Nonlinear parabolic problems with lower order terms and related integral estimates

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Abstract

We deal with the solutions to nonlinear parabolic equations of the form

$$u_t - \operatorname{div} a(x,t,Du) + g(x,t,u) = f(x,t) \ \text{ on } \Omega_T = \Omega \times (-T,0),$$

under standard growth conditions on g and a, with f only assumed to be integrable to the power $\gamma > 1$. We prove general local decay estimates for level sets of the solutions u and the gradient Du which imply very general estimates in rearrangement function spaces (Lebesgue, Orlicz, Lorentz) and non-rearrangement ones, up to Lorentz-Morrey spaces.

Keywords: Parabolic equations, lower-order term, absorption term, Morrey-Lorentz regularity, rearrangement function spaces
2010 MSC: Primary 35K55; Secondary 35K10, 35B65, 35D30

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[1] The authors have been supported by the ERC grant 207573 "Vectorial Problems".

Contents

1. Introduction

This paper deals with regularity properties of solutions to the following class of Cauchy-Dirichlet problems

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) + g(x,t,u) = f(x,t) & \text{in } \Omega_T = \Omega \times (-T,0) \\ u = 0 & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, $T > 0$, $\partial_{\mathrm{par}}\Omega_T$ is the usual parabolic boundary of Ω_T, f is an integrable function in Ω_T, g is a lower-order term and a is a Leray-Lions type operator. We immediately declare the specific assumptions we are considering: the vector field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory regular, i. e. measurable in $(x,t) \in \Omega_T$ for any fixed $z \in \mathbb{R}^n$ and continuous in $z \in \mathbb{R}^n$ for a. e. $(x,t) \in \Omega_T$. Moreover,

$$\begin{cases} \nu|z_2 - z_1|^2 \leq \langle a(x,t,z_2) - a(x,t,z_1), z_2 - z_1 \rangle \\ |a(x,t,z)| \leq L(\varsigma + |z|) \end{cases} \tag{1.2}$$

for every z_1, z_2, $z \in \mathbb{R}^n$ and $(x,t) \in \Omega_T$; the structure constants satisfy $0 < \nu \leq 1 \leq L$ and $\varsigma \geq 0$. The lower order term $g : \Omega_T \times \mathbb{R} \to \mathbb{R}$ will denote a Carathéodory function such that

$$\exists\, m, \alpha_0 > 0 \;:\; \text{for all } s \text{ and a. e. } (x,t) \in \Omega_T \;\; g(x,t,s)\operatorname{sgn}(s) \geq \alpha_0 |s|^m, \tag{1.3}$$

$$\forall\,\beta > 0 \text{ the function } \mathcal{G}_\beta(x,t) := \sup_{|s|\le\beta} |g(x,t,s)| \text{ belongs to } L^1_{\mathrm{loc}}(\Omega_T). \quad (1.4)$$

A typical example to keep in mind involves the Laplacean operator with coefficients:

$$\begin{cases} u_t - \operatorname{div}\big(c(x)Du\big) + |u|^{m-1}u = f & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega_T, \end{cases}$$

where $0 < \nu \le c(x) \le L$ is a measurable function.

We will focus mainly on the case when f belongs to the Lebesgue space $L^\gamma(\Omega_T)$ in a range of γ that does not necessarily permit to obtain the existence of finite energy solutions $u \in L^2(-T,0;W_0^{1,2}(\Omega))$ to problem (1.1). However, we can deal with the (very) weak solutions $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ obtained via the Boccardo-Gallouët standard approximation procedure ([11]) as exploited in [13] (see forthcoming Section 2.2).

The presence of lower order terms g in parabolic problems of type (1.1) is quite important in applications; usually, a lower order term represents an absorption or a reaction, depending on its sign. In view of (1.3)-(1.4), in this paper we are dealing with absorption zero order terms, that usually have a regularizing effect on the solutions to (1.1). This effect has been shown in the elliptic framework, by starting from measure data for regularity results on the Lebesgue scale (in [12, 17]) and on the Marcinkiewicz one ([9]). Recently, the previous cited results have been extended in all the most familiar function spaces of rearrangement (Lebesgue, Lorentz, Orlicz) and non-rearrangement one, up to Lorentz-Morrey[2], by the authors in [19]. Let us focus for a while on the elliptic analog of problem (1.1). In [19], we extend (to the case in which lower order terms are considered) some general estimates on level sets of the gradient of solutions, firstly obtained in [40] (see also [39]), where Mingione presents a non-linear potential theory version of the fundamental papers by Adams [3] and Adams & Lewis [5], providing optimal regularity results on the Morrey and also Lorentz-Morrey scale. Among other results, in [19] we prove the validity of the following implication for solutions $u \in W_0^{1,1}(\Omega)$ to the elliptic analog of equation (1.1)

$$f \in L^\theta(\gamma,q)(\Omega) \implies |Du| \in L^\theta\left(\frac{2m\gamma}{m+1}, \frac{2mq}{m+1}\right) \text{ locally in } \Omega, \qquad (1.5)$$

[2] We refer to Section 2.3, where the involved function spaces are defined in the parabolic framework. The analogous definitions in the elliptic case can be obtained by simply replacing parabolic cylinders by balls.

whenever $0 < q \le \infty$, $2 < \theta \le n$, $1 < \gamma \le 2\theta/(\theta+2)$ and $1 < m < 1/(\gamma-1)$.

The aim of this paper is to extend the results above to the case of parabolic equations of type (1.1) under the structural assumptions (1.2)–(1.4). It is worth mentioning that in the parabolic framework a non-linear analog of the classic Theorem of Adams has been recently obtained by Baroni & Habermann in [6] (see also [24, 22, 23, 14, 15, 34, 33]), following the potential approach in [40], that is when no lower order term is considered. On the other hand, Boccardo, Gallouët & Vazquez ([13]), by means of a priori techniques and classical approximating methods, analyzed the regularity properties of solutions to (1.1) starting from L^1-data; among other results, they showed that

$$f \in L^1(-T,0;L^1(\Omega)) \implies |Du| \in L^1(-T,0;L^q(\Omega)) \text{ with } q < 2m/(m+1). \tag{1.6}$$

Therefore, in the present paper we will extend this classical result in the more general Lorentz-Morrey spaces, as well as providing an extension to the results in [6] when no lower order term is considered; or, equivalently, to the parabolic analog of the results in [19]. Namely, our main result relies in general gradient estimates on level set (see Section 1.1 below) in turn implying the following

Theorem 1.1. *Let $q \in (0,\infty]$. Assume (1.2)-(1.3)-(1.4) and $f \in L^\theta(\gamma,q)(\Omega_T)$ with γ, θ such that $1 < \gamma \le 2\theta/(\theta+2)$, $2 < \theta \le N := n+2$. Then the solution $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ to (1.1), with*

$$1 < m < \frac{1}{\gamma-1}, \tag{1.7}$$

satisfies

$$|Du| \in L^\theta\Big(\frac{2m\gamma}{m+1}, \frac{2mq}{m+1}\Big) \text{ locally in } \Omega_T. \tag{1.8}$$

Moreover, the local estimate

$$\|Du\|_{L^\theta(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{C}_{R/2})} \le c\,R^{\frac{\theta(m+1)}{2m\gamma}-N}\||Du|+\varsigma\|_{L^1(\mathcal{C}_R)} + c\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_R)}$$

holds for any parabolic cylinder $\mathcal{C}_R \subseteq \Omega_T$, where c depends only on m, n, q, L/ν and γ.

Of course, in (1.8) we mean $2mq/(m+1) = \infty$ whenever $q = \infty$. Thus, by choosing $\theta = N$ and $q = \infty$, we can also deduce regularity results on the Marcinkiewicz scale. Furthermore, it is worth noticing that – as in the classic case (1.6) – Theorem 1.1 fails for the borderline choice $\gamma = 1$. This is classical even in the elliptic case, since one has to impose some further $L\log L$ integrability on the datum f in order to obtain the following implication

$$f \in L\log L(\Omega_T) \quad \Longrightarrow \quad |Du| \in L^{\frac{2m}{m+1}}_{\mathrm{loc}}(\Omega_T) \tag{1.9}$$

(see [25, Theorem 2.1]). Analogously, here we consider also this borderline case, again in a more general Morrey-Orlicz setting. We prove the following

Theorem 1.2. *Assume* (1.2)-(1.3)-(1.4) *and* $f \in L\log L^{\theta}(\Omega_T)$*, with* $2 < \theta \leq N$*. Then the solution* $u \in L^1(-T,0;W^{1,1}_0(\Omega))$ *to* (1.1)*, with* $1 < m < \infty$*, is such that*

$$|Du| \in L^{\frac{2m}{m+1},\theta} \text{ locally in } \Omega_T.$$

Moreover, the local estimate

$$\|Du\|_{L^{\frac{2m}{m+1},\theta}(\mathcal{C}_{R/2})} \leq c\,R^{\frac{(m+1)\theta}{2m}-N}\||Du|+\varsigma\|_{L^1(\mathcal{C}_R)} + c\|f\|^{\frac{m+1}{2m}}_{L\log L^{\theta}(\mathcal{C}_R)} \tag{1.10}$$

holds for every parabolic cylinder $\mathcal{C}_R \subseteq \Omega_T$*, where* c *depends only on* m*,* n *and* L/ν*.*

Clearly, by choosing $\theta = N$, Theorem 1.2 will cover the classical Sobolev implication in (1.9).

Now, we discuss a further extension of the results given in Theorem 1.1, that is, we consider a case when the lower order terms g verify some relaxed assumptions with respect to those considered until now. Namely, we will analyze the following Cauchy-Dirichlet problems

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) + h(x)|u|^{m-1}u = f(x,t) & \text{in } \Omega_T \\ u = 0 & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases} \tag{1.11}$$

where the function h is such that

$$0 < h(x) < 1 \text{ in } \Omega_T, \tag{1.12}$$

and

$$\frac{1}{h} \in L^p(\Omega) \quad \text{for some } p \geq 1. \tag{1.13}$$

Clearly, such $g(\cdot,s) = h(\cdot)s^{m-1}s$ satisfies (1.3) but not necessarily (1.4). In Theorem 1.3 below, we show that it is possible to recover general regularity results, even in spite of these different lower order terms g.

Theorem 1.3. *Let $q \in (0,\infty]$. Assume* (1.2)-(1.12)-(1.13) *and $f \in L^\theta(\gamma,q)(\Omega_T)$ with γ, θ such that $1<\gamma\le 2\theta/(\theta+2)$ and $2<\theta\le N$. Then the solution $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ to* (1.11)*, with*

$$\frac{p+1}{p} < m < \frac{1}{\gamma-1}, \tag{1.14}$$

where p is given by (1.13)*, satisfies*

$$|Du| \in L^\theta\Big(\frac{2m\gamma}{m+1}, \frac{2mq}{m+1}\Big) \text{ locally in } \Omega_T.$$

Moreover, the local estimate

$$\|Du\|_{L^\theta(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{C}_{R/2})} \le c\,R^{\frac{(m+1)\theta}{2m\gamma}-N}\||Du|+\varsigma\|_{L^1(\mathcal{C}_R)} + c\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_R)}$$

holds for every parabolic cylinder $\mathcal{C}_R \subseteq \Omega_T$, where c depends only on m, n, q, L/ν and γ.

Note that the interval in which m can vary depends on the integrability of the function h given by (1.13) and, as expected, the lower bound on the exponent m in (1.14) will converge to the one in (1.7) as p goes to infinity.

Furthermore, we will show that the techniques of establishing sharp estimates for the level sets of the maximal operator for the gradient Du of the solutions u to (1.1), in turn implying Theorems 1.1, 1.2 and 1.3, can be also extended, by the needed modifications, to the solutions u themselves to obtain Morrey/Lorentz-Morrey estimates for u. It is worth pointing out that, although in the elliptic case the regularity of u can be recovered by plainly combining the regularity of the gradient Du with the classic Sobolev embeddings, here we need to work in a separate way, by means of sharp estimates also involving some "fractional" maximal operator. We prove the following two theorems.

Theorem 1.4. *Let $q \in (0,\infty]$. Assume* (1.2)-(1.3)-(1.4) *and $f \in L^\theta(\gamma,q)(\Omega_T)$ with γ, θ such that $1<\gamma\le\theta/2$, $2<\theta\le N$. Then the solution $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ to* (1.1)*, with $1<m<1/(\gamma-1)$, satisfies*

$$u \in L^\theta\Big(\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}, \frac{2m\theta q}{m(\theta-2\gamma)+\theta}\Big) \text{ locally in } \Omega_T.$$

Moreover, the local estimate

$$\|u\|_{L^\theta(\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta},\frac{2m\theta q}{m(\theta-2\gamma)+\theta})(\mathcal{C}_{R/2})} \le c\,R^{\frac{m(\theta-2\gamma)+\theta}{2m\gamma}-N}\||u|+\varsigma R\|_{L^1(\mathcal{C}_R)} + c\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_R)}$$

holds for any parabolic cylinder $\mathcal{C}_R \subseteq \Omega_T$, where c depends only on m, n, q, L/ν and γ.

Theorem 1.5. *Assume* (1.2)-(1.3)-(1.4) *and $f \in L\log L^{\theta}(\Omega_T)$, with $2 < \theta \leq N$. Then the solution $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ to* (1.1)*, with $1 < m < \infty$, is such that*

$$u \in L^{\frac{2m\theta}{m(\theta-2)+\theta},\theta} \text{ locally in } \Omega_T.$$

Moreover, the local estimate

$$\|u\|_{L^{\frac{2m\theta}{m(\theta-2)+\theta},\theta}(\mathcal{C}_{R/2})} \leq c\, R^{\frac{m(\theta-2)+\theta}{2m}-N} \| |u| + \varsigma R\|_{L^1(\mathcal{C}_R)} + c\|f\|^{\frac{m+1}{2m}}_{L\log L^{\theta}(\mathcal{C}_R)}$$

holds for every parabolic cylinder $\mathcal{C}_R \subseteq \Omega_T$, where c depends only on m, n and L/ν.

Finally, we stress that all the results we obtained in the present paper hold for the weak solutions given by the approximation method described in forthcoming Section 2.2. It would be interesting to understand whether these results can be extended to some other notion of solutions. In this respect, a positive answer can be given when dealing with notions of solutions to measure data problems holding uniqueness in the case of integrable data, as in the case of the renormalized solutions in [44] (see [8] for the first definition of renormalized solutions in this framework, and also [21]).

1.1. Some ideas from the proofs

As already mentioned, we extend to the parabolic framework the techniques developed in [19], which in turn extends the potential approach introduced by Mingione in [40]. Roughly speaking, the proofs of Theorem 1.1, 1.2 and 1.3 rely on the fact that the integrability of the spatial gradient of the solutions to problem (1.1) is linked to a suitable choice of a potential operator. In this sense, the key-point will be the proof of a decay estimate that involves the level sets of the parabolic Hardy-Littlewood maximal operator[3] M^* of $|Du|$ in term of those of a suitable power of the maximal operator of the assigned datum f, up to a correction term which is negligible when considering the gradient regularity. We will obtain an estimate of the type

$$\left|\left\{M^*(|Du|) \geq S\lambda\right\}\right| \lesssim \frac{1}{S^{2\chi}}\left|\left\{M^*(|Du|) \geq \lambda\right\}\right| + \left|\left\{[M^*(|f|)]^{1/\sigma} \geq \lambda\right\}\right|, \tag{1.15}$$

[3] We refer to Section 2.4 for the definition of the parabolic maximal operators.

for every λ suitably large, and in which $S >> 1$ is a constant to be chosen, $\sigma = \sigma(m) \geq 1$ determines the regularity of the gradient of u, the exponent $\chi > 1$ is related to the higher integrability theory. Estimate (1.15) is fairly general and it will be relevant to deducing all the Lorentz and Lorentz-Morrey estimates stated in our theorems, also including the borderline case. In order to obtain the level set estimate (1.15) (whose precise version is given by forthcoming formula (4.6)), we apply the parabolic version of the classical Calderón-Zygmund covering lemma together with the Hölder continuity theory by De Giorgi-Nash-Moser and the higher integrability theory by Gehring. Therefore, we will work locally on basic estimates of the solutions u to (1.1) in comparison to the solutions v to the corresponding homogeneous problem (see Section 3). We will prove such comparison estimates, by means of classical truncation techniques going back to Boccardo & Gallouët, also used in the recent paper [24, 19], as well as by exploiting very recent contributions in the parabolic framework given in the forthcoming paper [36]. Clearly, the situation is complicated by the presence of the lower order terms g.

Analogously, a modified version of the level set estimate (1.15), by replacing Du by u, will permit to recover the desired estimates for the solutions u as given in Theorem 1.4 and 1.5. In this case - as stated in the previous section - different exponents will be involved and a fractional maximal operator will arise (see Section 4.3).

Finally, it is worth pointing out that maximal operators techniques have been used since the basic paper of Iwaniec [29]; see also [2, 20, 31, 30, 32, 24] for related nonlinear estimates and [1, 41, 42] for maximal function free techniques.

The paper is organized as follows. In Section 2, we fix notation; we give full details on the structure of the problem and we briefly recall the definitions and a few basic properties of the spaces and the operators we deal with, also providing some classical estimates for the solutions to nonlinear parabolic problems of type (1.1). In Section 3, we state and prove comparison regularity estimates and other preliminary results. Section 4 is devoted to the proof of the main result, to further extensions and results not covered by Theorem 1.1, and to Morrey/Lorentz-Morrey space estimates for the solutions u.

2. Preliminaries

In this section we fix notation and we provide definitions and some basic properties of the spaces and the operators we deal with. We also recall the solvability and a few classical results related to nonlinear parabolic problems of type (1.1).

2.1. Notation

In the present paper we follow the usual convention of denoting by c a general positive constant, possibly varying from line to line. Relevant dependencies on parameters will be emphasized by using parentheses; special constants will be denoted by c_0, c_1, ...

As customary, we denote by

$$B_R(x_0) \;=\; B(x_0;R) := \big\{x \in \mathbb{R}^n : |x - x_0| < R\big\}$$

the open ball centered in $x_0 \in \mathbb{R}^n$ with radius $R > 0$; and by

$$Q_R(x_0) \;=\; Q(x_0;R) := \big\{x \in \mathbb{R}^n : \max_{i=1,2,\dots,n} |x_i - x_{0,i}| < R\big\}$$

the open cube centered in $x_0 \in \mathbb{R}^n$ with sidelength $2R$. When not important and clear from the context, we shall use the shorter notation $B_R = B(x_0;R)$ and $Q_R = Q_R(x_0;R)$. Throughout the paper, all the cubes we consider are supposed to have side parallel to the coordinate axes in $\mathbb{R}^n$. We denote by

$$\mathcal{C}_R(x_0,t_0) \;=\; \mathcal{C}(x_0,t_0;R) := B(x_0;R) \times (t_0 - R^2, t_0)$$

the open parabolic cylinder centered in $(x_0,t_0) \in \mathbb{R}^{n+1}$ with height R^2, having a ball B_R as horizontal slice; and by

$$\mathcal{Q}_R(x_0,t_0) \;=\; \mathcal{Q}(x_0,t_0;R) := Q(x_0;R) \times (t_0 - R^2, t_0)$$

the open parabolic cylinder centered in $(x_0,t_0) \in \mathbb{R}^{n+1}$ with height R^2, having a cube Q_R as horizontal slice. Given a cylinder $\mathcal{C}$, we denote by $\sigma\mathcal{C}$ the concentric parabolic cylinder scaled by a factor $\sigma > 0$, that is

$$\sigma\mathcal{C}(x_0,t_0;R) \;=\; B(x_0;\sigma R) \times (t_0 - (\sigma R)^2, t_0);$$

similar notation will be used for $\sigma\mathcal{Q}$, σB and σQ.

Finally, we recall that, given a cylindrical domain of the type $\mathcal{C} = \Omega \times (t_0,t_1)$, with $\Omega \subset \mathbb{R}^n$ and $t_0, t_1 \in \mathbb{R}$, its parabolic boundary $\partial_{\rm par}\mathcal{C}$ is given by $\partial\mathcal{C} \setminus (\Omega \times \{t_1\})$.

2.2. Solvability of the problem

We give the natural definition of the solutions to problem (1.1) and we briefly recall the classical solvability of the nonlinear parabolic problems we are considering. Here and throughout the remaining of the paper, for the sake of simplicity we take $g(x,t,u) \equiv |u(x,t)|^{m-1}u(x,t)$, so we will consider the following Cauchy-Dirichlet problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) + |u|^{m-1}u = f & \text{in } \Omega_T \\ u = 0 & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases} \tag{2.1}$$

in which a verifies (1.2), $0 < \nu \leq 1 \leq L < +\infty$, $\varsigma \geq 0$, $f \in L^1(\Omega_T)$ and $1 < m < \infty$. A measurable function u is a distributional solution to (2.1) if $u \in L^1(-T,0;W_0^{1,1}(\Omega))$, $|u|^{m-1}u \in L^1(\Omega_T)$ and

$$\begin{aligned} -\int_{\Omega_T} u\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a(x,t,Du)D\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |u|^{m-1}u\varphi \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} f\varphi \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.2}$$

holds for any $\varphi \in C_0^\infty(\Omega_T)$. Also, while the lateral boundary condition can be formulated by prescribing the belonging of u to $L^1(-T,0;W_0^{1,1}(\Omega))$, the initial boundary condition $u(x,-T) = 0$ is understood in the L^1-sense, that is

$$\lim_{h\searrow 0} \frac{1}{h}\int_{-T}^{-T+h}\int_\Omega |u(x,t)| \,\mathrm{d}x\,\mathrm{d}t \;=\; 0.$$

As customary in the parabolic setting, one can provide a convenient "slice-wise" reformulation of equality (2.2) by mean of Steklov average (see for instance [18], and in particular Section I-3 and II-1 there). Indeed, for $h > 0$ and $t \in [-T,0)$, we can define

$$u_h(x,t) \;:=\; \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} u(x,\tilde{t}\,)\,\mathrm{d}\tilde{t} & \text{if } t+h < 0 \\ 0 & \text{if } t+h > 0 \end{cases}$$

and the following equality

$$\int_\Omega \Big(\partial_t u_h\varphi + \langle \big(a(\cdot,t,Du)\big)_h, D\varphi\rangle + \big(|u|^{m-1}u\big)_h\varphi\Big)\mathrm{d}x \;=\; \int_\Omega f\varphi \,\mathrm{d}x$$

holds for any $\varphi \in C_0^\infty(\Omega)$ and for a. e. $t \in (-T,0)$.

The existence of such a solution is obtained using a rather standard approximation method (see, e. g., [11, 10]). For the reader's convenience we report the results obtained in [13], in which the solvability of the problem (2.1) has been studied in the case $\Omega \equiv \mathbb{R}^n$; the modifications to obtain the same results in our case are minimals.

One considers a sequence of bounded functions $\{f_k\} \subset L^\infty(\Omega_T)$ such that $f_k \to f$ in $L^1(\Omega_T)$ as $k \to +\infty$. Then, by standard monotonicity arguments (see [38]), for each fixed k, there exists a unique solution

$$u_k \in C^0([-T,0]; L^2(\Omega)) \cap L^2(-T,0; W_0^{1,2}(\Omega))$$
$$(u_k)_t \in L^2(-T,0; W^{-1,2}(\Omega)), \quad |u_k|^{m-1}u_k \in L^1(\Omega_T)$$

to the Cauchy-Dirichlet problem

$$\begin{cases} (u_k)_t - \operatorname{div} a(x,t,Du_k) + |u_k|^{m-1}u_k = f_k & \text{in } \Omega_T \\ u_k = 0 & \text{on } \partial_{\mathrm{par}}\Omega_T. \end{cases} \tag{2.3}$$

The arguments in [13] permit to pass to the limit in the problem above and to prove, as in the case without lower order terms, the existence of a solution $u \in L^\infty(-T,0;L^1(\Omega)) \cap L^m(\Omega_T)$ and also $u \in L^r(-T,0;W_0^{1,q}(\Omega))$, under the restrictions

$$1 \le r < 2, \quad 1 \le q < \frac{n}{n-1} \quad \text{and} \quad \frac{2}{r} + \frac{n}{q} > n+1. \tag{2.4}$$

Taking into account the bound in $L^m(\Omega_T)$, u has an additional regularity, that is $u \in L^r(-T,0;W_0^{1,q}(\Omega))$ for every

$$1 \le r < 2, \quad 1 \le q < \frac{2m}{m+1} \quad \text{and} \quad (m-1)\frac{2}{r} + \frac{2}{q} > m+1. \tag{2.5}$$

We note that if m is large enough, precisely for

$$m+1 > \frac{2(n+1)}{n}$$

the admissible (q,r) region defined in (2.4) is extended. The maximal q-regularity is improved when $n/(n-1) < 2m/(m+1)$, that is when $m > n/(n-2)$ and in this case the new admissible region completely contains the previous one. So we have an "improved regularity" when m is sufficiently large.

We stress that, from now on, the sequence $\{u_k\} \subset C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ will be the one fixed in (2.3) by choosing $\{f_k\}$ as

$$f_k(x,t) := \max\big\{-k, \min\{f(x,t),k\}\big\}, \quad k \in \mathbb{N}$$

and by writing of "weak solutions to (1.1)" we will always mean the solution obtained via the approximating methods described in this section.

2.3. Relevant function spaces

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $T > 0$. Throughout this section, we denote by Ω_T the space time cylinder $\Omega \times (-T, 0)$.

Fix $q \in (0, \infty)$. A measurable map $f : \Omega_T \to \mathbb{R}$ belongs to the *Lorentz space* $L(\gamma, q)(\Omega_T)$ with $1 \le \gamma < \infty$ if and only if

$$\|f\|^q_{L(\gamma,q)(\Omega_T)} := q \int_0^\infty \left(\lambda^\gamma \big|\{(x,t) \in \Omega_T : |f(x,t)| > \lambda\}\big|\right)^{\frac{q}{\gamma}} \frac{\mathrm{d}\lambda}{\lambda} < +\infty. \tag{2.6}$$

In the case $q = \infty$, the Lorentz space $L(\gamma, \infty)$ with $\gamma \in [1, \infty)$ is the so-called *Marcinkiewicz space* and it is usually denoted by $\mathcal{M}^\gamma(\Omega_T)$. A measurable map $f : \Omega_T \to \mathbb{R}$ belongs to $\mathcal{M}^\gamma(\Omega_T)$ if and only if

$$\|f\|^\gamma_{\mathcal{M}^\gamma(\Omega_T)} \equiv \|f\|^\gamma_{L(\gamma,\infty)(\Omega_T)} := \sup_{\lambda>0} \lambda^\gamma \big|\{(x,t) \in \Omega_T : |f(x,t)| > \lambda\}\big| < +\infty. \tag{2.7}$$

By coupling the definitions in (2.6) and (2.7) with a density condition we obtain the so-called *parabolic Lorentz-Morrey spaces*. Precisely, a measurable map $f : \Omega_T \to \mathbb{R}$ belongs to $L^\theta(\gamma, q)(\Omega_T)$ for $\gamma \in [1, \infty)$, $q \in (0, \infty)$ and $\theta \in [0, N]$, if and only if

$$\|f\|_{L^\theta(\gamma,q)(\Omega_T)} := \sup_{\mathcal{C}_\rho \subseteq \Omega_T} \rho^{\frac{\theta-N}{\gamma}} \|f\|_{L(\gamma,q)(\mathcal{C}_\rho)} < +\infty, \tag{2.8}$$

where we recall that $N = n+2$. Accordingly, in the case $q = \infty$, a measurable map f belongs to $L^\theta(\gamma, \infty)(\Omega_T) = \mathcal{M}^{\gamma,\theta}(\Omega_T)$ if and only if

$$\|f\|_{\mathcal{M}^{\gamma,\theta}(\Omega_T)} \equiv \|f\|_{L^\theta(\gamma,\infty)(\Omega_T)} := \sup_{\mathcal{C}_\rho \subseteq \Omega_T} \rho^{\frac{\theta-N}{\gamma}} \|f\|_{\mathcal{M}^\gamma(\mathcal{C}_\rho)} < +\infty. \tag{2.9}$$

Note that when $\theta = N$, the space $L^N(\gamma, q)(\Omega_T)$ coincides with the space $L(\gamma, q)(\Omega_T)$. Also, by Fubini's Theorem one can see that for any $\gamma \in [1, \infty)$ $L(\gamma, \gamma)(\Omega_T) \equiv L^\gamma(\Omega_T)$.

A measurable map $f : \Omega_T \to \mathbb{R}$ belongs to the *Orlicz space* $L \log L(\Omega_T)$ if and only if

$$\|f\|_{L \log L(\Omega_T)} := \fint_{\Omega_T} |f(x,t)| \log\left(e + \frac{f(x,t)}{\fint_{\Omega_T} |f(y,\tau)|\, \mathrm{d}y\, \mathrm{d}\tau}\right) \mathrm{d}x\, \mathrm{d}t. \tag{2.10}$$

Fix $\theta \in [0, N]$, a measurable map $f : \Omega_T \to \mathbb{R}$ belongs to the *parabolic Morrey-Orlicz space* $L \log L^\theta(\Omega_T)$ if and only if

$$\|f\|_{L \log L^\theta(\Omega_T)} := \sup_{\mathcal{C}_\rho \subseteq \Omega_T} \rho^\theta \|f\|_{L \log L(\mathcal{C}_\rho)} < +\infty. \tag{2.11}$$

In the following, we will recall a few properties of the functionals introduced above. First, it is worth pointing out that, despite the notation, the functionals $\|\cdot\|_{L(\gamma,q)(\Omega_T)}$ defined in (2.6)-(2.7) are only quasi-norms. Nevertheless, by Fatou's Lemma, one can see that these functionals are lower semi-continuous with respect to the a. e. convergence. The same happens for the functionals $\|\cdot\|_{L^\theta(\gamma,q)(\Omega_T)}$ and $\|\cdot\|_{L\log L^\theta(\Omega_T)}$.

Moreover, the spaces defined above enjoy Hölder type inequalities. We only state a standard inequality for the Marcinkiewicz spaces $\mathcal{M}^\gamma(\Omega)$ in the form we will need it in the following of the paper.

Lemma 2.1. *Let $A \subseteq \mathbb{R}^{n+1}$ be a measurable set and let $f \in \mathcal{M}^\gamma(A)$ with $\gamma > 1$. Then, for any $q \in [1,\gamma)$, $f \in L^q(A)$ and*

$$\|f\|_{L^q(A)} \leq \left(\frac{\gamma}{\gamma - q}\right)^{\frac{1}{q}} |A|^{\frac{1}{q}-\frac{1}{\gamma}} \|f\|_{\mathcal{M}^\gamma(A)}.$$

Finally, we will state a lemma concerning the scaling properties of $\|\cdot\|_{L^\theta(\gamma,q)}$ and $\|\cdot\|_{L\log L^\theta}$, whose proof is an immediate consequence of the definitions in (2.8)-(2.9) and (2.10)-(2.11).

Lemma 2.2. *Let $f \in L^\theta(\gamma,q)(\mathcal{C}(x_0,t_0;\rho))$ with $1 \leq \gamma < \infty$ and $0 < q \leq \infty$. Then the map $\tilde{f}(\tilde{x},\tilde{t}) := f(x_0 + \rho\tilde{x}, t_0 + \rho^2\tilde{t})$, for $(\tilde{x},\tilde{t}) \in \mathcal{C}_1 \equiv \mathcal{C}(0,0;1)$, belongs to $L^\theta(\gamma,q)(\mathcal{C}_1)$ and it satisfies*

$$\|\tilde{f}\|_{L^\theta(\gamma,q)(\mathcal{C}_1)} \;=\; \rho^{-\frac{\theta}{\gamma}} \|f\|_{L^\theta(\gamma,q)(\mathcal{C}_\rho)}.$$

Similarly, if $f \in L\log L^\theta(\mathcal{C}_\rho)$ then $\tilde{f} \in L\log L^\theta(\mathcal{C}_1)$ and

$$\|\tilde{f}\|_{L\log L^\theta(\mathcal{C}_1)} \;=\; \rho^{-\theta} \|f\|_{L\log L^\theta(\mathcal{C}_\rho)}.$$

We conclude this section by recalling the definition of *Morrey spaces $L^{\gamma,\theta}$*. A measurable map $f : \Omega_T \to \mathbb{R}$ belongs to the Morrey space $L^{\gamma,\theta}(\Omega_T)$, with $\gamma \in [1,\infty)$ and $\theta \in [0,N]$, if and only if

$$\|f\|^\gamma_{L^{\gamma,\theta}(\Omega_T)} := \sup_{\mathcal{C}_\rho \subseteq \Omega_T} \rho^\theta \fint_{\mathcal{C}_\rho} |f|^\gamma \,\mathrm{d}x\,\mathrm{d}t \;<\; \infty.$$

Clearly, $L^{\gamma,N}(\Omega_T) \equiv L^\gamma(\Omega_T)$, $L^{\gamma,0}(\Omega_T) \equiv L^\infty(\Omega_T)$ and, also, $L^\theta(\gamma,\gamma)(\Omega_T) \equiv L^{\gamma,\theta}(\Omega_T)$.

For details and results about the theory of Lorentz, Morrey and Lorentz-Morrey spaces, we refer the interested reader to [45, 3, 4, 26].

2.4. Parabolic maximal operators

For any measurable function f, the (*restricted*) *fractional maximal operator* $M^*_{\beta,\mathcal{Q}_0}$, with $\beta \in [0,N]$, relative to a symmetric parabolic cylinder $\mathcal{Q}_0 = Q(x_0;R) \times (-T,0) \subset \mathbb{R}^{n+1}$ is defined by

$$M^*_{\beta,\mathcal{Q}_0}(f)(x,t) := \sup_{\mathcal{Q}\subseteq\mathcal{Q}_0,\ (x,t)\in\mathcal{Q}} |\mathcal{Q}|^{\frac{\beta}{N}} \fint_{\mathcal{Q}} |f(y,\tau)|\,\mathrm{d}y\,\mathrm{d}\tau,$$

where the cylinders $\mathcal{Q}$ have sides parallel to those of $\mathcal{Q}_0$. An equivalent definition can be provided by using parabolic cylinder $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ with balls as horizontal slice instead of cubes.

The boundedness of maximal operators in Marcinkiewicz spaces is classical (see, for instance, [16, 28]); i.e. for any $f \in L^\gamma(\mathcal{Q}_0)$

$$\left|\left\{x \in \mathcal{Q}_0 : M^*_{0,\mathcal{Q}_0}(f)(x,t) \geq \lambda\right\}\right| \leq \frac{c_0}{\lambda^\gamma}\int_{\mathcal{Q}_0} |f|^\gamma\,\mathrm{d}x\,\mathrm{d}t$$

holds for every $\lambda > 0$ and $\gamma \geq 1$; the constant c_0 depending only on n and γ.

More in general it holds a standard embedding result for the maximal function in Lorentz spaces, as given by the following theorem, whose proof is an application of Marcinkiewicz Theorem (see [7, IV.4.13, IV.4.18]) together with standard sublinear interpolation.

Theorem 2.3. *Let β, $\theta \in [0,N]$ and $\gamma > 1$ be such that $\beta\gamma < \theta$. Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a parabolic cylinder and denote by $\sigma\mathcal{C}$ the concentric parabolic cylinder scaled by a factor $\sigma > 1$.Then for every measurable function f in $\sigma\mathcal{C}$ and for any $q \in (0,\infty]$ it holds*

$$\|M^*_{\beta,\mathcal{C}}(f)\|_{L(\frac{\theta\gamma}{\theta-\beta\gamma},\frac{\theta q}{\theta-\beta\gamma})(\mathcal{C})} \leq c\|f\|^{\frac{\beta\gamma}{\theta}}_{L^\theta(\gamma,q)(\sigma\mathcal{C})}\|f\|^{\frac{\theta-\beta\gamma}{\theta}}_{L(\gamma,q)(\mathcal{C})},$$

where c is a constant depending only on β, γ, σ, θ, n and q. Moreover, if $|\sigma\mathcal{C}| \leq c(n)$ with $c(n)$ a positive constant sufficiently large, we have

$$\|M^*_{\beta,\mathcal{C}}(f)\|_{L(\frac{\theta\gamma}{\theta-\beta\gamma},\frac{\theta q}{\theta-\beta\gamma})(\mathcal{C})} \leq c\|f\|_{L^\theta(\gamma,q)(\sigma\mathcal{C})}. \tag{2.12}$$

The constant c blows up, i. e. $c \to \infty$ when $q \to 0$ or $\gamma \to 1$.

Also, in the borderline case $\gamma = 1$, we have the following

Theorem 2.4. *([6, Theorem 4.12]). Let $\beta, \theta \in [0, N]$ be such that $\beta < \theta$. Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a parabolic cylinder and consider the concentric parabolic cylinder $\sigma\mathcal{C}$ scaled by a factor $\sigma > 1$. Then there exists a constant $c = c(n, \beta, \sigma, \theta)$ such that, for any measurable function f in $\sigma\mathcal{C}$, it holds*

$$\|M^*_{\beta,\mathcal{C}}(f)\|_{L^{\frac{\theta}{\theta-\beta}}(\mathcal{C})} \leq c|\mathcal{C}|^{1-\frac{\beta}{\theta}} \|f\|^{\frac{\beta}{\theta}}_{L^{1,\theta}(\sigma\mathcal{C})} \|f\|^{1-\frac{\beta}{\theta}}_{L\log L(\mathcal{C})}.$$

2.5. Classical results

First, we recall some basic results from the Hölder regularity theory of De Giorgi-Nash-Moser as well from the higher integrability theory of Gehring. For the proofs we refer to [43, 37] (see also [6, Section 5.5] for a comprehensive sketch and further considerations), by observing that the presence of the lower order terms does not affect the proofs, because of the sign hypothesis (1.3).

Theorem 2.5. *Let $v \in C^0([-T, 0]; L^2(\Omega)) \cap L^2(-T, 0; W^{1,2}(\Omega))$ be a weak solution to the parabolic equation*

$$v_t - \operatorname{div} a(x, t, Dv) + |v|^{m-1}v = 0 \quad \text{in } \Omega_T$$

under the assumptions

$$|a(x, t, z)| \leq L(\varsigma + |z|), \quad \nu|z|^2 - \frac{L^2}{\nu}\varsigma^2 \leq \langle a(x, t, z), z\rangle$$

for every $(x, t) \in \Omega_T$ and $z \in \mathbb{R}^n$, where $0 < \nu \leq 1 \leq L < +\infty$, $\varsigma \geq 0$, and $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field.

Then there exists $\varpi = \varpi(n, L/\nu) \in (0, 1/2]$ such that for every $q \in (0, 2]$ there exists a constant $c = c(n, q, L/\nu)$ such that

$$\int_{\mathcal{C}_\rho} (|Dv|^q + \varsigma^q)\, \mathrm{d}x\, \mathrm{d}t \leq c\left(\frac{\rho}{R}\right)^{N-(1-\varpi)q} \int_{\mathcal{C}_R} (|Dv|^q + \varsigma^q)\, \mathrm{d}x\, \mathrm{d}t, \tag{2.13}$$

and

$$\int_{\mathcal{C}_\rho} (|v|^q + \varsigma^q\rho^q)\, \mathrm{d}x\, \mathrm{d}t \leq c\left(\frac{\rho}{R}\right)^{N} \int_{\mathcal{C}_R} (|v|^q + \varsigma^q R^q)\, \mathrm{d}x\, \mathrm{d}t, \tag{2.14}$$

whenever $\mathcal{C}_R \subseteq \Omega_T$ and $0 < \rho \leq R$.

Theorem 2.6. *Let the hypotheses of Theorem 2.5 hold. Then there exists $\chi = \chi(n, L/\nu) > 1$ such that $Dv \in L^{2\chi}_{\mathrm{loc}}(\Omega_T)$ and for any $q \in (0,2]$ there exists a constant $c = c(n, q, L/\nu)$ such that*

$$\left(\fint_{\mathcal{C}_{R/2}} (|Dv| + \varsigma)^{2\chi} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2\chi}} \leq c \left(\fint_{\mathcal{C}_R} (|Dv| + \varsigma)^{q} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{q}},$$

whenever $\mathcal{C}_R \subseteq \Omega_T$; moreover, there exists a constant $c = c(n, q, L/\nu)$ such that

$$\left(\fint_{\mathcal{C}_{R/2}} (|v| + \varsigma R)^{2\chi_0} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2\chi_0}} \leq c \left(\fint_{\mathcal{C}_R} (|v| + \varsigma R)^{q} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{q}}$$

holds for every $\chi_0 > 1$, $q \in (0,2]$ and $\mathcal{C}_R \subseteq \Omega_T$.

Furthermore, in the rest of the paper we will need the following parabolic version of the classical Calderón-Zygmund-Krylov-Safanov covering (see [6, Proposition 2.1] for the proof). Fix a parabolic cylinder $\mathcal{Q} = Q_0 \times (t_0 - R^2, t_0) \subset \mathbb{R}^{n+1}$ with a cube as horizontal slide. We denote by $\mathcal{D}(\mathcal{Q}_0)$ the class of all dyadic parabolic cylinders obtained from $\mathcal{Q}_0$ by a finite number of dyadic subdivisions; that is: we divide Q_0 into 2^n congruent sub-cubes Q' having sides parallel to Q_0 and $(t_0 - R^2, t_0)$ into four disjoint intervals I' of equal length $R^2/4$. The set of all parabolic sub-cylinders obtained by this dyadic subdivision consists of all cylinders of the form $\mathcal{Q}' \times I'$. We denote by $\tilde{\mathcal{Q}} \in \mathcal{D}(\mathcal{Q}_0)$ the predecessor of $\mathcal{Q}$, if $\mathcal{Q}$ has been obtained by exactly one dyadic subdivision from the parabolic cylinder $\tilde{\mathcal{Q}}$.

Proposition 2.7. *Let $\mathcal{Q}_0 \subset \mathbb{R}^n$ be a parabolic cylinder. Assume that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Q}_0$ are measurable sets satisfying the following properties*

(i) there exists $\delta > 0$ such that $|\mathcal{X}| < \delta|\mathcal{Q}_0|$;

(ii) if $\mathcal{Q} \in \mathcal{D}(\mathcal{Q}_0)$ then $|\mathcal{X} \cap \mathcal{Q}| > \delta|\mathcal{Q}|$ implies that $\tilde{\mathcal{Q}} \subset \mathcal{Y}$ where $\tilde{\mathcal{Q}}$ denotes the predecessor of $\mathcal{Q}$.

Then it follows that $|\mathcal{X}| < \delta|\mathcal{Y}|$.

To finish this section we recall an algebraic lemma, that we will need in the following. It is a classical result going back to Campanato, whose proof can be found in [27, Lemma 7.3]; see also [40, Lemma 1] and [42, Lemma 9.3].

Lemma 2.8. *Let $\Psi : [0, R_0] \to [0, \infty)$ be a non-decreasing function such that*

$$\Psi(r) \le c_0 \left(\frac{r}{R}\right)^{\delta_0} \Psi(R) + \mathcal{B} R^{\delta_1}, \quad \text{for every } r < R \le R_0, \text{ with } \mathcal{B} \ge 0,$$

where $0 < \delta_1 < \delta_0$, and c_0 is a given constant. Then there exists c_1 depending on c_0, δ_0 and δ_1 such that holds

$$\Psi(r) \le c_1 \left(\frac{r}{R}\right)^{\delta_1} \Psi(R) + c_1 \mathcal{B} r^{\delta_1}, \quad \text{for every } r \le R \le R_0.$$

3. Regularity estimates

In the rest of this section we consider

$$v \in C^0([t_0 - R^2, t_0]; L^2(B_R(x_0))) \cap L^2(t_0 - R^2, t_0; W^{1,2}(B_R(x_0)))$$

a weak solution to the following homogeneous Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x,t,Dv) + |v|^{m-1} v = 0 & \text{in } \mathcal{C}_R \\ v = u & \text{on } \partial_{\text{par}} \mathcal{C}_R, \end{cases} \tag{3.1}$$

in a fixed symmetric parabolic cylinder $\mathcal{C}_R = \mathcal{C}_R(x_0, t_0) \subseteq \Omega_T$, as defined in Section 2.1.

At certain points in the proofs of our results, we will need to scale from an arbitrary parabolic cylinder $\mathcal{C}_R(x_0, t_0)$ to $\mathcal{C}_1 = \mathcal{C}_1(0,0)$. So we introduce the following scaling procedure. For any $R > 0$ and any $(\tilde{x}, \tilde{t}) \in \mathcal{C}_1$, we consider the rescaled functions

$$\tilde{u}(\tilde{x}, \tilde{t}) := R^{\frac{2}{m-1}} \, u(x_0 + R\,\tilde{x}, t_0 + R^2\,\tilde{t}), \tag{3.2}$$

$$\tilde{v}(\tilde{x}, \tilde{t}) := R^{\frac{2}{m-1}} \, v(x_0 + R\,\tilde{x}, t_0 + R^2\,\tilde{t}), \tag{3.3}$$

$$\tilde{a}(\tilde{x}, \tilde{t}, z) := R^{\frac{m+1}{m-1}} \, a(x_0 + R\,\tilde{x}, t_0 + R^2\,\tilde{t}, R^{-\frac{m+1}{m-1}} z), \tag{3.4}$$

and

$$\tilde{f}(\tilde{x}, \tilde{t}) := R^{\frac{2m}{m-1}} \, f(x_0 + R\,\tilde{x}, t_0 + R^2\,\tilde{t}). \tag{3.5}$$

Then it is easy to see that $\tilde{u} = \tilde{v}$ on $\partial_{\text{par}} \mathcal{C}_1$ and the following equations weakly hold in $\mathcal{C}_1$

$$\tilde{u}_{\tilde{t}} - \operatorname{div} \tilde{a}(\tilde{x}, \tilde{t}, D\tilde{u}) + |\tilde{u}|^{m-1} \tilde{u} = \tilde{f} \quad \text{and} \quad \tilde{v}_{\tilde{t}} - \operatorname{div} \tilde{a}(\tilde{x}, \tilde{t}, D\tilde{v}) + |\tilde{v}|^{m-1} \tilde{v} = 0,$$

where the vector field $\tilde{a}$ satisfies (1.2).

Before starting, we want to underline that all the computations that we will do in this section are formal but they can be anyway made rigorous by a standard use of Steklov averages (see Section 2.2).

Our first result is the following lemma, in which, by means of suitable test functions, we will show that the L^γ norm of $|u-v|^m$ can be controlled by that of f.

Lemma 3.1. *Let $u \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ be the weak solution to problem* (2.3) *and v that to problem* (3.1)*, with $\mathcal{C}_R \subseteq \Omega_T$, then*

$$\int_{\mathcal{C}_R} |u-v|^{m\gamma} \,\mathrm{d}x\,\mathrm{d}t \le c \int_{\mathcal{C}_R} |f|^\gamma \,\mathrm{d}x\,\mathrm{d}t, \tag{3.6}$$

for any $\gamma \ge 1$; where $m > 1$ is the number appearing in (2.3) *and c depends only on m and γ.*

Proof. First, suppose that $\gamma = 1$. Consider a sequence of real smooth increasing functions $\{\Phi_h(s)\}$ that converges to the function $\Phi(s) \equiv \mathrm{sgn}(s)$ as $h \to \infty$. We test subtracted equations of u and v with $\Phi_h(u-v)$ to obtain

$$\begin{aligned}\int_{\mathcal{C}_R} &(u-v)_t \Phi_h(u-v)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\mathcal{C}_R} \langle a(x,t,Du) - a(x,t,Dv), D\Phi_h(u-v)\rangle \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\mathcal{C}_R} \big(|u|^{m-1}u - |v|^{m-1}v\big)\Phi_h(u-v)\,\mathrm{d}x\,\mathrm{d}t \;=\; \int_{\mathcal{C}_R} f\,\Phi_h \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.7}$$

Note that

$$(u-v)_t\Phi_h(u-v) = \partial_t\left[\int_0^{u-v} \Phi_h(s)\,\mathrm{d}s\right] + \Phi_h(0),$$

then

$$
\begin{aligned}
&\int_{\mathcal{C}_R} (u-v)_t \Phi_h(u-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{\mathcal{C}_R} \left(\partial_t \left[\int_0^{u-v} \Phi_h(s)\,\mathrm{d}s \right] + \Phi_h(0) \right) \mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{B_R} \int_{t_0-R^2}^{t_0} \partial_t \left[\int_0^{u-v} \Phi_h(s)\,\mathrm{d}s \right] \mathrm{d}x\,\mathrm{d}t + \int_{B_R} \int_{t_0-R^2}^{t_0} \Phi_h(0)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{B_R} \left[\int_0^{(u-v)(t_0)} \Phi_h(s)\,\mathrm{d}s - \int_0^{(u-v)(t_0-R^2)} \Phi_h(s)\,\mathrm{d}s \right] \mathrm{d}x \\
&\qquad + \int_{B_R} \int_{t_0-R^2}^{t_0} \Phi_h(0)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{B_R} \int_0^{(u-v)(t_0)} \Phi_h(s)\,\mathrm{d}s\,\mathrm{d}x + \int_{B_R} \int_{t_0-R^2}^{t_0} \Phi_h(0)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By the dominated convergence theorem we have, as $h \to \infty$,

$$
\int_{B_R} \int_0^{(u-v)(t_0)} \Phi_h(s)\,\mathrm{d}s\,\mathrm{d}x \;\to\; \int_{B_R} \int_0^{(u-v)(t_0)} \operatorname{sgn}(s)\,\mathrm{d}s\,\mathrm{d}x \geq 0
$$

$$
\int_{B_R} \int_{t_0-R^2}^{t_0} \Phi_h(0)\,\mathrm{d}x\,\mathrm{d}t \;\to\; 0.
$$

Moreover, by the first assumption in (1.2) and the fact that the test functions are increasing, we get

$$
\begin{aligned}
&\int_{\mathcal{C}_R} \langle a(x,t,Du) - a(x,t,Dv), D\Phi_h(u-v) \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad = \int_{\mathcal{C}_R} \langle a(x,t,Du) - a(x,t,Dv), D(u-v) \rangle \Phi_h' \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \geq \; c \int_{\mathcal{C}_R} |Du - Dv|^2 \, \Phi_h' \,\mathrm{d}x\,\mathrm{d}t \;\geq\; 0.
\end{aligned}
$$

So, dropping the nonnegative term and letting h goes to infinity, by Fatou's Lemma yields

$$
\begin{aligned}
\int_{B_R} \int_0^{(u-v)(t_0)} \operatorname{sgn}(s)\,\mathrm{d}s\,\mathrm{d}x \; + \int_{\mathcal{C}_R} \big(|u|^{m-1}u - |v|^{m-1}v \big) \operatorname{sgn}(u-v)\,\mathrm{d}x\,\mathrm{d}t \\
\leq \int_{\mathcal{C}_R} |f| \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Dropping again the nonnegative term, we arrive at

$$\int_{\mathcal{C}_R} \big(|u|^{m-1}u - |v|^{m-1}v\big)\mathrm{sgn}(u-v)\,\mathrm{d}x\,\mathrm{d}t \;\le\; \int_{\mathcal{C}_R} |f|\,\mathrm{d}x\,\mathrm{d}t.$$

Now, using the fact that

$$\big(|u|^{m-1}u - |v|^{m-1}v\big)\mathrm{sgn}(u-v) \;\ge\; c|u-v|^m, \quad \forall\, m>1, \tag{3.8}$$

we obtain

$$\int_{\mathcal{C}_R} |u-v|^m\,\mathrm{d}x\,\mathrm{d}t \le c\int_{\mathcal{C}_R} |f|\,\mathrm{d}x\,\mathrm{d}t,$$

that is (3.6) in the case $\gamma = 1$.

For the general case $\gamma > 1$, first we need to choose $|u-v|^{m(\gamma-1)-1}(u-v)$ as test function in the subtracted equations of u and v. Noting that

$$\begin{aligned}\int_{\mathcal{C}_R} &(u-v)_t|u-v|^{m(\gamma-1)-1}(u-v)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{B_R}\int_{t_0-R^2}^{t_0} \left(\partial_t\left[\int_0^{u-v}|s|^{m(\gamma-1)-1}s\,\mathrm{d}s\right]\right)\mathrm{d}t\,\mathrm{d}x\\ &= \int_{B_R}\left(\int_0^{(u-v)(t_0)}|s|^{m(\gamma-1)-1}s\,\mathrm{d}s\right)\mathrm{d}x \;\ge\; 0,\end{aligned}$$

we can drop the nonnegative terms and use (3.8) to arrive at

$$\int_{\mathcal{C}_R} |u-v|^{m\gamma}\,\mathrm{d}x\,\mathrm{d}t \;\le\; c\int_{\mathcal{C}_R} |f|\,|u-v|^{m(\gamma-1)}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.9}$$

Hence, estimate (3.6) plainly follows from (3.9) using the Hölder inequality on the right-hand side with exponents $\gamma > 1$ and $\gamma/(\gamma-1)$ and canceling the common terms. □

Remark 3.2. *We want to note that the previous lemma remains true also if* $0 < m \le 1$. *To prove the result for these values of* m *we have to use the following numerical inequality*

$$\big(|u|^{m-1}u - |v|^{m-1}v\big)\,sgn(u-v) \;\ge\; c\,\frac{|u-v|}{(|u|+|v|)^{1-m}}$$

instead of (3.8), *some usual tools as Hölder and Young inequalities and the fact that, since* u *is the solution to* (2.3), *by standard computation, we have*

$$\int_{\mathcal{C}_R} |u|^{m\gamma}\,\mathrm{d}x\,\mathrm{d}t \;\le\; c\int_{\mathcal{C}_R} |f|^{\gamma}\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\,\gamma\ge 1. \tag{3.10}$$

For more details see also the elliptic analog in [19].

Lemma 3.3. *Let* $u \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ *be the weak solution to problem* (2.3) *and* v *that to problem* (3.1)*, with* $\mathcal{C}_R \subseteq \Omega_T$*,* $m > 1$*, then*

$$\sup_{t_0 - R^2 < \tau < t_0} \int_{\mathcal{C}_R^\tau} |u - v| \, \mathrm{d}x \le \int_{\mathcal{C}_R} |f| \, \mathrm{d}x \, \mathrm{d}t, \tag{3.11}$$

where $\mathcal{C}_R^\tau := B_R(x_0) \times \{\tau\}$*. Moreover*

$$\int_{\mathcal{C}_R} \frac{|Du - Dv|^2}{(\alpha + |u - v|)^\xi} \, \mathrm{d}x \, \mathrm{d}t \le c \, \frac{\alpha^{1-\xi}}{\xi - 1} \int_{\mathcal{C}_R} |f| \, \mathrm{d}x \, \mathrm{d}t, \tag{3.12}$$

holds for $\alpha > 0$ *and* $\xi > 1$*, where* $c \equiv c(m, n, \nu/L) \ge 1$*.*

Proof. In this proof we extend the arguments used in that of [36, Lemma 4.1], by providing the needed modifications to handle the presence of the nonlinear lower order terms.

First, we can assume, without loss of generality, that the vertex of the cylinder (x_0, t_0) coincides with the origin and $R = 1$. To prove the estimates (3.11) and (3.12) for a general parabolic cylinder $\mathcal{C}_R(x_0, t_0)$ we have to use the scaling procedure introduced at the beginning of this section. For every $\varepsilon > 0$, we choose the test functions $\eta_{1,\varepsilon}^{\pm}$ given by

$$\eta_{1,\varepsilon}^{\pm}(x,t) := \pm \min\{1, (u - v)_{\pm}(x,t)/\varepsilon\} \phi(t), \tag{3.13}$$

where $\phi \in C^\infty(\mathbb{R})$ is a nonincreasing function such that $0 \le \phi \le 1$ and $\phi(t) = 0$ for all $t \ge \tau$, $\tau \in (-1, 0)$. In the following we will also need that $\int_{\mathbb{R}} |\phi_t| \, \mathrm{d}t = 1$. A direct computation gives

$$D\eta_{1,\varepsilon}^{\pm} = \frac{1}{\varepsilon} D(u - v) \chi_{\{0 < (u-v)_{\pm} < \varepsilon\}} \phi.$$

Using the weak formulation of u and v and testing the subtracted equations with $\eta_{1,\varepsilon}^{\pm}$, we obtain

$$\begin{aligned} \int_{\mathcal{C}_1} (u - v)_t \, \eta_{1,\varepsilon}^{\pm} \, \mathrm{d}x \, \mathrm{d}t &+ \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta_{1,\varepsilon}^{\pm} \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_{\mathcal{C}_1} (|u|^{m-1}u - |v|^{m-1}v) \, \eta_{1,\varepsilon}^{\pm} \, \mathrm{d}x \, \mathrm{d}t \; = \; \int_{\mathcal{C}_1} f \, \eta_{1,\varepsilon}^{\pm} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.14}$$

Noting that

$$(u - v)_t \min\big\{1, (u - v)_{\pm}/\varepsilon\big\} = \pm \partial_t \int_0^{(u-v)_{\pm}} \min\big\{1, s/\varepsilon\big\} \, \mathrm{d}s$$

and integrating by parts, we obtain for the first integral that appears in (3.14)

$$\int_{\mathcal{C}_1} (u-v)_t\, \eta^{\pm}_{1,\varepsilon}\, \mathrm{d}x\, \mathrm{d}t = \int_{\mathcal{C}_1} \left(\int_0^{(u-v)_\pm} \min\left\{1, s/\varepsilon\right\} \mathrm{d}s \right) (-\phi_t)\, \mathrm{d}x\, \mathrm{d}t.$$

Then, combining the equation above with (3.14) and using the fact that $|\eta^{\pm}_{1,\varepsilon}| \leq 1$, it follows

$$\begin{aligned} \int_{\mathcal{C}_1} \left(\int_0^{(u-v)_\pm} \min\{1, s/\varepsilon\}\, \mathrm{d}s \right) (-\phi_t)\, \mathrm{d}x\, \mathrm{d}t \\ + \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{1,\varepsilon} \rangle\, \mathrm{d}x\, \mathrm{d}t \\ + \int_{\mathcal{C}_1} (|u|^{m-1}u - |v|^{m-1}v)\, \eta^{\pm}_{1,\varepsilon}\, \mathrm{d}x\, \mathrm{d}t \;\leq\; \int_{\mathcal{C}_1} |f|\, \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{3.15}$$

Observe that the second term in the left-hand side of the previous inequality is nonnegative by (1.2); also, the third integral is nonnegative by the definitions of $\eta^{\pm}_{1,\varepsilon}$ and the monotony of the function $s \to |s|^{m-1}s$. Moreover, the dominated convergence theorem as $\varepsilon \to 0$ yields

$$0 \leq \int_{\mathcal{C}_1} \left(\int_0^{(u-v)_\pm} \min\{1, s/\varepsilon\}\, \mathrm{d}s \right) (-\phi_t)\, \mathrm{d}x\, \mathrm{d}t \;\to\; \int_{\mathcal{C}_1} (u-v)_\pm (-\phi_t)\, \mathrm{d}x\, \mathrm{d}t.$$

Thus, we arrive at

$$\int_{\mathcal{C}_1} (u-v)_\pm (-\phi_t)\, \mathrm{d}x\, \mathrm{d}t \leq \int_{\mathcal{C}_1} |f|\, \mathrm{d}x\, \mathrm{d}t,$$

that implies

$$\int_{\mathcal{C}_1} |u-v| (-\phi_t)\, \mathrm{d}x\, \mathrm{d}t \leq \int_{\mathcal{C}_1} |f|\, \mathrm{d}x\, \mathrm{d}t.$$

Letting ϕ approximate the characteristic function of $(-\infty, \tau)$, taking any $\tau \in (-1, 0)$, it yields

$$\int_{\mathcal{C}_1^\tau} |u-v|\, \mathrm{d}x \leq \int_{\mathcal{C}_1} |f|\, \mathrm{d}x\, \mathrm{d}t$$

and then (3.11) is proved for $\mathcal{C}_1$.

From (3.15), dropping the nonnegative terms, we also deduce

$$\int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{1,\varepsilon}\rangle \,\mathrm{d}x\,\mathrm{d}t \;\le\; \int_{\mathcal{C}_1} |f| \,\mathrm{d}x\,\mathrm{d}t,$$

that implies

$$\sup_{\varepsilon>0} \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{1,\varepsilon}\rangle \,\mathrm{d}x\,\mathrm{d}t \;\le\; \int_{\mathcal{C}_1} |f| \,\mathrm{d}x\,\mathrm{d}t. \tag{3.16}$$

Now we need to test the subtracted equations of u and v with a different choice of test function $\eta^{\pm}_{2,\varepsilon}$, given by

$$\eta^{\pm}_{2,\varepsilon} = \frac{\eta^{\pm}_{1,\varepsilon}}{(\alpha + (u-v)_{\pm})^{\xi-1}}, \quad \xi > 1, \quad \varepsilon,\, \alpha > 0.$$

We get

$$\int_{\mathcal{C}_1} (u-v)_t\, \eta^{\pm}_{2,\varepsilon} \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{2,\varepsilon}\rangle \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\mathcal{C}_1} (|u|^{m-1}u - |v|^{m-1}v)\, \eta^{\pm}_{2,\varepsilon} \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{C}_1} f\, \eta^{\pm}_{2,\varepsilon} \,\mathrm{d}x\,\mathrm{d}t. \tag{3.17}$$

For the first integral on the left-hand side, as for the first term in (3.15), by integration by parts, we have

$$\int_{\mathcal{C}_1} (u-v)_t\, \eta^{\pm}_{2,\varepsilon} \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{C}_1} \left(\int_0^{(u-v)_{\pm}} \frac{\min\{1, s/\varepsilon\}}{(\alpha+s)^{\xi-1}} \,\mathrm{d}s \right) (-\phi_t) \,\mathrm{d}x\,\mathrm{d}t.$$

Then, using (3.11), recalling that $\xi > 1$ and choosing ϕ such that $\int |\phi_t| \,\mathrm{d}t = 1$, we obtain

$$\begin{aligned}
\int_{\mathcal{C}_1} (u-v)_t\, \eta^{\pm}_{2,\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \;&\le\; \alpha^{1-\xi} \int_{\mathcal{C}_1} \left(\int_0^{(u-v)_{\pm}} \mathrm{d}s \right) (-\phi_t) \,\mathrm{d}x\,\mathrm{d}t \\
&=\; \alpha^{1-\xi} \int_{\mathcal{C}_1} (u-v)_{\pm} (-\phi_t) \,\mathrm{d}x\,\mathrm{d}t \\
&\le\; \alpha^{1-\xi} \int_{\mathcal{C}_1} |u-v||\phi_t| \,\mathrm{d}x\,\mathrm{d}t \\
&\le\; \alpha^{1-\xi} \sup_{\tau} \int_{\mathcal{C}_1^{\tau}} |u-v| \,\mathrm{d}x \int_{\mathbb{R}} |\phi_t| \,\mathrm{d}t \\
&\le\; \alpha^{1-\xi} \int_{\mathcal{C}_1} |f| \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.18}$$

For the second term in the left-hand side of (3.17), we have

$$
\begin{aligned}
\int_{\mathcal{C}_1} &\langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{2,\varepsilon}\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{1,\varepsilon}\rangle \frac{1}{(\alpha + (u-v)_{\pm})^{\xi-1}} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + (1-\xi) \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D(u-v)_{\pm}\rangle \\
&\qquad \times \frac{\eta^{\pm}_{1,\varepsilon}}{(\alpha + (u-v)_{\pm})^{\xi}} \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.19)
\end{aligned}
$$

By (3.16) and $\xi > 1$ the first integral in the right-hand side of the previous equality can be estimated as follows

$$
\begin{aligned}
\int_{\mathcal{C}_1} &\langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{1,\varepsilon}\rangle \frac{1}{(\alpha + (u-v)_{\pm})^{\xi-1}} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \alpha^{1-\xi} \sup_{\varepsilon>0} \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{1,\varepsilon}\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\le \alpha^{1-\xi} \int_{\mathcal{C}_1} |f| \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.20)
\end{aligned}
$$

Formula (3.17) also implies, by dropping the nonnegative term and using (3.18), the following estimate

$$
\begin{aligned}
\int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{2,\varepsilon}\rangle \,\mathrm{d}x\,\mathrm{d}t \;&\le\; \int_{\mathcal{C}_1} |f||\eta^{\pm}_{2,\varepsilon}| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \left| \int_{\mathcal{C}_1} (u-v)_t \, \eta^{\pm}_{2,\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\le\; 2\,\alpha^{1-\xi} \int_{Q} |f| \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

and so

$$
\left| \int_{\mathcal{C}_1} \langle a(x,t,Du) - a(x,t,Dv), D\eta^{\pm}_{2,\varepsilon}\rangle \,\mathrm{d}x\,\mathrm{d}t \right| \le 2\,\alpha^{1-\xi} \int_{\mathcal{C}_1} |f| \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.21)
$$

Using (3.20) and (3.21) in (3.19), we obtain

$$
\begin{aligned}
(\xi - 1) \int_{\mathcal{C}_1} &\langle a(x,t,Du) - a(x,t,Dv), D(u-v)_{\pm}\rangle \frac{\eta^{\pm}_{1,\varepsilon}}{(\alpha + (u-v)_{\pm})^{\xi}} \,\mathrm{d}x\,\mathrm{d}t \\
&\le 3\,\alpha^{1-\xi} \int_{\mathcal{C}_1} |f| \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By recalling the definition of $\eta^{\pm}_{1,\varepsilon}$ and by the first assumption in (1.2), we arrive at

$$\int_{\mathcal{C}_1} \frac{|Du-Dv|^2}{(\alpha+|u-v|)^{\xi}} \min\{1,|u-v|/\varepsilon\}\,\mathrm{d}x\,\mathrm{d}t \;\le\; c\,\frac{\alpha^{1-\xi}}{\xi-1}\int_{\mathcal{C}_1}|f|\,\mathrm{d}x\,\mathrm{d}t,$$

where we also used the definition of the approximating function ϕ. Letting $\varepsilon\to 0$ yields (3.12). The proof is complete. □

Lemma 3.4. *Let $u\in C^0([-T,0];L^2(\Omega))\cap L^2(-T,0;W^{1,2}_0(\Omega))$ be the weak solution to problem* (2.3) *and v that to problem* (3.1)*, with $1<m<\infty$. Then there exists a constant $c\equiv c(m,n,L/\nu)$ for which*

$$⨍_{\mathcal{C}_R}\big(R^{-1}|u-v|+|Du-Dv|\big)\,\mathrm{d}x\,\mathrm{d}t \le c\left(⨍_{\mathcal{C}_R}|f|\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{m+1}{2m}}. \tag{3.22}$$

Proof. We assume that the vertex (x_0,t_0) of the cylinder $\mathcal{C}_R$ coincides with $(0,0)$ and we fix $R=1$. Take

$$\alpha=\left(⨍_{\mathcal{C}_1}|u-v|^m\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{m}},$$

and $\xi=m>1$ in Lemma 3.3. We can suppose $\alpha>0$, otherwise $u=v$ and (3.22) follows trivially. Moreover, by Lemma 3.1 with $\gamma=1$, we have the following estimate for α,

$$\alpha\le c\left(⨍_{\mathcal{C}_1}|f|\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{m}}. \tag{3.23}$$

Using Cauchy-Schwarz inequality, together with the estimates (3.12) and (3.23), we obtain

$$\begin{aligned}
&⨍_{\mathcal{C}_1}|Du-Dv|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=⨍_{\mathcal{C}_1}\frac{|Du-Dv|}{(\alpha+|u-v|)^{\frac{m}{2}}}\,(\alpha+|u-v|)^{\frac{m}{2}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\le\left(⨍_{\mathcal{C}_1}\frac{|Du-Dv|^2}{(\alpha+|u-v|)^m}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\left(⨍_{\mathcal{C}_1}(\alpha+|u-v|)^m\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\\
&\qquad\le c\,\alpha^{(1-m)\frac12}\,\alpha^{\frac{m}{2}}\left(⨍_{\mathcal{C}_1}|f|\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\le\left(⨍_{\mathcal{C}_1}|f|\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{m+1}{2m}}
\end{aligned}$$

and the lemma is proved for $\mathcal{C}_1=\mathcal{C}_1(0,0)$. To show the result for a general cylinder $\mathcal{C}_R(x_0,t_0)$ we have to use the scaling procedure defined in (3.2)–(3.5) together with Poincaré inequality (that holds since $v=u$ on $\partial_{\mathrm{par}}\mathcal{C}_R$). □

Lemma 3.5. *Let the assumptions of Lemma* 3.4 *hold and suppose that* $f \in L^{1,\theta}(\mathcal{C}_R)$*,* $\theta \in [0, N]$*. Then*

$$\int_{\mathcal{C}_R} \big(R^{-1}|u-v| + |Du - Dv|\big)\,\mathrm{d}x\,\mathrm{d}t \le c\,R^{N-\frac{\theta(m+1)}{2m}}\|f\|_{L^{1,\theta}(\mathcal{C}_R)}^{\frac{m+1}{2m}}. \tag{3.24}$$

The constant $c > 0$ *depends only on* m*,* n*,* L/ν *and* γ*.*

Proof. We can easily deduce (3.24) from (3.22), since

$$\|f\|_{L^1(\mathcal{C}_R)} \le R^{N-\theta}\|f\|_{L^{1,\theta}(\mathcal{C}_R)},$$

for any $\theta \in [0, N]$. □

Using Lemma 2.1 together with the fact that the Marcinkiewicz space $\mathcal{M}^\gamma$ is continuously embedded in $L(\gamma, q)$, from (3.22) we obtain the following result.

Lemma 3.6. *Let the assumptions of Lemma* 3.4 *hold and suppose that* $f \in L^\theta(\gamma, q)(\mathcal{C}_R)$ *for some* $\gamma > 1$*,* $q \in (0, \infty]$ *and* $\theta \in [0, N]$*. Then,*

$$\int_{\mathcal{C}_R} \big(R^{-1}|u-v| + |Du - Dv|\big)\,\mathrm{d}x\,\mathrm{d}t \le c\,R^{N-\frac{\theta(m+1)}{2m\gamma}}\|f\|_{L^\theta(\gamma,q)(\mathcal{C}_R)}^{\frac{m+1}{2m}} \tag{3.25}$$

where the constant c *depends only on* m*,* n*,* L/ν *and* γ*.*

In Lemma 3.7 below, we state another tool that we will use in the proof of Theorem 1.1.

Let $\mathcal{Q}_0$ be a fixed parabolic cylinder such that $n^2\mathcal{Q}_0 \subset\subset \Omega_T$ and $|\mathcal{Q}_0| \le 1$. According to the definitions given in Section 2.4, we shall consider $M^* \equiv M^*_{0,n^2\mathcal{Q}_0}$. Thus, keeping in mind the properties of dyadic cubes given at the end of Section 2.5, we have the following lemma.

Lemma 3.7. *Let* $u \in C^0([-T, 0]; L^2(\Omega)) \cap L^2(-T, 0; W_0^{1,2}(\Omega))$ *be the solution to* (2.3)*, with* $1 < m < \infty$*. Then for every* $S > 1$ *there exists a number* $\varepsilon \equiv \varepsilon(m, n, L/\nu, S) \in (0, 1)$ *such that if* $\lambda > 1$ *and* $\mathcal{Q} \subset \mathcal{Q}_0$ *is a dyadic sub-cylinder of* $\mathcal{Q}_0$ *verifying*

$$\begin{aligned}\Big|\mathcal{Q} \cap \big\{(x,t) \in \mathcal{Q}_0 : M^*(|Du|+1)(x,t) > AS\lambda \\ \text{and } [M^*(f)]^{\frac{m+1}{2m}}(x,t) \le \varepsilon\,\lambda\big\}\Big| \;>\; S^{-2\chi}|\mathcal{Q}|\end{aligned}$$

then its predecessor $\tilde{\mathcal{Q}}$ *satisfies*

$$\tilde{\mathcal{Q}} \subseteq \{(x,t) \in \mathcal{Q}_0 : M^*(|Du|+1)(x,t) > \lambda\}.$$

Here $\chi \equiv \chi(n, L/\nu) > 1$ *is the higher integrability exponent as in Theorem* 2.6*, while* $A \equiv A(m, n, L/\nu) > 1$ *is an absolute constant.*

The proof of this lemma follows from [6, Lemma 6.2], by taking into account the modifications in [19, Lemma 3.6], where lower order terms are considered.

We conclude this section by presenting the following lemma that provides another important tool for the proof of our main result. This lemma could have its own interest, since an intermediate Morrey space regularity of $|Du|$ is shown.

Lemma 3.8. *Let $u \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ be the solution to (2.3), with $1 < m < \infty$. Assume that $f \in L^\theta(\gamma,q)(\Omega_T)$ with $1 < 2\gamma \leq \theta \leq N$ and $q \in (0,\infty]$, then the following inequality*

$$\||Du| + \varsigma\|_{L^{1,\frac{\theta(m+1)}{2m\gamma}}(\mathcal{C}_\sigma)} \leq c\,(\rho-\sigma)^{\frac{\theta(m+1)}{2m\gamma}-N}\||Du|+\varsigma\|_{L^1(\mathcal{C}_\rho)} + c\,\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_\rho)}, \tag{3.26}$$

holds for every couple of concentric parabolic cylinders $\mathcal{C}_\sigma \subset \mathcal{C}_\rho \subseteq \Omega_T$; where $c \equiv c(m,n,q,L/\nu,\gamma)$ is a positive constant.

Proof. Let us fix a couple of concentric parabolic cylinders $\mathcal{C}_\sigma \subset \mathcal{C}_\rho \subseteq \Omega_T$. Take $(x_0,t_0) \in \mathcal{C}_\sigma$ and $\mathcal{C}_R = \mathcal{C}_R(x_0,t_0)$ such that

$$0 < R \leq d_{\mathrm{par}}((x_0,t_0),\partial\mathcal{C}_\rho) := \inf_{(x,t)\in\partial\mathcal{C}_\rho}\Big\{\max\Big\{|x_0-x|,\sqrt{|t_0-t|}\Big\}\Big\},$$

so that $\mathcal{C}_R \subseteq \mathcal{C}_\rho$. Let v be the solution to (3.1). By means of the De Giorgi-Nash-Moser theory, that is estimate (2.13) with $q=1$, we get, for any $r \in (0,R]$

$$\int_{\mathcal{C}_r}(|Du|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t \leq c\left(\frac{r}{R}\right)^{N-1+\varpi}\int_{\mathcal{C}_R}(|Du|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{C}_R}|Du-Dv|\,\mathrm{d}x\,\mathrm{d}t,$$

where $c \equiv c(n,L/\nu)$ and $\varpi \equiv \varpi(n,L/\nu) \in (0,1/2]$. Combining the estimate above with (3.25) in Lemma 3.6, we get

$$\int_{\mathcal{C}_r}(|Du|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t \leq c\left(\frac{r}{R}\right)^{N-1+\varpi}\int_{\mathcal{C}_R}(|Du|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t + c\,R^{N-\frac{\theta(m+1)}{2m\gamma}}\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_R)}.$$

We can apply the algebraic Lemma 2.8 by choosing

$$\Psi(r) = \int_{\mathcal{C}_r} (|Du| + \varsigma)\,\mathrm{d}x\,\mathrm{d}t, \quad \mathcal{B} = c\,\|f\|_{L^\theta(\gamma,q)(\mathcal{C}_\rho)}^{\frac{m+1}{2m}}, \quad R_0 := d_{\mathrm{par}}((x_0,t_0),\partial\mathcal{C}_\rho)$$

and

$$\delta_0 := N - 1 + \varpi \,>\, N - \frac{\theta(m+1)}{2m\gamma} =: \delta_1 \quad (\text{since } 2\gamma < \theta).$$

By the fact that $R_0 > \rho - \sigma$, we have for any $\mathcal{C}_r \subseteq \mathcal{C}_\rho$ with center in $\mathcal{C}_\sigma$

$$\begin{aligned}&\int_{\mathcal{C}_r} (|Du| + \varsigma)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \le c\left((\rho-\sigma)^{\frac{\theta(m+1)}{2m\gamma}-N}\int_{\mathcal{C}_\rho}(|Du|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t + \|f\|_{L^\theta(\gamma,q)(\mathcal{C}_\rho)}^{\frac{m+1}{2m}}\right) r^{N-\frac{\theta(m+1)}{2m\gamma}},\end{aligned}$$

from which we can plainly deduce (3.26). □

4. Proofs of the main results

4.1. Integrability of Du

For the reader's convenience, we restate Theorem 1.1 from the Introduction.

Theorem 4.1. *Let $q \in (0,\infty]$. Assume (1.2) and $f \in L^\theta(\gamma,q)(\Omega_T)$ with γ, θ such that $1 < \gamma \le 2\theta/(\theta+2)$, $\;2 < \theta \le N = n+2$. Then the solution $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ to (2.1), with $1 < m < 1/(\gamma-1)$, satisfies*

$$|Du| \in L^\theta\Big(\frac{2m\gamma}{m+1}, \frac{2mq}{m+1}\Big) \text{ locally in } \Omega_T.$$

Moreover, the local estimate

$$\|Du\|_{L^\theta(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{C}_{R/2})} \,\le\, c\,R^{\frac{\theta(m+1)}{2m\gamma}-N}\||Du|+\varsigma\|_{L^1(\mathcal{C}_R)} + c\|f\|_{L^\theta(\gamma,q)(\mathcal{C}_R)}^{\frac{m+1}{2m}} \tag{4.1}$$

holds for any parabolic cylinder $\mathcal{C}_R \subseteq \Omega$, where c depends only on m, n, q, L/ν and γ.

Once established the regularity and comparison estimates in Section 3, the general strategy of the proof closely follows that of the proof of Theorem 6.1 in [6], where no lower order terms are considered (see also [24] and [36, 35]). Nevertheless as we expect recalling the elliptic case, [19], we have to make some modifications due to the different exponents that we are handling. We sketch the proof in a few steps.

Proof. Let $\mathcal{Q}_0$ be a fixed cylinder satisfying $n^2\,\mathcal{Q}_0 \subset\subset \Omega_T$ and $|\mathcal{Q}_0| \le 1$.

Step 1 - Application of Calderón-Zygmund-Krylov-Safonov covering theorem. We consider the maximal operator $M^* = M^*_{0,n^2\mathcal{Q}_0}$ and the sets $\mathcal{X}$ and $\mathcal{Y}$ defined for $k \in \mathbb{N}$ by

$$\mathcal{X} := \Big\{(x,t) \in \mathcal{Q}_0 : M^*(|Du| + \varsigma)(x,t) > (AS)^{k+1}\lambda_0 \\ \text{and } [M^*(f)]^{\frac{m+1}{2m}}(x,t) \le \varepsilon (AS)^k \lambda_0\Big\},$$

$$\mathcal{Y} := \Big\{(x,t) \in \mathcal{Q}_0 : M^*(|Du| + \varsigma)(x,t) > (AS)^k \lambda_0\Big\},$$

$$\text{with } \lambda_0 := 2\,\bar{c}\,n^{2N}\,S^{2\chi} \fint_{n^2\mathcal{Q}_0} (|Du| + \varsigma)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.2}$$

where $S > 1$, A, χ, ε are as in Lemma 3.7. By Lemma 3.7 the hypothesis (ii) in Proposition 2.7, with $\delta := S^{2\chi}$, is satisfied. In a similar way it is possible to prove that (i) holds, too. Thus, the application of Proposition 2.7, the definitions of $\mathcal{X}$ and $\mathcal{Y}$ and the multiplication by a factor $(AS)^{2m\gamma(k+1)/(m+1)}$ yield

$$\begin{aligned}(AS)^{\frac{2m\gamma(k+1)}{m+1}} \lambda_0^{\frac{2m\gamma}{m+1}} \mu_1((AS)^{k+1}\lambda_0) \\ \le (AS)^{\frac{2m\gamma k}{m+1}} A^{\frac{2m\gamma}{m+1}} S^{\frac{2m\gamma}{m+1} - 2\chi} \lambda_0^{\frac{2m\gamma}{m+1}} \mu_1((AS)^k\lambda_0) \\ + (AS)^{\frac{2m\gamma k}{m+1}} \left(\frac{AS}{\varepsilon}\right)^{\frac{2m\gamma}{m+1}} (\lambda_0\varepsilon)^{\frac{2m\gamma}{m+1}} \mu_2(\varepsilon(AS)^k\lambda_0),\end{aligned} \tag{4.3}$$

where, for any $K \ge 0$, we denoted by

$$\mu_1(K) := \big|\big\{(x,t) \in \mathcal{Q}_0 : M^*(|Du| + \varsigma)(x,t) > K\big\}\big| \tag{4.4}$$

and

$$\mu_2(K) := \big|\big\{(x,t) \in \mathcal{Q}_0 : [M^*(f)]^{\frac{m+1}{2m}}(x,t) > K\big\}\big|. \tag{4.5}$$

Now observe that, since $\chi > 1$ and $m < 1/(\gamma - 1)$,

$$d := 2\chi - \frac{2m\gamma}{m+1} \;=\; 2\left(\chi - \frac{\gamma m}{m+1}\right) > 0.$$

Therefore, we can choose S as follows

$$S := (4A^{\frac{2m\gamma}{m+1}})^{\frac{1}{d}}$$

and, by elementary estimates, inequality (4.3) provides the existence of a constant $c \equiv c(m,n,L/\nu)$ such that, for every $k \geq 0$,

$$
\begin{aligned}
(AS)^{\frac{2\gamma(k+1)}{m+1}} \lambda_0^{\frac{2m\gamma}{m+1}} \mu_1((AS)^{k+1}\lambda_0)& \\
\leq \frac{1}{4}(AS)^{\frac{2m\gamma k}{m+1}} \lambda_0^{\frac{2m\gamma}{m+1}} \mu_1((AS)^k\lambda_0)& \\
+ c(AS)^{\frac{2m\gamma k}{m+1}} (\varepsilon\lambda_0)^{\frac{2m\gamma}{m+1}} \mu_2((AS)^k\varepsilon\lambda_0)&.
\end{aligned} \tag{4.6}
$$

Step 2 - Level sets estimates. In order to establish some Lorentz spaces estimates on level sets we proceed as in the elliptic case; see Theorem 4.1 in [19]. Taking $0 < \beta < \infty$ and operating some manipulations, we get

$$
\begin{aligned}
&\int_0^\infty [\lambda^{\frac{2m\gamma}{m+1}} \mu_1(\lambda)]^{\frac{\beta(m+1)}{2m\gamma}} \frac{\mathrm{d}\lambda}{\lambda} \\
&\quad \leq \left(\frac{1}{\beta} + \tilde{c}^\beta (AS)^\beta \log(AS)\right) \lambda_0^\beta |\mathcal{Q}_0|^{\frac{\beta(m+1)}{2m\gamma}} + \tilde{c}^\beta (AS)^\beta \log(AS) J(\infty),
\end{aligned}
$$

where

$$
J(\infty) := \sum_{k=0}^\infty \left((AS)^{\frac{2m\gamma k}{m+1}} (\varepsilon\lambda_0)^{\frac{2m\gamma}{m+1}} \mu_2((AS)^k\varepsilon\lambda_0)\right)^{\frac{\beta(m+1)}{2m\gamma}}
$$

and the constant $\tilde{c} > 1$ is increasing in the variables m, n, L/ν and decreasing in β, such that $\tilde{c} \to \infty$ as $\beta \to 0$, while it remains bounded when β is bounded away from zero. From the previous inequality, we easily deduce

$$
\begin{aligned}
\int_0^\infty [\lambda^{\frac{2m\gamma}{m+1}} \mu_1(\lambda)]^{\frac{\beta(m+1)}{2m\gamma}} \frac{\mathrm{d}\lambda}{\lambda}& \\
\leq \left(\frac{1}{\beta} + 2\tilde{c}^\beta (AS)^\beta \log(AS)\right) \lambda_0^\beta |\mathcal{Q}_0|^{\frac{\beta(m+1)}{2m\gamma}}& \\
+ \tilde{c}^\beta (AS)^{2\beta} \int_0^\infty [\lambda^{\frac{2m\gamma}{m+1}} \mu_2(\lambda)]^{\frac{\beta(m+1)}{2m\gamma}} \frac{\mathrm{d}\lambda}{\lambda}&.
\end{aligned} \tag{4.7}
$$

Thus, recalling the definitions of μ_1 and μ_2 given in (4.4)-(4.5), by standard properties of maximal operators, choosing $\beta = 2mq/(m+1)$, $q \in (0,\infty)$, we get

$$
\|Du\|_{L(\frac{2m\gamma}{m+1}, \frac{2mq}{m+1})(\mathcal{Q}_0)} \leq \tilde{c}\, \lambda_0 |\mathcal{Q}_0|^{\frac{m+1}{2m\gamma}} + \tilde{c}\, \|M^*(f)\|_{L(\gamma,q)(\mathcal{Q}_0)}^{\frac{m+1}{2m}}, \tag{4.8}
$$

up to relabeling the constant $\tilde{c}$, by keeping the same properties as before. Now we can use (2.12) with $\beta = 0$, passing to the outer parabolic cylinder and taking $\sigma = 2$. We get

$$\|M^*(f)\|_{L(\gamma,q)(\mathcal{Q}_0)}^{\frac{m+1}{2m}} \le c\,\|f\|_{L^\theta(\gamma,q)(n^2\mathcal{Q}_0)}^{\frac{m+1}{2m}}. \tag{4.9}$$

Finally, by means of (4.8), (4.9) and the definition of λ_0 in (4.2), we obtain

$$\begin{aligned}\|Du\|_{L(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{Q}_0)} \;&\le\; c\left(\fint_{n^2\mathcal{Q}_0}(|Du|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t\right)|\mathcal{Q}_0|^{\frac{m+1}{2m\gamma}}\\ &\quad + c\,\|f\|_{L^\theta(\gamma,q)(n^2\mathcal{Q}_0)}^{\frac{m+1}{2m}}.\end{aligned} \tag{4.10}$$

Similarly, we can deal with the case $q = \infty$ (for more details see [19]) and we arrive at

$$\|Du\|_{\mathcal{M}^{\frac{2m\gamma}{m+1}}(\mathcal{Q}_0)} \le c\left(\fint_{n^2\mathcal{Q}_0}(|Du|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t\right)|\mathcal{Q}_0|^{\frac{m+1}{2m\gamma}} + c\,\|f\|_{\mathcal{M}^{\gamma,\theta}(n^2\mathcal{Q}_0)}^{\frac{m+1}{2m}}.$$

Step 3 - Morrey spaces regularity. First, we can write estimate (4.10) for the scaled functions $\tilde{u}$ and $\tilde{f}$ defined at the beginning of Section 3 in the spherical cylinder $\mathcal{C}_1$. Passing to inner and outer cylinders[4], we get

$$\begin{aligned}&\|D\tilde{u}\|_{L(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{C}_{1/n^4})}\\ &\qquad\le c\left(\||D\tilde{u}|+\varsigma\|_{L^1(\mathcal{C}_{9/10})} + \|\tilde{f}\|_{L^\theta(\gamma,q)(\mathcal{C}_1)}^{\frac{m+1}{2m}}\right)\\ &\qquad\le c\left(\||D\tilde{u}|+\varsigma\|_{L^{1,\frac{(m+1)\theta}{2m\gamma}}(\mathcal{C}_{9/10})} + \|\tilde{f}\|_{L^\theta(\gamma,q)(\mathcal{C}_1)}^{\frac{m+1}{2m}}\right),\end{aligned} \tag{4.11}$$

where we also used that the definitions of m, θ and γ yield $(m+1)\theta/2m\gamma < N$.

Scaling back to $\mathcal{C}_\rho$, as an immediate consequence of the definitions of the involved functional norms, we deduce

$$\rho^{\frac{m+1}{m-1}-\frac{(m+1)N}{2m\gamma}}\|Du\|_{L(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{C}_{\rho/n^4})} \le c\,\Theta(\mathcal{C}_\rho)\rho^{\frac{m+1}{m-1}-\frac{(m+1)\theta}{2m\gamma}}, \tag{4.12}$$

where

$$\Theta(\mathcal{C}_\rho) := \||Du|+\varsigma\|_{L^{1,\frac{(m+1)\theta}{2m\gamma}}(\mathcal{C}_{9\rho/10})} + \|f\|_{L^\theta(\gamma,q)(\mathcal{C}_\rho)}^{\frac{m+1}{2m}}, \quad \forall\, \mathcal{C}_\rho \subseteq \Omega_T.$$

[4] Note that $n^2\mathcal{Q}_0 = \mathcal{Q}_{1/n^2} \subseteq \mathcal{C}_{1/n}$.

By means of a covering argument (see [40, Lemma 11] and [6, Theorem 6.1]), from (4.12) it follows

$$\|Du\|_{L^\theta(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{C}_{R/2})} \leq c\,\Theta(\mathcal{C}_{3R/4}),$$

that, together with Lemma 3.8 (by choosing $\rho = R$ and $\sigma = 27R/40$ there), yields

$$\|Du\|_{L^\theta(\frac{2m\gamma}{m+1},\frac{2mq}{m+1})(\mathcal{C}_{R/2})} \leq cR^{\frac{(m+1)\theta}{2m\gamma}-N}\||Du|+\varsigma\|_{L^1(\mathcal{C}_R)} + c\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_R)}. \tag{4.13}$$

Step 4 - Conclusion of the proof. We recall that we proved the estimate in (4.13) for the approximating solutions $u \equiv u_k$ to problem (2.3) with f_k given by the truncation of f. In order to conclude, it suffices to use the lower semicontinuity of the Lorentz-Morrey norms together with the standard approximating arguments stated in Section 2.2. □

4.2. Further extensions

In this section we will show how to modify the proof of Theorem 1.1 to deal with further extensions of the results considered until now. In the borderline case, i. e. Theorem 1.2, we just need to use the boundedness of the maximal operators in $L\log L$ and to proceed with the choice $\gamma = 1$. Concerning Theorem 1.3, we will need to take care of the relaxed assumptions on the lower order terms g, by choosing suitable test functions in the revised versions of the estimates proved in Section 3. Anyway, in view of the strength of the level sets estimates of the maximal operator for the gradient Du given by (4.6), the modifications will be minimal. Also, they are in clear accordance with the results proved by the authors in the elliptic case [19]. For the reader's convenience, we will give some details.

Proof of Theorem 1.2.

Proof. We fix $\gamma = 1$ and we proceed as in the proof of Theorem 4.1 up to estimate (4.7). Hence, taking $\beta = 2m/(m+1)$ we arrive at

$$\int_{\mathcal{Q}_0} |Du|^{\frac{2m}{m+1}}\,\mathrm{d}x\,\mathrm{d}t \leq c\lambda_0^{\frac{2m}{m+1}}|\mathcal{Q}_0| + c\int_{\mathcal{Q}_0} |M^*(f)|\,\mathrm{d}x\,\mathrm{d}t.$$

Now, we recall that the boundedness of the maximal operators in $L\log L$ yields

$$\|M^*(f)\|_{L^1(\mathcal{Q}_0)} \leq c\|f\|_{L\log L(\mathcal{Q}_0)}, \tag{4.14}$$

and so, by considering the definition of λ_0 in (4.2), we get

$$\begin{aligned}\left(\fint_{\mathcal{Q}_0}|Du|^{\frac{2m}{m+1}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{m+1}{2m}} &\le c|\mathcal{Q}_0|^{\frac{m+1}{2m}}\left(\fint_{n^2\mathcal{Q}_0}(|Du|^2+\varsigma^2)\,\mathrm{d}x\,\mathrm{d}t\right)\\ &\quad + c|\mathcal{Q}_0|^{\frac{m+1}{2m}}\|f\|^{\frac{m+1}{2m}}_{L\log L(\mathcal{Q}_0)}.\end{aligned}$$

Passing in a standard way to the outer and inner parabolic cylinder having a ball B_R as horizontal slice, and rescaling everything to $\mathcal{C}_1$, we obtain the analog of (4.11), i. e.,

$$\|D\tilde{u}\|_{L^{\frac{2m}{m+1}}(\mathcal{C}_{1/n^4})} \le c\||D\tilde{u}|+\varsigma\|_{L^{1,\frac{(m+1)\theta}{2m}}(\mathcal{C}_{9/10})} + c\|\tilde{f}\|^{\frac{m+1}{2m}}_{L\log L^{\theta}(\mathcal{C}_1)},$$

where $\tilde{u}$ and $\tilde{f}$ are as in (3.2) and (3.5). Scaling back, we get

$$\rho^{\frac{(N-\theta)(m+1)}{2m}}\|Du\|_{L^{\frac{2m}{m+1}}(\mathcal{C}_{\rho/n^4})} \le c\,\Theta(\mathcal{C}_\rho)$$

with

$$\Theta(\mathcal{C}_\rho) = \||Du|+\varsigma\|_{L^{1,\frac{\theta(m+1)}{2m}}(\mathcal{C}_{9\rho/10})} + \|f\|_{L\log L^{\theta}(\mathcal{C}_\rho)}.$$

Arguing as in Theorem 4.1 by a standard covering argument we arrive at

$$\|Du\|_{L^{\frac{2m}{m+1},\theta}(\mathcal{C}_{R/2})} \le c\,\Theta(\mathcal{C}_{3R/4}), \quad \forall\, \mathcal{C}_R \subseteq \Omega_T.$$

To estimate $\Theta(\mathcal{C}_{3R/4})$ we use the following inequality

$$\begin{aligned}\||Du|+\varsigma\|_{L^{1,\frac{(m+1)\theta}{2m}}(\mathcal{C}_\sigma)} &\le c(\rho-\sigma)^{\frac{\theta(m+1)}{2m}-N}\||Du|+\varsigma\|_{L^1(\mathcal{C}_\rho)}\\ &\quad + c\|f\|^{\frac{m+1}{2m}}_{L^{1,\theta}(\mathcal{C}_\rho)}, \quad \forall\, \mathcal{C}_\sigma \subset \mathcal{C}_\rho \subseteq \Omega_T,\end{aligned}$$

with $\rho = R$, $\sigma = 27R/40$, that we can prove in the same way as in Lemma 3.8, using (3.24) instead of (3.25). Finally, by the fact that

$$\|f\|_{L^{1,\theta}(\mathcal{C}_R)} \le c\|f\|_{L\log L^{\theta}(\mathcal{C}_R)},$$

we obtain the estimate in (1.10). $\square$

Proof of Theorem 1.3.

Proof. First, we have to modify the proof of Lemma 3.1 by using the following test function $\phi = h^{\gamma-1}|u-v|^{m(\gamma-1)-1}(u-v)$ in the subtracted equations of u and v. In view of the fact that the additional contribution given by h is positive, we can use again the algebraic inequality (3.8). It follows

$$\int_{\mathcal{C}_R} h^\gamma |u-v|^{m\gamma}\,\mathrm{d}x\,\mathrm{d}t \;\le\; c\int_{\mathcal{C}_R} |f|^\gamma\,\mathrm{d}x\,\mathrm{d}t,$$

for any $m > 1$, $\gamma \geq 1$ and $\mathcal{C}_R \subseteq \Omega_T$. This estimate will permit to obtain an additional integrability on $u - v$. Then, by Hölder inequality we plainly deduce

$$\int_{\mathcal{C}_R} |u-v|^{\frac{pm\gamma}{p+\gamma}}\,\mathrm{d}x\,\mathrm{d}t \;\le\; c\left(\int_{\mathcal{C}_R} |f|^\gamma\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{p+\gamma}}. \tag{4.15}$$

As expected (recall the observation at page 6), we notice that

$$\frac{mp\gamma}{p+\gamma} < m\gamma \quad \forall\, p > 0 \quad \text{and} \quad \frac{mp\gamma}{p+\gamma} \to m\gamma \text{ as } p \to \infty.$$

In view of (4.15), inequality (3.22) in Lemma 3.4 holds. Similarly, we can deduce the validity of (3.24) in Lemma 3.5 and (3.25) in Lemma 3.6 for any

$$1 < \frac{(p+1)}{p} < m < \infty.$$

At this time, we can repeat the entire proof of Theorem 4.1 with slight modifications. □

4.3. Integrability of u

In this section we establish Lorentz-Morrey space estimates for the solution u to (2.1). We will use the same techniques used to prove the spatial regularity for the gradient. Therefore, we will obtain an estimate on the level sets of the maximal operator associated to u, in terms of the level sets of a power of a maximal operator of the assigned datum f, up to a correction term (the equivalent of (4.3) for u). This will allow to prove Theorem 1.4, stated in the Introduction. Since the proof of this theorem is very close to the one of Theorem 1.1, we confine ourselves to outline the necessary modifications. Also, we recall that we always deal with the approximating solutions $u \equiv u_k$ defined in Section 2.2, abbreviating $f \equiv f_k$. Keeping in mind the notation used in Theorem 1.1, we have the analog of Lemma 3.7 for u.

Lemma 4.2. *Let $u \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ be the solution to* (2.3)*, with $1 < m < \infty$. Then for every $S > 1$ and $\chi_0 > 1$ there exists a number $\varepsilon \equiv \varepsilon(m,n,L/\nu,S,\chi_0) \in (0,1)$, such that if $\lambda > 1$ and $\mathcal{Q} \subset \mathcal{Q}_0$ is a dyadic sub-cylinder of $\mathcal{Q}_0$ verifying*

$$\Big|\mathcal{Q} \cap \Big\{(x,t) \in \mathcal{Q}_0 : M^*(|u|+\varsigma)(x,t) > AS\lambda \\ \text{and } [M^*_{\frac{2m}{m+1}}(f)]^{\frac{m+1}{2m}}(x,t) \le \varepsilon\,\lambda\Big\}\Big| > S^{-2\chi_0}|\mathcal{Q}|$$

then its predecessor $\tilde{\mathcal{Q}}$ satisfies

$$\tilde{\mathcal{Q}} \subseteq \{(x,t) \in \mathcal{Q}_0 : M^*(|u|+\varsigma)(x,t) > \lambda\}.$$

Here $A \equiv A(m,n,L/\nu) > 1$ is an absolute constant.

In order to prove this lemma, it suffices to follow the proof of Lemma 3.7, by replacing $M^*_{0,n^2\mathcal{Q}_0}(|Du|+\varsigma)$ and $M^*_{0,n^2\mathcal{Q}_0}(f)$ by $M^*_{0,n^2\mathcal{Q}_0}(|u|+\varsigma)$ and $M^*_{\frac{2m}{m+1},n^2\mathcal{Q}_0}(f)$, respectively, and by using the fact that (3.25) yields

$$\int_{\mathcal{C}_R} |u-v|\,\mathrm{d}x\,\mathrm{d}r \le c\,R^{N+1-\frac{\theta(m+1)}{2m\gamma}}\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_R)} \quad \forall\, \mathcal{C}_R \subseteq \Omega_T.$$

Moreover, the higher integrability for u, stated in Theorem 2.6, will permit to introduce the parameter $\chi_0 > 1$ which can be chosen arbitrarily large. This is the main difference with respect to Lemma 3.7, in which, indeed, the quantity χ was fixed.

Another important tool that we will need in order to prove Theorem 1.4 is the following intermediate Morrey spaces regularity, which takes the place of Lemma 3.8 in the proof of the main result of this section.

Lemma 4.3. *Let $u \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ be the solution to* (2.3)*, with $1 < m < \infty$. Assume that $f \in L^\theta(\gamma,q)(\Omega_T)$ with $1 < 2\gamma \le \theta \le N$ and $q \in (0,\infty]$, then the following inequality*

$$\|u\|_{L^{1,\frac{m(\theta-2\gamma)+\theta}{2m\gamma}}(\mathcal{C}_\sigma)} \le c\,(\rho-\sigma)^{\frac{m(\theta-2\gamma)+\theta}{2m\gamma}-N}\|(|u|+\varsigma\rho)\|_{L^1(\mathcal{C}_\rho)} + c\,\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_\rho)},$$

holds for every couple of concentric parabolic cylinders $\mathcal{C}_\sigma \subset \mathcal{C}_\rho \subseteq \Omega_T$; where $c \equiv c(m,n,q,L/\nu,\gamma)$ is a positive constant.

Remark 4.4. *Following* [40, Remark 13]*, we note that when $\rho \approx \sigma \approx \rho-\sigma \approx R$, the previous inequality implies*

$$\|(|u|+\varsigma R)\|_{L^{1,\frac{m(\theta-2\gamma)+\theta}{2m\gamma}}(\mathcal{C}_\sigma)} \le c\,R^{\frac{m(\theta-2\gamma)+\theta}{2m\gamma}-N}\|(|u|+\varsigma R)\|_{L^1(\mathcal{C}_\rho)} + c\,\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(\mathcal{C}_\rho)}.$$

To prove of Lemma 4.3 we can argue as in that of Lemma 3.8, by using (2.14) instead of (2.13).

Proof of Theorem 1.4. For the reader's convenience, we will follow the same division by steps given in the proof of Theorem 1.1.

Proof. Step 1 - Application of Calderón-Zygmund-Krylov-Safonov covering theorem. We can apply Proposition 2.7, using Lemma 4.2, to

$$\begin{aligned}\mathcal{X} := \Big\{&(x,t) \in \mathcal{Q}_0 : M^*(|u| + \varsigma)(x,t) > (AS)^{k+1}\lambda_0 \\ &\qquad \text{and } [M^*_{\frac{2m}{m+1}}(f)]^{\frac{m+1}{2m}}(x,t) \le \varepsilon (AS)^k \lambda_0\Big\}, \\ \mathcal{Y} := \Big\{&(x,t) \in \mathcal{Q}_0 : M^*(|u| + \varsigma)(x,t) > (AS)^k \lambda_0\Big\},\end{aligned}$$

with

$$\lambda_0 := 2\,\bar{c}\, n^{2N}\, S^{2\chi_0} \fint_{n^2\mathcal{Q}_0} (|u| + \varsigma)\,\mathrm{d}x\,\mathrm{d}t,$$

to obtain, for every $k \ge 0$,

$$\begin{aligned}&(AS)^{\frac{2m\theta\gamma(k+1)}{m(\theta-2\gamma)+\theta}} \lambda_0^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}} \mu_1((AS)^{k+1}\lambda_0) \\ &\quad \le (AS)^{\frac{2m\theta\gamma k}{m(\theta-2\gamma)+\theta}} A^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}} S^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}-2\chi_0} \lambda_0^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}} \mu_1((AS)^k\lambda_0) \\ &\qquad + (AS)^{\frac{2m\theta\gamma k}{m(\theta-2\gamma)+\theta}} \left(\frac{AS}{\varepsilon}\right)^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}} (\lambda_0\varepsilon)^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}} \mu_2(\varepsilon(AS)^k\lambda_0),\end{aligned}$$

where, for any $K \ge 0$,

$$\mu_1(K) := \big|\big\{(x,t) \in \mathcal{Q}_0 : M^*(|u| + \varsigma)(x,t) > K\big\}\big|$$

and

$$\mu_2(K) := \big|\big\{(x,t) \in \mathcal{Q}_0 : [M^*_{\frac{2m}{m+1}}(f)]^{\frac{m+1}{2m}}(x,t) > K\big\}\big|.$$

At this stage we take advantage of the possibility to choose χ_0 large enough to satisfy

$$d := 2\chi_0 - \frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta} > 0;$$

thus, by elementary estimates, we arrive at

$$
\begin{aligned}
(AS)^{\frac{2m\theta\gamma(k+1)}{m(\theta-2\gamma)+\theta}}\lambda_0^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}}\mu_1((AS)^{k+1}\lambda_0)& \\
\leq \frac{1}{4}(AS)^{\frac{2m\theta\gamma k}{m(\theta-2\gamma)+\theta}}&\lambda_0^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}}\mu_1((AS)^k\lambda_0) \\
&+c(AS)^{\frac{2m\theta\gamma k}{m(\theta-2\gamma)+\theta}}(\lambda_0\varepsilon)^{\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta}}\mu_2(\varepsilon(AS)^k\lambda_0).
\end{aligned}
$$

Step 2 - Level sets estimates. Now, to establish Lorentz spaces estimates we can closely follow in Step 2 in the proof of Theorem 1.1, to get

$$
\begin{aligned}
&\|u\|_{L(\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta},\frac{2m\theta q}{m(\theta-2\gamma)+\theta})(\mathcal{Q}_0)} \\
&\qquad \leq \tilde{c}\,\lambda_0|\mathcal{Q}_0|^{\frac{m(\theta-2\gamma)+\theta}{2m\theta\gamma}}+c\,\|M^*_{\frac{2m}{m+1}}(f)\|^{\frac{m+1}{2m}}_{L(\frac{\gamma(m+1)\theta}{m(\theta-2\gamma)+\theta},\frac{q(m+1)\theta}{m(\theta-2\gamma)+\theta})(\mathcal{Q}_0)},
\end{aligned}
$$

Furthermore, it suffices to use (2.12), with $\beta = 2m/(m+1)$ and $\sigma = 2$, to arrive at

$$
\begin{aligned}
\|u\|_{L(\frac{2m\theta\gamma}{m(\theta-2\gamma)+\theta},\frac{2m\theta q}{m(\theta-2\gamma)+\theta})(\mathcal{Q}_0)} \leq\ & c\left(\fint_{n^2\mathcal{Q}_0}(|u|+\varsigma)\,\mathrm{d}x\,\mathrm{d}t\right)|\mathcal{Q}_0|^{\frac{m(\theta-2\gamma)+\theta}{2m\theta\gamma}} \\
&+c\,\|f\|^{\frac{m+1}{2m}}_{L^\theta(\gamma,q)(n^2\mathcal{Q}_0)};
\end{aligned}
$$

recalling the definition of λ_0.

Step 3 - Morrey spaces regularity. Having proved the local Lorentz integrability for u, in order to combine this with the Morrey spaces information, Lemma 4.3 (recall Remark 4.4), we can proceed exactly as in Step 3 of Theorem 1.1.

Step 4 - Conclusion of the proof. The proof ends with the usual approximation argument, again by means of the lower semicontinuity of the Lorentz-Morrey norms. □

Note that, in the Lebesgue case $\theta = N$ and $\gamma = q$, by equation (1.1) it is known that u belongs to $L^{m\gamma}(\Omega_T)$ (recall (3.10)). Hence, Theorem 1.4 also provides an improved integrability of u, since $m\gamma < 2mN\gamma/(m(N-2\gamma)+N)$ when $2N/(N+2) < \gamma < N/2$.

Proof of Theorem 1.5. The proof of the borderline regularity for the solutions u can be obtained following that of Theorem 1.4 by taking into account the modifications provided in the proof of Theorem 1.2. We just stress that one needs to use Theorem 2.4 instead of (4.14). □

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