A NOTE ON INTERIOR $W^{2,1+\varepsilon}$ ESTIMATES FOR THE MONGE-AMPÈRE EQUATION

G. DE PHILIPPIS, A. FIGALLI, AND O. SAVIN

Abstract. By a variant of the techniques introduced by the first two authors in [DF] to prove that second derivatives of solutions to the Monge-Ampère equation are locally in $L\log L$, we obtain interior $W^{2,1+\varepsilon}$ estimates.

1. Introduction

Interior $W^{2,p}$ estimates for solutions to the Monge-Ampère equation with bounded right hand side

$$\det D^2u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad 0 < \lambda \leq f \leq \Lambda, \tag{1.1}$$

were obtained by Caffarelli in [C] under the assumption that $|f-1| \leq \varepsilon(p)$ locally. In particular $u \in W^{2,p}_{\rm loc}$ for any $p < \infty$ if f is continuous.

Whenever f has large oscillation, $W^{2,p}$ estimates are not expected to hold for large values of p. Indeed Wang showed in [W] that for any $p > 1$ there are homogenous solutions to (1.1) of the type

$$u(tx, t^\alpha y) = t^{1+\alpha}u(x,y) \quad \text{for } t > 0,$$

which are not in $W^{2,p}$.

Recently the first two authors, motivated by a problem arising from the semi-geostrophic equation [ACDF, ACDF2], showed that interior $W^{2,1}$ estimates hold for the equation (1.1) [DF]. In fact they proved higher integrability in the sense that

$$\|D^2u\|\,\big|\log\|D^2u\|\big|^k \in L^1_{\rm loc} \qquad \forall\, k \geq 0.$$

In this short note we obtain interior $W^{2,1+\varepsilon}$ estimates for some small $\varepsilon = \varepsilon(n,\lambda,\Lambda) > 0$. In view of the examples in [W] this result is optimal. We use the same ideas as in [DF], which mainly consist in looking to the L^1 norm of $\|D^2u\|$ over the sections of u itself and prove some decay estimates. Below we give the precise statement.

Theorem 1.1. *Let $u : \overline{\Omega} \to \mathbb{R}$,*

$$u = 0 \quad \textit{on } \partial\Omega, \qquad B_1 \subset \Omega \subset B_n,$$

be a continuous convex solution to the Monge-Ampère equation

$$\det D^2u = f(x) \quad \textit{in } \Omega, \qquad 0 < \lambda \leq f \leq \Lambda, \tag{1.2}$$

for some positive constants λ, Λ. Then

$$\|u\|_{W^{2,1+\varepsilon}(\Omega')} \leq C, \qquad \textit{with} \quad \Omega' := \{u < -\|u\|_{L^\infty}/2\},$$

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where $\varepsilon, C > 0$ are universal constants depending on n, λ, and Λ only.

By a standard covering argument (see for instance [DF, Proof of (3.1)]), this implies that $u \in W^{2,1+\varepsilon}_{\mathrm{loc}}(\Omega)$.

Theorem 1.1 follows by slightly modifying the strategy in [DF]: We use a covering lemma that is better localized (see Lemma 3.1) to obtain a geometric decay of the "truncated" L^1 energy for $\|D^2u\|$ (see Lemma 3.3).

We also give a second proof of Theorem 1.1 based on the following observation: In view of [DF] the L^1 norm of $\|D^2u\|$ decays on sets of small measure:

$$|\{\|D^2u\| \geq M\}| \leq \frac{C}{M \log M},$$

for an appropriate universal constant $C > 0$ and for any M large. In particular, choosing first M sufficiently large and then taking $\varepsilon > 0$ small enough, we deduce (a localized version of) the bound

$$|\{\|D^2u\| \geq M\}| \leq \frac{1}{M^{1+\varepsilon}} |\{\|D^2u\| \geq 1\}|$$

Applying this estimate at all scales (together with a covering lemma) leads to the local $W^{2,1+\varepsilon}$ integrability for $\|D^2u\|$.

We believe that both approaches are of interest, and for this reason we include both. In particular, the first approach gives a direct proof of the $W^{2,1+\varepsilon}_{\mathrm{loc}}$ regularity without passing through the $L \log L$ estimate.

We remark that the estimate of Theorem 1.1 holds under slightly weaker assumptions on the right hand side. Precisely if

$$\det D^2u = \mu$$

with μ being a finite combination of measures which are bounded between two multiples of a nonnegative polynomial, then the $W^{2,1+\varepsilon}_{\mathrm{loc}}$ regularity still holds (see Theorem 3.7 for a precise statement).

The paper is organized as follows. In section 2 we introduce the notation and some basic properties of solution to the Monge-Ampère equation with bounded right hand side. Then, in section 3 we show both proofs of Theorem 1.1, together with the extension to polynomial right hand sides.

After the writing of this paper was completed, we learned that Schmidt [S] had just obtained the same result with related but somehow different techniques.

2. Notation and Preliminaries

Notation. Given a convex function $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$ bounded and convex, we define its section $S_h(x_0)$ centered at x_0 at height h as

$$S_h(x_0) = \{x \in \Omega : \quad u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\}.$$

We also denote by $\overline{S_h}(x_0)$ the closure of $S_h(x_0)$.

The norm $\|A\|$ of an $n \times n$ matrix A is defined as

$$\|A\| := \sup_{|x| \leq 1} Ax.$$

We denote by $|F|$ the Lebesgue measure of a measurable set F.

Positive constants depending on n, λ, Λ are called *universal constants*. In general we denote them by c, C, c_i, C_i.

Next we state some basic properties of solutions to (1.2).

2.1. **Scaling properties.** If $S_h(x_0) \subset\subset \Omega$, then (see for example [C]) there exists a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, with $\det A = 1$, such that

$$\sigma B_{\sqrt{h}} \subset A(S_h(x_0) - x_0) \subset \sigma^{-1} B_{\sqrt{h}}, \tag{2.1}$$

for some $\sigma > 0$, small universal.

Definition 2.1. We say that $S_h(x_0)$ has *normalized size* α if

$$\alpha := \|A\|^2$$

for some matrix A that satisfies the properties above. (Notice that, although A may not be unique, this definition fixes the value of α up to multiplicative universal constants.)

It is not difficult to check that if u is C^2 in a neighborhood of x_0, then $S_h(x_0)$ has normalized size $\|D^2u(x_0)\|$ for all small $h > 0$ (if necessary we need to lower the value of σ).

Given a transformation A as in (2.1), we define $\tilde{u}$ to be the rescaling of u

$$\tilde{u}(\tilde{x}) = h^{-1}u(x), \qquad \tilde{x} = Tx := h^{-1/2}A\,(x - x_0). \tag{2.2}$$

Then $\tilde{u}$ solves an equation in the same class

$$\det D^2\tilde{u} = \tilde{f}, \qquad \text{with} \quad \tilde{f}(\tilde{x}) := f(x), \qquad \lambda \le \tilde{f} \le \Lambda,$$

and the section $\tilde{S}_1(0)$ of $\tilde{u}$ at height 1 is normalized i.e

$$\sigma B_1 \subset \tilde{S}_1(0) \subset \sigma^{-1}B_1, \qquad \tilde{S}_1(0) = T(S_h(x_0)).$$

Also

$$D^2u(x) = A^T D^2\tilde{u}(\tilde{x})A,$$

hence

$$\|D^2u(x)\| \le \|A\|^2\|D^2\tilde{u}(\tilde{x})\|, \tag{2.3}$$

and

$$\gamma_1 I \le D^2\tilde{u}(\tilde{x}) \le \gamma_2 I \qquad \Rightarrow \qquad \gamma_1\|A\|^2 \le \|D^2u(x)\| \le \gamma_2\|A\|^2. \tag{2.4}$$

2.2. **Properties of sections.** Caffarelli and Gutierrez showed in [CG] that sections $S_h(x)$ which are compactly included in Ω have engulfing properties similar to the engulfing properties of balls. In particular we can find $\delta > 0$ small universal such that:

1) If $h_1 \le h_2$ and $S_{\delta h_1}(x_1) \cap S_{\delta h_2}(x_2) \neq \emptyset$ then
$$S_{\delta h_1}(x_1) \subset S_{h_2}(x_2).$$

2) If $h_1 \le h_2$ and $x_1 \in \overline{S_{h_2}}(x_2)$ then we can find a point z such that
$$S_{\delta h_1}(z) \subset S_{h_1}(x_1) \cap S_{h_2}(x_2).$$

3) If $x_1 \in \overline{S_{h_2}}(x_2)$ then
$$S_{\delta h_2}(x_1) \subset S_{2h_2}(x_2).$$

Now we also state a covering lemma for sections.

Lemma 2.2 (Vitali covering)**.** *Let D be a compact set in Ω and assume that to each $x \in D$ we associate a corresponding section $S_h(x) \subset\subset \Omega$. Then we can find a finite number of these sections $S_{h_i}(x_i)$, $i = 1, \dots, m$, such that*

$$D \subset \bigcup_{i=1}^{m} S_{h_i}(x_i), \qquad \text{with } S_{\delta h_i}(x_i) \text{ disjoint.}$$

The proof follows as in the standard case: we first select by comptactness a finite number of sections $S_{\delta h_j}(x_j)$ which cover D, and then choose a maximal disjoint set from these sections, selecting at each step a section which has maximal height among the ones still available (see the proof of [St, Chapter 1, §3, Lemma 1] for more details).

3. Proof of Theorem 1.1

We assume throughout that u is a normalized solution in $S_1(0)$ in the sense that

$$\det D^2 u = f \quad \text{in } \Omega, \qquad \lambda \le f \le \Lambda,$$

and

$$S_2(0) \subset\subset \Omega, \qquad \sigma B_1 \subset S_1(0) \subset \sigma^{-1} B_1.$$

In this section we show that

$$\int_{S_1(0)} \|D^2 u\|^{1+\varepsilon} dx \le C, \tag{3.1}$$

for some universal constants $\varepsilon > 0$ small and C large. Then Theorem 1.1 easily follows from this estimate and a covering argument based on the engulfing properties of sections. Without loss of generality we may assume that $u \in C^2$, since the general case follows by approximation.

3.1. **A direct proof of Theorem 1.1.** In this section we give a selfcontained proof of Theorem 1.1. As already mentioned in the introduction, the idea is to get a geometric decay for $\int_{\{\|D^2u\| \ge M\}} \|D^2 u\|$.

Lemma 3.1. *Assume $0 \in \overline{S_t}(y) \subset\subset \Omega$ for some $t \ge 1$ and $y \in \Omega$. Then*

$$\int_{S_1(0)} \|D^2 u\| dx \le C_0 \left| \left\{ C_0^{-1} I \le D^2 u \le C_0 I \right\} \cap S_\delta(0) \cap S_t(y) \right|,$$

for some C_0 large universal.

Proof. By convexity of u we have

$$\int_{S_1(0)} \|D^2 u\| dx \le \int_{S_1(0)} \Delta u \, dx = \int_{\partial S_1(0)} u_\nu \le C_1,$$

where the last inequality follows from the interior Lipschitz estimate of u in $S_2(0)$.

The second property in Subsection 2.2 gives

$$S_\delta(0) \cap S_t(y) \supset S_{\delta^2}(z)$$

for some point z, which implies that

$$|S_\delta(0) \cap S_t(y)| \ge c_1$$

for some $c_1 > 0$ universal. The last two inequalities show that the set

$$\left\{ \|D^2 u\| \le 2 C_1 c_1^{-1} \right\}$$

has at least measure $c_1/2$ inside $S_\delta(0) \cap S_h(y)$.

Finally, the lower bound on $\det D^2 u$ implies that

$$C_0^{-1} I \le D^2 u \le C_0 I \qquad \text{inside } \{\|D^2 u\| \le 2C_1 c_1^{-1}\},$$

and the conclusion follows provided that we choose C_0 sufficiently large. □

By rescaling we obtain:

Lemma 3.2. *Assume $S_{2h}(x_0) \subset\subset \Omega$, and $x_0 \in \overline{S_t}(y)$ for some $t \ge h$. If*

$$S_h(x_0) \ \ \textit{has normalized size } \alpha,$$

then

$$\int_{S_h(x_0)} \|D^2 u\| \, dx \le C_0 \alpha \, \left| \left\{ C_0^{-1} \alpha \le \|D^2 u\| \le C_0 \alpha \right\} \cap S_{\delta h}(x_0) \cap S_t(y) \right| .$$

Proof. The lemma follows by applying Lemma 3.1 to the rescaling $\tilde{u}$ defined in Section 2 (see (2.2)). More precisely, we notice first that by (2.3) we have

$$\|D^2 u(x)\| \le \alpha \|D^2 \tilde{u}(\tilde{x})\|, \qquad \tilde{x} = Tx,$$

hence

$$|\det T| \int_{S_h(x_0)} \|D^2 u\| \, dx \le \alpha \int_{\tilde{S}_1(0)} \|D^2 \tilde{u}\| \, d\tilde{x}.$$

Also, by (2.4) we obtain

$$\left\{ C_0^{-1} I \le D^2 \tilde{u} \le C_0 I \right\} \subset T \left(\left\{ C_0^{-1} \alpha \le \|D^2 u\| \le C_0 \alpha \right\} \right) .$$

which together with

$$\tilde{S}_\delta(0) = T(S_{\delta h}), \quad \tilde{S}_{t/h}(\tilde{y}) = T(S_t(y)),$$

implies that

$$\left| \left\{ C_0^{-1} I \le D^2 \tilde{u} \le C_0 I \right\} \cap \tilde{S}_\delta(0) \cap \tilde{S}_{t/h}(\tilde{y}) \right|$$

is bounded above by

$$|\det T| \, \left| \left\{ C_0^{-1} \alpha \le \|D^2 u\| \le C_0 \alpha \right\} \cap S_{\delta h}(x_0) \cap S_t(y) \right| .$$

The conclusion follows now by applying Lemma 3.1 to $\tilde{u}$. □

Next we denote by D_k, $k \ge 0$, the closed sets

$$D_k := \left\{ x \in \overline{S_1}(0) : \, \|D^2 u(x)\| \ge M^k \right\} , \tag{3.2}$$

for some large M. As we show now, Lemma 3.2 combined with a covering argument gives a geometric decay for $\int_{D_k} \|D^2 u\|$.

Lemma 3.3. *If $M = C_2$, with C_2 a large universal constant, then*

$$\int_{D_{k+1}} \|D^2 u\| \, dx \le (1 - \tau) \int_{D_k} \|D^2 u\| \, dx,$$

for some small universal constant $\tau > 0$.

Proof. Let $M \gg C_0$ (to be fixed later), and for each $x \in D_{k+1}$ consider a section

$$S_h(x) \text{ of normalized size } \alpha = C_0 M^k,$$

which is compactly included in $S_2(0)$. This is possible since for $h \to 0$ the normalized size of $S_h(x)$ converges to $\|D^2u(x)\|$ (recall that $u \in C^2$) which is greater than $M^{k+1} > \alpha$, whereas if $h = \delta$ the normalized size is bounded above by a universal constant and therefore by α.

Now we choose a Vitali cover for D_{k+1} with sections $S_{h_i}(x_i)$, $i = 1, \ldots, m$. Then by Lemma 3.2, for each i,

$$\int_{S_{h_i}(x_i)} \|D^2u\| dx \le C_0^2 M^k \left|\left\{M^k \le \|D^2u\| \le C_0^2 M^k\right\} \cap S_{\delta h_i}(x_i) \cap S_1(0)\right|.$$

Adding these inequalities and using

$$D_{k+1} \subset \bigcup S_{h_i}(x_i), \qquad S_{\delta h_i}(x_i) \text{ disjoint},$$

we obtain

$$\begin{aligned}\int_{D_{k+1}} \|D^2u\| dx &\le C_0^2 M^k \left|\left\{M^k \le \|D^2u\| \le C_0^2 M^k\right\} \cap S_1(0)\right| \\ &\le C \int_{D_k \setminus D_{k+1}} \|D^2u\| dx\end{aligned}$$

provided $M \ge C_0^2$. Adding $C \int_{D_{k+1}} \|D^2u\|$ to both sides of the above inequality, the conclusion follows with $\tau = 1/(1 + C)$. □

By the above result, the proof of (3.1) is immediate: indeed, by Lemma 3.3 we easily deduce that there exist $C, \varepsilon > 0$ universal such that

$$\int_{\{x \in S_1(0):\, \|D^2u(x)\| \ge t\}} \|D^2u\|\, dx \le C t^{-2\varepsilon} \qquad \forall\, t \ge 1.$$

Multiplying both sides by $t^{-(1-\varepsilon)}$ and integrating over $[1, \infty)$ we obtain

$$\int_1^\infty t^{-(1-\varepsilon)} \int_{\{x \in S_1(0):\, \|D^2u(x)\| \ge t\}} \|D^2u\|\, dx\, dt \le C \int_1^\infty t^{-1-\varepsilon} = \frac{C}{\varepsilon},$$

and we conclude using Fubini.

3.2. **A proof by iteration of the $L \log L$ estimate.** We now briefly sketch how (3.1) could also be easily deduced by applying the $L \log L$ estimate from [DF] inside every section, and then performing a covering argument.

First, any $K > 0$ we introduce the notation

$$F_K := \{\|D^2u\| \ge K\} \cap S_1(0).$$

Lemma 3.4. *Suppose u satisfies the assumptions of Lemma 3.1. Then there exist universal constants C_0 and C_1 such that, for all $K \ge 2$,*

$$|F_K| \le \frac{C_1}{K \log(K)} \left|\left\{C_0^{-1} I \le D^2u \le C_0 I\right\} \cap S_\delta(0) \cap S_t(y)\right|.$$

Indeed, from the proof of Lemma 3.1 the measure of the set appearing on the right hand side is bounded below by a small universal constant $c_1/2$, while by [DF] $|F_K| \le C/K \log(K)$ for all $K \ge 2$, hence

$$|F_K| \le \frac{2C}{c_1 K \log(K)} \left|\left\{C_0^{-1} I \le D^2u \le C_0 I\right\} \cap S_\delta(0) \cap S_t(y)\right|.$$

Exactly as in the proof of Lemma 3.2, by rescaling we obtain:

Lemma 3.5. *Suppose u satisfies the assumptions of Lemma 3.2. Then,*

$$|\{\|D^2u\| \geq \alpha K\} \cap S_h(x_0)| \leq \frac{C_1}{K\log(K)} \left|\{C_0^{-1}\alpha \leq \|D^2u\|\} \cap S_{\delta h}(x_0) \cap S_t(y)\right|,$$

for all $K \geq 2$.

Finally, as proved in the next lemma, a covering argument shows that the measure of the sets D_k defined in (3.2) decays as $M^{-(1+2\varepsilon)k}$, which gives (3.1).

Lemma 3.6. *There exist universal constants M large and $\varepsilon > 0$ small such that*

$$|D_{k+1}| \leq M^{-1-2\varepsilon}|D_k|.$$

Proof. As in the proof of Lemma 3.3, we use a Vitali covering of the set D_{k+1} with sections $S_h(x)$ of normalized size $\alpha = C_0M^k$, i.e.

$$D_{k+1} \subset \bigcup S_{h_i}(x_i), \qquad S_{\delta h_i}(x_i) \text{ disjoint sets.}$$

We then apply Lemma 3.5 above with

$$K := C_0^{-1}M$$

for some $M \geq 2C_0$ (to be fixed later). In this way $\alpha K = M^{k+1}$ and $C_0^{-1}\alpha = M^k$, and we find that, for each i,

$$|D_{k+1} \cap S_{h_i}(x_i)| \leq \frac{C_1}{M\log(M)}|D_k \cap S_{\delta h_i}(x_i)|.$$

Summing over i and choosing $M \geq e^{2C_1}$ we get

$$|D_{k+1}| \leq \frac{C_1}{M\log(M)}|D_k| \leq \frac{1}{2M}|D_k|,$$

and the lemma is proved by choosing $2\varepsilon = \log(2)/\log(M)$. □

3.3. **More general measures.** It is not difficult to check that our proof applies to more general right hand sides. Precisely we can replace f by any measure μ of the form

$$\mu = \sum_{i=1}^{N} g_i(x)|P_i(x)|^{\alpha_i}\,dx, \qquad 0 < \lambda \leq g_i \leq \Lambda, \quad P_i \text{ polynomial}, \quad \alpha_i \geq 0. \tag{3.3}$$

We state the precise estimate below.

Theorem 3.7. *Let $u : \overline{\Omega} \to \mathbb{R}$,*

$$u = 0 \quad \text{on } \partial\Omega, \qquad B_1 \subset \Omega \subset B_n,$$

be a continuous convex solution to the Monge-Ampère equation

$$\det D^2u = \mu \quad \text{in } \Omega, \qquad \mu(\Omega) \leq 1,$$

with μ as in (3.3). Then

$$\|u\|_{W^{2,1+\varepsilon}(\Omega')} \leq C, \qquad \text{with} \quad \Omega' := \{u < -\|u\|_{L^\infty}/2\},$$

where $\varepsilon, C > 0$ are universal constants.

The proof follows as before, based on the fact that for μ as above one can prove the existence of constants $\beta \geq 1$ and $\gamma > 0$, such that, for all convex sets S,[1]

$$\frac{\mu(E)}{\mu(S)} \geq \gamma \left(\frac{|E|}{|S|}\right)^\beta \qquad \forall\, E \subset S. \tag{3.4}$$

In this general situation, we need to write the scaling properties of u with respect to the measure μ. More precisely the scaling inclusion (2.1) becomes

$$\sigma\, h\, \mu(S_h(x_0))^{-\frac{1}{n}}\ B_1 \,\subset\, A(S_h(x_0)-x_0) \,\subset\, \sigma^{-1}\, h\, \mu(S_h(x_0))^{-\frac{1}{n}}\ B_1,$$

and

$$Tx := h^{-1}\mu(S_h(x_0))^{\frac{1}{n}}\, A\,(x-x_0).$$

Also we define the *normalized size* α of $S_h(x_0)$ (relative to the measure μ) as

$$\alpha := h^{-1}\mu(S_h(x_0))^{\frac{2}{n}}\, \|A\|^2.$$

With this notation the statements of the lemmas in Section 3 apply as before.

Indeed, first of all we observe that (3.4) implies that μ is doubling, so all properties of sections stated in Section 2.2 still hold.

Then, in the proof of Lemma 3.1, we simply apply (3.4) with $S = S_1(0)$ and $E = \{\det(D^2u) \leq c_2\}$ ($c_2 > 0$ small) to deduce that

$$\gamma|E|^\beta \leq C\mu(E) = C\int_E \det(D^2u) \leq Cc_2|E|.$$

This implies that, if $c_2 > 0$ is sufficiently small, the set

$$\left\{\|D^2u\| \leq 2C_1c_1^{-1}\right\} \cap \left\{\det(D^2u) > c_2\right\}$$

has at least measure $c_1/4$, and the result follows as before.

Moreover, since (3.4) is affinely invariant, Lemma 3.2 follows again from Lemma 3.1 by rescaling. Finally, the proof of Lemma 3.3 is identical.

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[1]Although this will not be used here, we point out for completeness that (3.4) is equivalent to the so-called *Condition* (μ_∞), first introduced by Caffarelli and Gutierrez in [CG]. Indeed, using (3.4) with $E = S\setminus F$ one sees that $|F|/|S| \ll 1$ implies $\mu(F)/\mu(S) \leq 1-\gamma/2$, and then an iteration and covering argument in the spirit of [CG, Theorem 6] shows that (3.4) is actually equivalent to *Condition* (μ_∞).

Scuola Normale Superiore di Pisa, 56126 Pisa, Italy
E-mail address: guido.dephilippis@sns.it

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712 USA
E-mail address: figalli@math.utexas.edu

Department of Mathematics, Columbia University, New York, NY 10027 USA
E-mail address: savin@math.columbia.edu