DENSITY LOWER BOUND ESTIMATES FOR LOCAL MINIMIZERS OF THE $2d$ MUMFORD-SHAH ENERGY

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ABSTRACT. We prove, using direct variational arguments, an explicit energy-treshold criterion for regular points of 2-dimensional Mumford-Shah energy minimizers. From this we infer an explicit constant for the density lower bound of De Giorgi, Carriero and Leaci.

1. Introduction

The Mumford-Shah model stands as a prototypical example of variational problem in image segmentation (see [14]). It consists in minimizing (adding either boundary or confinement conditions or fidelity terms) the energy

$$E(v,K) := \int_{\Omega\setminus K} |\nabla v|^2\,dx + \mathcal{H}^1(K),$$

where $\Omega \subset \mathbb{R}^2$ is a fixed open set, K is a rectifiable closed subset of Ω, and $v \in C^1(\Omega \setminus K)$. This energy has been then borrowed and conveniently modified in Fracture Mechanics, mainly to model quasi-static irreversible crack-growth for brittle materials (see [2, Section 4.6.6]).

One of the first existence theories for minimizers of E hinges upon a weak formulation in the space SBV of Special functions of Bounded Variation, the subspace of BV functions with singular part of the distributional derivative concentrated on a 1-rectifiable set. In this approach the set K is substituted by the (Borel) set S_v of approximate discontinuities of the function v (throughout the paper we will use standard notations and results concerning BV and SBV, following the book [2]). This is the reason for the terminology *free-discontinuity* problem introduced by De Giorgi. The Mumford-Shah energy of a function v in $SBV(\Omega)$ on an open subset $A \subseteq \Omega$ then reads as

$$\mathrm{MS}(v,A) = \int_A |\nabla v|^2 dx + \mathcal{H}^1(S_v \cap A). \tag{1.1}$$

In case $A = \Omega$ we drop the dependence on the set of integration. In what follows u will always denote a *local minimizer*, that is any $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < +\infty$ and such that

$$\mathrm{MS}(u) \le \mathrm{MS}(w) \quad \text{whenever } \{w \ne u\} \subset\subset \Omega.$$

The class of all local minimizers shall be denoted by $\mathcal{M}(\Omega)$.

As established in [10] in all dimensions (and proved alternatively in [6] in dimension two), if $u \in SBV$ is a minimizer of the energy MS, then the pair $(u, \overline{S_u})$ is a minimizer of E.

The main point is the identity $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$, which holds for every $u \in \mathcal{M}(\Omega)$. The groundbreaking paper [10] proves this identity via the following density lower bound

$$\frac{\mathrm{MS}(u, B_r(z))}{2r} \geq \theta \qquad \text{for all } z \in \overline{S_u}, \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)) \tag{1.2}$$

with θ a dimensional constant independent of u. Building upon the same ideas, in [5] it is proved that for some dimensional constant θ_0 independent of u it holds

$$\frac{\mathcal{H}^1(S_u \cap B_r(z))}{2r} \geq \theta_0 \qquad \text{for all } z \in \overline{S_u}, \text{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{1.3}$$

The argument for (1.2) used by De Giorgi, Carriero & Leaci in [10], and similarly in [5] for (1.3), is indirect: it relies on Ambrosio's SBV compactness theorem, an SBV Poincaré-Wirtinger type inequality and the asymptotic analysis of blow-ups of minimizers with vanishing Dirichlet energy. In this paper we give a simpler proof in 2 dimensions, which does not require any Poincaré-Wirtinger inequality, nor any compactness argument. Our argument differs from those used in [6] and [7] to derive (1.3) in the two dimensional case as well.

We first introduce some useful notation, which we borrow from [9]. Given $u \in \mathcal{M}(\Omega)$, $z \in \Omega$ and $r \in (0, \mathrm{dist}(z, \partial\Omega))$ let

$$e_z(r) := \int_{B_r(z)} |\nabla u|^2 dx, \quad \ell_z(r) := \mathcal{H}^1(S_u \cap B_r(z))$$

$$m_z(r) := \mathrm{MS}(u, B_r(z)), \quad \text{and} \quad h_z(r) := e_z(r) + \frac{1}{2}\ell_z(r).$$

Clearly $m_z(r) = e_z(r) + \ell_z(r) \leq 2h_z(r)$, with equality if and only if $e_z(r) = 0$.

Theorem 1.1. *Let $u \in \mathcal{M}(\Omega)$. Then*

$$\frac{m_z(r)}{r} \geq 1 \qquad \textit{for all } z \in \overline{S_u} \textit{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{1.4}$$

More precisely, the set $\Omega_u := \{z \in \Omega : (1.4) \text{ fails}\}$ is open and $\Omega_u = \Omega \setminus \overline{J_u} = \Omega \setminus \overline{S_u}$.

The quantity $m_z(\cdot)$ in Theorem 1.1 allows us to take advantage of a suitable monotonicity formula, discovered independently by David and Léger in [9] and Maddalena and Solimini in [13]. A simple iteration of Theorem 1.1 gives a density lower bound as in (1.3) with an explicit constant θ_0.

Corollary 1.2. *If $u \in \mathcal{M}(\Omega)$, then $\mathcal{H}^1(\overline{S_u} \setminus J_u) = 0$ and*

$$\frac{\ell_z(r)}{2r} \geq \frac{\pi}{2^{24}} \qquad \textit{for all } z \in \overline{S_u} \textit{ and all } r \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{1.5}$$

A natural question is the sharpness of the estimates (1.4) and (1.5). The analysis performed by Bonnet [3] suggests that $\frac{\pi}{2^{24}}$ in (1.5) should be replaced by $\frac{1}{2}$ and 1 in (1.4) by 2. Note that the square root function $u(r, \theta) = \sqrt{\frac{2}{\pi} r} \cdot \sin(\theta/2)$ satisfies $\ell_0(r) = e_0(r) = r$ for all $r > 0$. Thus both the constants conjectured above would be sharp by [8, Section 62]. Unfortunately, we cannot prove any of them.

Instead, in Corollary 1.3 below we prove an infinitesimal version of (1.4) for quasi-minimizers of the Mumford-Shah energy, that is any function v in $SBV(\Omega)$ with $\mathrm{MS}(v) < +\infty$ and satisfying for some $\omega \geq 0$ and $\alpha > 0$ and for all balls $B_\rho(z) \subset \Omega$

$$\mathrm{MS}(v, B_\rho(z)) \leq \mathrm{MS}(w, B_\rho(z)) + \omega\, \rho^{1+\alpha} \qquad \text{whenever } \{w \neq v\} \subset\subset B_\rho(z). \tag{1.6}$$

We denote the class of quasi-minimizers satisfying (1.6) by $\mathcal{M}_\omega(\Omega)$.

Corollary 1.3. *Let $v \in \mathcal{M}_\omega(\Omega)$, then*

$$\overline{S_u} = \overline{J_u} = \left\{ z \in \Omega : \liminf_{r \downarrow 0^+} \frac{m_z(r)}{r} \geq \frac{2}{3} \right\}. \tag{1.7}$$

Let us finally mention that Bucur & Luckhaus, independently from us, have used a similar idea to the main one of Theorem 1.1 (see [4]). Moreover, in their paper they improve remarkably on this key idea obtaining some results in the spirit of Theorem 1.1 and Corollary 1.3 without our dimensional limitation.

Plan of the paper. In section 2 we prove Theorem 1.1. The main ingredient, i.e. the David-Léger-Maddalena-Solimini monotonicity formula is proved in Appendix A. In section 3 we prove the Corollaries 1.2 and 1.3. The latter needs three additional tools: a Poincaré-Wirtinger type inequality, a technical lemma on sequences of MS minimizers and a decay lemma, proved in Appendices B, C and D, respectively. The technical lemma and the decay lemma are well-known facts. The Poincaré-Wirtinger inequality instead refines some results obtained in [12]: it is to our knowledge new and might be of independent interest.

2. Main result

As already mentioned, the main ingredient of Theorem 1.1 is the following monotonicity formula discovered independently in [9] and in [13] (cp. with [9, Proposition 3.5]).

Lemma 2.1. *Let $u \in \mathcal{M}(\Omega)$, then for every $z \in \Omega$ and for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(z, \partial\Omega))$*

$$\int_{\partial B_r(z)} \left(\left(\frac{\partial u}{\partial \nu}\right)^2 - \left(\frac{\partial u}{\partial \tau}\right)^2 \right) d\mathcal{H}^1 + \frac{\ell_z(r)}{r} = \frac{1}{r} \int_{J_u \cap \partial B_r(z)} |\langle \nu_u^\perp(x), x\rangle| d\mathcal{H}^0(x), \tag{2.1}$$

$\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ being the projections of ∇u in the normal and tangential directions to $\partial B_r(z)$, respectively.

We will also need the following elementary well-known facts.

Lemma 2.2. *Every $u \in \mathcal{M}(\Omega)$ is locally bounded and*

$$\mathrm{MS}(u, B_r(z)) \leq 2\pi r \qquad \text{for all } B_r(z) \subset \Omega. \tag{2.2}$$

We are now ready to prove the main result of the paper.

Proof of Theorem 1.1. Introduce the set $J_u^\star$ of points $x \in J_u$ for which

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(J_u \cap B_r(x))}{2r} = 1\,. \tag{2.3}$$

Since J_u is rectifiable, $\mathcal{H}^1(J_u \setminus J_u^\star) = 0$. Next let $z \in \Omega$ be such that

$$m_z(R) < R \qquad \text{for some } R \in (0, \operatorname{dist}(z, \partial\Omega)). \tag{2.4}$$

We claim that $z \notin J_u^\star$. W.l.o.g. we take $z = 0$ and drop the subscript z in e, ℓ, m and h.

In addition we can assume $e(R) > 0$. Otherwise, by the Co-Area formula and the trace theory of BV functions, we would find a radius $r < R$ such that $u|_{\partial B_r}$ is a constant. In turn, u would necessarily be constant in B_r because the energy decreases under truncations, thus implying $z \notin J_u^\star$. We can also assume $\ell(R) > 0$, since otherwise u would be harmonic in B_R and thus we would conclude $z \notin J_u^\star$.

We start next to compare the energy of u with that of an harmonic competitor on a suitable disk. The inequality $\ell(R) \leq m(R) < R$ is crucial to select good radii.

Step 1: For any fixed $r \in (0, R - \ell(R))$, there exists a set I_r of positive length in (r, R) such that

$$\frac{h(\rho)}{\rho} \leq \frac{1}{2} \cdot \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \qquad \text{for all } \rho \in I_r. \tag{2.5}$$

Define $J_r := \{t \in (r, R) : \mathcal{H}^0(S_u \cap \partial B_t) = 0\}$. We claim the existence of $J_r' \subseteq J_r$ with $\mathcal{L}^1(J_r') > 0$ and such that

$$\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \qquad \text{for all } \rho \in J_r'. \tag{2.6}$$

Indeed, we use the Co-Area formula for rectifiable sets (see [2, Theorem 2.93]) to find

$$\mathcal{L}^1((r,R) \setminus J_r) \leq \int_{(r,R)\setminus J_r} \mathcal{H}^0(S_u \cap \partial B_t) dt = \int_{S_u \cap (B_R \setminus \overline{B_r})} \left| \langle \nu_u^\perp(x), \frac{x}{|x|} \rangle \right| d\mathcal{H}^1(x) \leq \ell(R) - \ell(r).$$

In turn, this inequality implies $\mathcal{L}^1(J_r) \geq R - r - (\ell(R) - \ell(r)) > 0$, thanks to the choice of r. Then, define J_r' to be the subset of radii $\rho \in J_r$ for which

$$\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \leq \fint_{J_r} \left(\int_{\partial B_t} |\nabla u|^2 d\mathcal{H}^1 \right) dt\,.$$

Formula (2.6) follows by the Co-Area formula and the estimate $\mathcal{L}^1(J_r) \geq R - r - (\ell(R) - \ell(r))$.

We define I_r as the subset of radii $\rho \in J_r'$ satisfying both (2.1) and (2.6). Therefore

$$\int_{\partial B_\rho} \left(\frac{\partial u}{\partial \tau} \right)^2 d\mathcal{H}^1 = \frac{1}{2} \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 + \frac{\ell(\rho)}{2\rho} \qquad \forall \rho \in I_r. \tag{2.7}$$

Clearly, I_r has full measure in J_r', so that $\mathcal{L}^1(I_r) > 0$.

For any $\rho \in I_r$, we let w be the harmonic function in B_ρ with trace u on ∂B_ρ. Then, as $\frac{\partial w}{\partial \tau} = \frac{\partial u}{\partial \tau}\ \mathcal{H}^1$ a.e. on ∂B_ρ, the local minimality of u entails

$$m(\rho) \le \int_{B_\rho} |\nabla w|^2 dx \le \rho \int_{\partial B_\rho} \left(\frac{\partial u}{\partial \tau}\right)^2 d\mathcal{H}^1 \overset{(2.7)}{=} \frac{\rho}{2}\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 + \frac{\ell(\rho)}{2}.$$

The inequality (2.5) follows from the latter inequality and from (2.6):

$$h(\rho) = e(\rho) + \frac{\ell(\rho)}{2} \le \frac{\rho}{2}\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \le \frac{\rho}{2}\cdot \frac{e(R)-e(r)}{R-r-(\ell(R)-\ell(r))}.$$

Step 2: We now show that $0 \notin J_u^\star$.

Let $\varepsilon \in (0,1)$ be fixed such that $m(R) \le (1-\varepsilon)R$, and fix any radius $r \in (0, R - \ell(R) - \frac{1}{1-\varepsilon}e(R))$. Step 1 and the choice of r then imply

$$\frac{h(\rho)}{\rho} \le \frac{1}{2}\frac{e(R)-e(r)}{R-r-(\ell(R)-\ell(r))} \le \frac{e(R)}{2(R-\ell(R)-r)} < \frac{1-\varepsilon}{2},$$

in turn giving $m(\rho) \le 2h(\rho) < (1-\varepsilon)\rho$. Let $\rho_\infty := \inf\{t > 0 : m(t) \le (1-\varepsilon)t\}$, then $\rho_\infty \in [0,\rho]$. Note that if ρ_∞ were strictly positive then actually ρ_∞ would be a minimum. In such a case, we could apply the argument above and find $\widetilde{\rho} \in (r_\infty, \rho_\infty)$, with $r_\infty \in (0, \rho_\infty - \ell(\rho_\infty) - \frac{1}{1-\varepsilon}e(\rho_\infty))$, such that $m(\widetilde{\rho}) < (1-\varepsilon)\widetilde{\rho}$ contradicting the minimality of ρ_∞. Hence, there is a sequence $\rho_k \downarrow 0^+$ with $m(\rho_k) \le (1-\varepsilon)\rho_k$. Then, clearly condition (2.3) is violated, so that $0 \notin J_u^\star$.

Conclusion: We first prove that Ω_u is open. Let $z \in \Omega_u$ and let $R > 0$ and $\varepsilon > 0$ be such that $m_z(R) \le (1-\varepsilon)R$ and $B_{\varepsilon R}(z) \subset \Omega$. Let now $x \in B_{\varepsilon R}(z)$, then

$$m_x(R - |x-z|) \le m_z(R) \le (1-\varepsilon)R < R - |x-z|,$$

therefore $x \in \Omega_u$.

As $J_u^\star \cap \Omega_u = \emptyset$ by Step 2, we have $\mathcal{H}^1(J_u^\star \cap \Omega_u) = \mathcal{H}^1(J_u \cap \Omega_u) = \mathcal{H}^1(S_u \cap \Omega_u) = 0$. Hence, u is in $W^{1,2}$ of the open set Ω_u, and by minimality it is actually harmonic there. Thus, $S_u \cap \Omega_u = \emptyset$ and $\overline{S_u} \subseteq \Omega \setminus \Omega_u$. Moreover, let $z \notin \overline{J_u^\star}$ and $r > 0$ be such that $B_r(z) \subseteq \Omega \setminus \overline{J_u^\star}$. Since $\mathcal{H}^1(S_u \setminus \overline{J_u^\star}) = 0$, $u \in W^{1,2}(B_r(z))$ and thus u is an harmonic function in $B_r(z)$ by minimality. Therefore $z \in \Omega_u$, and in conclusion $\Omega \setminus \Omega_u = \overline{J_u^\star} = \overline{J_u} = \overline{S_u}$. □

Remark 2.3. The same arguments of Theorem 1.1 complemented by Theorem 3.1 show that

$$\Omega \setminus \overline{J_u} = \{z \in \Omega : m_z(R) \le R \text{ for some } R \in (0, d(z,\partial\Omega))\}. \tag{2.8}$$

Indeed, assuming $z = 0$ and dropping the subscript z, if $e(R) = 0$ or $\ell(R) = 0$, then $0 \in \Omega \setminus \overline{J_u}$. In the former case, the assertion follows since u is constant on B_ρ for some $\rho \in (0,R)$ by Theorem 3.1; in the latter case, u is harmonic on B_R by minimality. Hence, both $e(R)$ and $\ell(R)$ are in $(0,R)$. By Step 1 in Theorem 1.1 we have $h(\rho) \le \rho/2$ for some $\rho \in (0,R)$. If $e(\rho) = 0$ then $0 \in \Omega \setminus \overline{J_u}$, otherwise, $m(\rho) < 2h(\rho) \le \rho$. In the last instance, we are back to Theorem 1.1, so that $0 \in \Omega \setminus \overline{J_u}$. In any case, the set on the rhs of (2.8) is contained in $\Omega \setminus \overline{J_u}$. The opposite inclusion is trivial.

3. Proof of Corollaries 1.2 and 1.3

Proof of Corollary 1.2. Assume by contradiction that (1.5) fails for some $z \in \overline{S_u}$ and some $R_1 \in (0, \mathrm{dist}(z, \partial\Omega))$. W.l.o.g. we take $z = 0 \in S_u$ and drop the subscript z in e, ℓ, m and h.

Note that $R_1/4 - \ell(R_1) > R_1/8$ since $\ell(R_1) < 2\pi R_1/2^{24} < R_1/8$. Then, choosing $r_1 \in (R_1/8, R_1/4 - \ell(R_1))$ we have $2(R_1 - \ell(R_1) - r_1) > 3R_1/2$, and by applying Step 1 in Theorem 1.1 we infer, by (2.2),

$$\frac{h(\rho_1)}{\rho_1} \leq \frac{1}{2(R_1 - \ell(R_1) - r_1)} e(R_1) < \frac{2}{3} \frac{e(R_1)}{R_1} \leq \frac{4}{3}\pi$$

for some $\rho_1 \in (r_1, R_1)$. Note that

$$\frac{\ell(\rho_1)}{2\rho_1} \leq \frac{R_1}{\rho_1} \frac{\ell(R_1)}{2R_1} < 8 \frac{\ell(R_1)}{2R_1} < \frac{\pi}{2^{21}} < \frac{1}{16}.$$

Hence, we may use again Step 1 of Theorem 1.1 with the new radii $R_2 = \rho_1$, and r_2 satisfying $r_2 \in (R_2/8, R_2/4 - \ell(R_2))$ accordingly. Then, for some $\rho_2 \in (r_2, R_2)$ we get

$$\frac{h(\rho_2)}{\rho_2} \leq \frac{1}{2(R_2 - \ell(R_2) - r_2)} e(R_2) < \frac{2}{3} \frac{e(R_2)}{R_2} \Longrightarrow \frac{h(\rho_2)}{\rho_2} \leq \left(\frac{2}{3}\right)^2 2\pi.$$

In general, for $2 \leq k \leq 7$ given R_{k-1}, r_{k-1} and ρ_{k-1} set $R_k := \rho_{k-1}$, choose r_k such that $r_k \in (R_k/8, R_k/4 - \ell(R_k))$, and use Step 1 of Theorem 1.1 to find $\rho_k \in (r_k, R_k)$ satisfying

$$\frac{h(\rho_k)}{\rho_k} \leq \left(\frac{2}{3}\right)^j 2\pi.$$

Note that for any $2 \leq k \leq 6$

$$\frac{\ell(\rho_k)}{2\rho_k} < 8 \frac{\ell(\rho_{k-1})}{2\rho_{k-1}} < \frac{\pi}{2^{3(8-k)}} < \frac{1}{16},$$

and thus the construction is well defined. In addition,

$$\frac{h(\rho_7)}{\rho_7} \leq \left(\frac{2}{3}\right)^7 2\pi < \frac{1}{2} \Longrightarrow m(\rho_7) \leq 2h(\rho_7) < \rho_7.$$

From Theorem 1.1 we deduce that $0 \notin S_u$, which gives clearly a contradiction.

Eventually, standard density estimates imply $\mathcal{H}^1\left(\overline{S_u} \setminus S_u\right) = 0$ (cp. with [2, Theorem 2.56]), and being $\overline{S_u} = \overline{J_u}$ (see Theorem 1.1) we get $\mathcal{H}^1\left(\overline{S_u} \setminus J_u\right) = 0$. □

In the proof of Corollary 1.3 we will need a Poincaré-Wirtinger type inequality (see Appendix B), and a closure theorem for minimizers of the Mumford-Shah energy.

Theorem 3.1. *Let $u \in \mathcal{M}(B_R)$ with $\mathcal{H}^1(S_u) < 2R$, and let $\lambda \in (0,1)$. Then, $u \in L^\infty(B_\rho)$ for some $\rho \in (\lambda(R - \mathcal{H}^1(S_u)/2), R)$, and for any median* $\mathrm{med}(u)$ *of u on B_R we have*

$$\|u - \mathrm{med}(u)\|_{L^\infty(B_\rho)} \leq \frac{2}{2R - \mathcal{H}^1(S_u)} \|\nabla u\|_{L^1(B_R, \mathbb{R}^2)}.$$

Proposition 3.2. *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ be a sequence converging to some $u \in SBV(\Omega)$ strongly in L^2. Then $u \in \mathcal{M}(\Omega)$ and for all open sets $A \subseteq \Omega$ we have*

$$\lim_k \int_A |\nabla u_k|^2 dx = \int_A |\nabla u|^2 dx, \quad \lim_k \mathcal{H}^1(J_{u_k} \cap A) = \mathcal{H}^1(J_u \cap A). \tag{3.1}$$

Furthermore, $(\overline{J_{u_k}})_{k\in\mathbb{N}}$ converges locally in the Hausdorff distance to $\overline{J_u}$.

We will also take advantage of the following decay lemma inspired by [10, Lemma 4.9] (cp. also with [2, Lemma 7.14, Theorem 7.21]) and proved in Appendix D.

Lemma 3.3. *For all $\omega \geq 0$, $\beta \in (0,1]$ and $\tau \in (0,1)$ there exist $\varepsilon = \varepsilon(\beta,\tau) \in (0,1)$ and $R = R(\beta,\tau) > 0$ such that if $v \in \mathcal{M}_\omega(\Omega)$ satisfies*

$$\mathrm{MS}(v, B_\rho(z)) \leq \varepsilon\rho,$$

for some $z \in \Omega$ and $\rho \in (0, (R/\omega^{1/\alpha}) \wedge \mathrm{dist}(z, \partial\Omega))$, then for all $k \geq 1$

$$\mathrm{MS}(v, B_{\tau^k\rho}(z)) \leq \tau^{k+1-\beta}\,\varepsilon\rho.$$

Proof of Corollary 1.3. Denote by Ω_v the complement of the set on the rhs of (1.7). We first show that $\Omega_v = \Omega \setminus \overline{J_v^\star}$, where as usual $J_v^\star$ is the subset of points $z \in J_v$ for which

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(J_u \cap B_r(z))}{2r} = 1.$$

Let $z \in \Omega \setminus \overline{J_v^\star}$, then $v \in W^{1,2}(B_R(z))$ for some $R > 0$. Observe $v|_{\partial B_\rho(z)} \in W^{1,2}(\partial B_R(z))$ for $\mathcal{L}^1$ a.e. $\rho \in (0,R)$. Testing the quasi-minimality condition (1.6) with the harmonic extension φ of $v|_{\partial B_\rho(z)}$ to $B_\rho(z)$, Lemma 2.1 and the Co-Area formula yield

$$e_z(\rho) \leq \frac{\rho}{2}\, e_z'(\rho) + \omega\,\rho^{1+\alpha}.$$

Integrating this last inequality we get, for $\alpha \neq 1$,

$$e_z(\rho) \leq \left(\frac{\rho}{R}\right)^2 e_z(R) + \frac{2\omega}{\alpha-1}\,\rho^2\left(R^{\alpha-1} - \rho^{\alpha-1}\right), \tag{3.2}$$

from which we conclude $z \in \Omega_v$ since $m_z(\rho) = e_z(\rho) = o(\rho)$ as $\rho \downarrow 0^+$. Hence, $\Omega \setminus \overline{J_v^\star} \subseteq \Omega_v$. We can proceed analogously if $\alpha = 1$.

To prove the opposite inclusion, let $z \in \Omega_v$ and $r_k \downarrow 0^+$ be a sequence along which for some $\gamma \in (0, 2/3)$

$$\liminf_{r\downarrow 0^+} \frac{m_z(r)}{r} = \lim_{k\uparrow\infty} \frac{m_z(r_k)}{r_k} < \gamma. \tag{3.3}$$

Let m_k be a median of u on $B_{r_k}(z)$, and consider the functions $v_k : B_1 \to \mathbb{R}$ defined as $v_k(y) := r_k^{-1/2}(v(z + r_k y) - m_k)$. Note that $v_k \in \mathcal{M}_{\omega r_k^\alpha}(B_1)$. Let $\lambda \in (0,1)$ be a parameter whose choice will be specified later. Since $\mathcal{H}^1(J_{v_k}) < \gamma$ we apply Theorem B.6 to find functions $w_k : B_1 \to \mathbb{R}$ which are suitable truncations of v_k and such that, for all k,

$$\|w_k\|_{L^\infty(B_{\lambda(1-\gamma/2)})} \leq 2\|\nabla v_k\|_{L^1(B_1,\mathbb{R}^2)} \leq 2\pi^{1/2}\|\nabla v_k\|_{L^2(B_1,\mathbb{R}^2)} \overset{(2.2)}{\leq} 4\pi.$$

In particular, up to a subsequence, $(w_k)_{k\in\mathbb{N}}$ converges in $L^2(B_{\lambda(1-\gamma/2)})$ to a function w in $SBV(B_{\lambda(1-\gamma/2)})$ with $\mathrm{MS}(w, B_{\lambda(1-\gamma/2)}) < +\infty$ by Ambrosio's SBV compactness theorem (see [2, Theorems 4.7 and 4.8]).

We claim that for all open subsets A of B_1 it holds

$$0 \leq \mathrm{MS}(v_k, A) - \mathrm{MS}(w_k, A) \leq \omega\, r_k^\alpha. \tag{3.4}$$

Indeed, by the very definition of w_k we have $\{w_k \neq v_k\} \subset\subset B_1$ (cp. with formula (B.3) in Theorem B.6). Then, as $v_k \in \mathcal{M}_{\omega\, r_k^\alpha}(B_1)$, we get

$$\mathrm{MS}(v_k, B_1) - \mathrm{MS}(w_k, B_1) \leq \omega\, r_k^\alpha.$$

We conclude (3.4) by the latter estimate and since $\mathrm{MS}(w_k, B) \leq \mathrm{MS}(v_k, B)$ for all Borel subsets B of B_1 (recall that w_k is obtained from v_k by truncation).

Remark C.1 and (3.4) yield that $w \in \mathcal{M}(B_{\lambda(1-\gamma/2)})$, with

$$\mathrm{MS}(w, B_\rho) = \lim_{k\uparrow\infty} \mathrm{MS}(w_k, B_\rho) \qquad \text{for all } \rho \in (0, \lambda(1-\gamma/2)]. \tag{3.5}$$

By collecting (3.3), (3.4) and (3.5), we deduce for every $\rho \in (0, \lambda(1-\gamma/2)]$

$$\mathrm{MS}(w, B_\rho) = \lim_{k\uparrow\infty} \frac{m_z(\rho\, r_k)}{r_k} \leq \lim_{k\uparrow\infty} \frac{m_z(r_k)}{r_k} < \gamma \leq \lambda\left(1 - \frac{\gamma}{2}\right), \tag{3.6}$$

the last inequality holding true provided $\lambda \in (0,1)$ is suitably chosen (recall that $\gamma \in (0, 2/3)$).

In particular, if $\rho = \lambda(1-\gamma/2)$ from (3.6) we infer that $0 \notin \overline{S_w}$ in view of Remark 2.3. Hence, being w harmonic in $B_{\lambda(1-\gamma/2)}$ for every fixed $\rho \in (0, \lambda(1-\gamma/2)]$ we get

$$\frac{m_z(\rho\, r_k)}{\rho\, r_k} \leq 2\rho \qquad \text{for all } k \geq k_\rho, \tag{3.7}$$

so that $z \in \Omega \setminus J_v^\star$. Moreover, if $\varrho > 0$ is such that $4\varrho \leq \varepsilon \wedge (\lambda(1-\gamma/2)) \wedge (2/3)$ then $B_{\varrho\, r_{k_\varrho}/2}(z) \subseteq \Omega_v$. For, if $x \in B_{\varrho\, r_{k_\varrho}/2}(z)$, by Lemma 3.3 applied with $\tau = 1/2$, any $\beta \in (0,1)$ and $\rho = \varrho\, r_{k_\varrho}$, the choice of ϱ yields that

$$\frac{m_x(\varrho\, r_{k_\varrho}/2)}{\varrho\, r_{k_\varrho}/2} \leq 2\frac{m_z(\varrho\, r_{k_\varrho})}{\varrho\, r_{k_\varrho}} \stackrel{(3.7)}{\leq} 4\varrho \leq \varepsilon,$$

and thus we deduce $x \in \Omega_v$ by iterating Lemma 3.3 along the sequence $(2^{-i}\varrho\, r_{k_\varrho})_{i\in\mathbb{N}}$. Hence, Ω_v is an open set and $\Omega_v \cap J_v^\star = \emptyset$, in turn this implies $\Omega \setminus \overline{J_v^\star} = \Omega_v$.

Finally, being Ω_v open and v a quasi-minimizer of the Dirichlet energy on Ω_v then $v \in C^{1,1/2}(\Omega_v)$ by (3.2) and Campanato's estimates. In conclusion, $S_v \cap \Omega_v = \emptyset$, and then $\overline{S_v} = \overline{J_v} = \Omega \setminus \Omega_v$. □

Appendix A. The David-Léger-Maddalena-Solimini monotonicity formula

Proof of Lemma 2.1. We start by recalling the first variation formula for local minimizers of the Mumford-Shah energy (see [2, Section 7.4]): for every vector field $\eta \in \mathrm{Lip} \cap C_c(\Omega, \mathbb{R}^2)$

$$\int_\Omega \left(|\nabla u|^2 \mathrm{div}\eta - 2\langle \nabla u, \nabla u \nabla\eta\rangle\right) dx + \int_{J_u} \mathrm{div}^{J_u}\eta\, d\mathcal{H}^1 = 0. \tag{A.1}$$

With fixed a point $z \in \Omega$, $r > 0$ with $B_r(z) \subseteq \Omega$, we consider special radial vector fields $\eta_{r,s} \in \mathrm{Lip} \cap C_c(B_r(z), \mathbb{R}^2)$, $s \in (0, r)$, in formula above. For the sake of simplicity we assume $z = 0$, and drop the subscript z in what follows. Let

$$\eta_{r,s}(x) := x\, \chi_{[0,s]}(|x|) + \frac{|x| - r}{s - r}\, x\, \chi_{(s,r]}(|x|),$$

then routine calculations leads to

$$\nabla \eta_{r,s}(x) := \mathrm{Id}\, \chi_{[0,s]}(|x|) + \left(\frac{|x| - r}{s - r} \mathrm{Id} + \frac{1}{s - r} \frac{x}{|x|} \otimes x \right) \chi_{(s,r]}(|x|)$$

$\mathcal{L}^2$ a.e. in Ω. In turn, from the latter formula we infer for $\mathcal{L}^2$ a.e. in Ω

$$\mathrm{div} \eta_{r,s}(x) = 2\, \chi_{[0,s]}(|x|) + \left(2 \frac{|x| - r}{s - r} + \frac{|x|}{s - r} \right) \chi_{(s,r]}(|x|),$$

and, if $\nu_u(x)$ is a unit vector normal field in $x \in J_u$, for $\mathcal{H}^1$ a.e. $x \in J_u$

$$\mathrm{div}^{J_u} \eta_{r,s}(x) = \chi_{[0,s]}(|x|) + \left(\frac{|x| - r}{s - r} + \frac{1}{|x|(s - r)} |\langle x, \nu_u^\perp \rangle|^2 \right) \chi_{(s,r]}(|x|).$$

Consider the set $I := \{\rho \in (0, \mathrm{dist}(0, \partial\Omega)) : \mathcal{H}^1(J_u \cap \partial B_\rho) = 0\}$, then $(0, \mathrm{dist}(0, \partial\Omega)) \setminus I$ is at most countable being $\mathcal{H}^1(J_u) < +\infty$. If ρ and $s \in I$, by inserting η_s in (A.1) we find

$$\frac{1}{s - r} \int_{B_r \setminus B_s} |x| |\nabla u|^2 dx - \frac{2}{s - r} \int_{B_r \setminus B_s} |x| \langle \nabla u, \left(\mathrm{Id} - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \nabla u \rangle dx$$
$$= \ell(s) + \int_{J_u \cap (B_r \setminus B_s)} \frac{|x| - r}{s - r} d\mathcal{H}^1 + \frac{1}{s - r} \int_{J_u \cap (B_r \setminus B_s)} |x| |\langle \frac{x}{|x|}, \nu_u^\perp \rangle|^2 d\mathcal{H}^1.$$

Next we employ Co-Area formula and rewrite equality above as

$$\frac{1}{s - r} \int_s^r \rho\, d\rho \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 - \frac{2}{s - r} \int_s^r \rho\, d\rho \int_{\partial B_\rho} \left| \frac{\partial u}{\partial \tau} \right|^2 d\mathcal{H}^1$$
$$= \ell(s) + \int_{J_u \cap (B_r \setminus B_s)} \frac{|x| - r}{s - r} d\mathcal{H}^1 + \frac{1}{s - r} \int_s^r d\rho \int_{J_u \cap \partial B_\rho} |\langle x, \nu_u^\perp \rangle| d\mathcal{H}^0$$

where $\nu := x/|x|$ denotes the radial versor and $\tau := \nu^\perp$ the tangential one. Lebesgue differentiation theorem then provides a subset I' of full measure in I such that if $r \in I'$ and we let $s \uparrow t^-$ it follows

$$-r \int_{\partial B_r} |\nabla u|^2 d\mathcal{H}^1 + 2r \int_{\partial B_r} \left| \frac{\partial u}{\partial \tau} \right|^2 d\mathcal{H}^1 = \ell(r) - \int_{J_u \cap \partial B_r} |\langle x, \nu_u^\perp \rangle| d\mathcal{H}^0.$$

Formula (2.1) then follows straightforwardly. □

Appendix B. A Poincaré-Wirtinger type inequality

The arguments of this appendix refine a truncation procedure introduced by [12] (cp. with [12, Lemma 4.2, Theorem 4.1]). In what follows given any $\mathcal{L}^2$-measurable function $v : B_R \to \mathbb{R}$, for every $s \in \mathbb{R}$, we denote by $E_{v,s}$ the s sub-level of v in B_R, i.e.,

$$E_{v,s} := \{x \in B_R : v(x) \leq s\}, \tag{B.1}$$

and by $\mathrm{med}(v)$ a *median* of v in B_R, for instance we can take

$$\mathrm{med}(v) := \sup\{s \in \mathbb{R} : \mathcal{L}^2(E_{v,s}) \leq \mathcal{L}^2(B_R)/2\}. \tag{B.2}$$

Let us begin with the truncation procedure for functions in SBV with zero gradient.

Lemma B.1. *For every $v \in SBV(B_R)$ with $\nabla v = 0$ $\mathcal{L}^2$ a.e. B_R and $\mathcal{H}^1(S_v) < 2R$, the set $I = \{r \in (0,R) : \mathcal{H}^0(\partial B_t \cap S_v) = 0\}$ satisfies $\mathcal{L}^1(I) \geq R - \mathcal{H}^1(S_v)/2$.*

In addition, for $\mathcal{L}^1$ a.e. $r \in I$ the trace of v on ∂B_R is constant.

Proof. Set $J := \{r \in (0,R) : \mathcal{H}^0(\partial B_t \cap S_v) \geq 2\}$, and estimate $\mathcal{L}^1(J)$ by means of the Co-Area formula for rectifiable sets as follows

$$2\mathcal{L}^1(J) \leq \int_J \mathcal{H}^0(\partial B_t \cap S_v)\, dt \leq \mathcal{H}^1(S_v),$$

from which we infer $\mathcal{L}^1((0,R) \setminus J) \geq R - \mathcal{H}^1(S_v)/2$.

To conclude we prove the inequality $\mathcal{L}^1((0,R) \setminus J) \leq \mathcal{L}^1(I)$. To this aim note that for $\mathcal{L}^1$ a.e $r \in (0,R) \setminus J$ the slice v_r obtained by restricting v to ∂B_r belongs to $SBV(\partial B_r)$, it has zero approximate derivative and $\partial B_r \cap S_v = S_{v_r}$ (see [2, Section 3.11]). Finally, since $\#(\partial B_r \cap S_v) \leq 1$ as $r \in (0,R) \setminus J$, by taking into account that $v_r' = 0$ $\mathcal{H}^1$ a.e. on ∂B_r, we infer that actually $\partial B_r \cap S_v = \emptyset$. In conclusion, $\mathcal{L}^1((0,R) \setminus (I \cup J)) = 0$. □

Remark B.2. The estimate $\mathcal{L}^1(I) \geq R - \mathcal{H}^1(S_v)/2$ proved in Lemma B.1 above, clearly implies that $\mathcal{L}^1(I \cap (\lambda(R - \mathcal{H}^1(S_v)/2), R)) > 0$ for all $\lambda \in (0,1)$.

In what follows we identify any set of finite perimeter E with its $\mathcal{L}^2$-measure theoretic interior defined by $E^{(1)} := \{x \in \mathbb{R}^2 : \lim_{t \to 0+} (\pi t^2)^{-1} \mathcal{L}^2(B_t(x) \cap E) = 1\}$. Recall that $\partial^* E$ denotes the *essential boundary* of E, satisfying $\mathrm{Per}(E) = \mathcal{H}^1(\partial^* E)$ (see [2, Definition 3.60, Theorem 3.61]).

In particular, from Lemma B.1 we immediately deduce the following corollary.

Corollary B.3. *For every set of finite perimeter $E \subseteq B_R$ with $\mathrm{Per}(E) < 2R$ a set of positive $\mathcal{L}^1$ measure in $(0,R)$ exists such that either $\mathcal{H}^1(E \cap \partial B_t) = 0$ or $\mathcal{H}^1(E \cap \partial B_t) = \mathcal{H}^1(\partial B_t)$, for all t in this set.*

Under an additional smallness condition on the $\mathcal{L}^2$ measure of E, the previous result can be further improved (cp. to [12, Lemma 4.2]). To this aim we recall that a set of finite perimeter $E \subset \mathbb{R}^2$ is said to be *decomposable* if there exists a partition of E in two $\mathcal{L}^2$-measurable sets A, B with strictly positive measure such that $\mathrm{Per}(E) = \mathrm{Per}(A) + \mathrm{Per}(B)$. Accordingly, a set of finite perimeter is *indecomposable* otherwise. Notice that the properties of being decomposable or indecomposable depend only on the $\mathcal{L}^2$-equivalence class of E.

Lemma B.4. *If $E \subseteq B_R$ is such that $\mathcal{L}^2(E) \leq \mathcal{L}^2(B_R)/2$ and $\mathrm{Per}(E) < 2R$, the set $\mathcal{I} := \{t \in (0,R) : \mathcal{H}^1(\partial B_t \cap E) = 0\}$ satisfies $\mathcal{L}^1(\mathcal{I}) \geq R - \mathrm{Per}(E)/2$.*

Proof. According to [1, Theorem 1] there exists a unique and at most countable family of pairwise disjoint (maximal) indecomposable sets E_i, $i \in I \subseteq \mathbb{N}$, with $\mathcal{L}^2(E_i) > 0$ such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i\in I} E_i\right) = 0 \quad \text{and} \quad \mathrm{Per}(E) = \sum_{i\in I} \mathrm{Per}(E_i).$$

An elementary projection argument shows that $2d_i := 2\mathrm{diam}(E_i) \leq \mathrm{Per}(E_i)$, so that

$$2\sum_{i\in I} d_i \leq \sum_{i\in I} \mathrm{Per}(E_i) = \mathrm{Per}(E) < 2R.$$

Let now $\mathcal{I}_i := \{t \in (0,R) : \mathcal{H}^1(\partial B_t \cap E_i) = 0\}$, and note that $\mathcal{I} = \cap_{i\in I}\mathcal{I}_i$. In addition, since for all $\varepsilon > 0$ the sets E_i are contained in $B_{R_i+d_i+\varepsilon} \setminus \overline{B}_{R_i-\varepsilon}$ for some $R_i > 0$, we get that

$$\mathcal{L}^1((0,R) \setminus \mathcal{I}) \leq \sum_{i\in I} \mathcal{L}^1((0,R) \setminus \mathcal{I}_i) \leq \sum_{i\in I} d_i,$$

from which, finally, we infer

$$\mathcal{L}^1(\mathcal{I}) \geq R - \sum_{i\in I} d_i \geq R - \frac{\mathrm{Per}(E)}{2}.$$

□

Remark B.5. The estimate $\mathcal{L}^1(\mathcal{I}) \geq R - \mathrm{Per}(E)/2 > 0$ proved in Lemma B.4 above, clearly implies that $\mathcal{L}^1(\mathcal{I} \cap (\lambda(R - \mathrm{Per}(E)/2), R)) > 0$ for all $\lambda \in (0,1)$.

From Lemmata B.1 and B.4 we infer that SBV functions with suitably quantified short jump set enjoy a Poincaré-Wirtinger type inequality.

Theorem B.6 (A Poincaré-Wirtinger type inequality). *If $v \in SBV(B_R)$ with $\mathcal{H}^1(S_v) < 2R$, then there are truncation levels $s' \leq s''$ and for all $\lambda \in (0,1)$ radii $\rho' \leq \rho''$ belonging to $(\lambda(R - \mathcal{H}^1(S_v)/2), R)$ in a way that the function*

$$w := \begin{cases} v \vee s' \wedge s'' & B_{\rho'} \\ v \wedge s'' & B_{\rho''} \setminus B_{\rho'} \\ v & B_R \setminus B_{\rho''}, \end{cases} \tag{B.3}$$

satisfies $\mathcal{H}^1(S_w \setminus S_v) = 0$ and for any median $\mathrm{med}(v)$ of v on B_R

$$\|w - \mathrm{med}(v)\|_{L^\infty(B_{\rho'})} \leq \frac{2}{2R - \mathcal{H}^1(S_v)} \|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}.$$

Proof. First note that if $\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)} = 0$ we may apply Lemma B.1 and select $\rho \in (R/2 - \mathcal{H}^1(J_v)/4, R)$ (thanks to Remark B.2) such that the trace of v on ∂B_ρ is constant. In this case we take $s' = s''$ equal to such a value and $\rho = \rho' = \rho''$ to conclude.

Thus, we need to analyze only the case with $\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)} > 0$. To this aim set $\alpha := 2R - \mathcal{H}^1(S_v) > 0$, then the BV Co-Area Formula (see [2, Theorem 3.40]) implies

$$\int_{\mathrm{med}(v)-2\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}/\alpha}^{\mathrm{med}(v)} \mathcal{H}^1(\partial^* E_s \setminus S_v)\, ds \leq \int_{\mathbb{R}} \mathcal{H}^1(\partial^* E_s \setminus S_v)\, ds = \|\nabla v\|_{L^1(B_R,\mathbb{R}^2)},$$

where E_s is the sub-level of v in B_R defined in (B.1) and $\mathrm{med}(v)$ is defined in (B.2). Hence, by the Mean Value Theorem there exists $s' \in (\mathrm{med}(v) - 2\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}/\alpha, \mathrm{med}(v))$ such that $\mathcal{H}^1(\partial^* E_{s'} \setminus S_v) \leq \alpha/2$, and so

$$\mathcal{H}^1(\partial^* E_{s'}) \leq \mathcal{H}^1(\partial^* E_{s'} \setminus S_v) + \mathcal{H}^1(S_v) < 2R. \tag{B.4}$$

Analogously, we can find $s'' \in (\mathrm{med}(v), \mathrm{med}(v) + 2\|\nabla v\|_{L^1(B_R,\mathbb{R}^2)}/\alpha)$ such that

$$\mathcal{H}^1(\partial^* E_{s''}) < 2R. \tag{B.5}$$

The definition of median (B.2) and the choice $s' < \mathrm{med}(v)$ yield $\mathcal{L}^2(E_{s'}) \leq \mathcal{L}^2(B_R)/2$, and by arguing similarly, the same inequality holds for the set $B_R \setminus E_{s''}$ as well. By taking into account inequalities (B.4), (B.5) we may apply Lemma B.4 separately to the two sets $E_{s'}$, $B_R \setminus E_{s''}$ and find radii $\lambda(R - \mathcal{H}^1(S_v)/2) < \rho' \leq \rho'' < R$ with $\mathcal{H}^1(E_{s'} \cap \partial B_{\rho'}) = 0$ and $\mathcal{H}^1((B_R \setminus E_{s''}) \cap \partial B_{\rho''}) = 0$ (thanks to Remark B.5).

The conclusion then follows at once by the very definition of w in (B.3). □

In case v is a local minimizer of the Mumford-Shah energy we deduce Theorem 3.1.

Proof of Theorem 3.1. By keeping the notation of Theorem B.6, the function w defined in (B.3) turns out to be an admissible function to test the minimality of u on B_R. By construction $\mathcal{H}^1(S_w \setminus S_u) = 0$ and $|\nabla w| \leq |\nabla u|$ $\mathcal{L}^2$ a.e. in B_R, from this we infer that $u = w$ $\mathcal{L}^2$ a.e. in $B_{\rho'}$ being the Mumford-Shah energy decreasing under truncation. □

Remark B.7. If the length of the jump set exceeds $2R$ a similar Poincaré-Wirtinger type inequality does not hold. Take, for instance, $v = 1$ if $y > 0$ and -1 otherwise (see [2, Proposition 6.8] for a proof that such a function is in $\mathcal{M}(B_R)$ if R is sufficiently small).

Appendix C. Limits of sequences of local minimizers

In this section we prove that limits of converging sequences of local minimizers are local minimizers as well (cp. with [2, Theorem 7.7] in case the measure of the jump sets is vanishing, and with [11, Proposition 5.1] if the Dirichlet energies are infinitesimal).

Proof of Proposition 3.2. Let v be an admissible function to test the minimality of u, that is $v \in SBV(\Omega)$ and $\{v \neq u\} \subset\subset \Omega$. Moreover, let Ω' be an open set such that $\{v \neq u\} \subset\subset \Omega' \subset\subset \Omega$ and $\varphi \in C^1_c(\Omega)$ be such that $\varphi = 1$ on Ω' and $|\nabla\varphi| \leq 2/\mathrm{dist}(\Omega', \partial\Omega)$. Define

$v_k := \varphi\, v + (1-\varphi)u_k$. Then $v_k \in SBV(\Omega)$ and it is an admissible test function for u_k. Thus, for some fixed constant $C > 0$, routine calculations lead to

$$\mathrm{MS}(u_k) \le \mathrm{MS}(v_k) \le \mathrm{MS}(v) + C\,\mathrm{MS}(v, \Omega\setminus\overline{\Omega'}) + C\,\mathrm{MS}(u_k, \Omega\setminus\overline{\Omega'}) + C\int_{\Omega\setminus\overline{\Omega'}} |u-u_k|^2 dx. \quad \text{(C.1)}$$

To get the last term on the rhs above we have used the equality $v = u$ on $\Omega \setminus \overline{\Omega'}$.

Note that the sequence of Radon measures $(\mathrm{MS}(u_k, \cdot))_{k\in\mathbb{N}}$ is equi-bounded in mass in view of the energy upper bound (2.2). Hence, up to the extraction of a subsequence (not relabeled), $(\mathrm{MS}(u_k, \cdot))_{k\in\mathbb{N}}$ converges to some Radon measure μ on Ω. Without loss of generality we may also assume that $\mu(\partial\Omega') = 0$. Furthermore, we recall that, by Ambrosio's lower semicontinuity theorem, we have, for every open set $A \subseteq \Omega$,

$$\liminf_k \int_A |\nabla u_k|^2 dx \ge \int_A |\nabla u|^2 dx, \quad \liminf_k \mathcal{H}^1(J_{u_k} \cap A) \ge \mathcal{H}^1(J_u \cap A), \qquad \text{(C.2)}$$

(see [2, Theorems 4.7 and 4.8]). As $k \uparrow \infty$ in (C.1), thanks to condition $\mu(\partial\Omega') = 0$ and (C.2), we find

$$\mathrm{MS}(u) \le \liminf_k \mathrm{MS}(u_k) \le \limsup_k \mathrm{MS}(u_k) \le \mathrm{MS}(v) + C\,\mathrm{MS}(v, \Omega \setminus \overline{\Omega'}) + C\,\mu(\Omega \setminus \overline{\Omega'}).$$

Then, by letting Ω' increase to Ω (enforcing the condition $\mu(\partial\Omega') = 0$) we conclude

$$\mathrm{MS}(u) \le \liminf_k \mathrm{MS}(u_k) \le \limsup_k \mathrm{MS}(u_k) \le \mathrm{MS}(v). \qquad \text{(C.3)}$$

Hence, u belongs to $\mathcal{M}(\Omega)$. In addition, by choosing v equal to u itself, we can perform the same construction above for every open set $A \subseteq \Omega$ (with $\Omega' \subset\subset A$) and infer (C.3) localized onto A, so that equalities in (3.1) follow at once.

Finally, the density lower bound in Corollary 1.2 and the equalities in (3.1) imply easily the claimed local Hausdorff convergence. □

Remark C.1. The same conclusion of Proposition 3.2 holds provided we are given a sequence $(u_k)_{k\in\mathbb{N}}$ converging in $L^2(\Omega)$ to $u \in SBV(\Omega)$, with u_k satisfying, for some $\vartheta_k \downarrow 0^+$,

$$\mathrm{MS}(u_k) \le \mathrm{MS}(w) + \vartheta_k \quad \text{whenever } \{w \ne u_k\} \subset\subset \Omega.$$

Appendix D. A decay Lemma

We start off by proving a preliminary decay property of the energy.

Lemma D.1. *For all $\beta \in (0, 2)$ and $\tau \in (0, 1)$ there exist $\varepsilon = \varepsilon(\beta, \tau)$ and $\vartheta = \vartheta(\beta, \tau)$ in $(0, 1)$ such that if $v \in SBV(\Omega)$ satisfies, for some $z \in \Omega$ and $\rho > 0$,*

$$\mathrm{MS}(v, B_\rho(z)) \le \varepsilon\rho,$$

and

$$(1-\vartheta)\,\mathrm{MS}(v, B_\rho(z)) \le \mathrm{MS}(w, B_\rho(z)) \quad \textit{whenever } \{w \ne v\} \subset\subset B_\rho(z),$$

then

$$\mathrm{MS}(v, B_{\tau\rho}(z)) \le \tau^{2-\beta}\,\mathrm{MS}(v, B_\rho(z)).$$

Proof. We argue by contradiction and suppose that there are sequences $v_k \in SBV(\Omega)$, $\varepsilon_k \downarrow 0^+$, $\vartheta_k \downarrow 0^+$, $\rho_k \downarrow 0^+$ and $z_k \in \Omega$ with $B_{\rho_k}(z_k) \subset \Omega$ such that for some τ and $\beta \in (0,2)$

$$\mathrm{MS}(v_k, B_{\rho_k}(z_k)) = \varepsilon_k \rho_k, \tag{D.1}$$

$$(1-\vartheta_k)\,\mathrm{MS}(v_k, B_{\rho_k}(z_k)) \le \mathrm{MS}(w, B_{\rho_k}(z_k)) \tag{D.2}$$

for all $w \in SBV(\Omega)$ with $\{w \neq v_k\} \subset\subset B_{\rho_k}(z_k)$, but

$$\mathrm{MS}(v_k, B_{\tau\rho_k}(z_k)) > \tau^{2-\beta}\,\mathrm{MS}(v_k, B_{\rho_k}(z_k)). \tag{D.3}$$

Denote by $w_k : B_1 \to \mathbb{R}$ the functions $w_k(y) = (\varepsilon_k\rho_k)^{-1/2}(v_k(z_k+\rho_k y) - m_k)$ and by m_k a median of v_k on $B_{\rho_k}(z_k)$, so that, if we set,

$$F_k(v, B_\rho) := \int_{B_\rho} |\nabla v|^2 dy + \frac{1}{\varepsilon_k}\mathcal{H}^1(S_v \cap B_\rho),$$

then (D.1), (D.2) and (D.3) can be rewritten respectively as

$$F_k(w_k, B_1) = 1, \quad F_k(w, B_1) \ge 1 - \vartheta_k, \quad \text{and} \quad F_k(w_k, B_\tau) > \tau^{2-\beta}, \tag{D.4}$$

for all $w \in SBV(B_1)$ with $\{w \neq w_k\} \subset\subset B_1$.

In particular, from the first condition in (D.4) we infer that $\mathcal{H}^1(S_{w_k}) \le \varepsilon_k$. Thus, by applying Theorem B.6 to the w_k's, we find functions $\widetilde{w}_k \in SBV(B_1)$ satisfying, for all $r \in (0,1)$,

$$\{\widetilde{w}_k \neq w_k\} \subset\subset B_r, \quad \|\widetilde{w}_k\|_{L^\infty(B_r)} \le 2 \quad \text{for } k \ge k_r. \tag{D.5}$$

Then, Ambrosio's SBV compactness theorem and a diagonal argument provide a subsequence (not relabeled) and a function $\widetilde{w} \in W^{1,2} \cap L^\infty(B_1)$ such that $(\widetilde{w}_k)_{k\in\mathbb{N}}$ converges to $\widetilde{w}$ in $L^2_{loc}(B_1)$. Note that by lower semicontinuity and (D.4), we have

$$\int_{B_1} |\nabla \widetilde{w}|^2 dx \le \liminf_k F_k(\widetilde{w}_k, B_1) \le 1. \tag{D.6}$$

Next, we claim that $\widetilde{w}$ is harmonic in B_1 and that for all $r \in (0,1)$

$$\lim_k F_k(w_k, B_r) = \int_{B_r} |\nabla \widetilde{w}|^2 dx. \tag{D.7}$$

Given this for granted, we get a contradiction, since from (D.4) and (D.7)

$$\tau^{2-\beta} \le \int_{B_\tau} |\nabla \widetilde{w}|^2 dx,$$

but on the other hand the harmonicity of $\widetilde{w}$ on B_1 and (D.6) yield that

$$\int_{B_\tau} |\nabla \widetilde{w}|^2 dx \le \tau^2.$$

To prove (D.7), let $r < s \in (0,1)$ and $\varphi \in C_c^\infty(B_s)$ be such that $\varphi = 1$ on B_r. Define $\zeta_k = \varphi\widetilde{w} + (1-\varphi)\widetilde{w}_k$, since $w_k = \widetilde{w}_k$ on B_s for $k \ge k_s$ (see (D.5)), elementary computations, the first two conditions in (D.4), and the locality of the energy lead to

$$F_k(w_k, B_r) = F_k(\widetilde{w}_k, B_r) \leq F_k(\zeta_k, B_s) + \vartheta_k \leq F_k(\widetilde{w}, B_r) + C\, F_k(\widetilde{w}_k, B_s \setminus \overline{B_r}) + C\, F_k(\widetilde{w}, B_s \setminus \overline{B_r}) + C \int_{B_s \setminus \overline{B_r}} |\widetilde{w}_k - \widetilde{w}|^2 dx + \vartheta_k.$$

The sequence of Radon measures $(F_k(\widetilde{w}_k, \cdot))_{k\in\mathbb{N}}$ is equi-bounded in mass in view of (D.4). Hence, up to a subsequence not relabeled for convenience, $(F_k(\widetilde{w}_k, \cdot))_{k\in\mathbb{N}}$ converges to some Radon measure μ on B_1. Assume that $\mu(\partial B_s) = 0$, by passing to the limit as $k \uparrow \infty$ and by Ambrosio's lower semicontinuity result we find

$$\int_{B_r} |\nabla \widetilde{w}|^2 dx \leq \liminf_k F_k(w_k, B_r) \leq \limsup_k F_k(w_k, B_r) \leq \int_{B_r} |\nabla \widetilde{w}|^2 dx + C\, \mu(B_s \setminus \overline{B_r}) + C \int_{B_s \setminus \overline{B_r}} |\nabla \widetilde{w}|^2 dx.$$

Equality (D.7) then follows by letting $s \downarrow r^+$ along values satisfying $\mu(\partial B_s) = 0$.

Eventually, the harmonicity of $\widetilde{w}$ is easily deduced from its local minimality for the Dirichlet energy. This last property is obtained as above by modifying any test function $\zeta \in W^{1,2}(B_1)$ such that $\{\zeta \neq \widetilde{w}\} \subset\subset B_1$ into a test-function for $\widetilde{w}_k$ in order to exploit again the quasi-minimality condition satisfied by w_k in (D.4). □

We are now ready to prove Lemma 3.3.

Proof of Lemma 3.3. We argue as in [2, Theorem 7.21], and take $z = 0$ for the sake of simplicity. We claim that

$$\mathrm{MS}(v, B_{\tau\rho}) \leq \varepsilon\, \tau^{2-\beta} \rho \tag{D.8}$$

if we set $R := (\varepsilon\, \vartheta\, \tau^{2-\beta})^{1/\alpha}$, with $\varepsilon = \varepsilon(\beta, \tau)$ and $\vartheta = \vartheta(\beta, \tau)$ provided by Lemma D.1.

Indeed, either both the assumptions of Lemma D.1 are satisfied or not. In the former case the thesis of that lemma gives exactly inequality (D.8); otherwise for some $w \in SBV(\Omega)$ with $\{w \neq v\} \subset\subset B_\rho(z) \subset \Omega$ we have by the quasi-minimality of v

$$\mathrm{MS}(v, B_{\tau\rho}) \leq \mathrm{MS}(v, B_\rho) \leq \frac{1}{\vartheta} \left(\mathrm{MS}(v, B_\rho) - \mathrm{MS}(w, B_\rho)\right) \leq \frac{\omega}{\vartheta} \rho^{1+\alpha}.$$

Thus, (D.8) follows since $\rho \leq R/\omega^{1/\alpha}$.

Eventually, as $\tau \in (0, 1)$ we can repeat the previous argument, and conclude by induction. □

References

[1] L. Ambrosio, V. Caselles, S. Masnou and J. M. Morel. *Connected components of sets of finite perimeter and applications to image processing*, *J. Eur. Math. Soc.* **3** (2001), 39–92.

[2] L. Ambrosio, N. Fusco & D. Pallara. *Functions of bounded variation and free discontinuity problems*, in the *Oxford Mathematical Monographs*, The Clarendon Press Oxford University Press, New York, 2000.

[3] A. Bonnet. On the regularity of edges in image segmentation, *Ann. Inst. H. Poincaré, Analyse Non Linéaire* 13 (4) (1996), 485–528.

[4] D. Bucur & S. Luckhaus. *Monotonicity formula and regularity for general free discontinuity problems.* Preprint, 2012.

[5] M. Carriero & A. Leaci. Existence theorem for a Dirichlet problem with free discontinuity set. *Nonlinear Anal.* 15 (1990), no. 7, 661-677.

[6] G. Dal Maso, J.M. Morel & S. Solimini. *A variational method in image segmentation: existence and approximation results*, Acta Math. 168 (1992), no. 1-2, 89–151.

[7] G. David. *C^1-arcs for minimizers of the Mumford-Shah functional*, SIAM J. Appl. Math. 56 (1996), no. 3, 783-888.

[8] G. David. *Singular sets of minimizers for the Mumford-Shah functional.* Progress in Mathematics, 233. Birkhäuser Verlag, Basel, 2005. xiv+581 pp. ISBN: 978-3-7643-7182-1; 3-7643-7182-X

[9] G. David, J.C. Léger. *Monotonicity and separation for the Mumford-Shah problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), no. 5, 631–682.

[10] E. De Giorgi, M. Carriero & A. Leaci. *Existence theorem for a minimum problem with free discontinuity set*, Arch. Ration. Mech. Anal. 108 (1989), 195–218.

[11] C. De Lellis & M. Focardi. *Higher integrability of the gradient for minimizers of the 2d Mumford-Shah energy*, to appear on J. Math Pures Appl..

[12] M. Focardi, M.S. Gelli & M. Ponsiglione. *Fracture mechanics in perforated domains: a variational model for brittle porous media*, Math. Models Methods Appl. Sci. 19 (2009), 2065–2100.

[13] F. Maddalena & S. Solimini. *Blow-up techniques and regularity near the boundary for free discontinuity problems*, Advanced Nonlinear Studies 1 (2) (2001).

[14] D. Mumford & J. Shah. *Optimal approximations by piecewise smooth functions and associated variational problems*, Comm. Pure Appl. Math. 42 (1989), no. 5, 577–685.

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