Asymptotically regular problems I: Higher integrability

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Abstract

We consider weak solutions u of nonlinear systems of partial differential equations. Assuming that the system exhibits a certain kind of elliptic behavior near infinity we prove higher integrability results for the gradient Du. In particular, we establish Hölder continuity of u in low dimensions. Moreover, we obtain analogous results for vectorial minimizers of multi-dimensional variational integrals. Finally, we discuss an extension to minimizing sequences and applications to generalized minimizers.

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Contents

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1 Introduction

We consider nonlinear systems of partial differential equations

$$\operatorname{div} a(\cdot, u, Du) = 0 \qquad \text{on } \Omega \tag{1.1}$$

for vector-valued functions $u : \Omega \to \mathbb{R}^N$ with $N \in \mathbb{N}$. Here, Ω is an open and bounded subset of $\mathbb{R}^n$, where $2 \leq n \in \mathbb{N}$, and $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is a given structure function. Moreover, we study the minimization problem in Dirichlet classes for multidimensional variational integrals

$$F[u] := \int_\Omega f(\cdot, u, Du)\, dx \tag{1.2}$$

with a given integrand $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$. For both these problems we investigate the interior regularity properties of weak solutions u. Specifically, in the present paper we deal with the integrability properties of the first derivative Du, while in the second part [52] of our work on asymptotically regular problems we will focus on problems of (partial) Lipschitz regularity.

For the purposes of this introduction let us restrict our exposition to the simpler case of integrals

$$F[u] := \int_\Omega f(Du)\, dx \tag{1.3}$$

with a locally bounded Borel integrand $f : \mathbb{R}^{Nn} \to \mathbb{R}$ (the more general cases will be recovered in Section 2.3). A classical assumption on the integrand f is that f be strictly convex. We will say that f is regular (see Definition

2.1 for a precise statement) if this holds together with some supplementary assumptions ensuring that the natural space for the investigation of (1.3) is the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$ with $p \geq 2$. As a matter of fact, if f is regular then minimizers u exist in $W^{1,p}(\Omega, \mathbb{R}^N)$ and are actually more regular; precisely, Campanato [11] for $N > 1$ and Giaquinta & Giusti [31], Giaquinta & Modica [34] and Manfredi [46] for $N = 1$ proved

$$u \in W^{1,p^{\#}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N), \tag{1.4}$$

where we have set

$$p^{\#} := \begin{cases} \frac{np}{n-2} + \kappa & \text{if } n \geq 3, N \geq 2 \\ \infty & \text{if } n = 2 \text{ or } N = 1 \end{cases}$$

with some constant $\kappa > 0$. We record that (1.4) implies Hölder continuity of u in low dimensions, namely for $n \leq p + 2$.

Heuristically, it is plausible that the validity of (1.4) should depend only on the behavior of f near infinity in $\mathbb{R}^{Nn}$. Indeed, the aim of the present paper is to clarify this heuristic idea and to investigate whether (1.4) holds for a broader class of problems than just the regular ones. We will introduce this class, which we call the asymptotically regular problems, below. However, before providing more precise statements let us briefly comment on some previous results in the literature concerning variational problems with a quadratic or p-Laplacean structure at infinity:

First of all, Chipot & Evans [12] (see also [44,45]) proved that minimizers u of F from (1.3) satisfy

$$u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \tag{1.5}$$

provided f fulfills

$$\lim_{|z| \to \infty} D^2 f(z) = A$$

for some inner product A on $\mathbb{R}^{Nn}$. Clearly, (1.5) is stronger than (1.4) and can, in general, not even be expected for regular problems as demonstrated by recent counterexamples of Sverak & Yan [53]. However, (1.5) is explained by the fact that stronger regularity results are available for minimizers of the quadratic comparison functional $\frac{1}{2} \int_\Omega A(Dv, Dv)\, dx$ which can be partially carried over to minimizers of F.

Subsequently, Giaquinta & Modica [34] obtained an analogous result for integrands f with superquadratic growth; namely, they showed that (1.5) holds also if f satisfies

$$\lim_{|z| \to \infty} \frac{D^2 f(z) - D^2_z(\frac{1}{p}|z|^p)}{|z|^{p-2}} = 0 \tag{1.6}$$

for some $p \geq 2$ (see also [51,26,27,21,24]). Again, this result is based on an improved regularity theory for the comparison integral $\frac{1}{p}\int_\Omega |Dv|^p\,dx$, namely on Uhlenbeck's famous regularity result [55] for the p-Laplacean system.

Finally, merely requiring the weaker condition

$$\lim_{|z|\to\infty}\frac{|f(z)-\frac{1}{p}|z|^p|}{|z|^p}=0 \tag{1.7}$$

instead of (1.6) Fuchs [25] (see also [44]) proved

$$u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\qquad\text{for all }\alpha<1 \tag{1.8}$$

and Dolzmann & Kristensen [15] (see also [17]) showed that even

$$u\in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\qquad\text{for all }q<\infty \tag{1.9}$$

still holds. Moreover, Morrey space regularity for Du in a quite general setting has been proved recently in [22,23]. However, in the simple situation described here these results are already contained in (1.9).

In the present paper we will not impose any quadratic, p-Laplacean or other special structure near infinity. Instead, we will cover a broader class of problems which we call the asymptotically regular ones. Precisely, weakening (1.7) again we will only require that f is close to an arbitrary regular function g near ∞ in the sense of

$$\lim_{|z|\to\infty}\frac{|f(z)-g(z)|}{|z|^p}=0. \tag{1.10}$$

Then by Sverak & Yan's examples [53] one can no longer hope for (1.5), (1.8) or (1.9). However, as a particular case of our main results we will prove

$$u\in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\qquad\text{for all }q<p^{\#}$$

in this situation.

We stress that, in general, our assumptions do not imply the existence of minimizers (although minimizers may still exist in some cases, even if f is not convex). Therefore, we will also discuss the validity of our results for generalized minimizers: On the one hand, following ideas from [57,8,15] based on Ekeland's variational principle we will prove an equi-integrability result for minimizing sequences, which yields, in particular, higher integrability of Young measure minimizers and weak cluster points; compare [15,23]. On the other hand, we will apply our results in the context of relaxation.

We believe that our result can not be obtained following the blow-up strategies of [12,25,15] which make essential use of the homogeneity of the comparison functional. Instead, invoking some ideas of [34,25] we base our strategy of proof on Caffarelli & Peral's method [9,10] for obtaining gradient estimates. More precisely, we will use an extension of this method due to Acerbi & Mingione [3] and Kristensen & Mingione [38] (compare also [4] for another approach to gradient estimates and [18,40,41] for applications to boundary regularity). In addition, we will also present a more elementary method based on estimates in Morrey spaces, which allows to obtain somewhat weaker but related results. In particular, the latter method provides a self-contained proof for the Hölder continuity of minimizers in the case $n \leq p+2$.

Anyway, both these methods are not restricted to the case touched above or to proving higher integrability up to the exponent $p^\#$, which enters only through the estimates for the regular integral $G[v] := \int_\Omega g(Dv)\,dx$. Indeed, if for some reason minimizers of G are more regular then we can improve on our results. Recovering at the same time the general cases (1.1) and (1.2), we work out this idea in Section 2.3; we come out with a quite general statement unifying all the results mentioned before, apart from the fact that we cannot reach (1.5), but only (1.9) in some cases. However, by an example of [16] this loss cannot be avoided when passing from a condition for the second derivatives like (1.6) to conditions for the integrands themselves like (1.7) and (1.10). Anyway, we will come back to this point addressing questions of Lipschitz regularity and partial Lipschitz regularity in our forthcoming work [52].

Finally, we believe that it is natural to ask whether the existence of a regular integrand g with (1.10) can be characterized as a property of f itself. Indeed, we have obtained such a characterization of asymptotic regularity (Theorem 2.16), whose proof is elementary, but surprisingly non-trivial. More precisely, we exhibit a disturbed convexity condition for f near infinity which is equivalent to (1.10).

2 Statements

For the remainder of the paper we fix a growth exponent $\mathbf{2} \leq \boldsymbol{p} < \boldsymbol{\infty}$, dimensions $n, N \in \mathbb{N}$ with $n \geq 2$ and a non-empty bounded open set Ω in $\mathbb{R}^n$.

2.1 The autonomous case

Here, we are concerned with autonomous systems

$$\operatorname{div} a(Du) = 0 \qquad \text{on } \Omega \tag{2.1}$$

and autonomous integrals

$$F[u] := \int_\Omega f(Du)\,dx. \tag{2.2}$$

We start specifying our notion of regular problems.

Definition 2.1 *Let $m \in \mathbb{N}$. We say that a function $b : \mathbb{R}^m \to \mathbb{R}^m$ is* **regular** *if b is of class C^1 on $\mathbb{R}^m$ and satisfies the ellipticity and growth conditions*

$$\begin{gathered} Db(z)\xi \cdot \xi \geq \gamma(1+|z|)^{p-2}|\xi|^2 \\ |Db(z)| \leq \Gamma(1+|z|)^{p-2} \end{gathered}$$

for all $z, \xi \in \mathbb{R}^m$ and some positive constants γ and Γ. Similarly, we say that an integrand $g : \mathbb{R}^m \to \mathbb{R}$ is **regular** *if g is of class C^2 on $\mathbb{R}^m$ and satisfies the convexity and growth conditions*

$$\begin{gathered} D^2g(z)(\xi,\xi) \geq \gamma(1+|z|)^{p-2}|\xi|^2 \\ |D^2g(z)| \leq \Gamma(1+|z|)^{p-2} \end{gathered}$$

for all $z, \xi \in \mathbb{R}^m$ and some positive constants γ and Γ.

Remark 2.2 *In particular, regular functions b and g satisfy the coercivity and growth conditions*

$$b(z) \cdot z \geq l|z|^p - C, \tag{2.3}$$
$$|b(z)| \leq L(1+|z|)^{p-1} \tag{2.4}$$

and

$$l|z|^p - C \leq g(z) \leq L(1+|z|)^p, \tag{2.5}$$

respectively, for all $z \in \mathbb{R}^m$ with constants $L \geq l > 0$ and $C \in \mathbb{R}$.

With this terminology we state our main result for systems, which we will prove in Section 6.

Theorem 2.3 (Asymptotically elliptic systems) *We suppose that $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is Borel-measurable and locally bounded. Moreover, we assume that there exists a regular $b : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ such that*

$$\lim_{|z|\to\infty} \frac{|a(z)-b(z)|}{|z|^{p-1}} = 0 \tag{2.6}$$

Then every weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of (2.1) satisfies

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \textit{for all } q < p^\#,$$

where we have set

$$p^{\#} := \begin{cases} \frac{np}{n-2} + \kappa & \text{if } n \geq 3, N \geq 2 \\ \infty & \text{if } n = 2 \text{ or } N = 1 \end{cases} \tag{2.7}$$

with some constant $\kappa > 0$ depending only on n, p and $\frac{\Gamma}{\gamma}$. Moreover, for every $q < p^{\#}$ and for every cube Q with $4Q \subset \Omega$ we have the estimate

$$\fint_Q |Du|^q \, dx \leq C \left(1 + \fint_{4Q} |Du|^p \, dx\right)^{\frac{q}{p}}, \tag{2.8}$$

where C depends only on n, p and q, on the constants γ and Γ from Definition 2.1 and on $|a-b|$. More precisely, it depends only on an upper bound for $|a-b|$ determined by a function ω as in Theorem 2.21.

Here, we call $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ a weak solution of (2.1) iff

$$\int_\Omega a(Du) \cdot D\varphi \, dx = 0 \tag{2.9}$$

holds for all $\varphi \in C^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$. We stress that (2.3), (2.4) and (2.6) together with the local boundedness of a imply the coercivity and growth conditions

$$a(z) \cdot z \geq l|z|^p - C, \tag{2.10}$$
$$|a(z)| \leq L(1 + |z|)^{p-1} \tag{2.11}$$

for a, possibly with different constants. Thus, the integral in (2.9) is well-defined and finite for all $\varphi \in W^{1,p}(\Omega, \mathbb{R}^N)$ and (2.9) holds for all $\varphi \in W^{1,p}_0(\Omega, \mathbb{R}^N)$.

Next, we present our main result for integrals. For the proof we refer to Section 6 once more.

Theorem 2.4 (Asymptotically convex integrals) *We suppose that $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is a locally bounded Borel integrand. Moreover, we assume that there exists a regular $g : \mathbb{R}^{Nn} \to \mathbb{R}$ such that*

$$\lim_{|z| \to \infty} \frac{|f(z) - g(z)|}{|z|^p} = 0.$$

Then every minimizer $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of (2.2) satisfies

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \text{for all } q < p^{\#}$$

and the estimate (2.8), where $p^{\#}$ is defined in (2.7).

Here, we say that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a minimizer of (2.2) provided

$$F[u] \leq F[u + \varphi]$$

holds for all $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$. We record that our assumptions imply

$$l|z|^p - C \leq f(z) \leq L(1+|z|)^p \tag{2.12}$$

and thus $F[u]$ is well-defined and finite for all $u \in W^{1,p}(\Omega, \mathbb{R}^N)$.

Next, let us discuss the existence of weak solutions and minimizers. Proving the existence of weak solutions of (2.1) with given boundary values one usually requires a to be monotone; see [58, Chapter 26]. However, in Theorem 2.3 we have assumed monotonicity only near infinity and thus weak solutions need not exist in the full generality of our setting. Nevertheless, Theorem 2.3 covers some interesting cases, where also an existence theorem is available, such as systems with degenerate monotonicity or quasimonotone systems; see [59] for an existence theorem in the latter case.

The situation is similar in the case of integrals (2.2), where convexity is the classical assumption in proving the existence of minimizers in a given Dirichlet class. However, in the variational case several existence results for non-convex integrals have been established: Clearly, the most important ones deal with the quasiconvex case (see for instance [48,1] and [14, Chapter 8]), but there are also some results for non-quasiconvex integrals (see [14, Chapter 11] and the references given there). These existence theorems supply a number of applications for Theorem 2.4. Nevertheless, minimizers need not exist, in general, as can be seen already in the simple case $n = 2$, $N = 1$, $p = 4$ with zero boundary values, considering the integrand

$$f(z_1, z_2) = (z_1^2 - 1)^2 + z_2^4;$$

see [14, Example 11.28]. This non-existence result motivates us to provide extensions of Theorem 2.4 to different kinds of generalized minimizers — which always exist. Namely, we will discuss the validity of our result for minimizing sequences, Young measure minimizers, weak cluster points and relaxed minimizers.

For the remainder of this section we fix a Dirichlet class $\mathcal{D} := u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ with $u_0 \in W^{1,p}(\Omega, \mathbb{R}^N)$ and we consider F as in (2.2), where $f : \mathbb{R}^{Nn} \to \mathbb{R}$ will always be assumed to be Borel measurable and to fulfill (2.12). Then $\inf_{\mathcal{D}} F$ is finite and we may introduce the following notions of generalized minimizers:

2.1.1 Minimizing sequences

Definition 2.5 (Minimizing sequence) *We say that a sequence $(u_k)_{k\in\mathbb{N}}$ of functions in $\mathcal{D}$ is a* **minimizing sequence** *for F iff $F[u_k]$ converges to $\inf_{\mathcal{D}} F$ when k approaches ∞.*

Clearly, there is always a minimizing sequence for F in $\mathcal{D}$, and we have the following equi-integrability result:

Theorem 2.6 *In addition to the hypotheses of Theorem 2.4 we suppose that $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is lower semicontinuous, and consider a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ for F in $\mathcal{D}$. Then we can find another minimizing sequence $(v_k)_{k\in\mathbb{N}}$ for F in $\mathcal{D}$ such that $u_k - v_k$ converges to 0 strongly in $W_0^{1,p}(\Omega, \mathbb{R}^N)$ and such that for every $q < p^\#$ and every cube Q with $4Q \subset \Omega$ we have*

$$⨍_Q |Dv_k|^q \, dx \le C \left(1 + ⨍_{4Q} |Dv_k|^p \, dx \right)^{\frac{q}{p}}, \tag{2.13}$$

where C depends only on n, p, q, γ, Γ and on $|f-g|$, **but is independent of $\boldsymbol{k}$**. *In particular, the sequence $(Dv_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,q}(K, \mathbb{R}^{Nn})$ for every open set K with $K \subset\subset \Omega$.*

The proof of the above theorem is contained in Section 7.

2.1.2 Young measure minimizers

A fruitful idea in the calculus of variations, overcoming the possible lack of minimizers in $\mathcal{D}$, is to search for a minimizer in a larger class, namely among the gradient p-Young measures; see for instance [7,50,36] for definitions, notation and general properties of Young measures.

Definition 2.7 (Young measure minimizer) *We write $\mathcal{Y}$ for the family of all gradient p-Young measures generated by sequences in $\mathcal{D}$. For $\nu = \int_\Omega \delta_x \otimes \nu_x \, dx \in \mathcal{Y}$ we let*

$$F[\nu] := \int_{\Omega\times\mathbb{R}^{Nn}} f(X) \, d\nu(x, X) = \int_\Omega \int_{\mathbb{R}^{Nn}} f \, d\nu_x \, dx$$

and we call ν a **Young measure minimizer** *of F iff*

$$F[\nu] \le F[u] \qquad \text{holds for all } u \in \mathcal{D}.$$

With this definition, [36, Theorem 2.4] implies that every minimizing sequence for F in $\mathcal{D}$ generates a Young measure minimizer of F in $\mathcal{Y}$ and, in particular, that there always exists a Young measure minimizer of F in $\mathcal{Y}$. Moreover, taking into account [36, Corollary 1.8 and Theorem 2.4], we note:

Remark 2.8 *For every Young measure minimizer $\nu \in \mathcal{Y}$ of F we even have*

$$F[\nu] = \inf_{\mathcal{D}} F = \min_{\mathcal{Y}} F.$$

Following [15] we note that Theorem 2.6 implies a higher integrability result for Young measure minimizers:

Corollary 2.9 *Under the assumptions of Theorem 2.6, we consider a Young measure minimizer $\nu = \int_\Omega \delta_x \otimes \nu_x \, dx \in \mathcal{Y}$ of F. Then for every $q < p^\#$ and every cube Q with $4Q \subset \Omega$ we have*

$$\fint_Q \int_{\mathbb{R}^{Nn}} |\cdot|^q \, d\nu_x \, dx \leq C \left(1 + \fint_{4Q} \int_{\mathbb{R}^{Nn}} |\cdot|^p \, d\nu_x \, dx\right)^{\frac{q}{p}},$$

where C is the constant from Theorem 2.6. In particular, ν has a finite q-th moment on $K \times \mathbb{R}^{Nn}$ for every set $K \subset\subset M$.

Proof. We choose a generating sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$ for ν. Via Theorem 2.6 we find another generating sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$ for ν for which (2.13) holds. In particular, (2.13) implies that $(v_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,q}$ and thus $(|Dv_k|^p)_{k\in\mathbb{N}}$ is equi-integrable away from the boundary of Ω. Consequently, applying [36, Theorem 2.4] twice we get

$$\begin{aligned}\fint_Q \int_{\mathbb{R}^{Nn}} |\cdot|^q \, d\nu_x \, dx &\leq \liminf_{k\to\infty} \fint_Q |Dv_k|^q \, dx \\ &\leq C \left(1 + \lim_{k\to\infty} \fint_{4Q} |Dv_k|^p \, dx\right)^{\frac{q}{p}} = C \left(1 + \fint_{4Q} \int_{\mathbb{R}^{Nn}} |\cdot|^p \, d\nu_x \, dx\right)^{\frac{q}{p}},\end{aligned}$$

for all cubes Q with $4Q \subset\subset \Omega$. However, since C is independent of Q, the resulting inequality still holds if we merely require $4Q \subset \Omega$ as claimed. $\square$

2.1.3 Weak cluster points

By (2.12) minimizing sequences always possess a weak cluster point, and from Theorem 2.6 we deduce a corresponding regularity result:

Corollary 2.10 *Under the assumptions of Theorem 2.6, we consider a cluster point u — with respect to the weak $W^{1,p}$-topology — of a minimizing sequence for F in $\mathcal{D}$. Then we have*

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \text{for all } q < p^\#.$$

Let us add a brief comment on the relationship between Corollary 2.9 and Corollary 2.10: Starting from a weak cluster point u as in Corollary 2.10, we find a minimizing sequence in $\mathcal{D}$, weakly convergent to u, which generates a Young measure minimizer $\nu = \int_\Omega \delta_x \otimes \nu_x \, dx \in \mathcal{Y}$. Then we have $Du(x) = \int_{\mathbb{R}^{Nn}} X \, d\nu_x(X)$ for a.e. $x \in \Omega$ and consequently, Jensen's inequality gives $|Du(x)|^q \leq \int_{\mathbb{R}^{Nn}} |\cdot|^q \, d\nu_x$ for a.e. $x \in \Omega$. Thus, Corollary 2.10 may also be deduced from Corollary 2.9, together with the additional estimate

$$\fint_Q |Du|^q \, dx \leq C \left(1 + \fint_{4Q} \int_{\mathbb{R}^{Nn}} | \cdot |^p \, d\nu_x \, dx \right)^{\frac{q}{p}}$$

for every cube Q with $4Q \subset \Omega$ and all $q < p^{\#}$.

2.1.4 Quasiconvexity and Strong Local Minimizers

Now we point out applications of Theorem 2.3 and Theorem 2.4 in the theory of quasiconvex integrals.

We start by recalling that $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is said to be quasiconvex (in the sense of Morrey) iff

$$\fint_{B_1} f(z + D\varphi) \, dx \geq f(z)$$

holds for all $z \in \mathbb{R}^{Nn}$ and $\varphi \in W_0^{1,p}(B_1, \mathbb{R}^N)$. Quasiconvexity is a generalization of convexity and it was first observed by Morrey [48] (see also [1]) that quasiconvexity of f is equivalent to sequential weak lower semicontinuity of F and thus implies the existence of a minimizer in $\mathcal{D}$; see [14, Chapter 8] for further information on quasiconvexity, weak lower semicontinuity and existence theorems in the calculus of variations. Moreover, quasiconvexity and the related notions of polyconvexity and rank-one convexity are also crucial in nonlinear elasticity as recognized by Ball [6]. Additionally, inspired by previous results in the setting of geometric measure theory, Evans [20] (see also [28,33,2,35]) demonstrated that quasiconvexity is an appropriate notion for proving the partial regularity of minimizers; that is, regularity outside a negligible set. While it is known from a celebrated example of Müller & Sverak [49] that partial regularity fails for mere solutions of the Euler equation, Kristensen & Taheri [42, Theorem 4.1] have shown that Evans' result extends to an intermediate case, namely to strong local minimizers; that is, local minima of F with respect to the $W^{1,q}$-topology for some $p < q < \infty$. However, they needed to assume that the minimizer is a priori in $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, a condition which need not even be satisfied for absolute minimizers. Actually, it even seems difficult to obtain any everywhere regularity results for general quasiconvex integrals [1]. Thus, a twofold interest arises in the question whether higher integrability results can be obtained at least in some particular situations: On the one hand, such results for absolute minimizers are of interest in themselves; on the other hand, results for weak solutions of systems cover the case of strong local minimizers, and may be used to verify the assumptions of the partial regularity theorem [42, Theorem 4.1].

[1] Indeed, the only results for quasiconvex integrals exceeding almost-everywhere regularity are Gehring's improvement and the dimension reduction in [39].

Clearly, the results of [12,34,15] mentioned in the introduction apply to absolute minimizers provided f is of special structure near infinity, and analogous results for systems have been established in [42, Proposition 5.1]. With Theorem 2.3 and Theorem 2.4 we extend all these results covering a much broader class of integrands f. For instance we include such genuine examples as

$$f(z) := g(z) + |\det z|^2$$

with a regular g and $p > 2n = 2N$. In exchange, we pay for this generality with the restriction $q < p^{\#}$.

2.1.5 Relaxation

Next, we discuss another approach of defining generalized minimizers. If f fails to be quasiconvex, one may consider its quasiconvex envelope Qf defined by

$$Qf(z) := \sup \Big\{ g(z) \,:\, g \text{ is quasiconvex with } g \le f \text{ on } \mathbb{R}^{Nn} \Big\}.$$

Clearly, Qf is quasiconvex and thus a minimizer of

$$QF[u] := \int_\Omega Qf(Du)\,dx$$

in $\mathcal{D}$ always exists. Moreover, by Dacorogna's relaxation theorem [13,1] we have

$$QF[u] = \inf \left\{ \liminf_{k\to\infty} F[u_k] \,:\, u_k - u \underset{k\to\infty}{\rightharpoonup} 0 \text{ weakly in } W_0^{1,p}(\Omega, \mathbb{R}^N) \right\} \tag{2.14}$$

for all $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and thus

$$\inf_{\mathcal{D}} F = \min_{\mathcal{D}} QF \tag{2.15}$$

holds. In particular, if a minimizer of F exists, it is also a minimizer of QF. Hence, it is reasonable to introduce the following terminology:

Definition 2.11 (Relaxed Minimizer) *We say that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a* **relaxed minimizer** *of F iff it is a minimizer of QF.*

The following simple lemma, proved in Appendix A, implies that Theorem 2.4 holds for relaxed minimizers.

Lemma 2.12 *Under the assumptions of Theorem 2.4 we have*

$$\lim_{|z|\to\infty} \frac{|f(z) - Cf(z)|}{|z|^p} = 0,$$

where Cf denotes the convex envelope of f.

Consequently, if f satisfies the assumptions of Theorem 2.4, then also Qf satisfies the same assumptions and we get:

Corollary 2.13 *Suppose that f is as in Theorem 2.4. Then every relaxed minimizer $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of F satisfies*

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \textit{for all } q < p^{\#},$$

and the estimate (2.8)*, where $p^{\#}$ is defined in* (2.7)*.*

Finally, we stress that starting from (2.14), (2.15) and (2.12) one can show that the relaxed minimizers of F are exactly the weak cluster points of minimizing sequences for F. Hence, Corollary 2.10 and Corollary 2.13 are in fact equivalent.

2.2 A characterization of asymptotic regularity

We say that a function $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is asymptotically regular iff it satisfies the assumptions of Theorem 2.3, and that an integrand $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is asymptotically regular iff it satisfies the assumptions of Theorem 2.4. In the following twin theorems we provide the characterization of asymptotic regularity that we have announced in the introduction. The theorems show that asymptotic regularity is in fact equivalent to certain weakened forms of the usual monotonicity, convexity and growth conditions. We will state them for functions $a : \mathbb{R}^m \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ with an arbitrary $m \in \mathbb{N}$. The proofs will be given in Section 9.

Theorem 2.14 *We consider a measurable function $a : \mathbb{R}^m \to \mathbb{R}^m$. Then the following assertions are* **equivalent***:*

(i) There exists a regular function $b : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\lim_{|z|\to\infty} \frac{|b(z) - a(z)|}{|z|^{p-1}} = 0.$$

(ii) There exist positive constants γ, Γ and M_0 and a function $\omega : [M_0, \infty) \to [0, \infty)$ with

$$\lim_{t\to\infty} \omega(t) = 0$$

such that the inequalities

$$(a(z_2) - a(z_1)) \cdot (z_2 - z_1) \geq \gamma(|z_1| + |z_2|)^{p-2}|z_2 - z_1|^2 - \Omega(|z_1|, |z_2|)|z_2 - z_1|,$$

$$|a(z_2) - a(z_1)| \leq \Gamma(|z_1| + |z_2|)^{p-2}|z_2 - z_1| + \Omega(|z_1|, |z_2|)$$

hold for all $z_1, z_2 \in \mathbb{R}^m$ with $|z_1| \geq M_0$ and $|z_2| \geq M_0$. Here, we used the abbreviation $\Omega(s, t) := \omega(\min\{s, t\})(s + t)^{p-1}$.

(iii) *There exist positive constants* γ, Γ *and* M_0 *and a function* $\varphi : [M_0, \infty) \to (0, \infty)$ *with*

$$\lim_{t\to\infty} \varphi(t) = 0$$

such that the following holds: Whenever the conditions

$$|z_1| \geq t, \quad |z_2| \geq t \quad \text{and} \quad |z_2 - z_1| \geq \varphi(t)(|z_1| + |z_2|) \tag{2.16}$$

are satisfied for some $t \geq M_0$ *and* $z_1, z_2 \in \mathbb{R}^m$, *we have*

$$\begin{gathered}(a(z_2) - a(z_1)) \cdot (z_2 - z_1) \geq \gamma(|z_1| + |z_2|)^{p-2}|z_2 - z_1|^2,\\ |a(z_2) - a(z_1)| \leq \Gamma(|z_1| + |z_2|)^{p-2}|z_2 - z_1|.\end{gathered}$$

Remark 2.15 *All three conditions imply the coercivity and growth conditions* (2.10) *and* (2.11) *for* $|z| \gg 1$ *and some constants* $L \geq l > 0$ *and* $C \in \mathbb{R}$. *Clearly, if* a *is locally bounded these conditions hold for all* $z \in \mathbb{R}^m$.

Theorem 2.16 *We consider a measurable and locally bounded function* $f : \mathbb{R}^m \to \mathbb{R}$. *Then the following assertions are* **equivalent***:*

(i) *There exists a regular integrand* $g : \mathbb{R}^m \to \mathbb{R}$ *such that we have*

$$\lim_{|z|\to\infty} \frac{|g(z) - f(z)|}{|z|^p} = 0.$$

(ii) *There exist positive constants* γ, Γ *and* M_0 *and a function* $\omega : [M_0, \infty) \to [0, \infty)$ *with*

$$\lim_{t\to\infty} \omega(t) = 0$$

such that for all $\lambda \in [0, 1]$, $z_1, z_2 \in \mathbb{R}^m$ *and* $z := \lambda z_1 + (1 - \lambda)z_2$ *with* $|z| \geq M_0$ *the following inequalities hold:*

$$\begin{aligned}\lambda f(z_1) + (1 - \lambda)f(z_2) \geq f(z) &+ \gamma(|z_1| + |z_2|)^{p-2}\lambda(1 - \lambda)|z_2 - z_1|^2\\ &- \omega(|z|)(|z_1| + |z_2|)^{p-2}|z|^2,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\lambda f(z_1) + (1 - \lambda)f(z_2) \leq f(z) &+ \Gamma(|z_1| + |z_2|)^{p-2}\lambda(1 - \lambda)|z_2 - z_1|^2\\ &+ \omega(|z|)(|z_1| + |z_2|)^{p-2}|z|^2.\end{aligned} \tag{2.18}$$

(iii) *There exist positive constants* γ, Γ *and* M_0 *and a function* $\varphi : [M_0, \infty) \to (0, \infty)$ *with*

$$\lim_{t\to\infty} \varphi(t) = 0$$

such that the following holds: Whenever the conditions

$$|z| \geq t \quad \text{and} \quad \lambda(1 - \lambda)|z_2 - z_1|^2 \geq \varphi(t)|z|^2 \tag{2.19}$$

are satisfied for some $t \geq M_0$, $\lambda \in [0, 1]$ *and* $z_1, z_2 \in \mathbb{R}^m$ *with* $z := \lambda z_1 + (1 - \lambda)z_2$, *we have*

$$\lambda f(z_1) + (1 - \lambda)f(z_2) \geq f(z) + \gamma(|z_1| + |z_2|)^{p-2}\lambda(1 - \lambda)|z_2 - z_1|^2 \tag{2.20}$$

and

$$\lambda f(z_1) + (1-\lambda)f(z_2) \leq f(z) + \Gamma(|z_1| + |z_2|)^{p-2}\lambda(1-\lambda)|z_2 - z_1|^2. \quad (2.21)$$

Remark 2.17 *We note* $\lambda(1-\lambda)|z_2 - z_1|^2 = |z_2 - z||z_1 - z|$.

Remark 2.18 *Keeping Remark 2.17 and the local boundedness of f in mind, all three conditions can be shown to imply the coercivity and growth conditions* (2.12) *for all $z \in \mathbb{R}^m$ and some constants $L \geq l > 0$ and $C \in \mathbb{R}$.*

2.3 The general case

Finally, we provide more general versions of Theorem 2.3, Theorem 2.4 and Theorem 2.6, which cover the general cases (1.1) and (1.2) from the very beginning. However, an essential difference to Section 2.1 is that we now simply postulate certain regularity results for the comparison problems; see the end of the section for a discussion of this hypothesis.

We start with a technical definition.

Definition 2.19 (Admissibility) *We write $\mathcal{L}$ for the σ-algebra of measurable subsets of Ω, $\mathcal{B}$ for the σ-algebra of Borel subsets of $\mathbb{R}^N \times \mathbb{R}^{Nn}$ and $\mathcal{L} \otimes \mathcal{B}$ for their product σ-algebra. We say that $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is an* **admissible structure function** *iff a is $\mathcal{L} \otimes \mathcal{B}$-measurable. Similarly, we call $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ an* **admissible integrand** *iff f is $\mathcal{L} \otimes \mathcal{B}$-measurable.*

Remark 2.20 *Admissibility ensures that the compositions $a(\cdot, u, Du) : \Omega \to \mathbb{R}^{Nn}$ and $f(\cdot, u, Du) : \Omega \to \mathbb{R}$, respectively, are still measurable for every weakly differentiable function $u : \Omega \to \mathbb{R}^N$. Moreover, it should be noted that a function is admissible if and only if it coincides with a Borel function outside $E \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ for some negligible set $E \subset \Omega$; see the discussion after Definition 5.5 and Exercise 5.4 in [5].*

Next, we present the generalization of Theorem 2.3 to systems of type (1.1). We refer to Section 6 for the proof.

Theorem 2.21 *We suppose that $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ and $b : \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ are admissible structure functions such that we have*

$$|a(x, y, z) - b(x, z)| \leq \omega(|z|)(1 + |z|)^{p-1} \quad (2.22)$$

for all $x \in \Omega$, $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^{Nn}$ and some bounded function $\omega : [0, \infty) \to \mathbb{R}$ with

$$\lim_{t \to \infty} \omega(t) = 0.$$

Moreover, we require b to be C^1 in its last argument and we impose the growth and ellipticity conditions

$$|b(x,z)| \leq \Psi(x) + L|z|^{p-1}, \tag{2.23}$$
$$D_z b(x,z)(\xi,\xi) \geq \gamma(1+|z|)^{p-2}|\xi|^2 \tag{2.24}$$

for some positive constants γ and L, some function $0 < \Psi \in L^{\frac{p}{p-1}}(\Omega)$ and for all $x \in \Omega$ and $z, \xi \in \mathbb{R}^{Nn}$. Finally, we assume that every weak solution $v \in W^{1,p}(\Omega, \mathbb{R}^N)$ of

$$\operatorname{div} b(\cdot, Dv) = 0 \qquad \text{on } \Omega \tag{2.25}$$

satisfies $v \in W^{1,q^\#}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and

$$\fint_Q |Dv|^{q^\#}\,dx \leq H\left(1 + \fint_{2Q} |Dv|^p\,dx\right)^{\frac{q^\#}{p}}$$

on all cubes Q with $2Q \subset \Omega$, with some positive constant H and some exponent $p < q^\# < \infty$. Then every weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of

$$\operatorname{div} a(\cdot, u, Du) = 0 \qquad \text{on } \Omega \tag{2.26}$$

satisfies $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with

$$\fint_Q |Du|^q\,dx \leq C\left(1 + \fint_{4Q} |Du|^p\,dx\right)^{\frac{q}{p}},$$

for all $q < q^\#$ and all cubes Q with $4Q \subset \Omega$. Here, the constant C depends only on $n, p, q, q^\#, H, \gamma$ and ω.

Here, the notion of a weak solution is defined analogously to (2.9) by testing with C^∞_{cpt}-functions; clearly, keeping (2.23) in mind, one may also test (2.25) with $W^{1,p}_0$-functions. Moreover, we record that (2.23) and (2.24) imply the coercivity condition

$$b(x,z) \cdot z \geq l|z|^p - C\Psi(x)^{\frac{p}{p-1}} \tag{2.27}$$

with constants $l > 0$ and $C \in \mathbb{R}$. In addition, taking into account (2.22), the conditions (2.23) and (2.27) can be carried over to a, possibly with different constants, and we deduce that (2.26) may also be tested with $W^{1,p}_0$-functions.

Next, generalizing Theorem 2.4 and Theorem 2.6, we turn our attention to integrals of type (1.2). We begin with the generalization of Theorem 2.4, which we will establish in Section 6.

Theorem 2.22 *We suppose that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}$ are admissible integrands such that we have*

$$|f(x,y,z) - g(x,z)| \leq \omega(|z|)(1+|z|)^p \tag{2.28}$$

for all $x \in \Omega$, $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^{Nn}$ and some bounded function $\omega : [0,\infty) \to \mathbb{R}$ with

$$\lim_{t\to\infty} \omega(t) = 0.$$

Moreover, we require g to be C^2 in its last argument and we impose the coercivity and convexity conditions

$$g(x,z) \geq l|z|^p - \Psi(x) \tag{2.29}$$

$$D_z^2 g(x,z)(\xi,\xi) \geq \gamma(1+|z|)^{p-2}|\xi|^2 \tag{2.30}$$

for some positive constants l and γ, some function $0 < \Psi \in L^1(\Omega)$ and for all $x \in \Omega$ and $z,\xi \in \mathbb{R}^{Nn}$. Finally, we assume that every minimizer $v \in W^{1,p}(\Omega,\mathbb{R}^N)$ of

$$G[v] := \int_\Omega g(\cdot, Dv)\,dx \tag{2.31}$$

satisfies $v \in W^{1,q^\#}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and

$$\fint_Q |Dv|^{q^\#}\,dx \leq H\left(1 + \fint_{2Q} |Dv|^p\,dx\right)^{\frac{q^\#}{p}},$$

on all cubes Q with $2Q \subset \Omega$, with some positive constant H and some exponent $p < q^\# < \infty$. Then every minimizer $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ of

$$F[u] := \int_\Omega f(\cdot, u, Du)\,dx$$

satisfies

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$$

and (2.8) for all $q < q^\#$.

Here, we have called $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ a minimizer of F iff we have $F[u] < \infty$ and $F[u] \leq F[u+\varphi]$ for all $\varphi \in W^{1,p}_0(\Omega,\mathbb{R}^N)$. Clearly, a minimizer of G is defined analogously. The reader should note that this definition is consistent with the one given in Section 2.1, where — recalling the growth condition in (2.12) — the requirement $F[u] < \infty$ was trivially satisfied. However, in Theorem 2.22 we have not imposed any upper bounds for the growth of f or g.

In contrast, we have assumed the lower bound (2.29) for g and invoking (2.28) we get the analogous lower bound

$$f(x,y,z) \geq \frac{l}{2}|z|^p - \Psi(x) - C \tag{2.32}$$

for f, with some $C \in \mathbb{R}$. In particular, the functionals F and G are bounded from below and for every Dirichlet class $\mathcal{D}$ we have $\inf_{\mathcal{D}} F > -\infty$.

Finally, assuming that the Dirichlet class $\mathcal{D}$ satisfies $\inf_{\mathcal{D}} F < \infty$, let us briefly discuss an extension to minimizing sequences.

Definition 2.23 (Normal integrand) *We say that an admissible integrand* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ *is a* **normal integrand** *iff for a.e.* $x \in \Omega$ *the function* $f(x, \cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ *is lower semicontinuous.*

Theorem 2.24 *In addition to the hypotheses of Theorem 2.22 we require* f *to be a normal integrand. Then starting from a minimizing sequence (in the sense of Definition 2.5)* $(u_k)_{k\in\mathbb{N}}$ *for* F *in* $\mathcal{D}$*, we can find another minimizing sequence* $(v_k)_{k\in\mathbb{N}}$ *for* F *in* $\mathcal{D}$ *such that* $u_k - v_k$ *converges to* 0 *strongly in* $W_0^{1,p}(\Omega, \mathbb{R}^N)$ *and such that we have* (2.13) *for every* $q < q^{\#}$ *with* C *independent of* k.

For the proof see Section 7.

Clearly, most remarks we have made in Section 2.1 apply also in the current situation. In particular, starting from Theorem 2.24, higher integrability can be carried over to the various kinds of generalized minimizers. Instead of elaborating on further details, we rather discuss the crucial hypothesis of this section:

On the regularity theory for the comparison problems: In contrast to Section 2.1 we have now simply assumed higher integrability properties of solutions v. This assumption is justified by the fact that such results are available in several particular situations. Let us record only some of them without entering into the details of the corresponding estimates:

- For functions b or g of special structure, e.g. a linear/quadratic or p-Laplacean one, adequate integrability results are available (see e.g. [55,34,54,29]) and we regain most of the results mentioned in the introduction, but also more general results allowing some x- and y-dependence.
- For $N = 1$ or $n = 2$, imposing natural growth and continuity conditions on b or g, we have $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ (cf. [31,34,46,35] for $N = 1$ and [38, Section 9] for $n = 2$) and consequently the theorems give $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for all $q < \infty$.
- Considering — in view of Section 4 — regular functions b and g which are independent of x, we recover the results of Section 2.1.
- More generally, if b or Dg is Hölder continuous in x with exponent $\alpha < 1$ and satisfies some additional growth conditions, then by [47, Proposition 3.1] and the fractional Sobolev embedding we have

  $$v \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \text{for all } q < \frac{np}{n - 2\alpha}$$

  and the theorems guarantee the same degree of integrability for u.

Finally, let us mention that one may think about considering even more general comparison problems; for instance, one might weaken the smoothness assumptions on b and g or allow an additional dependence on y. However, we believe that the treatment of such generalizations would result in further technicalities and lies beyond the scope of the present paper.

3 Preliminaries

Notation.

Constants. We use the notations c and C for positive constants, possibly varying from line to line. The dependences of such constants will only occasionally be highlighted. Anyway, we widely follow the convention that large constants will be denoted by capital letters, and small constants by lowercase letters.

Balls and spheres. By $B_r(x)$ we denote the open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$. Similarly, we write $S_r(x)$ for the $(n-1)$-dimensional sphere with center $x \in \mathbb{R}^n$ and radius r. Here, the centers will be omitted if they are 0. In addition, the volume of the unit ball B_1 will be abbreviated by ω_n and if B denotes a ball we will occasionally write $2B$ for the ball with the same center and twice the radius.

Cubes. In the following a cube will always denote an open cube in $\mathbb{R}^n$ with edges parallel to the axes; more precisely, the cube with edges of length $l > 0$ and center $x \in \mathbb{R}^n$ is the set $x + \left]-\frac{l}{2}, \frac{l}{2}\right[^n$. If Q is a cube with edges of length l, we write rQ for the cube with the same center and edges of length rl. Finally, by a subcube of Q we simply mean another cube which is contained in Q.

Mean values. We use the common notations f_A and $⨍_A f\,dx$ for the mean value $\frac{1}{|A|}\int_A f\,dx$ of f on A, where $|A|$ is the Lebesgue measure of A. In particular, in the case of balls we abbreviate $f_{x,r} := f_{B_r(x)}$ and $f_r := f_{0,r}$.

Function spaces. As usual we write L^p, $W^{k,p}$ and $C^{k,\alpha}$ for Lebesgue, Sobolev and Hölder spaces, respectively. Moreover, for $\lambda \geq 0$ we write $L^{p,\lambda}(\Omega, \mathbb{R}^N)$ for the Morrey space consisting of all functions $u \in L^p(\Omega, \mathbb{R}^N)$ with

$$\sup_{\substack{x\in\Omega \\ 0<\rho<\operatorname{diam}\Omega}} \rho^{-\lambda} \int_{B_\rho(x)\cap\Omega} |u|^p\,dx < \infty.$$

Finally, we introduce localized function spaces: Let $\mathcal{F} \in \{L^p, W^{k,p}, C^{k,\alpha}, L^{p,\lambda}\}$. Then, we write $\mathcal{F}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for the space of all functions $u : \Omega \to \mathbb{R}^N$ with $u \in \mathcal{F}(K, \mathbb{R}^N)$ for every open set K with $\varnothing \neq K \subset\subset \Omega$.

The function V. For $z \in \mathbb{R}^{Nn}$ we let

$$V(z) := (1 + |z|^2)^{\frac{p-2}{4}} z. \tag{3.1}$$

Some inequalities.
We recall the standard inequality (see for instance [33, Lemma 2.1])

$$\int_0^1 (1+|z_0+s(z-z_0)|)^{p-2}\,ds \ge c(|z_0|+|z|)^{p-2} \ge c|z-z_0|^{p-2} \tag{3.2}$$

for all $z_0, z \in \mathbb{R}^{Nn}$ with some positive constant c depending only on p. In particular, we deduce

$$\int_0^1 \int_0^1 (1+|z_0+t\xi+stz|)^{p-2}\,ds\,t\,dt \ge c\int_0^1 |tz|^{p-2}t\,dt = \frac{c}{p}|z|^{p-2}. \tag{3.3}$$

for all $z_0, z, \xi \in \mathbb{R}^{Nn}$.

The maximal function.
Denote by Q a cube in $\mathbb{R}^n$. We introduce the Hardy-Littlewood maximal function restricted to Q:

$$M_Q\varphi(x) := \sup_{x\in W\subset Q} \fint_W |\varphi|\,dx \qquad \text{for } \varphi \in L^1(Q),$$

where the supremum ranges over all subcubes W of Q containing x. We recall that M is sublinear, in particular

$$M_Q(\varphi_1+\varphi_2) \le M_Q\varphi_1 + M_Q\varphi_2 \qquad \text{on } Q, \tag{3.4}$$

and bounded in the sense of

$$|Q\cap\{M_Q\varphi > \lambda\}| \le \frac{C}{\lambda^q}\int_Q |\varphi|^q\,dx \tag{3.5}$$

for all $q \in [1,\infty)$ and all $\lambda > 0$, where C depends only on n and q. In the case $q = 1$ the above inequality holds with $C = 4^n$.

Dyadic decomposition of cubes and Calderón-Zygmund coverings.
Next, we introduce some terminology concerning dyadic decompositions of cubes: For a cube $Q = y+]0,l[^n$ (with center $y+\frac{1}{2}(l,l,\ldots,l)$ this time) the cubes $y+2^{-k}lz+]0,2^{-k}l[^n$ with $k \in \mathbb{N}_0$ and $z \in \{0,1,2,\ldots,2^k-1\}^n$ are called the dyadic subcubes of Q. Moreover, for a dyadic subcube W of Q the predecessor of W is the smallest (with respect to inclusion) dyadic subcube of Q which strictly contains W. For $W \neq Q$ the predecessor of W exists and is unique and we will always denote it by W^*.

With this terminology we restate the following covering lemma, which will play a crucial role in our proofs:

Lemma 3.1 ([10, Lemma 1.2]) *We consider a cube Q in $\mathbb{R}^n$ and measurable sets $A \subset B \subset Q$ such that for some $\varsigma \in (0,1)$ we have*

$$|A| \le \varsigma|Q|$$

and the following property:

(P) For each dyadic subcube W of Q with $|A \cap W| > \varsigma |W|$ one has $W^* \subset B$.

Then, there holds $|A| \le \varsigma |B|$.

The proof of Lemma 3.1 is elementary. It is based on a Calderón-Zygmund covering technique, i.e. on a covering of A with certain disjoint dyadic subcubes of Q; see [10, Lemma 1.1] for details.

Ekeland's variational principle.
The main tool in proving our regularity theorems for minimizing sequences will be Ekeland's variational principle. We state a version of this principle, which can be found for instance in [19, Theorem 1.1] or [35, Theorem 5.6 and Remark 5.5]. It allows to pass from an almost-minimizer to a minimizer of some disturbed functional, which will be very convenient later.

Lemma 3.2 *We consider a functional $F : X \to (-\infty, \infty]$ on a complete metric space X and we assume that F is lower semicontinuous and bounded from below. Then, for every $\delta > 0$ and every $u \in X$ with*

$$F[u] \le \inf_X F + \delta$$

there is a $v \in X$ with $d(u, v) \le \sqrt{\delta}$ and $F[v] \le F[u]$ such that

$$F[v] \le F[w] + \sqrt{\delta}\, d(v, w)$$

holds for all $w \in X$.

4 Regular problems

In this section we collect some regularity results for regular problems. We start with a standard result concerning the existence of second derivatives:

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ denote either a ball with radius l or a cube with edges of length l, and consider a regular structure function $b : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$. Then, for every weak solution $v \in W^{1,p}(2\Omega, \mathbb{R}^N)$ of*

$$\operatorname{div} b(Dv) = 0 \qquad \text{on } 2\Omega \tag{4.1}$$

the function

$$\bar{V} := V(Dv)$$

with V from (3.1) satisfies

$$\bar{V} \in W^{1,2}(\Omega, \mathbb{R}^N)$$

and

$$\int_\Omega |D\bar V|^2\,dx \le C\int_{2\Omega\setminus\Omega}\left|\frac{\bar V-\xi}{l}\right|^2\,dx \tag{4.2}$$

for all $\xi\in\mathbb{R}^{Nn}$*, where* C *depends only on* p *and* $\frac{\Gamma}{\gamma}$*.*

To prove Theorem 4.1 one uses the difference quotient method to differentiate the system (4.1). This is usually done using balls (see [11]), but works in the same manner for cubes.

Once the Caccioppoli inequality (4.2) is obtained, Giaquinta & Modica's version [32] of Gehring's higher integrability lemma [30] gives $W^{1,2+\kappa}$-integrability for $\bar V$, where κ is some positive constant depending only on n, p and $\frac{\Gamma}{\gamma}$:

Theorem 4.2 ([11, Theorem 1.V]) *There is a positive number* $\kappa>0$ *depending only on* n*,* p *and* $\frac{\Gamma}{\gamma}$ *with the following property: Under the assumptions of Theorem 4.1 we have*

$$\bar V\in W^{1,2+\kappa}(\Omega,\mathbb{R}^N)$$

and

$$\fint_\Omega |D\bar V|^{2+\kappa}\,dx \le C\left(\fint_{2\Omega}\left|\frac{\bar V-\bar V_{2\Omega}}{l}\right|^2\,dx\right)^{\frac{2+\kappa}{2}},$$

where C *depends only on* n*,* p *and* $\frac{\Gamma}{\gamma}$*.*

The reader should note that one of the main features of Theorem 4.2 is that it implies Hölder continuity of Dv for $n=2$ and of v for $n\le p+2$, while Theorem 4.1 only gives Hölder continuity of v for $n<p+2$.

Recently, Kristensen & Melcher [37] have established a refined version of Theorem 4.2. Precisely, in the case $p=2$ they proved that κ can be chosen depending only on the dispersion ratio $\frac{\Gamma}{\gamma}$ — precisely $\kappa=\frac{1}{50}\frac{\gamma}{\Gamma}$ —, but independent of the dimension n.

Finally, in the scalar case $N=1$ the above results can be considerably strengthened and weak solutions are Lipschitzian [2]; see [31,34,46,35]. Combining this with the above results and Sobolev's embedding we get:

Corollary 4.3 *Under the assumption of Theorem 4.1 we have*

$$v\in W^{1,p^\#}(\Omega,\mathbb{R}^N)$$

and

$$\left(\fint_\Omega |Dv|^{p^\#}\,dx\right)^{\frac{1}{p^\#}} \le C\left(1+\fint_{2\Omega}|Dv|^p\,dx\right)^{\frac{1}{p}},$$

[2] In fact, weak solutions are even $C^{1,\alpha}_{\mathrm{loc}}$-regular, but this will not be relevant for the purposes of this paper.

where $p^{\#}$ is defined in (2.7)*, κ is the constant from Theorem 4.2 and C depends only on n, p and $\frac{\Gamma}{\gamma}$. Here, in the case $p^{\#}=\infty$ the left-hand side of* (2.7) *should be interpreted as* $\sup_\Omega |Dv|$.

Proof. We start with the case $n \geq 3$, $N \geq 2$: First we note $(2+\kappa)^* \geq \frac{2n}{n-2}+\frac{2}{p}\kappa$, where $(2+\kappa)^*$ denotes the Sobolev exponent of $2+\kappa$. Thus, from Theorem 4.2 and Sobolev's embedding we deduce

$$\fint_\Omega |\bar V - \bar V_\Omega|^{\frac{2n}{n-2}+\frac{2}{p}\kappa}\,dx \leq C\left(\fint_{2\Omega} |\bar V - \bar V_{2\Omega}|^2\,dx\right)^{\frac{n}{n-2}+\frac{\kappa}{p}}.$$

Clearly, this implies

$$\fint_\Omega |\bar V|^{\frac{2n}{n-2}+\frac{2}{p}\kappa}\,dx \leq C\left(|\bar V_\Omega|^2 + \fint_{2\Omega} |\bar V|^2\,dx\right)^{\frac{n}{n-2}+\frac{\kappa}{p}}.$$

Finally, recalling the definition of $\bar V$ we have $|Dv|^p \leq |\bar V|^2 \leq C(1+|Dv|^p)$ and $|\bar V_\Omega|^2 \leq C\,(1+\fint_{2\Omega}|Dv|^p\,dx)$; thus, we infer

$$\fint_\Omega |Dv|^{\frac{np}{n-2}+\kappa}\,dx \leq C\left(1 + \fint_{2\Omega} |Dv|^p\,dx\right)^{\frac{n}{n-2}+\frac{\kappa}{p}},$$

which proves the claim.

Replacing integrals by supremums, the case $n=2$ is analogous, but simpler. Finally, for the case $N=1$ we refer to [35, Theorem 8.2]. □

Remark 4.4 *In particular, taking into account the Euler equation*

$$\operatorname{div} Dg(Dv) = 0 \qquad \text{on } 2\Omega,$$

all the results of this section apply to minimizers v of regular integrals

$$G[v] := \int_{2\Omega} g(Dv)\,dx.$$

5 Comparison estimates

In this section we prove that solutions of asymptotically regular problems can be approximated, close to infinity, by solutions of regular problems.

Lemma 5.1 *Assume that structure functions a and b are given which satisfy the assumptions of Theorem 2.21. Then for every $\varepsilon > 0$, there is a constant $K(\varepsilon)$, depending only on p, γ, ω and ε with the following property: Whenever $u, v \in W^{1,p}(\Omega, \mathbb{R}^N)$ are weak solutions of the systems* (2.26) *and* (2.25), *respectively, with $u - v \in W_0^{1,p}(\Omega, \mathbb{R}^N)$, then the assumption*

$$\fint_\Omega |Du|^p \, dx > K^p(\varepsilon)$$

implies

$$\fint_\Omega |Du - Dv|^p \, dx \le \varepsilon \fint_\Omega |Du|^p \, dx.$$

Proof. We choose a constant $\varepsilon_1 \in (0, 1)$ to be fixed later and observe that by the assumption (2.22) we may choose a constant M_1 so large that

$$\sup_{x \in \Omega,\, y \in \mathbb{R}^N} |a(x, y, \xi) - b(x, \xi)| \le \varepsilon_1 (1 + |\xi|^{p-1}) \qquad \text{for } |\xi| \ge M_1. \tag{5.1}$$

We let $S_\varepsilon := \|\omega\|_{L^\infty}(1 + M_1^{p-1}) < \infty$. Next we observe that by the condition (2.24), we have the pointwise estimate

$$\begin{aligned}
&b(\cdot, Du) \cdot (Du - Dv) - b(\cdot, Dv) \cdot (Du - Dv) \\
&\quad = \int_0^1 D_z b(\cdot, Dv + t(Du - Dv)) \, dt \, (Du - Dv, Du - Dv) \\
&\quad \ge \gamma |Du - Dv|^2 \int_0^1 (1 + |Dv + t(Du - Dv)|)^{p-2} \, dt \\
&\quad \ge c\gamma \, |Du - Dv|^p,
\end{aligned}$$

where we used the inequality (3.2) in the last step. The constant c depends only on p. Recalling (2.23) we see that $u - v \in W_0^{1,p}(B_R(x), \mathbb{R}^N)$ is an admissible test function in the weak formulation of (2.25). Therefore, integrating the above inequality yields

$$\begin{aligned}
c\gamma \int_\Omega |Du - Dv|^p \, dx &\le \int_\Omega b(\cdot, Du) \cdot (Du - Dv) \, dx \\
&= \int_\Omega [b(\cdot, Du) - a(\cdot, u, Du)] \cdot (Du - Dv) \, dx,
\end{aligned}$$

where here we used the equation (2.26) in the last step. By the choice of M_1 according to (5.1) and by the definition of S_ε, we conclude

$$\begin{aligned}
&c\gamma \int_{B_R(x)} |Du - Dv|^p \, dx \\
&\le \varepsilon_1 \int_{\{|Du| \ge M_1\}} (1 + |Du|^{p-1}) |Du - Dv| \, dx + S_\varepsilon \int_{\{|Du| \le M_1\}} |Du - Dv| \, dx.
\end{aligned}$$

Applying Hölder's inequality, we arrive at

$$
\begin{aligned}
&\fint_\Omega |Du-Dv|^p\,dx\\
&\le \left[\frac{\varepsilon_1}{c\gamma}\left(\fint_\Omega |Du|^p\,dx\right)^{1-\frac{1}{p}}+\frac{S_\varepsilon+1}{c\gamma}\right]\left(\fint_\Omega |Du-Dv|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

The last estimate implies

$$
\fint_\Omega |Du-Dv|^p\,dx \le C\left(\frac{\varepsilon_1}{\gamma}\right)^{\frac{p}{p-1}}\fint_\Omega |Du|^p\,dx + C\left(\frac{S_\varepsilon+1}{\gamma}\right)^{\frac{p}{p-1}}. \tag{5.2}
$$

Now if we assume that for some $K>0$ there holds

$$
\fint_\Omega |Du|^p\,dx > K^p,
$$

then the estimate (5.2) implies

$$
\fint_\Omega |Du-Dv|^p\,dx \le C\left[\left(\frac{\varepsilon_1}{\gamma}\right)^{\frac{p}{p-1}}+\left(\frac{S_\varepsilon+1}{\gamma}\right)^{\frac{p}{p-1}}\frac{1}{K^p}\right]\fint_\Omega |Du|^p\,dx.
$$

Thus, we have established the claim choosing first $\varepsilon_1>0$ small enough and then K large enough to ensure that the factor preceding the last integral does not exceed ε. $\square$

Next, we state a version of the comparison lemma for minimizers. It holds also for the disturbed functionals that come into play by Ekeland's variational principle applied to minimizing sequences. Precisely, given the minimizer $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ we consider the functional

$$
F_\delta[w] := \int_\Omega f(\cdot,w,Dw)\,dx + \sqrt{\delta}|\Omega|\left(\fint_\Omega |Dw-Du|^p\,dx\right)^{\frac{1}{p}}
$$

with $0\le\delta\le 1$. Clearly, since F_0 equals F we include the case of minimizers of F.

Lemma 5.2 *Assume that integrands f and g are given which satisfy the assumptions of Theorem 2.22 and let $0\le\delta\le 1$. Then for every $\varepsilon>0$, there is a $K(\varepsilon)>0$, depending only on p, γ, ω and ε, such that the following holds: For a minimizer $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ of F_δ and a minimizer $v\in u+W^{1,p}_0(\Omega,\mathbb{R}^N)$ of G from* (2.31)*, the property*

$$
\fint_\Omega |Du|^p\,dx > K^p(\varepsilon)
$$

implies

$$
\fint_\Omega |Du-Dv|^p\,dx \le \varepsilon\fint_\Omega |Du|^p\,dx.
$$

Proof. Let $\varepsilon_1 \in (0,1)$ be a constant which will be chosen later in dependence on ε. By the assumption (2.28) we can choose M_1 large enough so that

$$\sup_{x\in\Omega,\, y\in\mathbb{R}^N} |f(x,y,\xi) - g(x,\xi)| \le \varepsilon_1(1+|\xi|^p) \qquad \text{for } |\xi| \ge M_1. \tag{5.3}$$

and let $S_\varepsilon := \|\omega\|_{L^\infty}(1+M_1^p) < \infty$. Introducing the auxiliary map $w := \frac{1}{2}(u+v)$, we get from the minimizing property of v

$$\int_\Omega \Big[g(\cdot,Du) - g(\cdot,Dw)\Big]\,dx \ge \int_\Omega \Big[g(\cdot,Dv) + g(\cdot,Du) - 2g(\cdot,Dw)\Big]\,dx. \tag{5.4}$$

The integrand on the right-hand side can be written as

$$\begin{aligned}
& g(\cdot,Dv) + g(\cdot,Du) - 2g(\cdot,Dw) \\
&= \frac{1}{2}\int_0^1 \Big[D_z g(\cdot, Dw + t(Dv{-}Dw)) - D_z g(\cdot, Dw + t(Du{-}Dw))\Big]\,dt\,(Dv{-}Du) \\
&= \frac{1}{2}\int_0^1\int_0^1 D_z^2 g(\cdot, Dw + t(Du{-}Dw) + st(Dv{-}Du))\,ds\,t\,dt\,(Dv{-}Du, Dv{-}Du) \\
&\overset{(2.30)}{\ge} \frac{\gamma}{2}|Du{-}Dv|^2 \int_0^1\int_0^1 (1+|Dw + t(Du{-}Dw) + st(Dv{-}Du)|)^{p-2}\,ds\,t\,dt \\
&\ge c\gamma|Dv{-}Du|^p,
\end{aligned}$$

where we used (3.3) in the last step and where c depends only on p. Integrating the last estimate and using (5.4), we arrive at

$$\begin{aligned}
& c\gamma \fint_\Omega |Du-Dv|^p\,dx \\
&\le \fint_\Omega (g(\cdot,Du) - g(\cdot,Dw))\,dx \\
&\le \fint_\Omega [f(\cdot,u,Du) - f(\cdot,w,Dw)]\,dx \\
&\quad + \fint_\Omega \Big[|g(\cdot,Du) - f(\cdot,u,Du)| + |(g(\cdot,Dw) - f(\cdot,w,Dw)|\Big]\,dx \\
&\le \sqrt{\delta}\left(\fint_\Omega |Du-Dw|^p\,dx\right)^{1/p} + 2S_\varepsilon + \varepsilon_1 \fint_\Omega (2+|Du|^p+|Dw|^p)\,dx \\
&\le \sqrt{\delta}\left(\fint_\Omega |Du-Dv|^p\,dx\right)^{1/p} + 2S_\varepsilon + C\varepsilon_1 \fint_\Omega (1+|Du|^p+|Dv|^p)\,dx,
\end{aligned}$$

where we used in turn the minimizing property of u, the definition of S_ε, the estimate (5.3) and the definition of w. By Young's inequality with exponents p and $\frac{p}{p-1}$, we conclude that there is a constant C, depending only on p, such that

$$
\begin{aligned}
&\gamma \fint_\Omega |Du - Dv|^p \, dx \\
&\le C\Big(S_\varepsilon + \varepsilon_1^{1/(1-p)}\Big) + C\varepsilon_1 \fint_\Omega |Du|^p \, dx + C\varepsilon_1 \fint_\Omega |Du - Dv|^p \, dx.
\end{aligned}
$$

For sufficiently small $\varepsilon_1 > 0$, we can absorb the last integral on the left-hand side. Thus, assuming

$$\fint_\Omega |Du|^p \, dx > K^p$$

for some $K > 0$, we deduce

$$\fint_\Omega |Du - Dv|^p \, dx \le \frac{C\Big(S_\varepsilon + \varepsilon_1^{1/(1-p)}\Big)}{\gamma K^p} \fint_\Omega |Du|^p \, dx + \frac{C\varepsilon_1}{\gamma} \fint_\Omega |Du|^p \, dx.$$

We have thus established the claim if we choose $\varepsilon_1 > 0$ so small that $\frac{C\varepsilon_1}{\gamma} \le \frac{\varepsilon}{2}$ and then K large enough to ensure that the factor in front of the penultimate integral is not larger than $\frac{\varepsilon}{2}$ either. $\square$

6 Calderón-Zygmund estimates

This section contains the proofs of Theorems 2.3, 2.4, 2.21 and 2.22. All of them can be deduced from the following result.

Theorem 6.1 *Consider a cube Q in $\mathbb{R}^n$ and assume that the integrand $f : 4Q \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ satisfies the assumptions of Theorem 2.22 (with $\Omega = 4Q$). In particular, for some $q^\# > p$ and all cubes W with $4W \subset 4Q$ the minimizers $v \in W^{1,p}(4W, \mathbb{R}^N)$ of the comparison functional G are assumed to satisfy $v \in W^{1,q^\#}(2W, \mathbb{R}^N)$ and*

$$\fint_{2W} |Dv|^{q^\#} \, dx \le H \left(1 + \fint_{4W} |Dv|^p \, dx\right)^{\frac{q^\#}{p}}$$

for some constant $H > 0$. Let $u \in W^{1,p}(4Q, \mathbb{R}^N)$ be a minimizer of the functional

$$F_\delta[w] := \int_{4Q} f(\cdot, w, Dw) \, dx + \sqrt{\delta}|4Q| \left(\fint_{4Q} |Dw - Du|^p \, dx\right)^{\frac{1}{p}}$$

in the Dirichlet class $u + W_0^{1,p}(4Q, \mathbb{R}^N)$, where $0 \le \delta \le 1$. Then for every $q \in [1, q^\#)$ we have $u \in W^{1,q}(Q, \mathbb{R}^N)$ and

$$\fint_Q |Du|^q \, dx \le C \left(1 + \fint_{4Q} |Du|^p \, dx\right)^{\frac{q}{p}}, \tag{6.1}$$

where C depends only on n, p, q, $q^\#$, H, γ and ω.

Analogously, if $u \in W^{1,p}(4Q, \mathbb{R}^N)$ is a solution to the system

$$\operatorname{div} a(\cdot, u, Du) = 0 \qquad \text{on } 4Q, \tag{6.2}$$

where the structure function $a : 4Q \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ satisfies the assumptions of Theorem 2.21, then there holds $u \in W^{1,q}(Q, \mathbb{R}^N)$ for every $q \in [1, q^\#)$ and the estimate (6.1).

The proof of Theorem 6.1 will follow ideas of [9,10,3,38] applying the Calderón-Zygmund covering lemma (Lemma 3.1) to the superlevel sets of the maximal function $M_{4Q}(|Du|^p)$. The crucial point is contained in the following proposition verifying the property (P) from Lemma 3.1.

Proposition 6.2 *Under the assumptions of Theorem 6.1 there is a constant $L > 1$ depending only on n, p, $q^\#$ and H, and for every $K > 1$ there is a $\lambda_0 > 1$ depending only on n, p, $q^\#$, H, γ, ω and on K such that the following holds: If $u \in W^{1,p}(4Q, \mathbb{R}^N)$ is either a minimizer of F_δ from Theorem 6.1 or a solution to the system* (6.2)*, where Q is an arbitrary cube in $\mathbb{R}^n$, then*

$$|W \cap \{M_{4Q}(|Du|^p) > KL\lambda\}| > K^{-\frac{q^\#}{p}} |W|$$

implies

$$W^* \subset \{M_{4Q}(|Du|^p) > \lambda\}$$

for every dyadic subcube W of Q with $W \neq Q$ and every $\lambda \geq \lambda_0$.

Proof. We fix $K > 1$. For λ_0 to be chosen later we assume the above statement to be wrong. Then, there are a $\lambda \geq \lambda_0$ and a dyadic subcube W of Q with $W \neq Q$ such that we have

$$|W \cap \{M_{4Q}(|Du|^p) > KL\lambda\}| > K^{-\frac{q^\#}{p}} |W|, \tag{6.3}$$

but

$$M_{4Q}(|Du|^p)(x_0) \leq \lambda \qquad \text{for some } x_0 \in W^*. \tag{6.4}$$

We choose the comparison map $v \in u + W_0^{1,p}(4W, \mathbb{R}^N)$ to be the solution of the system (2.25) or the minimizer of G from (2.31) on $4W$, respectively. Indeed, such a v exists — and by the way is unique — as one proves by Galerkin's method for monotone operators (see [58, Chapter 26]) and the direct method in the calculus of variations (see [35, Chapter 4]), respectively. By assumption we have $v \in W^{1,q^\#}(2W, \mathbb{R}^N)$ with the corresponding estimate (6.1).

We begin the proof with the observation

$$M_{4Q}(|Du|^p)(y) \leq \max\{M_{2W}(|Du|^p)(y), 5^n\lambda\} \qquad \text{for all } y \in W. \tag{6.5}$$

To verify this claim we fix $y \in W$ and consider a subcube Z of $4Q$ containing y. In the case $Z \subset 2W$, we obviously have $\fint_Z |Du|^p \leq M_{2W}(|Du|^p)(y)$, while in the case $Z \not\subset 2W$, there holds $|Z| \geq 2^{-n}|W| = 4^{-n}|W^*|$, which enables us to find another cube $\tilde{Z}$ with $Z \cup W^* \subset \tilde{Z} \subset 4Q$ and $|\tilde{Z}| \leq 5^n|Z|$. Hence, (6.4) implies $\fint_Z |Du|^p\,dx \leq 5^n \fint_{\tilde{Z}} |Du|^p\,dx \leq 5^n\lambda$ in the latter case and (6.5) follows. Assuming $L \geq 5^n$ and recalling $K > 1$ we deduce from (6.5)

$$|W \cap \{M_{4Q}(|Du|^p) > KL\lambda\}| \leq |W \cap \{M_{2W}(|Du|^p) > KL\lambda\}|.$$

With an $\varepsilon \in (0,1)$ to be fixed later we apply Lemma 5.1 in the case of systems and Lemma 5.2 in the case of minimizers to infer that we have either

$$\fint_{4W} |Du|^p\,dx \leq K^p(\varepsilon) \tag{6.6}$$

or

$$\fint_{4W} |Du - Dv|^p\,dx \leq \varepsilon \fint_{4W} |Du|^p\,dx. \tag{6.7}$$

We will derive an estimate for $|W \cap \{M_{2W}(|Du|^p) > KL\lambda\}|$ distinguishing the above cases. In the first case, we have by (3.5) and (6.6)

$$|W \cap \{M_{2W}(|Du|^p) > KL\lambda\}| \leq \frac{4^n}{KL\lambda}\int_{2W} |Du|^p\,dx \leq \frac{4^{2n}}{KL\lambda_0}K^p(\varepsilon)|W|.$$

In the second case (6.7) we easily conclude, since $x_0 \in W^* \subset 4W \subset 4Q$

$$\fint_{4W} |Du - Dv|^p\,dx \leq \varepsilon\lambda, \tag{6.8}$$

$$\fint_{4W} |Dv|^p\,dx \leq 2^{p-1}(1+\varepsilon)\fint_{4W} |Du|^p\,dx \leq 2^p\lambda. \tag{6.9}$$

By the assumption (6.1) we deduce from (6.9), since $\lambda > 1$,

$$\left(\fint_{2W} |Dv|^{q^\#}\,dx\right)^{\frac{p}{q^\#}} \leq C(p,H)\lambda. \tag{6.10}$$

Applying (3.4), we find the estimate

$$\begin{aligned}
&|W \cap \{M_{2W}(|Du|^p) > KL\lambda\}| \\
&\qquad \leq |W \cap \{M_{2W}(|Du - Dv|^p) + M_{2W}(|Dv|^p) > 2^{1-p}KL\lambda\}| \\
&\qquad \leq |W \cap \{M_{2W}(|Du - Dv|^p) > 2^{-p}KL\lambda\}| \\
&\qquad\qquad\qquad + |W \cap \{M_{2W}(|Dv|^p) > 2^{-p}KL\lambda\}|.
\end{aligned}$$

Next we note that due to (3.5) and (6.10), we can control the last term on the right-hand side in the following way.

$$\begin{aligned}
|W \cap \{M_{2W}(|Dv|^p) > 2^{-p}KL\lambda\}| &\leq C(n,p,q^\#)(KL\lambda)^{-\frac{q^\#}{p}}\int_{2W} |Dv|^{q^\#}\,dx \\
&\leq C(n,p,q^\#,H)(KL)^{-\frac{q^\#}{p}}|W|.
\end{aligned}$$

In order to estimate the other term, we use (3.5) and (6.8) with the result

$$|W \cap \{M_{2W}(|Du - Dv|^p) > 2^{-p}KL\lambda\}| \\ \leq \frac{4^n 2^p}{KL\lambda} \int_{2W} |Du - Dv|^p \, dx \leq \frac{4^{2n}2^p}{KL}\varepsilon|W|.$$

Collecting all the estimates, we infer either

$$|W \cap \{M_{4Q}(|Du|^p) > KL\lambda\}| \leq \frac{4^{2n}}{KL\lambda_0} K^p(\varepsilon)|W|$$

or

$$|W \cap \{M_{4Q}(|Du|^p) > KL\lambda\}| \leq C(n, p, q^\#, H)(KL)^{-\frac{q^\#}{p}}|W| + \frac{4^{2n}2^p}{KL}\varepsilon|W|.$$

Now we fix L, ε and λ_0. First we choose $L \geq 5^n$ such that

$$C(n, p, q^\#, H)L^{-\frac{q^\#}{p}} \leq \frac{1}{2},$$

then ε such that

$$\frac{4^{2n}2^p}{KL}\varepsilon \leq \frac{1}{2}K^{-\frac{q^\#}{p}}$$

and, finally, λ_0 such that

$$\frac{4^{2n}}{KL\lambda_0} K^p(\varepsilon) \leq K^{-\frac{q^\#}{p}}.$$

In view of these choices we have

$$|W \cap \{M_{4Q}(|Du|^p) > KL\lambda\}| \leq K^{-\frac{q^\#}{p}}|W|$$

in any case, which contradicts (6.3), thus completing the proof. $\square$

Proof of Theorem 6.1. We abbreviate $h := M_{4Q}(|Du|^p)$ and $\mu_h(\lambda) := |Q \cap \{h > \lambda\}|$. For $q \in (p, q^\#)$, we fix $K > 1$ such that

$$L^q K^{q - q^\#} < 1 \tag{6.11}$$

holds, where L denotes the constant from Proposition 6.2. In particular, this fixes λ_0. Moreover, we choose

$$\lambda_1 := \max\left\{\lambda_0, \frac{4^n K^{\frac{q^\#}{p}}}{|Q|} \int_{4Q} |Du|^p \, dx\right\},$$

which implies in particular, by (3.5) with $q = 1$,

$$\mu_h((KL)^k \lambda_1) \leq K^{-\frac{q^\#}{p}}|Q| \tag{6.12}$$

for all $k \in \mathbb{N}_0$. Keeping (6.12) in mind, we apply Lemma 3.1 with $\varsigma := K^{-\frac{q^\#}{p}}$ to the sets $Q \cap \{h > KL\lambda_1\}$ and $Q \cap \{h > \lambda_1\}$. This is possible since property (P) is satisfied by Proposition 6.2 and we come out with

$$\mu_h(KL\lambda_1) \leq K^{-\frac{q^\#}{p}} \mu_h(\lambda_1).$$

In the next step, again in view of (6.12) and Proposition 6.2, we apply Lemma 3.1 to $Q \cap \{h > (KL)^2\lambda_1\}$ and $Q \cap \{h > KL\lambda_1\}$ getting

$$\mu_h((KL)^2\lambda_1) \leq K^{-\frac{q^\#}{p}} \mu_h(KL\lambda_1) \leq K^{-2\frac{q^\#}{p}} \mu_h(\lambda_1).$$

Continuing inductively we arrive at

$$\mu_h((KL)^k\lambda_1) \leq K^{-k\frac{q^\#}{p}} \mu_h(\lambda_1) \tag{6.13}$$

for all $k \in \mathbb{N}_0$. This yields an $L^{q/p}$-estimate for h in the following way.

$$\begin{aligned}
\int_Q h^{\frac{q}{p}}\, dx &\leq \lambda_1^{\frac{q}{p}} |Q \cap \{h \leq \lambda_1\}| \\
&\qquad + \sum_{k=0}^{\infty} ((KL)^{k+1}\lambda_1)^{\frac{q}{p}} |Q \cap \{(KL)^k\lambda_1 < h \leq (KL)^{k+1}\lambda_1\}| \\
&\leq \lambda_1^{\frac{q}{p}} |Q \cap \{h \leq \lambda_1\}| + (KL\lambda_1)^{\frac{q}{p}} \sum_{k=0}^{\infty} (KL)^{k\frac{q}{p}} \mu_h((KL)^k\lambda_1) \\
&\leq \lambda_1^{\frac{q}{p}} |Q| + (KL\lambda_1)^{\frac{q}{p}} \sum_{k=0}^{\infty} (L^q K^{q-q^\#})^{\frac{k}{p}} |Q|,
\end{aligned}$$

where we used (6.13) in the last step. By the choice of K according to (6.11), the last series converges and we have proved $h \in L^{q/p}(Q)$ with $⨍_Q h^{q/p}\, dx \leq c\lambda_1^{q/p}$, where c depends on n, p, q, $q^\#$, K and L. By Lebesgue's differentiation theorem we have $|Du|^p \leq h$ almost everywhere on $4Q$, which gives $Du \in L^q(Q, \mathbb{R}^{Nn})$ and

$$⨍_Q |Du|^q\, dx \leq C\lambda_1^{\frac{q}{p}}.$$

Taking into account the choice of λ_1 and the dependences of K, L and λ_0, we finally arrive at (6.1). The claim $u \in W^{1,q}(Q, \mathbb{R}^N)$ follows via Poincaré's inequality. □

Finally, the regularity theorems of Section 2 follow from Theorem 6.1:

Proof of Theorem 2.21 and Theorem 2.22. The claims readily follow from the above Theorem 6.1 (taking $\delta = 0$ in case of Theorem 2.22). □

Proof of Theorem 2.3 and Theorem 2.4. Taking into account Corollary 4.3 and (2.4) we see that Theorem 2.3 is a special case of Theorem 2.21, applied with $q^\# = p^\#$ in the case $p^\# < \infty$ and with any $q^\# < \infty$ in the case $p^\# = \infty$.

Similarly, recalling Remark 4.4 and (2.5), Theorem 2.4 follows from Theorem 2.22. □

7 Minimizing Sequences

In this section, we will prove Theorem 2.6 and Theorem 2.24.

Proof of Theorem 2.24. We will apply Ekeland's variational principle on the Dirichlet class $\mathcal{D} = u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$, equipped with the metric

$$d(u,v) := |\Omega| \left(⨍_\Omega |Du - Dv|^p \, dx \right)^{\frac{1}{p}} \qquad \text{for } u, v \in \mathcal{D},$$

which makes $\mathcal{D}$ a complete metric space. According to our assumptions, the integrand $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is lower semicontinuous in the two last arguments and recalling (2.32), it is additionally bounded from below. Thus, as a consequence of Fatou's lemma the functional

$$F[u] = \int_\Omega f(\cdot, u, Du) \, dx$$

is lower semicontinuous with respect to the metric d. Furthermore, F is bounded from below, so that Ekeland's principle is applicable.

Now let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence for F in $\mathcal{D}$ as in the theorem, that is,

$$\delta_k := F[u_k] - \inf_{\mathcal{D}} F \underset{k\to\infty}{\longrightarrow} 0.$$

Then, in view of the above discussion we may apply Lemma 3.2 coming out with a sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$, where v_k minimizes the functional

$$F_k[w] := F[w] + \sqrt{\delta_k}\, d(v_k, w)$$

for every $k \in \mathbb{N}$ and satisfies furthermore

$$d(u_k, v_k) \le \sqrt{\delta_k} \underset{k\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad F[v_k] \le F[u_k] \quad \text{for all } k \in \mathbb{N}.$$

By this last property, $(v_k)_{k\in\mathbb{N}}$ is itself a minimizing sequence for F in $\mathcal{D}$. Clearly, it is not restrictive to assume $\delta_k \le 1$ for all $k \in \mathbb{N}$. Then, since v_k minimizes the functional F_k, Theorem 6.1 yields the higher integrability $v_k \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for every $q < q^\#$ with the estimates

$$⨍_Q |Dv_k|^q \, dx \le C \left(1 + ⨍_{4Q} |Dv_k|^p \, dx \right)^{\frac{q}{p}}$$

for every cube Q with $4Q \subset \Omega$ and all $k \in \mathbb{N}$, where the constant C does not depend on k. This completes the proof. □

Proof of Theorem 2.6. Recalling Corollary 4.3, Remark 4.4 and (2.5) once more, we see that Theorem 2.6 is a special case of Theorem 2.24. $\square$

8 An alternative approach: Morrey estimates and Hölder continuity

In this section we present an elementary method, related to ideas of [34,25], which is based on the iteration of Morrey space estimates. This method enables us to give a short and widely self-contained proof of parts of the results of Section 2 and avoids some of the more technical tools like Calderòn-Zygmund coverings and Gehring's lemma. However, it does not allow to prove higher integrability of the gradient Du, but only weaker regularity properties, namely Morrey and Campanato regularity for Du and Hölder continuity of u in low dimensions.

To simplify our presentation we will abandon some of the technical features of Section 2 here, restricting ourselves to the simpler setting of Theorem 2.3 and Theorem 2.4. We establish the following result, which is a particular case of these theorems:

Theorem 8.1 *Under the assumptions of either Theorem 2.3 or Theorem 2.4 there is a constant $\kappa > 0$, depending only on n, p, γ and Γ, such that we have*

$$Du \in L^{p,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn}) \qquad \text{for all } 0 \le \lambda < \min\{2 + \kappa p, n\}.$$

In particular, for $\kappa > \frac{n-p-2}{p}$ — which is guaranteed in the low-dimensional case $n \le p+2$ — this implies

$$u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \text{for all } 0 \le \alpha < \min\left\{\frac{p+2-n}{p} + \kappa, 1\right\}.$$

Let us start with some comments on the alternative proof announced above: We stress that the only ingredients from the preceding sections are Theorem 4.1 and the comparison estimates of Section 5. In particular, we will not rely on the Gehring improvement of Theorem 4.2 and Corollary 4.3. Instead, starting from Theorem 4.1 we will apply Widman's hole filling trick [56] to obtain an analogous improvement on the scale of Campanato and Morrey spaces. Clearly, this application of the hole filling trick is close to Widman's original ideas and well-known to experts; nevertheless, let us briefly sketch it:

Lemma 8.2 *Consider a regular structure function $b : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$. Then, there is a constant $\kappa > 0$, depending only on n, p, γ and Γ, such that for* *every weak solution* $v \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$ *of*

$$\operatorname{div} b(Dv) = 0 \qquad \text{on } B_R(x_0) \tag{8.1}$$

the function

$$\bar{V} := V(Dv)$$

with V *from* (3.1) *satisfies the Campanato estimate*

$$\int_{B_\rho(x_0)} |\bar{V} - \bar{V}_{x_0,\rho}|^2 \, dx \le C \left(\frac{\rho}{R}\right)^{2+\kappa p} \int_{B_R(x_0)} |\bar{V} - \bar{V}_{x_0,R}|^2 \, dx$$

for all $0 < \rho \le \frac{1}{2}R$. *Here,* C *depends only on* n, p, γ *and* Γ.

Proof. We assume $x_0 = 0$. Combining Theorem 4.1 with the Poincaré inequality on the annulus $B_\rho \setminus B_{\rho/2}$ we find

$$\int_{B_{\rho/2}} |D\bar{V}|^2 \, dx \le C \int_{B_\rho \setminus B_{\rho/2}} |D\bar{V}|^2 \, dx.$$

Next we use the hole filling trick, i.e. we add $C \int_{B_{\rho/2}} |D\bar{V}|^2 \, dx$ on both sides and divide by $C+1$. Choosing $\kappa > 0$ with $2^{-\kappa p} = \frac{C}{C+1}$, we arrive at

$$\int_{B_{\rho/2}} |D\bar{V}|^2 \, dx \le 2^{-\kappa p} \int_{B_\rho} |D\bar{V}|^2 \, dx.$$

Now we recall that the previous inequality holds for all radii $\rho \le R$. Thus, we may iterate it, coming out with

$$\int_{B_\rho} |D\bar{V}|^2 \, dx \le \left(\frac{2\rho}{R}\right)^{\kappa p} \int_{B_R} |D\bar{V}|^2 \, dx$$

for all $\rho \le R$. Now, Poincaré's inequality gives for $\rho \le \frac{1}{2}R$

$$\begin{aligned}
\int_{B_\rho} |\bar{V} - \bar{V}_\rho|^2 \, dx &\le C\rho^2 \int_{B_\rho} |D\bar{V}|^2 \, dx \\
&\le C\rho^2 \left(\frac{\rho}{R}\right)^{\kappa p} \int_{B_{R/2}} |D\bar{V}|^2 \, dx \\
&\le C \left(\frac{\rho}{R}\right)^{2+\kappa p} \int_{B_R} |\bar{V} - \bar{V}_R|^2 \, dx,
\end{aligned}$$

where we applied the Caccioppoli inequality (4.2) from Theorem 4.1 once more in the last step. This completes the proof of the lemma. $\square$

Next, we will convert the Campanato estimate for $\bar{V}$ into the following Morrey estimate for Dv. The essential idea here is to exploit the well-known equivalence of the Morrey spaces $L^{p,\lambda}$ and the Campanato spaces $\mathcal{L}^{p,\lambda}$ for $\lambda < n$; see for instance [43, Theorem 4.6.1].

Corollary 8.3 *In the situation of Lemma 8.2, there holds furthermore*

$$\int_{B_\rho(x_0)} (1 + |Dv|^p)\, dx \le C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R(x_0)} \left(1 + |Dv|^p\right) dx$$

for all $0 < \rho \le \frac{1}{2}R$ *and for every* $\lambda \in [0, n)$ *with* $\lambda \le 2 + \kappa p$*. Here,* C *depends only on* n*,* p*,* γ*,* Γ *and* λ*.*

Proof. We assume $x_0 = 0$. For an arbitrary $0 < r \le \frac{1}{2}R$ we conclude from the excess estimate of Lemma 8.2

$$|\bar{V}_{r/2} - \bar{V}_r| \le 2^n \left(\fint_{B_r} |\bar{V} - \bar{V}_r|^2\, dx\right)^{\frac{1}{2}} \le C \left(\frac{r}{R}\right)^{\frac{\lambda - n}{2}} \left(\fint_{B_R} |\bar{V}|^2\, dx\right)^{\frac{1}{2}}.$$

Applying the last inequality with $r = \rho$, $r = 2\rho$, $r = 2^2\rho$, ..., 2^{k_0}, where $k_0 \in \mathbb{N} \cup \{0\}$ is such that $\tilde{R} := 2^{k_0}\rho \in (\frac{1}{2}R, R]$, yields

$$|\bar{V}_\rho| \le |\bar{V}_{\tilde{R}}| + C \left(\frac{\rho}{R}\right)^{\frac{\lambda - n}{2}} \sum_{k=0}^{\infty} 2^{k(\lambda - n)/2} \left(\fint_{B_R} |\bar{V}|^2\, dx\right)^{\frac{1}{2}}.$$

Since $\lambda < n$, the series on the right-hand side converges. Thus, noting $|\bar{V}_{\tilde{R}}|^2 \le 2^n \fint_{B_R} |\bar{V}|^2\, dx$ we may combine the preceding estimates in the following way

$$\begin{aligned}
\int_{B_\rho} \left(1 + |\bar{V}|^2\right) dx &\le 2 \int_{B_\rho} |\bar{V} - \bar{V}_\rho|^2\, dx + C\rho^n \left(1 + |\bar{V}_\rho|^2\right) \\
&\le C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |\bar{V}|^2\, dx + C\rho^n \left(1 + |\bar{V}_{\tilde{R}}|^2\right) \\
&\le C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R} \left(1 + |\bar{V}|^2\right) dx,
\end{aligned}$$

where we used $\lambda < n$ again. Finally, recalling that we have $|Dv|^p \le |\bar{V}|^2 \le C(1 + |Dv|^p)$ by the very definition of $\bar{V}$, we arrive at the claim. $\square$

Finally, we will carry over the estimate of Corollary 8.3 to solutions of asymptotically regular problems. To this aim we assume that u is as in Theorem 8.1 and we fix a ball $B_R(x_0) \subset \Omega$. Then, for $0 < r \le R$, we introduce the excess

$$\Phi(r) := \int_{B_r(x_0)} \left(1 + |Du|^p\right) dx.$$

The core of the proof of Theorem 8.1 is now contained in the following decay estimates for Φ:

Lemma 8.4 *Let* $\kappa > 0$ *denote the constant from Lemma 8.2. For every* $0 \le \lambda < \min\{2 + \kappa p, n\}$*, there are constants* $0 < \tau \le \frac{1}{2}$ *and* $L > 0$*, such that we* *have either*

$$\Phi(\tau R) \le \tau^\lambda \Phi(R) \qquad \text{or} \qquad \Phi(R) \le L|B_R|.$$

Here, τ depends only on λ, n, p, γ and Γ and L depends additionally on $|a-b|$ and $|f-g|$, respectively.

Proof. As in the proof of Proposition 6.2 we find a solution v of the comparison problem in $u+W_0^{1,p}(B_R(x_0),\mathbb{R}^N)$; that is, v is either a weak solution of (8.1) or a minimizer of $G[v] := \int_{B_R(x_0)} g(Dv)\,dx$. Then, the estimates of Lemma 8.2 and Corollary 8.3 hold for v; compare with Remark 4.4 in the case of minimizers. Next, for a given $0 < \lambda < \min\{2+\kappa p, n\}$ we may choose a $\lambda^\# \in [0,n)$, depending only on n, p, γ, Γ and λ such that we have $\lambda < \lambda^\# \le 2+\kappa p$. For $0<\tau\le \frac{1}{2}$ to be fixed later, we estimate

$$\begin{aligned}\Phi(\tau R) &\le 2^{p-1}\int_{B_{\tau R}(x_0)} (1+|Dv|^p)\,dx + 2^{p-1}\int_{B_{\tau R}(x_0)} |Du-Dv|^p\,dx\\ &\le C\tau^{\lambda^\#}\int_{B_R(x_0)} (1+|Dv|^p)\,dx + 2^{p-1}\int_{B_{\tau R}(x_0)} |Du-Dv|^p\,dx\\ &\le C\tau^{\lambda^\#}\int_{B_R(x_0)} (1+|Du|^p)\,dx + C\int_{B_R(x_0)} |Du-Dv|^p\,dx,\end{aligned} \tag{8.2}$$

where we applied Corollary 8.3 with $\rho = \tau R$ in the second step. Here, the constant C depends only on n, p, γ, Γ and λ. Next, let $\varepsilon > 0$ be given and suppose, for the moment,

$$⨍_{B_R(x_0)} |Du|^p\,dx > K^p(\varepsilon). \tag{8.3}$$

Then combining (8.2) with Lemma 5.1 and Lemma 5.2, respectively, we conclude

$$\begin{aligned}\Phi(\tau R) &\le C\tau^{\lambda^\#}\int_{B_R(x_0)} (1+|Du|^p)\,dx + C\varepsilon\int_{B_R(x_0)} |Du|^p\,dx\\ &\le C(\tau^{\lambda^\#}+\varepsilon)\Phi(R).\end{aligned}$$

Choosing $0<\tau\le\frac{1}{2}$ so small that $2C\tau^{\lambda^\#} \le \tau^\lambda$ and $\varepsilon := \tau^{\lambda^\#}$, we have proven that (8.3) implies the first alternative of the lemma. On the other hand, if (8.3) fails to hold then the second alternative is satisfied with $L := 1+K^p(\varepsilon)$. □

Proof of Theorem 8.1. Let $0 \le \lambda < \min\{2+\kappa p, n\}$, where κ still denotes the constant from Lemma 8.2. We define $r_k := \tau^k R$, where τ is the constant from Lemma 8.4. Distinguishing the two cases $\Phi(r_{k-1}) \le L|B_{r_{k-1}}|$ and $\Phi(r_{k-1}) >$

$L|B_{r_{k-1}}|$, Lemma 8.4 yields

$$\begin{aligned}\Phi(r_k) &\le \max\Big(\tau^{nk}\tau^{-n}L|B_R|, \tau^{\lambda}\Phi(r_{k-1})\Big)\\ &\le \max\Big(\tau^{\lambda k}\tau^{-n}L|B_R|, \tau^{\lambda}\Phi(r_{k-1})\Big),\end{aligned}$$

where we used $\lambda < n$ once more. Iterating this inequality we find

$$\Phi(r_k) \le \tau^{\lambda k}\max(\tau^{-n}L|B_R|, \Phi(R))$$

and using a standard argument we arrive at

$$\Phi(\rho) \le C(\tau)\left(\frac{\rho}{R}\right)^{\lambda}\max(L|B_R|, \Phi(R)) \qquad \text{for all } 0 < \rho \le R.$$

In particular, considering $\emptyset \neq K \subset\subset \Omega$ the last inequality holds for every $x_0 \in K$ with $R = \delta_K := \operatorname{dist}(K, \mathbb{R}^n \setminus \Omega)$. Hence, we have

$$\sup_{\substack{x\in K\\ 0<\rho\le\delta_K}} \rho^{-\lambda}\int_{B_\rho(x)} |Du|^p\,dx \le \frac{C}{\delta_K^{\lambda}}\max\Big(L\delta_K^n, \delta_K^n + \|Du\|^p_{L^p(\Omega,\mathbb{R}^{Nn})}\Big). \tag{8.4}$$

Finally, (8.4) implies $Du \in L^{p,\lambda}(K, \mathbb{R}^{Nn})$ and we arrive at the claim $Du \in L^{p,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$. The remaining claims in Theorem 8.1 concerning the Hölder continuity of u follow from the Dirichlet growth theorem. $\square$

9 Asymptotic regularity

This section is devoted to the proofs of Theorem 2.14 and Theorem 2.16. Here, in both theorems the main challenge is to prove that the property (iii) implies the property (i); that is, to construct regular functions b and g from a and f, respectively. Before going into the details, let us briefly highlight the main idea of this construction: We smooth a and f, respectively, with a variable smoothing radius constructed from the modulus φ in (iii). In some sense this procedure smears the values of a and f, getting back the usual growth, monotonicity and convexity conditions from the disturbed ones in (iii). Unfortunately, the implementation of this idea turns out to be quite technical:

Proof of Theorem 2.14. First we assume that (i) holds with a map b that is regular with structure constants γ and Γ as in Definition 2.1. Then, there are a $M_0 > 0$ and a function $\omega : [M_0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty}\omega(t) = 0$ such that $|b(z) - a(z)| \le \omega(|z|)|z|^{p-1}$ holds for $|z| \ge M_0$. We may assume that ω is non-increasing. In addition, we note that the ellipticity of b and (3.2) give the monotonicity condition

$$\begin{aligned}(b(z_2)-b(z_1))\cdot(z_2-z_1) &= \int_0^1 Db(z_1+t(z_2-z_1))\,dt(z_2-z_1)\cdot(z_2-z_1)\\ &\ge c\gamma(|z_1|+|z_2|)^{p-2}|z_2-z_1|^2,\end{aligned}$$

where c depends only on p. Thus, for $|z_1|,|z_2|\ge M_0$ we have

$$\begin{aligned}&(a(z_2)-a(z_1))\cdot(z_2-z_1)\\ &\quad\ge (b(z_2)-b(z_1))\cdot(z_2-z_1)-\Big[\omega(|z_1|)|z_1|^{p-1}+\omega(|z_2|)|z_2|^{p-1}\Big]|z_2-z_1|\\ &\quad\ge c\gamma(|z_1|+|z_2|)^{p-2}|z_2-z_1|^2-\omega(\min\{|z_1|,|z_2|\})(|z_1|+|z_2|)^{p-1}|z_2-z_1|.\end{aligned}$$

Similarly, we see

$$|a(z_2)-a(z_1)|\le C\Gamma(|z_1|+|z_2|)^{p-2}|z_2-z_1|+\omega(\min\{|z_1|,|z_2|\})(|z_1|+|z_2|)^{p-1}.$$

Thus, (ii) is valid with the constant $c\gamma$ and $C\Gamma$ instead of γ and Γ, respectively.

Now suppose that (ii) holds with constants γ, Γ and M_0 and a function ω. We assume that ω is non-increasing and set $\varphi(t):=\frac{2\omega(t)}{\gamma}$ for $t\ge M_0$. Then, if $z_1,z_2\in\mathbb{R}^m$ satisfy (2.16) for some $t\ge M_0$, we have

$$\begin{aligned}&(a(z_2)-a(z_1))\cdot(z_2-z_1)\\ &\qquad\qquad\ge\left(\gamma-\frac{\omega(\min\{|z_1|,|z_2|\})}{\varphi(t)}\right)(|z_1|+|z_2|)^{p-2}|z_2-z_1|^2\\ &\qquad\qquad\ge\frac{\gamma}{2}(|z_1|+|z_2|)^{p-2}|z_2-z_1|^2.\end{aligned}$$

Analogously, we get

$$|a(z_2)-a(z_1)|\le\left(\Gamma+\frac{1}{2}\gamma\right)(|z_1|+|z_2|)^{p-2}|z_2-z_1|.$$

Thus, (iii) holds with constants $\frac{1}{2}\gamma$ and $\Gamma+\frac{1}{2}\gamma$.

Finally, we suppose that (iii) holds with constants γ, Γ and M_0 and a function φ. We may assume that φ is decreasing with $\varphi\le\frac{1}{64}$ on $[M_0,\infty)$. In the following we will need a smooth function $\Phi:[\exp(M_0+2),\infty)\to(0,\frac{1}{8}]$ with the following properties: Φ is decreasing with $\Phi>\varphi$ and

$$\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\frac{\varphi(t)^2}{\Phi(t)^3}=\lim_{t\to\infty}t\Phi'(t)=0.$$

Indeed, such a function Φ can be constructed from φ. For instance, choosing a smooth kernel $0\le\theta\in C_0^\infty(-1,1)$ with $\int_{-1}^1\theta(s)\,ds=1$, one checks that $\Phi(t):=\int_{-1}^1\theta(s)\sqrt{\varphi(\log t-s-1)}\,ds$ has all the desired properties. Next, we let

$$\begin{aligned} r(z) &:= 4\varphi(|z|/2)|z| \\ R(z) &:= 4\Phi(|z|/2)|z| \end{aligned} \qquad \text{for } |z| \gg 1.$$

Then, we clearly have

$$r(z) < R(z) < \frac{1}{2}|z| \qquad \text{for } |z| \gg 1.$$

In addition, from the above features of φ and Φ we get

$$\lim_{|z|\to\infty} \frac{R(z)}{|z|} = \lim_{|z|\to\infty} \frac{|z|r(z)^2}{R(z)^3} = \lim_{|z|\to\infty} |\nabla R(z)| = 0. \tag{9.1}$$

The function r has been chosen in such a way that as a consequence of (iii), for any $z_1, z_2 \in B_{R(z)}(z)$ with $|z| \gg 1$, the condition

$$|z_2 - z_1| \geq r(z)$$

implies

$$\begin{aligned} (a(z_2) - a(z_1)) \cdot (z_2 - z_1) &\geq \gamma(|z_1| + |z_2|)^{p-2}|z_2 - z_1|^2, \\ |a(z_2) - a(z_1)| &\leq \Gamma(|z_1| + |z_2|)^{p-2}|z_2 - z_1|. \end{aligned} \tag{9.2}$$

This property will be used extensively in the remainder of the proof. After these preparations we introduce

$$\tilde{b}(z) := \fint_{B_{R(z)}} a(z+w)\, dw = \fint_{B_{R(z)}(z)} a(w)\, dw$$

for $|z| \gg 1$. Using (9.2) at the points where $|w| \geq r(z)$ and keeping in mind that by Remark 2.15, we have the growth condition

$$|a(z)| \leq L|z|^{p-1} \qquad \text{for } |z| \gg 1, \tag{9.3}$$

we estimate

$$\begin{aligned} \frac{|\tilde{b}(z) - a(z)|}{|z|^{p-1}} &\leq \frac{1}{|z|^{p-1}} \fint_{B_{R(z)}} |a(z+w) - a(z)|\, dw \\ &\leq \frac{1}{\omega_m R(z)^m |z|^{p-1}} \Bigg[\int_{B_{R(z)} \setminus B_{r(z)}} |a(z+w) - a(z)|\, dw \\ &\qquad\qquad + \int_{B_{r(z)}} |a(z+w) - a(z)|\, dw \Bigg] \\ &\leq C(p)\Gamma \frac{R(z)}{|z|} + C(p)L \left(\frac{r(z)}{R(z)} \right)^m . \end{aligned}$$

In view of (9.1) we conclude

$$\lim_{|z|\to\infty} \frac{|\tilde{b}(z) - a(z)|}{|z|^{p-1}} = 0. \tag{9.4}$$

It remains to prove that $\tilde{b}$ is regular at least in a neighborhood of ∞. Actually, by the growth condition (9.3), a is locally bounded and hence $\tilde{b}$ is locally Lipschitzian outside a large ball. Calculating the derivative of the BV-function $z \mapsto R(z)^{-m}\mathbb{1}_{B_{R(z)}}(z-w)$ we find

$$\begin{aligned} D\tilde{b}(z) &= \frac{m}{R(z)}\Bigg[\fint_{S_{R(z)}} a(z+w)\otimes\frac{w}{R(z)}\,d\mathcal{H}^{m-1}(w) \\ &\qquad\qquad + \fint_{S_{R(z)}} a(z+w)\,d\mathcal{H}^{m-1}(w)\otimes\nabla R(z) \\ &\qquad\qquad - \fint_{B_{R(z)}} a(z+w)\,dw\otimes\nabla R(z)\Bigg] \\ &=: \frac{m}{R(z)}\Big[I_1+I_2+I_3\Big] \end{aligned}$$

for a.e. $z\in\mathbb{R}^m$ with $|z|\gg 1$. Here, $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure on $\mathbb{R}^m$. We will estimate I_1, I_2 and I_3 for fixed $z\in\mathbb{R}^m$ with $|z|\gg 1$ and $\zeta\in\mathbb{R}^m$ with $|\zeta|=1$. To simplify the notation we write simply r and R for $r(z)$ and $R(z)$, respectively. Introducing the abbreviations $S_R^+ := \{w\in S_R \,:\, w\cdot\zeta>0\}$ for the halfsphere and $w^* := w-2(w\cdot\zeta)\zeta$ for the reflection of w at the hyperplane $\{\zeta\}^\perp := \{w\in\mathbb{R}^m \,:\, w\cdot\zeta=0\}$, we have

$$\frac{m}{R}I_1\zeta\cdot\zeta = \frac{m}{2R}\fint_{S_R^+}\Big(a(z+w)-a(z+w^*)\Big)\cdot\zeta\left(\frac{w}{R}\cdot\zeta\right)d\mathcal{H}^{m-1}(w).$$

Now we introduce the disjoint subsets

$$A^1 := \left\{w\in S_R \,:\, w\cdot\zeta>\frac{1}{\sqrt{2}}R\right\} \quad\text{and}\quad A^2 := \left\{w\in S_R \,:\, 0<w\cdot\zeta<\frac{1}{2}r\right\}$$

of S_R^+. We note $\mathcal{H}^{m-1}(A^1)\geq m\omega_m(\frac{1}{\sqrt{2}}R)^{m-1}$. Moreover, A^2 is empty for $m=1$ and $\mathcal{H}^{m-1}(A^2)\leq CrR^{m-2}$ holds for $m\geq 2$ and some constant C depending only on m. Since $w-w^*$ is parallel to ζ with $|w-w^*|=2w\cdot\zeta$ for all $w\in S_R^+$, we can use the lower bound in (9.2) to estimate the integrand from below outside A^2. Keeping these facts in mind and recalling (9.3), we get for $|z|\gg 1$

$$\begin{aligned} &\frac{m}{R}I_1\zeta\cdot\zeta \\ &\quad\geq \frac{c}{R^m}\int_{A^1}|z|^{p-2}|w-w^*|\,d\mathcal{H}^{m-1}(w) - C\frac{r}{R^{m+1}}\int_{A^2}|z|^{p-1}\,d\mathcal{H}^{m-1}(w) \\ &\quad\geq \left(c-C\frac{|z|r^2}{R^3}\right)|z|^{p-2}, \end{aligned}$$

where c and C depend only on m, p, γ and L. In addition, we have

$$\frac{m}{R}[I_2+I_3] = \frac{m}{R}\fint_{S_R}\int_0^1\Big[a(z-w)-a(z+tw)\Big]mt^{m-1}\,dt\,d\mathcal{H}^{m-1}(w)\otimes\nabla R$$

and by the upper estimate in (9.2) we get

$$\left|\frac{m}{R}[I_2 + I_3]\right| \le C|\nabla R|\,|z|^{p-2},$$

where C depends only on m, p and Γ. Collecting the estimates for I_1, I_2 and I_3, we have shown

$$D\tilde{b}(z)\zeta \cdot \zeta \ge c|z|^{p-2} - C\left[\frac{|z|r(z)^2}{R(z)^3} + |\nabla R(z)|\right]|z|^{p-2}$$

for a.e. $z \in \mathbb{R}^m$ with $|z| \gg 1$ and all $\zeta \in \mathbb{R}^m$ with $|\zeta| = 1$. In view of (9.1) this implies

$$D\tilde{b}(z)\zeta \cdot \zeta \ge c|z|^{p-2}|\zeta|^2 \tag{9.5}$$

for a.e. $z \in \mathbb{R}^m$ with $|z|$ large enough and all $\zeta \in \mathbb{R}^m$. Arguing similarly we find that the upper estimate

$$|D\tilde{b}(z)| \le C|z|^{p-2} \tag{9.6}$$

holds for a.e. $z \in \mathbb{R}^m$ with $|z|$ large enough. Finally, we define b_* as a standard mollification of $\tilde{b}$, for instance with smoothing radius 1. Then b_* is C^∞ outside a large ball and (9.5) and (9.6) are easily seen to hold also for b_*. Moreover, using (9.6) we see $\lim_{|z|\to\infty} \frac{|b_*(z)-\tilde{b}(z)|}{|z|^{p-1}} = 0$ and consequently, (9.4) holds also with b_* instead of $\tilde{b}$. Finally, recalling (9.3) we apply [52, Corollary 4.6] deducing that there is a regular function b in the sense of Definition 2.1 such that $b(z)$ coincides with $b_*(z)$ for large values of $|z|$. Thus, (i) is valid. □

Proof of Theorem 2.16. We begin by proving that (i) implies (ii). Assume that the functions f and g satisfy (i), where g is regular with structure constants γ and Γ as in Definition 2.1. Then there is a bounded function $\omega_0 : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \omega_0(t) = 0$ such that

$$|f(z) - g(z)| \le \omega_0(|z|)(1 + |z|^p) \qquad \text{for all } z \in \mathbb{R}^m.$$

We may assume that ω_0 is decreasing by choosing a larger function if necessary. For $\lambda \in [0,1]$, $z_1, z_2 \in \mathbb{R}^m$ and $z := \lambda z_1 + (1-\lambda)z_2$, we infer from the strict convexity of g that

$$\lambda g(z_1) + (1-\lambda)g(z_2) - g(z) \ge c_p\gamma\lambda(1-\lambda)(|z_1| + |z_2|)^{p-2}|z_1 - z_2|^2$$

holds with some positive constant c_p depending only on p. Putting together the last two estimates, we conclude

$$\begin{aligned}\lambda f(z_1) &+ (1-\lambda)f(z_2) - f(z)\\ &\ge c_p\gamma\lambda(1-\lambda)(|z_1| + |z_2|)^{p-2}|z_1 - z_2|^2 - \omega_0(|z|)(1 + |z|^p)\\ &\qquad - \Big[\lambda\omega_0(|z_1|)(1 + |z_1|^p) + (1-\lambda)\omega_0(|z_2|)(1 + |z_2|^p)\Big].\end{aligned} \tag{9.7}$$

We claim that there is a constant $M_0 > 1$ such that $|z| \geq M_0$ implies

$$\begin{aligned} X &:= \lambda\omega_0(|z_1|)(1+|z_1|^p) + (1-\lambda)\omega_0(|z_2|)(1+|z_2|^p) \\ &\leq (|z_1|+|z_2|)^{p-2}\Big[\tilde{\omega}(|z|)|z|^2 + \frac{c_p\gamma}{2}\lambda(1-\lambda)|z_1-z_2|^2\Big] \end{aligned} \tag{9.8}$$

for some function $\tilde{\omega} : [M_0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \tilde{\omega}(t) = 0$, namely for

$$\tilde{\omega}(t) := \omega_0\big(\sqrt{t}\big)\frac{1+t^2}{t^2} + \Big(\max_{[0,\infty)} \omega_0\Big)\frac{1+t^{p/2}}{t^p}.$$

For the proof of (9.8), we assume $|z_1| \leq |z_2|$ and $|z| > 1$ and distinguish the cases $|z_1| > \sqrt{|z|}$ and $|z_1| \leq \sqrt{|z|}$.

In the first case we infer from the monotonicity of ω_0

$$\begin{aligned} X \leq\, &\omega_0\Big(\sqrt{|z|}\Big)\big[1 + \lambda|z_1|^p + (1-\lambda)|z_2|^p\big] \\ \leq\, &\omega_0\Big(\sqrt{|z|}\Big)(|z_1|+|z_2|)^{p-2}\Big[1+\lambda|z_1|^2 + (1-\lambda)|z_2|^2\Big] \\ =\, &\omega_0\Big(\sqrt{|z|}\Big)(|z_1|+|z_2|)^{p-2}\Big[1+|z|^2 + \lambda(1-\lambda)|z_1-z_2|^2\Big], \end{aligned} \tag{9.9}$$

where the last equality follows from the definition of z by a straightforward calculation.

In the case $|z_1| \leq \sqrt{|z|}$, we observe $|z_1| \leq |z|$ and thus $|z_2| \geq |z| \geq \sqrt{|z|}$, since z is a convex combination of z_1 and z_2. Using the monotonicity and the boundedness of ω_0, we estimate in this case

$$\begin{aligned} X &\leq \omega_0\Big(\sqrt{|z|}\Big)\Big(1+(1-\lambda)|z_2|^p\Big) + \Big(\max_{[0,\infty)}|\omega_0|\Big)(1+|z|^{p/2}) \\ &\leq \omega_0\Big(\sqrt{|z|}\Big)|z_2|^{p-2}\Big[1+|z|^2+\lambda(1-\lambda)|z_1-z_2|^2\Big] \\ &\qquad\qquad + \Big(\max_{[0,\infty)}|\omega_0|\Big)|z_2|^{p-2}\frac{1+|z|^{p/2}}{|z|^{p-2}}, \end{aligned}$$

where we used the same equality as in (9.9).

Finally, recalling the definition of $\tilde{\omega}$ and choosing $M_0 > 1$ so large that $\omega_0(\sqrt{M_0}) \leq \frac{c_p\gamma}{2}$ we arrive in both cases at the claim (9.8).

Combining (9.8) with (9.7), we find

$$\begin{aligned} &\lambda f(z_1) + (1-\lambda)f(z_2) - f(z) \\ &\geq \frac{c_p\gamma}{2}\lambda(1-\lambda)|z_1-z_2|^2 - \Big[\omega_0(|z|)\frac{1+|z|^p}{|z|^p} + \tilde{\omega}(|z|)\Big](|z_1|+|z_2|)^{p-2}|z|^2 \end{aligned}$$

for $|z| \geq M_0 \geq 1$. This implies the claim (2.17) with the constant $c_p\gamma/2$ and the function $\omega(t) := \omega_0(t)\frac{1+t^p}{t^p} + \tilde{\omega}(t)$.

The proof of (2.18) proceeds analogously and we omit the details.

Now assume that (ii) is satisfied with constants γ, Γ, M_0 and a function $\omega : [M_0, \infty) \to [0, \infty)$. As above, we may assume that ω is decreasing. We claim that (iii) is satisfied with constants $\frac{\gamma}{2}, \Gamma + \frac{\gamma}{2}, M_0$ and the function $\varphi(t) := \frac{2}{\gamma}\omega(t)$ for $t \geq M_0$. For $\lambda \in [0, 1]$ and $z_1, z_2 \in \mathbb{R}^m$, we write $z := \lambda z_1 + (1 - \lambda)z_2$. Suppose that (2.19) is satisfied for some $t \geq M_0$. Combining this with the assumption (2.17), we infer

$$\begin{aligned}\lambda f(z_1) &+ (1 - \lambda)f(z_2) - f(z)\\ &\geq \left(\gamma - \frac{\omega(|z|)}{\varphi(t)}\right)\lambda(1 - \lambda)(|z_1| + |z_2|)^{p-2}|z_2 - z_1|^2.\end{aligned}$$

By the choice of φ and the monotonicity of ω, the first factor on the right-hand side is bounded from below by $\gamma/2$. This proves (2.20). The claim (2.21) follows analogously.

Finally, we assume that (iii) holds with constants γ, Γ and M_0 and a function φ. In order to show the validity of (i), we start with some preparations: We let

$$\eta(z) := \begin{cases} \frac{1}{4}N_m(1 - |z|^2)^2 & \text{for } |z| \leq 1 \\ 0 & \text{for } |z| \geq 1 \end{cases},$$

where N_m is chosen in such a way that

$$\int_{\mathbb{R}^m} \eta(z)\, dz = 1.$$

We observe that η is C^1 and twice weakly differentiable on $\mathbb{R}^m$ with support in $\overline{B_1}$. We compute

$$\nabla\eta(z) = \begin{cases} -N_m(1 - |z|^2)z & \text{for } |z| < 1 \\ 0 & \text{for } |z| > 1 \end{cases}$$

and

$$\nabla^2\eta(z) = \begin{cases} N_m\Big(2z \otimes z - (1 - |z|^2)\,\mathrm{I}_m\Big) & \text{for } |z| < 1 \\ 0 & \text{for } |z| > 1 \end{cases}. \tag{9.10}$$

Here, I_m denotes the $m\times m$-unit matrix. Next, we introduce the scaled kernels

$$\psi(R, z) := \frac{1}{R^m}\eta\left(\frac{z}{R}\right) \qquad \text{for } R > 0 \text{ and } z \in \mathbb{R}^m.$$

Clearly, ψ is C^1 on $]0, \infty[\times\mathbb{R}^m$ and its second derivatives exist in the weak sense. Moreover, we have

$$\int_{\mathbb{R}^m} \psi(R,z)\,dz = 1 \tag{9.11}$$

for every $R > 0$. In particular, $\int_{\mathbb{R}^m} \psi(R,z)\,dz$ is independent of R and thus we get

$$\int_{\mathbb{R}^m} \partial_1\psi(R,z)\,dz = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^m} \partial_1^2\psi(R,z)\,dz = 0 \tag{9.12}$$

for every $R > 0$, where ∂_1 denotes the partial derivative with respect to the first argument R. Moreover, partial integration gives

$$\int_{\mathbb{R}^m} \partial_1\nabla_2\psi(R,z)\,dz = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^m} \nabla_2^2\psi(R,z)\,dz = 0 \tag{9.13}$$

for every $R > 0$, where ∇_2 is the total derivative with respect to the second argument z.

Now, we get back to the function φ. Enlarging φ if necessary we may assume that φ is decreasing. Similarly as in the proof of Theorem 2.14 we will need a smooth decreasing Φ with $\Phi(t) > \sqrt{\varphi(t)}$ for $t \gg 1$ and

$$\lim_{t\to\infty} \Phi(t) = \lim_{t\to\infty} \frac{\sqrt{\varphi(t)}}{\Phi(t)^3} = \lim_{t\to\infty} t\Phi'(t) = \lim_{t\to\infty} t^2\Phi''(t) = 0.$$

This time, choosing a smooth kernel $0 \le \theta \in C_0^\infty(-1,1)$ with $\int_{-1}^1 \theta(s)\,ds = 1$ one finds that $\Phi(t) := \int_{-1}^1 \theta(s)\sqrt[7]{\varphi(\log t - s - 1)}\,ds$ works. Next, we let

$$\begin{rcases} r(z) := 2\sqrt{\varphi(|z|/2)}|z| \\ R(z) := 2\Phi(|z|/2)|z| \end{rcases} \quad \text{for } |z| \gg 1$$

Then we clearly have

$$2r(z) < R(z) < \frac{1}{2}|z| \qquad \text{for } |z| \gg 1.$$

In addition, the above features of φ and Φ imply

$$\begin{aligned} \lim_{|z|\to\infty} \frac{R(z)}{|z|} &= \lim_{|z|\to\infty} \frac{|z|^2 r(z)}{R(z)^3} \\ &= \lim_{|z|\to\infty} |\nabla R(z)| = \lim_{|z|\to\infty} |z||\nabla^2 R(z)| = 0. \end{aligned} \tag{9.14}$$

From the above construction, the defining property of φ and Remark 2.17, we conclude that the conditions

$$|w_2 - w| \ge r(z) \qquad \text{and} \qquad |w_1 - w| \ge r(z) \tag{9.15}$$

for

$$w = \lambda w_1 + (1-\lambda) w_2 \in B_{R(z)}(z)$$

with $0 < \lambda < 1$ and $|z| \gg 1$ imply

$$\begin{aligned}\lambda f(w_1) + (1-\lambda) f(w_2) &\geq f(w) + \gamma(|w_1|+|w_2|)^{p-2}|w_2-w|\,|w_1-w|,\\ \lambda f(w_1) + (1-\lambda) f(w_2) &\leq f(w) + \Gamma(|w_1|+|w_2|)^{p-2}|w_2-w|\,|w_1-w|.\end{aligned} \tag{9.16}$$

We will use this property extensively in the remainder of the proof. Furthermore, we recall that by Remark 2.18, the function f satisfies the growth condition

$$0 \leq f(z) \leq C|z|^p \qquad \text{for } |z| \gg 1. \tag{9.17}$$

Finally, for $|z| \gg 1$ we construct

$$\tilde{g}(z) := \int_{\mathbb{R}^m} \psi(R(z), w) f(z-w)\, dw = \int_{\mathbb{R}^m} \psi(R(z), z-w) f(w)\, dw.$$

Then, employing in turn (9.11), $|\psi(R,w)| \leq CR^{-m}$, (9.16) and (9.17) we calculate

$$\begin{aligned}\frac{|\tilde{g}(z) - f(z)|}{|z|^p} &= \frac{1}{|z|^p}\left|\int_{B^+_{R(z)}} \psi(R(z), w)\big[f(z-w) + f(z+w) - 2f(z)\big]\, dw\right|\\ &\leq \frac{C}{R(z)^m |z|^p}\Bigg[\int_{B^+_{R(z)}\setminus B^+_{r(z)}} |f(z-w) + f(z+w) - 2f(z)|\, dw\\ &\qquad + \int_{B^+_{r(z)}} |f(z-w) + f(z+w) - 2f(z)|\, dw\Bigg]\\ &\leq C\left[\left(\frac{R(z)}{|z|}\right)^2 + \left(\frac{r(z)}{R(z)}\right)^m\right] \xrightarrow[|z|\to\infty]{} 0,\end{aligned}$$

where the superscript $^+$ indicates the intersection of the ball with the upper halfspace $\mathbb{R}^{m-1} \times (0, \infty)$. Moreover, a straightforward computation gives

$$\begin{aligned}D^2\tilde{g}(z) &= \int_{\mathbb{R}^m} \partial_1\psi(R(z), w) f(z-w)\, dw\, \nabla^2 R(z)\\ &\quad + \int_{\mathbb{R}^m} \partial_1^2\psi(R(z), w) f(z-w)\, dw\, \nabla R(z) \otimes \nabla R(z)\\ &\quad + \int_{\mathbb{R}^m} \partial_1\nabla_2\psi(R(z), w) f(z-w)\, dw \odot \nabla R(z)\\ &\quad + \int_{\mathbb{R}^m} \nabla_2^2\psi(R(z), w) f(z-w)\, dw\\ &=: I + II + III + IV,\end{aligned}$$

where $a \odot b := a \otimes b + b \otimes a$ denotes the symmetric product of $a, b \in \mathbb{R}^m$. In the remainder of the proof we will rewrite these four terms such that (9.15) is satisfied and thus (9.16) can be applied. Simplifying our notation again by writing r and R for $r(z)$ and $R(z)$, we start with I: Since ψ and consequently also $\partial_1\psi$ are even in their second argument with (9.12) we may write

$$I = \left[\int_{B_R^+\setminus B_r^+} \partial_1\psi(R,w)[f(z-w)+f(z+w)-2f(z)]\,dw \right.$$
$$\left. + \int_{B_r} \partial_1\psi(R,w)[f(z-w)-f(z)]\,dw\right]\nabla^2 R.$$

Noting $|\partial_1\psi(R,w)| \le CR^{-m-1}$ we estimate the first integral with (9.16) and the second one with (9.17) coming out with

$$|I| \le C|z|^{p-2}\left[1+\frac{|z|^2 r^m}{R^{m+2}}\right]R|\nabla^2 R| \le C|z|^{p-2}\left[1+\frac{|z|^2 r}{R^3}\right]|z||\nabla^2 R|. \tag{9.18}$$

Relying on (9.12) and (9.13) and on the fact that $\partial_1^2\psi$ and $\nabla_2^2\psi$ are also even in their second argument, estimates for II and IV, namely

$$|II| \le C|z|^{p-2}\left[1+\frac{|z|^2 r}{R^3}\right]|\nabla R|^2$$

and

$$|IV| \le C|z|^{p-2}\left[1+\frac{|z|^2 r}{R^3}\right],$$

can be obtained analogously and we omit the details.
The same reasoning does not work for III since $\nabla_2\psi(R,w)$ and $\partial_1\nabla_2\psi(R,w)$ are not even in w. Therefore, we will use a more sophisticated argument: Fixing a $\zeta \in \mathbb{R}^m$ with $|\zeta| = 1$ and using (9.16), we observe that for all $R < s < 2R$ there holds

$$\begin{aligned}
&\int_{\mathbb{R}^m} \zeta\cdot\partial_1\nabla_2\psi(R,w)f(z-w)\,dw \\
&\le \int_{\mathbb{R}^m} \zeta\cdot\partial_1\nabla_2\psi(R,w)\left[\frac{1}{2}f(z-s\zeta-w)+\frac{1}{2}f(z+s\zeta-w)\right]dw \\
&\qquad\qquad + C\frac{s^2}{R^2}|z|^{p-2} \\
&\le \frac{1}{2}\zeta\cdot\int_{\mathbb{R}^m}\left[\partial_1\nabla_2\psi(R,z-s\zeta-w)+\partial_1\nabla_2\psi(R,z+s\zeta-w)\right]f(w)\,dw \\
&\qquad\qquad + C|z|^{p-2} \\
&=: \frac{1}{2}III_s + C|z|^{p-2}.
\end{aligned} \tag{9.19}$$

Integrating the last integral with respect to the parameter s, we calculate

$$
\begin{aligned}
&\int_R^{2R} III_s\, ds \\
&= \Big[\int_{\mathbb{R}^m} \partial_1\psi(R, z - R\zeta - w) f(w)\, dw - \int_{\mathbb{R}^m} \partial_1\psi(R, z - 2R\zeta - w) f(w)\, dw \\
&\quad + \int_{\mathbb{R}^m} \partial_1\psi(R, z + 2R\zeta - w) f(w)\, dw - \int_{\mathbb{R}^m} \partial_1\psi(R, z + R\zeta - w) f(w)\, dw \Big].
\end{aligned}
$$

Each of the latter integrals can be estimated in the same way as I in (9.18), and consequently,

$$
\int_R^{2R} III_s\, ds \le C|z|^{p-2}\left[1 + \frac{|z|^2 r}{R^3}\right] R.
$$

We conclude that we can find a parameter $s \in (R, 2R)$ with

$$
III_s \le C|z|^{p-2}\left[1 + \frac{|z|^2 r}{R^3}\right].
$$

Applying (9.19) with this value of s, we deduce

$$
\int_{\mathbb{R}^m} \zeta \cdot \partial_1 \nabla_2 \psi(R, w) f(z - w)\, dw \le C|z|^{p-2}\left[1 + \frac{|z|^2 r}{R^3}\right].
$$

and we arrive at the estimate

$$
|III| \le C|z|^{p-2}\left[1 + \frac{|z|^2 r}{R^3}\right] |\nabla R|.
$$

Finally, we will provide a finer estimate from below for $IV(\zeta, \zeta)$: Recalling (9.10), we have

$$
\nabla_2^2 \psi(R, w) = N_m R^{-m-2}\left[2\, \frac{w \otimes w}{R^2} - \left(1 - \frac{|w|^2}{R^2}\right) \mathrm{I}_m\right] \qquad \text{for } |w| < R.
$$

Now we fix - once more - a $\zeta \in \mathbb{R}^m$ with $|\zeta| = 1$ and use the superscripts $^+$ for the intersection of a ball or a sphere with the halfspace $\{w \in \mathbb{R}^m : w \cdot \zeta > 0\}$ and * for the reflection at $\{\zeta\}^\perp$ as in the proof of Theorem 2.14. Moreover, for $y \in \{\zeta\}^\perp \cap B_R$ we abbreviate $\rho_y := \frac{1}{\sqrt{3}}\sqrt{R^2 - |y|^2}$, so that every $w \in B_R$ can be written as $w = y + \sigma\zeta$ with $y \in \{\zeta\}^\perp \cap B_R$ and $\sigma \in (-\sqrt{3}\rho_y, \sqrt{3}\rho_y)$. Thus, Fubini's theorem yields

$$
\begin{aligned}
&IV(\zeta,\zeta)\\
&= \int_{\mathbb{R}^m} \nabla_2^2\psi(R,w)(\zeta,\zeta) f(z-w)\,dw\\
&= \frac{N_m}{R^{m+2}} \int_{B_R} \left[2\,\frac{(w\cdot\zeta)^2}{R^2} - \left(1-\frac{|w|^2}{R^2}\right)\right] f(z-w)\,dw\\
&= \frac{N_m}{R^{m+2}}\int_{\{\zeta\}^\perp\cap B_R} \int_{-\sqrt{3}\rho_y}^{\sqrt{3}\rho_y} \left[2\frac{\sigma^2}{R^2} - \left(1-\frac{|y|^2+\sigma^2}{R^2}\right)\right] f(z-y-\sigma\zeta)\,d\sigma\,d\mathcal{H}^{m-1}(y)\\
&=: \frac{N_m}{R^{m+4}} \int_{\{\zeta\}^\perp\cap B_R} IV_y\,d\mathcal{H}^{m-1}(y).
\end{aligned}
\tag{9.20}
$$

In addition, we have

$$
\begin{aligned}
IV_y &= \int_{-\sqrt{3}\rho_y}^{\sqrt{3}\rho_y} 3(\sigma^2-\rho_y^2) f(z-y-\sigma\zeta)\,d\sigma\\
&= \left(\int_{A_0} + \int_{A_1} + \int_{A_2}\right) 3(\sigma^2-\rho_y^2)\Big(f(z-y-\sigma\zeta) - f(z-y)\Big)\,d\sigma\\
&=: IV_{0;y} + IV_{1;y} + IV_{2;y},
\end{aligned}
$$

where we decomposed the interval $[-\sqrt{3}\rho_y, \sqrt{3}\rho_y]$ into the three sets

$$
\begin{aligned}
A_0 &:= \Big[-\sqrt{3}\rho_y, -\sqrt{2}\rho_y\Big] \cup \Big[\sqrt{2}\rho_y, \sqrt{3}\rho_y\Big],\\
A_1 &:= \Big[-\rho_y, 0\Big] \cup \Big[\rho_y, \sqrt{2}\rho_y\Big] \quad \text{and} \quad A_2 := -A_1.
\end{aligned}
$$

We first consider the case $\rho_y \geq r$. In tis case, we can use (9.16) to estimate

$$
\begin{aligned}
IV_{0;y} &= \int_{\sqrt{2}\rho_y}^{\sqrt{3}\rho_y} 3(\sigma^2-\rho_y^2)\Big(f(z-y-\sigma\zeta) + f(z-y+\sigma\zeta) - 2f(z-y)\Big)\,d\sigma\\
&\geq c\rho_y^2 \int_{\sqrt{2}\rho_y}^{\sqrt{3}\rho_y} \sigma^2|z|^{p-2}\,d\sigma = c\rho_y^5|z|^{p-2}.
\end{aligned}
$$

In order to estimate the integrals over A_1 and A_2, we define the decreasing bijections

$$
T_+ : \Big[-\sqrt{2}\rho_y, -\rho_y\Big] \to \Big[0,\rho_y\Big] \qquad \text{and} \qquad T_- : \Big[\rho_y, \sqrt{2}\rho_y\Big] \to \Big[-\rho_y, 0\Big]
$$

by

$$
T_\pm(\sigma) := \pm\sqrt{\rho_y^2 \pm \sigma\sqrt{2\rho_y^2-\sigma^2}}.
$$

These transformations have been chosen in such a way that

$$3(T_\pm^2(\sigma)-\rho_y^2)T_\pm'(\sigma) = -3(\sigma^2-\rho_y^2)\frac{\sigma}{T_\pm(\sigma)}.$$

Thus, the transformation rule yields

$$\begin{aligned} IV_{1;y} = \int_{\rho_y}^{\sqrt{2}\rho_y} 3(\sigma^2-\rho_y^2)\frac{T_-(\sigma)-\sigma}{T_-(\sigma)}\biggl(&\frac{-\sigma}{T_-(\sigma)-\sigma} f(z-y-T_-(\sigma)\zeta) \\ &+\frac{T_-(\sigma)}{T_-(\sigma)-\sigma} f(z-y-\sigma\zeta) - f(z-y)\biggr)\,d\sigma. \end{aligned}$$

Recalling that we are in the case $\rho_y \geq r$ we can employ (9.16) to infer that the integrand is non-negative for $T_-(\sigma) \geq r$. On the other hand, if $T_-(\sigma) < r$, we use the growth estimates (9.17). We conclude, with $a := T_-^{-1}(r) = \sqrt{\rho_y^2 + r\sqrt{2\rho_y^2 - r^2}}$,

$$IV_{1;y} \geq -C|z|^p \left(\int_{-r}^{0} (\sigma^2-\rho_y^2)\,d\sigma + \int_{\rho_y}^{a} (\sigma^2-\rho_y^2)\,d\sigma\right) \geq -C|z|^p r R^2.$$

The term $IV_{2;y}$ can be treated in the same way, using the transformation T_+ instead of T_-. Combining the estimates for $IV_{0;y}$, $IV_{1;y}$ and $IV_{2;y}$ we have proven

$$IV_y \geq c\rho_y^5|z|^{p-2} - C|z|^p r R^2 \qquad \text{provided } \rho_y \geq r.$$

In the case $\rho_y < r$ starting from the growth estimate (9.17) one finds $IV_y \geq -C|z|^p r R^2$.

Inserting the estimates for both cases into (9.20) we arrive at

$$\begin{aligned} IV(\zeta,\zeta) &\geq c|z|^{p-2}R^{-m-4}\int_{\{y\in\{\zeta\}^\perp : |y|^2 < R^2-3r^2\}} (R^2-|y|^2)^{5/2}\,d\mathcal{H}^{m-1}(y) \\ &\quad - CR^{-m-4}R^{m-1}|z|^p r R^2 \\ &\geq \left(c - C\frac{|z|^2 r}{R^3}\right)|z|^{p-2}. \end{aligned}$$

Here, recalling $2r < R$ and thus $\sqrt{R^2-3r^2} \geq \frac{1}{2}R$ we used that the integrand in the previous formula can be estimated from below by cR^5 for $|y| < \frac{1}{2}R$. Finally, collecting all our estimates for I, II, III and IV and invoking (9.14) we have proved that

$$\begin{aligned} |D^2\tilde{g}(z)| &\leq C|z|^{p-2}, \\ D^2\tilde{g}(z)(\zeta,\zeta) &\geq c|z|^{p-2}|\zeta|^2, \end{aligned}$$

holds for $|z| \gg 1$ and all $\zeta \in \mathbb{R}^m$. Now, (i) follows as in the proof of Theorem 2.14: Smoothing again we may replace $\tilde{g}$ by a smooth function g_* with the same properties and [52, Corollary 4.3] finally yields the claim. $\square$

A The convex envelope

Proof of Lemma 2.12. Setting

$$e_p(z) := \frac{1}{p}(1+|z|^2)^{\frac{p}{2}}$$

we have

$$D^2 e_p(z)(\xi,\xi) \le (p-1)(1+|z|)^{p-2}|\xi|^2 \qquad \text{for all } z,\xi \in \mathbb{R}^{Nn}.$$

Now we fix an arbitrary $0 < \varepsilon \le \frac{\gamma}{2(p-1)}$ and let

$$\tilde{g}(z) := g(z) - \varepsilon e_p(z).$$

Then our asumptions give

$$\tilde{g}(z) \le f(z) \qquad \text{for } |z| \gg 1$$

and

$$\begin{aligned} D^2\tilde{g}(z)(\xi,\xi) &= D^2 g(z)(\xi,\xi) - \varepsilon\, D^2 e_p(z)(\xi,\xi) \\ &\ge [\gamma - \varepsilon(p-1)](1+|z|)^{p-2}|\xi|^2 \ge \frac{\gamma}{2}(1+|z|)^{p-2}|\xi|^2 \end{aligned}$$

for all $z,\xi \in \mathbb{R}^{Nn}$. Keeping in mind the preceding two properties of $\tilde{g}$ we may apply [52, Lemma 4.1] to $\min\{f,\tilde{g}\}$ and obtain $C(\min\{f,\tilde{g}\})(z) = \tilde{g}(z)$ for $|z| \gg 1$. In particular, this implies $Cf(z) \ge \tilde{g}(z)$ for $|z| \gg 1$ and we infer

$$f(z) - 2\varepsilon|z|^p \le g(z) - \varepsilon|z|^p \le Cf(z) \le f(z)$$

for sufficiently large values of $|z|$. Now, the claim is obvious. $\square$

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