $\{u=i\}\subset\{v=i\}$ for every index $i\in S$.*

Let us remark that if (F_h) is a sequence of (B)-convex functionals, $u_h \to u$ and $|\{u=i\}| = 0$ for some i, then by proposition 2.9 there exists a sequence (v_h) such that $|\{v_h=i\}|=0$ for all h, $v_h\to u$ and $F_h(v_h)\le F_h(u_h)$. Proceeding as in the proof of theorem 3.3 of [AB], this shows that minimizing sequences of (B)-convex semicontinuous integral functionals can be chosen with the same volume constraints as their limit (in [AB] theorem 3.3 this result was proven only for strictly positive volume constraints).

We say that a Borel function $\varphi : T \times T \times \mathbf{S}^{n-1}$ is (B)-convex if

$$p \quad \to \quad \varphi(i,j,\frac{p}{|p|})|p|$$

is a convex and positively 1-homogeneous function in $\mathbf{R}^n$ and the associated functional (2.1) is (B)-convex. (B)-convex integral functionals are lower semicontinuous, as the following proposition shows:

Proposition 2.10. *Every (B)-convex integrand φ is BV- elliptic.*

Proof. Let $i, j \in T$, $\nu \in \mathbf{S}^{n-1}$, $\Omega = B_1(0)$, $u \in BV(\Omega, T)$ such that $u = h$ on the boundary $\partial\Omega$, where $h(x) = i$ if $\langle x, \nu \rangle > 0$, $h(x) = j$ otherwise. We define

$$F(u) = \int_{S_u} \varphi(u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x).$$

By Proposition 2.9, taking $S = \{i, j\}$, there exists $v \in BV(\Omega; \{i, j\})$ such that $F(v) \le F(u)$ and satisfies the same boundary conditions. By the convexity of $\varphi(i, j, \cdot)$ and by Jensen's inequality we get

$$F(h) \le F(v) \le F(u).$$

As (i, j, ν) are arbitrary, the proposition is proven. **q.e.d.**

The inverse of Proposition 2.10 is false. Let us take $T = \{0, 1, 2, 3, 4\}$, $\Omega = B_1(0) \subset \mathbf{R}^2$,

$$\sigma(i,j) = \begin{cases} 1 & \text{if } i \text{ or } j \text{ are equal to } 0; \\ \\ 2 & \text{otherwise,} \end{cases}$$

and

$$F(u) = \int_{S_u} \sigma(u^+, u^-)\, d\mathcal{H}_{n-1}(x) \qquad u \in BV(\Omega, T).$$

By example 2.8, F is a lower semicontinuous functional. Now, let us define

$$u(x,y) = \begin{cases} 0 & \text{if } |x| + |y| \le 1; \\ \\ i & \text{if } |x| + |y| > 1 \text{ and } x \text{ lies in the } i\text{-th quadrant.} \end{cases}$$

If $i = 0$, for all functions $v \in BV(\Omega, T \setminus \{0\})$ verifying (2.14) we have

$$F(v) \ge 8 > 4\sqrt{2} = F(u),$$

so that, F is not (B)-convex.

3. Relaxation and Γ-convergence of Dirichlet problems.

Let $\psi : T \times T \times \mathbf{S}^{n-1} \to [0, +\infty[$ be a function. We define $E\psi$ as the greatest BV-elliptic function less than ψ. Since BV-elliptic functions are a lattice, the definition makes sense. We claim that

$$E\psi(i,j,\nu) = \inf\Big\{ \int_{\Omega\cap S_u} \psi\big(u^+, u^-, \nu_u\big)\, d\mathcal{H}_{n-1}(x) : u \in BV(\Omega, T),\ u^* = u_0^* \text{ on } \partial\Omega \Big\} \tag{3.1}$$

where u_0 is defined as in (2.2) and Q is a unit cube. Inequality $\le$ in (3.1) is trivial, because $E\psi$ is BV-elliptic. On the other hand, by Theorem 1.1 and (1.13), the functional

$$\inf\left\{\liminf_{h\to+\infty}\int\limits_{S_{u_h}}\psi(u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}_{n-1}(x):u_h\to u\text{ in measure}\right\}\tag{3.2}$$

admits integral representation by means of a function φ which is, by Theorem 2.1, BV-elliptic. By the same truncation argument of the sufficiency part of Theorem 2.1, we see that

$$\inf\left\{\liminf_{h\to+\infty}\int\limits_{S_{u_h}}\psi(u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}_{n-1}(x):u_h\to u\text{ in measure}\right\}=$$

$$=\inf\left\{\liminf_{h\to+\infty}\int\limits_{S_{u_h}}\psi(u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}_{n-1}(x):u_h\to u\text{ in measure, }u_h^*=u^*\text{ on }\partial\Omega\right\},$$

for every function $u\in BV(\Omega,T)$. Applying this equality to the function $u_0(i,j,\nu)$ in (2.2), we get

$$\inf\left\{\int\limits_{\Omega\cap S_u}\psi(u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x):u\in BV(\Omega,T),\ u^*=u_0^*\text{ on }\partial\Omega\right\}\le\varphi(i,j,\nu)\le E\psi(i,j,\nu).$$

Actually, we have proved (3.1) and also that $E\psi$ guarantees representation to the relaxed functional (3.2). This is true also for integrands $\varphi(x,i,j,\nu)$, as the following theorem shows.

Theorem 3.1. *Let $c>0$, and let $\varphi:\Omega\times T\times T\times\mathbf{S}^{n-1}\to[c,+\infty[$ be a bounded continuous function. Then, we have*

$$\inf\left\{\liminf_{h\to+\infty}\int\limits_{S_{u_h}}\varphi(x,u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}_{n-1}(x):u_h\to u\text{ in measure}\right\}=\int\limits_{S_u}E\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x)\tag{3.3}$$

where $E\varphi(x,\cdot,\cdot,\cdot)=E[\varphi(x,\cdot,\cdot,\cdot)]$ for every $x\in\Omega$.

Proof. By formula (3.1) we easily get that $E\varphi$ is a continuous function, hence

$$u\quad\to\quad\int\limits_{S_u}E\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x)$$

is a lower semicontinuous functional. Thus, inequality $\ge$ in (3.3) is verified. To prove the opposite inequality, we recall that by Theorem 1.1 the relaxed functional in the left hand side of (3.3) admits integral representation by a BV-elliptic continuous integrand ϕ. Since

$$\int\limits_{S_u}\phi(x,u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x)\le\int\limits_{S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x)\qquad\forall u\in BV(\Omega,T),$$

we have in particular $\phi\le\varphi$, so that $\phi\le E\varphi$ by the definition of $E\varphi$. **q.e.d.**

Now we investigate the Γ-convergence of Dirichlet problems. Let $\Omega\subset\mathbf{R}^n$ be a bounded open set with C^2 boundary. Under these assumptions on Ω, there exists $t_0>0$ such that ([GI], Appendix C)

$$\big\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)=s\big\}=\big\{x+s\nu_\Omega(x):x\in\partial\Omega\big\}\tag{3.4}$$

for every $s<t_0$, where ν_Ω is the inner normal to Ω.

Let $C > c > 0$ be constants, and let $\varphi_h : \Omega \times T \times T \times \mathbf{S}^{n-1} \to [c, C]$, $\phi : \partial\Omega \to T$ be Borel functions. We define

$$F_h(u) = \int_{S_u} \varphi_h(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \qquad \forall u \in BV(\Omega, T),$$

and

$$\tilde{F}_h(u) = \begin{cases} F_h(u) & \text{if } u^* = \phi \quad \mathcal{H}_{n-1}\text{-a.e. in } \partial\Omega; \\ +\infty & \text{otherwise,} \end{cases} \qquad \forall u \in BV(\Omega, T).$$

We shall prove the following result.

Theorem 3.2. *Assume that the functions $\varphi_h(\cdot, i, j, \nu)$ are equi-uniformly continuous in Ω, and assume that*

$$\Gamma(d_\Omega^-) \lim_{h \to +\infty} F_h(u)$$

exists for every $u \in BV(\Omega, T)$. Then, the Γ-limit is representable by integration by a function φ, and

$$\Gamma(d_\Omega^-) \lim_{h \to +\infty} \tilde{F}_h(u) = \int_{S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) + \int_{\partial\Omega} \varphi(x, u^*, \phi, \nu_\Omega)\, d\mathcal{H}_{n-1}(x) \qquad \forall u \in BV(\Omega, T).$$

Lemma 3.3. *(i) Let $\phi : \partial\Omega \to T$ be a Borel function. There exists $u_\phi \in BV(\mathbf{R}^n; T)$ such that*

$$\big(u_\phi|_\Omega\big)^* = \phi = \big(u_\phi|_{\mathbf{R}^n \setminus \Omega}\big)^* \qquad \mathcal{H}_{n-1}\text{-a.e. in } \partial\Omega.$$

(ii) Let $u \in BV(\Omega, T)$, and let u_t^* be the trace on $\partial\Omega_t$ of $u|_{\Omega_t}$ for every $t < t_0$. Then,

$$\lim_{t \to 0^+} \mathcal{H}_{n-1}\big(\{x \in \partial\Omega : u^*(x + t\nu_\Omega(x)) \neq u_t^*(x)\}\big) = 0.$$

Proof. (i) Since T is only an index set, we can assume that $T = \{1, \ldots, m\}$ with $m \in \mathbf{N}$, $m \geq 2$. By [GI], Theorem 2.16, there exists a function $v \in W^{1,1}(\mathbf{R}^n)$ such that

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |v(y) - \phi(x)|\, dy = 0$$

for $\mathcal{H}_{n-1}$-almost every $x \in \partial\Omega$. By Fleming-Rishel formula ([FE1], 4.5.9) we can find real numbers $z_i \in]i-1, i[$ such that $\{u > z_i\}$ has finite perimeter in $\mathbf{R}^n$ for $i = 1, \ldots, m$. The function $u_\phi : \mathbf{R}^n \to T$ such that

$$\big\{u_\phi = i\big\} = \big\{z_{i-1} < v \leq z_i\big\}$$

meets the requirements of the lemma.

(ii) Let $S_{a,b} \subset \mathbf{R}^n$ be the strip defined by

$$\big\{x \in \mathbf{R}^n : x = (x_1, \ldots, x_{n-1}, x_n), |(x_1, \ldots, x_{n-1})| < a,\ 0 < x_n < b\big\}.$$

For every function $u \in BV(S_{a,b}; T)$ it is possible to obtain by [GI] Lemma 2.4 formula (2.8) and Theorem 2.11 the inequality

$$|\big\{y \in \mathbf{R}^{n-1} : |y| < a : u^*(y) \neq u^*(y + b\mathbf{e}_n)\big\}| \leq 2\mathcal{H}_{n-1}(S_{a,b} \cap S_u).$$

Using (3.4) we find that $\Omega \setminus \overline{\Omega}_t$ is locally diffeomorphic to a strip, hence the proof follows by using change of coordinates. **q.e.d.**

Proof of Theorem 3.2. Let $u_\phi \in BV(\mathbf{R}^n;T)$ be given by Lemma 3.3(i) , and let $\omega : [0,+\infty[\to [0,+\infty[$ be a bounded continuous function such that $\omega(0)=0$ and

$$|f_h(x,i,j,\nu) - f_h(y,i,j,\nu)| < \omega(|x-y|) \qquad \forall x,\, y \in \Omega,\, i,\, j \in T,\, \nu \in \mathbf{S}^{n-1}, h \in \mathbf{N}.$$

The functions

$$\tilde{f}_h(x,i,j,\nu) = \min\bigl\{f_h(y,i,j,\nu) + \omega(|x-y|) : y \in \Omega\bigr\}$$

are equicontinuous extensions of f_h to all $\mathbf{R}^n \times T \times T \times \mathbf{S}^{n-1}$. By Theorem 1.1, passing eventually to subsequences, we can assume that the functionals

$$\mathcal{F}_h(u,A) = \int_{A\cap S_u} \tilde{f}_h(x,u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x)$$

Γ-converge in $BV(\mathbf{R}^n;T)$ to the functional

$$\mathcal{F}(u,A) = \int_{A\cap S_u} \tilde{\varphi}(x,u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x)$$

for every open set $A \subset \mathbf{R}^n$, for a suitable continuous function $\tilde{\varphi}$ such that $\tilde{\varphi}(x,\cdot,\cdot,\cdot) = \varphi(x,\cdot,\cdot,\cdot)$ for every $x \in \Omega$.
Let $(u_h) \subset BV(\Omega,T)$ be a sequence converging in measure to $u \in BV(\Omega,T)$, such that $u_h^* = \phi$ $\mathcal{H}_{n-1}$-almost everywhere on $\partial\Omega$. The functions

$$\tilde{u}_h(x) = \begin{cases} u_h(x) & \text{if } x \in \Omega; \\ & \\ u_\phi(x) & \text{otherwise}\ , \end{cases}$$

belong to $BV(\mathbf{R}^n;T)$ by (1.9), and converge in measure to the function

$$\tilde{u} = \begin{cases} u(x) & \text{if } x \in \Omega; \\ & \\ u_\phi(x) & \text{otherwise} \end{cases}\ .$$

Moreover, the boundary condition implies $\mathcal{H}_{n-1}(S_{\tilde{u}_h} \cap \overline{\Omega} \setminus S_{u_h}) = 0$, so that

$$\liminf_{h\to+\infty} \tilde{F}_h(u_h) \geq \liminf_{h\to+\infty} \mathcal{F}_h(\tilde{u}_h, B) - \|\varphi\|_\infty \mathcal{H}_{n-1}(S_{u_\phi} \cap B \setminus \overline{\Omega}) \geq$$

$$\geq \mathcal{F}(\tilde{u},B) - \|\varphi\|_\infty \mathcal{H}_{n-1}(S_{u_\phi} \cap B \setminus \overline{\Omega}) = \int_{S_{\tilde{u}}\cap B} \varphi(x,\tilde{u}^+,\tilde{u}^-,\nu_{\tilde{u}})\, d\mathcal{H}_{n-1}(x) - \|\varphi\|_\infty \mathcal{H}_{n-1}(S_{u_\phi} \cap B \setminus \overline{\Omega})$$

for every open set $B \supset \overline{\Omega}$. By letting $B \downarrow \overline{\Omega}$, we get

$$\liminf_{h\to+\infty} \tilde{F}_h(u_h) \geq \int_{S_u} \varphi(x,u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x) + \int_{\partial\Omega} \varphi(x,u^*,\phi,\nu_\Omega)\, d\mathcal{H}_{n-1}(x),$$

because $(\tilde{u}^+,\tilde{u}^-,\nu_{\tilde{u}}) = (u^*,\phi,\nu_\Omega)$ $\mathcal{H}_{n-1}$-almost everywhere in $\partial\Omega$. We have proved that

$$\Gamma(d_\Omega^-) \liminf_{h\to+\infty} \tilde{F}_h(u) \geq \int_{S_u} \varphi(x,u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x) + \int_{\partial\Omega} \varphi(x,u^*,\phi,\nu_\Omega)\, d\mathcal{H}_{n-1}(x) \quad \forall u \in BV(\Omega,T).$$

To prove the inequality

$$\int_{S_u} \varphi(x,u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x) + \int_{\partial\Omega} \varphi(x,u^*,\phi,\nu_\Omega)\, d\mathcal{H}_{n-1}(x) \leq \Gamma(d_\Omega^-) \limsup_{h\to+\infty} \tilde{F}_h(u), \tag{3.5}$$

we set for every $t < t_0$

$$u_t = \begin{cases} u(x) & \text{if } x \in \Omega_t; \\ \\ u_\phi(x) & \text{otherwise.} \end{cases}$$

By the same argument of Theorem 2.1, we get

$$\inf\left\{\limsup_{h\to+\infty} F_h(u_h) : u_h \to u \quad \text{in measure}\right\} =$$

$$= \inf\left\{\limsup_{h\to+\infty} F_h(u_h) : u_h \to u \quad \text{in measure}, u_h^* = u^* \ \mathcal{H}_{n-1}\text{-a.e. in } \partial\Omega\right\},$$

for every $u \in BV(\Omega, T)$, so that

$$\Gamma(d_\Omega^-) \limsup_{h\to+\infty} \tilde{F}_h(u_t) \le \int_{S_{u_t}} \varphi(x, u_t^+, u_t^-, \nu_{u_t})\, d\mathcal{H}_{n-1}(x)$$

for every $t < t_0$, because u_t has trace ϕ on the boundary. Recalling that the Γ-limits are lower semicontinuous, we achieve (3.5) by letting $t \downarrow 0$, using lemma 3.3(ii). **q.e.d.**

4. Homogenization.

Let $\varphi : \Omega \times T \times T \times \mathbf{S}^{n-1} \to [0, +\infty[$ be a continuous bounded function. We would give some conditions on f for the existence of the limit

$$\Gamma\text{-} \lim_{\epsilon\to 0^+} \int_{\Omega\cap S_u} f\big(\frac{x}{\epsilon}, u^+, u^-, \nu_u\big)\, d\mathcal{H}_{n-1}(x)$$

for every open set $\Omega \subset \mathbf{R}^n$ and every $u \in BV(\mathbf{R}^n; T)$. The hypothesis of periodicity in the first variable does not seem quite natural, for, since we consider $(n-1)$-dimensional surfaces, we would like the periodic structure of f to be mantained on plane $(n-1)$-dimensional surfaces. If f is periodic (take for example $f(x) = f(x_1, x_2) = \sin x_1 \sin x_2$ on the plane) its restriction can be not periodic (for example in this case on the line $x_1 - \sqrt{2}x_2 = 0$). So we would better take a weaker condition than periodicity: almost periodicity.

Definition.(Besicovitch) A function $g : \mathbf{R}^n \to \mathbf{R}$ is said to be almost periodic if it is the uniform limit of trigonometric polynomials

$$p_h(x) = \mathrm{Re}\Big[\sum_{k=1}^{m_h} a_{hk} \exp i\langle\lambda_{hk}, x\rangle\Big]$$

with $a_{hk} \in \mathbf{R}$ and $\lambda_{hk} \in \mathbf{R}^n$.

If g is almost periodic, then any restriction to a linear surface of $\mathbf{R}^n$ is almost periodic.

Almost periodic functions can be characterized in a more complicated but also more handy way by the following theorem of Besicovitch.

Theorem 4.1. *Let $g : \mathbf{R}^n \to \mathbf{R}$ be a continuous function. Then g is almost periodic if and only if for every $\epsilon > 0$ there exists an inclusion length $L_\epsilon > 0$ and a set $T_\epsilon \subset \mathbf{R}^n$ such that*

(i) $$\big(a + [0, L_\epsilon]^n\big) \cap T_\epsilon \neq \emptyset \qquad \forall a \in \mathbf{R}^n;$$

(ii) $$|g(x+\tau) - g(x)| < \epsilon \qquad \forall x \in \mathbf{R}^n,\ \tau \in T_\epsilon.$$

The members of the set T_ϵ will be called ϵ-quasi periods of g. We will say that $f = f(x, i, j, \nu)$ is almost periodic (in x) if the choice of L_ϵ and T_ϵ in Theorem 4.1 can be uniformly made with respect to i, j, ν for every $\epsilon > 0$. Such functions have been studied by Fink in [FI] and can be characterized by means of trygonometric polynomials depending on parameters.

Let us return to our homogeneization problem. For every $\epsilon > 0$ we set

$$\mathcal{F}_\epsilon(u,\Omega) = \int\limits_{\Omega\cap S_u} \varphi(\frac{x}{\epsilon}, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \tag{4.1}$$

whenever $\Omega \subset \mathbf{R}^n$ is an open set and $u \in BV(\mathbf{R}^n; T)$, and

$$g_\epsilon(i,j,\nu) = \inf\Big\{ \int\limits_{Q\cap S_u} \varphi(\frac{x}{\epsilon}, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) : u \in BV(\Omega, T),\ u^* = u_0^* \text{ on } \partial Q \Big\}, \tag{4.2}$$

where Q is a unitary n–cube with sides either orthogonal or parallel to ν. We want to show the following result.

Theorem 4.2. *Let $c > 0$, and let $\varphi : \mathbf{R}^n \times T \times T \times \mathbf{S}^{n-1} \to [c, +\infty[$ be a continuous, bounded, almost periodic function. Then, there exists the limit*

$$\Gamma\text{-}\lim_{\epsilon\to 0^+} \int\limits_{\Omega\cap S_u} \varphi(\frac{x}{\epsilon}, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) = \int\limits_{\Omega\cap S_u} \psi(u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x)$$

for every open set $\Omega \subset \mathbf{R}^n$ and every $u \in BV(\Omega, T)$. Moreover, the function ψ is BV-elliptic and satisfies the asymptotic formula

$$\psi(i,j,\nu) = \lim_{\epsilon\to 0^+} g_\epsilon(i,j,\nu)$$

for every $(i, j, \nu) \in T \times T \times \mathbf{S}^{n-1}$.

To prove Theorem 4.2, we begin with showing that the Γ-limit of every Γ-converging subsequence of our sequence (4.1) admits integral representation by an integrand $\psi(i, j, \nu)$, which a priori depends on the subsequence.

Proposition 4.3. *Let $(\epsilon_h) \subset]0, +\infty[$ be a sequence converging to 0, such that the functionals $\mathcal{F}_{\epsilon_h}(\cdot, A)$ Γ-converge as $h \to +\infty$ to a functional $\mathcal{F}(\cdot, A)$ for every function $u \in BV(\mathbf{R}^n; T)$. Then, there exists a BV-elliptic continuous function $\psi(i, j, \nu)$ such that*

$$\mathcal{F}(u, A) = \int\limits_{A\cap S_u} \psi(u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x)$$

for every function $u \in BV(\mathbf{R}^n; T)$ and every open set $A \subset \mathbf{R}^n$.

Proof. By Theorem 1.1, it will be sufficient to show that

$$\mathcal{F}(u, A) = \mathcal{F}(v, A + z) \tag{4.3}$$

for every $z \in \mathbf{R}^n \setminus \{0\}$ and every pair of functions u, $v \in BV(\mathbf{R}^n; T)$ such that $u(x) = v(x - z)$ in A. Let $\epsilon > 0$ be given, let

$$A_k = \{x \in A : \mathrm{dist}(x, \partial A) > \frac{1}{k}\}$$

and let τ_h be a sequence of ϵ-quasi periods of φ such that $z_h = \epsilon_h \tau_h \to z$ as $h \to +\infty$. Let $(u_h) \subset BV(\mathbf{R}^n; T)$ be a sequence converging in measure to u in A and such that

$$\lim_{h\to+\infty} \mathcal{F}_{\epsilon_h}(u_h, A) = \int\limits_{A\cap S_u} \psi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x).$$

In particular,

$$L = \limsup_{h\to+\infty} \mathcal{H}_{n-1}(A\cap S_{u_h}) < +\infty.$$

If h is large enough, $A_k + z \subset z_h + A$, and then we obtain, having set $\hat u_h(x) = u_h(x - z_h)$,

$$\mathcal{F}(u,A) \geq \liminf_{h\to+\infty} \int_{A\cap S_{u_h}} \varphi(\frac{x}{\epsilon_h} + \tau_h, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x) - \epsilon\frac{L}{c} =$$

$$= \liminf_{h\to+\infty} \int_{A+z_h\cap S_{\hat u_h}} \varphi(\frac{x}{\epsilon_h}, \hat u_h^+, \hat u_h^-, \nu_{\hat u_h})\, d\mathcal{H}_{n-1}(x) - \epsilon\frac{L}{c} \geq$$

$$\geq \liminf_{h\to+\infty} \mathcal{F}_{\epsilon_h}(\hat u_h, A_k + z) - \epsilon\frac{L}{c} \geq \mathcal{F}(v, A_k + z) - \epsilon\frac{L}{c}.$$

By letting first $\epsilon\downarrow 0$ then $k\uparrow+\infty$ we get inequality $\geq$ in (4.3). The opposite inequality can be proved by a symmetric argument. **q.e.d.**

The next step is the following.

Proposition 4.4. *The limit*

$$\lim_{\epsilon\to 0^+} g_\epsilon(i,j,\nu)$$

exists for every $(i,j,\nu)\in T\times T\times \mathbf{S}^{n-1}$.

Proof. Let us fix (i,j,ν), and set $g_r = g_{1/r}(i,j,\nu)$, $M = \|\varphi\|_\infty$. Let $\epsilon > 0$, let $t > 0$ and let $u_t \in BV(Q,T)$ equal to $u_0(i,j,\nu)$ (defined by (2.2))on ∂Q such that

$$\int_{Q\cap S_u} \varphi(tx, u_t^+, u_t^-, \nu_{u_t})\, d\mathcal{H}_{n-1}(x) \leq (1+\epsilon) g_t.$$

Let $L_\epsilon > 0$ the inclusion length of φ related to ϵ. If $s > 2(t + L_\epsilon)$, we can construct $u_s \in BV(Q;T)$ in the following way: for every $(n-1)$-tuple of integers $(i) = (i_1,\dots,i_{n-1}) \in \mathbf{Z}^{n-1}$ with

$$2|i_h|(t+L_\epsilon) < s \qquad h = 1,\dots,n-1,$$

let

$$\tau_i \in \left[\sum_{h=1}^{n-1}(t+L_\epsilon) i_h \nu_h + [0, L_\epsilon]^n\right] \cap T_\epsilon$$

be an almost period of φ ($(\nu_1,\dots,\nu_{n-1})$ are the $n-1$ linearly independent edges of Q other than ν); then set

$$C(s,t) = Q\setminus \bigcup_{(i)} \frac{\tau_{(i)}}{s} + \frac{t}{s}Q$$

and

$$u_s(x) = \begin{cases} u_t(\frac{sx}{t} - \tau_{(i)}) & \text{if } x \in \frac{\tau_{(i)}}{s} + \frac{t}{s}Q; \\ i & \text{if } x\in C(s,t),\ \langle x,\nu\rangle \geq 0; \\ j & \text{if } x \in C(s,t),\ \langle x,\nu\rangle < 0. \end{cases}$$

Using u_s we can give an estimate of g_s.

$$g_s \leq \int_{Q\cap S_{u_s}} \varphi(sx, u_s^+, u_s^-, \nu_{u_s})\, d\mathcal{H}_{n-1}(x) \leq \sum_{(i)} \int_{(\frac{1}{s}\tau_{(i)} + (\frac{t}{s})Q)\cap S_{u_s}} \varphi(sx, u_s^+, u_s^-, \nu_{u_s})\, d\mathcal{H}_{n-1}(x) +$$

$$+\sum_{(i)} \int_{\partial(\frac{1}{s}\tau_{(i)}+(\frac{t}{s})Q)\cap S_{u_s}} \varphi(sx, u_s^+, u_s^-, \nu_{u_s})\, d\mathcal{H}_{n-1}(x) + \int_{C(s,t)\cap S_{u_s}} \varphi(sx, u_s^+, u_s^-, \nu_{u_s})\, d\mathcal{H}_{n-1}(x).$$

The first sum is given by the part of S_{u_s} internal to the n- cubes $(1/s)\tau_{(i)} + (t/s)Q$ and can be estimated, thanks to the almost periodicity of φ, with

$$\sum_{(i)} \int_{(\frac{t}{s}Q)\cap S_{u_s}} [\varphi(x, u_s^+, u_s^-, \nu_{u_s}) + \epsilon]\, d\mathcal{H}_{n-1}(x) =$$

$$\sum_{(i)} \big(\frac{t}{s}\big)^{n-1} \int_{Q\cap S_{u_t}} [\varphi(x, u_t^+, u_t^-, \nu_{u_t}) + \epsilon]\, d\mathcal{H}_{n-1}(x) \le \big(2[\frac{s}{2(t+L_\epsilon)}] + 1\big)\big(\frac{t}{s}\big)^{n-1}(1 + \frac{\epsilon(1+\epsilon)}{c})g_t$$

($[z]$ denotes the integer part of z). The second sum is given by the contribution on the sides of the n-cubes and can be estimated with

$$\sum_{(i)} 2M(n-1)\frac{L_\epsilon}{s}\big(\frac{t}{s}\big)^{n-2} = 2M(n-1)\frac{L_\epsilon}{s}\big(\frac{t}{s}\big)^{n-2}\big(2[\frac{s}{2(t+L_\epsilon)}] + 1\big)^{n-1}.$$

The last term is due to the integration on the part of the hyperplane $\langle x, \nu\rangle = 0$ which does not intersect the n-cubes and can be estimated with

$$M\big(1 - (2[\frac{s}{2(t+L_\epsilon)}] + 1)^{n-1}\big(\frac{t}{s}\big)^{n-1}\big).$$

Using these estimates and taking the limit first in s and then in t, we obtain

$$\limsup_{s\to+\infty} g_s \le \big(1 + \frac{\epsilon(1+\epsilon)}{c}\big) \liminf_{t\to+\infty} g_t;$$

as ϵ can be chosen arbitrarily, the existence of the limit is proved. **q.e.d.**

Now we can prove Theorem 4.2. By the properties of Γ-convergence (see for instance [DMO], Proposition 4.8), it is sufficient to show that for every sequence (ϵ_h) as in Proposition 4.3, the function ψ is equal to the limit function g of Proposition 4.4. In fact, since ψ is BV-elliptic, we have by Theorem 3.2 and (1.12)

$$\psi(i,j,\nu) = \min\Big\{ \int_{Q\cap S_u} \psi(u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) : u \in BV(Q;T),\ u^* = u_0^* \text{ on } \partial Q \Big\} =$$

$$= \min\Big\{ \int_{Q\cap S_u} \psi(u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) + \int_{\partial Q} \psi(u^*, u_0^*, \nu_Q)\, d\mathcal{H}_{n-1}(x) : u \in BV(Q;T) \Big\}.$$

Applying again Theorem 3.2, recalling the definition of g_ϵ and (1.12), we get

$$\psi(i,j,\nu) = \lim_{h\to+\infty} g_{\epsilon_h}(i,j,\nu) = \lim_{\epsilon\to 0^+} g_\epsilon(i,j,\nu) = g(i,j,\nu),$$

and the theorem is proved.

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