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Measure data problems, lower order terms and interpolation effects

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Abstract We deal with the solutions to nonlinear elliptic equations of the form

$$-\operatorname{div} a(x, Du) + g(x, u) = f,$$

with f being just a summable function, under standard growth conditions on g and a. We prove general local decay estimates for level sets of the gradient of solutions in turn implying very general estimates in rearrangement and non-rearrangement function spaces, up to Lorentz-Morrey spaces. The results obtained are in clear accordance with the classical Gagliardo-Nirenberg interpolation theory.

Keywords Nonlinear elliptic problems · Lower order term · Morrey-Lorentz regularity · Rearrangement function spaces · Gagliardo-Nirenberg interpolation inequalities

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Contents

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1 Introduction

This paper deals with regularity properties of the solutions to the following class of Dirichlet problems

$$\begin{cases} -\mathrm{div}\, a(x, Du) + g(x, u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

in which Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, f belongs to $L^1(\Omega)$, g is a lower order term and a is a Leray-Lions type operator on $W^{1,p}_0(\Omega)$, with standard growth and monotonicity properties. The specific assumptions we are considering are now listed as follows. The vector field $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory regular and satisfies standard monotonicity and p-growth conditions, i. e.,

$$\begin{cases} \nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_2 - z_1|^2 \leq \langle a(x, z_2) - a(x, z_1), z_2 - z_1 \rangle \\ |a(x, z)| \leq L(s^2 + |z|^2)^{\frac{p-1}{2}} \end{cases} \tag{2}$$

for every z_1, z_2, $z \in \mathbb{R}^n$ and $x \in \Omega$; the structure constants satisfy $0 < \nu \leq 1 \leq L$ and $s \geq 0$. In particular, we stress that the function $x \mapsto a(x, \cdot)$ is measurable.

The lower order term $g : \Omega \times \mathbb{R} \to \mathbb{R}$ will denote a Carathéodory function such that

$$\exists\, m, \alpha_0 > 0 \ : \ \text{for all } t \text{ and a. e. } x \in \Omega \ \ g(x, t)\,\mathrm{sgn}(t) \geq \alpha_0 |t|^m, \tag{3}$$

$$\forall\, \tau > 0 \ \text{ the function } \mathcal{G}_\tau(x) := \sup_{|t| \leq \tau} |g(x, t)| \text{ belongs to } L^1_{\mathrm{loc}}(\Omega). \tag{4}$$

Finally, the right hand side datum f is in the most general case considered to be a measure. A typical example involves the p-Laplacean operator with coefficients:

$$\begin{cases} -\mathrm{div}\, \big(c(x)|Du|^{p-2} Du\big) + |u|^{m-1} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \nu \leq c(x) \leq L$ is a measurable function.

At this point we preliminary observe that the dependence of the vector field $a(\cdot)$ with respect to x is just measurable, and this restricts our analysis to the so-called subdual range (see for instance [38]). This means that f is not in general considered to be in the dual $W^{-1,p'}(\Omega)$ and the maximal regularity expected for Du does not in general go beyond

$$Du \in L^{p+\delta}(\Omega) \tag{5}$$

for some small $\delta \equiv \delta(n, p, \nu, L)$. This is in general a consequence of Gehring's Lemma when f is integrable enough. We refer to Sections 2 for more notation, definitions and for the type of solutions considered in this paper.

In the last decades a wide literature has dealt with elliptic equations with measure data. We just cite the pioneering paper [9], in which it has been firstly used a priori estimates techniques and standard approximating methods, in order to obtain the existence of solutions to (1). After [9], a plenty of interesting regularity results have been obtained for solutions to (1) with and without the lower order term g.[1] In particular, the presence of the lower order terms usually brings new features in this kind of problems. Basically, lower order terms, with sign and standard growth properties, do have a regularizing effect on the solutions to elliptic problems with measure data. This is proved via the approach towards subdual problems as mentioned above, in [12] and [11], in which it is shown the existence and some basic regularity properties of the solutions to (1) starting from L^1-data. By using a similar approach, the case when f belongs to $L^\gamma(\Omega)$ has been treated in [16], namely,

$$\text{for } 1 < p < n, \quad 1 < \gamma < np/(np - n + p) \quad \text{and} \quad p - 1 < m < 1/(\gamma - 1) \tag{6}$$

it holds

$$f \in L^\gamma(\Omega) \implies Du \in L^{\frac{mp\gamma}{m+1}}(\Omega). \tag{7}$$

Recently, the case when f belongs to the Marcinkiewicz space $\mathcal{M}^\gamma(\Omega)$ has been dealt, again by means of a priori estimates and approximating method. In [8, Theorem 2.4], if $p = 2$, $a = Du$ and the assumptions in (6) hold, then

$$f \in \mathcal{M}^\gamma(\Omega) \implies Du \in \mathcal{M}^{\frac{2m\gamma}{m+1}}(\Omega). \tag{8}$$

In the present paper, we will suggest a general approach to the regularity of solutions to such problems, extending the one for measure data problems introduced in [37,38]. We will in general obtain an estimate on the level sets of the maximal operator M of the gradient of solutions in terms of the level sets of the assigned datum f, up to a correction term which is negligible when

[1] The literature is really too wide to attempt any reasonable comprehensive treatment in a single paper. We refer the interested reader for instance to [13,17,18,43,47] and the references therein.

considering the gradient regularity. Roughly speaking, we will obtain estimates of the type

$$\big|\{M(|Du|) \geq T\lambda\}\big| \lesssim \frac{1}{T^{p+\delta}}\big|\{M(|Du|) \geq \lambda\}\big| + \big|\{[M(|f|)]^{1/\sigma} \geq \lambda\}\big|, \tag{9}$$

for every λ suitably large, and in which $T >> 1$ is a constant to be chosen, $\sigma \equiv \sigma(m,p) \geq 1$ determines the regularity of the gradient of u, $\delta \equiv \delta(n,p,\nu,L)$ is the higher integrability exponent determined in (5); see (84) below. The presence of the factor $T^{-(p+\delta)}$ makes the intermediate term in (9) negligible whenever (9) is used to determine an estimate that *does not violate the maximal regularity in* (5). Estimate (9) is fairly general and implies local estimates in virtually all the most familiar function spaces of rearrangement (Lebesgue, Orlicz, Lorentz) and non-rearrangement ones (Morrey). In particular, the known results (7) and (8) follow as a corollary. Moreover, as described in Section 1.1 below, estimate (9) is a sort of *nonlinear analog of the classical Gagliardo-Nirenberg Interpolation inequalities*, in turn giving a sort of extension of these estimates to more general spaces.

We remark that maximal operators techniques have been used since the basic paper of T. Iwaniec [27]; see for instance [19,14,30,1,21,31,37,32,34], and for related nonlinear estimates [45,28,46,29,33,15,20]; see also [2,39,40,22] for maximal function free techniques.

We give an explicit example of an application of estimate (9) to a regularity result in so-called Lorentz-Morrey spaces, in turn generalizing the main result in [38] (here we also consider the subquadratic case, on the contrary of [38]; see also [39]), where no lower order term is considered. Such extremely general spaces, considered since the basic work of Adams and Lewis in the linear case [6], allow in turn to get general results in Lebesgue, weak-Lebesgue, Morrey and Lorentz spaces. We refer the reader to Section 2.3 for the precise definitions. We will prove the following

Theorem 1 *Let $q \in (0,\infty]$. Assume* (2), (3), (4) *and*

$$f \in L^{\theta}(\gamma,q)(\Omega), \tag{10}$$

with γ, θ such that

$$1 < \gamma \leq \frac{\theta p}{\theta p - \theta + p}, \qquad 1 < p < \theta \leq n \tag{11}$$

and

$$p - 1 < m < \frac{1}{\gamma - 1} \tag{12}$$

Then the solution $u \in W^{1,1}_0(\Omega)$ to (1) *satisfies*

$$Du \in L^{\theta}\Big(\frac{mp\gamma}{m+1}, \frac{mpq}{m+1}\Big) \text{ locally in } \Omega. \tag{13}$$

Moreover, the local estimate

$$
\begin{aligned}
&\||Du|^{p-1}\|_{L^{\theta}\left(\frac{mp\gamma}{(m+1)(p-1)},\frac{mpq}{(m+1)(p-1)}\right)(B_{R/2})}\\
&\qquad\le C\,R^{\frac{\theta(m+1)(p-1)}{mp\gamma}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)}+C\|f\|_{L^{\theta}(\gamma,q)(B_R)}^{\frac{(m+1)(p-1)}{mp}} \quad (14)
\end{aligned}
$$

holds for every ball $B_R \subseteq \Omega$, where C depends only on n, m, p, q and on the structure constants of a.

We immediately refer to Section 5 for further extensions and results not covered by the previous theorem; results on measure data follow as well (see Remark 1).

Clearly, one may see that, by choosing $\theta = n$ and $q = \gamma$ in (10), Theorem 1 covers the classical Sobolev implication as in (7). Moreover, we notice that we mean of course $mpq/(m+1) = \infty$ whenever $q = \infty$; thus, by choosing $\theta = n$ and $q = \infty$ in (10), we can also deduce the regularity results on the Marcinkiewicz scale as in (8). We also point out that in both the cited cases the assumptions on m, p, γ given in Theorem 1 coincide with the ones in [12], [16] and [8]. We will come back on such assumptions in Section 2.2.

As in the classical cases ([12,16,8]), if we do not assume (3) but rather, we consider only a sign condition on g, then no regularity improvement appears and the regularity results are in accordance to those of the cases when no lower terms appear.

Finally, it is worth noticing that the properties of the solutions to the analogous of problem (1) without lower order term in a limiting space, that is $\gamma = 1$, have been studied in [9] provided that $p > 2-1/n$. In the contrary, every result we are proving in the present paper is valid in the full range $1 < p < n$.

1.1 Interpolation effects

Another interesting consequence of our investigation is an arisen connection between the regularity results in Theorem 1 and the Gagliardo-Nirenberg interpolation theory. Indeed, the classical interpolation inequality yields

$$
\|Du\|_{L^r(\Omega)} \le c_1\|D^2u\|_{L^{r_0}(\Omega)}^{\ell}\|u\|_{L^{r_1}(\Omega)}^{1-\ell}+c_2\|u\|_{L^{r_1}(\Omega)}, \tag{15}
$$

where $\ell \in [1/2, 1)$ is the interpolation parameter and the Sobolev exponent r is given by

$$
\frac{1}{r}=\frac{1}{n}+\ell\left(\frac{1}{r_0}-\frac{2}{n}\right)+(1-\ell)\frac{1}{r_1} \tag{16}
$$

(see [23,42]). Now, fix $p = 2$ and consider the solution u to problem (1) starting from $f \in L^{\gamma}(\Omega)$. We can read the nonlinear equation in a separate way, in the sense that the first contribution will give $|D^2u| \in L^{\gamma}$, while the second contribution will give $|u|^m \in L^{\gamma}$. At this point, by choosing $r_0 = \gamma$ and

$r_1 = m\gamma$ in (15) and balancing the interpolation with $\ell = 1/2$, we can compute r by (16) and we obtain

$$r = \frac{2m\gamma}{m+1},$$

that is exactly the exponent given by (13). Hence, the regularity result in Theorem 1, and the level set estimate (9) can be also seen as the proof of a nonlinear interpolation effect. In other words the method proposed here allows to decouple the equation

$$-\mathrm{div}\, a(x, Du) + g(x, u) = f(x) \in L^{\gamma}$$

in the separate inclusions

$$D^2 u \in L^{\gamma} \qquad \text{and} \qquad u \in L^{m\gamma}$$

and gives the same result of the available interpolation inequalities. We remark that this effect is non-trivial since the equation is non-linear, and, especially, non-differentiable since coefficients are measurable. As a further matter, inequality (9) allows to extend such an interpolation effect to the case of degenerate operators on one side, and on another side, to get similar interpolation-type inequalities for solutions in Orlicz, Lorentz and other spaces. Moreover, interpolation effects in weighted spaces also follow by considering different assumptions on $g(\cdot)$; see Section 5.2.

1.2 Ideas from the proof

Now, let us focus on the proof of Theorem 1. It is worth noticing that, in order to obtain the regularity results in (13) and (14), we need to change the classical approach to the problem. Namely, we will extend the techniques recently introduced by Mingione in [38], which deals with the solutions to problem (1) without any lower order terms. The author presents a non-linear potential theory version of the fundamental papers by Adams [3] and Adams and Lewis [6], providing optimal regularity results on the Morrey and the Lorentz-Morrey scale, too. Among other results, in [38, Theorem 11] it is shown that, for any $0 < q \leq \infty$, $2 \leq p < \theta \leq n$, $1 < \gamma \leq \theta p/(\theta p - \theta + p)$, the solutions u to the analogous of (1) without the lower order term g satisfy

$$f \in L^{\theta}(\gamma, q)(\Omega) \implies Du \in L^{\theta}_{\mathrm{loc}}\left(\frac{\theta\gamma(p-1)}{\theta - \gamma}, \frac{\theta q(p-1)}{\theta - \gamma}\right)(\Omega). \tag{17}$$

As stated, the strength of the proofs in [38] relies on elegant estimates on the level sets of sharp fractional maximal operators. In the proof of Theorem 1, we can extend such original arguments, but we need to operate various modifications due to the presence of the lower order term g. Moreover, while the gradient integrability stated in (17) and the related results in [38] are proved only in the case $p \geq 2$, here we will obtain regularity results and the corresponding local estimates also in the subquadratic case $1 < p < 2$. For the latter extension, we will combine the mentioned ingredients together with some arguments in [40], which proposes a purely PDE approach to deal with Lorentz-Morrey estimates for solutions to elliptic problems without lower order terms for any $2 - 1/n < p < n$ (see, in particular, Theorem 4.3 there).

Finally, we will show that Theorem 1 can be extended in a natural way in order to deal with the borderline case $\gamma = 1$. In this case, to obtain (13)-(14) one has to impose some further $L \log L$ integrability on the datum f, that is working in the scale of the Morrey-Orlicz spaces (see Theorem 7). Moreover, in Theorem 8 we will provide the needed modifications to the proof of Theorem 1 in order to handle the solutions to the analogous of problem (1) with a class of different lower order terms g.

The paper is organized as follows. In Section 2, we give full details on the structure of the problem and we briefly recall the definitions and some basic properties of the spaces we deal with, also providing some classical estimates for the solutions to nonlinear elliptic problems of type (1). In Section 3, we state and prove basic regularity estimates and other preliminary results. Section 4 is devoted to the proof of Theorem 1. Finally, in Section 5 we analyze further extensions and results not covered by Theorem 1.

2 Setting of the problem

In this section we analyze the structure of the problem, by recalling solvability and scaling properties. Also, we briefly recall the definitions and some basic properties of the spaces and the operators we deal with, as well as a few classical results.

2.1 Solvability of the problem

We recall the natural definition of the solutions and we discuss the classical solvability of nonlinear elliptic problems (1). Here and throughout the remaining of the paper, for the sake of simplicity we take $g(t) = |t|^{m-1}t$, so we will consider the following problem:

$$\begin{cases} -\operatorname{div} a(x, Du) + |u|^{m-1}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

in which a verifies (2), $0 < \nu \le 1 \le L$, $s \ge 0$, $1 < p < n$, $f \in L^1(\Omega)$ and $0 < m < \infty$. Note that the general case with g satisfying (3)-(4) will follow plainly with no significant modification.

A measurable function u is a distributional solution to (18) if it satisfies

$$u \in W_0^{1,1}(\Omega), \quad a(x, Du) \in L^1(\Omega), \quad |u|^{m-1}u \in L^1(\Omega)$$

and

$$\int_\Omega a(x,Du)D\phi\,\mathrm{d}x+\int_\Omega |u|^{m-1}u\phi\,\mathrm{d}x=\int_\Omega f\phi\,\mathrm{d}x,\quad \forall\,\phi\in C_0^\infty(\Omega).$$

Now, for any $k\in\mathbb{N}$ and $f_k\in W^{-1,p/(p-1)}(\Omega)$ such that $f_k\to f$ in $L^1(\Omega)$, we consider

$$\begin{cases}-\mathrm{div}\,a(x,Du_k)+T_k(|u_k|^{m-1}u_k)=f_k & \text{in }\Omega\\ u_k=0 & \text{on }\partial\Omega,\end{cases}\tag{19}$$

where the truncation function T_k is such that

$$T_k(t)=\min\big\{k,\max\big\{-k,\,t\big\}\big\}.\tag{20}$$

It is known (see [35, Théorème 1]) that, for any $1<p<n$, there exists the (unique) weak solution u_k to (19) which belongs to $W_0^{1,p}(\Omega)$ and such that

$$\int_\Omega a(x,Du_k)D\psi\,\mathrm{d}x+\int_\Omega T_k(|u_k|^{m-1}u_k)\psi\,\mathrm{d}x=\int_\Omega f_k\psi\,\mathrm{d}x,\quad \forall\;\psi\in W_0^{1,p}(\Omega).$$

Also, [10, Theorem 2], yields

$$\int_{\{|u_k|>t\}}|u_k|^m\,\mathrm{d}x\le\int_{\{|u_k|>t\}}|f_k|\,\mathrm{d}x,\quad \forall\,k\in\mathbb{N}\text{ and }\forall\,t\in\mathbb{R}^+,$$

where, as usual, we denoted by $\{|u_k|>t\}$ the t-level set of $|u_k|$ in Ω, that is $\{x\in\Omega:|u_k(x)|>t\}$. Moreover, since the sequence $\{u_k\}$ is relatively compact in $W_0^{1,q}(\Omega)$ for all $q\in[1,(n(p-1))/(n-1))$, we can assume (up to subsequences, still denoted by $\{u_k\}$) the following statements as k goes to ∞

$$u_k\to u\text{ in }W_0^{1,q}(\Omega),\quad 1\le q<\frac{n(p-1)}{n-1},\quad\text{and in a.e. in }\Omega$$
$$a(\cdot,Du_k)\to a(\cdot,Du)\text{ in }L^r(\Omega),\quad 1\le r<\frac{n}{n-1}.$$

We point out that the last assertion is a consequence of the second assumption in (2).

Now, the a. e. convergence of $\{u_k\}$ implies that $|u_k|^{m-1}u_k$ converges a. e. to $|u|^{m-1}u$, too. In addition $|u_k|^{m-1}u_k$ is equi-integrable, since the sequence $\{f_k\}$ is equi-integrable on Ω, $|\{|u_k|>t\}|$ converges to zero as t goes to ∞ uniformly respect to k and (4) holds. Then, by Vitali's Theorem, we deduce that $|u_k|^{m-1}u_k\to|u|^{m-1}u$ in $L^1_{\mathrm{loc}}(\Omega)$ and so we can pass to the limit in (19) to obtain a distributional solution u to (18).[2] Moreover, the function u satisfies

$$u\in W_0^{1,q}(\Omega)\cap L^m(\Omega)\quad\forall\,1\le q<\frac{n(p-1)}{n-1}$$

[2] Note that in order to pass to the limit in (19) we only need a weaker assumption on g with respect to (3); that is, $g(x,t)\mathrm{sgn}(t)\ge 0$. This will guarantee the existence of the solutions to (1) also in the generalized case studied in the forthcoming Section 5.2.

$$a(\cdot, Du) \in L^r(\Omega) \quad \forall\, 1 \le r < \frac{n}{n-1}.$$

We note that since $n(p-1)/(n-1) > 1 \Leftrightarrow p > 2 - 1/n$, we have to assume $2 - 1/n < p < n$.

In the case $1 < p \le 2 - 1/n$, we will again be able to pass to the limit in (19), but we have to use the extra structure assumption on the lower order term g, i. e. (3). By arguing as in [12, Theorem 6], if $m > 1/(p-1)$, we have

$$\int_\Omega |u_k|^m \, \mathrm{d}x \le \int_\Omega |f| \, \mathrm{d}x \quad \text{and} \quad \int_\Omega |Du_k|^q \, \mathrm{d}x \le C, \quad \forall\, 1 \le q < \frac{mp}{m+1}. \tag{21}$$

Thus, we can assume $u_k \to u$ a. e. in $W_0^{1,q}(\Omega)$ for any $1 \le q < mp/(m+1)$, since $u_k = 0$ on $\partial\Omega$, and so $|u_k|^{m-1}u_k \to |u|^{m-1}u$ a. e. in Ω. As before, $|u_k|^{m-1}u_k \to |u|^{m-1}u$ in $L^1_{\mathrm{loc}}(\Omega)$ and, consequently, Fatou's Lemma gives $|u|^{m-1}u \in L^1(\Omega)$.

By standard computations, thanks to the L^q-bound of $|Du_k|$ for some $1 \le q < mp/(m+1)$, we can also prove the a. e. convergence of Du_k to Du. Thus, $Du_k \to Du$ in $L^q(\Omega)$, for $1 \le q < mp/(m+1)$ and, since $p - 1 < 1$, $a(x, Du_k) \to a(x, Du)$ in $L^1(\Omega)$. This will give the existence of a distributional solution also in the case $1 < p \le 2-1/n$. Clearly, if, for any $k \in \mathbb{N}$, $f_k \in L^\infty(\Omega)$, then also $u_k \in L^\infty(\Omega)$, for any $k \in \mathbb{N}$. Also, note that if we take a different approximating sequence $f_k \to f$ in $L^1(\Omega)$, then we obtain the same limiting solution.

From now on, the sequence $\{u_k\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ will be the one fixed in (19), by choosing $f_k \in L^\infty(\Omega)$ as follows

$$f_k(x) = T_k(f(x)), \quad k \in \mathbb{N}. \tag{22}$$

where T_k is given by (20).

2.2 More on the structure of the problem

We complement the assumptions on the exponents m, p and γ that appear in Theorem 1, specifying the range of such structural parameters. First, we emphasize that the bound $p < n$ is not a limitation, otherwise one can use the theory of operators acting between Sobolev spaces in duality.

The natural inequalities $\gamma \le np/(np - n + p) = (pn/(n-p))'$ and $m < 1/(\gamma - 1)$ are due to the fact that we are treating with the case of infinity energy solutions. Also, we have $\theta p/(\theta p - \theta + p) \le \theta/p \;\Leftrightarrow\; 1 < p \le \theta$, and so (11) implies $p\gamma \le \theta$. Moreover,

$$p - 1 < \frac{1}{\gamma - 1} \quad \text{since } 1 < \gamma < \frac{np}{np - n + p} < \frac{p}{p-1}$$

and so (12) is not empty. Finally, the lower bound on m is necessary in order to guarantee that $a(x, Du)$ belongs to $L^1_{\mathrm{loc}}(\Omega)$.

We conclude this section by recalling that in [10] it is proved that Du belongs to $L^{\frac{n\gamma(p-1)}{n-\gamma}}(\Omega)$. Hence, since

$$\frac{mp\gamma}{m+1} > \frac{n\gamma(p-1)}{n-\gamma} \;\Leftrightarrow\; m > \frac{n(p-1)}{n-p\gamma}, \tag{23}$$

the results obtained in [12,16] provide the so-called "improved regularity" with respect to [10]. In analogy, by choosing again $\theta = n$, $q \in (1,\infty]$, if we take m in Theorem 1 as in (23), we will obtain the improved regularity with respect to the results in [38, Theorem 11] also in the scale of Lorentz spaces $\big($see (17)$\big)$. Note that the interval $\left(\frac{n(p-1)}{(n-p\gamma)}, \frac{1}{(\gamma-1)}\right)$ is not empty, since we are assuming that $\gamma \leq np/(np-n+p)$ because of (11).

2.3 Lorentz-Morrey spaces

In this section we recall the definitions and few basic proprieties of some relevant spaces we deal with.

Fix $q \in (0,\infty)$. A measurable function $f : \Omega \to \mathbb{R}$ belongs to the *Lorentz space* $L(\gamma,q)(\Omega)$, with $\gamma \in [1,\infty)$, if and only if

$$\|f\|^q_{L(\gamma,q)} := q \int_0^\infty \Big(\lambda^\gamma \big|\{x\in\Omega : |f(x)|>\lambda\}\big|\Big)^{\frac{q}{\gamma}} \frac{\mathrm{d}\lambda}{\lambda} < \infty. \tag{24}$$

In the case $q=\infty$, the Lorentz space $L(\gamma,\infty)(\Omega)$, with $\gamma\in[1,\infty)$, is the so-called *Marcinkiewicz space* and it is usually denoted by $\mathcal{M}^\gamma(\Omega)$. A measurable function $f:\Omega\to\mathbb{R}$ belongs to $\mathcal{M}^\gamma(\Omega)$ if and only if

$$\|f\|^\gamma_{\mathcal{M}^\gamma(\Omega)} \equiv \|f\|^\gamma_{L(\gamma,\infty)(\Omega)} := \sup_{\lambda>0} \lambda^\gamma \big|\{x\in\Omega : |f(x)|>\lambda\}\big| < \infty. \tag{25}$$

These spaces were introduced by Lorentz in [36] as a generalization of the classic Lebesgue spaces $L^\gamma(\Omega)$. Indeed, for any $\gamma\in[1,\infty)$, by Fubini's Theorem, one can see that $L(\gamma,\gamma)(\Omega) = L^\gamma(\Omega)$. Moreover, for any $0<r<\gamma<q\leq\infty$, the following continuous embeddings hold: $L^q \equiv L(q,q) \to L(\gamma,r) \to L^\gamma \equiv L(\gamma,\gamma) \to L(\gamma,q) \to L(r,r) \equiv L^r$.

It is worth pointing out that, despite the notation, the functionals $\|\cdot\|_{L(\gamma,q)}$ defined in (24) and (25), are not norms, because they do not satisfy the triangle inequality for the whole range of γ and q (see the forthcoming formula (26)). However, one can introduce equivalent functionals which do have this property (see, for instance, [25, Theorem 4.18-4.19]).[3] Although not being a norm, the

[3] We prefer to keep the definitions in (24) and (25), since it will be more convenient in order to obtain Lorentz estimates by directly manipulating level sets of functions, as we will do in the following (see Sections 3 and 4).

functionals $\|\cdot\|_{L(\gamma,q)}$ are still additive, in the sense stated in the following formula. Fix $k \in \mathbb{N}$ and assume that $\Omega \subseteq \bigcup_{j=1}^{k} \Omega_j$, then

$$\|f\|_{L(\gamma,q)(\Omega)} \leq \begin{cases} k^{\frac{1}{q}-1} \displaystyle\sum_{j=1}^{k} \|f\|_{L(\gamma,q)(\Omega_j)} & \text{if } 0 < q < 1, \\ \displaystyle\sum_{j=1}^{k} \|f\|_{L(\gamma,q)(\Omega_j)} & \text{if } 1 \leq q \leq \gamma \text{ or } q = \infty, \\ k^{\frac{q-\gamma}{\gamma q}} \displaystyle\sum_{j=1}^{k} \|f\|_{L(\gamma,q)(\Omega_j)} & \text{if } q > \gamma. \end{cases} \tag{26}$$

As well known, Lorentz spaces enjoy Hölder type inequalities. We just state a standard inequality for the Marcinkiewicz spaces $\mathcal{M}^\gamma(\Omega)$ in the form we will need it in the following of the paper.

Lemma 1 *Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set and let $f \in \mathcal{M}^\gamma(\Omega)$ with $\gamma > 1$. Then, for any $q \in [1, \gamma)$, $f \in L^q(\Omega)$ and*

$$\|f\|_{L^q(\Omega)} \leq \left(\frac{\gamma}{\gamma - q}\right)^{\frac{1}{q}} |\Omega|^{\frac{1}{q}-\frac{1}{\gamma}} \|f\|_{\mathcal{M}^\gamma(\Omega)}.$$

Now, we are ready to introduce the so-called *Lorentz-Morrey spaces* $L^\theta(\gamma, q)$, by coupling the definitions in (24) and (25) with a density condition. Precisely, a measurable function f belongs to $L^\theta(\gamma, q)(\Omega)$, for $\gamma \in [1, \infty)$, $q \in (0, \infty)$ and $\theta \in [0, n]$, if and only if

$$\|f\|_{L^\theta(\gamma,q)(\Omega)} := \sup_{B_R \subseteq \Omega} R^{\frac{\theta-n}{\gamma}} \|f\|_{L(\gamma,q)(B_R)} < \infty.$$

Accordingly, in the case $q = \infty$, a measurable function f belongs to $L^\theta(\gamma, \infty)(\Omega) \equiv \mathcal{M}^{\gamma,\theta}(\Omega)$ if and only if

$$\|f\|_{\mathcal{M}^{\gamma,\theta}(\Omega)} \equiv \|f\|_{L^\theta(\gamma,\infty)(\Omega)} := \sup_{B_R \subseteq \Omega} R^{\frac{\theta-n}{\gamma}} \|f\|_{\mathcal{M}^\gamma(B_R)} < \infty. \tag{27}$$

Clearly, when $\theta = n$, the space $L^n(\gamma, q)(\Omega)$ coincides with the space $L(\gamma, q)(\Omega)$. Moreover, it is worth noticing that, by means of Fatou's Lemma, one can prove that the functionals $\|\cdot\|_{L(\gamma,q)(\Omega)}$ as well as $\|\cdot\|_{L^\theta(\gamma,q)(\Omega)}$ are lower semicontinuous with respect to the a. e. convergence. For details and results about the theory of Lorentz and Lorentz-Morrey spaces, we refer the interested reader to [44,3,4].

We conclude this section by recalling the definition of *Morrey spaces* $L^{\gamma,\theta}$, introduced by Morrey in [41]; we refer also to book [24]. A measurable function $f : \Omega \to \mathbb{R}$ belongs to the Morrey space $L^{\gamma,\theta}(\Omega)$, with $\gamma \in [1, \infty)$ and $\theta \in [0, n]$, if and only if

$$\|f\|^\gamma_{L^{\gamma,\theta}(\Omega)} := \sup_{B_R \subseteq \Omega} R^\theta \fint_{B_R} |f|^\gamma \, \mathrm{d}x \; < \; \infty.$$

Clearly, $L^{\gamma,n}(\Omega) \equiv L^\gamma(\Omega)$, $L^{\gamma,0}(\Omega) \equiv L^\infty(\Omega)$ and, also, $L^\theta(\gamma, \gamma)(\Omega) \equiv L^{\gamma,\theta}(\Omega)$.

2.4 Maximal operators

Let Q_0 be a cube in $\mathbb{R}^n$ and denote by Q any cube contained in Q_0 with its sides parallel to those of Q_0. For any measurable function f, the (*restricted*) *fractional maximal operator* M^*_{β,Q_0}, with $\beta \in [0,n]$, is defined by

$$M^*_{\beta,Q_0}(f)(x) := \sup_{Q\subseteq Q_0,\ x\in Q} |Q|^{\frac{\beta}{n}} \fint_Q |f(y)|\,\mathrm{d}y.$$

An equivalent definition can be provided by using balls $B \subseteq \mathbb{R}^n$ instead of cubes.

The boundedness of maximal operators in Marcinkiewicz spaces is classical and

$$|\{x\in B : M^*_{0,B}(f)(x) \geq \lambda\}| \leq \frac{\bar{C}}{\lambda^t}\int_B |f|^t\,\mathrm{d}x \tag{28}$$

holds for every $\lambda > 0$ and $t \geq 1$, and it is valid for any $f \in L^t(B)$; the constant $\bar{C}$ depends only on n and t. More in general it holds the boundedness of maximal operators in Lorentz spaces. It is given by the following theorem, whose proof is an application of Marcinkiewicz Theorem (see [7, IV.4.13, IV.4.18]) together with standard sublinear interpolation.

Theorem 2 *Let $B \subseteq \mathbb{R}^n$ be a ball, $\beta\in[0,n]$ and $\gamma > 1$, such that $\beta\gamma < n$. Then for every function $f \in L(\gamma,q)(B)$, with $q\in(0,\infty]$, it holds*

$$\|M^*_{\beta,B}(f)\|_{L(\frac{n\gamma}{n-\beta\gamma},q)(B)} \leq C\|f\|_{L(\gamma,q)(B)},$$

with C depending only on β,γ,n and q.

Combining the previous theorem with Lemma 1 and an Hedberg type inequality ([3, Page 768]), we can control the maximal operators in Lorentz spaces when working on magnified domain.

Theorem 3 ([38, Theorem 9]). *Let $B \subset \mathbb{R}^n$ be a ball and denote by δB its scaling with $\delta > 1$. Let $\beta,\theta\in[0,n]$ and $\gamma > 1$ be such that $\beta\gamma<\theta$. Then, for every measurable function f in δB, for any $q\in(0,\infty]$, it holds*

$$\|M^*_{\beta,B}(f)\|_{L(\frac{\theta\gamma}{\theta-\beta\gamma},\frac{\theta q}{\theta-\beta\gamma})(B)} \leq C\|f\|^{\frac{\beta\gamma}{\theta}}_{L^\theta(\gamma,q)(\delta B)}\|f\|^{\frac{\theta-\beta\gamma}{\theta}}_{L(\gamma,q)(B)},$$

where C is a constant depending only on $\beta,\gamma,\delta,\theta,n$ and q.
Moreover, if $|\delta B|\leq 100^n$, it holds

$$\|M^*_{\beta,B}(f)\|_{L(\frac{\theta\gamma}{\theta-\beta\gamma},\frac{\theta q}{\theta-\beta\gamma})(B)} \leq C_\gamma\|f\|_{L^\theta(\gamma,q)(\delta B)},$$

where $C_\gamma\to\infty$ as $\gamma\to 1$.

For a comprehensive treatment of maximal operators and related results, we refer to [3,7,5].

2.5 Scaling

Let $u(\equiv u_k) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the weak solution to the regularized problems (19). Taking $B_R \equiv B(x_0, R) \subseteq \Omega$, we consider the function $v \in W^{1,p}(B_R)$, defined as the weak solution to the related homogeneous problem

$$\begin{cases} -\operatorname{div} a(x, Dv) + |v|^{m-1}v = 0 & \text{in } B_R \\ v = u & \text{on } \partial B_R. \end{cases} \tag{29}$$

The aim of this section is to establish a useful scaling procedure that will allow to pass from problems (19) and (29) to the analogous ones in the unit ball $B_1 \equiv B(0, 1)$ with a normalized datum.

Fix α and η as follows

$$\alpha := \frac{p}{m - p + 1} \quad \text{and} \quad \eta := \alpha m = \frac{mp}{m - p + 1}. \tag{30}$$

For any $R > 0$ and any $y \in B_1$, consider the following rescaled functions

$$\begin{cases} \bar{u}(y) := \dfrac{R^\alpha u(x_0 + Ry)}{\mathcal{A}}, & \bar{v}(y) := \dfrac{R^\alpha v(x_0 + Ry)}{\mathcal{A}}, \\ \bar{a}(y, z) := \dfrac{R^{\alpha m - 1} a(x_0 + Ry, R^{-(\alpha+1)}\mathcal{A} z)}{\mathcal{A}^m}, & \bar{f}(y) := \dfrac{R^\eta f(x_0 + Ry)}{\mathcal{A}^m}, \end{cases} \tag{31}$$

where the constant $\mathcal{A}$ is given by

$$\mathcal{A} := \left(R^{\eta\gamma} \fint_{B_R} |f|^\gamma \, dx \right)^{\frac{1}{m\gamma}}. \tag{32}$$

In view of the previous choices, it is easy to see that we have $\bar{u} = \bar{v}$ on ∂B_1 and the following equations weakly hold in B_1

$$-\operatorname{div} \bar{a}(y, D\bar{u}) + |\bar{u}|^{m-1}\bar{u} = \bar{f} \quad \text{and} \quad -\operatorname{div} \bar{a}(y, D\bar{v}) + |\bar{v}|^{m-1}\bar{v} = 0,$$

where $\bar{f}$ is such that $\|\bar{f}\|_{L^\gamma(B_1)} = 1$ and the vector field $\bar{a}(y, z)$ satisfies (2) with s, ν and L replaced by $s/\mathcal{A}$, $\nu\mathcal{A}^{-m-1+p}$ and $L\mathcal{A}^{-m-1+p}$, respectively.

Finally, as an immediate consequence of the definitions given in (24) and (25), we have the following lemma.

Lemma 2 *Let $u \in L^\theta(\gamma, q)(B_R)$, with $1 \le \gamma < \infty$ and $0 < q \le \infty$, and let $\bar{u}$ be the rescaled function defined by (31). Then $\bar{u} \in L^\theta(\gamma, q)(B_1)$ and*

$$\|\bar{u}\|_{L^\theta(\gamma,q)(B_1)} = \frac{R^{\alpha - \frac{\theta}{\gamma}}}{\mathcal{A}} \|u\|_{L^\theta(\gamma,q)(B_R)}.$$

2.6 A few preliminary results

In order to obtain the fundamental level set estimate (9) in the precise form given by the forthcoming (84), we will work locally on basic estimates of the solution u to (1) in comparison to the solution v to the corresponding homogeneous problem (29) (see Section 3 below). To do this, we also make use of some classical results from the theory of De Giorgi-Nash-Moser of Hölder continuity, as well as from the theory of Gehring's higher integrability. These are collected in the following two theorems; see, for instance, [26, Chapter 7] and [37, Lemma 3.3] for the proofs, by observing that the presence of the lower order terms does not affect the proof, because of the sign hypothesis (3).

Theorem 4 *Let $v \in W^{1,p}(\Omega)$, with $p \in (1,n]$, be the weak solution of the equation*

$$-\mathrm{div}\, a(x,Dv) + |v|^{m-1}v = 0 \quad \textit{in the open subset } \Omega \subset \mathbb{R}^n,$$

under the assumptions

$$|a(x,z)| \le c(s^2+|z|^2)^{\frac{p-1}{2}}, \quad c^{-1}|z|^p - cs^p \le \langle a(x,z), z\rangle \tag{33}$$

for every $x \in \Omega$ and $z \in \mathbb{R}^n$, where $c = c(L/\nu) > 0$, ν, L are the numbers given in (2) and $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field.

Then there exists $\varpi = \varpi(n,p,L/\nu) \in (0,1/2]$, such that for every $q \in (0,p]$ there exists a constant $C = C(n,p,q,L/\nu)$ such that

$$\int_{B_\rho} (|Dv|^q + s^q)\,\mathrm{d}x \le C\left(\frac{\rho}{R}\right)^{n-(1-\varpi)q} \int_{B_R} (|Dv|^q+s^q)\,\mathrm{d}x, \tag{34}$$

whenever $B_R \subseteq \Omega$ and $0 < \rho \le R$.

Theorem 5 *Let the hypotheses of Theorem 4 hold. Then there exists $\chi = \chi(n,p,L/\nu) > 1$ such that $Dv \in L^{p\chi}_{\mathrm{loc}}(\Omega)$ and for any $q \in (0,p]$ there exists a constant $C = C(n,p,q,L/\nu)$ such that*

$$\left(\fint_{B_{R/2}} (|Dv|+s)^{p\chi}\,\mathrm{d}x\right)^{\frac{1}{p\chi}} \le C\left(\fint_{B_R} (|Dv|^q+s^q)\,\mathrm{d}x\right)^{\frac{1}{q}},$$

whenever $B_R \subseteq \Omega$.

Furthermore, we will need a Calderón-Zygmund-Krylov-Safonov covering result, as stated in Proposition 1 below. For the proof, see, e. g., [14, Lemma 1.2].

Fix a cube Q_0 in $\mathbb{R}^n$ and denote with $\mathcal{D}(Q_0)$ the class of all dyadic cubes obtained from Q_0, that is the class of those cubes, with sides parallel to those of Q_0, having been obtained by a positive, finite number of dyadic subdivisions of the cube Q_0. We denote by $\tilde{Q} \in \mathcal{D}(Q_0)$ the predecessor of Q if Q has been obtained by exactly one dyadic subdivision from the original cube $\tilde{Q}$. We have the following

Proposition 1 *Let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $X \subset Y \subset Q_0$ are measurable sets satisfying the following properties*

(i) there exists $\delta > 0$ such that $|X| < \delta|Q_0|$;
(ii) if $Q \in \mathcal{D}(Q_0)$ then $|X \cap Q| > \delta|Q|$ implies that $\tilde{Q} \subset Y$ where $\tilde{Q}$ denotes the predecessor of Q.

Then it follows that $|X| < \delta|Y|$.

3 Some regularity estimates and other preliminary results

In this section we show some comparison estimates between u_k, the solution to problem (19) with f_k as in (22), and v, the solution to the homogeneous problem (29). Note that in the remaining of the paper we will write u instead of u_k. We will show how to recover regularity and estimates for the original solutions only in the conclusion of the proof of Theorem 1 (see Step 4 at page 31).

First, for any nonnegative $s \in \mathbb{R}$, we introduce the auxiliary function V defined by

$$V(z) = V_s(z) := (s^2 + |z|^2)^{\frac{p-2}{4}} z, \quad z \in \mathbb{R}^n. \tag{35}$$

The function V is a locally bi-Lipschitz bijection of $\mathbb{R}^n$ which verifies the following properties for any $s \geq 0$. For any $z_1,\, z_2 \in \mathbb{R}^n$,

$$C^{-1}(s^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}} \leq \frac{|V(z_2) - V(z_1)|^2}{|z_2 - z_1|^2} \leq C(s^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}, \tag{36}$$

where $C \equiv C(n,p) > 0$ is independent of s.
Also, for any $z \in \mathbb{R}^n$,

$$V_{s/\mathcal{A}}(z/\mathcal{A}) = \mathcal{A}^{-p/2} V_s(z) \quad \forall\, \mathcal{A} > 0, \tag{37}$$

(see [26, Chapter 9] for more details). The main feature of the function in (35) relies in the fact that it can be used to reformulate the monotonicity properties of $z \mapsto a(\cdot, z)$. Namely, as a consequence of (2) and (36), we get

$$C^{-1}\nu|V(z_2) - V(z_1)|^2 \leq \langle a(x,z_2) - a(x,z_1), z_2 - z_1\rangle, \quad \forall z_1, z_2 \in \mathbb{R}^n. \tag{38}$$

Moreover, in the case $p \geq 2$, the first assumption in (2) also implies

$$C^{-1}\nu|z_2 - z_1|^p \leq \langle a(x,z_2) - a(x,z_1), z_2 - z_1\rangle, \quad \forall z_1, z_2 \in \mathbb{R}^n. \tag{39}$$

Now, we state an algebraic lemma, that we will need in the following of this section. It is a classical result going back to Campanato, whose proof can be obtained by simply modifying [26, Lemma 7.3]; see also [38, Lemma 1] and [40, Lemma 9.3].

Lemma 3 *Let $\Psi : [0, \bar{R}] \to [0, \infty)$ be a non-decreasing function such that*

$$\Psi(\rho) \le c_0 \left(\left(\frac{\rho}{R}\right)^{\delta_0} + \varepsilon \right) \Psi(R) + \mathcal{B} R^{\delta_1}, \quad \textit{for every } \rho < R \le \bar{R}, \textit{ with } \mathcal{B} \ge 0,$$

where $0 < \delta_1 < \delta_0$, and c_0 is a given constant. Then there exists $\varepsilon_0 \equiv \varepsilon_0(c_0, \delta_0, \delta_1)$ such that if $\varepsilon \le \varepsilon_0$ it holds

$$\Psi(\rho) \le c_1 \left(\frac{\rho}{R}\right)^{\delta_1} \Psi(R) + c_1 \mathcal{B} \rho^{\delta_1}, \quad \textit{for every } \rho \le R \le \bar{R},$$

where c_1 is a constant depending only on c_0, δ_0 and δ_1.

In the following lemma, by means of suitable test functions, we will show that the L^γ norm of $|u - v|^m$ can be controlled by that of f.

Lemma 4 *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution to problem (19) and $v \in W^{1,p}(B_R)$ that to problem (29), with $B_R \subseteq \Omega$, then*

$$\int_{B_R} |u - v|^{m\gamma} \,\mathrm{d}x \le C \int_{B_R} |f|^\gamma \,\mathrm{d}x, \tag{40}$$

for any $m > 0$ and any $\gamma \ge 1$.

Proof First, suppose that $\gamma = 1$. Consider a sequence of smooth increasing functions $\{\Phi_h(t)\}$ that converges to the function $\Phi(t) \equiv \mathrm{sgn}(t)$ and choose $\phi = \Phi_h(u - v) \in W_0^{1,p}(B_R)$ as test function in

$$\int_{B_R} \langle a(x, Du) - a(x, Dv), D\phi \rangle \,\mathrm{d}x + \int_{B_R} \big(|u|^{m-1}u - |v|^{m-1}v \big) \phi \,\mathrm{d}x \;=\; \int_{B_R} f\,\phi \,\mathrm{d}x. \tag{41}$$

Using the first assumption in (2) and dropping the nonnegative term, we obtain

$$\int_{B_R} \big(|u|^{m-1}u - |v|^{m-1}v \big) \Phi_h(u - v) \,\mathrm{d}x \;\le\; \int_{B_R} |f| |\Phi_h(u - v)| \,\mathrm{d}x.$$

Letting h go to infinity in the inequality above, Fatou's Lemma yields

$$\int_{B_R} \big(|u|^{m-1}u - |v|^{m-1}v \big) \mathrm{sgn}(u - v) \,\mathrm{d}x \;\le\; \int_{B_R} |f| \,\mathrm{d}x. \tag{42}$$

Now, we state the following algebraic inequality

$$\big(|u|^{m-1}u - |v|^{m-1}v \big) \mathrm{sgn}(u - v) \;\ge\; C \begin{cases} |u - v|^m & \text{if } m \ge 1, \\ \dfrac{|u - v|}{(|u| + |v|)^{1-m}} & \text{if } 0 < m < 1. \end{cases} \tag{43}$$

Clearly, if $m \ge 1$, it suffices to combine the inequality above with (42) to obtain

$$\int_{B_R} |u - v|^m \,\mathrm{d}x \le C \int_{B_R} |f| \,\mathrm{d}x,$$

that is (40) in the case $\gamma = 1$.

When $0 < m < 1$, we use the Hölder inequality (with exponents $1/m$ and $1/(1-m)$) as follows

$$\int_{B_R} |u-v|^m \,\mathrm{d}x = \int_{B_R} \frac{|u-v|^m}{(|u|+|v|)^{m(1-m)}} \big(|u|+|v|\big)^{m(1-m)} \,\mathrm{d}x \\ \leq \left(\int_{B_R} \frac{|u-v|}{\big(|u|+|v|\big)^{1-m}} \,\mathrm{d}x\right)^m \left(\int_{B_R} \big(|u|+|v|\big)^m \,\mathrm{d}x\right)^{1-m} \tag{44}$$

Thus, putting together (42), (43) and (44), we obtain

$$\int_{B_R} |u-v|^m \,\mathrm{d}x \leq C\left(\int_{B_R} |f| \,\mathrm{d}x\right)^m \left(\int_{B_R} \big(|u|+|v|\big)^m \,\mathrm{d}x\right)^{1-m} \\ \leq C\left(\int_{B_R} |f| \,\mathrm{d}x\right)^m \left(\int_{B_R} \big(|u|^m + |u-v|^m\big) \,\mathrm{d}x\right)^{1-m}. \tag{45}$$

Now, we can use the Young inequality (with exponents $1/m$ and $1/(1-m)$ and $\varepsilon > 0$) in the right-hand side of (45) to get

$$\int_{B_R} |u-v|^m \,\mathrm{d}x \;\leq\; C\bigg(C(\varepsilon)\int_{B_R} |f| \,\mathrm{d}x + \varepsilon \int_{B_R} |u|^m \,\mathrm{d}x + \varepsilon \int_{B_R} |u-v|^m \,\mathrm{d}x\bigg). \tag{46}$$

At this point, it suffices to choose a suitably small ε in order to absorb the last term in the right-hand side of the inequality above; it follows

$$\int_{B_R} |u-v|^m \,\mathrm{d}x \;\leq\; C\bigg(\int_{B_R} |f| \,\mathrm{d}x + \int_{B_R} |u|^m \,\mathrm{d}x\bigg), \tag{47}$$

up to relabeling the constant C.

Since u is the solution to (19), by standard computation (see, for instance, [9] and also [16, Lemma 2]), we have

$$\int_{B_R} |u|^{m\gamma} \,\mathrm{d}x \;\leq\; C\int_{B_R} |f|^\gamma \,\mathrm{d}x, \quad \forall\, \gamma \geq 1, \tag{48}$$

and thus estimate (40) with $\gamma = 1$ plainly follows by (47).

For the general case $\gamma > 1$, first we need to choose $\phi = |u-v|^{m(\gamma-1)-1}(u-v)$ as test function in (41). Note that this is admissible since $m(\gamma-1) > 0$, by the assumptions on m and γ. Again, in the case $m > 1$, we can drop the nonnegative term and we can use the first inequality in (43). We have

$$\int_{B_R} |u-v|^{m\gamma} \,\mathrm{d}x \;\leq\; \int_{B_R} |f|\,|u-v|^{m(\gamma-1)} \,\mathrm{d}x. \tag{49}$$

Hence, estimate (40) plainly follows from (49) using the Hölder inequality on the right-hand side with exponents $\gamma > 1$ and $\gamma/(\gamma-1)$ and canceling the common terms.

When $0 < m < 1$, we use the second inequality in (43) to obtain

$$\int_{B_R} |u-v|^{m\gamma}\,\mathrm{d}x \leq \left(\int_{B_R} |f|\,|u-v|^{m(\gamma-1)}\,\mathrm{d}x\right)^{\frac{m\gamma}{m(\gamma-1)+1}} \left(\int_{B_R} \big(|u|+|v|\big)^{m\gamma}\,\mathrm{d}x\right)^{\frac{1-m}{m(\gamma-1)+1}}.$$

Then, by using the Hölder inequality on the right-hand side with exponents $\gamma > 1$ and $\gamma/(\gamma-1)$, we get

$$\int_{B_R} |u-v|^{m\gamma}\,\mathrm{d}x \leq C\left(\int_{B_R} |f|^{\gamma}\,\mathrm{d}x\right)^{\frac{1}{\gamma}\frac{m\gamma}{m(\gamma-1)+1}} \left(\int_{B_R} |u-v|^{m\gamma}\,\mathrm{d}x\right)^{\frac{\gamma-1}{\gamma}\frac{m\gamma}{m(\gamma-1)+1}} \times\left(\int_{B_R} \big(|u|^{m\gamma}+|u-v|^{m\gamma}\big)\,\mathrm{d}x\right)^{\frac{1-m}{m(\gamma-1)+1}},$$

which gives

$$\int_{B_R} |u-v|^{m\gamma}\,\mathrm{d}x \leq C\left(\int_{B_R} |f|^{\gamma}\,\mathrm{d}x\right)^{m} \left(\int_{B_R} \big(|u|^{m\gamma}+|u-v|^{m\gamma}\big)\,\mathrm{d}x\right)^{1-m}.$$

Finally, by arguing as in (46)-(48), we will obtain estimate (40). □

Now, we define

$$\sigma(t) := \frac{m\,p\,t}{(m+1)(p-1)}, \quad \forall\, t \geq 0. \tag{50}$$

Lemma 5 *Let $1 < p < n$, $1 < \gamma \leq np/(np-n+p)$, $u \in W^{1,p}_0(\Omega)$ be the weak solution to (19) and $v \in W^{1,p}(B_R)$ be the weak solution to (29), with*

$$0 < m < \frac{1}{\gamma-1}.$$

Then there exists a constant $C \equiv C(m,n,p,L/\nu,\gamma)$ for which, for any $0 < q \leq \sigma(\gamma)(p-1)$,

$$\begin{aligned}\fint_{B_R} \Big[|V(Du)-V(Dv)|^{\frac{2q}{p}} + |Du-Dv|^{q}\Big]\mathrm{d}x &\leq C\left(\fint_{B_R} |f|^{\gamma}\mathrm{d}x\right)^{\frac{q}{\sigma(\gamma)(p-1)}} \\ &\quad + C\,\chi_{\{p<2\}}\left(\fint_{B_R} |f|^{\gamma}\mathrm{d}x\right)^{\frac{q}{\sigma(\gamma)(p-1)}\frac{p}{2}} \left(\fint_{B_R} (|Du|+s)^{q}\right)^{1-\frac{p}{2}},\end{aligned} \tag{51}$$

where V and σ are defined by (35) and (50), respectively, and we denote by $\chi_{\{p<2\}}$ the usual characteristic function of the set $\{p<2\}$, that is $\chi_{\{p<2\}} = 1$ if $p < 2$ and $\chi_{\{p<2\}} = 0$ if $p \geq 2$.

Proof First, we assume that $R = 1$ and take

$$q = \sigma(\gamma)(p-1) \equiv \frac{m\gamma p}{m+1}. \tag{52}$$

The key of the proof relies in the following estimate.

$$\|f\|_{L^\gamma(B_1)} \leq 1$$
$$\Downarrow \tag{53}$$
$$\int_{B_1} \Big[|V(Du)-V(Dv)|^{\frac{2q}{p}} + |Du-Dv|^q\Big] \mathrm{d}x$$
$$\leq \frac{C}{\nu^{\frac{q}{p}}} + \chi_{\{p<2\}} \frac{C}{\nu^{\frac{q}{2}}} \left(\int_{B_1} (|Du|+s)^q \,\mathrm{d}x\right)^{1-\frac{p}{2}},$$

where $C \equiv C(n,p,\gamma)$ and ν is the ellipticity constant given by (2).

In order to show the result above, we will need a basic estimate on the gradient of the functions u and v (see the forthcoming formula (55)), that we will obtain by modifying the classical arguments introduced by Boccardo and Gallouët in [10,11] (see also [16]).[4]

To do this, for any $t \in \mathbb{R}$ and any $k \in \mathbb{N}$, consider the standard truncation operators

$$T_k(t) := \min\big\{k, \max\{-k,t\}\big\} \quad \text{and} \quad \Phi_k(t) := T_1(t - T_k(t)). \tag{54}$$

and the level sets

$$C_k := \big\{x \in B_1 : k \leq |u(x)-v(x)|\big\},$$
$$D_k := \big\{x \in B_1 : k < |u(x)-v(x)| \leq k+1\big\}.$$

Since u and v are the solutions to (19) and (29), respectively, by subtracting the corresponding weak formulations, we get

$$\int_{B_1} \langle a(x,Du) - a(x,Dv), D\phi\rangle \,\mathrm{d}x + \int_{B_1} \big[|u|^{m-1}u - |v|^{m-1}v\big]\phi \,\mathrm{d}x = \int_{B_1} f\phi \,\mathrm{d}x.$$

By testing with $\phi = \Phi_k(u-v)$, dropping the nonnegative term and taking (38) into account, we obtain

$$\int_{D_k} |V(Du)-V(Dv)|^2 \,\mathrm{d}x \leq \frac{C}{\nu}\int_{C_k} |f| \,\mathrm{d}x, \tag{55}$$

where $C > 0$ depends only on n and p.

[4] We note that in [10,11,16], as well in [38], one can get rid of the ellipticity constant ν in the estimates. In contrast, since the problem we are dealing with does not satisfy a homogeneous scaling property, due to the presence of the lower order term g, we have to take care of such a constant in (53) throughout the proof of Lemma 5 and 6.

Now, we set

$$G = G(u,v) := |V(Du) - V(Dv)|^{2/p} + |Du - Dv|,$$

Let us focus firstly on the case $p \geq 2$. By arguing as in (55) and using (39), we get

$$\int_{D_k} G^p \, \mathrm{d}x \;\leq\; \frac{C}{\nu} \int_{C_k} |f| \mathrm{d}x.$$

Furthermore, for any $q < p$ $\big($since $m < 1/(\gamma-1)\big)$, we can use the Hölder inequality with exponents $p/q > 1$ and $p/(p-q)$ in the inequality above, together with the definitions of sets D_k and C_k and with the properties of the series with positive terms; we get

$$\begin{aligned}
\int_{B_1} G^q \, \mathrm{d}x &\leq \left(\sum_{j=0}^{\infty} \int_{D_j} \frac{G^p}{(1+|u-v|)^t} \, \mathrm{d}x \right)^{\frac{q}{p}} \left(\int_{B_1} (1+|u-v|)^{\frac{tq}{p-q}} \, \mathrm{d}x \right)^{1-\frac{q}{p}} \\
&\leq \frac{C}{\nu^{\frac{q}{p}}} \left(\sum_{k=0}^{\infty} \int_{D_k} |f| \, \mathrm{d}x \sum_{j=0}^{k} \frac{1}{(1+j)^t} \right)^{\frac{q}{p}} \left(\int_{B_1} (1+|u-v|)^{\frac{tq}{p-q}} \, \mathrm{d}x \right)^{1-\frac{q}{p}} \quad (56)
\end{aligned}$$

where we denoted by

$$t := 1 - m(\gamma - 1). \tag{57}$$

Notice that $\gamma > 1$ yields $t < 1$; thus, using the definition of D_k in (56), we get

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_{D_k} |f| \, \mathrm{d}x \sum_{j=0}^{k} \frac{1}{(1+j)^t} &\leq C \sum_{k=0}^{\infty} \int_{D_k} |f| \big(1 + |u-v|^{1-t}\big) \, \mathrm{d}x \\
&\leq C \int_{B_1} |f| \big(1 + |u-v|^{1-t}\big) \, \mathrm{d}x.
\end{aligned}$$

By using again the Hölder inequality, together with the fact that $\|f\|_{L^\gamma(B_1)} \leq 1$ (recall (53)), we get

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_{D_k} |f| \, \mathrm{d}x \sum_{j=0}^{k} \frac{1}{(1+j)^t} &\leq C\|f\|_{L^\gamma(B_1)} + C\|f\|_{L^\gamma(B_1)} \||u-v|^{1-t}\|_{L^{\frac{\gamma}{\gamma-1}}(B_1)} \\
&\leq C + C \left(\int_{B_1} |u-v|^{\frac{(1-t)\gamma}{\gamma-1}} \, \mathrm{d}x \right)^{1-\frac{1}{\gamma}}.
\end{aligned} \tag{58}$$

Now we notice that the definitions of q in (52) and t in (57) yield

$$\frac{tq}{p-q} = \frac{(1-t)\gamma}{\gamma-1} = m\gamma.$$

Hence, putting together (56) and (58) with estimate (40) in Lemma 4, we arrive at

$$\int_{B_1} G^q \, \mathrm{d}x \;\leq\; \frac{C}{\nu^{\frac{q}{p}}} + \frac{C}{\nu^{\frac{q}{p}}} \left(\int_{B_1} |u-v|^{m\gamma} \, \mathrm{d}x \right)^{1-\frac{q}{p}+\frac{q(\gamma-1)}{p\gamma}} \;\leq\; \frac{C}{\nu^{\frac{q}{p}}},$$

that is the desired inequality in (53).

In the case $1 < p < 2$, we do the same calculations as in (56) to (58); we arrive at

$$\int_{B_1} |V(Du) - V(Dv)|^{\frac{2q}{p}} \leq \frac{C}{\nu^{\frac{q}{p}}}, \tag{59}$$

in which q is defined by (52). When $|Du| \neq |Dv|$, we write

$$|Du - Dv| = \big((|Du|^2 + |Dv|^2 + s^2)^{\frac{p-2}{2}} |Du - Dv|^2\big)^{\frac{1}{2}} \big(|Du|^2 + |Dv|^2 + s^2\big)^{\frac{2-p}{4}}$$

and so by (36) it follows

$$\begin{aligned} |Du - Dv| &\leq C|V(Du) - V(Dv)|\big(|Du|^2 + |Dv|^2 + s^2\big)^{\frac{2-p}{4}} \\ &\leq C|V(Du) - V(Dv)|\big(|Du - Dv|^{\frac{2-p}{2}} + (|Du| + s)^{\frac{2-p}{2}}\big). \end{aligned}$$

Therefore, using the Young inequality we can choose a suitable $\varepsilon > 0$ to obtain

$$\begin{aligned} |Du - Dv|^q \leq C|V(Du) - V(Dv)|^{\frac{2q}{p}} + \frac{1}{2}|Du - Dv|^q \\ + C|V(Du) - V(Dv)|^q \big(|Du| + s\big)^{\frac{(2-p)q}{2}} \end{aligned}$$

and then

$$|Du - Dv|^q \leq C|V(Du) - V(Dv)|^{\frac{2q}{p}} + C|V(Du) - V(Dv)|^q \big(|Du| + s\big)^{\frac{(2-p)q}{2}}.$$

By using the estimate above together with the Hölder inequality (with exponents $2/p > 1$ and $2/(2-p)$), we get

$$\begin{aligned} &\int_{B_1} |Du - Dv|^q \,\mathrm{d}x \\ &\quad \leq C \int_{B_1} |V(Du) - V(Dv)|^{\frac{2q}{p}} \,\mathrm{d}x \\ &\qquad + C \left(\int_{B_1} |V(Du) - V(Dv)|^{\frac{2q}{p}} \,\mathrm{d}x\right)^{\frac{p}{2}} \left(\int_{B_1} (|Du| + s)^q \,\mathrm{d}x\right)^{1-\frac{p}{2}}. \end{aligned} \tag{60}$$

At this time, it suffices to combine (59) with (60) to get

$$\int_{B_1} |Du - Dv|^q \,\mathrm{d}x \leq \frac{C}{\nu^{\frac{q}{p}}} + \frac{C}{\nu^{\frac{q}{2}}} \left(\int_{B_1} (|Du| + s)^q \,\mathrm{d}x\right)^{1-\frac{p}{2}}$$

and we arrive at (53). We note that (53) also follows for any other $q \leq \sigma(\gamma)(p-1)$, as one can see by the Hölder and Young inequalities.

The last step of this proof consists into recovering the general case for arbitrary $R > 0$ and $f \in L^\gamma(\Omega)$, in order to obtain estimate (51). To do this, we use the scaling argument introduced in Section 2.5 and we write

$$\int_{B_1} \left[|V_{s/\mathcal{A}}(D\bar{u}) - V_{s/\mathcal{A}}(D\bar{v})|^{\frac{2q}{p}} + |D\bar{u} - D\bar{v}|^q\right] \mathrm{d}x \leq \frac{C}{\nu^{\frac{q}{p}}},$$

with ν replaced by $\nu\mathcal{A}^{-m-1+p}$, $\mathcal{A}$ as in (32). Re-scaling back the last inequality, that is passing from $\bar{u},\bar{v}$ to u,v $\big($recall (31)$\big)$, we finally obtain

$$\begin{aligned}
&\fint_{B_R}\Big[|V(Du)-V(Dv)|^{\frac{2q}{p}}+|Du-Dv|^q\Big]\,\mathrm{d}x\\
&\qquad\qquad \le C\,R^{\eta\frac{q(m+1)}{mp}-(\alpha+1)q}\left(\fint_{B_R}|f|^\gamma\,\mathrm{d}x\right)^{\frac{(m+1)q}{mp\gamma}}\\
&\qquad\qquad\quad +C\chi_{\{p<2\}}R^{\frac{\eta(m+1)q}{2m}-\frac{(\alpha+1)pq}{2}}\left(\fint_{B_R}|f|^\gamma\,\mathrm{d}x\right)^{\frac{(m+1)q}{2m\gamma}}\\
&\qquad\qquad\quad \times\left(\fint_{B_R}(|Du|+s)^q\,\mathrm{d}x\right)^{1-\frac{p}{2}},
\end{aligned}$$

in which we also used (37). Note that

$$\eta\,\frac{(m+1)q}{mp}-(\alpha+1)q=0\quad\text{and}\quad\frac{\eta(m+1)q}{2m}-\frac{(\alpha+1)pq}{2}=0$$

with α and η as in (30). This completes the proof. □

Lemma 6 *Let $1<p<n$, $u\in W_0^{1,p}(\Omega)$ be the weak solution to (19) and $v\in W^{1,p}(B_R)$ be the weak solution to (29), $p-1<m<\infty$. Then*

$$\begin{aligned}
&\fint_{B_R}\big[R^{1-p}|u-v|^{p-1}+|Du-Dv|^{p-1}\big]\,\mathrm{d}x\\
&\qquad\le C\left(\fint_{B_R}|f|\,\mathrm{d}x\right)^{\frac{1}{\sigma(1)}}\\
&\qquad\quad +C\,\chi_{\{p<2\}}\left(\fint_{B_R}|f|\,\mathrm{d}x\right)^{\frac{p}{2\sigma(1)}}\left(\fint_{B_R}(|Du|+s)^{p-1}\,\mathrm{d}x\right)^{1-\frac{p}{2}}.
\end{aligned}\tag{61}$$

Moreover, for any $\theta\in[0,n]$,

$$\begin{aligned}
&\int_{B_R}|Du-Dv|^{p-1}\,\mathrm{d}x\\
&\qquad\le\ C\,R^{n-\frac{\theta}{\sigma(1)}}\|f\|_{L^{1,\theta}(B_R)}^{\frac{1}{\sigma(1)}}\\
&\qquad\quad +C\chi_{\{p<2\}}R^{\frac{p}{2}\left(n-\frac{\theta}{\sigma(1)}\right)}\|f\|_{L^{1,\theta}(B_R)}^{\frac{p}{2\sigma(1)}}\left(\int_{B_R}(|Du|+s)^{p-1}\,\mathrm{d}x\right)^{1-\frac{p}{2}}.
\end{aligned}\tag{62}$$

The constant $C>0$ depends only on m, n, p, L/ν and γ; σ is given by (50).

Proof In the case $p\ge 2$, we can deduce the estimate in (61) following the strategy used to obtain (51) in the previous lemma, but we need to choose different exponents to be able to consider also the case $\gamma=1$. For this, we take $t=m(p-q)/q$ and thus, for any $\|f\|_{L^1(B_1)}\le 1$, we obtain

$$\int_{B_1}G^q\,\mathrm{d}x\ \le\ \frac{C}{\nu^{\frac{q}{p}}}\left(\int_{B_1}(1+|u-v|)^m\,\mathrm{d}x\right)^{1-\frac{q}{p}},$$

where $C \equiv C(n,p) > 0$ and we also used the fact that $\sum_{j=0}^{\infty} 1/(1+j)^t$ is finite, since, for $q = p-1$ and $m > p-1$, we have $t > 1$. Also, (40) holds for $\gamma = 1$ and then (53) follows.

If $1 < p < 2$, we can argue as in Lemma 5 and we obtain (53). Finally, the scaling argument in (30) and (31) together with Poincaré inequality (that holds since $v = u$ on ∂B_R) yield (61).

Furthermore, since $\|f\|_{L^1(B_R)} \le R^{n-\theta}\|f\|_{L^{1,\theta}(B_R)}$ for any $\theta \in [0,n]$, we can easily deduce (62) from (61). □

By combining (62) with the Hölder inequality together with the fact that the Marcinkiewicz space $\mathcal{M}^\gamma$ is continuously embedded in $L(\gamma,q)$, we obtain the following result.

Lemma 7 *Let the assumptions of* Lemma 6 *hold and suppose that* $f \in L^\theta(\gamma,q)(B_R)$ *for some* $\gamma > 1$, $q \in (0,\infty]$ *and* $\theta \in [0,n]$. *Then,*

$$\begin{aligned}
\int_{B_R} &[R^{1-p}|u-v|^{p-1} + |Du - Dv|^{p-1}]\,\mathrm{d}x \\
&\le C\,R^{n-\frac{\theta}{\sigma(\gamma)}}\|f\|_{L^\theta(\gamma,q)(B_R)}^{\frac{1}{\sigma(1)}} \\
&\quad + C\chi_{\{p<2\}}R^{\frac{p}{2}\left(n-\frac{\theta}{\sigma(\gamma)}\right)}\|f\|_{L^\theta(\gamma,q)(B_R)}^{\frac{p}{2\sigma(1)}}\left(\int_{B_R}(|Du|+s)^{p-1}\,\mathrm{d}x\right)^{1-\frac{p}{2}},
\end{aligned} \tag{63}$$

where the constant C *depends only on* m, n, p, L/ν, γ *and* σ *is given by* (50).

In Lemma 8 below, we prove one of the main tools that we will use in the proof of Theorem 1. Although the proof of this result is a little bit technical and it follows strongly the first step of the proof of [38, Theorem 11], we give full details, since here we also deal with the case $1 < p < 2$.

Let Q_0 be a fixed cube such that $n^2 Q_0 \subset\subset \Omega$ and $|Q_0| \le 1$. According to the definitions given in Section 2.4, we shall consider $M^* \equiv M^*_{0,n^2Q_0}$. Thus, keeping in mind the properties of dyadic cubes given at the end of Section 2.6, we have the following lemma.

Lemma 8 *Let* $u \in W_0^{1,p}(\Omega)$ *be the solution to* (19), *with* $p-1 < m < \infty$. *Then for every* $T > 1$ *there exists a number* $\varepsilon \equiv \varepsilon(m,n,p,L/\nu,T) \in (0,1)$, *such that if* $\lambda > 1$ *and* $Q \subset Q_0$ *is a dyadic sub-cube of* Q_0 *such that*

$$\begin{aligned}
\big|Q \cap \big\{x \in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) > AT\lambda \\
\text{and } [M^*(f)]^{\frac{1}{\sigma(1)}}(x) \le \varepsilon\,\lambda\big\}\big| > T^{-\frac{p\chi}{p-1}}|Q|
\end{aligned} \tag{64}$$

then its predecessor $\tilde{Q}$ *satisfies*

$$\tilde{Q} \subseteq \{x \in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) > \lambda\}. \tag{65}$$

Here $\chi \equiv \chi(n,p,L/\nu) > 1$ *is the higher integrability exponent as in Theorem* 5, *while* $A \equiv A(m,n,p,L/\nu) > 1$ *is an absolute constant.*

Proof We argue by contradiction. Since we assume that (65) fails, there exists $\tilde{x} \in \tilde{Q}$ such that

$$M^*((|Du|^2 + s^2)^{\frac{p-1}{2}})(\tilde{x}) \le \lambda \tag{66}$$

and, thus, by recalling that $\tilde{Q} \subset 3Q$ because $\tilde{Q}$ is the predecessor of Q, we have

$$\fint_{3Q} (|Du|^2 + s^2)^{\frac{p-1}{2}} \,\mathrm{d}x \le \lambda. \tag{67}$$

In view of (64), there exists $\bar{x} \in Q$ such that

$$M^*(f)(\bar{x}) \le (\varepsilon\,\lambda)^{\sigma(1)}. \tag{68}$$

Let $B \subset n^2 Q_0$ be the unique ball having $3Q$ as inner cube and $v \in W^{1,p}(B)$ be the solution to the following problem

$$\begin{cases} -\operatorname{div} a(x, Dv) + |v|^{m-1} v = 0 & \text{in } B \\ v = u & \text{on } \partial B. \end{cases} \tag{69}$$

Considering the outer cube to B which satisfies $Q_{\mathrm{out}}(B) \subset n^2 Q_0$, then (68) yields

$$\fint_B |f| \,\mathrm{d}x \le \frac{|Q_{\mathrm{out}}(B)|}{|B|} \fint_{Q_{\mathrm{out}}(B)} |f| \,\mathrm{d}x \le C(n)(\varepsilon\,\lambda)^{\sigma(1)}, \tag{70}$$

since $\bar{x}$ belongs to $Q \subset Q_{\mathrm{out}}(B)$. Using (61) with $B_R = B$, (67) and (70), noticing that $\tilde{x} \in B \subset n^2 Q_0$, we get

$$\begin{aligned} &\fint_{3Q} |Du - Dv|^{p-1} \,\mathrm{d}x \\ &\le C(n) \fint_B |Du - Dv|^{p-1} \,\mathrm{d}x \\ &\le C \left(\fint_B |f| \,\mathrm{d}x \right)^{\frac{1}{\sigma(1)}} + C\chi_{\{p<2\}} \left(\fint_B |f| \,\mathrm{d}x \right)^{\frac{p}{2\sigma(1)}} \left(\fint_B (|Du| + s)^{p-1} \,\mathrm{d}x \right)^{1-\frac{p}{2}} \\ &\le C\varepsilon\lambda + C\chi_{\{p<2\}} \varepsilon^{\frac{p}{2}} \lambda, \end{aligned} \tag{71}$$

for a positive constant C depending only on m, n, p and ν.

Since v solves the homogeneous Dirichlet problem (69), the Gehring's higher integrability theory holds - as noticed in Section 2.6 - and thus we can use Theorem 5 with $q = p - 1$ to get

$$\left(\fint_{2Q} (|Dv|^2 + s^2)^{\frac{p\chi}{2}} \,\mathrm{d}x \right)^{\frac{1}{p\chi}} \le C \left(\fint_{3Q} (|Dv|^2 + s^2)^{\frac{p-1}{2}} \right)^{\frac{1}{p-1}},$$

with $\chi \equiv \chi(n, p, L/\nu) > 1$ as in Theorem 5. Therefore, combining (67) with (71) together with the fact that $\varepsilon \le 1$, we get

$$\fint_{3Q} (|Dv|^2 + s^2)^{\frac{p-1}{2}} \,\mathrm{d}x \le C \fint_{3Q} (|Du|^2 + s^2)^{\frac{p\chi}{2}} \,\mathrm{d}x + C \fint_{3Q} |Du - Dv|^{p-1} \,\mathrm{d}x \le C\,\lambda$$

that implies

$$\fint_{2Q} (|Dv|^2 + s^2)^{\frac{p\chi}{2}}\, \mathrm{d}x \;\le\; C\, \lambda^{\frac{p\chi}{p-1}}. \tag{72}$$

Let $M^{**} = M^*_{0,2Q}$ be the maximal operator restricted to $2Q$. Using the boundedness of maximal operators in Marcinkiewicz spaces given in (28), both in the case $t = p\chi/(p-1) \ge 1$ and $t = 1$, we get for every A and $T > 1$,

$$\begin{aligned}
&|\{x \in Q : M^{**}((|Du|^2 + s^2)^{\frac{p-1}{2}})(x) > AT\lambda\}| \\
&\qquad \le |\{x \in Q : M^{**}((|Dv|^2 + s^2)^{\frac{p-1}{2}})(x) > AT\lambda/4^p\}| \\
&\qquad\quad + |\{x \in Q : M^{**}(|Du - Dv|^{p-1})(x) > AT\lambda/4^p\}| \\
&\qquad \le \frac{C}{(AT\lambda)^{\frac{p\chi}{p-1}}} \int_{2Q} (|Dv|^2 + s^2)^{\frac{p\chi}{2}}\, \mathrm{d}x \qquad (73)\\
&\qquad\quad + \frac{C}{AT\lambda} \int_{2Q} |Du - Dv|^{p-1}\, \mathrm{d}x.
\end{aligned}$$

Also, by (72) we have

$$\frac{C}{(AT\lambda)^{\frac{p\chi}{p-1}}} \int_{2Q} (|Dv|^2 + s^2)^{\frac{p\chi}{2}}\, \mathrm{d}x \;\le\; \frac{C_2}{(AT)^{\frac{p\chi}{p-1}}} |Q|.$$

Taking

$$A := 10^n \big(C_2(n, p, m, L/\nu) + 1\big), \tag{74}$$

we finally obtain

$$\frac{C}{(AT\lambda)^{\frac{p\chi}{p-1}}} \int_{2Q} (|Dv|^2 + s^2)^{\frac{p\chi}{2}}\, \mathrm{d}x \;\le\; \frac{1}{4T^{\frac{p\chi}{p-1}}} |Q|. \tag{75}$$

Now we use (71) to estimate the second term in the right-hand side of (73). We have

$$\frac{C}{AT\lambda} \int_{2Q} |Du - Dv|^{p-1}\, \mathrm{d}x \;\le\; \frac{C_3}{AT}\, \varepsilon |Q|, \quad \text{for } p \ge 2$$

and

$$\frac{C}{AT\lambda} \int_{2Q} |Du - Dv|^{p-1}\, \mathrm{d}x \;\le\; \frac{C_3}{AT}\, \varepsilon^{\frac{p}{2}} |Q|, \quad \text{for } 1 < p < 2.$$

We fix the value of ε so that

$$\varepsilon := \frac{1}{\big(C_3(m, n, p, L/\nu) + 1\big) T^{\frac{p\chi}{p-1} - 1}} < 1, \quad \text{if } p \ge 2, \tag{76}$$

and

$$\varepsilon := \left(\frac{1}{\big(C_3(m, n, p, L/\nu) + 1\big) T^{\frac{p\chi}{p-1} - 1}} \right)^{\frac{2}{p}} < 1, \quad \text{if } 1 < p < 2. \tag{77}$$

As a consequence, we obtain

$$\frac{C}{AT\lambda} \int_{2Q} |Du - Dv|^{p-1}\, \mathrm{d}x \;\le\; \frac{|Q|}{4T^{\frac{p\chi}{p-1}}}.$$

Thus, by merging the inequality above with (73) and (75), we obtain

$$\left|\left\{x \in Q : M^{**}((|Du|^2+s^2)^{\frac{p-1}{2}})(x) > AT\lambda\right\}\right| < T^{-\frac{p\chi}{p-1}}|Q|. \tag{78}$$

By following the same argument in [38, Theorem 11, Step 1], we can use again (66) passing to arbitrary cubes and we get

$$M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) \le \max\left\{M^{**}(|Du|^2+s^2)^{\frac{p-1}{2}})(x),\, 9^n\lambda\right\}, \quad \forall\, x \in Q. \tag{79}$$

Finally, (78), (79) and the choice of A in (74) yield

$$\left|\left\{x \in Q : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) > AT\lambda\right\}\right| < T^{-\frac{p\chi}{p-1}}|Q|,$$

which is a contradiction to (64). The proof is complete. □

The following lemma provides another important tool for the proof of our main result. We think that this lemma could have its own interest, since it shows an intermediate Morrey space regularity of $|Du|^{p-1}$.

Lemma 9 *Let $u \in W^{1,p}_0(\Omega)$ be the solution to (19), with $p-1 < m < \infty$. Assume that $f \in L^\theta(\gamma,q)(\Omega)$ with $1 < p\gamma \le \theta \le n$ and $q \in (0,\infty]$, then the following inequality holds*

$$\begin{aligned}\|(|Du|+s)^{p-1}\|_{L^{1,\frac{\theta}{\sigma(\gamma)}}(B_t)} &\le C\,(d-t)^{\frac{\theta}{\sigma(\gamma)}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_d)}\\ &\quad + C\,\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_d)},\end{aligned} \tag{80}$$

for every couple of concentric balls $B_t \subset B_d \subseteq \Omega$; where $C \equiv C(m,n,p,q,L/\nu,\gamma)$ is a positive constant.

Proof Let us take $x_c \in B_t$ and a ball B_R centered in x_c, and such that $R \le \operatorname{dist}(x_c,\partial B_d)$, so that $B_R \subseteq B_d$ and let v be the solution to (69). Following [38, Lemma 11], since (2) implies (33), we can use the De Giorgi-Nash-Moser theory, that is estimate (34) with $q = p-1$, and we get

$$\begin{aligned}\int_{B_\rho}(|Dv|+s)^{p-1}\,\mathrm{d}x &\le C\left(\frac{\rho}{R}\right)^{n-(1-\varpi)(p-1)}\int_{B_R}(|Du|+s)^{p-1}\,\mathrm{d}x\\ &\quad + C\int_{B_R}|Du-Dv|^{p-1}\,\mathrm{d}x, \quad \forall\,\rho \in (0,R),\end{aligned}$$

where $C \equiv C(n,p,L/\nu)$ and $\varpi \equiv \varpi(n,p,L/\nu) \in (0,1/2]$. Combining the estimate above with (63) in Lemma 7, we get

$$\begin{aligned}&\int_{B_\rho}(|Du|+s)^{p-1}\,\mathrm{d}x\\ &\qquad \le C\left(\frac{\rho}{R}\right)^{n-(1-\varpi)(p-1)}\int_{B_R}(|Du|+s)^{p-1}\,\mathrm{d}x\\ &\qquad\quad + C\,R^{n-\frac{\theta}{\sigma(\gamma)}}\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}\\ &\qquad\quad + C\chi_{\{p<2\}}R^{\frac{p}{2}\left(n-\frac{\theta}{\sigma(\gamma)}\right)}\|f\|^{\frac{p}{2\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}\left(\int_{B_R}(|Du|+s)^{p-1}\,\mathrm{d}x\right)^{1-\frac{p}{2}}.\end{aligned} \tag{81}$$

In the case $1 < p < 2$, we can use the Young inequality $\big($with exponents $2/p > 1$, $2/(2-p)$ and $\varepsilon > 0\big)$ in the second term in the right-hand side of (81) and we get

$$\begin{aligned}
&\int_{B_\rho} (|Du| + s)^{p-1}\,\mathrm{d}x \\
&\quad \le C\left(\frac{\rho}{R}\right)^{n-(1-\varpi)(p-1)} \int_{B_R} (|Du| + s)^{p-1}\,\mathrm{d}x + C\,R^{n-\frac{\theta}{\sigma(\gamma)}} \|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)} \\
&\qquad + C(\varepsilon)\chi_{\{p<2\}} R^{n-\frac{\theta}{\sigma(\gamma)}} \|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)} + \varepsilon \int_{B_R} (|Du| + s)^{p-1}\,\mathrm{d}x \\
&\quad = C\left(\left(\frac{\rho}{R}\right)^{n-(1-\varpi)(p-1)} + \varepsilon\right) \int_{B_R} (|Du| + s)^{p-1}\,\mathrm{d}x \\
&\qquad + C(\varepsilon) R^{n-\frac{\theta}{\sigma(\gamma)}} \|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}.
\end{aligned}$$

Thus, for any $p > 1$, we can apply the algebraic Lemma 3 by choosing

$$\Psi(\rho) = \int_{B_\rho} (|Du| + s)^{p-1}\,\mathrm{d}x, \quad \mathcal{B} = C\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}, \quad \bar{R} = \mathrm{dist}(x_c, \partial B_d)$$

and

$$\delta_0 := n - (1-\varpi)(p-1) \; > \; n - \frac{\theta}{\sigma(\gamma)} =: \delta_1 \quad (\text{since } p\gamma < \theta).$$

It follows

$$\int_{B_\rho} (|Du| + s)^{p-1}\,\mathrm{d}x \;\le\; C\left(\frac{\rho}{\bar{R}}\right)^{\delta_1} \int_{B_{\bar{R}}} (|Du| + s)^{p-1}\,\mathrm{d}x + C\mathcal{B}\rho^{\delta_1}.$$

Since $\bar{R} > d - t$, for any $B_\rho \subseteq B_d$ centered in B_t, we have that (80) follows by

$$\begin{aligned}
&\int_{B_\rho} (|Du| + s)^{p-1}\,\mathrm{d}x \\
&\qquad \le C\left((d-t)^{\frac{\theta}{\sigma(\gamma)}-n} \int_{B_d} (|Du| + s)^{p-1}\,\mathrm{d}x + \|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_d)}\right) \rho^{n-\frac{\theta}{\sigma(\gamma)}}.
\end{aligned}$$

□

4 Proof of the main result

For the reader's convenience, we restate Theorem 1 from the Introduction. We also recall that, for any $t > 0$, the exponent $\sigma(t)$ is defined by

$$\sigma(t) := \frac{m\,p\,t}{(m+1)(p-1)}, \quad \forall\; 1 < p < n \;\text{ and }\; \forall\; 0 < m < \infty. \tag{82}$$

Theorem 6 *Let $q \in (0,\infty]$. Assume (2) and*

$$f \in L^\theta(\gamma,q)(\Omega)$$

with γ, θ such that

$$1<\gamma\leq \frac{\theta p}{\theta p-\theta+p}, \quad 1<p<\theta\leq n.$$

Then the solution $u\in W^{1,1}_0(\Omega)$ to (18), with

$$p-1<m<\frac{1}{\gamma-1},$$

satisfies

$$|Du|^{p-1}\in L^\theta\big(\sigma(\gamma),\sigma(q)\big) \text{ locally in } \Omega.$$

Moreover, the local estimate

$$\begin{aligned}\||Du|^{p-1}\|_{L^\theta(\sigma(\gamma),\sigma(q))(B_{R/2})}\\ \leq C\,R^{\frac{\theta}{\sigma(\gamma)}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)}+C\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}\end{aligned}$$

holds for every ball $B_R\subseteq\Omega$, where C depends only on m, n, p, q, L/ν and γ.

Before starting, we want to observe that most of the differences between the problem we are dealing with and the analogous without lower order terms analyzed in [38] has arisen in the previous section. At this stage, the general strategy in the proof below will follow that of [38, Theorem 11]. We prefer to give some details for the reader's convenience, emphasizing the different exponents we have to handle.

Proof Keeping in mind the same terminology introduced in Lemma 8 and the definitions of dyadic cubes given in Section 2.6, we divide the proof in few steps.

Step 1 - Application of Calderón-Zygmund-Krylov-Safonov covering theorem. We want to apply Proposition 1 with $\delta = T^{-\frac{p\chi}{p-1}}$,

$$X:=\Big\{x\in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x)>(AT)^{k+1}\lambda_0 \\ \text{and } [M^*(f)]^{\frac{1}{\sigma(1)}}(x)\leq\varepsilon(AT)^k\lambda_0\Big\},$$

$$Y:=\Big\{x\in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x)>(AT)^{k}\lambda_0\Big\}$$

and

$$\lambda_0 := 2\,\bar C\,n^{2n}\,T^{\frac{p\chi}{p-1}}\fint_{n^2Q_0}(|Du|^2+s^2)^{\frac{p-1}{2}}\,\mathrm{d}x, \tag{83}$$

where $k\in\mathbb{N}$, $\bar C$ is as in (28), T and A are as in Lemma 8.

First, Lemma 8 guarantees the validity of hypothesis (ii) in Proposition 1. Moreover, using the boundedness of the maximal operators in Marcinkiewicz spaces and some considerations as in Lemma 8, it easily follows that, for any $k \in \mathbb{N}$,

$$\begin{aligned}&\left|\left\{x \in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) > (AT)^k\lambda_0\right\}\right| \\ &\qquad \le \left|\left\{x \in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) \ge \lambda_0\right\}\right| < T^{-\frac{p\chi}{p-1}}|Q_0|\end{aligned}$$

and so hypothesis (i) in Proposition 1 holds, too. Hence, we arrive at

$$\begin{aligned}&\left|\left\{x \in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) > (AT)^{k+1}\lambda_0\right\}\right| \\ &\quad \le T^{-\frac{p\chi}{p-1}}\left|\left\{x \in Q_0 : M^*((|Du|^2+s^2)^{\frac{p-1}{2}})(x) > (AT)^k\lambda_0\right\}\right| \\ &\quad + \left|\left\{x \in Q_0 : [M^*(f)]^{\frac{1}{\sigma(1)}}(x) > \varepsilon(AT)^k\lambda_0\right\}\right|, \quad \forall\, k \ge 0,\end{aligned} \tag{84}$$

that can be rewritten as follows

$$\begin{aligned}&(AT)^{(k+1)\sigma(\gamma)}\lambda_0^{\sigma(\gamma)}\mu_1((AT)^{k+1}\lambda_0) \\ &\qquad \le (AT)^{k\sigma(\gamma)}A^{\sigma(\gamma)}T^{\sigma(\gamma)-\frac{p\chi}{p-1}}\lambda_0^{\sigma(\gamma)}\mu_1((AT)^k\lambda_0) \\ &\qquad + (AT)^{k\sigma(\gamma)}\left(\frac{AT}{\varepsilon}\right)^{\sigma(\gamma)}(\lambda_0\varepsilon)^{\sigma(\gamma)}\mu_2(\varepsilon(AT)^k\lambda_0).\end{aligned} \tag{85}$$

where, for any $K \ge 0$, we denoted $\mu_1(K) := \left|\left\{x \in Q_0 : M^*((|Du|^2 + s^2)^{\frac{p-1}{2}})(x) > K\right\}\right|$ and $\mu_2(K) := \left|\left\{x \in Q_0 : [M^*(f)]^{\frac{1}{\sigma(1)}}(x) > K\right\}\right|$. Now observe that, since $\chi > 1$ and $m < 1/(\gamma-1)$,

$$d := \frac{p\chi}{(p-1)} - \sigma(\gamma) = \frac{p}{p-1}\left(\chi - \frac{\gamma m}{m+1}\right) > 0.$$

Therefore, with A given by (74), we can choose T as follows

$$T := (4A^{\sigma(\gamma)})^{\frac{1}{d}}$$

and, taking into account (76), (77), the definitions of A and T, inequality (85) gives that there exists a constant $C \equiv C(m,n,p,L/\nu)$ such that, for every $k \ge 0$,

$$\begin{aligned}&(AT)^{(k+1)\sigma(\gamma)}\lambda_0^{\sigma(\gamma)}\mu_1((AT)^{k+1}\lambda_0) \\ &\qquad \le \frac{1}{4}(AT)^{k\sigma(\gamma)}\lambda_0^{\sigma(\gamma)}\mu_1((AT)^k\lambda_0) \\ &\qquad + C(AT)^{k\sigma(\gamma)}(\varepsilon\lambda_0)^{\sigma(\gamma)}\mu_2((AT)^k\varepsilon\lambda_0).\end{aligned} \tag{86}$$

Step 2 - Level sets estimates. In order to establish some Lorentz spaces estimates on level sets, we can proceed as in Step 3 of [38, Theorem 11]. Taking

$0 < t < \infty$ and operating some manipulations, we arrive at

$$\int_0^\infty [\lambda^{\sigma(\gamma)}\mu_1(\lambda)]^{\frac{t}{\sigma(\gamma)}} \frac{\mathrm{d}\lambda}{\lambda} \leq \left(\frac{1}{t} + 2\tilde{C}^t (AT)^t \log(AT)\right) \lambda_0^t |Q_0|^{\frac{t}{\sigma(\gamma)}} + \tilde{C}^t (AT)^{2t} \int_0^\infty [\lambda^{\sigma(\gamma)}\mu_2(\lambda)]^{\frac{t}{\sigma(\gamma)}} \frac{\mathrm{d}\lambda}{\lambda},$$

where the constant $\tilde{C} > 1$ is increasing in the variables $m, n, p, L/\nu$ and decreasing in t, such that $\tilde{C} \to \infty$ as $t \to 0$, while it remains bounded when t is bounded away from zero.

Thus, the following inequality

$$|Du(x)|^{p-1} \leq M^*((|Du|^2 + s^2)^{\frac{p-1}{2}})(x), \quad \text{a. e. } x \in \Omega,$$

yields

$$\||Du|^{p-1}\|_{L(\sigma(\gamma),t)(Q_0)} \leq C\lambda_0 |Q_0|^{\frac{1}{\sigma(\gamma)}} + C\|M^*(f)\|^{\frac{1}{\sigma(1)}}_{L\left(\frac{\sigma(\gamma)}{\sigma(1)}, \frac{t}{\sigma(1)}\right)(Q_0)}, \tag{87}$$

where the constant $C \equiv C(m, n, p, L/\nu, t)$ is bounded with respect to t as long as t is bounded away from zero. By choosing $t = \sigma(q)$ in (87), with $q \in (0, \infty)$, we obtain

$$\||Du|^{p-1}\|_{L(\sigma(\gamma),\sigma(q))(Q_0)} \leq C\lambda_0 |Q_0|^{\frac{1}{\sigma(\gamma)}} + C\|M^*(f)\|^{\frac{1}{\sigma(1)}}_{L(\gamma,q)(Q_0)}. \tag{88}$$

Now we can use Theorem 3 with $\beta = 0$, $t = \gamma$, eventually passing to the outer ball B, and taking $\delta = 2$. We get

$$\|M^*(f)\|^{\frac{1}{\sigma(1)}}_{L(\gamma,q)(Q_0)} \leq C\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(n^2 Q_0)}. \tag{89}$$

Finally, by means of (88), (89) and the definition of λ_0 in (83), we otain

$$\||Du|^{p-1}\|_{L(\sigma(\gamma),\sigma(q))(Q_0)} \leq C\left(\fint_{n^2 Q_0} (|Du|^2 + s^2)^{\frac{p-1}{2}}\,\mathrm{d}x\right) |Q_0|^{\frac{1}{\sigma(\gamma)}} + C\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(n^2 Q_0)}. \tag{90}$$

Similarly, we can deal with the case $q = \infty$ and we arrive at

$$\||Du|^{p-1}\|_{\mathcal{M}^{\sigma(\gamma)}(Q_0)} \leq C\left(\fint_{n^2 Q_0} (|Du|^2 + s^2)^{\frac{p-2}{2}}\,\mathrm{d}x\right) |Q_0|^{\frac{1}{\sigma(\gamma)}} + C\|f\|^{\frac{1}{\sigma(1)}}_{\mathcal{M}^{\gamma,\theta}(n^2 Q_0)}.$$

Step 3 - Morrey spaces regularity. We want to use the intermediate Morrey spaces regularity of $|Du|^{p-1}$ proved in Lemma 9. Let $B_\rho \subseteq \Omega$ be a ball; we write (90) for $\bar{u}$ and $\bar{f}$ defined in B_1 as in (31) with $\mathcal{A} = 1$, by choosing $R \equiv \rho$, and we pass to inner and outer balls. We have

$$\||D\bar{u}|^{p-1}\|_{L(\sigma(\gamma),\sigma(q))(B_{1/n^4})} \leq C\|(|D\bar{u}|+s)^{p-1}\|_{L^{1,\frac{\theta}{\sigma(\gamma)}}(B_{9/10})} + C\|\bar{f}\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_1)}, \quad (91)$$

where we used that the definitions of m, θ and γ yield $\theta/\sigma(\gamma) < n$.

Now, we scale back to B_ρ. Lemma 2 with $\mathcal{A} = 1$ yields

$$\rho^{(\alpha+1)(p-1)-\frac{n}{\sigma(\gamma)}}\||Du|^{p-1}\|_{L(\sigma(\gamma),\sigma(q))(B_{\rho/n^4})} \leq C\Theta(B_\rho)\rho^{(\alpha+1)(p-1)-\frac{\theta}{\sigma(\gamma)}}, \quad (92)$$

with α as in (30) and

$$\Theta(B_\rho) := \|(|Du|+s)^{p-1}\|_{L^{1,\frac{\theta}{\sigma(\gamma)}}(B_{9\rho/10})} + \|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_\rho)}$$

for every ball $B_\rho \subseteq \Omega$.

In view of the definition of the Lorentz-Morrey norm and by means of a covering argument (see Step 5 in [38, Theorem 11] and in particular Page 612 there), from (92) we can deduce

$$\||Du|^{p-1}\|_{L^\theta(\sigma(\gamma),\sigma(q))(B_{R/2})} \leq C\,\Theta(B_{3R/4}),$$

together with Lemma 9 (with $d = R$ and $t = 27R/40$), yields

$$\||Du|^{p-1}\|_{L^\theta(\sigma(\gamma),\sigma(q))(B_{R/2})} \leq C\,R^{\frac{\theta}{\sigma(\gamma)}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)} + C\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}, \quad (93)$$

where $C \equiv C(n, p, m, L/\nu, \gamma, q) > 0$.

Step 4 - Conclusion of the proof. We recall that estimate (93) holds for $u \equiv u_k$ solution to (19) with $f \equiv f_k = T_k(f)$, where T_k is the truncation operator defined by (54); i. e.,

$$\||Du_k|^{p-1}\|_{L^\theta(\sigma(\gamma),\sigma(q))(B_{R/2})} \leq C\,R^{\frac{\theta}{\sigma(\gamma)}-n}\|(|Du_k|+s)^{p-1}\|_{L^1(B_R)} + C\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}, \quad (94)$$

where the constant C does not depend on k. Note that in (94) we have used the fact that

$$\|f_k\|_{L^\theta(\gamma,q)(B_R)} \leq \|f\|_{L^\theta(\gamma,q)(B_R)},$$

since by the definition of f_k it holds that $|f_k| \leq |f|$. Hence, in order to pass to the limit on $k \to \infty$ in (94), it suffices to use the lower semicontinuity of the Lorentz-Morrey norms together with the approximating arguments stated in Section 2.2 (recall, in particular, (21)). □

Remark 1 It is worth noticing that it is possible to extend the result of Theorem 6 in the case of f being a measure μ verifying the following density condition

$$|\mu|(B_R) \,\leq\, CR^{n-\theta}, \quad p \leq \theta \leq n,$$

for a nonnegative constant C. In view of the property above, one can repeat the proof of Theorem 6 modulo minor changes (see, also, [40, Theorem 4.3]).

5 Further extensions

In the following we analyze some possible extensions of Theorem 1. First, we deal with the solutions to (1) when the datum belongs to the Morrey-Orlicz spaces. Next, we will show that the regularity results in Theorem 1 can be obtained also when the lower order terms g verify some relaxed assumptions with respect to those considered until now.

5.1 Orlicz regularity

We want to study the case in which the given function f in (1) belongs to an Orlicz space or to a Morrey-Orlicz space. For this, we firstly recall the definition of these spaces.

A measurable function $f : \Omega \to \mathbb{R}$ belongs to the *Orlicz space* $L\log L(\Omega)$ if and only if

$$\|f\|_{L\log L(\Omega)} := \fint_{\Omega} |f| \log\left(e + \frac{f}{\fint_{\Omega}|f(y)|\,\mathrm{d}y}\right)\mathrm{d}x \,<\, \infty.$$

Fix $\theta \in [0,n]$, a measurable function $f : \Omega \to \mathbb{R}$ belongs to the *Morrey-Orlicz space* $L\log L^{\theta}(\Omega)$ if and only if

$$\begin{aligned}\|f\|_{L\log L^{\theta}(\Omega)} &:= \sup_{B_R\subseteq\Omega} R^{\theta}\|f\|_{L\log L(B_R)} \\ &= \sup_{B_R\subseteq\Omega} R^{\theta-n}\int_{B_R} |f| \log\left(e + \frac{f}{\fint_{B_R}|f(y)|\,\mathrm{d}y}\right)\mathrm{d}x < \infty.\end{aligned}$$

We note that, as in the Morrey-Lorentz case, the functional $\|\cdot\|_{L\log L(\Omega)}$ is lower semicontinuous with respect to the a. e. convergence.

We are ready to state and prove the following regularity result.

Theorem 7 *Assume* (2) *and*

$$f \in L\log L^{\theta}(\Omega), \quad \theta \in (p,n].$$

Then the solution $u \in W^{1,1}_0(\Omega)$ *to* (18)*, with*

$$p-1 < m < \infty,$$

satisfies

$$|Du|^{p-1} \in L^{\sigma(1),\theta} \text{ locally in } \Omega.$$

Moreover, the local estimate

$$\||Du|^{p-1}\|_{L^{\sigma(1),\theta}(B_{R/2})} \leq C\, R^{\frac{\theta}{\sigma(1)}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)} + C\|f\|^{\frac{1}{\sigma(1)}}_{L\log L^\theta(B_R)} \tag{95}$$

holds for every ball $B_R \subseteq \Omega$, where C depends only on m, n, p and L/ν; σ is given by (82).

Proof We can repeat the proof of Theorem 6 up to (87) with $\gamma = 1$. Hence, taking $t = \sigma(1)$ we arrive at

$$\int_{Q_0} |Du|^{(p-1)\sigma(1)}\,\mathrm{d}x \leq C\lambda_0^{\sigma(1)}|Q_0| + C\int_{Q_0}|M^*(f)|\,\mathrm{d}x.$$

So by the boundedness of the maximal operators in $L\log L$, $\|M^*(f)\|_{L^1(Q_0)} \leq C\|f\|_{L\log L(Q_0)}$, and recalling the definition of λ_0 in (83), we get

$$\left(⨍_{Q_0}|Du|^{(p-1)\sigma(1)}\,\mathrm{d}x\right)^{\frac{1}{\sigma(1)}} \leq C|Q_0|^{\frac{1}{\sigma(1)}}\left(⨍_{n^2Q_0}(|Du|^2+s^2)^{\frac{p-1}{2}}\,\mathrm{d}x\right) + C|Q_0|^{\frac{1}{\sigma(1)}}\|f\|^{\frac{1}{\sigma(1)}}_{L\log L(Q_0)}.$$

Passing to the outer and inner balls in a standard way, and rescaling everything to B_1, we obtain the analog of (91), i. e.,

$$\||D\overline{u}|^{(p-1)}\|_{L^{\sigma(1)}(B_{1/n^4})} \leq C\|(|D\overline{u}|+s)^{p-1}\|_{L^{1,\frac{\theta}{\sigma(1)}}(B_{9/10})} + C\|\overline{f}\|^{\frac{1}{\sigma(1)}}_{L\log L^\theta(B_1)},$$

where $\overline{u}$ and $\overline{f}$ are as in (31) with $\mathcal{A} = 1$. Scaling back, we get

$$\rho^{\frac{n-\theta}{\sigma(1)}}\||Du|^{p-1}\|_{L^{\sigma(1)}(B_{\rho/n^4})} \leq C\Theta(B_\rho)$$

with

$$\Theta(B_\rho) = \|(|Du|+s)^{p-1}\|_{L^{1,\frac{\theta}{\sigma(1)}}(B_{9\rho/10})} + \|f\|_{L\log L^\theta(B_\rho)}.$$

Arguing as in Theorem 6 by a standard covering argument we arrive at

$$\||Du|^{p-1}\|_{L^{\sigma(1),\theta}(B_{R/2})} \leq C\Theta(B_{3R/4}), \quad \forall\, B_R \subseteq \Omega.$$

To estimate $\Theta(B_{3R/4})$ we use the following inequality

$$\|(|Du|+s)^{p-1}\|_{L^{1,\frac{\theta}{\sigma(1)}}(B_t)} \leq C(d-t)^{\frac{\theta}{\sigma(1)}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_d)} + C\|f\|^{\frac{1}{\sigma(1)}}_{L^{1,\theta}(B_d)}, \quad \forall\, B_t \subset B_d \subseteq \Omega,$$

with $d = R$, $t = 27R/40$, that we can prove in the same way as in Lemma 9, using (62) instead of (61). Finally, by the fact that $\|f\|_{L^{1,\theta}(B_R)} \leq C\|f\|_{L\log L^\theta(B_R)}$, we obtain the estimate in (95). □

5.2 Weighted lower order terms

In this section we deal with an extension of problem (1) involving a class of lower order terms g satisfying weaker assumptions with respect to those considered until now, that is

$$\begin{cases} -\mathrm{div}\, a(x, Du) + h(x)|u|^{m-1}u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{96}$$

where h is such that $0 < h(x) < 1$ and

$$1/h \in L^{\kappa}(\Omega) \quad \text{for some } \kappa \geq 1. \tag{97}$$

Clearly, such $g(\cdot, t) = h(\cdot)|t|^{m-1}t$ satisfies (3), but not necessarily (4).

In the following, we will show how to recover a regularity result, as in Theorem 6, even in spite of such different lower order term g. Indeed, we can again recover additional information about the summability of the solutions coming from the structure of the equation, but, in this case, we will have to reduce the interval in which m can vary, depending on the integrability of h given by (97).

First, we have to modify the proof of Lemma 4 by using $\phi = h^{\gamma-1}|u - v|^{m(\gamma-1)-1}(u - v)$ as test function in (41). In view of the fact that the additional function h is positive, we can again use the algebraic inequality (43), by carefully distinguishing the correspondent range on validity in dependance of m. It follows

$$\int_{B_R} h^{\gamma}|u - v|^{m\gamma}\, \mathrm{d}x \,\leq\, C \int_{B_R} |f|^{\gamma}\, \mathrm{d}x,$$

for any $m > 0$, $\gamma \geq 1$ and $B_R \subseteq \Omega$. This estimate will permit to obtain an additional summability of $u - v$. As a matter of fact, by the Hölder inequality we plainly deduce

$$\int_{B_R} |u - v|^{\frac{\kappa m\gamma}{\kappa+\gamma}}\, \mathrm{d}x \,\leq\, C \left(\int_{B_R} |f|^{\gamma}\, \mathrm{d}x \right)^{\frac{\kappa}{\kappa+\gamma}}. \tag{98}$$

As expected, notice that

$$\frac{\kappa m\gamma}{\kappa+\gamma} < m\gamma \quad \forall\, \kappa > 0 \quad \text{and} \quad \frac{\kappa m\gamma}{\kappa+\gamma} \to m\gamma \text{ as } \kappa \to \infty.$$

In view of (98), inequality (51) in Lemma 5 holds for any $0 < q \leq \kappa mp\gamma/(\kappa(m+1)+\gamma)$. Similarly, we can deduce the validity of (61) and (62) in Lemma 6 and (63) in Lemma 7 for any

$$p - 1 < \frac{(\kappa+1)(p-1)}{\kappa} < m < \infty.$$

At this time, we can repeat the entire proof of Theorem 6 with slight modifications and we arrive at

Theorem 8 *Let $q \in (0,\infty]$. Assume (2) and $f \in L^\theta(\gamma,q)(\Omega)$ with γ, θ such that*

$$1 < \gamma \leq \frac{\theta p}{\theta p - \theta + p}, \qquad 1 < p < \theta \leq n.$$

Then the solution $u \in W_0^{1,1}(\Omega)$ to (96), with

$$\frac{(\kappa+1)(p-1)}{\kappa} < m < \frac{1}{\gamma - 1},$$

where κ is given by (97), satisfies $|Du|^{p-1} \in L^\theta\big(\sigma(\gamma),\sigma(q)\big)$ locally in Ω. Moreover, the local estimate

$$\||Du|^{p-1}\|_{L^\theta(\sigma(\gamma),\sigma(q))(B_{R/2})} \leq C\, R^{\frac{\theta}{\sigma(\gamma)}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)} + C\|f\|^{\frac{1}{\sigma(1)}}_{L^\theta(\gamma,q)(B_R)}$$

holds for every ball $B_R \subseteq \Omega$, where C depends only on m, n, p, q, L/ν and γ.

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