# VARIATIONAL TECHNIQUES FOR ASSESSING THE TECHNOLOGICAL SIGNATURE OF FLAT SURFACES

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ABSTRACT. The quality assessment of manufacturing operations performed to obtain given flat surfaces is always a problem of comparing the substitute model (approximating the features of the true manufactured part) to the nominal specifications, at any stage of the manufacturing cycle. A novel methodology, based on applications of classical tools of Calculus of Variations, is here presented with the aim of assessing the output quality of manufactured flat surfaces based on the information available on transformation imposed by technological processes. By assuming that any manufacturing process operates under equilibrium states, the proposed variational methodology allows to account for the traces left by different stages of manufacturing processes. A simple two-dimensional case is here discussed, to give the flavor of the methodology and its future potential developments.

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## 1. INTRODUCTION

The quality assessment of manufacturing operations performing geometrical transformations is, to a great extent, a problem of comparing tolerances specifications with workpiece geometries (macro-geometries such as straight lines, planes, circumferences, cylinders, etc.), its feature forms (e.g., flatness, straightness, roundness, run-out), and workpiece

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micro-geometries (e.g., roughness, surface texture). This task, which is a metrological problem (say, inspection planning, sampling strategy, measuring devices, etc.), is strongly conditioned by the knowledge of transformations experienced by the workpiece undergoing different technological operations. Since the early studies of Eppinger [12], the recognition of the importance of the *signature* left on workpieces by manufacturing operations has been considered as a relevant question to be faced, to improve the metrological quality-assessment. Representing a signature means to define a "simple representation of an object or a process in the form of a mathematical function, a feature vector, a geometric shape, or some other representations" ([13]). Since the signature changes according to changes in process state, it allows to uniquely capture the significant characteristics of an object, such as the manufactured workpiece, at a certain state.

Following this stream, other authors ([18]; [7]; [8]; and [9]) have further developed this point by introducing the concepts of *"technological signature"* or *"technological imprint"* or *"technological fingerprint"*. The idea is that all the available information related to the manufacturing operations performed on the workpiece and their process parameters should not be ignored, since they are extremely useful to understand the quality of the output and to define adequate measurement strategies and feedbacks for quality improvement actions.

Whenever a discrete number of measured points are available on the workpiece, it is possible to derive a model of the substitute geometry adherent to the real one; the wider use of coordinate measuring machines (CMMs) justifies the interest in more accurate mathematical models of recent times. Evaluating quality output of a manufacturing process with discrete measurement approaches then becomes a matter of estimating accurate parameters of the mathematical model from the measured points, and then comparing the model with the desired geometrical specifications. Unfortunately, several uncertainty factors affect this estimation process, mainly "hard" factors and "soft" ones. "Hard" uncertainty factors come from the nature of the measurement data-sets derived, their completeness and meaningfulness with respect to their use. Measuring strategies adopted, as well as the physical devices utilized, contribute to this uncertainty: the knowledge of technological information on the workpiece should allow the minimization of these "hard" uncertainty components, since it allows to select relevant information for estimation. "Soft" factors of uncertainty, on the other hand, come from the model adopted to represent the real workpiece: the substitute geometry build knowing the technological signature should guarantee the best representation of the measured workpiece. Relevant data points can in fact be selected appropriately for fitting algorithms [14], also with respect to the functional critical conditions. The variational methodology here proposed characterizes deviations from a reference geometry and allows their minimization despite the occurrence of errors or noises.

The proposed methodology, in fact, embeds the technological signature by virtues of its founding principle, the minimization of a functional, thus assuring in principle the best explanation of the soft causes of deviation of the substitute geometries. The method adopted is to assume that manufacturing operations always satisfy the energy minimization principle: the question then is how to write the correct functional for each manufacturing operation. This formal explanation of operation characteristics leads to two different implications: firstly it is possible to recognize the depart from optimal condition of any manufacturing process (due to several cases) [25]; secondly, the suggested formalization methodology already provides a rationale to compare the nature of geometrical deviations due to different technological processes.

1.1. **Assessing flatness of manufactured surfaces.** Assessing quality of manufactured flat surfaces is usually referred to the "flatness" surface feature, as defined in ASME Y14.5M-1994 Standard [3], "the condition of a surface having all elements in one plane". According to this Standard a flatness tolerance specifies a tolerance zone defined by two parallel planes within which the surface must lie.

Dealing with flatness error assessment, two main fitting methods are adopted to evaluate the substitute geometries: the Least Square Method (LSM) and the Minimum Zone Method (MZM).

The LSM is based on finding the minimum sum of the squared errors of the measured points from the nominal feature: *the perpendicular signed deviation of each point from the fitted surface is calculated and the difference between the maximum deviation and the minimum deviation represents the flatness error* (see [6]). Although this method is simple and characterized by a unique solution response and fast computing time, it does not guarantee the minimum zone solution specified in the standard ([6];[22]). Furthermore, the deviation values and geometric tolerances, as determined by LSM, are generally larger than the actual ones, leading to rejection of good parts [22].

The MZM is formulated as an optimization problem and the flatness error is defined as the minimum distance between parallel planes containing all the measured points. It can be implemented adopting several techniques: the simplex search method ([21]), the Monte Carlo search method, the spiral search method, the tabu search and hybrid search methods ([4]), the convex hull procedure ([26]; [19]), methods based on Tchebyshev approximations, genetic algorithms ([20]), fuzzy logic, classical deterministic methods based on sensitivity information. MZM tends to underestimate the form error and is very sensitive to asperities which, if undetected, can lead to poor results ([22]; [4]).

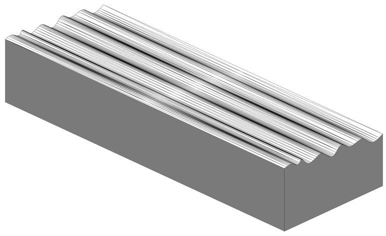

FIGURE 2.1. Outline of a generic output workpiece

The common application of the above mentioned approaches always consider data sets without asking any further information, i.e., ignoring the technological signature on workpieces: this fact brings to a meaningless substitute geometry due to the wrong geometrical errors appreciation in the data sets.

The methodology proposed here is intended to set an objective methodology to better recognize the signatures left on the workpiece by different technological operations' stage in order to minimize flatness evaluation errors. The approach is quite new since it is based on (classical) variational techniques (see [5] or [11]) to fit data and to model the *technological signature*, i.e., the set of informations representing the signature of manufacturing conditions experienced by a workpiece.

## 2. THE VARIATIONAL MODEL FOR ASSESSING FLATNESS OF SURFACES

2.1. **A general variational model.** To model the flatness of a given machined surface, the variational point of view assumed here starts from the hypotheses that any removal manufacturing process, in ideal conditions, experiences only equilibrium states. More precisely, we ask that this process realizes a minimum of an energy-like functional, or at least a critical point, among all other possible geometric configurations with the same amount of material removed.

In establishing a variational framework for a given technological process, the first crucial step consists in identifying the energy-like functional, close enough to the physics of the manufacturing operation performed. Whenever a flat surface has to be manufactured by material removal, it is reasonable to assume that this energy-like functional depends on the amount of material removed, as well as on the output geometry.

Let $B$ an open bounded subset of $\mathbb{R}^3$ containing the unit cube $[0,1]^3$. As initial shape we consider the set $\Omega = \overline{B} \cap ([0,1]^2 \times \mathbb{R}^+)$. The problem of assessing the flatness of the manufactured workpiece consists in finding a subset $M \subset \Omega$ such that $\partial M$ is a (piece-wise) regular manifold, $[0,1]^2 \times \{0\} \subset M$ and $|M| = \text{const}$, minimizing the functional

$$\mathcal{F}(M) = \int_M \varphi(x)\mathrm{d}x + \int_{\partial M} \tilde{\sigma}(p, N(p))\mathrm{d}\mathcal{H}^2(p). \tag{2.1}$$

The functional in equation (2.1) results from the addition of a volume contribution and a surface one. The first represents the energy spent in removing material, while the second one represents the energy spent in flattening the manufactured (upper) surface.

The density function $x \mapsto \varphi(x)$ will be called *specific forming energy* and $p \mapsto \sigma(p, N(p))$ will be called *specific shaping energy*, where $N$ denotes the outward unit normal to the surface $\partial M$. The specific forming energy $\varphi : M \to \mathbb{R}_+$ is assumed to be continuous and is related to the density $d$ of the material through the *technological constant* $c_{\text{Tech}}$ by

$$\varphi(x) = c_{\text{Tech}} d(x). \tag{2.2}$$

The technological constant $c_{\text{Tech}}$ represents the amount of energy required to remove a unit amount of mass and depends on the particular technology used.

A reasonable choice for the shaping energy is given by

$$\tilde{\sigma}(p, N(p)) = \psi(|N^{\|}(p)|), \tag{2.3}$$

where $N^{\|}(p) = (I - e_3 \otimes e_3)N(p)$ is the horizontal component of $N(p)$ and $\psi : [0,1] \to \mathbb{R}_+$ is a continuous and increasing function, such that $\psi(0) = 0$.

A detailed analysis of the mathematical problem of the existence of solutions and their regularity properties is out of the scope of the present paper. It will be the subject of a future work. For a general overview on existence and regularity in Calculus of Variations we refer to [5] and [15]. Some problems related to particular forms of (2.1) are investigated in [16, 17]. Moreover, the above variational approach needs a precise specification of the initial shape $\Omega$, of the forming and shaping energies $\varphi$ and $\sigma$, to be a really predictive mathematical theory. Note that $\Omega$ is only accessible through an approximate description. In the next section all these aspects will be discussed for a two-dimensional flat model.

**Example 2.1.** Assume $\Omega = [0,1] \times [0,1]$ and $d(x) = 1$, in such a case the volume term in (2.1) reduces to a constant, say $c$. Then consider the variational problem

$$\int_{\partial M} |N^{\|}(p)|\mathrm{d}\mathcal{H}^2(p) \to \min.$$

Now, if the amount of removable material is sufficiently large, a minimizer surface is achieved by taking the upper plane of the parallelepiped.

**Example 2.2.** If the worked piece $M$ is assigned through a function $f : U \to \mathbb{R}$, i.e.

$$M = M_f = \left\{ x \in \mathbb{R}^3 \mid x_3 \leq f(x_1, x_2),\ \ x_1, x_2 \in U \right\},$$

then $N = (\nabla f(x_1, x_2), -1)^T \in \mathbb{R}^3$ and so $N^{\parallel} = (f_{x_1}, f_{x_2}, 0)^T$. In such a case, if $d(x) = 1$, the variational problem, discarding the volume and the boundary's reminder contributions, takes the form

$$\mathrm{Min}_{f \in \mathcal{A}} \int_{\Omega} \psi(|\nabla f|) \sqrt{1 + |\nabla f|^2}\ \mathrm{d}x_1 \mathrm{d}x_2,$$

where $\mathcal{A} = \{f \in C^1(\Omega) \mid M_f \subset \Omega\}$.

2.2. **A two-dimensional flatness model.** The previous variational setting is here investigated in the simple case of a two-dimensional flatness model, with the aim of providing an analytic and numerical evidence of the method proposed here by means of experimental tests. As previously observed, the initial shape $\Omega$ is unknown. Therefore, given a function $f : [0, 1] \to \mathbb{R}$, without loss of generality, the flat manufactured surface is defined by

$$\Omega_f = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq f(x),\ x \in [0, 1]\}.$$

Assuming the material to be homogeneous, the fact that the amount of material utilized in the production is unchanged is given by the requirement:

$$\int_0^1 f(x)\ \mathrm{d}x = K \tag{2.4}$$

for a fixed constant $K$. Let $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ be a continuous function representing the specific forming energy, then the total forming energy spent to obtain the configuration $\Omega_f$ is given by

$$J_1(f) = \int_{\Omega_f} W(x, y)\ \mathrm{d}x \mathrm{d}y. \tag{2.5}$$

The term $J_1(f)$ is built as a function of the type of technological process performed to shape material (e.g., turning, milling), not including the specific operating conditions (e.g., machine, process parameters, environmental conditions).

It is also required that the resource consumption to remove the material depends on the geometric configuration assumed by the machined surface at the end of the manufacturing operation, namely on the normal vector, and hence on the first derivative $f'(x)$ of the upper boundary of $\Omega_f$. Moreover, for modeling the technological signature, we introduce a history dependent shaping energy allowing to incorporate in the energetic term the energy needed to reach the current configuration, starting from a given one. More precisely, let $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$

be a $C^1$ function denoting the specific shaping energy, then the total shaping energy spent to obtain the configuration $\Omega_f$ is given by

$$J_2(f) = \int_0^1 \int_0^{f'(x)} \sigma(x,t)\ \mathrm{d}t\mathrm{d}x. \tag{2.6}$$

Recalling the notations of Section 2.1, $\tilde{\sigma} \equiv 0$ on the flat sides of the boundary of $\Omega_f$, while on the upper boundary the *specific shaping energy* amounts to

$$\tilde{\sigma}(x, N(x)) = \int_0^{-N(x)^{\parallel}\sqrt{1+f'(x)^2}} \sigma(x,t)\ dt,$$

where

$$N(x) = \frac{1}{\sqrt{1+f'(x)^2}}\left(-f'(x), 1\right),$$

so that $N(x)^{\parallel} = \frac{-f'(x)}{\sqrt{1+f'(x)^2}}$ .

Note that $\sigma$ does not depend on the variable $y$, which means that the shaping energy is independent from the thickness of the manufactured workpiece. It may be interesting to consider such a $\sigma$ depending also on $y$, but, in this paper we will only consider the slightly simpler case. The term $J_2(f)$ represents, to some extent, the technological signature left by the manufacturing process on the workpiece, indeed the integral term $\int_0^{f'(x)} \sigma(x,t)\ \mathrm{d}t$ keeps track of all the energetic contributions up to the final state represented by $f'(x)$.

**Remark 2.3.** Note that the shaping energies in (2.1) and (2.6) are quite different mathematical objects. Indeed, while the first one depends only on the intrinsic properties of the surface $\partial M$ and therefore it is called a *geometric integral* (see [1]), the second one, keeping the memory of the energetic history of the shaping, delivers a term which cannot be represented by a geometric integral. We believe that both these points of view contribute to provide a rationale framework for the problems related to flatness manufacturing operations.

Therefore the total energy spent to realize the configuration $\Omega_f$ is given by

$$J(f) = J_1(f) + J_2(f). \tag{2.7}$$

In both the energy terms (2.5) and (2.6), for any given technology, the expression of $f$ and $f'$ can be easily determined by deriving appropriate information on the machined workpiece (namely set of points of the $\Omega_f$). Once this information are known, it is possible to derive $W$ and $\sigma$ by imposing the necessary conditions, expressing the state of equilibrium of the process, and using the basic tools of the Calculus of Variations. The proposed methodology consists of deriving a condition to recognize the equilibrium state of any manufacturing operation as below. The matter is then to study the behavior of the *energy-type functional* (2.7) under the constraint (2.4). This means to characterize those functions $f : [0,1] \to \mathbb{R}$, in some functions space, which realize the minimum or, at least critical points of $J$ under the constraint (2.4). To this aim, consider a function $\eta : [0,1] \to \mathbb{R}$ such that $\eta(0) = \eta(1) = 0$ and $\int_0^1 \eta(x)\mathrm{d}x = 0$. For $\varepsilon > 0$ we consider the perturbations $f_\varepsilon(x) = f(x) + \varepsilon\eta(x)$. By definition, the first variation of the functional $J$ is given by

$$\delta J(f, \eta) := \lim_{\varepsilon \to 0^+} \frac{J(f_\varepsilon) - J(f)}{\varepsilon}.$$

Observe that

$$\begin{aligned}
&\frac{J(f_\varepsilon) - J(f)}{\varepsilon} = \\
&= \frac{1}{\varepsilon}\left(\int_0^1 \left(\int_0^{f_\varepsilon(x)} W(x,y)\mathrm{d}y - \int_0^{f(x)} W(x,y)\mathrm{d}y \, + \right.\right.\\
&+ \left.\left.\int_0^1 \int_0^{f'_\varepsilon(x)} \sigma(x,t)\mathrm{d}t - \int_0^{f'(x)} \sigma(x,t)\mathrm{d}t\right) \mathrm{d}x\right) = \\
&= \int_0^1 \left(\frac{1}{\varepsilon}\int_{f(x)}^{f_\varepsilon(x)} W(x,y)\mathrm{d}y + \frac{1}{\varepsilon}\int_{f'(x)}^{f'_\varepsilon(x)} \sigma(x,t)\mathrm{d}t\right) \mathrm{d}x.
\end{aligned}$$

Let $\varepsilon \to 0^+$, the Fundamental Theorem of Calculus allows to deduce that

$$\delta J(f, \eta) = \int_0^1 \left(W(x, f(x))\eta(x) + \sigma(x, f'(x))\eta'(x)\right) dx.$$

Integrating by parts we obtain

$$\delta J(f, \eta) = \int_0^1 \left(W(x, f(x)) - \frac{d}{dx}\sigma(x, f'(x))\right) \eta(x) dx.$$

As a consequence, since the perturbations $\eta$ are area preserving, if the following Euler equation is satisfied

$$W(x, f(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\sigma(x, f'(x)) = \text{constant}, \tag{2.8}$$

then $\delta J(f, \eta) = 0$, that is, $f$ is a critical point for $J$. Vice versa, as it is standard in Calculus of Variations (see [5] or [11]), if $\delta J(f, \eta) = 0$ for every $\eta$, condition (2.8) holds. The condition (2.8) could be taken as a characterization of the technological process performed, once the energy density functions are given either by physical inspection or by experimental derivation. In general, the equation (2.8) can be explicitly solved just for special forms of $W$ and $\sigma$ and thus leads just to local solutions, while (2.8) is a global condition. It is worth to remark that if the functions $W$ and $\sigma$ depend only on the local thickness $f$ of the workpiece, then the perfectly-flat surface $f(x) = h$ satisfies the condition (2.8). Therefore, the condition of a perfectly-flat surface can be regarded as a critical point for an energy functional. In this sense, this variational approach extends the condition usually considered in literature. We also remark that the presence of the shaping term $\sigma$ makes the problem more involved since it introduces the derivative of $f$ which is not directly accessible by measurements. Condition (2.8) can be used as a check of the manufacturing output, i.e., whenever the substitute geometry $f(x)$ of a manufactured workpiece does not satisfy (2.8), i.e., the equilibrium state is lost, one can suspect a modification occurred in the manufacturing process (say, e.g., decay of machine tools, modified energy levels, and so on). The above variational paradigm makes sense only if equilibrium states actually exist. By applying the *direct methods* of the Calculus of Variations, it is not difficult to find conditions ensuring the existence of minimum configurations. Basically, the point is to have a compactness condition on the space of admissible functions $f$ and lower semi-continuity (l.s.c.) of the functional $J$. In such a case, given any minimizing sequence, i.e., a sequence $f_n$ such that

$$\lim_{n\to+\infty} J(f_n) = \inf J(f) = m,$$

by compactness we find (by passing to a subsequence) an admissible function $f$ such that $f_n \to f$ as $n \to +\infty$. Then, by l.s.c. we have

$$J(f) \le \liminf_{n\to+\infty} J(f_n) = m.$$

Hence $f$ is a minimizer for the functional $J$. To this aim we set

$$L(x, u, v) = \int_0^u W(x, s)\mathrm{d}s + \int_0^v \sigma(x, t)\mathrm{d}s,$$

then the functional $J$ can be written in the standard form

$$J(f) = \int_0^1 L(x, f(x), f'(x))\mathrm{d}x. \tag{2.9}$$

Typically, to get existence results we have to impose conditions on the Lagrangian density $L(x, u, v)$ and/or on the space of admissible functions $f$. A first classical existence result (see for instance [5] or [11]) states that if $L(x, u, v)$ is continuous and satisfies the following conditions:

- (Growth condition) There exists constants $a > 0, b, c \in \mathbb{R}$ and exponents $p > q \ge 1$ such that

$$L(x, u, v) \ge a|v|^p + b|u|^q + c; \tag{2.10}$$

- (Convexity condition) Fixed $(x, u)$, the function

$$L(x, u, v) \text{ is convex with respect to } v; \tag{2.11}$$

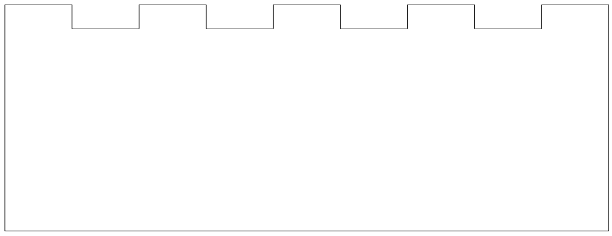

FIGURE 2.2. Geometry excluded by $\|f'\|_\infty \leq C_1$ avoids discontinuities in the boundary $z(x)$

then, the functional $J(f)$ admits a minimizer among the functions such that $\int_0^1 f(x)\mathrm{d}x = K$ and $f \in W^{1,p}([0,1])$, i.e., the Sobolev space of weakly differentiable functions such that both $|f|^p$ and $|f'|^p$ are integrable. Moreover, every solution of (2.8) is in fact a minimizer of $J$. Finally, if $L(x, u, \cdot)$ is strictly convex, we have a unique minimizer of $J$.

Observe that the previous assumptions are conditions on the integrand function $L(x, u, v)$ imposing some restrictions on the functions $W$ and $\sigma$. This means to make assumptions on the behavior of the manufacturing process: for instance, to produce perfectly flat surfaces the process needs to have an homogeneous energy density (i.e., independent on the $x-$variable), otherwise the necessary condition (2.8) cannot be satisfied.

The forming and shaping densities $W$ and $\sigma$ of the model are usually unknown: it could then be preferable to make some more assumptions on the space of admissible functions. To get compactness, independently on $W$ and $\sigma$, some more conditions on the admissible functions are necessary. More precisely, one may consider the following set of functions

$$X = \left\{ f \in C^1([0,1]) \ : \ \|f\|_\infty \leq C_0, \|f'\|_\infty \leq C_1, \mathrm{Lip}(f') \leq C_2 \right\}, \tag{2.12}$$

where

$$\mathrm{Lip}(f') = \sup_{x \neq y} \left| \frac{f'(x) - f'(y)}{x - y} \right| .$$

The first constraint $\|f\|_\infty \leq C_0$ corresponds to handle with equi-bounded work-pieces. On the other hand, the constraints on the derivatives correspond to restricting the possible geometries produced by the work process. For instance these exclude geometries like those of Figure (2.2) and Figure (2.3).

The restriction of $J$ to the space $X$ introduces restrictions on the geometry of the surfaces produced, but no restrictions on the densities $W$ and $\sigma$ are involved. Actually, by considering continuous density functions $W$ and $\sigma$, the functional $J$ admits minimizers in the space $X$. Indeed, let $\{f_n\}$ be any minimizing sequence, since the functions $\{f'_n\}$ are equi-bounded and equi-continuous, by Ascoli-Arzelà Theorem,

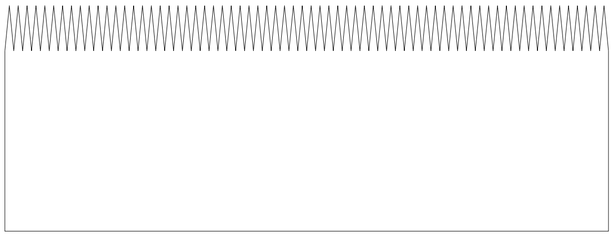

FIGURE 2.3. Geometry excluded by $\mathrm{Lip}(f') \le C_2$

there exists $v \in C([0,1])$ such that (up to a subsequence) $f_n' \to v$ uniformly. Observe that $f_n$ is equi-bounded and then we may assume that $f_n(0)$ converges to a value $u(0)$. Since

$$f_n(x) = f_n(0) + \int_0^x f_n'(t)\mathrm{d}t,$$

by the uniform convergence of the functions involved, passing to the limit under the integral sign we have

$$u(x) := \lim_{n\to+\infty} f_n(x) = u(0) + \int_0^x v(t)\mathrm{d}t,$$

that is $u' = v$. It is also straightforward to check that actually $f_n \to u$ uniformly. Indeed, it is enough to observe that

$$|f_n(x)-u(x)| \le |f_n(0)-u(0)|+\int_0^x |f_n'(t)-v(t)|\mathrm{d}t \le |f_n(0)-u(0)|+\|f_n'-v\|_\infty.$$

Moreover, since

$$|f_n'(x) - f_n'(y)| \le C_2|x-y|,$$

it results (just by pointwise convergence)

$$|v(x) - v(y)| \le C_2|x-y|.$$

Then, setting $f(x) = u(x)$ we obtain a function $f \in X$ such that (up to a subsequence) $f_n \to f$ and $f_n' \to f'$ uniformly. Since the Lagrangian $L(x, f(x), f'(x))$ is a continuous function, passing to the limit under the integral sign we get

$$J(f) = \int_0^1 L(x, f, f')\mathrm{d}x = \lim_{n\to+\infty} \int_0^1 L(x, f_n, f_n')\mathrm{d}x.$$

Therefore $f$ is a minimum of $J$ in the space $X$.

Although in such a case there are almost no restrictions on $W$ and $\sigma$, and it is possible to find regular minimizers of the functional $J$, in general terms in this case (2.8) it is a sufficient condition for equilibrium states but not a necessary condition. Indeed, in such a case the variations $f_\varepsilon = f + \varepsilon\eta$ could not satisfy the constraints on the derivatives.

2.3. **Selection of $\sigma$ and $W$: the application to flat surfaces.** In this section we address the question of finding explicit forms of $W$ and $\sigma$, useful to work coherently for any given technological process having as an output flat surfaces. An example of a numerical procedure is provided, to show how it works in testing whether a technological process satisfies the equilibrium condition (2.8) or not. The aim of this section is to explain how the variational methodology could be applied to assess the technological signature of a workpiece, and thus to assess the quality of any manufacturing process. Let $(x_i, f_i), i = 0, 1, \dots, n$ be the 2D coordinates of $(n+1)$ measured points on the manufactured workpiece. Since the condition (2.8) has to be satisfied on the whole interval $[0, 1]$, it could be preferable to handle it with a global expression of the upper boundary $f(x)$. Moreover, condition (2.8) involves the derivatives of $f(x)$. A first step of the methodology proposed is to infer a global expression of $f(x)$ from the sampled points $(x_i, f_i)$. A reasonable choice could be to work with approximating polynomials. For instance, it is possible to construct the polynomial $f_n(x) \in \mathbb{R}[x]$ of degree $n$ such that $f_n(x_i) = f_i$:

$$f_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 = \sum_{j=0}^{n} a_j x^j.$$

To avoid numerical complications (see for instance [24]), provided we are free to choose the $x$-coordinates, a better choice would be to work with Bernstein polynomials given by

$$f_n(x) = B_n(x) := \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} f_k x^k (1-x)^{n-k}, \tag{2.13}$$

where $f_k$ corresponds to the $y$-coordinate evaluated at the point $\frac{k}{n}$. Bernstein polynomials converge to the profile $f(x)$: the approximation could be thus improved by increasing the number of points measured and hence the degree of the polynomial.

A second step is to infer the expression of the cost densities $W$ and $\sigma$. These functions are certainly linked with the physics of the manufacturing process, but, in general, the physics underlying the process is very complex and a reasonable description of the essential features is not easy to derive. Accordingly, we here propose to infer $W$ and $\sigma$ from the process output, thus avoiding the description of the physics of the process themselves. In other words, we assume that, whatever the characteristics of the technological process actually are, the workpiece is produced under equilibrium states, namely satisfying condition (2.8). In order to simplify computations, we choose a particular polynomial expression for $W(x, s)$ and $\sigma(x, t)$. We set

$$W(x, s) = (s - h) x^{n-3},$$

where $h$ is the thickness of the workpiece to be manufactured. The factor $x^{n-3}$ is assumed just for technical reasons which simplify computations. Observe that we are assuming that $W$ is in some sense a known function. Let $\sigma(x,t)$ be

$$\sigma(x,t) = p(x) + Dt^2, \tag{2.14}$$

where $p$ and $D$ are respectively a polynomial of degree $(2n-3)$ and a constant which will be determined through the measured points on the workpiece. Observe that with these choices, the Lagrangian

$$L(x,u,v) = \frac{1}{2}(u-h)^2 x^{n-3} + p(x)v + \frac{D}{3}v^3 \tag{2.15}$$

does not satisfy the conditions (2.10) and (2.11). Observe that conditions (2.10) and (2.11) require the knowledge of the global behavior of the density functions $W$ and $\sigma$. On the other hand, by searching minimizers of $J$ on the space $X$ defined by (2.12), the functions $W$ and $\sigma$ are completely undetermined. However, as observed, in this case the equation (2.8) is not a necessary equilibrium condition. In this case one would carefully estimates the minimum value $m = \min_X J$ and studying the equilibrium of a configuration $f \in X$ by estimating $|J(f) - m|$. Here, to illustrate an example application, we have chosen an intermediate state in which the functions $W$ and $\sigma$ are just partially determined. In such a way, we are able to directly equilibrium configurations (Euler equation solutions) without passing through any minimization process. Therefore, in particular we will choose the parameters of the Lagrangian (2.15) in order to reach solutions of equation (2.8), namely, to handle with equilibrium configurations. As stated in the previous section, a sufficient condition for equilibrium states is given by (2.8). This can be rewritten as

$$W(x, f(x)) - \frac{\partial \sigma}{\partial x}(x, f'(x)) - \frac{\partial \sigma}{\partial f}(x, f'(x)) f''(x) = constant.$$

Approximating the map $x \mapsto f(x)$ with the polynomial $f_n(x)$, we obtain:

$$(f_n(x) - h)x^{n-3} - p'(x) - 2D f_n'(x) f_n''(x) = constant. \tag{2.16}$$

The analysis of the involved polynomials degree gives

$$\underbrace{(f_n(x) - h)x^{n-3}}_{\deg = 2n-3} - \underbrace{p'(x)}_{\deg = 2n-4} - \underbrace{2D f_n'(x) f_n''(x)}_{\deg = 2n-3} = \underbrace{constant}_{\deg = 0}. \tag{2.17}$$

The coefficient of the highest power of $x$ is given by

$$a_n - 2Dn^2(n-1)a_n^2. \tag{2.18}$$

Since it has to be zero, for the Identity Principle of Polynomials, this allows to determine the value of D.

The polynomial $p(x)$ can be determined up to a constant by

$$p'(x) = (f_n(x) - h)x^{n-3} - 2Df_n'(x)f_n''(x) - constant. \quad (2.19)$$

Since the minimization, or in general equilibrium configurations, of $J$ are not affected by the addition of a constant term to $W$, we may assume this arbitrary constant to be equal to zero. Accordingly, by the first data set we can determine the cost functions $W$ and $\sigma$ by using (2.18) and (2.19). Once $W$ and $\sigma$ are obtained, condition (2.16) could be assumed as a reference for the given manufacturing process. Precisely, for other data sets corresponding to workpieces processed by the same manufacturing process, we expect that condition (2.16) holds as well, since the process is in an equilibrium state. On the other hand, if for some reason the process leaves this equilibrium state, the condition (2.16) has to fail. Vice versa, whenever the condition (2.16) is not satisfied by a workpiece one can suspect a modification occurring in the technological process.

In order to assess the sensitivity of the proposed variational methodology, a simulation of the cutting process was performed through a VBA Macro created in Microsoft Excel$^{TM}$. The input to be processed by the simulation were a data set of 50 equally-spaced measurement points on the $x$-dimension of the workpiece, in the range $x \in [0, 1]$ with a step of $0.01mm$. Setting the level of three different factors, namely tool wear, tool holder vibrations, and spindle axes loose, the simulation provided the $y$-coordinate of the manufactured part corresponding to each $x$-coordinate. Setting specific level to each factor, only to extreme levels were selected for each one of them: "low" corresponds to an ideal condition, while "high" corresponds to the worst operational case. These two levels correspond to different equilibrium states of the manufacturing process. A three factors two-levels factorial experiment, with five replications for each trial, was performed having the $z$-dimension measure as a response; Table 1 summarizes the experimental layout selected for the purpose.

A numerical code was implemented in Matlab$^{TM}$ version 7 to compute the algorithm of Figure 3.1, reported and described in the Appendix, built on the basis of the proposed variational approach for data points provided by the simulation model.

Concerning the method to check the presence of arbitrary constant trends in the left side of (2.16)(see step 7 in the description of the algorithm of Figure 3.1), the non-parametric Spearman's rank-correlation coefficient, denoted by $\rho$, has been chosen. We refer to [10] for a preliminary discussion on the subject. The $\rho$ coefficient satisfies the inequality $-1 \leq \rho \leq 1$ and gives a measure of how well an arbitrary monotonic function describes the relationship between two variables. Here we confine ourself to observe that values of $|\rho|$ close to 1 denotes a monotonic trend, otherwise we assume that no trend is detectable. Therefore, a

| | Factors Combination | | |
|---|---|---|---|
| Trials n | *A* | *B* | *C* |
| *1* | high | high | high |
| *2* | high | high | low |
| *3* | high | low | high |
| *4* | high | low | low |
| *5* | low | high | high |
| *6* | low | high | low |
| *7* | low | low | high |
| *8* | low | low | low |
| *9* | NA | NA | NA |

TABLE 1. Experimental layout: A= tool wear, B= tool-holder vibrations, and C= spindle axes loose. (Note that NA means the factor did not affect the cutting process)

low value of $|\rho|$ denotes a geometry compatible with a flat surface. As an alternative, a threshold value can be appropriately selected according to the specific manufacturing process considered. Here we evaluate the $\rho$ coefficient by the simplified formula

$$\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} \tag{2.20}$$

where $d_i = x_i - y_i$ is the difference between the ranks of the corresponding values of the data under considerations, while $n$ is the total number of couples $(x_i, y_i)$ of the data set. The values of $z$-dimension of the machined surface are plotted for each of the five replications of the same run in which all the factors are at their lowest levels (Figure 2.4) and at their highest levels (Figure 2.5). In the same figures the approximations of the datasets by means of Bernstein's polynomials are also provided.

Note that, according to the previously reported steps, the first replication - with the factors at their lowest levels- represents the *master workpiece* and the polynomial $p'_m(x)$ and the constant $D$ computed for the master workpiece are fixed and used for the other computations. In Figure 2.6 and 2.7 the values of the polynomials obtained by the left side of (2.16) are plotted for the trials LLL and HHH, respectively.

In Figure 2.8 a comparison of the values obtained by the left side of (2.16) for trials LLL and HHH calculated in the $x_i$ coordinate of a workpiece is provided. It is possible to notice a clear distinction between the lines representing values of different trials. Lines representing the same trial have a similar trend. Moreover, by changing the parameters of the manufacturing process, the corresponding trend, whose values are given by the left side of (2.16), departs even more from the constant

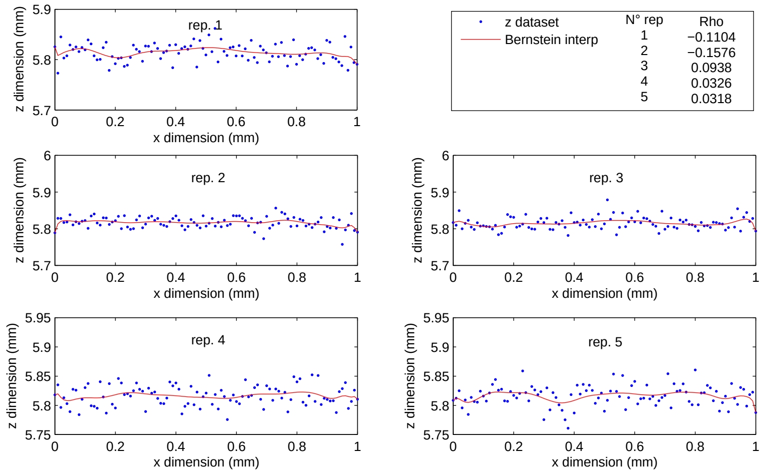


FIGURE 2.4. The profile of part after the manufacturing process for the trial of the experimental layout in which all the factors are at their lowest levels

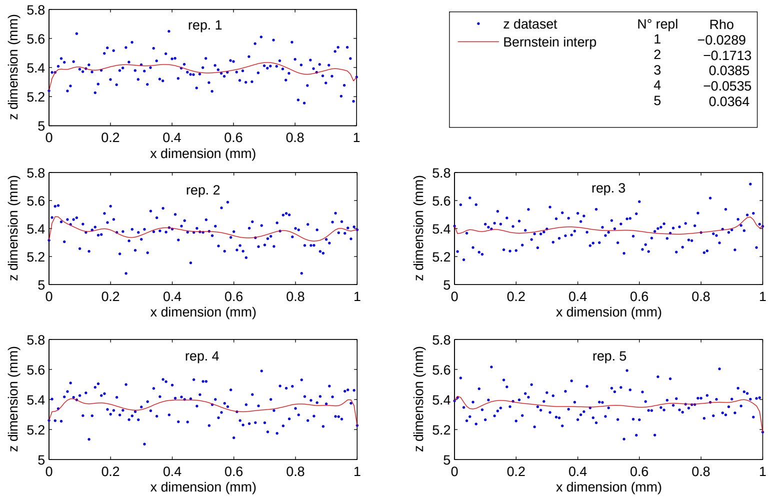


FIGURE 2.5. The profile of part after the manufacturing process for the trial of the experimental layout in which all the factors are at their highest levels

trend. These trends are shown in detail in Figure 2.9 where only the right part of the previous figure is provided. It is possible to observe that all these plots (Figures 2.6-2.9) show a fast growth of the graphs whenever $x$ is close to $x = 1$. This phenomenon is due to the high degree of the approximating polynomial, since the condition (2.16) involves the second derivatives. To reduce this phenomenon it would be

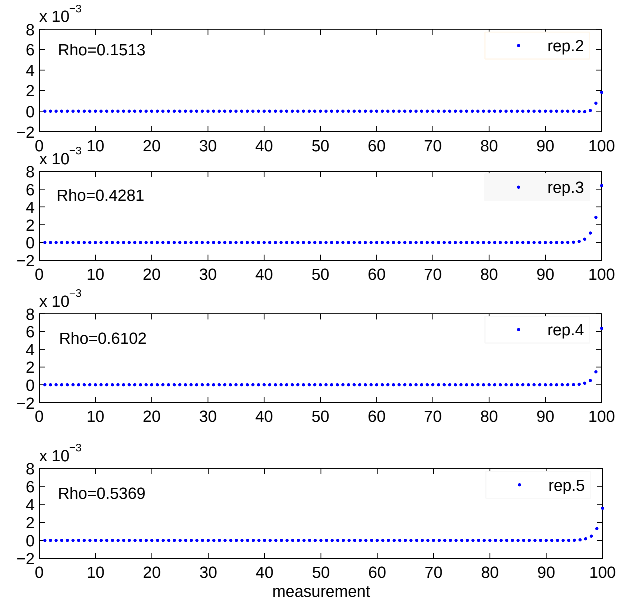


FIGURE 2.6. The values of the polynomials obtained by the left side of (2.16) for trials LLL

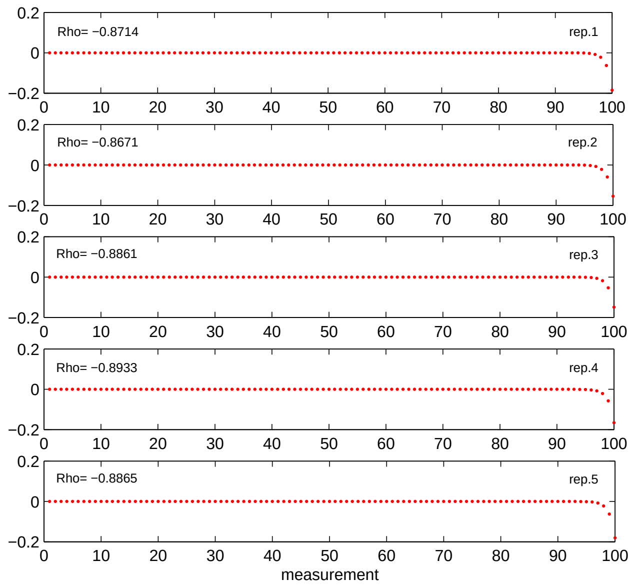


FIGURE 2.7. The values of the polynomials obtained by the left side of (2.16) for trials HHH

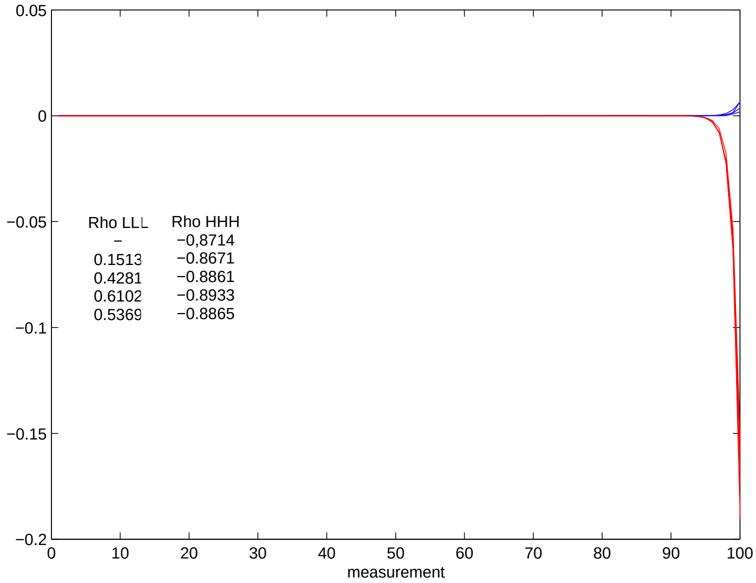


Figure 2.8. Comparison of the values of the polynomials obtained by the left side of (2.16) for trials LLL and HHH

useful to work with low-degree polynomial or by different interpolating functions.

Therefore, as a qualitative conclusion, it is possible to state that the methodology proposed seems able to detect changes in the technological process even when no trends are detectable in the micro/macro geometry of the workpieces, as shown in Figure (2.5).

## 3. Concluding remarks

Several approaches have been proposed so far to assess the quality features of manufactured surfaces. Despite simple in principle, this task is affected by a strong uncertainties: this fact justifies the amount of different approaches developed so far, in particular for flat surfaces. Amongst the most promising ones, those taking into account information on past manufacturing process seems to be the most promising. Recognizing the technological signature, i.e., the trace left by previous manufacturing operations on the workpiece, can in fact give significant advantages to this aim. To make explicit all the causal relationships existing among different technological process-stage, experienced by a workpiece, is an ideal condition in assessing the quality of the outputs of technological processes. In this paper we address a new method to assess the technological signature, by providing a formal representation of the state transitions experienced by a manufactured workpiece using the Calculus of Variation. The variational methodology introduced in

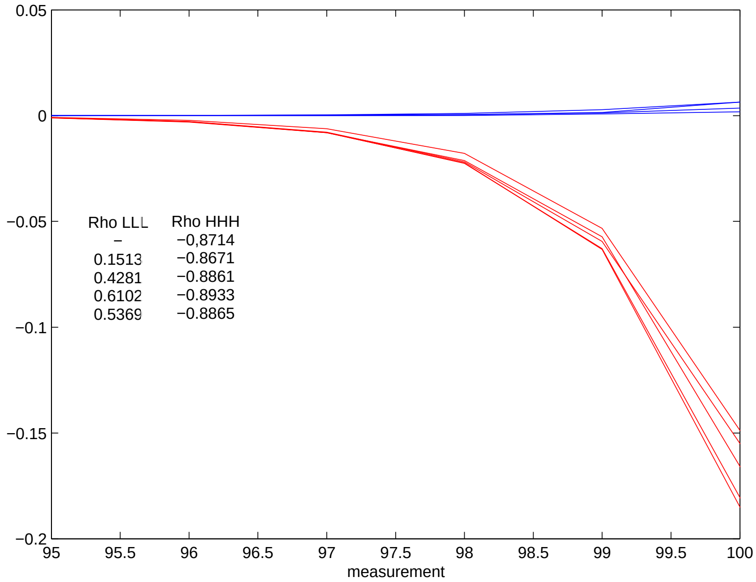


FIGURE 2.9. Comparison of the values of the polynomials obtained by the left side of (2.16) for trials LLL and HHH

this paper is intended to improve quality estimation of flat surfaces by capturing the essence of manufacturing processes. Despite the paper concentrates on flat surfaces mostly, the concepts introduced are general enough to be easily extended to other assessment problems of different geometrical features.

Future steps of this approach will be the extension to data sampled from a real case, to test complexity of adoption as well as the extension to a three-dimensional case.

The exploratory numerical procedure provided for flat geometries, although rudimentary, is complex enough to give a taste of the potentialities of the approach. The outcomes from the analysis performed on the specific case seems promising in the Author's opinion. The position here maintained is that as far as the functional $J$ is able to embed the evolution of the system, it is then possible to understand the sequence of different transformations occurred on the manufactured part and their particular features. This means to reproduce the "technological memory" of the workpiece.

It is the author's belief that the variational methodology proposed deserves to be investigated more deeply both from the theoretical and the applicative point of view in the near future for quality assessment problems. Further efforts can be devoted in a near future also extend the model to different geometries.

## Appendix

The flowchart can be described as follow:

- *Step 1*: Consider $i(i \geq 5)$ manufactured workpieces within the same process conditions (namely, for the simulation model it is possible to consider the five replications of the same combination of factors' levels; the first of this workpiece will be regarded as a reference workpiece for other manufactured workpieces.
- *Step 2*: Insert the $y$-coordinate of $n+1$ measured points on the manufactured workpiece master and construct the polynomial $f_n^m(x)$ where the superscript $m = 1$ means master.
- *Step 3*: Insert the thickness $h$ of the workpiece and compute the polynomial $W_m(x, f)$ of degree $2n - 3$ and the constant $D = \frac{1}{2n^2(n-1)a_n}$ for the master workpiece.
- *Step 4*: Compute the polynomial $p'_m(x)$ for the master workpiece as in (2.16) using the constant $D$.
- *Step 5*: Insert the $z$-coordinate of $n+1$ measured points on the manufactured workpiece 2 and construct the polynomial $f_n^2(x)$.
- *Step 6*: Insert the thickness $h$ of the workpiece and compute the left side of (2.16) by using the polynomial $p'_m(x)$ and the value $D$ of the master workpiece.
- *Step 7*: Check if the values computed in the previous step are disposed according to an arbitrary constant trend.
- *Step 8*: Repeat steps from step 5 to 7 for all $i$ workpieces.
- *Step 9*: Repeat steps from step 5 to 7 for workpieces coming from a different technological processes.

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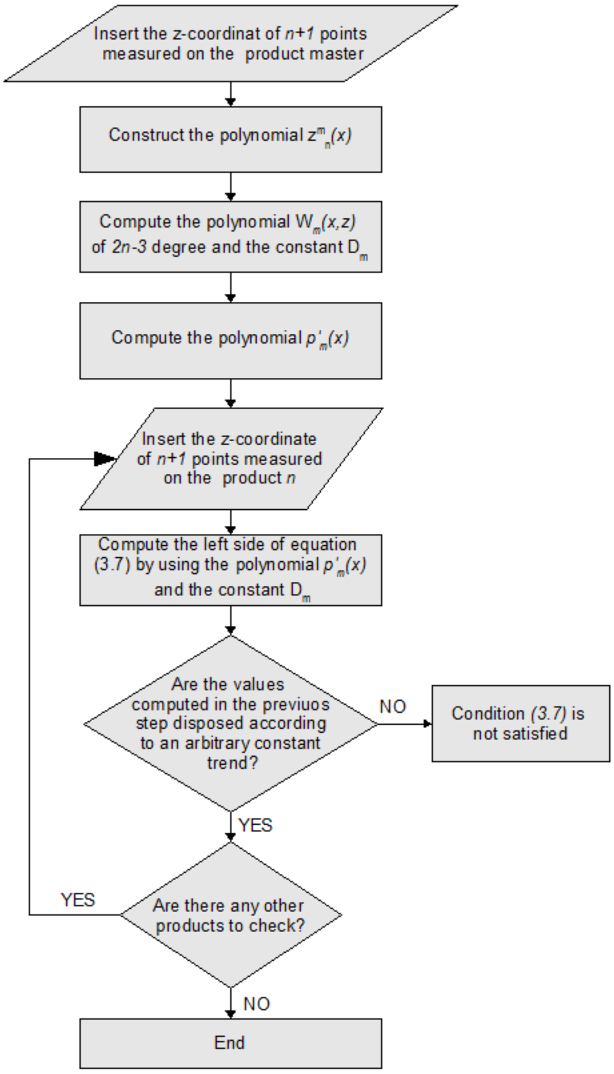


FIGURE 3.1. Flowchart of the algorithm for the variational approach


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