# On the relation between the curvature-dimension condition and the Bochner inequality

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**Abstract**

We show that on smooth manifolds there exists a direct link between the Bochner inequality and the (reduced) curvature-dimension condition: they can be seen, respectively, as the Eulerian and the Lagrangian point of view on Ricci curvature bounds.

Aim of this short note is to provide a new and simple proof of the following theorem:

**Theorem 1** *Let $M$ be a smooth Riemannian manifold which satisfies the $CD^*(K,N)$ condition, $K \in \mathbb{R}$, $N \in (1,\infty]$. Then for every $\varphi \in C^\infty(M)$ it holds*

$$\Delta\frac{|\nabla\varphi|^2}{2} \geq \frac{(\Delta\varphi)^2}{N} + \nabla\Delta\varphi\cdot\nabla\varphi + K|\nabla\varphi|^2. \tag{1}$$

This statement is obviously well known, as both the $CD^*(K,N)$ condition and the Bochner inequality are known to be equivalent to 'the Ricci curvature of $M$ is bounded from below by $K$ and its dimension above by $N$'. The key point here is the methodology of the proof (no Jacobi fields calculus is involved), its interpretation and the fact that the link between the $CD^*(K,N)$ condition and the Bochner inequality is direct and does not require to call into play the notion of Ricci curvature nor that of dimension.

We recall that the $CD^*(K,N)$ condition, introduced by Bacher-Sturm in [2] as a sort of local variant of the $CD(K,N)$ condition (the definition of $CD(K,N)$ spaces was independently given by Lott-Villani in [4] and by Sturm in [6]), in this setting reads as:

$$\partial_{tt}\mathcal{U}_N(\mu_t)_{|t=0} \geq \frac{K}{N}\int \rho_0^{1-\frac{1}{N}}|\nabla\varphi|^2\,\mathrm{dvol}, \tag{2}$$

where vol is the volume measure on $M$, $t\mapsto\mu_t=\rho_t\mathrm{vol}$ is a constant speed geodesic on $(\mathscr{P}_2(M),W_2)$ made of measures concentrated on some bounded subset of $M$, $\mathcal{U}_N(\mu_t):=\int u_N(\rho_t)\,\mathrm{dvol}$, $u_N(z):=-z^{1-\frac{1}{N}}$ (resp. $u_\infty(z):=z\log z$) and $\varphi$ is any locally Lipschitz Kantorovich potential from $\mu_0$ to $\mu_1$.

*proof* Clearly, it is sufficient to prove (1) for $\varphi\in C_c^\infty(M)$. Fix such $\varphi$, let $\rho$ be any smooth probability density with compact support and recall (see for instance Lemma 1.34 in [1]), that for $\varepsilon>0$ sufficiently small the function $\psi:=\varepsilon\varphi$ is $\frac{\mathsf{d}^2}{2}$-concave, $\mathsf{d}$ being the Riemannian distance on $M$, and $\exp(-t\nabla\psi):M\to M$ has smooth inverse for any $t\in[0,1]$. Fix such $\varepsilon$,

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let $\mu_0 := \rho\mathrm{vol}$, and $\mu_t := (\exp(-\nabla\psi))_\sharp\mu_0 = \rho_t\mathrm{vol}$, $t \in [0,1]$. Recall (see for instance Chapter 2 of [1]) that the evolution of $\rho_t$ is driven by

$$\partial_t\rho_t + \nabla\cdot(\rho_t\nabla\psi_t) = 0, \tag{3}$$

where $[0,1] \ni t \mapsto \psi_t$ is the only smooth solution of

$$\partial_t\psi_t + \frac{|\nabla\psi_t|^2}{2} = 0, \tag{4}$$

with $\psi_0 := \psi$. Using (3), (4) one easily gets, by explicit computation, that

$$\partial_{tt}\mathcal{U}_N(\mu_t) = \int \mathrm{p}_{2,N}(\rho_t)(\Delta\psi_t)^2 - \mathrm{p}_N(\rho_t)\nabla(\Delta\psi_t)\cdot\nabla\psi_t - \mathrm{p}_N(\rho_t)\partial_t\Delta\psi_t\,\mathrm{dvol},$$

where $\mathrm{p}_N, \mathrm{p}_{2,N} : [0,\infty) \to [0,\infty)$ are given by $\mathrm{p}_N(z) := zu_N'(z) - u_N(z)$, $\mathrm{p}_{2,N}(z) := z\mathrm{p}_N'(z) - \mathrm{p}_N(z)$. Hence if (2) holds we must have, for $N < \infty$, that

$$\int \rho^{1-\frac{1}{N}}\left(-\frac{(\Delta\psi)^2}{N^2} - \frac{\nabla(\Delta\psi)\cdot\nabla\psi}{N} - \frac{\partial_t\Delta\psi_t|_{t=0}}{N}\right)\mathrm{dvol} \geq \frac{K}{N}\int\rho^{1-\frac{1}{N}}|\nabla\psi|^2\,\mathrm{dvol}. \tag{5}$$

Now recall that $\rho$ is non negative and chosen independently on $\psi$, thus, taking into account the linearity of $\Delta$ and (4), we get the the conclusion. Similarly for $N = \infty$. $\square$

The proof of the converse implication looks technically more involved, as in general Kantorovich potentials are not smooth, nor they can be approximated by smooth ones. It is out of the scope of this paper to investigate in this direction.

**Remark 1 (Weighted manifolds)** The very same proof works also if the volume measure is replaced by a different one $\mathfrak{m}$: this change affects, in an obvious way, both the definition of the functional $\mathcal{U}_N$ and the one of the Laplacian. $\blacksquare$

**Remark 2 (The Finsler case)** Equations (3), (4) remain valid in a Finsler manifold endowed with smooth - outside 0 - and strictly convex norms on the tangent spaces. Then the very same computations can be done (we are deliberately neglecting to discuss the delicate regularity issues coming from the lack of smoothness in 0 of the squared norms) and they lead to

$$-\partial_t\Delta\psi_t|_{t=0} \geq \frac{(\Delta\psi)^2}{N} + D(\Delta\psi)(\nabla\psi) + K|\nabla\psi|^2, \tag{6}$$

where $(\psi_t)$ evolves from $\psi_0 = \psi$ according to (4). In a Finsler manifold the Laplacian in general is not a linear operator, so we can't swap $\Delta$ and the time derivation at the left hand side to deduce that the Bochner inequality can be written as in (1). Notice that (6) is a different formulation of the Bochner inequality in a Finlser setting w.r.t. that obtained in [5]. $\blacksquare$

The correct way to write the $CD^*(K,N)$ condition (i.e. the one which makes sense also in a non smooth setting) is

$$\mathcal{U}_N((\mathrm{e}_t)_\sharp\boldsymbol{\pi}) \leq -\int \sigma_{K,N}^{(1-t)}\big(\mathsf{d}(\gamma_0,\gamma_1)\big)\rho^{-\frac{1}{N}}(\gamma_0) + \sigma_{K,N}^{(t)}\big(\mathsf{d}(\gamma_0,\gamma_1)\big)\eta^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\boldsymbol{\pi}(\gamma), \tag{7}$$

for every $t \in [0,1]$, under the same assumptions on $(\mu_t)$ as in (2), where $\boldsymbol{\pi}$ is any plan concentrated on the geodesics of $M$ such that $(\mathrm{e}_t)_\sharp\boldsymbol{\pi} = \mu_t$ for any $t \in [0,1]$ and $\iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) = W_2^2(\mu_0,\mu_1)$, $\mathrm{e}_t : C([0,1],M) \to M$ being the evaluation maps defined by $\mathrm{e}_t(\gamma) := \gamma_t$. Here the distortion coefficients $\sigma^{(t)}_{K,N}$ are given by

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} +\infty, & \text{if } K\theta^2 \geq N\pi^2,\\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2,\\ t & \text{if } K\theta^2 = 0,\\ \frac{\sinh(t\theta\sqrt{K/N})}{\sinh(\theta\sqrt{K/N})} & \text{if } K\theta^2 < 0. \end{cases}$$

(actually, (7) is required to hold with $N$ replaced by an arbitrary $N' \geq N$, but on non branching spaces this makes no difference).

Now observe that inequality (7) is an inequality concerning the distribution of masses at different times along a (Wasserstein) geodesic. As such, we can think at it as a Lagrangian point of view on Ricci bounds. Opposed to this, there should be a Eulerian point of view which gives the same information read at the level of velocity vector fields. This is exactly the point of view adopted in the proof of Bochner inequality just provided: as we learned from Otto's interpretation of the Wasserstein space as infinite dimensional Riemannian manifold, for any $\varphi \in C^\infty_c(M)$ and any $\mu \in \mathscr{P}_2(M)$, the vector field $\nabla\varphi$ can be seen as the initial velocity of a Wasserstein geodesic starting from $\mu$ (this is made rigorous by (3) and (4)). From this perspective, Bochner inequality should be regarded as an inequality on gradients of functions, rather than on functions themselves. All this also works in a Finsler setting.

As pointed out to me by Sturm, it is not surprising that to deduce Bocher inequality the $CD^*(K,N)$ condition is sufficient, as opposed to the stronger $CD(K,N)$. This is due to the fact that (1) is local in nature, exactly as $CD^*(K,N)$ (while the locality of the $CD(K,N)$ condition in the non smooth setting is an open problem).

We conclude observing that taking just one derivative of $\mathcal{U}_N$ along a geodesic and using the $CD(K,N)$ condition, one proves, instead of Bocher inequality, the sharp Laplacian comparison estimates for the distance function. This computation is perfectly justifiable on a general metric measure space, as shown in [3]. The hope would then be to find a way to justify the computations done here to state and prove the Bochner inequality in a non smooth setting, but as of today it is not clear - at least to me - how to overcome in full generality the severe technical issues that occur.

# References

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