Γ-CONVERGENCE ANALYSIS OF SYSTEMS OF EDGE DISLOCATIONS: THE SELF ENERGY REGIME

L. DE LUCA, A. GARRONI, AND M. PONSIGLIONE

Abstract. This paper deals with the elastic energy induced by systems of straight edge dislocations in the framework of linearized plane elasticity. The dislocations are introduced as point topological defects of the displacement-gradient fields. Following the core radius approach, we introduce a parameter $\varepsilon > 0$ representing the lattice spacing of the crystal, we remove a disc of radius ε around each dislocation and compute the elastic energy stored outside the union of such discs, namely outside the core region. Then, we analyze the asymptotic behaviour of the elastic energy as $\varepsilon \to 0$, in terms of Γ-convergence. We focus on the self energy regime of order $\log \frac{1}{\varepsilon}$; we show that configurations with logarithmic diverging energy converge, up to a subsequence, to a finite number of multiple dislocations and we compute the corresponding Γ-limit.

Keywords: Γ-convergence, dislocations, topological singularities, plasticity.

2000 Mathematics Subject Classification: 49J45, 58K45, 74C05.

1. Introduction

Dislocations are the most common defects in crystals and their presence is considered the main mechanism of plastic deformations in metals. They are classically modeled as lines to which is associated a vector called the *Burgers vector*. Straight dislocations are classified into two main types: *edge* if the Burgers vector is orthogonal to the dislocation line, and *screw* if it is parallel. Dislocations found in real materials typically are mixed, meaning that they might be not straight and have characteristics of both types.

This paper deals with energy minimization methods proposed to model static elastic properties of edge dislocations. More precisely, we study the asymptotic behaviour of the elastic energy induced by a system of straight edge dislocations, as the atomic scale ε tends to zero. We focus on a low-energy regime, corresponding to a finite number of defects.

We consider the setting of plane elasticity. In this setting the elastic body is assumed to have a cylindrical symmetry, so that the mathematical formulation involves only a two-dimensional problem set on the cross section Ω of the crystal. In classical linear elasticity, a planar displacement is a regular vector field $u : \Omega \to \mathbb{R}^2$. The equilibrium equations have the form $\operatorname{Div} \mathbb{C}[e(u)] = 0$, where $e(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$ is the infinitesimal strain tensor, and $\mathbb{C}$ is a linear operator from $\mathbb{M}^{2\times 2}$ into itself usually referred to as the *elasticity tensor*, incorporating the material properties of the crystal. It satisfies

$$c_1|\xi^{\mathrm{sym}}|^2 \leq \mathbb{C}\,\xi : \xi \leq c_2|\xi^{\mathrm{sym}}|^2 \qquad \text{for any } \xi \in \mathbb{M}^{2\times 2}, \tag{1.1}$$

where c_1 and c_2 are two given positive constants and $\xi^{\mathrm{sym}} := \frac{1}{2}(\xi + \xi^{\mathrm{T}})$. The corresponding elastic energy, in absence of dislocations, is given by

$$\int_\Omega W(\beta)\,\mathrm{d}x, \tag{1.2}$$

where $\beta = \nabla u$ is the displacement gradient field, and $W(\xi) = \frac{1}{2}\mathbb{C}\,\xi : \xi = \frac{1}{2}\mathbb{C}\,\xi^{\mathrm{sym}} : \xi^{\mathrm{sym}}$ is the elastic energy density.

We now describe the presence of edge dislocations in our model assuming that they are straight line orthogonal to the cross section Ω. We work in a continuum setting, but dislocations have a microscopic nature, so that their definition involves discrete quantities, having memory of the lattice structure of the crystal. According to the so-called discrete dislocation model, we identify each dislocation lines with its intersection x^i with Ω and a vector $\xi^i \in \mathbb{S}$. Here $\mathbb{S}$ is a discrete lattice representing the class of all the horizontal translations under which the crystal is invariant. It is generated by a set $S := \{v_1, v_2\} \subset \mathbb{R}^2$, where v_i are called *primitive vectors*, i.e., $\mathbb{S} = \mathrm{span}_{\mathbb{Z}} S$ (we are implicitly assuming that Ω lies on a slip plane of the crystal). Accordingly, a configuration of dislocations can be represented by a vector valued measure $\mu = \sum_i \xi^i \delta_{x^i}$, where $x^i \in \Omega$ and $\xi^i \in \mathbb{S}$. It turns out, by our analysis (see Remark 6.1) that the relevant configurations, i.e., optimal in energy, exploit only a finite subset $\mathfrak{B}$ of $\mathbb{S}$ with $\mathbb{S} = \mathrm{span}_{\mathbb{Z}}\mathfrak{B}$. We identify this class $\mathfrak{B}$ with the class of Burgers vectors of the crystal. The Burgers vectors in $\mathfrak{B}$ are determined only by $\mathbb{S}$ and by the elasticity tensor $\mathbb{C}$. They are the translations in $\mathbb{S}$ that store less energy (see Definition 2.3), and hence represent the preferred slip directions of the crystal. This notion agrees (at least in the isotropic case), with the standard notion of the Burgers vectors in crystallography as the translations in $\mathbb{S}$ with minimal length.

The class of admissible fields β associated with any configuration of dislocations μ is given by the matrix valued fields whose circulation around the dislocations x^i is equal to ξ^i. These fields by definition have a singularity at each x^i and are not in $L^2(\Omega; \mathbb{M}^{2\times 2})$. To set up a variational formulation we then follow the so called *core region approach.* More precisely, we introduce a scale parameter ε, proportional to the lattice spacing of the crystal, and we compute the energy outside the core region $\cup_i B_\varepsilon(x^i)$. Indeed, considerations at a discrete level show that the elastic energy stored in the core region can be neglected. Since the elastic distortion due to the presence of dislocations decays as the inverse of the distance from dislocations, it is commonly accepted in the literature that the linearized elasticity provides a good approximation of the stored elastic energy outside the core region (see [20] for a justification of these arguments in terms of Γ-convergence). We are now in a position to introduce the elastic energy induced by an arbitrary configuration of dislocations μ and an admissible field β

$$E_\varepsilon(\mu,\beta) := \int_{\Omega_\varepsilon(\mu)} W(\beta)\,\mathrm{d}x, \qquad (\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_i B_\varepsilon(x^i)). \tag{1.3}$$

By minimizing the elastic energy (1.3) among all admissible fields, we obtain the elastic energy induced by μ.

This variational formulation has been considered in [8] by Cermelli and Leoni who study the limit of the elastic energy induced by a fixed configuration of dislocations as the atomic scale ε tends to zero. They exploit the analogy between this formulation and the core radius approach for vortices in superconductivity described in [6]. In particular they show that a finite number of dislocations has an elastic energy of order $\log \frac{1}{\varepsilon}$. In the framework of Γ-convergence, the asymptotic analysis as $\varepsilon \to 0$ has been done in [17] in the scalar case of screw dislocations. It is proved that (up to the logarithmic pre-factor) the limit energy is given, as in the Ginzburg-Landau setting (see [15]), by the number of defects. Such energy is called the self energy, since each dislocation gives a quantum of energy, whose density concentrates around its core. This equivalence of vortices and screw dislocations can be pushed to any $|\log \varepsilon|^h$ energy regime and justified in terms of Γ-convergence (see [2]). Note that in the regimes $|\log \varepsilon|^h$, $h > 1$, the number of defects N_ε increases, tending to infinity as $\varepsilon \to 0$, and the interaction between singularities becomes relevant; in the critical $|\log \varepsilon|^2$ energy regime (that corresponds to $N_\varepsilon \approx \log \frac{1}{\varepsilon}$), the two effects of interaction energy and self-energy are balanced. In the context of vortices this regime has been first analyzed in [19] and [16] for the 2-dimensional Ginzburg-Landau energy and recently considered by [5] in the 3-dimensional setting. The vectorial case of edge dislocations in the critical $|\log \varepsilon|^2$ energy regime has been considered in [12] under the assumption that the dislocations are well separated. The limit energy is of the form

$$\int_\Omega W(\beta)\,\mathrm{d}x + \int_\Omega \varphi\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right)\,\mathrm{d}|\mu|, \tag{1.4}$$

where φ is a positively 1-homogeneous density function defined by a suitable cell problem formula, determined only by the elasticity tensor $\mathbb{C}$ and the geometric structure of the crystal. This structure of the limit energy is set in the framework of so called *strain gradient theories* for plasticity (see [11],[13], and [9]). We remark that in the vectorial case of edge dislocations, although there is still a strong analogy with the Ginzburg-Landau setting, the precise relation between the two frameworks appears less clear. Indeed, in the asymptotics of edge dislocations, relaxation effects, that are encoded in the definition of φ, take place. Moreover, the fact that the energy is not coercive, depending only on the symmetric part of the strain, introduces specific difficulties in the analysis. In particular compactness of sequences with bounded energy is a challenging task. In [12] this problem was bypassed assuming well separation of dislocations.

In this paper, we study the asymptotic behaviour of the elastic energy induced by edge dislocations in terms of Γ-convergence, in the self energy $\log 1\varepsilon$ regime, without assuming the dislocations to be fixed, uniformly bounded in mass nor well separated. In order to perform this analysis we first introduce the rescaled energy associated with μ

$$\mathcal{F}_\varepsilon(\mu) := \frac{1}{\log \frac{1}{\varepsilon}}\left(\min_\beta E_\varepsilon(\mu,\beta) + |\mu|(\Omega)\right).$$

The term $|\mu|(\Omega)$ represents the energy stored in the core region. Computations at a discrete level (see [17]) show that such energy is indeed of order 1. We remark that this term is essential in order to have a meaningful energy $\mathcal{F}_\varepsilon(\mu)$; indeed, without the core energy any configuration μ such that $\Omega_\varepsilon(\mu) = \emptyset$ would induce no energy. On the other hand, even if this term is essential, its specific choice does not affect the Γ-limit (see [17]). In this respects, it plays a role similar to the double well potential in Ginzburg-Landau functionals. In Theorem 2.4 we prove that the Γ-limit of the functionals $\mathcal{F}_\varepsilon$ is given by the functional

$$\mathcal{F}(\mu) := \int_\Omega \varphi\Big(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\Big)\,\mathrm{d}|\mu|,$$

where φ is obtained through a cell problem formula as for (1.4). The main difficulty in this analysis is to prove compactness properties for sequences with bounded energy. Indeed, it turns out that the elastic energy of a given distributions of dislocations does not control the number of dislocations due to the possible presence of many short dipoles that may have very small energy. This means that almost minimizers are not precompact in general with respect to the weak* topology. Therefore, it seems natural to perform the analysis using a weaker topology for which annihilating dipoles converge to zero; this is the *flat topology*, that is the weak star topology in the dual of the space of Lipschitz continuous functions (see Section 2). These considerations are nowadays very well understood in the context of vorticity modeled by Ginzburg-Landau functionals (see e.g., [6], [14], [18], [19], [1]).

A very efficient tool for lower bounds for the Ginzburg-Landau functionals is the *ball construction*, for which we refer to [18]. Actually, very recently in [3] it has been shown that also compactness of vortices can be easily deduced just running this powerful machine. The idea of the ball construction is to build a family of growing and merging balls, that identify a family of annuli, where most of the energy is concentrated. In our setting of plane elasticity we have to deal with an extra difficulty: in our lower bounds we need Korn's inequality in such annuli, and clearly we need uniform constants. It means that we have to perform the ball construction avoiding too tiny annuli (where the Korn's constant blows up). This will be done in Section 3 where we construct an *ad hoc* discrete version of the ball construction. Once this ball construction is done, we deduce a lower bound with a pre-factor error due to the use of Korn's inequality. Then, compactness is easily deduced in Section 4 using arguments similar to [17], [3]. Finally, to get the Γ-liminf inequality we have to get rid of these Korn's constants, estimating the energy on fat annuli of the type $B_R(x) \setminus B_r(x)$ with $R/r \to +\infty$ as $\varepsilon \to 0$. To this end we have first to suitably remove clusters of dipoles. After this procedure we end up with a finite (i.e., independent of ε) number of clusters of dislocations, concentrated around the support of the limit configuration μ. In view of this preliminary analysis, we can easily find the required annuli where the energy concentrates, providing the optimal lower bound. This will be done in Section 5. Finally, in Section 6 we provide the upper bound, concluding the proof of our Γ-convergence result.

2. The main result

In this section we state the main result of the paper and introduce the required preliminaries and notations. We recall that Ω is a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary, which represents a cross section of the cylindrical crystal.

Let $S := \{v_1, v_2\} \subset \mathbb{R}^2$ be such that $\mathbb{S} := \operatorname{span}_{\mathbb{Z}} S$ is a discrete lattice, representing the class of horizontal slips (translations) under which the crystal is invariant. For instance, in the case of cubic crystals we would choose $S = \{e_1, e_2\}$, while for fcc crystals S can be chosen as $S = \{e_1, \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2\}$.

The space of finite distributions of edge dislocations X is given by

$$X := \{\mu \in \mathcal{M}(\Omega, \mathbb{R}^2) : \mu = \sum_{i=1}^{N} \xi^i \delta_{x^i},\, N \in \mathbb{N},\, x^i \in \Omega,\, \xi^i \in \mathbb{S}\},$$

where $\mathcal{M}(\Omega, \mathbb{R}^2)$ denotes the set of vector valued Radon measures on Ω. Each of the point x^i in the support of μ represents the cross section of a straight line dislocation with the domain Ω and the corresponding ξ^i its vector multiplicity. We endow X with the *flat norm* $\|\mu\|_{\text{flat}}$

defined by

$$\|\mu\|_{\text{flat}} := \sup_{\|\phi\|_{W_0^{1,\infty}(\Omega)}\leq 1} \int_\Omega \phi \, \mathrm{d}\mu;$$

in particular, we can consider X as a subspace of $W^{-1,1}(\Omega)$. We will denote by $\mu_h \overset{\text{flat}}{\to} \mu$ the flat convergence of μ_h to μ.

Fix $\varepsilon > 0$. Given $\mu \in X$, we denote by

$$\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_{x^i \in \text{supp}\,(\mu)} B_\varepsilon(x^i).$$

With a little abuse of terminology we will call admissible strain associated with μ any field $\beta \in \mathcal{AS}_\varepsilon(\mu)$, where

$$\begin{aligned} \mathcal{AS}_\varepsilon(\mu) \quad := \quad & \Big\{\beta \in L^2(\Omega_\varepsilon(\mu);\mathbb{M}^{2\times 2}) : \ \operatorname{Curl}\beta = 0 \text{ in } \Omega_\varepsilon(\mu), \\ & \int_{\partial A} \beta(s)\cdot t(s)\, ds = \mu(A) \text{ for every open set } A \subset \Omega \\ & \text{with } \partial A \text{ smooth: } \partial A \subset \Omega_\varepsilon(\mu), \text{ and } \int_{\Omega_\varepsilon(\mu)} (\beta - \beta^T)\,\mathrm{d}x = 0\Big\}. \end{aligned}$$

Here t denotes the tangent vector to ∂A and the integrand $\beta \cdot t$ is intended in the sense of traces (see Theorem 2 page 204 in [7]).

The elastic energy associated with a strain $\beta \in \mathcal{AS}_\varepsilon(\mu)$ is defined by

$$E_\varepsilon(\mu,\beta) := \int_{\Omega_\varepsilon(\mu)} W(\beta)\,\mathrm{d}x,$$

where $W(\beta) = \frac{1}{2}\mathbb{C}\beta : \beta$. The elastic energy $\mathcal{E}_\varepsilon : X \to \mathbb{R}$ induced by the distribution of dislocations μ is given by

$$\mathcal{E}_\varepsilon(\mu) := \min_{\beta\in\mathcal{AS}_\varepsilon(\mu)} E_\varepsilon(\mu,\beta) + |\mu|(\Omega).$$

The rescaled energy functionals $\mathcal{F}_\varepsilon : X \to \mathbb{R}$ are defined by

$$\mathcal{F}_\varepsilon(\mu) := \frac{1}{\log\frac{1}{\varepsilon}}\mathcal{E}_\varepsilon(\mu). \tag{2.1}$$

The main result of this paper is the study in terms of Γ-convergence with respect to the flat topology of the functionals $\mathcal{F}_\varepsilon(\mu)$. We show in Theorem 2.4 that the Γ-limit is obtained by a suitable relaxation of the so-called prelogarithmic factor ψ, that we define as follows: given $\xi \in \mathbb{R}^2$, we set, according with [12]

$$\begin{aligned} \psi(\xi) := \min\Big\{ \int_{\partial B_1} W(\Gamma(\theta))\,\mathrm{d}\theta \ : \ \Gamma \in L^2(\partial B_1, \mathbb{M}^{2\times 2}), \operatorname{Curl}\frac{1}{\rho}\Gamma(\theta) = 0, \\ \int_{\partial B_1} \Gamma(\theta)\cdot t(\theta)\,\mathrm{d}\theta = \xi\Big\}, \end{aligned} \tag{2.2}$$

where (ρ,θ) are polar coordinates in $\mathbb{R}^2$, $t(\theta)$ denotes the tangent vector to ∂B_1, and the equation $\operatorname{Curl}\frac{1}{\rho}\Gamma(\theta) = 0$ has to be understood in the sense of distributions in $\mathbb{R}^2 \setminus \{0\}$. The minimum in (2.2) is attained by a function denoted by Γ_ξ which is unique up to additive skew matrices.

The displacement $u_{\mathbb{R}^2}(\xi)$ induced on the whole plane by a straight infinite dislocation centered at 0 with multiplicity ξ is computed explicitly in the literature (see e.g.,[4, formula (4.1.25)]) and it is of the form

$$u^{\xi}_{\mathbb{R}^2}(\rho,\theta) = F_{\xi}(\theta) + g_{\xi}\log\rho\,,$$

where $g_{\xi} \in \mathbb{R}^2$ and the function F_{ξ} is given by $F_{\xi}(\theta) = \int_0^{\theta} f_{\xi}(\omega)\,\mathrm{d}\omega$ for a suitable function $f_{\xi} \in C^0(\partial B_1;\mathbb{R}^2)$, with $\int_0^{2\pi} f_{\xi}(\omega)\,\mathrm{d}\omega = \xi$. The corresponding strain field is given by

$$\beta^{\xi}_{\mathbb{R}^2}(\rho,\theta) := \frac{1}{\rho}\left(f_{\xi}(\theta)\otimes(-\sin\theta,\cos\theta) + g_{\xi}\otimes(\cos\theta,\sin\theta)\right). \tag{2.3}$$

The equations satisfied by $\beta^{\xi}_{\mathbb{R}^2}$ are

$$\begin{cases} \operatorname{Curl}\beta^{\xi}_{\mathbb{R}^2} = \xi\delta_0 & \text{in } \mathbb{R}^2;\\ \operatorname{Div}\mathbb{C}\beta^{\xi}_{\mathbb{R}^2} = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{2.4}$$

It can be proved that a field satisfying (2.3) and (2.4) is unique up to additive skew matrices and it is indeed given by

$$\beta^{\xi}_{\mathbb{R}^2}(\rho,\theta) = \frac{1}{\rho}\Gamma_{\xi}(\theta)\,, \tag{2.5}$$

where Γ_{ξ} is a minimizer of (2.2) (see [12]). In particular

$$\psi(\xi) = \int_{\partial B_1} W(\Gamma_{\xi}(\theta))\,\mathrm{d}\theta = \lim_{\varepsilon\to 0}\frac{1}{\log\frac{1}{\varepsilon}}\int_{B_1\setminus B_{\varepsilon}} W(\beta^{\xi}_{\mathbb{R}^2})\,\mathrm{d}x, \tag{2.6}$$

where B_{ρ} denotes the ball of radius ρ and center 0.

Let us introduce for any given $\xi \in \mathbb{R}^2$ and for $0 < r < R$, the space

$$\begin{aligned} \mathcal{AS}_{r,R}(\xi) := \{\beta \in L^2(B_R\setminus B_r;\mathbb{M}^{2\times 2}) : \operatorname{Curl}\beta = 0, \int_{\partial B_r}\beta\cdot t\,\mathrm{d}s = \xi,\\ \int_{B_R\setminus B_r}(\beta-\beta^{\mathrm{T}})\,\mathrm{d}x = 0\}. \end{aligned} \tag{2.7}$$

The relation between the prelogarithmic factor defined in (2.2) and our energy is clarified by the following proposition (proved in [12], Corollary 6).

Proposition 2.1. *There exists a constant $C_0 > 0$ such that*

$$|\psi(\xi) - \psi_{\varepsilon}(\xi)| \le C_0\frac{|\xi|^2}{\log\frac{1}{\varepsilon}}\,,$$

where

$$\psi_{\varepsilon}(\xi) := \frac{1}{\log\frac{1}{\varepsilon}}\min_{\beta\in\mathcal{AS}_{\varepsilon,1}(\xi)}\int_{B_1\setminus B_{\varepsilon}} W(\beta)\,\mathrm{d}x\,.$$

Remark 2.2. In our analysis it will be convenient to introduce the following notation for the elastic energy of a dislocation in the annulus $B_R\setminus B_r$

$$\psi_{r,R}(\xi) := \frac{1}{\log R - \log r}\min_{\beta\in\mathcal{AS}_{r,R}(\xi)}\int_{B_R\setminus B_r} W(\beta)\,\mathrm{d}x\,. \tag{2.8}$$

Using a change of variables we clearly have $\psi_{r,R}(\xi) = \psi_{\frac{r}{R}}(\xi)$, and hence

$$|\psi(\xi) - \psi_{r,R}(\xi)| \le C_0 \frac{|\xi|^2}{\log R - \log r}.$$

In particular

$$\lim_{\frac{r}{R}\to 0} \psi_{r,R}(\xi) = \psi(\xi)\,.$$

We introduce the density function $\varphi : \mathbb{S} \mapsto [0, +\infty)$ of the energy $\mathcal{F}$ through the following relaxation procedure

$$\varphi(\xi) := \inf \left\{ \sum_{k=1}^{N} |\lambda^k| \psi(\xi^k) : \ \sum_{k=1}^{N} \lambda^k \xi^k = \xi,\, N \in \mathbb{N}, \lambda^k \in \mathbb{Z}, \xi^k \in \mathbb{S} \right\}. \tag{2.9}$$

It can be easily proved (see [12]) that the infimum in (2.9) is in fact a minimum.

Definition 2.3. We say that $b \in \mathbb{S}$ is a *Burgers vector* if $\varphi(b) = \psi(b)$, and denote by $\mathfrak{B}$ the class of such vectors.

It is easy to see that $\mathbb{S} = \operatorname{span}_{\mathbb{Z}} \mathfrak{B}$ and that for every $\lambda^1, \dots, \lambda^k \in \mathbb{Z}$, $b^1, \dots, b^k \in \mathfrak{B}$

$$\psi\left(\sum_{i=1}^{k} \lambda^i b^i \right) \ge \sum_{i=1}^{k} |\lambda^i| \psi(b^i).$$

Therefore, in the relaxation in (2.9) we can replace $\mathbb{S}$ with $\mathfrak{B}$, namely for every $\xi \in \mathbb{S}$ we have

$$\varphi(\xi) = \min \left\{ \sum_{i=1}^{k} |\lambda^i| \psi(b^i) : \xi = \sum_{i=1}^{k} \lambda^i b^i,\ \lambda^i \in \mathbb{Z},\ b^i \in \mathfrak{B} \right\}. \tag{2.10}$$

The limit energy induced by a configuration μ is the functional

$$\mathcal{F}(\mu) := \sum_{i=1}^{N} \varphi(\xi^i) \qquad \text{for any } \mu = \sum_{i=1}^{N} \xi^i \delta_{x^i} \in X\,, \tag{2.11}$$

The following Γ-convergence result holds.

Theorem 2.4. *Let $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ be defined by (2.1) and (2.11):*

(i) **Compactness**. *Let $\varepsilon_h \to 0$ and let $\{\mu_h\}$ be a sequence in X such that $\mathcal{F}_{\varepsilon_h}(\mu_h) \le M$ for some positive constant M independent of h. Then, (up to a subsequence) $\mu_h \overset{\text{flat}}{\to} \mu \in X$.*

(ii) **Γ-convergence**. *The functionals $\mathcal{F}_{\varepsilon_h}$ Γ-converge to $\mathcal{F}$, as $\varepsilon_h \to 0$, with respect to the flat norm, i.e., the following inequalities hold.*

Γ-liminf inequality*: $\mathcal{F}(\mu) \le \liminf_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h)$ for any $\mu \in X$, and $\mu_h \overset{\text{flat}}{\to} \mu$.*

Γ-limsup inequality*: given $\mu \in X$, there exists $\{\mu_h\} \subset X$, with $\mu_h \overset{\text{flat}}{\to} \mu$, such that $\limsup_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \le \mathcal{F}(\mu)$.*

The proofs of the compactness and the Γ-liminf inequality are quite technical and are based on the so-called "ball construction" (see [18] and [6]), which is used in the context of superconductivity. As explained in the Introduction, a specific difficulty of our context of plane elasticity is due to the fact that the energy depends only on the symmetric part of the field β. Moreover, the optimal Korn's inequality constant blows up on thin annuli, and the function $\psi_{r,R}$ defined in (2.8) vanishes as $R/r \to 1$ (see Example A.1). It is then not clear how to estimate the energy from below on thin annuli. For this reason, in the implementation of the ball construction technique, we will work only with annuli whose ratio of the radii is given by a constant $c > 1$. To this purpose we have to revisit the standard ball construction in [18]. We will introduce the needed discrete ball construction in the next section.

3. Revised Ball Construction

Here we revisit the ball construction introduced in [18]. The main goal is to provide the key lower bounds (see Proposition 3.2) on annular sets, needed in the proof of the Γ-liminf inequality and of the compactness. First, we give a lower bound for the energy on a single annulus $B_R \setminus B_r$.

Lemma 3.1. *Given $0 < r < R$ and $\xi \in \mathbb{R}^2$, for any admissible configuration $\beta \in \mathcal{AS}_{r,R}(\xi)$ (defined in (2.7)) we have*

$$\int_{B_R \setminus B_r} |\beta^{\mathrm{sym}}|^2 \,\mathrm{d}x \geq \frac{|\xi|^2}{2\pi} \frac{1}{K(R/r)} \log \frac{R}{r},$$

where $K(R/r)$ is the Korn's constant defined according with (A.1).

Proof. We introduce a cut L on the annulus $B_R \setminus B_r$ so that $(B_R \setminus B_r) \setminus L$ is simply connected, and exploit the fact that β is a curl free field in $B_R \setminus B_r$. More precisely, there exists a function $u \in H^1((B_R \setminus B_r) \setminus L; \mathbb{R}^2)$ with $\nabla u = \beta$ in $(B_R \setminus B_r) \setminus L$. From the circulation condition in (2.7), applying Jensen inequality, it is easy to see that, for any given skew symmetric matrix A, we have

$$\int_{B_R \setminus B_r} |\nabla u|^2 \,\mathrm{d}x \geq \int_r^R \frac{1}{2\pi\rho} \left| \int_0^{2\pi} \nabla u \cdot t \,\mathrm{d}\theta \right|^2 \mathrm{d}\rho = \frac{|\xi|^2}{2\pi} \log \frac{R}{r};$$

the thesis follows directly by applying classical Korn's inequality (Theorem A.1). □

For any given $C > 0$, let $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(r, R, t) := Ct \log \frac{R}{r}. \tag{3.1}$$

Clearly f satisfies the following properties

i) $f(r, \rho, t) + f(\rho, R, t) = f(r, R, t)$ for every $t > 0$ and $0 < r < \rho < R$;
ii) if $f(r^i, R^i, 1) = \alpha$ for every $i = 1, \dots, m$, for some $\alpha \in \mathbb{R}^+$, then

$$\alpha = f(\sum_{i=1}^m r^i, \sum_{i=1}^m R^i, 1).$$

Fix $\mu = \sum_{i=1}^N \xi^i \delta_{x^i} \in X$, and set

$$\omega_\varepsilon := \bigcup_{i=1}^N B_\varepsilon(x^i). \tag{3.2}$$

Proposition 3.2. *Let $c > 1$ be fixed and let f be defined as in (3.1). Let F be a positive additive set function on the open subsets of Ω that satisfies*

$$F(B_R(x)) \geq f(r, R, |\mu(B_R(x))|) + F(B_r(x)), \tag{3.3}$$

for every $x \in \mathbb{R}^2$ and every r, $R \in \mathbb{R}^+$ with $\frac{R}{r} = c$ such that $B_R(x) \setminus B_r(x) \subset \Omega \setminus \omega_\varepsilon$. Finally, let $\rho > 0$ and let A be an open subset of Ω such that $\mathrm{dist}(x^i, \partial A) \geq \rho$ for all $x^i \in A$. Then,

$$F(A) \geq |\mu(A)| f(c^N \varepsilon N, \frac{\rho}{2c}, 1). \tag{3.4}$$

The statement of Proposition 3.2 is proved by computing a lower bound for the energy on a sequence of larger and larger annuli in which the main part of the energy is stored. We follow closely the strategy of the ball construction introduced by Sandier in [18]. The main difference is that we need to construct annular sets with radii satisfying $R/r = c$. To this purpose, our ball construction consists in a discrete rather than continuous process in which at each step either all the balls expand or some of them merge together. We proceed by introducing our discrete ball construction.

Discrete Ball Construction

Let $\{x^i\}_{i=1,\dots,N}$ be a set of points in $\mathbb{R}^2$, $c > 1$, and $\varepsilon > 0$. We set $N_0 := N$, $x^i_0 = x^i$, $R^i_0 = r^i_0 = \varepsilon$, for every $1 \leq i \leq N_0$ and $\mathcal{B}_0 = \{B_{R^i_0}(x^i_0)\}_{i=1,\dots,N_0}$. Given x^i_{n-1}, R^i_{n-1}, r^i_{n-1} for $i = 1, \dots, N_{n-1}$, we construct recursively x^i_n, R^i_n, r^i_n, for $i = 1, \dots, N_n$, as follows. First, we consider the family of balls $\{B_{cR^i_{n-1}}(x^i_{n-1})\}$. If these balls are pairwise disjoint, we say that n is an *expansion time*. In this case, we set $N_n = N_{n-1}$, and

$$x^i_n = x^i_{n-1}, \quad R^i_n = cR^i_{n-1}, \quad r^i_n = r^i_{n-1} \qquad \text{for all } i = 1, \dots, N_n.$$

If, otherwise, the balls in $\{B_{cR^i_{n-1}}(x^i_{n-1})\}$ are not pairwise disjoint, we say that n is a *merging time*. The merging consists in identifying a suitable partition $\{S_j\}_{j=1,\dots,N_n}$ of the family $\{B_{cR^i_{n-1}}(x^i_{n-1})\}$ and, for each subclass S_j, in finding a ball $B_{R^j_n}(x^j_n)$ which contains all the balls in S_j with the following properties:

i) the balls $B_{R^j_n}(x^j_n)$ of the new family are pairwise disjoint;

ii) R^j_n is not larger than the sum of all the radii of the balls $B_{cR^i_{n-1}}(x^i_{n-1}) \in S_j$, i.e., contained in $B_{R^j_n}(x^j_n)$.

Such a construction can be always done by an induction argument, for more details we refer to [18]. After the merging, we reset all the quantities introduced above as follows: x^j_n and R^j_n for $j = 1, \dots, N_n$ are determined by the merging construction, while the parameters r^j_n, referred to as the *seed sizes*, are defined so that, for all $1 \leq i \leq N_{n-1}$ and $1 \leq j \leq N_n$, we have

$$\frac{R^j_n}{r^j_n} = \frac{R^i_{n-1}}{r^i_{n-1}},$$

and hence

$$f(r^j_n, R^j_n, 1) = f(r^i_{n-1}, R^i_{n-1}, 1). \tag{3.5}$$

Furthermore, at any step n, we define a parameter τ_n that counts the number of merging occurred until the n-th step. More precisely, if n is an expansion time $\tau_n = \tau_{n-1}$ whereas if it is a merging time $\tau_n = \tau_{n-1} + 1$. In this way, at time n we have made $n - \tau_n$ expansions and τ_n merging.

Definition 3.3. We refer to the construction above as the *Discrete Ball Construction* associated with the points $\{x^i\}_{i=1,\dots,N}$. In particular, for every $n \in \mathbb{N}$ we have defined a family of balls

$$\mathcal{B}_n = \{B_{R_n^i}(x_n^i)\}_{i=1,\dots,N_n},$$

a family of seed sizes $\{r_n^i\}_{i=1,\dots,N_n}$ and the merging counter τ_n.

We are now in a position to prove Proposition 3.2.

Proof of Proposition 3.2. Consider the Discrete Ball Construction associated to the points $x^i \in A$. The balls in $\mathcal{B}_n$ satisfy

$$R_n^j \leq c^n \varepsilon \,\sharp\{i : \ B_\varepsilon(x^i) \subset B_{R_n^j}(x_n^j)\} \tag{3.6}$$

$$r_n^j \leq c^{\tau_n} \varepsilon \,\sharp\{i : \ B_\varepsilon(x^i) \subset B_{R_n^j}(x_n^j)\}. \tag{3.7}$$

We first prove (3.6) by induction arguing as follows. If n is an expansion time, then we clearly have $R_n^j = c\, R_{n-1}^j$. While if n is a merging time, by construction (namely, by property ii)) we have

$$R_n^j \leq c \sum_{i:\, B_{R_{n-1}^i}(x_{n-1}^i) \subset B_{R_n^j}(x_n^j)} R_{n-1}^i.$$

As for the proof of (3.7), notice that

$$\begin{aligned} \frac{R_n^j}{r_n^j} &= c\frac{R_{n-1}^i}{r_{n-1}^i} \quad \text{if } n \text{ is an expansion step, for any } j \in N_n = N_{n-1},\ i \in N_{n-1}, \\ \frac{R_n^j}{r_n^j} &= \frac{R_{n-1}^i}{r_{n-1}^i} \quad \text{if } n \text{ is a merging step, for any } j \in N_n,\ i \in N_{n-1}\ . \end{aligned}$$

We deduce that $\frac{R_n^j}{r_n^j} = c^{n-\tau_n}$. Therefore, (3.7) follows by (3.6) since

$$r_n^j = \frac{R_n^j}{c^{n-\tau_n}} \leq \frac{c^n}{c^{n-\tau_n}} \varepsilon \,\sharp\{i : \ B_\varepsilon(x^i) \subset B_{R_n^j}(x_n^j)\} = c^{\tau_n}\, \varepsilon \,\sharp\{i : \ B_\varepsilon(x^i) \subset B_{R_n^j}(x_n^j)\}.$$

The main point of this construction is that it provides the following lower bound: for every $n \in \mathbb{N}$ and for every $j = 1, \dots, N_n$

$$F(B_n^j) \geq |\mu(B_n^j)| f(r_n^j, R_n^j, 1), \tag{3.8}$$

where, for sake of simplicity, we have set $B_n^j := B_{R_n^j}(x_n^j)$.

We prove (3.8) by an induction argument. For $n = 0$ there is nothing to prove. Suppose that the inequality is true at time $n - 1$. If n is an expansion time, then

$$\begin{aligned} F(B_n^j) &= F(B_n^j \setminus B_{n-1}^j) + F(B_{n-1}^j) \geq f(R_{n-1}^j, R_n^j, |\mu(B_n^j)|) + f(r_{n-1}^j, R_{n-1}^j, |\mu(B_{n-1}^j)|) \\ &= |\mu(B_n^j)| f(r_{n-1}^j, R_n^j, 1) = |\mu(B_n^j)| f(r_n^j, R_n^j, 1), \end{aligned}$$

where we have used (3.3), the induction hypothesis, the fact that the quantity $|\mu(B_{n-1}^j)|$ does not vary during the expansion times and that, since n is an expansion time, $r_{n-1}^j = r_n^j$.

It remains to prove that inequality (3.8) is preserved during a merging time. Let n be a merging time and let $\{B_{n-1}^i\}_{i \in I} \subset B_n^j$. Since $\mu(B_n^j) = \sum_{i \in I} \mu(B_{n-1}^i)$, we have $|\mu(B_n^j)| \leq$

$\sum_{i\in I}|\mu(B^i_{n-1})|$. Then, using (3.5), we conclude

$$\begin{aligned} F(B^j_n) &\geq \sum_{i\in I} F(B^i_{n-1}) \\ &\geq \sum_{i\in I}|\mu(B^i_{n-1})|f(r^i_{n-1},R^i_{n-1},1)\geq|\mu(B^j_n)|f(r^j_n,R^j_n,1). \end{aligned}$$

Finally, let $\bar n\in\mathbb{N}$ be the first integer such that at least one ball in $\mathcal{B}_{\bar n}$ intersects ∂A. Clearly $\sum_{i=1}^{N_{\bar n}} R^i_{\bar n}\geq\rho/2$; moreover, by (3.7), we immediately deduce $\sum_{i=1}^{N_{\bar n}} r^i_{\bar n}\leq c^N\varepsilon N$. Now we distinguish two cases. If $\bar n$ is an expansion time, then using (3.8) and property ii) of f, we get

$$\begin{aligned} F(A) &\geq \sum_{i=1}^{N_{\bar n-1}} F(B^i_{\bar n-1})\geq\sum_{i=1}^{N_{\bar n-1}}|\mu(B^i_{\bar n-1})|f(r^i_{\bar n-1},R^i_{\bar n-1},1)=\sum_{i=1}^{N_{\bar n}}|\mu(B^i_{\bar n})|f(r^i_{\bar n},\frac{R^i_{\bar n}}{c},1) \\ &= \sum_{i=1}^{N_{\bar n}}|\mu(B^i_{\bar n})|f(\sum_{k=1}^{N_{\bar n}} r^k_{\bar n},\frac{1}{c}\sum_{k=1}^{N_{\bar n}}R^k_{\bar n},1)\geq|\mu(A)|f(c^N\varepsilon\, N,\frac{\rho}{2c},1). \end{aligned}$$

If otherwise n is a merging time, then we conclude

$$\begin{aligned} F(A) &\geq \sum_{i=1}^{N_{\bar n-1}} F(B^i_{\bar n-1})\geq\sum_{i=1}^{N_{\bar n-1}}|\mu(B^i_{\bar n-1})|f(r^i_{\bar n-1},R^i_{\bar n-1},1)\geq\sum_{j=1}^{N_{\bar n}}|\mu(B^j_{\bar n})|f(r^j_{\bar n},R^j_{\bar n},1) \\ &= \sum_{j=1}^{N_{\bar n}}|\mu(B^j_{\bar n})|f(\sum_{k=1}^{N_{\bar n}} r^k_{\bar n},\sum_{k=1}^{N_{\bar n}}R^k_{\bar n},1)\geq|\mu(A)|f(c^N\varepsilon\, N,\frac{\rho}{2},1). \end{aligned}$$

Since $c>1$, the conclusion follows. □

Remark 3.4. Notice that, in order to prove (3.4), we gained indeed the following stronger estimate: for every $n\in\mathbb{N}$, we have

$$F(A)\geq\sum_{\substack{B^i_n\in\mathcal{B}_n\\ B^i_n\subset A}}|\mu(B^i_n)|f(c^N\varepsilon\, N,\sum_{k=1}^{N_n}R^k_n,1)\,.$$

4. Compactness

The first step in order to prove the compactness and the Γ-liminf inequality is to show a lower bound for the elastic energy of a "cluster" of dislocations. Let $\mu:=\sum_{i=1}^N\xi^i\delta_{x^i}\in X$ and $\varepsilon>0$. We recall that ω_ε is defined in (3.2) and that $K(c)$ is the Korn's constant for an annulus with a cut, whose ratio of the radii is c (see (A.1)). Finally, we recall that c_1 is the constant in (1.1).

Lemma 4.1. *Fix $\varepsilon>0$, let $\mu:=\sum_{i=1}^N\xi^i\delta_{x^i}\in X$ for some $x^i\in\Omega$ and $\xi^i\in\mathbb{S}$, and let $\beta\in\mathcal{AS}_\varepsilon(\mu)$. Finally, let $0<\delta<1$ and $A\subset\Omega$ be open. If $\operatorname{dist}(x^i,\partial A)\geq\varepsilon^\delta$ for all $x^i\in A$, then, for every constant $c>1$, we have*

$$\int_{A\setminus\omega_\varepsilon}W(\beta)\,\mathrm{d}x\geq c_1\frac{|\mu(A)|}{2\pi K(c)}((1-\delta)\log\tfrac{1}{\varepsilon}-(N+1)\log c-\log 2N). \tag{4.1}$$

Proof. We apply Proposition 3.2 for f defined as in (3.1) with $C = \frac{c_1}{2\pi K(c)}$ and

$$F(U) = E_\varepsilon(\mu, \beta, U) := \int_{U \setminus \omega_\varepsilon} W(\beta)\, \mathrm{d}x, \tag{4.2}$$

for all open subsets U of Ω. By Lemma 3.1 and (1.1) we deduce that (3.3) holds. Setting $\rho = \varepsilon^\delta$, from (3.4) we conclude

$$\begin{aligned} \int_{A\setminus\omega\varepsilon} W(\beta)\, \mathrm{d}x \quad &\geq \quad |\mu(A)| f(c^N \varepsilon\, N, \frac{\varepsilon^\delta}{2c}, 1) = c_1 \frac{|\mu(A)|}{2\pi K(c)} \log \frac{\varepsilon^\delta}{2c^{N+1}\varepsilon\, N} \\ &= c_1 \frac{|\mu(A)|}{2\pi K(c)} ((1-\delta) \log \tfrac{1}{\varepsilon} - (N+1)\log c - \log 2N). \end{aligned}$$

□

We are now in a position to prove the compactness result. The idea is to modify a sequence of measures $\{\mu_h\}$ with equi-bounded energy by identifying clusters of dislocations with Dirac masses whose multiplicity is given by the effective Burgers vector of the cluster, i.e. the total mass of the cluster. Applying our lower bound, we show that the modified sequence $\{\tilde\mu_h\}$ is bounded in variation and then weakly* converges, up to a subsequence, to some $\mu \in X$. We deduce the convergence of μ_h to μ with respect to the flat norm by the fact that $\mu_h - \tilde\mu_h$ has vanishing flat norm.

Proof of the compactness property. Let $\varepsilon_h \to 0$ as $h \to +\infty$ and let $\mu_h = \sum_{i=1}^{N_h} \xi_h^i \delta_{x_h^i}$ be a sequence such that $\sup_h \mathcal{F}_{\varepsilon_h}(\mu_h) \leq M$ for some positive constant M. We have to prove that (up to a subsequence) $\{\mu_h\} \stackrel{\text{flat}}{\to} \mu$ for some $\mu \in X$.

Fix $0 < \delta < 1$ and let

$$A_{\varepsilon_h^\delta}(\mu_h) = \bigcup_{x_h^i \in \operatorname{supp}(\mu_h)} B_{\varepsilon_h^\delta}(x_h^i)\,.$$

Notice in particular that $\operatorname{dist}(x_h^i, \partial A_{\varepsilon_h^\delta}) \geq \varepsilon_h^\delta$. Let $\{C_{\delta,h}^l\}_{l=1}^{L_h}$ be the family of the connected components of $A_{\varepsilon_h^\delta}(\mu_h)$ which are contained in Ω and satisfy $|\mu_h(C_{\delta,h}^l)| > 0$. By Lemma 4.1 we deduce that for every $l = 1, \dots, L_h$ and $\beta_h \in \mathcal{AS}_{\varepsilon_h}(\mu_h)$

$$\int_{C_{\delta,h}^l \setminus \omega_{\varepsilon_h}} W(\beta_h)\, \mathrm{d}x \geq c_1 \frac{|\mu_h(C_{\delta,h}^l)|}{2\pi K(c)} ((1-\delta)\log \frac{1}{\varepsilon_h} - (N_h + 1)\log c - \log 2N_h).$$

Since $N_h \leq |\mu_h|(\Omega) \leq \mathcal{E}_{\varepsilon_h}(\mu_h) \leq M \log \frac{1}{\varepsilon_h}$, we deduce

$$\mathcal{E}_{\varepsilon_h}(\mu_h) \geq c_1 \sum_{l=1}^{L_h} \frac{|\mu_h(C_{\delta,h}^l)|}{2\pi K(c)} \Big((1 - \delta - M\log c) \log \frac{1}{\varepsilon_h} - \log(2cM \log \frac{1}{\varepsilon_h}) \Big) \tag{4.3}$$

If $c - 1$ is small enough we deduce that $L_h \leq \tilde L$ for some $\tilde L$ independent of h, so that, up to a subsequence, we have $L_h \equiv L \in \mathbb{N}$. For any $l = 1, \dots, L$, let $\tilde x_{\delta,h}^l \in C_{\delta,h}^l$ be fixed and set

$$\tilde\mu_h = \sum_{l=1}^{L} \mu_h(C_{\delta,h}^l) \delta_{\tilde x_{\delta,h}^l}.$$

From (4.3) we easily see that $|\tilde\mu_h|(\Omega)$ is uniformly bounded; hence the sequence $\{\tilde\mu_h\}$ is precompact in X with respect to the weak* topology, and therefore also with respect to the flat topology. It remains to prove that $\|\mu_h - \tilde{\mu}_h\|_{\mathrm{flat}} \to 0$ as $h \to +\infty$. Fix $\phi \in W_0^{1,\infty}(\Omega)$ with $\|\phi\|_{W_0^{1,\infty}(\Omega)} \le 1$. Let $D^l_{\delta,h}$, $l = 1, \dots \tilde{N}_h$ be the connected components of $A_{\varepsilon_h^\delta}$ which are not contained in Ω, and let $E^l_{\delta,h}$, $l = 1, \dots \hat{N}_h$ be the remaining ones, i.e., contained in Ω. Since $\phi = 0$ on $\partial\Omega$ and $\|\phi\|_{W_0^{1,\infty}(\Omega)} \le 1$ we have

$$|\phi(x)| \le \operatorname{diam}(D^l_{\delta,h}) \le 2N_h\varepsilon_h^\delta \le 2M\varepsilon_h^\delta \log\frac{1}{\varepsilon_h} \qquad \text{for all } x \in D^l_{\delta,h}, \tag{4.4}$$

and so

$$\int_{D^l_{\delta,h}} \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h) \le \sup_{D^l_{\delta,h}} |\phi| \int_{D^l_{\delta,h}} \mathrm{d}(|\mu_h| + |\tilde{\mu}_h|) \le (|\mu_h| + |\tilde{\mu}_h|)(D^l_{\delta,h})\, 2M\varepsilon_h^\delta \log\frac{1}{\varepsilon_h}.$$

Set $\bar{\phi}_l = \dfrac{1}{|E^l_{\delta,h}|}\displaystyle\int_{E^l_{\delta,h}} \phi \,\mathrm{d}x$. As in (4.4), we deduce $|\phi - \bar{\phi}_l| \le 2M\varepsilon_h^\delta \log\frac{1}{\varepsilon_h}$ for all $x \in E^l_{\delta,h}$. Therefore, for every $l = 1, \dots \hat{N}_h$ we have

$$\begin{aligned}\int_{E^l_{\delta,h}} \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h) &= \int_{E^l_{\delta,h}} (\phi - \bar{\phi}_l) \,\mathrm{d}(\mu_h - \tilde{\mu}_h) + \int_{E^l_{\delta,h}} \bar{\phi}_l \,\mathrm{d}(\mu_h - \tilde{\mu}_h)\\ &\le (|\mu_h| + |\tilde{\mu}_h|)(E^l_{\delta,h}) \operatorname{diam}(E^l_{\delta,h}) \le (|\mu_h| + |\tilde{\mu}_h|)(E^l_{\delta,h})\, 2M\varepsilon_h^\delta \log\frac{1}{\varepsilon_h}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_\Omega \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h) &= \sum_{l=1}^{\tilde{N}_h} \int_{D^l_{\delta,h}} \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h) + \sum_{l=1}^{\hat{N}_h} \int_{E^l_{\delta,h}} \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h)\\ &\le (|\mu_h| + |\tilde{\mu}_h|)(\Omega)\Big(4M\varepsilon_h^\delta \log\frac{1}{\varepsilon_h}\Big) \le C\Big(\log\frac{1}{\varepsilon_h}\Big)^2 \varepsilon_h^\delta,\end{aligned} \tag{4.5}$$

which tends to zero as $\varepsilon_h \to 0$. By the very definition of the flat norm it follows that $\|\mu_h - \tilde{\mu}_h\|_{\mathrm{flat}} \to 0$ as h tends to infinity. □

5. Lower bound

In the proof of the Γ-liminf inequality we will first suitably remove the clusters of dislocations with zero multiplicity. To this purpose we need a lemma providing upper bounds for the energy on suitable annuli surrounding such clusters. We will use the notation of the discrete ball construction (see Definition 3.3).

Lemma 5.1. *For any given $\varepsilon > 0$, let $\mu \in X$ and $\beta \in \mathcal{AS}_\varepsilon(\mu)$ be fixed. Let $0 < \gamma < \alpha < 1$ and let $c > 1$ be such that $\log c < \frac{\log\frac{1}{\varepsilon}(\alpha-\gamma)}{|\mu|(\Omega)+1}$.*

Then there exists $\bar{n} \in \mathbb{N}$ such that

(i) $\varepsilon^\alpha \le \displaystyle\sum_{i=1}^{N_{\bar{n}}} R^i_{\bar{n}} \le \varepsilon^\gamma$;

(ii) *$\bar{n}$ is not a merging time;*

(iii) $\displaystyle\int_{\Omega\cap\cup_i B_{c R^i_{\bar{n}}}(x^i_{\bar{n}})\setminus B_{R^i_{\bar{n}}}(x^i_{\bar{n}})} W(\beta)\ \mathrm{d}x \le \frac{\log c\, E_\varepsilon(\mu,\beta)}{\log\frac{1}{\varepsilon}(\alpha-\gamma) - \log c\,|\mu|(\Omega) - \log c}.$

Proof. We denote by n_α the first step n in the ball construction such that $\sum_{i=1}^{N_n} R_n^i \geq \varepsilon^\alpha$ and similarly we set n_γ, so that for every $n_\alpha \leq n \leq n_\gamma - 1$ (i) holds true. Notice that in the ball construction

$$\sum_{i=1}^{N_n} R_n^i \leq c \sum_{i=1}^{N_{n-1}} R_{n-1}^i.$$

By a straightforward computation we get

$$\varepsilon^\gamma \leq c^{n_\gamma - n_\alpha + 1} \varepsilon^\alpha,$$

and so $n_\gamma - n_\alpha \geq \frac{(\alpha-\gamma)\log\frac{1}{\varepsilon}}{\log c} - 1$. Recalling that the total number of merging is smaller than $|\mu|(\Omega)$, we deduce that

$$n_\gamma - 1 - \tau_{n_\gamma - 1} - (n_\alpha - 1 - \tau_{n_\alpha - 1}) \geq \frac{(\alpha-\gamma)\log\frac{1}{\varepsilon}}{\log c} - 1 - |\mu|(\Omega),$$

where the left hand side represents the number of expansion times between n_α and $n_\gamma - 1$. The thesis follows by the mean value theorem since

$$E_\varepsilon(\mu,\beta) \geq \sum_{\substack{n_\alpha \leq n \leq n_\gamma - 1\\ n \text{ is an expansion time}}} \int_{\Omega \cap \cup_i B_{c\,R_n^i}(x_n^i) \setminus B_{R_n^i}(x_n^i)} W(\beta)\ \mathrm{d}x.$$

$\square$

Proof of the Γ*-liminf inequality.* Let $\varepsilon_h \to 0$ as $h \to +\infty$. For any $h \in \mathbb{N}$, let $\mu_h = \sum_{i=1}^{N_h} \xi_h^i \delta_{x_h^i} \in X$ such that $\mu_h \stackrel{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^{N} \xi^i \delta_{x^i} \in X$. We have to prove that

$$\mathcal{F}(\mu) \leq \liminf_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h).$$

By a standard localization argument we can assume $\mu = \xi^0 \delta_{x^0}$ for some $\xi^0 \in \mathbb{S}$, $x^0 \in \Omega$. Moreover, we can assume that $\liminf_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) = \lim_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \leq M$, for some positive constant M.

Let $\beta_h \in \mathcal{AS}_{\varepsilon_h}(\mu_h)$ be the strain that realizes the minimum in (1.3), namely $E_{\varepsilon_h}(\mu_h,\beta_h) = \min_{\beta \in \mathcal{AS}_{\varepsilon_h}(\mu_h)} E_{\varepsilon_h}(\mu_h,\beta)$. The idea is to give a lower bound for the energy on a finite number of shrinking balls where both the energy and the flat norm concentrate. To this purpose fix $0 < \gamma < \alpha < 1$, $c > 1$ such that

$$\log c < \min\left\{\frac{\alpha-\gamma}{M+1}, \frac{1-\alpha}{M}\right\}. \tag{5.1}$$

Since

$$N_h = |\mu_h|(\Omega) \leq M \log\frac{1}{\varepsilon_h}, \tag{5.2}$$

we can apply Lemma 5.1; in particular, let $\bar n$ be such that $\varepsilon_h^\alpha \leq \sum_{i=1}^{N_{\bar n}} R_{\bar n}^i \leq \varepsilon_h^\gamma$. Consider the family of balls $B_{\bar n}^i := B_{R_{\bar n}^i}(x_{\bar n}^i)$ in $\mathcal{B}_{\bar n}$ such that $B_{cR_{\bar n}^i}(x_{\bar n}^i) \subset \Omega$. We denote by $J_h \subset \{1,\dots,N_{\bar n}\}$ the set of indices i such that $B_{cR_{\bar n}^i}(x_{\bar n}^i) \subset \Omega$ and $\mu_h(B_{\bar n}^i) = 0$, and by $I_h \subset \{1,\dots,N_{\bar n}\}$ the set of indices i such that $B_{cR_{\bar n}^i}(x_{\bar n}^i) \subset \Omega$ and $\mu_h(B_{\bar n}^i) \neq 0$.

We prove that I_h is finite. Recalling the definition of E_{ε_h} in (4.2) and in view of Remark 3.4 applied with $f(r,R,t)=\frac{c_1}{2\pi K(c)}t\log\frac{R}{r}$ we obtain

$$\begin{aligned}E_{\varepsilon_h}(\mu_h,\beta_h,\cup_{i\in I_h}B^i_{\bar n}) &\geq \sum_{i\in I_h}|\mu_h(B^i_{\bar n})|f(c^{N_h}\varepsilon_h N_h,\sum_{i=1}^{N_{\bar n}}R^i_{\bar n},1)\\ &\geq \sum_{i\in I_h}c_1\frac{|\mu_h(B^i_{\bar n})|}{2\pi K(c)}\Big((1-\alpha-M\log c)\log\frac{1}{\varepsilon_h}-\log\big(M\log\frac{1}{\varepsilon_h}\big)\Big),\end{aligned}$$

where we have used $\sum_{i=1}^{N_{\bar n}}R^i_{\bar n}\geq\varepsilon_h^\alpha$ and (5.2). Since $E_{\varepsilon_h}(\mu_h,\beta_h,\cup_{i\in I_h}B^i_{\bar n})\leq M\log\frac{1}{\varepsilon_h}$, and $1-\alpha-M\log c>0$ (see (5.1)), we conclude that $\sharp I_h$ is uniformly bounded. Up to a subsequence, we have $\sharp I_h=L$ for every $h\in\mathbb N$, for some $L\in\mathbb N$.

Consider now $i\in J_h$. Recalling that $\operatorname{Curl}\beta_h=0$ in the annulus $C^i_{\bar n}:=B_{c\,R^i_{\bar n}}(x^i_{\bar n})\setminus B_{R^i_{\bar n}}(x^i_{\bar n})$ and $\mu_h(B_{c\,R^i_{\bar n}}(x^i_{\bar n}))=0$, we get that $\beta_h=\nabla v^i_{h,\bar n}$ for some $v^i_{h,\bar n}\in H^1(C^i_{\bar n};\mathbb R^2)$. Thus, applying Korn's inequality (Theorem A.1) to $v^i_{h,\bar n}$, we deduce that

$$\int_{C^i_{\bar n}}|\nabla v^i_{h,\bar n}-A^i_{h,\bar n}|^2\,\mathrm dx\leq K(c)\int_{C^i_{\bar n}}|(\nabla v^i_{h,\bar n})^{\rm sym}|^2\,\mathrm dx=K(c)\int_{C^i_{\bar n}}|\beta_h^{\rm sym}|^2\,\mathrm dx,$$

where $A^i_{h,\bar n}$ is a suitable skew-symmetric matrix. By a standard extension argument, there exists a function $u^i_{h,\bar n}\in H^1(B_{c\,R^i_{\bar n}}(x^i_{\bar n});\mathbb R^2)$ such that $\nabla u^i_{h,\bar n}=\nabla v^i_{h,\bar n}-A^i_{h,\bar n}$ in $C^i_{\bar n}$ and

$$\int_{B_{c\,R^i_{\bar n}}(x^i_{\bar n})}|\nabla u^i_{h,\bar n}|^2\,\mathrm dx\leq C_1\int_{C^i_{\bar n}}|\nabla v^i_{h,\bar n}-A^i_{h,\bar n}|^2\,\mathrm dx\leq C_1K(c)\int_{C^i_{\bar n}}|\beta_h^{\rm sym}|^2\,\mathrm dx,\tag{5.3}$$

for some positive constant C_1. Consider the field $\tilde\beta_h:\Omega\to\mathbb M^{2\times 2}$ defined by

$$\tilde\beta_h(x):=\begin{cases}\nabla u^i_{h,\bar n}(x)+A^i_{h,\bar n} & \text{if } x\in B^i_{\bar n}\text{ with } i\in J_h,\\ \beta_h(x) & \text{otherwise in }\Omega_{\varepsilon_h}(\mu_h).\end{cases}$$

It follows, by the definition of $\tilde\beta_h$ and by (5.3), that for every $i\in J_h$ the following inequalities hold

$$\begin{aligned}\int_{B_{cR^i_{\bar n}}(x^i_{\bar n})}W(\tilde\beta_h)\,\mathrm dx &\leq c_2\int_{B_{cR^i_{\bar n}}(x^i_{\bar n})}|\tilde\beta_h^{\rm sym}|^2\,\mathrm dx\\ &\leq c_2\int_{C^i_{\bar n}}|\beta_h^{\rm sym}|^2\,\mathrm dx+c_2\int_{B^i_{\bar n}}|\tilde\beta_h^{\rm sym}|^2\,\mathrm dx\\ &\leq \frac{c_2}{c_1}(1+C_1K(c))\int_{C^i_{\bar n}}W(\beta_h)\,\mathrm dx\,,\end{aligned}$$

where c_1 and c_2 are the constants in (1.1). Applying Lemma 5.1, we deduce

$$\frac{1}{\log\frac{1}{\varepsilon_h}}\int_{\bigcup_{i\in J_h}B_{cR^i_{\bar n}}(x^i_{\bar n})}W(\tilde\beta_h)\,\mathrm dx\leq\frac{c_2}{c_1}(1+C_1K(c))\frac{M\log c}{\log\frac{1}{\varepsilon_h}(\alpha-\delta-M\log c)-\log c},\tag{5.4}$$

which vanishes as $\varepsilon_h\to0$.

Let us introduce the modified measure

$$\hat\mu_h=\sum_{i\in I_h}\mu_h(B^i_{\bar n})\delta_{x^i_{\bar n}}.$$

Arguing as in the proof of the compactness property, and more precisely of estimate (4.5), we deduce that $\hat{\mu}_h - \mu_h \xrightarrow{\text{flat}} 0$, and hence, up to a subsequence, $\hat{\mu}_h \overset{*}{\rightharpoonup} \xi^0 \delta_{x^0}$.

The points $x^i_{\bar{n}}$, $i \in I_h$ converge, up to a subsequence, to some point in a finite set of points $\{y^0 = x^0, y^1, \dots, y^{L'}\}$ contained in $\bar{\Omega}$. Let $\rho > 0$ be such that $B_{2\rho}(x_0) \subset\subset \Omega$ and $B_{2\rho}(y^j) \cap B_{2\rho}(y^k) = \emptyset$ for all $j \neq k$. Then,

$$x^i_{\bar{n}} \in B_\rho(y^j) \qquad \text{for some } j \text{ and for } h \text{ large enough}.$$

Thus, using the convergence of $\hat{\mu}_h$ to $\xi^0 \delta_{x^0}$, one can show that for h large enough

$$\sum_{x^i_{\bar{n}} \in B_\rho(x^0)} \mu_h(B^i_{\bar{n}}) = \xi^0 . \tag{5.5}$$

We finally introduce the measure

$$\tilde{\mu}_h = \mu_h \llcorner \cup_{i \in I_h(\rho)} B^i_{\bar{n}} ,$$

where we have introduced the notation $I_h(\rho) = \{i \in I_h \;:\; x^i_{\bar{n}} \in B_\rho(x^0)\}$; by (5.4), it follows that

$$\mathcal{E}_{\varepsilon_h}(\mu_h) = \int_{\Omega_{\varepsilon_h}(\mu_h)} W(\beta_h)\, \mathrm{d}x \geq \int_{\Omega_{\varepsilon_h}(\tilde{\mu}_h) \cap B_{2\rho}(x^0)} W(\tilde{\beta}_h)\, \mathrm{d}x + \mathrm{o}(1). \tag{5.6}$$

It remains to prove the lower bound for the right hand side of (5.6). Fix $0 < \eta < \gamma$. Let us denote by $g_h : [\eta, \gamma] \to \{1, \dots, L\}$ the function which associates with any $\delta \in (\eta, \gamma)$ the number $g_h(\delta)$ of the connected components of $\cup_{i \in I_h(\rho)} B_{\varepsilon_h^\delta}(x^i_{\bar{n}})$. For every $h \in \mathbb{N}$, the function g_h is monotone so that it can have at most L discontinuities. Let us denote by δ^i_h for $i = 1, \dots, \hat{L} \leq L$ such points of discontinuity, with

$$\eta \leq \delta^1_h < \ldots < \delta^{\hat{L}}_h \leq \gamma.$$

It is easy to see that there exists a finite set $\Delta = \{\delta^0, \delta^1, \dots, \delta^{\tilde{L}}\}$ with $\delta^i < \delta^{i+1}$, such that, up to a subsequence $\{\delta^i_h\}_{h \in \mathbb{N}}$ converges to some point in Δ, as $h \to +\infty$, for every $i = 1, \dots, \hat{L}$. We may always assume $\delta^0 = \eta$, $\delta^{\tilde{L}} = \gamma$ and $\tilde{L} \leq \hat{L} + 2$.

Now, for any fixed $\sigma > 0$ small enough and for h large enough (i.e., such that for any $j = 1, \dots, \hat{L}$, $|\delta^j_h - \delta^i| < \sigma$ for some $\delta^i \in \Delta$) the function g_h is constant in the interval $[\delta^i + \sigma, \delta^{i+1} - \sigma]$. Thus for every $i = 0, \dots, \tilde{L} - 1$ we can construct a finite family of $N_{i,h}$ annuli $C^{j,h}_i = B^{j,h}_{\varepsilon_h^{\delta^i + \sigma}} \setminus B^{j,h}_{\varepsilon_h^{\delta^{i+1} - \sigma}}$ with $j = 1, \dots, N_{i,h}$, such that $C^{j,h}_i$ are pairwise disjoint for all i and all j and

$$\bigcup_{k \in I_h(\rho)} B^k_{\bar{n}} \subseteq \bigcup_{j=1}^{N_{i,h}} B^{j,h}_{\varepsilon_h^{\delta^{i+1} - \sigma}} \tag{5.7}$$

for all $i = 0, \dots, \hat{L}$. Note that, for h large enough, $C^{j,h}_i \subset B_{2\rho}(x^0)$ for all i and j. Recalling (2.8) and in view of Remark 2.2, the following estimate holds

$$\begin{aligned} \int_{C^{j,h}_i} W(\tilde{\beta}_h)\, \mathrm{d}x \;\; &\geq \;\; \log \frac{1}{\varepsilon_h} (\delta^{i+1} - \delta^i + 2\sigma) \psi_{\varepsilon_h^{\delta^{i+1} - \sigma}, \varepsilon_h^{\delta^i + \sigma}} (\tilde{\mu}_h(B^{j,h}_{\varepsilon_h^{\delta^i + \sigma}})) \\ &\geq \;\; \log \frac{1}{\varepsilon_h} (\delta^{i+1} - \delta^i + 2\sigma) \psi(\tilde{\mu}_h(B^{j,h}_{\varepsilon_h^{\delta^i + \sigma}})) - C_0 |\tilde{\mu}_h(B^{j,h}_{\varepsilon_h^{\delta^i + \sigma}})|^2 . \end{aligned}$$

Notice that in view of (5.7) and the weak* convergence of $\{\hat{\mu}_h\}$, we have

$$|\tilde{\mu}_h(B^{j,h}_{\varepsilon_h^{\delta^i+\sigma}})| \leq \sum_{k\in I_h(\rho)} |\tilde{\mu}_h(B^k_{\tilde{n}})| \leq |\hat{\mu}_h|(\Omega) \leq C_2\,,$$

for some $C_2 > 0$. Summing over $i = 0, \dots, \tilde{L}-1$ and $j = 1, \dots, N_{h,i}$, we obtain the following chain of inequalities

$$\int_{\Omega_{\varepsilon_h}(\tilde{\mu}_h)\cap B_{2\rho}(x^0)} W(\tilde{\beta}_h)\,\mathrm{d}x \geq \sum_{i=0}^{\tilde{L}-1}\sum_{j=1}^{N_{h,i}} \int_{C_i^{j,h}} W(\tilde{\beta}_h)\,\mathrm{d}x$$

$$\geq \sum_{i=0}^{\tilde{L}-1}\sum_{j=1}^{N_{h,i}} \left(\log\frac{1}{\varepsilon_h}(\delta^{i+1}-\delta^i+2\sigma)\psi(\tilde{\mu}_h(B^{j,h}_{\varepsilon_h^{\delta^i+\sigma}})) - C_0|\tilde{\mu}_h(B^{j,h}_{\varepsilon_h^{\delta^i+\sigma}})|^2\right)$$

$$\geq \log\frac{1}{\varepsilon_h}\sum_{i=0}^{\tilde{L}-1}(\delta^{i+1}-\delta^i+2\sigma)\varphi(\xi^0) - C_0L^2C_2^2\,,$$

where the last inequality is a consequence of (5.5), recalling the definition of φ (see (2.9)). Finally we get

$$\int_{\Omega_{\varepsilon_h}(\tilde{\mu}_h)\cap B_{2\rho}(x^0)} W(\tilde{\beta}_h)\,\mathrm{d}x \geq (\gamma-\eta+2\sigma\tilde{L})\log\frac{1}{\varepsilon_h}\varphi(\xi^0) - C_0L^2C_2^2,$$

and hence using (5.6) we have

$$\liminf_{h\to+\infty}\mathcal{F}_{\varepsilon_h}(\mu_h) \geq (\gamma-\eta+2\sigma\tilde{L})\varphi(\xi^0).$$

The Γ-liminf inequality follows by taking the limits $\sigma\to 0$, $\eta\to 0$ and $\gamma\to 1$. □

6. Upper Bound

In this section we will prove the Γ-limsup inequality, namely we will show that for every $\mu\in X$ there exists a *recovery sequence* $\{\mu_h\}\subset X$ that converges to μ in the flat topology and satisfies

$$\limsup_{h\to+\infty}\mathcal{F}_{\varepsilon_h}(\mu_h) \leq \mathcal{F}(\mu).$$

We first assume that μ belongs to the subclass $\mathcal{D}$ of X defined by

$$\mathcal{D} := \{\mu\in X \mid \mu = \sum_{i=1}^N b^i\delta_{x^i},\, b^i\in\mathfrak{B},\, x^i\neq x^j \text{ for } i\neq j\}.$$

where $\mathfrak{B}$ is the class of Burgers vectors defined in Definition 2.3. The general case is obtained by a standard diagonal argument.

Let $\mu = \sum_{i=1}^N b^i\delta_{x^i}$ in $\mathcal{D}$; then $\mathcal{F}(\mu) = \sum_{i=1}^N \varphi(b^i) = \sum_{i=1}^N \psi(b^i)$. In this case, the recovery sequence is given by the constant sequence $\mu_h\equiv\mu$ for every $h\in\mathbb{N}$. To show this, for every $i = 1,\dots,N$, let $\beta^{b^i}_{\mathbb{R}^2}$ be the planar strain field defined in the whole of $\mathbb{R}^2$ corresponding to the dislocation centered at x^i with Burgers vector b^i. Recalling (2.3), we set

$$\beta^i(x) := \beta^{b^i}_{\mathbb{R}^2}(x-x^i) = \frac{1}{|x-x^i|}\Gamma_{b^i}(\theta) \quad \text{where } \theta = \arctan\frac{x_2-x_2^i}{x_1-x_1^i}\,,$$

and define $\beta_\mu := \sum_{i=1}^N \beta^i$. Clearly $\beta_\mu \in \mathcal{AS}_{\varepsilon_h}(\mu_h)$ for every $h \in \mathbb{N}$. Then

$$\begin{aligned}
\mathcal{F}_{\varepsilon_h}(\mu_h) &= \frac{1}{\log\frac{1}{\varepsilon_h}} \min_{\beta\in\mathcal{AS}_{\varepsilon_h}(\mu_h)} \int_{\Omega_{\varepsilon_h}(\mu_h)} W(\beta)\,\mathrm{d}x \\
&\le \frac{1}{\log\frac{1}{\varepsilon_h}} \int_{\Omega_{\varepsilon_h}(\mu_h)} W(\beta_\mu)\,\mathrm{d}x = \frac{1}{\log\frac{1}{\varepsilon_h}} \int_{\Omega_{\varepsilon_h}(\mu_h)} W\big(\sum_{i=1}^N \beta^i\big)\,\mathrm{d}x \\
&\le \frac{1}{\log\frac{1}{\varepsilon_h}} \sum_{i=1}^N \int_{B_R(x^i)\setminus B_{\varepsilon_h}(x^i)} W(\beta^i)\,\mathrm{d}x \qquad (6.1)\\
&\quad + \frac{2}{\log\frac{1}{\varepsilon_h}} \sum_{i=1}^N \sum_{j=i+1}^N \int_{(\Omega\setminus B_{\varepsilon_h}(x^i))\setminus B_{\varepsilon_h}(x^j)} \mathbb{C}\beta^i : \beta^j\,\mathrm{d}x, \qquad (6.2)
\end{aligned}$$

where $R > \mathrm{diam}(\Omega)$. As for the integrals in (6.1), from (2.6) we have that for every $i = 1,\dots,N$

$$\lim_{h\to+\infty} \frac{1}{\log\frac{1}{\varepsilon_h}} \int_{B_R(x^i)\setminus B_{\varepsilon_h}(x^i)} W(\beta^i)\,\mathrm{d}x = \psi(b^i).$$

In order to conclude, it suffices to prove that each term of the sum in (6.2) tends to 0 as $h \to +\infty$. To this purpose, for every $i, j = 1,\dots,N$ with $i \ne j$ set $\rho_{ij} := \frac{|x^i - x^j|}{2}$. Then

$$\begin{aligned}
\int_{(\Omega\setminus B_{\varepsilon_h}(x^i))\setminus B_{\varepsilon_h}(x^j)} \mathbb{C}\beta^i : \beta^j\,\mathrm{d}x &= \int_{B_{\rho_{ij}}(x^i)\setminus B_{\varepsilon_h}(x^i)} \mathbb{C}\beta^i : \beta^j\,\mathrm{d}x + \int_{B_{\rho_{ij}}(x^j)\setminus B_{\varepsilon_h}(x^j)} \mathbb{C}\beta^i : \beta^j\,\mathrm{d}x \\
&\quad + \int_{(\Omega\setminus B_{\rho_{ij}}(x^i))\setminus B_{\rho_{ij}}(x^j)} \mathbb{C}\beta^i : \beta^j\,\mathrm{d}x.
\end{aligned}$$

Since $\beta^i \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{x^i\})$ the last term in the right hand side is bounded. As for the first two integrals, it is enough to apply Hölder's inequality in order to obtain

$$\begin{aligned}
\int_{B_{\rho_{ij}}(x^i)\setminus B_{\varepsilon_h}(x^i)} \mathbb{C}\beta^i : \beta^j\,\mathrm{d}x &\le \left\|\mathbb{C}\beta^i\right\|_{L^2(B_{\rho_{ij}}(x^i)\setminus B_{\varepsilon_h}(x^i))} \left\|\beta^j\right\|_{L^2(B_{\rho_{ij}}(x^i)\setminus B_{\varepsilon_h}(x^i))} \\
&\le C \left\|\beta^i\right\|_{L^2(B_{\rho_{ij}}(x^i)\setminus B_{\varepsilon_h}(x^i))} \left\|\beta^j\right\|_{L^2(\Omega\setminus B_{\rho_{ij}}(x^j))};
\end{aligned}$$

here and in the following lines C denotes a positive constant that may change from line to line. By (2.5) we get

$$\int_{B_{\rho_{ij}}(x^i)\setminus B_{\varepsilon_h}(x^i)} |\beta^i|^2\,\mathrm{d}x \le C \log\frac{1}{\varepsilon_h},$$

and hence

$$\int_{(\Omega\setminus B_{\varepsilon_h}(x^i))\setminus B_{\varepsilon_h}(x^j)} \mathbb{C}\beta^i : \beta^j\,\mathrm{d}x \le C\sqrt{\log\frac{1}{\varepsilon_h}}$$

for every $i, j = 1,\dots,N$ with $i \ne j$. Therefore,

$$\lim_{h\to+\infty} \frac{1}{\log\frac{1}{\varepsilon_h}} \int_{(\Omega\setminus B_{\varepsilon_h}(x^i))\setminus B_{\varepsilon_h}(x^j)} \mathbb{C}\beta^i_h : \beta^j_h\,\mathrm{d}x = 0,$$

and so

$$\limsup_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \le \sum_{i=1}^N \psi(b^i) = \sum_{i=1}^N \varphi(b^i) = \mathcal{F}(\mu).$$

We have proved that the Γ-limsup inequality holds for any $\mu \in \mathcal{D}$. Now we conclude noticing that $\mathcal{D}$ is dense in X with respect to the weak* topology, and hence with respect to the flat topology. More precisely, for any $\mu = \sum_{i=1}^N \xi^i \delta_{x^i}$, with $\xi^i \in \mathbb{S} = \operatorname{span}_{\mathbb{Z}} \mathfrak{B}$ $(i = 1, \dots, N)$, we can construct a sequence $\{\mu_k\} \subset \mathcal{D}$ such that $\mathcal{F}(\mu_k) = \mathcal{F}(\mu)$ and $\mu_k \overset{*}{\rightharpoonup} \mu$. Indeed, by (2.10), for every $i = 1, \dots, N$ we can find a decomposition of $\xi^i = \sum_{j=1}^{s_i} \alpha_{ij} b^j$ such that $\varphi(\xi^i) = \sum_{j=1}^{s_i} |\alpha_{ij}| \psi(b^j)$. Now, for every $k \in \mathbb{N}$ we define

$$\mu_k = \sum_{i=1}^{N} \sum_{j=1}^{s_i} b^j \sum_{l=1}^{|\alpha_{ij}|} \delta_{x^i_{jl}(k)},$$

where for every k $x^i_{jl}(k)$ are distinct points in Ω, and $|x^i_{jl}(k) - x^i| \to 0$ as $k \to +\infty$. Clearly $\{\mu_k\} \subset \mathcal{D}$ and $\mu_k \overset{*}{\rightharpoonup} \mu$. Moreover

$$\mathcal{F}(\mu_k) = \sum_{i=1}^{N} \sum_{j=1}^{s_i} \sum_{l=1}^{|\alpha_{ij}|} \varphi(b^j) = \sum_{i=1}^{N} \sum_{j=1}^{s_i} |\alpha_{ij}| \psi(b^j) = \sum_{i=1}^{N} \varphi(\xi^i) = \mathcal{F}(\mu).$$

The thesis follows using a standard diagonal argument. Indeed, since for any measure in $\mathcal{D}$, the recovery sequence is given by the constant sequence, we have

$$\limsup_{h \to \infty} \mathcal{F}_{\varepsilon_h}(\mu_k) \le \mathcal{F}(\mu_k) = \mathcal{F}(\mu).$$

Therefore, there exists a sequence $k_h \to \infty$ as $h \to \infty$ such that $\mu_h := \mu_{k_h}$ is a recovery sequence, i.e.,

$$\limsup_{h \to \infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \le \mathcal{F}(\mu).$$

□

Remark 6.1. In the proof of the Γ-limsup inequality we have shown that configurations of dislocations optimal in energy belong to the class $\mathcal{D}$. As a consequence, we get the same Γ-limit if we start from an energy for which the only admissible dislocations are those whose multiplicity belongs to $\mathfrak{B}$, i.e. to the set of Burgers vectors. Precisely, if we define

$$\mathcal{G}_\varepsilon(\mu) = \begin{cases} \mathcal{F}_\varepsilon(\mu) & \text{if } \mu \in \mathcal{D}, \\ +\infty & \text{otherwise,} \end{cases}$$

then $\mathcal{G}_\varepsilon$ still Γ-converge to the functional $\mathcal{F}$ defined in (2.11). In this respects, the class of Burgers vectors in $\mathfrak{B}$ are the building blocks to describe multiple dislocations in $\mathcal{S}$.

Appendix A. Korn's inequality in thin annuli

Here we revisit some results concerning the Korn's inequality in thin domains. First, we recall the Korn's inequality on annular sets with a cut.

Theorem A.1 (Korn's inequality). *Let $0 < r < R$, let $L := \{0\} \times (r, R)$, and let $u \in H^1((B_R \setminus B_r) \setminus L; \mathbb{R}^2)$ be such that $\int_{(B_R \setminus B_r) \setminus L} (\nabla u - \nabla u^{\mathrm{T}})\, \mathrm{d}x = 0$. Then, there exists a positive constant $K = K(R/r)$ such that*

$$\int_{(B_R \setminus B_r) \setminus L} |\nabla u|^2 \le K\Big(\frac{R}{r}\Big) \int_{(B_R \setminus B_r) \setminus L} |(\nabla u)^{\mathrm{sym}}|^2 \, \mathrm{d}x, \tag{A.1}$$

where $(\nabla u)^{\mathrm{sym}} := \dfrac{\nabla u + \nabla u^{\mathrm{T}}}{2}$.

The proof of such theorem can be proved for instance covering the annulus $(B_R \setminus B_r) \setminus L$ with two open overlapping sets A_1, $A_2 \subset (B_R \setminus B_r)$ with Lipschitz boundary, and applying classical Korn's inequality on each A_i, see for instance [20].

The best constant K of the Korn's inequality on annular sets (without cuts) has been explicitly computed in [10]. In this context it's important to remark that such Korn's constant depends only on the ratio of the radii, and tends to infinity when this parameter tends to 1. In particular, we deduce that also $K(R/r) \to \infty$ as $R/r \to 1$.

A natural question is whether the best (i.e., the lower) Korn' s inequality blows up on thin annuli also in the class of our admissible strains $\mathcal{AS}_{r,R}(\xi)$. Let us show that, actually, this is the case. More precisely, let $\xi \in \mathbb{R}^2$ and let $r_n \to 1$. Then, there exists a sequence of strains $\beta_n \in \mathcal{AS}_{r_n,1}(\xi)$ such that

$$\int_{B_1 \setminus B_{r_n}} |\beta_n|^2\, dx \geq c_n \int_{B_1 \setminus B_{r_n}} |\beta_n^{\mathrm{sym}}|^2\, dx, \tag{A.2}$$

for some $c_n \to \infty$ as $n \to \infty$. Indeed, by [10] there exists a sequence $u_n \in H^1(B_1 \setminus B_{r_n}; \mathbb{R}^2)$ such that

$$\int_{B_1 \setminus B_{r_n}} |\nabla u_n|^2\, dx \geq \tilde{c}_n \int_{B_1 \setminus B_{r_n}} |\nabla u_n^{\mathrm{sym}}|^2\, dx$$

with $\tilde{c}_n \to \infty$ as $n \to \infty$. By homogeneity we may assume

$$\int_{B_1 \setminus B_{r_n}} |\nabla u_n^{\mathrm{sym}}|^2\, dx = 1.$$

Let $\beta(\rho, \theta) := \frac{\xi}{2\pi\rho} \otimes (-\sin\theta, \cos\theta)$, and notice that $\beta \in \mathcal{AS}_{r_n,1}(\xi)$ for every n. Finally, set $\beta_n = \nabla u_n + \beta \in \mathcal{AS}_{r_n,1}(\xi)$; a straightforward computation shows that (A.2) holds.

The sequence β_n just constructed is such that its symmetric part is bounded in L^2, while its skew part blows up as $n \to \infty$. In particular, the linearized energy induced by β_n on the annuli $B_1 \setminus B_{r_n}$ is larger than $1 - r_n$. In the next example we construct a strain $\beta \in \mathcal{AS}_{r,1}(\xi)$ for every $0 < r < 1$ whose linearized energy density vanishes on thin annuli $B_1 \setminus B_r$ (as $r \to 1$), showing that the function $\psi_{r,R}$ defined in (2.8) vanishes as $R/r \to 1$.

Example A.1. Let $S(x, y) : \mathbb{R}^2 \mapsto \mathbb{M}^{2\times 2}$ be defined by

$$S(x, y) := \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}.$$

Notice that curl $S = (1, 0)$. Set

$$f(\rho, \theta) := \frac{\rho^2}{4} - \frac{1}{2}\log\rho.$$

Notice that $\Delta f = 1$, and hence curl $(-f_y, f_x) = 1$. Finally, set

$$\beta(x, y) := S(x, y) - \begin{pmatrix} -\frac{\partial f}{\partial y} & \frac{\partial f}{\partial x} \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\beta \in \mathcal{AS}_{r,1}((\pi, 0))$ for every $0 < r < 1$ and $|\beta^{\mathrm{sym}}|^2 \leq |\nabla f|^2$. Moreover, $|\nabla f| = 0$ on ∂B_1; a straightforward computation shows that

$$\lim_{r\to 1} \frac{1}{\log\frac{1}{r}} \int_{B_1 \setminus B_r} |\nabla f|^2\, \mathrm{d}x = 0,$$

so that the density of the linearized elastic energy vanishes on thin annuli $B_1 \setminus B_r$ as $r \to 1$.

References

[1] Alberti G., Baldo S., Orlandi G.: Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J.* **54** (2005), 1411-1472.

[2] Alicandro R., Cicalese M., Ponsiglione M.: Variational equivalence between Ginzburg- Landau, XY spin systems and screw dislocations energies. *Indiana Univ. Math. J.*, in press.

[3] Alicandro R., Ponsiglione M.: Ginzburg-Landau functionals and renormalized energy: A revised Γ-convergence approach. Preprint 2011.

[4] Bacon D.J., Barnett D.M., Scattergood R.O.: Anisotropic continuum theory of lattice defects.*Progress in Material Science* **23** (1978), 51-262.

[5] Baldo S., Jerrard R., Soner H.M., Orlandi G.: Convergence of Ginzburg-Landau functionals in 3D superconductivity. Submitted paper.

[6] Bethuel F., Brezis H., Hélein F.: *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and Their Applications, vol.13, Birkhäuser Boston, Boston (MA), 1994.

[7] Dautray R., Lions J.L.: *Mathematical analysis and numerical methods for science and technology*, vol.3, Springer, Berlin, 1988.

[8] Cermelli P., Leoni G.: Renormalized energy and forces on dislocations. *SIAM J. Math. Anal.* **37** (2005), no. 4, 1131-1160.

[9] Conti S., Ortiz M.: Dislocation microstructures and the effective behaviour of single crystals. *Arch. Rat. Mech. Anal.* **176** (2005), 103-147.

[10] Dafermos C.M.: Some remarks on Korn's inequality. *Z. Angew. Math. Phys.* (ZAMP) **19** (1968), 913-920.

[11] Fleck N.A., Hutchinson J.W.: A phenomenological theory for strain gradient effects in plasticity. *J. Mech. Phys. Solids* **51** (2003), 2057-2083.

[12] Garroni A., Leoni G., Ponsiglione M.: Gradient theory for plasticity via homogenization of discrete dislocations, *J. Eur. Math. Soc.* **12** (2010), no. 5, 1231-1266.

[13] Gurtin M.E., Anand L.: A theory of strain gradient plasticity for isotropic, plastically irrotational materials. I. Small deformations. *J. Mech. Phys. Solids* **53** (2005), 1624-1649.

[14] Jerrard R.L.: Lower bounds for generalized Ginzburg-Landau functionals, *SIAM J. Math. Anal.* **30** (1999), no. 4, 721-746.

[15] Jerrard R.L., Soner H.M.: Limiting behaviour of the Ginzburg-Landau functional, *J. Funct. Anal.* **192** (2002), no. 2, 524561.

[16] Jerrard R.L., Soner H.M.: The Jacobian and the Ginzburg-Landau energy, *Calc. Var. Partial Differential Equations* **14** (2002), no. 2, 151-191.

[17] Ponsiglione M.: Elastic energy stored in a crystal induced by screw dislocations: from discrete to continuous, *SIAM J. Math. Anal.* **39** (2007), no. 2, 449-469.

[18] Sandier E.: Lower bounds for the energy of unit vector fields and applications, *J. Funct. Anal.* **152** (1998), no. 2, 379-403.

[19] Sandier E., Serfaty S.: *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and Their Applications, vol. 70, Birkhäuser Boston, Boston (MA), 2007.

[20] Scardia L., Zeppieri C.I: Gradient theory for plasticity as the Γ-limit of a nonlinear dislocation energy. Preprint 2011.

(Lucia De Luca) Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma, P.le Aldo Moro 5, I-00185 Roma, Italy

E-mail address, L. De Luca: l.deluca@mat.uniroma1.it

(Adriana Garroni) Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma, P.le Aldo Moro 5, I-00185 Roma, Italy

E-mail address, A. Garroni: garroni@mat.uniroma1.it

(Marcello Ponsiglione) Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma, P.le Aldo Moro 5, I-00185 Roma, Italy

E-mail address, M. Ponsiglione: marcello.ponsiglione@mat.uniroma1.it