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Functionals defined on partitions of sets of finite perimeter, I: integral representation and Γ-convergence

1. Introduction.

During the last years a renewed interest has arisen towards problems in the Calculus of Variations related to partitions of sets in regions of finite perimeter. Problems of this type have been studied in connection with phase transition problems in Cahn-Hilliard fluids ([AL], [B], [M01], [MO2], [ST]) and also with the research of equilibrium positions of mixtures of fluids and liquid crystals.
Given a bounded domain $\Omega \subset \mathbf{R}^n$, with Lipschitz continuous boundary, it is possible to define the functional which assigns to every partition $\{E_1, \dots, E_m\}$ of Ω in sets of finite perimeter the real number

$$\mathcal{F}\big(E_1, \dots, E_m\big) = \sum_{i,j=1\,, i\neq j}^{m} \int_{\partial^* E_i \cap \partial^* E_j} f(x, i, j, \nu_i)\, d\mathcal{H}_{n-1}(x) \tag{1.1}$$

where $\partial^* E_i$ is the essential boundary of the set E_i, ν_i is the inner normal to E_i, and $f = f(x, i, j, \nu)$ is an interface energy which may also depend on x and on the orientation of the surface.
The aim of this paper is to include this kind of functionals in the framework of the Calculus of Variations, and particularly in the theory of Γ-convergence.
The problem we deal with is to clarify whether the limit of minimum problems with isovolumetrical constraints

$$\inf\Big\{ \sum_{i,j=1\,, i\neq j}^{m} \int_{\partial^* E_i \cap \partial^* E_j} f_h(x, i, j, \nu_i)\, d\mathcal{H}_{n-1}(x) : \ |E_i| = \alpha_i, \ i = 1, \dots, m \Big\} \tag{1.2}$$

is still a problem of the same type. In other words, we want to prove some closure properties of the class of functionals (1.1) with respect to variational convergence (Γ-convergence).
A standard technique, already exploited in variational problems in Sobolev spaces, is based on the localization of functionals, setting

$$\mathcal{F}_h\big((E_1, \dots, E_m); A\big) = \sum_{i,j=1\, i\neq j}^{m} \int_{A \cap \partial^* E_i \cap \partial^* E_j} f_h(x, i, j, \nu_i)\, d\mathcal{H}_{n-1}(x) \tag{1.3}$$

for every open set $A \subset \Omega$ and proving that the functionals $\mathcal{F}_h(\cdot, A)$ Γ-converge to some functional $\mathcal{F}(\cdot, A)$ for every A. Then, one looks for necessary and sufficient conditions for a functional $\mathcal{F}(\cdot, A)$ which ensure the integral representation

$$\mathcal{F}\big((E_1, \dots, E_m); A\big) = \sum_{i,j=1\, i\neq j}^{m} \int_{A \cap \partial^* E_i \cap \partial^* E_j} \psi(x, i, j, \nu_i)\, d\mathcal{H}_{n-1}(x) \tag{1.4}$$

for a suitable integrand ψ. Eventually, one proves that the functional obtained as limit of the functionals $\mathcal{F}_h$ fulfils such conditions. In such a case, the convergence of minimum problems (1.2) to the problem

$$\min\Big\{ \sum_{i,j \in I, i\neq j} \int_{\partial^* E_i \cap \partial^* E_j} \psi(x, i, j, \nu_i)\, d\mathcal{H}_{n-1}(x) : \ |E_i| = \alpha_i, \ i = 1, \dots, m \Big\} \tag{1.5}$$

is a straightforward consequence of well known Γ-convergence theorems and equi-coercivity assumptions, because it can be shown that the Γ-limits is not affected by the volume constraint (Theorem 3.3). Assured in such a way the existence of an integrand which allows representation of the limit problem, the natural question arises of its characterization in terms of the functions f_h. In the case $f_h \equiv f$ for every h the Γ-limit is also called relaxed functional, and it might be expected that the integrand ψ in (1.5) is the "greatest" function ϕ such that $\phi \le f$ and the functional

$$\sum_{i,j=1\, i\neq j}^{m} \int_{\partial^* E_i \cap \partial^* E_j} \phi(x,i,j,\nu_i)\, d\mathcal{H}_{n-1}(x)$$

is lower semicontinuous with respect to convergence in measure.
Another interesting problem is posed by homogeneization, which deals with limits of periodic structures. Mathematically, one considers a function $f(x,i,j,\nu)$ periodic in x and defines

$$f_h(x,i,j,\nu) = f(hx,i,j,\nu).$$

In this paper we prove an integral representation theorem for functionals $\mathcal{F}(E_1,\dots,E_m;A)$ and we apply this result to prove that, under suitable equi-coerciveness and equi-continuity assumptions on the functions f_h, the Γ-limit admits integral representation. Moreover, we show that these results can be applied to give a representation to the limit of problems (1.2).
The study of semicontinuity conditions and of homogeneization problems will be carried on in a forthcoming paper.

2. Notation and results about sets of finite perimeter.

Let $\Omega \subset \mathbf{R}^n$ be an open set; we denote by $\mathbf{A}(\Omega)$ the class of open subsets of Ω and by $\mathbf{B}(\Omega)$ the class of Borel subsets of Ω. For every set $E \in \mathbf{B}(\mathbf{R}^n)$ we denote by $|E|$ its Lebesgue n-dimensional measure and by $\mathcal{H}_{n-1}(E)$ its Hausdorff $(n-1)$-dimensional measure. We also denote by $\mathbf{S}^{n-1}$ the unit sphere in $\mathbf{R}^n$ and we set $\omega_{n-1} = \mathcal{H}_{n-1}(\mathbf{S}^{n-1})$. For every Borel set E we denote by $\partial^* E$ its essential boundary, i.e.,

$$\partial^* E = \{x \in \mathbf{R}^n : \limsup_{\rho\to 0^+} \frac{|B_\rho(x)\setminus E|}{\rho^n} > 0 \quad \text{and} \quad \limsup_{\rho\to 0^+} \frac{|B_\rho(x)\cap E|}{\rho^n} > 0\}. \tag{2.1}$$

Let $E \in \mathbf{B}(\Omega)$ such that $|E| < +\infty$. We say that E is a set of finite perimeter in Ω if

$$\sup\Big\{\int_E \mathrm{div} g\, dx : \ g \in C^1_0(\Omega;\mathbf{R}^n),\ |g| \le 1\Big\} < +\infty.$$

We denote by $\mathcal{P}(\Omega)$ the class of sets $E \subset \Omega$ such that E has finite perimeter in Ω. It can be shown that if $E \in \mathcal{P}(\Omega)$ then there exists a unique vector Radon measure in $\mathbf{B}(\Omega)$, denoted by $D1_E$, which is the distributional derivative of 1_E, that is

$$\int_\Omega \langle g, D1_E\rangle = -\int_E \mathrm{div} g\, dx \qquad \forall g \in C^1_0(\Omega;\mathbf{R}^n).$$

If E is an open set with smooth boundary, the Gauss-Greeen theorem implies that $D1_E = \nu_E \cdot \mathcal{H}_{n-1}|_{\partial E}$, where ν_E is the inner normal to E. This representation of the distributional derivative was generalized by E. De Giorgi, Federer ([DG1], [DG2], [FE2]) who proved that in $\mathcal{H}_{n-1}$-almost every $x \in \partial^* E$ the limit

$$\nu_E(x) = \lim_{\rho\to 0^+} \frac{D1_E(B_\rho(x)}{|D1_E|(B_\rho(x))}$$

exists and belongs to $\mathbf{S}^{n-1}$; in addition,

$$D1_E = \nu_E \cdot \mathcal{H}_{n-1}|_{\partial^* E}$$

and, in particular, for every set $E \in \mathcal{P}(\Omega)$ it is $\mathcal{H}_{n-1}(\partial^* E) < +\infty$. Also the opposite implication is true ([FE1], 4.5.11). Moreover, if we set

$$E_t = \{x \in \Omega : \lim_{\rho \to 0^+} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = t\}$$

for every $t \in [0,1]$, then

$$\mathcal{H}_{n-1}(\partial^* E \setminus E_{1/2}) = 0 \tag{2.2}$$

Let T be a finite set, endowed with the discrete topology. We denote by $BV(\Omega, T)$ the class of Borel functions $u : \Omega \to T$ such that $\{u = i\}$ is a set of finite perimeter in Ω for every $i \in T$. For every function $u \in BV(\Omega, T)$ we set also

$$S_u = \bigcup_{i \in T} \partial^* \{u = i\} = \bigcup_{i,j \in T} \partial^* \{u = i\} \cap \partial^* \{u = j\}. \tag{2.3}$$

By definition, each point in $\Omega \setminus S_u$ is a set of density 1 for a unique set $\{u = i\}$, and we denote by $\tilde{u}(x) = i$ this essential value. By (2.2) it follows (see for instance [V1], [V2]) that in $\mathcal{H}_{n-1}$ almost every $x \in S_u$ there exists a triplet $(u^+, u^-, \nu_u) \in T \times T \times \mathbf{S}^{n-1}$ such that

(2.4)
$$\lim_{\rho \to 0^+} \frac{|\{y \in B_\rho(x) : \langle y - x, \nu_u \rangle > 0,\ u(y) \neq u^+\}|}{\rho^n} = \lim_{\rho \to 0^+} \frac{|\{y \in B_\rho(x) : \langle y - x, \nu_u \rangle < 0,\ u(y) \neq u^-\}|}{\rho^n} = 0.$$

The triplet (u^+, u^-, ν_u) is uniquely determined up to a change of sign of ν_u and of an interchange of u^+, u^-. Henceforth, we set

$$(i, j, \nu) \qquad \sim \qquad (i', j', \nu') \tag{2.5}$$

if $i = i'$, $j = j'$, $\nu = \nu'$ or $i = j'$, $j = i'$, $\nu = -\nu'$. Then the functional in (1.1) can be written as

$$\int_{S_u} \tilde{f}(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x)$$

where

$$\tilde{f}(x, i, j, \nu) = f(x, i, j, \nu) + f(x, j, i, -\nu).$$

In the following we shall always work with this representation of the functional (1.1). By (2.2) it follows also

$$\mathcal{H}_{n-1}(\partial^* \{u = i\} \cap \partial^* \{u = j\} \cap \partial^* \{u = k\}) = 0 \tag{2.6}$$

whenever $\mathrm{card}(\{i, j, k\}) = 3$. To join minimizing sequences, we shall need a decomposability property of $BV(\Omega, T)$ ([V1], [V2]): for every pair of functions u, $v \in BV(\Omega, T)$ and for every set $E \in \mathcal{P}(\Omega)$, the function

$$w(x) = \begin{cases} u(x) & \text{if } x \in E \\ \\ v(x) & \text{if } x \in \Omega \setminus E \end{cases}$$

belongs to $BV(\Omega, T)$ and

$$S_w \subset (S_u \cap \Omega \setminus E_0) \cup (S_v \cap \Omega \setminus E_1) \cup \{x \in \partial^* E \setminus (S_u \cup S_v) : \tilde{u}(x) \neq \tilde{v}(x)\}. \tag{2.7}$$

We shall need also a particular case of Fleming-Rishel formula ([FE1], 4.5.9): for every function $\phi \in W^{1,1}(\Omega)$ the set

$$\{t \in \mathbf{R} : \{\phi > t\} \notin \mathcal{P}(\Omega)\}$$

is negligible in $\mathbf{R}$ and

$$\int_\Omega h |\nabla \phi|\, dx = \int_{-\infty}^{+\infty} \int_{\partial^* \{\phi > t\}} h\, d\mathcal{H}_{n-1}\, dt \tag{2.8}$$

for every bounded Borel function $h : \Omega \to \mathbf{R}$.

3.Statement of the main results.

In the following, we denote by Λ a fixed positive constant. We say that a set function $\alpha : \mathbf{A}(\Omega) \to [0,+\infty[$ is a measure if it is the trace on $\mathbf{A}(\Omega)$ of a Radon measure in $\mathbf{B}(\Omega)$. A set function α is a measure if and only if the following conditions are satisfied (see for instance [DL]):

$$A \subset B \qquad \Rightarrow \qquad \alpha(A) \le \alpha(B); \tag{3.1}$$

$$A \cap B = \emptyset \qquad \Rightarrow \qquad \alpha(A \cup B) \ge \alpha(A) + \alpha(B); \tag{3.2}$$

$$\alpha(A \cup B) \le \alpha(A) + \alpha(B); \tag{3.3}$$

$$\alpha(A) = \sup_{B \subset\subset A} \alpha(B). \tag{3.4}$$

Since the extension to $\mathbf{B}(\Omega)$ if exists is unique, we adopt the same notation for set functions on $\mathbf{A}(\Omega)$ and for their extensions.
The following theorem shows that under suitable assumptions a functional $\mathcal{F} : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0,+\infty[$ can be represented by means of a continuous function in the following way

$$\mathcal{F}(u, A) = \int_{A \cap S_u} f(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \qquad \forall u \in BV(\Omega, T),\ \forall A \in \mathbf{A}(\Omega) \tag{3.5}$$

Theorem 3.1: *Let $\mathcal{F} : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0,+\infty[$ be a functional satisfying the following conditions:*

(i) $$0 \le \mathcal{F}(u, A) \le \Lambda \mathcal{H}_{n-1}(A \cap S_u) \qquad \forall u \in BV(\Omega, T), \quad \forall A \in \mathbf{A}(\Omega);$$

(ii) $$\mathcal{F}(u, \cdot) \text{ is a measure for every } u \in BV(\Omega, T);$$

(iii) $$\mathcal{F}(u, A) = \mathcal{F}(v, A) \text{ whenever } u = v \text{ almost everywhere in } A;$$

(iv) $$u_h \to u \quad \text{almost everywhere in } A \qquad \Rightarrow \qquad \mathcal{F}(u, A) \le \liminf_{h \to +\infty} \mathcal{F}(u_h, A)$$

for every open set $A \in \mathbf{A}(\Omega)$;
(v) for every open set $A \subset\subset \Omega$ there exists a continuous function $\omega_A : [0,+\infty[\to [0,+\infty[$ such that $\omega_A(0) = 0$ and

$$|\mathcal{F}(u, B) - \mathcal{F}(v, B + z)| \le \omega_A(|z|)\mathcal{H}_{n-1}(B \cap S_u)$$

whenever $B \in \mathbf{A}(A)$, $z \in \mathbf{R}^n$, $|z| < \operatorname{dist}(A, \partial\Omega)/2$ and $v(x + z) = u(x)$ in B.
Then, there exists a unique continuous function $f : \Omega \times T \times T \times \mathbf{S}^{n-1} \to [0, \Lambda]$ such that $f(x, i, j, \nu) = f(x, j, i, -\nu)$,

$$p \to f(x, i, j, \frac{p}{|p|})|p| \text{ is convex in } \mathbf{R}^n \text{ for every } x \in \Omega,\ i,\ j \in T; \tag{3.6}$$

and $\mathcal{F}(u, A)$ is representable as in (3.5).

Now we see how Theorem 3.1 can be applied to show that the Γ- limit of a sequence of integral functionals is still an integral functional.
Let us recall some basic definitions and results about Γ- convergence (we refer to [DF1], [DF2], [DM] for a wide bibliography on the subject). Let (X, d) be a separable metric space, and let (F_h) be a sequence of real extended valued functions defined in X. We set

$$\Gamma(d^-) - \limsup_{h\to+\infty} F_h(u) = \inf\bigl\{\limsup_{h\to+\infty} F_h(u_h) : (u_h) \subset X,\ u_h \to u\bigr\}, \tag{3.7}$$

$$\Gamma(d^-) - \liminf_{h\to+\infty} F_h(u) = \inf\bigl\{\liminf_{h\to+\infty} F_h(u_h) : (u_h) \subset X,\ u_h \to u\bigr\}, \tag{3.8}$$

for every $u \in X$. The functions in (3.7), (3.8) are both lower semicontinuous. We say that the sequence (F_h) Γ-converges to F_∞ if

$$\Gamma(d^-) - \liminf_{h\to+\infty} F_h(u) = F_\infty(u) = \Gamma(d^-) - \limsup_{h\to+\infty} F_h(u) \qquad \forall u \in X.$$

The Γ-limit if exists is unique; moreover, by every sequence (F_h) it is possible to extract a subsequence (F_{h_k}) which Γ -converges.
The property which motivates the introduction of Γ-convergence in Calculus of Variations is the following: assume that (F_h) Γ- converges to F_∞ and

$$\inf_X F_h = \inf_K F_h \qquad \forall h \in \mathbf{N}$$

for a suitable compact set $K \subset X$. Then

$$\lim_{h\to+\infty} \inf_X F_h = \min\{F_\infty(x) : x \in X\} \tag{3.9}$$

and every sequence $(x_h) \subset K$ such that

$$\lim_{h\to+\infty} F_h(x_h) = \lim_{h\to+\infty} \inf_X F_h$$

admits a subsequence converging to a minimizer of F_∞.
In the following, we are interested to study Γ-convergence of functionals defined on $BV(\Omega, T)$, endowed with the distance

$$d_\Omega(u, v) = \sum_{k=1}^\infty 2^{-k}\bigl(1 \wedge |\{x \in \Omega_k : u(x) \neq v(x)\}|\bigr)$$

where $\Omega_k = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 2^{-k}\}$. This distance induces (local) convergence in measure. We recall that every sequence converging almost everywhere locally converges in measure, and sequences converging locally in measure admit subsequences converging almost everywhere. Since we deal with localized functionals, we set also

$$\Gamma(d_A^-) - \limsup_{h\to+\infty} \mathcal{F}_h(u, A) = \inf\bigl\{\limsup_{h\to+\infty} \mathcal{F}_h(u_h, A) : (u_h) \subset X,\ d_A(u_h, u) \to 0\bigr\}, \tag{3.10}$$

$$\Gamma(d_A^-) - \liminf_{h\to+\infty} \mathcal{F}_h(u, A) = \inf\bigl\{\liminf_{h\to+\infty} \mathcal{F}_h(u_h, A) : (u_h) \subset X,\ d_A(u_h, u) \to 0\bigr\}, \tag{3.11}$$

for every $u \in BV(\Omega, T)$, $A \in \mathbf{A}(\Omega)$, where

$$d_A(u, v) = \sum_{k=1}^\infty 2^{-k}\bigl(1 \wedge |\{x \in A_k : u(x) \neq v(x)\}|\bigr)$$

and $A_k = \{x \in A : \mathrm{dist}(x, \partial A) > 2^{-k}\}$.

Theorem 3.2: *Let $\mathcal{F}_h : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0, +\infty[$ be a sequence of functionals satisfying conditions (i), (ii), (iii) of Theorem 3.1. Then, there exists an increasing sequence of integers (h_k) and a functional $\mathcal{F} : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0, \Lambda]$ satisfying the same conditions such that*

$$\mathcal{F}(\cdot, A) = \Gamma(d_A^-) - \lim_{k \to +\infty} \mathcal{F}_{h_k}(\cdot, A) \qquad \forall A \in \mathbf{A}(\Omega).$$

If, in addition, condition (v) of Theorem 3.1 is uniformly satisfied by $\mathcal{F}_h$ and if

$$\lambda \mathcal{H}_{n-1}(A \cap S_u) \le \mathcal{F}_h(u, A) \le \Lambda \mathcal{H}_{n-1}(A \cap S_u) \qquad \forall u \in BV(\Omega, T), \forall A \in \mathbf{A}(\Omega) \tag{3.12}$$

for some constant $\lambda > 0$, then $\mathcal{F}$ satisfies all the hypotheses of Theorem 3.1, and admits integral representation.

Assume that Ω is a bounded set with Lipschitz continuous boundary; in this case, the sets

$$\{u \in BV(\Omega, T) : \mathcal{H}_{n-1}(S_u) \le C\}$$

are compact with respect to convergence in measure ([GI]). Assume also that T is endowed with a distance, let $\varphi : \Omega \to T$ be a Borel function and let

$$\mathcal{F}_h(u, A) = \int_{A \cap S_u} f_h(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) + \int_A d(u, \varphi)\, dx.$$

Since Γ-convergence is stable under continuous perturbations, by Theorem 3.1, Theorem 3.2 and (3.9) we get a subsequence f_{h_k} such that the problems

$$\inf \Big\{ \int_{A \cap S_u} f_{h_k}(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) + \int_A d(u, \varphi)\, dx : u \in BV(\Omega, T) \Big\}$$

converge for every open set $A \subset \Omega$ to the problem

$$\min \Big\{ \int_{A \cap S_u} f(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) + \int_A d(u, \varphi)\, dx : u \in BV(\Omega, T) \Big\}$$

for some continuous function f which depends only on (f_{h_k}), provided

$$0 < \lambda \le f_h(x, i, j, \nu) \le \Lambda \qquad \forall x \in \Omega,\ i, j \in T,\ \nu \in \mathbf{S}^{n-1}, \tag{3.14}$$

and the functions $f_{h_k}(\cdot, i, j, \nu)$ are equicontinuous.

Since volume constraints can not be localized, we have considered in Theorem 3.1 and Theorem 3.2 only functionals defined on the whole $BV(\Omega, T)$. However, in many problems such constraints are present. This is the motivation of the following theorem , which shows that the Γ-limit "commutes" with the volume constraint.

Theorem 3.3: *Let $T = \{z_1, \ldots, z_m\}$, let $V_1, \ldots, V_m$ be m strictly positive real numbers such that*

$$V_1 + \ldots + V_m = |\Omega|,$$

and let $(\mathcal{F}_h)$ be a sequence of functionals defined in $BV(\Omega,T)\times\mathbf{A}(\Omega)$ which satisfy the conditions (i), (ii), (iii) of Theorem 3.1. Assume that

$$\mathcal{F}_\infty(\cdot,A)=\Gamma(d_A^-)\lim_{h\to+\infty}\mathcal{F}_h(\cdot,A)$$

for every open set $A\subset\Omega$, and let us define for $h\in\mathbf{N}\cup\{\infty\}$

$$\tilde{\mathcal{F}}_h(u)=\begin{cases}\mathcal{F}_h(u,\Omega) & \text{if } |\{u=z_i\}|=V_i,\quad i=1,\dots,m;\\ \\ +\infty & \text{otherwise.}\end{cases}$$

We have then

$$\tilde{\mathcal{F}}_\infty(u)=\Gamma(d_\Omega^-)\lim_{h\to+\infty}\tilde{\mathcal{F}}_h(u)\qquad\forall u\in BV(\Omega,T).$$

Let (f_h) as in (3.14) and $(\mathcal{F}_h)$ as in (3.13); Theorem 3.2, Theorem 3.3 and (3.9) give a subsequence (f_{h_k}) such that the problems

$$\inf\Big\{\int\limits_{A\cap S_u}f_{h_k}(x,u^+,u^-,\nu_u):|u=z_i|=V_i,\ i=1,\dots,m\Big\}$$

converge for every open set $A\subset\Omega$ to the problem

$$\min\Big\{\int\limits_{A\cap S_u}f(x,u^+,u^-,\nu_u):|u=z_i|=V_i,\ i=1,\dots,m\Big\}$$

for a suitable continuous integrand f, provided the functions $f_h(\cdot,i,j,\nu)$ are equicontinuous.

4. Preliminary lemmas.

Our first lemma shows that also the extension of local functionals to $BV(\Omega,T)\times\mathbf{B}(\Omega)$ satisfies locality properties. For a similar result in Sobolev spaces, compare with [BD3].

Lemma 4.1: *Let $\mathcal{F}:BV(\Omega,T)\times\mathbf{A}(\Omega)\to[0,+\infty[$ be a functional such that*

(i) $$0\le\mathcal{F}(u,A)\le\Lambda\mathcal{H}_{n-1}(A\cap S_u)\qquad\forall u\in BV(\Omega,T),\quad\forall A\in\mathbf{A}(\Omega);$$

(ii) $$\mathcal{F}(u,A)=\mathcal{F}(v,A)\ \text{whenever } u=v \text{ almost everywhere in } A;$$

(iii) $$\mathcal{F}(u,\cdot)\ \text{is a measure for every } u\in BV(\Omega,T);$$

(iv) $$u_h\to u\quad\text{almost everywhere in } A\qquad\Rightarrow\qquad\mathcal{F}(u,A)\le\liminf_{h\to+\infty}\mathcal{F}(u_h,A)$$

for every open set $A\in\mathbf{A}(\Omega)$. Then we have

(4.1) $$\mathcal{F}(u,B)=0\qquad\forall B\in\mathbf{B}(\Omega)\ \text{with } \mathcal{H}_{n-1}(B)=0.$$

Moreover, $\mathcal{F}(u,B)=\mathcal{F}(v,B)$ whenever $B\in\mathbf{B}(\Omega)$, $\mathcal{H}_{n-1}((S_u\Delta S_v)\cap B)=0$ and

(4.2) $$(u^+,u^-,\nu_u)\sim(v^+,v^-,\nu_v)\qquad\mathcal{H}_{n-1}\text{-a.e. in } S_u\cap S_v.$$

Proof. Formula (4.1) is a straightforward consequence of (2.4). The proof of (4.2) is based essentially on decomposability property (2.7) and on (2.2). We refer to [AMT], Proposition 4.4, Step 1, where the same locality property is proved in a more general context. **q.e.d.**

Now we prove that our functionals are uniquely determined by their values on sets on smooth boundary.

Lemma 4.2: *Let $T = \{0,1\}$, and set $F(E,A) = F(\chi_E, A)$. Let $\mathcal{F}$, $\mathcal{G}$ be functionals satisfying conditions (i), (ii), (iii), (iv) of Lemma 4.1. Then $\mathcal{F} = \mathcal{G}$ if and only if*

$$\mathcal{F}(E,A) = \mathcal{G}(E,A)$$

for all pairs (E,A) such that $A \cap \partial^* E$ is a C^1 hypersurface.

Proof. Let $E \in \mathcal{P}(\Omega)$, $A \in \mathbf{A}(\Omega)$; to prove equality $\mathcal{F}(E,A) = \mathcal{G}(E,A)$ it is not restrictive to assume that $A \subset\subset \Omega$. By the De Giorgi rectifiability theorem ([DG2], [GI]), it is possible to find functions $f_h \in C^1_0(\Omega)$, compact sets $K_h \subset \partial^* E$ such that $\nu_E(x)$ exists for every $x \in K_h$, and

$$K_h \subset \{f_h = 0\}, \qquad \nabla f_h(x) = \nu_E(x) \quad \forall x \in K_h.$$

Moreover, $K_h \subset K_k$ if $h \leq k$ and

$$\lim_{h \to +\infty} \mathcal{H}_{n-1}(A \cap \partial^* E \setminus K_h) = 0.$$

By our assumptions, the equality

$$\mathcal{F}(\{f_h > 0\}, B) = \mathcal{G}(\{f_h > 0\}, B)$$

holds for every open set $B \subset A \cap \{\nabla f_h \neq 0\}$; since $\mathcal{F}$, $\mathcal{G}$ are measures, we obtain

$$\mathcal{F}(\{f_h > 0\}, A \cap K_h) = \mathcal{G}(\{f_h > 0\}, A \cap K_h).$$

By Lemma 4.1 we get

$$\mathcal{F}(E, A \cap K_h) = \mathcal{F}(\{f_h > 0\}, A \cap K_h) = \mathcal{G}(\{f_h > 0\}, A \cap K_h) = \mathcal{G}(E, A \cap K_h).$$

By letting $h \to +\infty$ we get the stated equality. **q.e.d.**

We recall an integral representation result for functionals defined on $W^{1,1}(\Omega)$ which can be desumed by [DB3].

Theorem 4.3: *Let $\mathcal{F} : W^{1,1}(\Omega) \times \mathbf{A}(\Omega) \to [0, +\infty[$ be a functional satisfying the following conditions:*

$$(i) \qquad 0 \leq \lambda \int_A |\nabla u|\,dx \leq \mathcal{F}(u,A) \leq \Lambda \int_A |\nabla u|\,dx \qquad \forall u \in W^{1,1}(\Omega), \quad \forall A \in \mathbf{A}(\Omega);$$

$$(ii) \qquad \mathcal{F}(u,\cdot) \text{ is a measure for every } u \in W^{1,1}(\Omega);$$

$$(iii) \qquad \mathcal{F}(u,A) = \mathcal{F}(v,A) \text{ whenever } u = v \text{ almost everywhere in } A;$$

$$(iv) \qquad u_h \to u \quad \text{almost everywhere in } A \qquad \Rightarrow \qquad \mathcal{F}(u,A) \leq \liminf_{h \to +\infty} \mathcal{F}(u_h, A)$$

for every open set $A \in \mathbf{A}(\Omega)$;

$$(v) \qquad \mathcal{F}(u+t,A)=\mathcal{F}(u,A) \quad \forall t\in \mathbf{R}, \qquad \mathcal{F}(tu,A)=t\mathcal{F}(u,A) \quad \forall t\in \mathbf{R}^+$$

for every $u\in W^{1,1}(\Omega)$, $A\in \mathbf{A}(\Omega)$;
(vi) for every open set $A\subset\subset\Omega$ there exists a continuous function $\omega_A:[0,+\infty[\to[0,+\infty[$ such that $\omega_A(0)=0$ and

$$|\mathcal{F}(u,B)-\mathcal{F}(v,B+z)|\le \omega_A(|z|)\int_B |\nabla u|\,dx$$

whenever $B\in\mathbf{A}(A)$, $z\in\mathbf{R}^n$, $|z|<\operatorname{dist}(A,\partial\Omega)/2$ and $v(x+z)=u(x)$ in B.
Then, there exists a unique continuous function $f(x,p):\Omega\times\mathbf{R}^n\to[0,+\infty[$, convex and positively 1-homogeneous in p, such that

$$\mathcal{F}(u,A)=\int_A f(x,\nabla u(x))\,dx \qquad \forall u\in W^{1,1}(\Omega),\ A\in\mathbf{A}(\Omega)$$

and

$$0\le\lambda|p|\le f(x,p)\le\Lambda|p| \qquad \forall x\in\Omega,\ p\in\mathbf{R}^n.$$

To prove that the Γ-limit of a sequence of functionals is a measure, we need to join minimizing sequences on different open sets; this is done in the following lemma.

Lemma 4.4: *Let K, B, $A\in\mathbf{A}(\Omega)$ such that $K\subset\subset B\subset\subset A$. Then, there exists a constant c depending only on K, B, A such that, for every pair of functions $u,v\in BV(\Omega,T)$ and for every functional $\mathcal{F}:BV(\Omega,T)\times\mathbf{A}(\Omega)\to[0,+\infty[$ satisfying the hypotheses (i), (ii), (iii) of Lemma 4.1 it is possible to find $w\in BV(\Omega,T)$ with the following properties*

$$\mathcal{F}(w,A)\le\mathcal{F}(u,B)+\mathcal{F}(v,A\setminus\overline{K})+c\Lambda|\{x\in B\setminus\overline{K}:u(x)\neq v(x)\}|;$$

$$w=u \text{ in } A\setminus B, \quad w=v \text{ in } K, \quad w(x)\in\{u(x),v(x)\} \text{ almost everywhere in } A.$$

Proof. To simplify our notations, we set $d(i,j)=1$ if $i=j\in T$ and $i\neq j$, $d(i,i)=0$. Let $\varphi\in C^1(\Omega)$ such that $0\le\varphi\le1$, $\varphi=0$ in K and $\varphi=1$ in $\Omega\setminus\overline{B}$. We claim that $c=\|\nabla\varphi\|_\infty$ satisfies the conditions of the lemma. In fact, let $u,v\in BV(\Omega,T)$, and let $w_t\in BV(\Omega,T)$ be the functions defined by

$$w_t(x)=\begin{cases} u(x) & \text{if } \varphi(x)<t;\\ v(x) & \text{if } \varphi(x)\ge t,\end{cases}$$

for every $t\in]0,1[$. By Fleming-Rishel formula (2.8), it is possible to find $t\in]0,1[$ such that

$$\{x\in A:\varphi(x)<t\}\in\mathcal{P}(\Omega), \quad \mathcal{H}_{n-1}(S_u\cap\{x\in A:\varphi(x)=t\})=\mathcal{H}_{n-1}(S_v\cap\{x\in A:\varphi(x)=t\})=0,$$

and

$$\int_{B\cap\partial^*\{\varphi<t\}\setminus\overline{K}} d(\tilde u,\tilde v)\,d\mathcal{H}_{n-1}\le\int_{B\setminus\overline{K}} d(u,v)|\nabla\varphi|\,dx\le c|\{x\in B\setminus\overline{K}:u(x)\neq v(x)\}|.$$

By (2.7) we get

$$\mathcal{F}(w_t,A)\le\mathcal{F}(w_t,\{\varphi<t\})+\mathcal{F}(w_t,\{\varphi>t\})+\mathcal{F}(w_t,\{\varphi=t\})\le\mathcal{F}(u,\{\varphi<t\})+\mathcal{F}(v,\{\varphi>t\})+$$

$$+\Lambda|Dw_t|(\{\varphi=t\}) \le \mathcal{F}(u,B)+\mathcal{F}(v,A\setminus\overline{K})+\Lambda\int\limits_{B\cap\partial^*\{\varphi<t\}\setminus\overline{K}} d(\tilde u,\tilde v)\,d\mathcal{H}_{n-1}\le$$

$$\le \mathcal{F}(u,B)+\mathcal{F}(v,A\setminus\overline{K})+c\Lambda|\{x\in B\setminus\overline{K}: u(x)\neq v(x)\}|$$

and the statement is proved. **q.e.d.**

5. Proof of the theorems.

In the case $T=\{0,1\}\subset\mathbf{R}$, the set $BV(\Omega,T)$ can be identified in a natural way with the class $\mathcal{P}(\Omega)$, via the bijection

$$E\in\mathcal{P}(\Omega)\qquad\longleftrightarrow\qquad 1_E\in BV(\Omega,T).$$

In this case, Theorem 3.1 can be stated in the equivalent form:

Theorem 5.1: *Let $\mathcal{F}:\mathcal{P}(\Omega)\times\mathbf{A}(\Omega)\to[0,+\infty[$ be a functional such that*

$$(i)\qquad 0\le\mathcal{F}(E,A)\le\Lambda\mathcal{H}_{n-1}(\partial^*E\cap A)\qquad\forall E\in\mathcal{P}(\Omega),\ \forall A\in\mathbf{A}(\Omega);$$

$$(ii)\qquad \lim_{h\to+\infty}|(E_h\Delta E)\cap A|=0\qquad\Rightarrow\qquad\mathcal{F}(E,A)\le\liminf_{h\to+\infty}\mathcal{F}(E_h,A)$$

for every sequence $\{E_h\}\subset\mathcal{P}(\Omega)$, $E\in\mathcal{P}(\Omega)$;

$$(iii)\qquad \mathcal{F}(E,\cdot)\quad\text{is a measure for every } E\in\mathcal{P}(\Omega);$$

$$(iv)\qquad \mathcal{F}(E,A)=\mathcal{F}(F,A)\quad\text{whenever } |(E\Delta F)\cap A|=0;$$

(v) for every open set $A\subset\subset\Omega$ there exists a continuous function $\omega_A:[0,+\infty[\to[0,+\infty[$ such that $\omega(0)=0$ and

$$|\mathcal{F}(E,B)-\mathcal{F}(F,B+z)|\le\omega_A(|z|)\mathcal{H}_{n-1}(\partial^*E\cap B)$$

whenever $B\in\mathbf{A}(A)$, $z\in\mathbf{R}^n$, $|z|<\operatorname{dist}(A,\partial\Omega)/2$ and $1_{F-z}=1_E$ in B.
Then, there exists a unique continuous function $f:\Omega\times\mathbf{S}^{n-1}\to[0,\Lambda]$ such that

$$(5.1)\qquad p\to f(x,\frac{p}{|p|})|p|\quad\text{is convex in } \mathbf{R}^n \text{ for every } x\in\Omega;$$

and

$$(5.2)\qquad \mathcal{F}(E,A)=\int\limits_{\partial^*E\cap A} f(x,\nu_E)\,d\mathcal{H}_{n-1}\qquad\forall E\in\mathcal{P}(\Omega),\ \forall A\in\mathbf{A}(\Omega).$$

Theorem 3.1 is an easy consequence of Theorem 5.1 and Lemma 4.1:

Proof of Theorem 3.1. Let $(i,j)\in T\times T$, and let

$$\mathcal{F}_{ij}(E,A)=\mathcal{F}(i1_E+j1_{\Omega\setminus E},A).$$

The functional $\mathcal{F}_{ij}$ satisfies all the hypotheses of Theorem 5.1, hence there exists a unique continuous function $f_{ij}:\Omega\times\mathbf{S}^{n-1}\to[0,\Lambda]$ such that

$$\mathcal{F}_{ij}(E,A)=\int\limits_{\partial^*E\cap A} f_{ij}(x,\nu_E)\,d\mathcal{H}_{n-1}\qquad\forall E\in\mathcal{P}(\Omega),\ \forall A\in\mathbf{A}(\Omega).$$

We remark that since f_{ij} is unique, necessarily $f_{ij}(x,\nu) = f_{ji}(x,-\nu)$. Let $f(x,i,j,\nu) = f_{ij}(x,\nu)$, and let

$$u_{ij}(x) = \begin{cases} i & \text{if } u(x) = i; \\ \\ j & \text{if } u(x) \neq i. \end{cases}$$

Let ν_{E_i} be the inner normal to E_i. By the definition of (u^+, u^-, ν_u) and by (2.2) we get

$$(i, j, \nu_{E_i}) = (u_{ij}^+, u_{ij}^-, \nu_{E_i}) \sim (u^+, u^-, \nu_u) \qquad \mathcal{H}_{n-1}\text{-a.e. in } \partial^*\{u = i\} \cap \partial^*\{u = j\}.$$

By Lemma 4.1 we get, recalling (2.6),

$$2\mathcal{F}(u, A) = \sum_{i,j\in T,\, i\neq j} \mathcal{F}(u, A \cap \partial^*\{u = i\} \cap \partial^*\{u = j\}) = \sum_{i,j\in T,\, i\neq j} \mathcal{F}(u_{ij}, A \cap \partial^*\{u = i\} \cap \partial^*\{u = j\}) =$$

$$= \sum_{i,j\in T,\, i\neq j} \int_{A\cap\partial^*\{u=i\}\cap\partial^*\{u=j\}} f(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) = 2 \int_{A\cap S_u} f(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x)$$

and the theorem is proved. **q.e.d.**

Proof of Theorem 5.1. The idea is to use integral representation theorem 4.3, setting

$$\tilde{\mathcal{F}}(u, A) = \int_{-\infty}^{+\infty} \mathcal{F}(\{u < t\}, A)\, dt \qquad u \in W^{1,1}(\Omega),\ A \in \mathbf{A}(\Omega). \tag{5.3}$$

We shall assume for simplicity that Ω is a bounded open set. By the local character of the functionals, this assumption is not restrictive. Moreover, we shall assume that

$$\lambda\mathcal{H}_{n-1}(A \cap \partial^* E) \leq \mathcal{F}(u, A)$$

for some constant $\lambda > 0$. The proof in the general case is easily achieved considering the functionals

$$\mathcal{F}(u, A) + \delta\mathcal{H}_{n-1}(A \cap \partial^* E)$$

and letting $\delta \downarrow 0$. Recalling (2.8), it is easy to check that the functional defined by (5.3) satisfies all the hypotheses of Theorem 4.3, therefore there exists a continuous function $f(x,p) : \Omega \times \mathbf{R}^n \to [0, +\infty[$ which is positively 1-homogeneous and convex in p, satisfying the conditions

$$\tilde{\mathcal{F}}(u, A) = \int_A f(x, \nabla u(x))\, dx \qquad \forall u \in W^{1,1}(\Omega),\ A \in \mathbf{A}(\Omega)$$

and

$$\lambda|p| \leq f(x,p) \leq \Lambda|p| \qquad \forall x \in \Omega,\ p \in \mathbf{R}^n. \tag{5.4}$$

We claim that f is the required integrand in (5.2). The functional

$$\int_A f(x, \nu_E)\, d\mathcal{H}_{n-1}(x)$$

satisfies, by Reshetnyak lower semicontinuity theorem ([RE]), the condition (iv) of Lemma 4.1. Hence, by Lemma 4.2, to prove our statement it will be sufficient to show (5.2) for all pairs (E, A) such that $A \cap \partial^* E$

is is a C^1 hypersurface. Let $\Gamma = A \cap \partial^* E$, and let $B \subset\subset A$ be an open ball such that $B \cap \Gamma$ is the graph of a C^1 function. To fix the ideas, we assume that $\mathcal{H}_{n-1}(\Gamma \cap \partial B) = 0$ and

$$B \cap \Gamma = \{x \in B : (x_1, \ldots, x_{n-1}) \in U, \quad x_n = \varphi(x_1, \ldots, x_{n-1})\},$$

$$B \cap E = \{x \in B : (x_1, \ldots, x_{n-1}) \in U, \quad x_n < \varphi(x_1, \ldots, x_{n-1})\}$$

for some bounded open set $U \subset \mathbf{R}^{n-1}$, for some function $\varphi \in C^1_0(\mathbf{R}^{n-1})$. The functions

$$u(x) = \varphi(x_1, \ldots, x_{n-1}) - x_n, \qquad u_s(x) = \big(s \wedge u(x)\big) \vee -s, \quad s > 0$$

belong to $W^{1,1}(\Omega)$ and it can be easily verified that

$$\lim_{s \to 0^+} \frac{1}{s} \int_B f(x, \nabla u_s(x))\, dx = \int_{B \cap \Gamma} f(x, \nu_E(x))\, d\mathcal{H}_{n-1}(x) = \int_{B \cap \partial^* E} f(x, \nu_E(x))\, d\mathcal{H}_{n-1}(x). \tag{5.5}$$

Moreover, the continuity hypothesis (v) and the assumption $\mathcal{H}_{n-1}(\Gamma \cap \partial B) = 0$ yield

$$\lim_{s \to 0^+} \mathcal{F}(\{u > s\}, B) = \lim_{s \to 0^+} \mathcal{F}(\{u > 0\}, B + s\mathbf{e}_n) = \mathcal{F}(\{u > 0\}, B) = \mathcal{F}(E, B). \tag{5.6}$$

Since

$$\int_B f(x, \nabla u_s(x))\, dx = \tilde{\mathcal{F}}(u_s, B) = \int_0^s \mathcal{F}(\{u > t\}, B)\, dt$$

the equality

$$\mathcal{F}(E, B) = \int_B f(x, \nu_E(x))\, d\mathcal{H}_{n-1}(x)$$

follows by (5.5), (5.6). Taking as B sufficiently small open balls, we get

$$\liminf_{\rho \to 0^+} \frac{\mathcal{F}(E, B_\rho(x))}{\mathcal{H}_{n-1}(B_\rho(x) \cap \partial^* E)} = \liminf_{\rho \to 0^+} \frac{\int_{B_\rho(x)} f(x, \nu_E(x))\, d\mathcal{H}_{n-1}(x)}{\mathcal{H}_{n-1}(B_\rho(x) \cap \partial^* E)} \qquad \forall x \in A$$

and since $\mathcal{F}(E, \cdot)$ is a measure, (5.2) follows. **q.e.d.**

Now we prove that the Γ-limit of a sequence of functionals $\mathcal{F}_h(u, A)$ with the same control from above is a measure if all the functionals $\mathcal{F}_h$ are measures.

Lemma 5.2: *Let $\mathcal{F}_h : BV(\Omega, T) \times \mathbf{A}(\Omega) \to [0, +\infty[$ be a sequence of functionals satisfying conditions (i), (ii), (iii) of Theorem 3.1, and let us define, for $u \in BV(\Omega, T)$ and $A \in \mathbf{A}(\Omega)$,*

$$\mathcal{F}_+(u, A) = \Gamma(d_A^-) \limsup_{h \to +\infty} \mathcal{F}_h(u, A), \qquad \mathcal{F}_-(u, A) = \Gamma(d_A^-) \liminf_{h \to +\infty} \mathcal{F}_h(u, A).$$

as in (3.10), (3.11). We have then

(i) $$\mathcal{F}_+(u, A) \le \mathcal{F}_+(u, B), \qquad \mathcal{F}_-(u, A) \le \mathcal{F}_-(u, B) \qquad \text{whenever } A \subset B;$$

(ii) $$\mathcal{F}_-(u, A) + \mathcal{F}_-(u, B) \le \mathcal{F}_-(u, A \cup B) \qquad \text{whenever } A \cap B = \emptyset;$$

(iii) $$0 \le \mathcal{F}_-(u, A) \le \mathcal{F}_+(u, A) \le \Lambda \mathcal{H}_{n-1}(A \cap S_u);$$

$$(iv) \qquad \mathcal{F}_+(u,A) = \sup_{B\subset\subset A} \mathcal{F}_+(u,B) \qquad \mathcal{F}_-(u,A) = \sup_{B\subset\subset A} \mathcal{F}_-(u,B);$$

$$(v) \qquad \mathcal{F}_+(u,A\cup B) \le \mathcal{F}_+(u,A) + \mathcal{F}_+(u,B), \qquad \mathcal{F}_-(u,A\cup B) \le \mathcal{F}_-(u,A) + \mathcal{F}_+(u,B).$$

Proof. (i), (ii) are trivial, since all $\mathcal{F}_h$ are increasing functionals, and (iii) follows by the upper bound on $\mathcal{F}_h$, taking the sequence $u_h = u$.
(iv) We first prove that if $K \subset\subset B \subset\subset A$, we have

$$(5.7) \qquad \mathcal{F}_+(u,A) \le \mathcal{F}_+(u,B) + \mathcal{F}_+(u,A\setminus\overline{K}), \qquad \mathcal{F}_-(u,A) \le \mathcal{F}_-(u,B) + \mathcal{F}_+(u,A\setminus\overline{K}).$$

Infact, let (u_h), $(v_h) \subset BV(\Omega,T)$ be sequences of functions locally converging in measure to u in B and $A\setminus\overline{K}$ respectively, such that

$$\mathcal{F}_+(u,B) = \limsup_{h\to+\infty} \mathcal{F}_h(u_h,B), \qquad \mathcal{F}_+(u,A\setminus\overline{K}) = \limsup_{h\to+\infty} \mathcal{F}_h(u_h,A\setminus\overline{K}).$$

By joint lemma 4.3, we can find a sequence $(w_h) \subset BV(\Omega,T)$ locally converging to u in measure in A, such that

$$\mathcal{F}_h(w_h,A) \le \mathcal{F}_h(u_h,B) + \mathcal{F}_h(v_h,A\setminus\overline{K}) + c\Lambda|\{x\in B\setminus\overline{K} : u_h(x)\ne v_h(x)\}| \qquad \forall h\in\mathbf{N}.$$

where $c>0$ is a constant depending only on A, B, C. We have then

$$\mathcal{F}_+(u,A) \le \limsup_{h\to+\infty} \mathcal{F}_h(w_h,A) \le \limsup_{h\to+\infty} \mathcal{F}_h(u_h,B) + \limsup_{h\to+\infty} \mathcal{F}_h(v_h,A\setminus\overline{K}) +$$

$$+ c\Lambda \limsup_{h\to+\infty} |\{x\in A\setminus\overline{K} : u_h(x)\ne v_h(x)\}| \le \mathcal{F}_+(u,A) + \mathcal{F}_-(v,B)$$

that proves the first of (5.7). The second one can be proved similarly. As a consequence of (iii), (5.7), we have

$$\mathcal{F}_+(u,A) \le \mathcal{F}_+(u,B) + \Lambda\mathcal{H}_{n-1}(A\setminus\overline{K}), \qquad \mathcal{F}_-(u,A) \le \mathcal{F}_-(u,B) + \mathcal{H}_{n-1}(A\setminus\overline{K}).$$

As we can take $\mathcal{H}_{n-1}(A\setminus\overline{K})$ arbitrarily small and $\mathcal{F}_+(u,\cdot)$, $\mathcal{F}_-(u,\cdot)$ are increasing set functions, inequality $\le$ in (iv) is proved. Since the opposite inequality is trivial, we obtain (iv).
(v) The inequalities are trivial if A and B are disjoint. Let

$$A_t = \{x\in A\cup B : t\mathrm{dist}(x,A\setminus B) < (1-t)\mathrm{dist}(x,B\setminus A)\}$$

and

$$B_t = \{x\in A\cup B : t\mathrm{dist}(x,A\setminus B) > (1-t)\mathrm{dist}(x,B\setminus A)\}$$

for every $t \in]0,1[$. Since the sets $S_t = (A\cup B)\setminus(A_t\cup B_t)$ are disjoint, we can find $t\in]0,1[$ such that $\mathcal{H}_{n-1}(S_t\cap S_u) = 0$. Let $\epsilon>0$ and let $M \subset\subset M' \subset\subset A_t$, $N \subset\subset N' \subset\subset B_t$ such that $\mathcal{H}_{n-1}(S_u\cap[(A\cup B)\setminus(M\cup N)]) < \epsilon$. By (iii), (iv) we get

$$\mathcal{F}_+(u,A\cup B) \le \mathcal{F}_+(u,M'\cup N') + \mathcal{F}_+(u,(A\cup B)\setminus(M\cup N)) \le$$

$$\le \mathcal{F}_+(u,M') + \mathcal{F}_+(u,N') + \Lambda\epsilon \le \mathcal{F}_+(u,A) + \mathcal{F}_+(u,B) + \Lambda\epsilon.$$

Since ϵ is arbitrary, the inequality is proved for $\mathcal{F}_+$. The proof for $\mathcal{F}_-$ is similar. **q.e.d.**

Proof of Theorem 3.2. Let $\mathcal{B}$ be a countable base for $\mathbf{A}(\Omega)$, stable under finite union. By the compactness of Γ-convergence we can find, by a diagonalization procedure, an increasing sequence of integers (h_k) such that (we adopt the notations of previous lemma)

$$\mathcal{F}_+(u,A) = \mathcal{F}_-(u,A)$$

whenever $u \in BV(\Omega, T)$ and $A \in \mathcal{B}$. By (iv) of the lemma,

$$\mathcal{F}_+(u, A) = \sup_{B \subset\subset A} \mathcal{F}_+(u, B) = \sup_{B \subset\subset A,\, B \in \mathcal{B}} \mathcal{F}_+(u, B) = \sup_{B \subset\subset A,\, B \in \mathcal{B}} \mathcal{F}_-(u, B) = \sup_{B \subset\subset A} \mathcal{F}_-(u, B) = \mathcal{F}_-(u, A)$$

so that the Γ-limit exists for every $u \in BV(\Omega, T)$, $A \in \mathbf{A}(\Omega)$. Lemma 5.2 yields that $\mathcal{F}_+ = \mathcal{F}_-$ satisfies conditions (i), (ii), (iii) of Theorem 3.1.
Let us assume now that condition (v) of Theorem 3.1 is uniformly satisfied by $\mathcal{F}_h$ and

$$\lambda \mathcal{H}_{n-1}(A \cap S_u) \leq \mathcal{F}_h(u, A) \leq \Lambda \mathcal{H}_{n-1}(A \cap S_u) \qquad \forall u \in BV(\Omega, T), \forall A \in \mathbf{A}(\Omega)$$

for some constant $\lambda > 0$. Let $A \subset\subset \Omega$, ω_A as in (v), $z \in \mathbf{R}^n$ such that $|z| < \mathrm{dist}(A, \partial\Omega)/2$, and let $u \in BV(\Omega, T)$, $B \in \mathbf{A}(A)$. Let $(u_h) \subset BV(\Omega, T)$ be a sequence locally converging in measure in B to u, such that

$$\limsup_{h \to +\infty} \mathcal{F}_h(u_h, B) \leq \mathcal{F}_+(u, B) + \epsilon$$

and let $v \in BV(\Omega, T)$ such that $v(x + z) = u(x)$ in B. It is easy to find a sequence $(v_h) \subset BV(\Omega, T)$ such that $v_h(x + z) = u_h(x)$ in B, hence

$$\mathcal{F}_-(v, B + z) \leq \limsup_{h \to +\infty} \mathcal{F}_h(v_h, B + z) \leq \liminf_{h \to +\infty} \mathcal{F}_h(u_h, B) + \Lambda \omega_A(|z|) \limsup_{h \to +\infty} \mathcal{H}_{n-1}(B \cap S_u) \leq$$

$$\leq \mathcal{F}_-(u, B) + \epsilon + \Lambda \frac{\mathcal{F}_+(u, B) + \epsilon}{\lambda} \omega_A(|z|).$$

By letting $\epsilon \downarrow 0$ and replacing z by $-z$, we find that $\mathcal{F}$ satisfies condition (v) of Theorem 3.1 with $\omega'_A = \Lambda^2/\lambda\omega_A$, therefore, it admits integral representation. **q.e.d.**

Proof of Theorem 3.3. Let us define

$$BV_0 = \{u \in BV(\Omega, T) : |\{u = z_i\}| = V_i \quad \text{for } i = 1, \ldots, m\}.$$

If $u \in BV(\Omega, T)$ and (u_h) is a sequence in $BV(\Omega, T)$ converging to u in almost everywhere, we have

$$\lim_{h \to +\infty} |\{u_h = z_i\}| = |\{u = z_i\}|,$$

so that if $u \notin BV_0$ we have $\tilde{\mathcal{F}}_h(u_h) \to +\infty$ and, by (1.9), $\tilde{\mathcal{F}}(u) = +\infty$. We need now to prove the inequality

$$\tilde{\mathcal{F}}_\infty(u) \geq \Gamma(d_\Omega^-) - \limsup_{h \to +\infty} \tilde{\mathcal{F}}_h(u) \qquad \forall u \in BV(\Omega, T).$$

Let us suppose $u \in BV_0$; we have to find a sequence (u_h) in BV_0 converging to u in measure, such that

$$\mathcal{F}_\infty(u, \Omega) = \lim_{h \to +\infty} \mathcal{F}_h(u_h, \Omega).$$

We define $E_i^h = \{u_h = z_i\}$ for $h \in \mathbf{N}$, $i = 1, \ldots, m$. We shall find the sequence (u_h) first under the additional assumption that there exist $m(m-1)/2$ points $x_{ij} \in \Omega$ such that

$$B(x_{ij}, 2\eta) \subset E_i^h \qquad \forall i, j \in 1, \ldots, m,\ i \neq j,\ \forall h \in \mathbf{N}, \tag{5.8}$$

for a suitable constant $\eta > 0$. Let $(v_h) \subset BV(\Omega, T)$ be a sequence converging to u in measure such that

$$\mathcal{F}_\infty(u, \Omega) = \lim_{h \to +\infty} \mathcal{F}_h(v_h, \Omega).$$

We first modify the sequence (v_h), setting

$$w_h(x) = \begin{cases} z_i & \text{if } x \in B(x_{ij}, \eta_{ijh}); \\ v_h(x) & \text{otherwise.} \end{cases}$$

where $\eta_{ijh} \in [\eta, 2\eta]$ are chosen in such a way that (apply (2.8) with $\phi(x) = |x - x_{ij}|$)

$$\mathcal{H}_{n-1}\big(\partial B(x_{ij}, \eta_{ijh}) \cap (S_{v_h} \cup S_v)\big) = 0$$

and

$$\eta\mathcal{H}_{n-1}(\{x \in \partial B(x_{ij}, \eta_{ijh}) : \tilde v_h(x) \neq \tilde u(x)\}) \leq |\{x \in B(x_{ij}, 2\eta) \setminus B(x_{ij}, \eta) : v_h(x) \neq u(x)\}| \tag{5.9}$$

The sequence w_h still converges to u almost everywhere, and by (5.9), (2.7) we get

$$\mathcal{F}_h(w_h, \Omega) \leq \mathcal{F}_h(v_h, \Omega) + 1/\eta \sum_{i,j=1,\, i\neq j}^{m} |\{x \in B(x_{ij}, 2\eta) \setminus B(x_{ij}, \eta) : v_h(x) \neq u(x)\}|$$

so that

$$\mathcal{F}_\infty(u, \Omega) = \lim_{h\to+\infty} \mathcal{F}_h(w_h, \Omega). \tag{5.10}$$

By construction, we have

$$B(x_{ij}, \eta) \subset \bigcap_{h\in\mathbf{N}} \{u_h = i\} \cap \{u = i\} \qquad \forall i, j \in \{1, \ldots, m\},\ i \neq j. \tag{5.11}$$

Let us define

$$\eta_i^h = |E_i^h| - V_i, \qquad \eta^h = \sum_{i=1}^m |\eta_i^h|,$$

and

$$P_h = \{j : \eta_j^h > 0\}, \qquad N_h = \{i : \eta_i^h \leq 0\},$$

so that

$$\eta_h \to 0 \qquad \text{and} \qquad \frac{\eta_h}{2} = \sum_{j\in P_h} \eta_j^h = -\sum_{i\in N_h} \eta_i^h.$$

If $\eta_h = 0$ let us define $u_h = w_h$; if not, let

$$r_{ij}^h = \left(-\frac{2n\eta_i^h\eta_j^h}{\omega_{n-1}\eta^h}\right)^{\frac{1}{n}} \qquad B_{ij}^h = B(x_{ij}, r_{ij}^h)$$

for $i \in P_h$, $j \in N_h$, and

$$u_h(x) = \begin{cases} z_j & \text{if } x \in B_{ij}^h,\ i \in P_h,\ j \in N_h; \\ w_h(x) & \text{otherwise.} \end{cases}$$

Since for h large enough all r_{ij}^h are less than η, we have

$$|\{u_h = i\}| = |E_i^h| - \sum_{j\in N_h} |B_{ij}^h| = |E_i^h| - \sum_{j\in N_h} \frac{-2\eta_i^h\eta_j^h}{\eta_h} = |E_i^h| - \eta^h = V_i.$$

Similarly, if $i \in N_h$ we have for h large enough

$$|\{u_h = i\}| = |E_i^h| + \sum_{j \in P_h} |B_{ij}^h| = |E_i^h| + \sum_{j \in P_h} \frac{-2\eta_i^h \eta_j^h}{\eta_h} = |E_i^h| - \eta_i^h = V_i,$$

so that $u_h \in BV_0$; moreover $u_h \to u$ almost everywhere and

$$|\mathcal{F}_h(u_h, \Omega) - \mathcal{F}_h(w_h, \Omega)| \leq \sum_{i \in P_h,\, j \in N_h} \Lambda \mathcal{H}_{n-1}(\partial B_{ij}^h) \leq$$

$$\leq \Lambda \sum_{i \in P_h,\, j \in N_h} \omega_{n-1} \left(\frac{2n\eta_i^h \eta_j^h}{\omega_{n-1}\eta_h} \right)^{\frac{n-1}{n}} \leq \Lambda \omega_{n-1}^{\frac{1}{n}} m^2 \left(\frac{n\eta_h}{2} \right)^{\frac{n-1}{n}}$$

and (5.10) yileds that (u_h) is the required sequence.
Now we show that assumption (5.8) is not restrictive. Let $x \in S_u$ be a point of density $1/2$ for two level sets of u ; in particular, there exists $\delta > 0$ such that

$$|B_\rho(x) \cap \{u = i\}| < |B_\rho(x)| \qquad \forall \rho < \delta,\ \forall i \in \{1, \ldots, m\}.$$

Let $C_1, \ldots, C_m$ be a partition of $B_\rho(x)$ in m mutually disjoint sets of finite perimeter with nonempty interior such that

$$|C_i| = |\{u = i\} \setminus B_\rho(x)| > 0$$

and

$$\mathcal{H}_{n-1}(\partial^* C_i) \leq 2\omega_{n-1}\rho^{n-1}$$

for every index i (one can take for instance concentric spherical regions). The function u_ρ whose level sets are

$$\{u = i\} \setminus B_\rho(x) \cup C_i$$

belongs, by construction to BV_0 and satisfies condition (5.8), so that

$$\Gamma(d_\Omega^-) \limsup_{\rho \to 0^+} \tilde{\mathcal{F}}_h(u_\rho) \leq \tilde{\mathcal{F}}_\infty(u_\rho)$$

for every $\rho < \delta$. On the other hand,

$$\limsup_{\rho \to 0^+} \tilde{\mathcal{F}}_\infty(u_\rho) \leq \limsup_{\rho \to 0^+} \mathcal{F}_\infty(u_\rho, \Omega \setminus \overline{B}_\rho(x)) + \limsup_{\rho \to 0^+} \mathcal{F}_\infty(u_\rho, B_\rho(x)) \leq$$

$$\leq \mathcal{F}_\infty(u, \Omega) + \limsup_{\rho \to 0^+} \Lambda(2m+1)\omega_{n-1}\rho^{n-1} = \tilde{\mathcal{F}}_\infty(u).$$

Since the Γ-limits are lower semicontinuous, we get

$$\Gamma(d_\Omega^-) \limsup_{\rho \to 0^+} \tilde{\mathcal{F}}_h(u) \leq \Gamma(d_\Omega^-) \limsup_{\rho \to 0^+} \tilde{\mathcal{F}}_h(u_\rho) \leq \limsup_{\rho \to 0^+} \tilde{\mathcal{F}}_\infty(u_\rho) \leq \tilde{\mathcal{F}}_\infty(u)$$

and our statement is proved also in the general case. **q.e.d.**

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