Regularity Theorems for Degenerate Quasiconvex Energies with (p, q)-Growth

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Abstract. We study autonomous integrals

$$F[u] := \int_\Omega f(Du)\,dx \qquad \text{for } u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$$

in the multidimensional calculus of variations, where the integrand f is a strictly quasiconvex function satisfying the (p, q)-growth conditions

$$\gamma|\xi|^p \le f(\xi) \le \Gamma(1 + |\xi|^q)$$

with exponents $1 < p \le q < p + \frac{\min\{2,p\}}{2n}$. Imposing the additional assumption that f resembles the degenerate behavior of the p-energy density, we establish a partial $C^{1,\alpha}$-regularity theorem for F-minimizers and a similar theorem for minimizers of a relaxed functional.

Our results cover the model case of polyconvex integrands

$$f(\xi) := \frac{1}{p}|\xi|^p + h(\det \xi),$$

where h is a smooth convex function with $\frac{q}{n}$-growth.

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1 Introduction

For $n, N \in \mathbb{N}$ with $n \ge 2$ and a bounded open subset Ω of $\mathbb{R}^n$, we consider multidimensional variational integrals

$$F[u] := \int_\Omega f(Du)\,dx \qquad \text{for vector-valued functions } u \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^N),$$

where throughout this paper $f : \mathbb{R}^{nN} \to [0, \infty[$ denotes a continuous function. We are interested in the minimization of F with respect to Dirichlet boundary values specifying the following notion of a minimizer.

Definition 1.1 (Minimizer). Let $p \in [1,\infty[$. We say that $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a $W^{1,p}$-*minimizer* of F on Ω iff we have $F[u] < \infty$ and $F[u] \leq F[u+\varphi]$ for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. In the following p will be fixed and we will omit the prefix $W^{1,p}$ calling u just a *minimizer* of F on Ω.

It is well known that to ensure the existence and a certain amount of regularity of minimizers one has to impose growth, coercivity, and convexity conditions on f. In the case of standard growth conditions these issues are nowadays well understood. We mention the semicontinuity and existence results of [Mo, Me, AF1, Ma1] and the regularity results in [E, FH1, GM1, AF2, CFM], and refer to the recent monograph [Giu] and the survey [Mi2] for an exhaustive list of references. Here we will focus on the more general (p,q)-growth conditions

$$\gamma|\xi|^p \leq f(\xi) \leq \Gamma(1+|\xi|^q) \qquad \text{for all } \xi \in \mathbb{R}^{nN}, \tag{1.1}$$

where $1 < p \leq q < \infty$ are the growth exponents and $\gamma, \Gamma > 0$ are constants. Note that the case of standard growth conditions corresponds to taking $p = q$ in (1.1).

Now, if (1.1) is complemented with a convexity condition, the existence of minimizers can be proven by means of the direct method. Let us summarize the known results, which are relevant for our setting:

Theorem 1.2 (Existence, [S1, S2]). *We assume that f satisfies* (1.1), *where $1 < p \leq q < \infty$, and $u_0 \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $F[u_0] < \infty$ is given. Then, we have the following existence results:*

(I) If f is convex, then there is a minimizer $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ of F on Ω;

(II) If f is $W^{1,p}$-quasiconvex with $q < \frac{np}{n-1}$, then there is a minimizer $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ of F on Ω;

(III) If Ω has a C^0-boundary, then there is a minimizer $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ of $\mathscr{F}_{\mathrm{loc}}$ on Ω.

Next, we will comment on this theorem introducing by the way the terminology, which has already been used in the preceding statement.

We start recalling Morrey's concept of quasiconvexity.

Definition 1.3 (Quasiconvexity, [Mo]). f is called *quasiconvex* iff

$$⨍_{B_1} f(\xi + D\varphi)\,dx \geq f(\xi) \tag{1.2}$$

holds for all $\xi \in \mathbb{R}^{nN}$ and all φ in the space $C^\infty_{\mathrm{cpt}}(B_1;\mathbb{R}^N)$ of smooth functions $B_1 \to \mathbb{R}^N$ with compact support in the open unit ball B_1 in $\mathbb{R}^n$.

By Jensen's inequality quasiconvexity is a generalization of the classical notion of convexity. Nowadays, it has become a standard assumption in the multidimensional calculus of variations and is well known to be a key condition for both, the existence and the regularity of minimizers. However, in the presence of the (p, q)-growth conditions (1.1) quasiconvexity is a quite delicate notion and the following variant has turned out to be crucial.

Definition 1.4 ($W^{1,p}$-Quasiconvexity, [BM]). f is called $W^{1,p}$-*quasiconvex* iff (1.2) holds for all $\xi \in \mathbb{R}^{nN}$ and all $\varphi \in W_0^{1,p}(B_1; \mathbb{R}^N)$.

Note that in the presence of the growth condition (1.1) $W^{1,q}$-quasiconvexity and quasiconvexity are equivalent. In particular, in the case $p = q$ of standard growth conditions $W^{1,p}$-quasiconvexity is the same as quasiconvexity, while in general these notions are known to be different. Further information concerning $W^{1,p}$-quasiconvexity and its relations to lower semicontinuity and existence can be found in [BM].

Now we are ready for some comments on the proof of Theorem 1.2: All the statements of the theorem can be proven by means of the direct method and the crucial ingredient in the proofs is sequential weak lower semicontinuity on $W^{1,p}(\Omega; \mathbb{R}^N)$. In the convex case (I) such a semicontinuity result for F is classical, while in the $W^{1,p}$-quasiconvex case (II) it has been proven in [S1, Theorem 4.4], relying heavily on methods from [FMal].

However, in more general situations than (I) and (II) it may occur - actually, this happens in the case (III) - that F fails to have adequate semicontinuity properties for the application of the direct method. Then, the basic idea of relaxation is to replace F by a related functional exhibiting better semicontinuity properties. For $p = q$ this issue is well understood and we refer to [D1, D2] for the basic facts of relaxation. In the more general setting of the (p, q)-growth conditions (1.1) the semicontinuity theorems of [FMal, K] suggest to investigate the following relaxed functional appearing in the third part of Theorem 1.2.

Definition 1.5 ([FMal]). The *relaxed functional* $\mathscr{F}_{\mathrm{loc}}$ is defined by

$$\mathscr{F}_{\mathrm{loc}}[u] := \inf \left\{ \liminf_{k\to\infty} F[u_k] \;:\; \begin{array}{c} u_k \in W^{1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^N) \cap W^{1,p}(\Omega; \mathbb{R}^N), \\ u_k \underset{k\to\infty}{\rightharpoonup} u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^N) \end{array} \right\}$$

for $u \in W^{1,p}(\Omega; \mathbb{R}^N)$.

With this definition at hand we specify the concept of a *minimizer* of $\mathscr{F}_{\mathrm{loc}}$ in an evident way:

Definition 1.6 (Minimizer). We say that $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a minimizer of $\mathscr{F}_{\mathrm{loc}}$ on Ω iff we have $\mathscr{F}_{\mathrm{loc}}[u] < \infty$ and $\mathscr{F}_{\mathrm{loc}}[u] \le \mathscr{F}_{\mathrm{loc}}[u + \varphi]$ for all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N)$.

Next, let us describe some properties of $\mathscr{F}_{\mathrm{loc}}$ (assuming $1 < p \le q < \frac{np}{n-1}$ for the moment): If some convexity condition is imposed on f, then $\mathscr{F}_{\mathrm{loc}}$ should be related to F. More precisely it has been shown in [FMal, S2] (see also [K, Ma2]) that $\mathscr{F}_{\mathrm{loc}}$ coincides with F for a convex f and that for a quasiconvex f one has

$$\begin{aligned}\mathscr{F}_{\mathrm{loc}}[u] &\geq F[u] \qquad \text{for all } u \in W^{1,p}(\Omega;\mathbb{R}^N),\\ \mathscr{F}_{\mathrm{loc}}[u] &= F[u] \qquad \text{for all } u \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \cap W^{1,p}(\Omega;\mathbb{R}^N).\end{aligned}$$

Therefore, in the latter case, $\mathscr{F}_{\mathrm{loc}}$ can be understood as a way to extend F from a smaller class of functions to a larger one by means of its semicontinuity properties. This corresponds to a classical idea in the calculus of variations, namely to the concept of the Lebesgue-Serrin extension; see for instance [Ma2] for an introduction. We mention that a similar representation of $\mathscr{F}_{\mathrm{loc}}$ involving the quasiconvex envelope of f holds even if f fails to be quasiconvex (see [FMal, S2]).

It is important to notice that $\mathscr{F}_{\mathrm{loc}}$ does not behave completely like an integral anymore. Indeed, it has been proven in [FMal] that $\mathscr{F}_{\mathrm{loc}}$ depends on Ω like a Radon measure. Furthermore, the results of [BFM] imply that the absolutely continuous part of this measure coincides with F in the quasiconvex case. However, the singular part does not necessarily vanish (see [AD, FMal, FMar]).

Finally, we return to the proof of Theorem 1.2: Since $\mathscr{F}_{\mathrm{loc}}$ is designed to have better semicontinuity properties than F, it is not surprising that $\mathscr{F}_{\mathrm{loc}}$ turns out to be sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^N)$ under quite weak assumptions. In fact, this can be proven by simple arguments, which have been formulated in [S2, Lemma 5.1].

Recapitulating, adequate semicontinuity properties are available in all the cases of Theorem 1.2 and the above statement follows easily by means of the direct method.

With these existence results at hand it is natural to ask for the regularity of the minimizers. To this aim we will suppose, in addition, $f \in C^2(\mathbb{R}^{nN})$ and

$$|D^2 f(\xi)| \leq \Lambda \left[(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-2}{2}} \right] \tag{1.3}$$

for all $\xi \in \mathbb{R}^{nN}$, where $\Lambda > 0$ is a constant and $\mu \in [0,\infty[$ is a parameter controlling the degeneration of $D^2 f$. Note that the right inequality in (1.1) is a consequence of (1.3) and that (1.3) should be combined with a strict convexity assumption linked to p and μ. Precisely, this assumption reads

$$D^2 f(\xi)(\sigma,\sigma) \geq \lambda (\mu^2 + |\xi|^2)^{\frac{p-2}{2}} |\sigma|^2 \tag{1.4}$$

in the convex case and

$$\fint_{B_1} f(\xi + D\varphi)\, dx \geq f(\xi) + \lambda \fint_{B_1} (\mu^2 + |\xi|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, dx \tag{1.5}$$

in the ($W^{1,p}$-)quasiconvex case, where $\lambda > 0$ is the ellipticity constant.

The nondegenerate case $\mu > 0$ can be reduced to the case $\mu = 1$, possibly with different constants. Note that in this case the condition (1.3) reads $|D^2 f(\xi)| \leq \Lambda(1 + |\xi|^2)^{\frac{q-2}{2}}$. Meanwhile there are quite a lot of known regularity results under such growth conditions including everywhere regularity results in the scalar case $N = 1$ and under additional structure conditions. We mention that the investigation of such conditions started with a series of papers by Marcellini and refer to [Bi, Mi2] for an overview. Here, however, we will not discuss such issues concentrating on the vector-valued case and partial regularity of minimizers. We mention the following results:

Theorem 1.7 (Partial Regularity in the Nondegenerate Case, [PS, BF, S1, S2]). *Assume* $\mu = 1$, $1 < p \leq q < \infty$, *and* $f \in C^2(\mathbb{R}^{nN})$ *with* (1.3). *Furthermore let one of the following conditions be fulfilled:*

- *(I)* f *satisfies the strict convexity condition* (1.4) *for all* $\xi, \sigma \in \mathbb{R}^{nN}$ *with* $q < \frac{n+2}{n} p$ *and* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *is a minimizer of* F*;*
- *(II)* f *satisfies the strict* $W^{1,p}$*-quasiconvexity condition* (1.5) *for all* $\xi \in \mathbb{R}^{nN}$ *and all* $\varphi \in W_0^{1,p}(B_1; \mathbb{R}^N)$ *with* $q < p + \frac{\min\{2,p\}}{2n}$ *and* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *is a minimizer of* F*;*
- *(III)* f *satisfies* (1.1) *and the strict quasiconvexity condition* (1.5) *for all* $\xi \in \mathbb{R}^{nN}$ *and all* $\varphi \in C^\infty_{\mathrm{cpt}}(B_1; \mathbb{R}^N)$ *with* $q < p + \frac{\min\{2,p\}}{2n}$ *and* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *is a minimizer of* $\mathscr{F}_{\mathrm{loc}}$.

Then, there is an open subset Ω_0 *of* Ω *with* $|\Omega \setminus \Omega_0| = 0$ *such that one has* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0; \mathbb{R}^N)$ *for all* $\alpha \in]0, 1[$.

The case (I) of the theorem is a result of [BF, Theorem 1.1], which has been previously established in [PS, Theorem 2.1] under more restrictive conditions on p and q, and (II) is essentially a restatement of [S1, Theorem 5.2]. Let us make a few remarks on (III): Indeed, it is not usual in the calculus of variations to prove regularity for minimizers of a relaxed functional. Nevertheless, higher integrability of such minimizers in a somewhat different situation has already been investigated in [ELM2]. Concerning $C^{1,\alpha}$-regularity the first result seems to be (III) above which was obtained in [S2, Theorem 6.3]. There, one of the crucial points was the validity of Euler's equation for $\mathscr{F}_{\mathrm{loc}}$-minimizers, which we restate in Lemma 5.2.

Finally, in the situation (I) one can even prove that Ω_0 can be chosen such that the Hausdorff dimension of $\Omega \setminus \Omega_0$ does not exceed $n - 2$, while no analog is known in the quasiconvex cases; see [KM] for an interesting result in this context.

In the present paper we deal with the degenerate case $\mu = 0$. From (1.3), (1.4), and (1.5) respectively we see $D^2 f(0) = 0$ for $p > 2$, which means that the ellipticity of $D^2 f$ degenerates at 0, while $D^2 f$ is forced to behave singularly near the origin in the case $p < 2$. Clearly, for $p = 2$ degeneration cannot occur. Therefore, we will assume henceforth that $p \neq 2$ holds. At this stage we should mention that the setting described above is not completely consistent. Indeed, in the case $p < 2$ the assumption

$f \in C^2(\mathbb{R}^{nN})$ prevents the singular behavior of $D^2 f$ mentioned before. Therefore we shall weaken that assumption, replacing it by

$$f \in \begin{cases} C^2(\mathbb{R}^{nN}) & \text{for } p > 2 \\ C^2(\mathbb{R}^{nN} \setminus \{0\}) \cap C^1(\mathbb{R}^{nN}) & \text{for } p < 2 \end{cases}. \tag{1.6}$$

For degenerate integrands, no direct analog of Theorem 1.7 is known, not even in the case $p = q$ of standard growth conditions. However, assuming, in addition, the structure condition

$$f(\xi) = g\left(\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij} b^{\alpha\beta} \xi_\alpha^i \xi_\beta^j\right) \tag{1.7}$$

for some $g : [0, \infty[\to \mathbb{R}$ and positive definite matrices (a_{ij}), $(b^{\alpha\beta})$ and the Hölder condition

$$|D^2 f(\xi_1) - D^2 f(\xi_2)| \leq \begin{cases} L(|\xi_1|^2 + |\xi_2|^2)^{\frac{p-2-\beta}{2}} |\xi_1 - \xi_2|^\beta & \text{for } p > 2 \\ L|\xi_1|^{p-2} |\xi_2|^{p-2} (|\xi_1|^2 + |\xi_2|^2)^{\frac{2-p-\beta}{2}} |\xi_1 - \xi_2|^\beta & \text{for } p < 2 \end{cases} \tag{1.8}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^{nN} \setminus \{0\}$ and some $\beta \in]0, \max\{p-2, 2-p\}[$, minimizers are $C^{1,\alpha}_{\mathrm{loc}}$-regular everywhere on Ω for some $\alpha \in]0, 1[$. The theorem just described has essentially been established in Uhlenbeck‘s celebrated paper [U] (see also [GM2, AF3, H1, C] and Theorem 5.7). Note that this was the first regularity result covering the case of the p-energy density

$$e_p(\xi) := \frac{1}{p} |\xi|^p. \tag{1.9}$$

Recently, it has been shown in [DM2], still assuming standard growth, that for the purpose of proving partial regularity (1.7) can be replaced by an asymptotic condition near the origin, where the degeneration occurs. Let us restate this result:

Theorem 1.8 ([DM2]). *Assume* $2 \neq p = q \in]1, \infty[$, (1.6), (1.3), *and the quasiconvexity condition* (1.5) *for all* $\xi \in \mathbb{R}^{nN}$ *and all* $\varphi \in C^\infty_{\mathrm{cpt}}(B_1; \mathbb{R}^N)$ *with* $\mu = 0$. *Furthermore, assume the validity of the Hölder condition* (1.8) *and of the asymptotic condition*

$$\frac{Df(\xi) - De_p(\xi)}{|\xi|^{p-1}} \xrightarrow[\xi \to 0]{} 0. \tag{1.10}$$

Then, there is an open subset Ω_0 *of* Ω *with* $|\Omega \setminus \Omega_0| = 0$ *such that* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0; \mathbb{R}^N)$ *holds for some* $\alpha \in]0, 1[$ *depending only on* n, N, *and* p.

In the present paper we will prove a degenerate counterpart of Theorem 1.7, relying on an assumption like (1.10). This regularity result can also be understood as the generalization of Theorem 1.8 to (p, q)-growth conditions.

Let us briefly outline the plan of the paper:

In Section 2 we state our assumptions and our main result concerning partial regularity. Afterwards, in Section 3 we discuss some simple polyconvex model integrands. Subsequently, in Section 4 we introduce some terminology and collect a few preliminaries, while the regularity proof is carried out in Section 5.

2 Assumptions and main result

We restate the assumptions explained in the introduction for the degenerate case $\mu = 0$. We begin recalling that throughout this paper f is assumed to be a continuous function on $\mathbb{R}^{nN}$ with nonnegative real values. We require the following smoothness, growth, and coercivity conditions for all $\xi \in \mathbb{R}^{nN}$:

$$f \in \begin{cases} C^2(\mathbb{R}^{nN}) & \text{for } p > 2 \\ C^2(\mathbb{R}^{nN} \setminus \{0\}) \cap C^1(\mathbb{R}^{nN}) & \text{for } p < 2 \end{cases}, \tag{2.1}$$

$$f(\xi) \leq \Gamma(1 + |\xi|^q), \tag{2.2}$$

$$f(\xi) \geq \gamma|\xi|^p. \tag{2.3}$$

Here, γ and Γ are positive constants. Next, we impose a condition on the second derivatives of f. However, (1.3) would rule out some of model integrands of Section 3. Therefore, we impose the following weakened version: There is a $\varepsilon > 0$ such that for $|\xi| \leq \varepsilon$ we have

$$|D^2 f(\xi)| \leq \Lambda|\xi|^{p-2}. \tag{2.4}$$

We stress that (2.4) is assumed only for small values of $|\xi|$. Contrary, for large values of $|\xi|$ we impose only (2.2), while the growth of $D^2 f$ is completely unrestricted. The technique, which allows to deal with such weak growth conditions has been introduced in [AF2]. Note that (2.4) implies $D^2 f(0) = 0$ for $p > 2$.

In, addition we will impose one of the following three convexity conditions for all $M > 0$, all $\xi \in \mathbb{R}^{nN}$ with $|\xi| \leq M + 1$, all $\sigma \in \mathbb{R}^{nN}$, and all $\varphi : B_1 \to \mathbb{R}^N$ of the specified class:

$$D^2 f(\xi)(\sigma, \sigma) \geq \lambda|\xi|^{p-2}|\sigma|^2, \tag{2.5}$$

$$⨍_{B_1} f(\xi + D\varphi)\, dx \geq f(\xi) + \lambda_M ⨍_{B_1} (|\xi|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\, dx \quad (\varphi \in W_0^{1,p}), \tag{2.6}$$

$$⨍_{B_1} f(\xi + D\varphi)\, dx \geq f(\xi) + \lambda_M ⨍_{B_1} (|\xi|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\, dx \quad (\varphi \in C^\infty_{\mathrm{cpt}}). \tag{2.7}$$

As discussed in the introduction these conditions are degenerate p-versions of strict convexity, strict $W^{1,p}$-quasiconvexity, and strict quasiconvexity respectively with positive convexity constants λ and λ_M. Note that in the convex case (2.5) we do not allow the convexity constant to depend on M. Nevertheless, henceforth we will write λ_M instead of λ even if we are working with (2.5). We mention that, adding a constant to f if required, (2.3) is a consequence of (2.5), but not of (2.6) or (2.7). However, we will not use this fact.

As suggested by the occurrence of (1.10) we need an asymptotic condition near the origin:

$$\frac{Df(\xi) - Df(0) - De_p(\xi)}{|\xi|^{p-1}} \underset{\xi\to 0}{\longrightarrow} 0. \tag{2.8}$$

Note that we could replace e_p in (2.8) by every integrand satisfying (1.3), (1.4), (1.7), and (1.8) with $p = q$ and $\mu = 0$. Indeed, these integrands can be treated in exactly the same fashion as e_p noting that versions of the p-harmonic approximation lemma and Theorem 5.7 hold in this more general setting. However, for sake of simplicity we restrict ourselves to e_p.

Finally, we need to control the continuity properties of $D^2 f$ near the origin. Here, we use a generalization of (1.8), which, however, stays in the same spirit: There are an $\varepsilon > 0$ and a modulus of continuity ν (i. e. $\nu : [0,\infty[\to [0,\infty[$ with $\lim_{\omega\searrow 0}\nu(\omega) = 0$) such that for $|\xi_1|, |\xi_2| \le \varepsilon$ we have

$$|D^2 f(\xi_1) - D^2 f(\xi_2)| \le \begin{cases} (|\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} \nu\left(\frac{|\xi_1-\xi_2|^2}{|\xi_1|^2+|\xi_2|^2}\right) & \text{for } p > 2 \\ \left(\frac{|\xi_1|^2+|\xi_2|^2}{|\xi_1|^2|\xi_2|^2}\right)^{\frac{2-p}{2}} \nu\left(\frac{|\xi_1-\xi_2|^2}{|\xi_1|^2+|\xi_2|^2}\right) & \text{for } p < 2 \end{cases}. \tag{2.9}$$

We will assume in the following that the constants ε occurring in relation with (2.4) and (2.9) coincide. Clearly, this is not restrictive. Furthermore, note that (2.4), (2.5), and (2.9) are intended to hold for $p < 2$ only if the arguments of $D^2 f$ are not zero.

We find it worth remarking that the assumptions (2.4), (2.8) and (2.9) can all be deduced from the condition

$$\frac{D^2 f(\xi) - D^2 e_p(\xi)}{|\xi|^{p-2}} \underset{\xi\to 0}{\longrightarrow} 0. \tag{2.10}$$

For (2.4) and (2.8) it is easy to show this. For (2.9) things are a bit more involved and we will not give a detailed proof. Actually, the crucial observations are that $D^2 e_p$ and $|-|^{p-2}$ satisfy (1.8) (because of their homogeneity) and that $\frac{D^2 f - D^2 e_p}{|-|^{p-2}}$ is uniformly continuous near the origin.

Next, we come to the statement of our main results.

Assumption 2.1. Assume $1 < p \le q < \infty$, $p \neq 2$ and that one of the following sets of conditions holds:

(I) f satisfies the convexity condition (2.5) with $q < \frac{np}{n-1}$ and $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a minimizer of F on Ω;

(II) f satisfies the $W^{1,p}$-quasiconvexity condition (2.6) with $q < p + \frac{\min\{2,p\}}{2n}$ and $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a minimizer of F on Ω;

(III) f satisfies (2.3) and the quasiconvexity condition (2.7) with $q < p + \frac{\min\{2,p\}}{2n}$ and $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a minimizer of $\mathscr{F}_{\mathrm{loc}}$ on Ω.

Theorem 2.2 (Partial Regularity in the Degenerate Case). *We assume* (2.1), (2.2), (2.4), (2.8), (2.9)*, and Assumption 2.1 with $q < p + 1$. Then, there is an open subset Ω_0 of Ω with $|\Omega \setminus \Omega_0| = 0$ such that we have $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0; \mathbb{R}^N)$ for some $\alpha \in]0, 1[$ depending only on n, N, and p.*

Remark 2.3. More precisely, Du is locally Hölder continuous on Ω_0 with every exponent $\alpha \in]0, \tilde{\alpha}[$, where $\tilde{\alpha}$ is the Hölder exponent occurring for p-harmonic functions themselves. This can be seen comparing the estimates of Theorem 5.7 and Lemma 5.11.

Remark 2.4. Moreover, our proof of Theorem 2.2 will provide an explicit characterization of Ω_0: Indeed, one can choose $\Omega_0 = R_1 \cap R_2$, where

$$R_1 := \{x \in \Omega \,:\, \liminf_{\varrho \searrow 0} \Phi(u, x, \varrho) = 0\},$$

$$R_2 := \{x \in \Omega \,:\, \limsup_{\varrho \searrow 0} |(Du)^V_{x,\varrho}| < \infty\}.$$

Here, we used some terminology that will be explained in Section 4.

Remark 2.5. Requiring the bound $|D^2 f(\xi)| \leq \Lambda\big[|\xi|^{p-2} + |\xi|^{q-2}\big]$ for all $\xi \in \mathbb{R}^{nN}$, in the convex case (I) of Theorem 2.2 we can bound the Hausdorff dimension of $\Omega \setminus \Omega_0$ by $n - 2$. This is a routine consequence of Remark 2.4 and [ELM1, Theorem 2.2]; see [Giu, Proposition 2.7], [Mi1], and the references given there.

Furthermore, we believe that in the case (I) of Assumption 2.1 the theorem and the remarks hold under more general conditions for the exponents p and q. This conjecture seems natural looking at case (I) of Theorem 1.7 and [ELM1]. However, we do not pursue this idea, since our techniques are tailored out, primarily, for the quasiconvex cases (II) and (III).

Let us mention that the degeneration of the integrand f takes place near the origin in $\mathbb{R}^{nN}$, while the (p, q)-growth conditions are relevant far away from 0. Therefore, at first glance it seems obvious that there should be few interactions between degeneration and (p, q)-growth and that Theorem 2.2 is a simple modification of Theorem 1.7 and Theorem 1.8. We stress, however, that this kind of heuristic reasoning can not be made precise without knowing anything about continuity (or at least local boundedness) of the gradient Du. Indeed, a careful and quite technical adaption of all the techniques involved is necessary. We believe that some arguments presented below, especially around Lemma 5.1, Lemma 5.5, and Proposition 5.10 have not been used in the same way before.

Moreover, note that even under standard growth conditions our proof yields two slight technical improvements of Theorem 1.8. On the one hand we have weakened the conditions imposed on $D^2 f$ assuming just (2.4) and (2.9). On the other hand Remark 2.3 is new for $p > 2$. The latter fact is an outcome of a refinement in Proposition 5.10.

3 A model case

Throughout this section we take

$$1 < p \leq q < p + \frac{\min\{2,p\}}{2n}, \quad p \neq 2, \quad \text{and} \quad q \geq n = N. \tag{3.1}$$

This paper is largely motivated by the model integrands

$$f(\xi) := e_p(\xi) + h(\det\, \xi) \qquad \text{for } \xi \in \mathbb{R}^{n^2}, \tag{3.2}$$

where e_p is defined in (1.9) and h satisfies the following assumptions:

$$h \in C^2(\mathbb{R}), \tag{3.3}$$

$$0 \leq h(d) \leq C_1(1 + |d|^{\frac{q}{n}}) \qquad \text{for all } d \in \mathbb{R} \text{ and some } C_1 > 0, \tag{3.4}$$

$$h \text{ is convex.} \tag{3.5}$$

$$\frac{|h'(d)|}{|d|^{\frac{p}{n}-1}} \underset{d\to 0}{\longrightarrow} 0 \tag{3.6}$$

$$\frac{|h''(d)|}{|d|^{\frac{p}{n}-2}} \underset{d\to 0}{\longrightarrow} 0 \tag{3.7}$$

Note that some interest in such integrands arises from problems motivated by nonlinear elasticity (see [Ba, BM]). We restate our main result in this setting.

Corollary 3.1 (Partial Regularity in the Model Case). *Assume* (3.1) *and let f be defined by* (3.2)*, where h satisfies* (3.3), (3.4), *and* (3.5)*. In the case $p \geq n$ assume in addition* (3.6)*, and in the case $p \geq 2n$ assume further* (3.7)*. Then, if $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ is either a minimizer of F on Ω with $p \geq n$ or a minimizer of $\mathscr{F}_{\mathrm{loc}}$ on Ω, we have $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0;\mathbb{R}^n)$ for some $\alpha \in]0,1[$ and some open subset Ω_0 of Ω with $|\Omega \setminus \Omega_0| = 0$.*

Proof. To derive the corollary from Theorem 2.2 we will show that (2.1), (2.2), (2.3), (2.6) and (2.7) respectively, and (2.10) are fulfilled in the present setting.

It is obvious that (3.2), (3.3), and (3.4) imply (2.1), (2.2), and (2.3). Furthermore, it is well known that $h \circ \det$ is polyconvex and hence quasiconvex for a convex h. Using this together with the convexity properties of e_p we find that f from (3.2) with (3.5) satisfies (2.7). In addition, for $p \geq n$ we have from [BM, Theorem 4.1] that $h \circ \det$ is even $W^{1,p}$-quasiconvex. With this observation at hand, (2.6) can be easily verified in this case.

Thus, it remains to discuss the validity of (2.10): Differentiating (3.2) by the chain and product rule we find

$$\frac{|D^2 f(\xi) - D^2 e_p(\xi)|}{|\xi|^{p-2}} \leq c \left[|h'(\det\, \xi)||\xi|^{n-p} + |h''(\det\, \xi)||\xi|^{2n-p} \right]. \tag{3.8}$$

From (3.6) we see that the first term on the right-hand side of (3.8) tends to 0 as $\xi \to 0$, provided $p \geq n$. However, for $p < n$ the same conclusion holds since $h'(\det \xi)$ remains bounded as $\xi \to 0$ by (3.3). Distinguishing the cases $p \geq 2n$ (where (3.7) is used) and $p < 2n$ one finds analogously that the second term tends to 0 as $\xi \to 0$. Hence, (2.10) holds in any case, which completes the proof. ◻

We believe that the case $p < q = n = N$, in which h has linear growth and minimizers may be discontinuous, is of special interest. We highlight that Corollary 3.1 covers $\mathscr{F}_{\mathrm{loc}}$-minimizers in this case, solely assuming (3.2), (3.3), (3.4), and (3.5) with $\frac{8}{5} < p < 2$ for $n = 2$ and $n - \frac{1}{n} < p < n$ for $n \geq 3$.

Finally, we mention that a regularity theory for polyconvex integrands is known if the integrand has standard growth in the vector of all minors of ξ (see [FH2, FH3, P, EM, H2]). These results are related to taking $q = np$ in the above setting, which is quite far from our condition (3.1), and, analyzing the relevant methods, they seem to have more in common with the standard growth theory. Nevertheless, Corollary 3.1 should be compared with the results of [EM], where integrands with a similar degeneration structure are treated.

4 Notation and preliminaries

For the remainder of the paper we fix some abbreviations:

$$V_\xi^\beta(\sigma) := (|\xi|^2 + |\sigma|^2)^{\frac{\beta-1}{2}}\sigma,$$
$$V_\xi(\sigma) := V_\xi^{\frac{p}{2}}(\sigma).$$

Here, β is an arbitrary positive real number, while p is the fixed lower growth exponent of f. Furthermore, ξ and σ denote arbitrary matrices or vectors. Note that in the following $V_0^\beta(0)$ should be interpreted as 0 even for $0 < \beta \leq 1$. We recall the triangle inequality

$$|V_\xi^\beta(\sigma_1 + \sigma_2)| \leq c\left(|V_\xi^\beta(\sigma_1)| + |V_\xi^\beta(\sigma_2)|\right), \tag{4.1}$$

where c depends only on $\beta > 0$ (see [CFM, Lemma 2.1]), the Young inequality

$$|V_\xi^{p-1}(\sigma_1)|\,|\sigma_2| \leq |V_\xi(\sigma_1)|^2 + |V_\xi(\sigma_2)|^2, \tag{4.2}$$

for $p > 1$ (see [DM2, Lemma 1]), and the integral estimate

$$\int_0^1 (\mu^2 + |\xi + t\sigma|^2)^{\frac{p-2}{2}}\,dt \leq c(\mu^2 + |\xi|^2 + |\sigma|^2)^{\frac{p-2}{2}}, \tag{4.3}$$

for $\mu \in [0, \infty[$ with $\mu^2 + |\xi|^2 + |\sigma|^2 > 0$, where c depends only on $p > 1$ (see [AF3, Lemma 2.1]). Next we introduce the excess

$$\Phi(u, x_0, \varrho, \xi) := \fint_{B_\varrho(x_0)} |V_\xi(Du - \xi)|^2 \, dx,$$
$$\Phi(u, x_0, \varrho) := \Phi(u, x_0, \varrho, (Du)^V_{x_0,\varrho}),$$

where $B_\varrho(x_0)$ is a ball in $\mathbb{R}^n$ and u a function in $W^{1,p}(B_\varrho(x_0); \mathbb{R}^N)$. Here and in the following $(Du)^V_{x_0,\varrho}$ is such that we have $(Du)^V_{x_0,\varrho} = (Du)_{x_0,\varrho} = \fint_{B_\varrho(x_0)} Du \, dx$ for $p > 2$ and $V_0((Du)^V_{x_0,\varrho}) = [V_0(Du)]_{x_0,\varrho} = \fint_{B_\varrho(x_0)} V_0(Du) \, dx$ for $p < 2$. We will often omit the first two arguments of Φ in the following, which should then always be understood as u and x_0. Note that our definition of $\Phi(u, x_0, \varrho, \xi)$ is equivalent with

$$\fint_{B_\varrho(x_0)} |V_0(Du) - V_0(\xi)|^2$$

up to a constant depending only on p (see [AF3, Lemma 2.2]). Therefore, our excess essentially coincides with the one used in [AF3, DM2].

In the remainder of this section we collect a few estimates for the integrand f.

Lemma 4.1 ([Ma1]). *Assume that $f \in C^1(\mathbb{R}^{nN})$ is quasiconvex with* (2.2). *Then, one has*

$$|Df(\xi)| \leq c(1 + |\xi|^{q-1}) \tag{4.4}$$

for all $\xi \in \mathbb{R}^{nN}$, where c depends only on n, N, q, and Γ.

For a proof of the previous lemma we refer to [Ma1] and [Giu, Lemma 5.2].

Remark 4.2. Let $M > 0$. From (2.1) and (2.4) we deduce that there is a constant $\Lambda_M > 0$ such that for all $\xi \in \mathbb{R}^{nN}$ we have

$$0 < |\xi| \leq M + 2 \implies |D^2 f(\xi)| \leq \Lambda_M |\xi|^{p-2}.$$

The next lemma is the analog of [AF2, Lemma II.3] in the degenerate case.

Lemma 4.3. *We assume that f is quasiconvex with* (2.1), (2.2), *and* (2.4) *and consider $\xi, \sigma \in \mathbb{R}^{nN}$ with $|\xi| \leq M + 1$. Then the following estimates hold:*

$$|f(\xi + \sigma) - f(\xi) - Df(\xi)\sigma| \leq c \left[|V_\xi(\sigma)|^2 + |V_\xi^{\frac{q}{2}}(\sigma)|^2 \right],$$
$$|Df(\xi + \sigma) - Df(\xi)| \leq c \left[|V_\xi^{p-1}(\sigma)| + |V_\xi^{q-1}(\sigma)| \right],$$

where c depends only on n, N, p, q, Γ, M, and Λ_M.

Proof. The proof follows the lines of [AF2, Lemma II.3]. For $|\sigma| \geq 1$ we get by (4.4) and $|\xi| \leq M+1$

$$|Df(\xi+\sigma) - Df(\xi)| \leq c(1 + |\xi|^{q-1} + |\sigma|^{q-1}) \leq c|\sigma|^{q-1} \leq c|V_\xi^{q-1}(\sigma)|$$

For $0 < |\sigma| \leq 1$ we have by Remark 4.2 and (4.3) (with $\mu = 0$)

$$\begin{aligned}|Df(\xi+\sigma) - Df(\xi)| &\leq \int_0^1 |D^2 f(\xi + t\sigma)|\, dt\, |\sigma| \\ &\leq \Lambda_M \int_0^1 |\xi + t\sigma|^{p-2}\, dt\, |\sigma| \leq c|V_\xi^{p-1}(\sigma)|.\end{aligned}$$

This proves the second inequality of the lemma. The first one can be verified in a similar fashion. □

5 Proof of Theorem 2.2

Our proof of Theorem 2.2 follows [DM2] and is divided into 6 subsections. We briefly outline the plan of our approach:

(1) We start proving a **Caccioppoli type inequality** with some additional perturbation terms on the right-hand side, which has been introduced in [S1].

(2) Next, we state the **Euler equation** for the minimizer. Note that in the case of an $\mathscr{F}_{\mathrm{loc}}$-minimizer this is a result from [S2] relying on some heavy tools from [FMal] and [BFM].

(3) We recall that the **Legendre-Hadamard condition** is valid for $D^2 f$.

(4) We prove approximate A-harmonicity and use the ***A*-harmonic approximation** method of [DS] to establish a decay estimate for the excess in the **nondegenerate case**.

(5) Approximate p-harmonicity enables us to derive an analogous excess estimate in the **degenerate case** via the ***p*-harmonic approximation** lemma of [DM1].

(6) We note that the **iteration procedure** of [DM2] enables us to combine these excess estimates and to derive partial regularity.

5.1 A Caccioppoli Inequality

Lemma 5.1. *We assume* (2.1), (2.2), (2.4), *and Assumption 2.1. Then, for every* $M > 0$ *the following Caccioppoli type inequality holds for every ball* $B_\varrho(x_0) \subset \Omega$, *all* $\xi \in$

$\mathbb{R}^{nN}$ *with* $|\xi| \le M+1$*, and all* $\zeta \in \mathbb{R}^N$*:*

$$\Phi\left(\frac{\varrho}{2},\xi\right) \le c\left[\fint_{B_\varrho(x_0)} \left|V_\xi\left(\frac{v}{\varrho}\right)\right|^2 dx + \left(\fint_{B_\varrho(x_0)} \left|V_\xi\left(\frac{v}{\varrho}\right)\right|^2 dx\right)^{\frac{q}{p}} + \Phi(\varrho,\xi)^{\frac{q}{p}}\right]. \quad (5.1)$$

Here we have set $v(x) := u(x) - \zeta - \xi x$ *and* c *depends only on* n*,* p*,* q*,* Γ*,* M*,* λ_M*, and* Λ_M*. Furthermore, in the case (III) of Assumption 2.1 we have an additional dependence on* γ*.*

Proof. We start with the case (II) of Assumption 2.1. In this case the proof is a combination of arguments from [DM2, Proposition 2] and [S1, Lemma 7.3] relying also on previous work in [E, FMal]. We assume $x_0 = 0$ and consider arbitrary radii $\frac{\varrho}{2} \le r < s \le \varrho$. Employing [FMal, Lemma 2.3] for

$$\Xi(t) := \begin{cases} \int_{B_t} \left[|Dv|^p + \left|\frac{v}{s-r}\right|^p\right] dx & \text{for } p > 2 \\ \int_{B_t} \left[|V_\xi(Dv)|^2 + \left|V_\xi\left(\frac{v}{s-r}\right)\right|^2\right] dx & \text{for } p < 2 \end{cases}$$

we find further radii $\tilde{r} \in \left[r, \frac{2r+s}{3}\right]$ and $\tilde{s} \in \left[\frac{r+2s}{3}, s\right]$ such that we have

$$\frac{\Xi(t)-\Xi(\tilde{r})}{t-\tilde{r}} \le 3\frac{\Xi(s)-\Xi(r)}{s-r},$$
$$\frac{\Xi(\tilde{s})-\Xi(t)}{\tilde{s}-t} \le 3\frac{\Xi(s)-\Xi(r)}{s-r}$$

for all $t \in]\tilde{r},\tilde{s}[$. Now we choose a cut-off function $\eta \in C^\infty_{\mathrm{cpt}}(B_{\tilde{s}})$ satisfying $\eta \equiv 1$ in a neighborhood of $B_{\tilde{r}}$ and $0 \le \eta \le 1$, $|\nabla\eta| \le \frac{2}{\tilde{s}-\tilde{r}} \le \frac{6}{s-r}$ on $B_{\tilde{s}}$. We set

$$\psi := T_{\tilde{r},\tilde{s}}[(1-\eta)v] \qquad \text{and} \qquad \varphi := v - \psi,$$

where the smoothing operator $T_{\tilde{r},\tilde{s}}$ is defined by

$$T_{\tilde{r},\tilde{s}}w(x) := \fint_{B_1} w(x+\vartheta(x)y)\,dy \quad \text{with} \quad \vartheta(x) := \frac{1}{2}\max\Big\{\min\{|x|-\tilde{r},\tilde{s}-|x|\},0\Big\}.$$

We note

$$\psi = 0, \quad \varphi = v \quad \text{on } B_{\tilde{r}},$$
$$Du - \xi = Dv = D\varphi + D\psi \quad \text{on } B_{\tilde{s}},$$
$$\psi = v \quad \text{near } \partial B_{\tilde{s}}, \qquad \varphi \in W^{1,p}_0(B_{\tilde{s}};\mathbb{R}^N).$$

Applying (2.6) and the minimality of u we get

$$\begin{aligned}\lambda_M \int_{B_{\tilde r}} |V_\xi(Dv)|^2\,dx &\le \int_{B_{\tilde s}} [f(\xi + D\varphi) - f(\xi)]\,dx \\ &\le \int_{B_{\tilde s}} [f(\xi + D\varphi) - f(Du)]\,dx + \int_{B_{\tilde s}} [f(Du - D\varphi) - f(\xi)]\,dx.\end{aligned}$$

This estimate can rewritten as

$$\begin{aligned}&\lambda_M \int_{B_{\tilde r}} |V_\xi(Dv)|^2\,dx \\ &\qquad \le \int_{B_{\tilde s}} [f(Du - D\psi) - f(Du)]\,dx + \int_{B_{\tilde s}} [f(\xi + D\psi) - f(\xi)]\,dx. \qquad (5.2)\end{aligned}$$

By elementary integration we deduce

$$\begin{aligned}&\lambda_M \int_{B_{\tilde r}} |V_\xi(Dv)|^2\,dx \\ &\le \int_{B_{\tilde s}} \left[\int_0^1 \Big(Df(\xi) - Df(Du - tD\psi)\Big)\,dt\,D\psi + f(\xi + D\psi) - f(\xi) - Df(\xi)D\psi\right]\,dx.\end{aligned}$$

Now controlling the right-hand side by means of Lemma 4.3 we find

$$\begin{aligned}&\int_{B_r} |V_\xi(Dv)|^2\,dx \\ &\quad \le c\int_{B_{\tilde s}} \Bigg[\int_0^1 \Big(|V_\xi^{p-1}(Dv - tD\psi)|\,|D\psi| + |V_\xi^{q-1}(Dv - tD\psi)|\,|D\psi|\Big)\,dt \\ &\qquad\qquad + |V_\xi(D\psi)|^2 + |V_\xi^{\frac{q}{2}}(D\psi)|^2\Bigg]\,dx. \qquad (5.3)\end{aligned}$$

Our next aim is to estimate the right-hand side of (5.3) in terms of

$$\Delta := \int_{B_s\setminus B_r} \left[|V_\xi(Dv)|^2 + \left|V_\xi\left(\frac{v}{s-r}\right)\right|^2\right]\,dx.$$

For that purpose we distinguish the cases $p > 2$ and $p < 2$.

We start with the case $p > 2$: Using (4.1) and $|\xi| \le M+1$ we infer from (5.3)

$$\begin{aligned}
&\int_{B_r} |V_\xi(Dv)|^2\,dx \\
&\le c\int_{B_{\tilde{s}}} \Big[|V_\xi(D\psi)|^2 + |V_\xi^{p-1}(Dv)|\,|D\psi| + |V_\xi^{\frac{q}{2}}(D\psi)|^2 + |V_\xi^{q-1}(Dv)|\,|D\psi|\Big]\,dx \\
&\le c(M)\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}} \Big[|V_\xi(D\psi)|^2 + |V_\xi^{p-1}(Dv)|\,|D\psi| + |D\psi|^q + |Dv|^{q-1}|D\psi|\Big]\,dx.
\end{aligned} \tag{5.4}$$

Next recalling $\psi = T_{\tilde{r},\tilde{s}}[(1-\eta)v]$ we will use some ideas from [FMal, Lemma 2.4] (see also [S1, Lemma 6.3]) concerning the smoothing operator $T_{\tilde{r},\tilde{s}}$: First we note that $T_{\tilde{r},\tilde{s}}$ is L^p-bounded and compatible with derivatives. As in [S1, Lemma 7.3] this gives the estimate

$$\int_{B_s\setminus B_r} |D\psi|^p\,dx \le c\int_{B_s\setminus B_r} |D[(1-\eta)v]|^p\,dx \le c\int_{B_s\setminus B_r}\left[|Dv|^p + \left|\frac{v}{s-r}\right|^p\right]dx,$$

where c depends only on n and p. The same inequalities hold with the exponent 2 instead of p and we get

$$\int_{B_s\setminus B_r} |V_\xi(D\psi)|^2\,dx \le c\Delta.$$

Thus, using also (4.2) we can estimate the first two terms on the right-hand side of (5.4) by $c\Delta$. To treat the remaining terms we exploit the fact that $T_{\tilde{r},\tilde{s}}$ even improves the integrability properties. Actually, relying on [FMal, Lemma 2.4] again, $\|D\psi\|_{\kappa;B_{\tilde{s}}\setminus B_{\tilde{r}}}$ can be controlled by $\|D[(1-\eta)v]\|_{p;B_s\setminus B_r}$ for all $p \le \kappa < \frac{np}{n-1}$. Precisely from the proof of [S1, Lemma 7.3] we retrieve

$$\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}} |D\psi|^\kappa\,dx \le c(s-r)^n\left(\frac{\Delta}{(s-r)^n}\right)^{\frac{\kappa}{p}} \qquad \text{for all } p \le \kappa < \frac{np}{n-1}, \tag{5.5}$$

where c depends only on n, p, and κ. Note that the derivation of (5.5) makes use of the choice of $\tilde{r}$ and $\tilde{s}$. Using (5.5) with $\kappa = q$ we estimate the third term on the right-hand side of (5.4). Finally, the last term can be handled by Hölder's inequality and (5.5) with $\kappa = \frac{p}{p+1-q}$. Note that the treatment of this last term makes use of the bounds $q < p+1$ and $\frac{p}{p+1-q} < \frac{np}{n-1}$, which are equivalent with $q < p + \frac{1}{n}$. Collecting the estimates we arrive at

$$\int_{B_r} |V_\xi(Dv)|^2\,dx \le c\left[\Delta + (s-r)^n\left(\frac{\Delta}{(s-r)^n}\right)^{\frac{q}{p}}\right]. \tag{5.6}$$

Next we briefly sketch the case $p < 2$, which is a bit a more involved. Note that the following computations are similar to the ones of [S1, Lemma 7.3], where a more detailed exposition can be found. We start from (5.3) and apply (4.1) and (4.3) getting

$$\int_{B_r} |V_\xi(Dv)|^2\,dx \le c\int_{B_{\tilde s}} \Big[|V_\xi(D\psi)|^2 + |V_\xi^{p-1}(Dv)|\,|D\psi| + |V_\xi^{\frac{q}{2}}(D\psi)|^2 + (|\xi|^2+|Dv|^2+|D\psi|^2)^{\frac{q-2}{2}}(|Dv|+|D\psi|)|D\psi|\Big]\,dx.$$

Using the elementary estimate

$$(|\xi|^2+|\sigma_1|^2+|\sigma_2|^2)^{\frac{q-p}{2}} \le c\left[|\xi|^{q-p} + \Big[(|\xi|^2+|\sigma_1|^2+|\sigma_2|^2)^{\frac{p-2}{2}}(|\sigma_1|^2+|\sigma_2|^2)\Big]^{\frac{q-p}{p}}\right]$$

and $|\xi| \le M+1$ some calculations lead to

$$\int_{B_r} |V_\xi(Dv)|^2\,dx \le c(M)\int_{B_{\tilde s}\setminus B_{\tilde r}} \Big[|V_\xi(D\psi)|^2 + |V_\xi^{p-1}(Dv)|\,|D\psi| + |V_\xi(D\psi)|^{\frac{2q}{p}} + |V_\xi(Dv)|^{\frac{2q}{p}-1}|V_\xi(D\psi)|\Big]\,dx. \tag{5.7}$$

For the estimation of the right-hand side of (5.7) we follow [FMal, S1] again. The properties of the smoothing operator reveal the inequality

$$\int_{B_{\tilde s}\setminus B_{\tilde r}} |V_\xi(D\psi)|^{\frac{2\kappa}{p}}\,dx \le c(s-r)^n\left(\frac{\Delta}{(s-r)^n}\right)^{\frac{\kappa}{p}} \qquad \text{for all } p\le\kappa<\frac{np}{n-1}. \tag{5.8}$$

Note that (5.8) is a variant of (5.5) and, adopting [S1, Lemma 6.5], it can easily be derived from the properties of $T_{\tilde r,\tilde s}$ and the convexity of $\sigma \mapsto (|\xi|+|\sigma|)^{\frac{p-2}{p}}|\sigma|^{\frac{2}{p}}$. Next, we will use (5.8) to estimate the four terms on the right-hand side of (5.7): Taking $\kappa = p$ we control the first term by $c\Delta$ and involving, in addition, (4.2) we handle the second one. For the third term we choose $\kappa = q$. Finally, the last term can be estimated by Hölder's inequality and (5.8) with $\kappa = \frac{p^2}{3p-2q}$. Here, the last estimate makes use of the bounds $q < \frac{3}{2}p$ and $\frac{p^2}{3p-2q} < \frac{np}{n-1}$, which are equivalent with $q < p + \frac{p}{2n}$. Arguing in this way we get (5.6), also in the subquadratic case.

Thus, (5.6) is valid for all p. At this stage, the rest of the proof consists essentially of standard arguments: (5.1) follows by filling the hole and an iteration lemma; see for instance [DM2, Proposition 2]. This ends the proof in the case (II).

Next we come to the case (I) in Assumption 2.1. Following [S1, Remark 7.5] we can use the convexity of f to get the following improved version of (5.2):

$$\lambda\int_{B_r} |V_\xi(Dv)|^2\,dx \le \int_{B_{\tilde s}} [f(\xi+D\tilde\varphi)-f(\xi)-Df(\xi)D\tilde\varphi]\,dx + \int_{B_{\tilde s}} [f(\xi+D\psi)-f(\xi)-Df(\xi)D\psi]\,dx,$$

where $\tilde{\varphi}$ is defined by $\tilde{\varphi} := T_{\tilde{r},\tilde{s}}[\eta v]$. Arguing essentially as supplied above we derive a variant of (5.3):

$$\int_{B_r} |V_\xi(Dv)|^2\,dx \leq c\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}} \Big[|V_\xi(D\psi)|^2+|V_\xi(D\tilde{\varphi})|^2+|V_\xi^{\frac{q}{2}}(D\psi)|^2+|V_\xi^{\frac{q}{2}}(D\tilde{\varphi})|^2\Big]\,dx.$$

The terms containing only ψ have already been treated above. However, we can now treat the terms containing $\tilde{\varphi}$ completely analogous, using only the bound $q < \frac{np}{n-1}$, and arrive at (5.1).

Finally let us say a few words about the case (III). Here, the proof of (II) works once we establish (5.2). To this aim we proceed as for [S2, Lemma 7.13]. The only alteration which has to be made in the proof of the lemma concerns the auxiliary integrand g: In our case assuming $\lambda_M < pC_2\gamma$ we need to define $g(\sigma) := f(\sigma) - \frac{\lambda_M}{C_2}e_p(\sigma)$ for $\sigma \in \mathbb{R}^{nN}$. Note that we use e_p in the sense of (1.9), which is different from the meaning in [S2]. It is not difficult to show that there is a constant $C_2 \geq 1$ depending only on p such that one has

$$C_2^{-1}\fint_O |V_\xi(D\varphi)|^2\,dx \leq \fint_O [e_p(\xi+D\varphi)-e_p(\xi)]\,dx \leq C_2\fint_O |V_\xi(D\varphi)|^2\,dx$$

for all open bounded subsets O of $\mathbb{R}^n$ and all $\varphi \in W_0^{1,p}(O;\mathbb{R}^N)$. We observe that by (2.3), (2.7), and the preceding estimate g is quasiconvex at ξ with $\left(\gamma - \frac{\lambda_M}{pC_2}\right)|\sigma|^p \leq g(\sigma) \leq \Gamma(1+|\sigma|^q)$ for all $\sigma \in \mathbb{R}^{nN}$. Relying on these observations the modified g can be used exactly as demonstrated in [S2]. ◻

5.2 Euler's equation

Lemma 5.2. *Consider a quasiconvex $f \in C^1(\mathbb{R}^{nN})$ with (2.2) for some $q \in [1,\infty[$. In addition, assume that one of the following sets of conditions holds:*

(I) $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ *is a minimizer of F on Ω with $1 \leq p \leq q \leq p+1$;*

(II) (2.3) *holds and $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a minimizer of $\mathscr{F}_{\mathrm{loc}}$ on Ω with $1 < p \leq q < \min\left\{\frac{np}{n-1}, p+1\right\}$.*

Then, u satisfies the Euler equation of F in a weak formulation:

$$\int_\Omega Df(Du)D\varphi\,dx = 0 \qquad \text{for all } \varphi \in C^\infty_{\mathrm{cpt}}(\Omega;\mathbb{R}^N). \tag{5.9}$$

Noting that (4.4) holds, the result is classical in the case of a F-minimizer. The case of a $\mathscr{F}_{\mathrm{loc}}$-minimizer is more delicate and has been treated in [S2, Lemma 7.3].

5.3 The Legendre-Hadamard Condition

We recall the validity of the Legendre-Hadamard condition for D^2f; see for instance [Giu, Proposition 5.2].

Lemma 5.3. *Let f satisfy* (2.1) *and the quasiconvexity condition* (2.7). *Then, for all $M > 0$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\xi \in \mathbb{R}^{nN}$ with $0 < |\xi| \leq M$ we have*

$$D^2 f(\xi)(\zeta x^T, \zeta x^T) \geq \lambda_M |\xi|^{p-2} |x|^2 |\zeta|^2. \tag{5.10}$$

In particular, (5.10) is also available, if (2.5) or (2.6) holds.

5.4 The Nondegenerate Case

Remark 5.4. Let $M > 0$. From (2.1) we deduce that $D^2 f$ is uniformly continuous on $\{\xi \in \mathbb{R}^{nN} : \frac{\varepsilon}{2} \leq |\xi| \leq 2M\}$. Using this together with (2.9) one can show that there is a modulus of continuity $\nu_M : [0, \infty[\to [0, \infty[$ with $\lim_{\omega \searrow 0} \nu_M(\omega) = 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^{nN}$ with $0 < |\xi_1| \leq M$ and $0 < |\xi_2| \leq 2M$ we have

$$|D^2 f(\xi_1) - D^2 f(\xi_2)| \leq \begin{cases} (|\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} \nu_M\left(\frac{|\xi_1 - \xi_2|^2}{|\xi_1|^2 + |\xi_2|^2}\right) \\ \left(\frac{|\xi_1|^2 + |\xi_2|^2}{|\xi_1|^2 |\xi_2|^2}\right)^{\frac{2-p}{2}} \nu_M\left(\frac{|\xi_1 - \xi_2|^2}{|\xi_1|^2 + |\xi_2|^2}\right) \end{cases}.$$

Furthermore, we may assume the following properties of ν_M:

(I) ν_M is nondecreasing;
(II) ν_M^2 is concave.

Next we establish approximate A-harmonicity of minimizers u.

Lemma 5.5. *Assume that $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a weak solution of the Euler equation* (5.9). *Furthermore, assume* (2.1), (2.2), (2.4), *and* (2.9) *with $1 < p \leq q < p + 1$, $p \neq 2$. Then, for every ball $B_\varrho(x_0) \subset \Omega$, all $M > 0$, all $\xi \in \mathbb{R}^{nN}$ with $0 < |\xi| \leq M$, and all $\varphi \in C^\infty_{\mathrm{cpt}}(B_\varrho(x_0); \mathbb{R}^N)$ we have*

$$\left| \fint_{B_\varrho(x_0)} A(|\xi|^{\frac{p-2}{2}}(Du - \xi), D\varphi)\, dx \right| \\ \leq c\sqrt{\Phi(\varrho, \xi)} \left[\nu_M\left(\frac{\Phi(\varrho,\xi)}{|\xi|^p}\right) + \left(\frac{\Phi(\varrho,\xi)}{|\xi|^p}\right)^{\kappa - \frac{1}{2}} \right] \sup_{B_\varrho(x_0)} |D\varphi|,$$

where $A := \frac{D^2 f(\xi)}{|\xi|^{p-2}}$, $\kappa \in]\frac{1}{2}, 1[$ is defined by $\kappa := \begin{cases} \frac{q-1}{p} & \text{for } q \geq 2 \\ \frac{1}{p} & \text{for } q \leq 2 \end{cases}$, and c depends only on n, N, p, q, Γ, M, and Λ_M.

Proof. We assume $x_0 = 0$ and $\sup_{B_\varrho} |D\varphi| = 1$. Setting $v(x) := u(x) - \xi x$ we get from (5.9)

$$\left| \fint_{B_\varrho} D^2 f(\xi)(Dv, D\varphi)\, dx \right| \leq \fint_{B_\varrho} |D^2 f(\xi)(Dv, D\varphi) + Df(\xi)D\varphi - Df(Du)D\varphi|\, dx. \quad (5.11)$$

Next we estimate the integrand on the right-hand side of (5.11). On $\{x \in B_\varrho : |Dv| < |\xi|\}$ we get from Remark 5.4 and (4.3) (with $\mu = 0$)

$$\begin{aligned} &|D^2 f(\xi)(Dv, D\varphi) + Df(\xi)D\varphi - Df(Du)D\varphi| \\ &\quad \leq \int_0^1 |D^2 f(\xi) - D^2 f(\xi + tDv)|\, dt\, |Dv| \\ &\quad \leq \begin{cases} c|\xi|^{p-2}|Dv|\nu_M\left(\frac{|Dv|^2}{|\xi|^2}\right) & \text{for } p > 2 \\ c\int_0^1 |\xi + tDv|^{p-2}\, dt |Dv|\nu_M\left(\frac{|Dv|^2}{|\xi|^2+|Dv|^2}\right) & \text{for } p < 2 \end{cases} \\ &\quad \leq c|\xi|^{\frac{p-2}{2}}|V_\xi(Dv)|\nu_M\left(\frac{|V_\xi(Dv)|^2}{|\xi|^p}\right), \end{aligned}$$

while on $\{x \in B_\varrho : |Dv| \geq |\xi|\}$ we have from Remark 4.2, Lemma 4.3, and $|\xi| \leq M$

$$\begin{aligned} &|D^2 f(\xi)(Dv, D\varphi) + Df(\xi)D\varphi - Df(Du)D\varphi| \\ &\quad \leq c\left[|\xi|^{p-2} + (|\xi|^2 + |Dv|^2)^{\frac{p-2}{2}} + (|\xi|^2 + |Dv|^2)^{\frac{q-2}{2}}\right] |Dv| \\ &\quad \leq \begin{cases} c|\xi|^{p-q}|Dv|^{q-1} \leq c|\xi|^{p-q}|V_\xi(Dv)|^{2\frac{q-1}{p}} & \text{for } q \geq 2 \\ c|\xi|^{p-2}|Dv| \leq c|\xi|^{p-2}|V_\xi(Dv)|^{\frac{2}{p}} & \text{for } q \leq 2 \end{cases} \\ &\quad \leq c|\xi|^{p-p\kappa-1}|V_\xi(Dv)|^{2\kappa}. \end{aligned}$$

Collecting the estimates and using the inequalities of Hölder and Jensen we find

$$\begin{aligned}
&\left| \fint_{B_\varrho} D^2 f(\xi)(Dv, D\varphi)\,dx \right| \\
&\le c \left[|\xi|^{\frac{p-2}{2}} \fint_{B_\varrho} |V_\xi(Dv)| \nu_M \left(\frac{|V_\xi(Dv)|^2}{|\xi|^p} \right) dx + |\xi|^{p-p\kappa-1} \fint_{B_\varrho} |V_\xi(Dv)|^{2\kappa}\,dx \right] \\
&\le c \Bigg[|\xi|^{\frac{p-2}{2}} \left(\fint_{B_\varrho} |V_\xi(Dv)|^2\,dx \right)^{\frac{1}{2}} \nu_M \left(\fint_{B_\varrho} \frac{|V_\xi(Dv)|^2}{|\xi|^p}\,dx \right) \\
&\qquad\qquad + |\xi|^{p-p\kappa-1} \left(\fint_{B_\varrho} |V_\xi(Dv)|^2\,dx \right)^{\kappa} \Bigg] \\
&= c|\xi|^{\frac{p-2}{2}} \sqrt{\Phi(\varrho,\xi)} \left[\nu_M \left(\frac{\Phi(\varrho,\xi)}{|\xi|^p} \right) + \left(\frac{\Phi(\varrho,\xi)}{|\xi|^p} \right)^{\kappa-\frac{1}{2}} \right].
\end{aligned}$$

Dividing the last inequality by $|\xi|^{\frac{p-2}{2}}$ we obtain the claim. □

Now we are ready to prove an excess estimate in certain nondegenerate cases.

Proposition 5.6. *We assume* (2.1), (2.2), (2.4), (2.9), *and Assumption 2.1 with* $q < p+1$. *Furthermore, we consider arbitrary* $M > 0$ *and* $\alpha \in]0,1[$. *Then, there are constants* $\varepsilon_0 > 0$ *and* $\theta \in]0,1[$ *such that for every ball* $B_\varrho(x_0) \subset \Omega$ *the conditions*

$$\Phi(\varrho) \le \varepsilon_0 |(Du)^V_{x_0,\varrho}|^p \qquad \textit{and} \qquad |(Du)^V_{x_0,\varrho}| \le M \tag{5.12}$$

imply

$$\Phi(\theta\varrho) \le \theta^{2\alpha}\Phi(\varrho).$$

Here θ *depends only on* n, N, p, q, Γ, M, λ_M, Λ_M, *and* α, *and* ε_0 *depends additionally on* ν_M. *Again we have an additional dependence on* γ *in the case (III).*

Proof. We assume $x_0 = 0$, $\Phi(\varrho) \ne 0$ and set $\xi := (Du)^V_\varrho$. Then, (5.12) implies $\xi \ne 0$. The proof is based on Lemma 5.1, Lemma 5.3, and Lemma 5.5 and follows closely the lines of [DM2, Proposition 3 rsp. 4] using the method of A-harmonic approximation with $A := \frac{D^2 f(\xi)}{|\xi|^{p-2}}$. In contrast to [DM2] we are restricted to the case $|\xi| \le M$ in Remark 4.2 and Lemma 5.3. Hence, in our setting many constants like the ellipticity constant and the upper bound for A depend on M, but this does not cause any problems. Moreover, our estimate in Lemma 5.5 is a bit different from the corresponding estimate in [DM2, Lemma 9], but can be used in exactly the same fashion. Therefore, the only difference remaining concerns the application of the Caccioppoli inequality from Lemma 5.1.

We will now explain this last point in detail in the superquadratic case $p > 2$: It has been proven in [DM2, (5.17)] that

$$\fint_{B_{2\theta\varrho}} \left| V_{DP_{2\theta\varrho}} \left(\frac{u - P_{2\theta\varrho}}{2\theta\varrho} \right) \right|^2 \leq c\theta^2 \Phi(\varrho) \tag{5.13}$$

holds, where $\theta \in]0, \frac{1}{4}]$ will be fixed later and the affine functions $P_{2\theta\varrho}$ are defined in [DM2, Section 3]. We will apply this estimate to control the first and second term on the right-hand side of (5.1). For the third one we will supply some additional arguments. We note that [DM2, (3.8) in Lemma 6] implies

$$\begin{aligned} \fint_{B_{2\theta\varrho}} |Du - DP_{2\theta\varrho}|^p \, dx &\leq c \left[\fint_{B_{2\theta\varrho}} |Du - (Du)_{2\theta\varrho}|^p \, dx + |(Du)_{2\theta\varrho} - DP_{2\theta\varrho}|^p \right] \\ &\leq c \fint_{B_{2\theta\varrho}} |Du - (Du)_{2\theta\varrho}|^p \, dx \leq c \fint_{B_{2\theta\varrho}} |Du - \xi|^p \, dx. \end{aligned}$$

Using this and the analogous estimate with 2 instead of p we find

$$\Phi(2\theta\varrho, DP_{2\theta\varrho}) \leq c \fint_{B_{2\theta\varrho}} |V_{DP_{2\theta\varrho}}(Du - \xi)|^2 \, dx.$$

Now working with $\frac{1}{\max\{M,2\}}$ instead of $\frac{1}{2}$ on the right-hand side of [DM2, (5.16)] we obtain $M|DP_{2\theta\varrho} - \xi| \leq |\xi|$ and

$$|DP_{2\theta\varrho}| \leq \frac{M+1}{M} |\xi| \leq M + 1. \tag{5.14}$$

We conclude

$$\Phi(2\theta\varrho, DP_{2\theta\varrho}) \leq c \fint_{B_{2\theta\varrho}} |V_\xi(Du - \xi)|^2 \, dx \leq c\theta^{-n} \Phi(\varrho). \tag{5.15}$$

In view of (5.14) we can combine the Caccioppoli inequality from Lemma 5.1 with (5.13) and (5.15) arriving at

$$\Phi(\theta\varrho, DP_{2\theta\varrho}) \leq c[\theta^2 \Phi(\varrho) + (\theta^2 \Phi(\varrho))^{\frac{q}{p}} + (\theta^{-n} \Phi(\varrho))^{\frac{q}{p}}].$$

Involving $\theta \leq 1$ and the second part of (5.12) we deduce

$$\Phi(\theta\varrho, DP_{2\theta\varrho}) \leq c \left[\theta^2 \Phi(\varrho) + \theta^{-n\frac{q}{p}} \left(\frac{\Phi(\varrho)}{|\xi|^p} \right)^{\frac{q-p}{p}} \Phi(\varrho) \right].$$

Note that the last term on the right-hand side does not occur in the case $q = p$ as explained in [S1, Remark 7.4]. In the case $q > p$ we impose the additional smallness assumption

$$\theta^{-n\frac{q}{p}} \left(\frac{\Phi(\varrho)}{|\xi|^p} \right)^{\frac{q-p}{p}} \leq \theta^2 \tag{5.16}$$

and arrive - in any case - at

$$\Phi(\theta\varrho, DP_{2\theta\varrho}) \leq c\theta^2\Phi(\varrho),$$

which coincides with the inequality before [DM2, (5.18)]. The remaining proof is now exactly the same as in [DM2]. Note that the constants ε_0 and θ will be determined later in the proof. At this point the assumption (5.16) can be deduced from the first part of (5.12) choosing ε_0 small enough.

Now we come to the subquadratic case $p < 2$. Here, [DM2, p. 762] states

$$\fint_{B_{2\theta\varrho}} \left| V_{\tilde{\xi}}\left(\frac{u(x) - s|\xi|h(0) - \tilde{\xi}x)}{2\theta\varrho}\right)\right| \, dx \leq c\theta^2\Phi(\varrho), \tag{5.17}$$

where s coincides with $\sqrt{\frac{\Phi(\varrho)}{|\xi|^p}}$ up to a constant and h is the approximating A-harmonic function (see [DM2] for more precise definitions). Furthermore, we have used the abbreviation $\tilde{\xi} := \xi + s|\xi|Dh(0)$. Once more, our goal is to derive an estimate for the third term on the right-hand side of (5.1) in this setting. Employing the estimate $|Dh(0)| \leq c$ from [DM2, p. 762] we see

$$|\xi - \tilde{\xi}| \leq C_3 s|\xi| \tag{5.18}$$

for a constant $C_3(n, N, p, q, M, \lambda_M, \Lambda_M) > 0$. Imposing the smallness condition

$$C_3 s \leq \frac{1}{\max\{2, M\}} \tag{5.19}$$

we get

$$\frac{1}{2}|\xi| \leq |\tilde{\xi}| \leq M + 1. \tag{5.20}$$

Now, by (4.1), (5.20), and (5.18) we have

$$\begin{aligned}
\Phi(2\theta\varrho, \tilde{\xi}) &\leq c\left[\fint_{B_{2\theta\varrho}} |V_{\tilde{\xi}}(Du - \xi)| \, dx + |V_{\tilde{\xi}}(\xi - \tilde{\xi})|\right] \\
&\leq c\left[\fint_{B_{2\theta\varrho}} |V_{\xi}(Du - \xi)| \, dx + |V_{\xi}(s\xi)|\right] \\
&\leq c[\theta^{-n}\Phi(\varrho) + s^2|\xi|^p] \\
&\leq c\theta^{-n}\Phi(\varrho).
\end{aligned} \tag{5.21}$$

Next, in view of (5.20) we can combine (5.1), (5.17), and (5.21) attaining

$$\Phi(\theta\varrho, \tilde{\xi}) \leq c[\theta^2\Phi(\varrho) + (\theta^2\Phi(\varrho))^{\frac{q}{p}} + (\theta^{-n}\Phi(\varrho))^{\frac{q}{p}}].$$

As for $p > 2$, imposing (5.16) if $q > p$, we simplify the right-hand side getting

$$\Phi(\theta\varrho, \tilde{\xi}) \leq c\theta^2\Phi(\varrho),$$

which coincides with [DM2, (5.53)]. The remaining proof follows [DM2] again, deducing (5.16) and (5.19) from the first part of (5.12). □

5.5 The Degenerate Case

In this subsection we will carry over the excess estimates of Uhlenbeck's theorem for p-harmonic functions to our minimizer u. Let us begin restating the result for p-harmonic functions (see [GM2, AF3] for a similar exposition):

Theorem 5.7. *Consider $1 < p < \infty$ and a p-harmonic function $h \in W^{1,p}(B_\varrho(x_0);\mathbb{R}^N)$. Then, there is a constant $\tilde{\alpha} \in]0,1[$ depending only on n, N, and p such that we have*

$$\sup_{B_{\varrho/2}(x_0)} |Dh|^p \leq c \fint_{B_\varrho(x_0)} |Dh|^p \, dx,$$

$$\Phi(h, x_0, r) \leq c \left(\frac{r}{\varrho}\right)^{2\tilde{\alpha}} \Phi(h, x_0, \varrho) \qquad \textit{for all } r \in]0, \varrho],$$

where c depends only on n, N, p.

Remark 5.8. The asymptotic condition (2.8) implies the existence of a modulus $\eta :]0,\infty[\to]0,1]$ with $\lim_{\mu\searrow 0} \eta(\mu) = 0$ such that for all $\mu > 0$ and $\xi \in \mathbb{R}^{nN}$ we have

$$|\xi| \leq \eta(\mu) \implies |Df(\xi) - Df(0) - De_p(\xi)| \leq \mu|\xi|^{p-1}.$$

Setting

$$E(u, x_0, \varrho) := \fint_{B_\varrho(x_0)} |Du|^p \, dx$$

we state approximate p-harmonicity of u:

Lemma 5.9. *Assume that $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a weak solution of the Euler equation* (5.9)*. Furthermore, assume* (2.1), (2.8)*, and* (4.4) *with $1 < p \leq q < p+1$, $p \neq 2$. Then, for every ball $B_\varrho(x_0) \subset \Omega$, all $\mu > 0$, and all $\varphi \in C^\infty_{\mathrm{cpt}}(B_\varrho(x_0);\mathbb{R}^N)$ we have*

$$\left| \fint_{B_\varrho(x_0)} De_p(Du) D\varphi \, dx \right| \leq \left[\mu E(u, x_0, \varrho)^{\frac{p-1}{p}} + 3c \frac{E(u, x_0, \varrho)}{\eta(\mu)^p} \right] \sup_{B_\varrho(x_0)} |D\varphi|,$$

where c is the constant from (4.4)*.*

Proof. The proof is a straightforward modification of [DM2, Lemma 10]. We assume $x_0 = 0$ and $\sup_{B_\varrho} |D\varphi| = 1$ and observe that (5.9) implies

$$\left| \fint_{B_\varrho} De_p(Du) D\varphi \, dx \right| \leq \fint_{B_\varrho} |Df(Du) - Df(0) - De_p(Du)| \, dx.$$

Now we set $S := \{x \in B_\varrho \,:\, |Du| > \eta(\mu)\}$ and estimate the last integral separately over $B_\varrho \setminus S$ and S. By Remark 5.8 and Hölder's inequality we get

$$\frac{1}{|B_\varrho|}\int_{B_\varrho\setminus S} |Df(Du) - Df(0) - De_p(Du)|\,dx$$
$$\leq \mu \fint_{B_\varrho} |Du|^{p-1}\,dx \leq \mu\left(\fint_{B_\varrho}|Du|^p\right)^{\frac{p-1}{p}}.$$

To deal with the integral over S we observe $\frac{|S|}{|B_\varrho|} \leq \eta(\mu)^{-p}\fint_{B_\varrho}|Du|^p\,dx$ and use (4.4) and Hölder's inequality

$$\frac{1}{|B_\varrho|}\int_S |Df(Du) - Df(0) - De_p(Du)|\,dx$$
$$\leq \frac{3c}{|B_\varrho|}\int_S (1 + |Du|^{q-1})\,dx$$
$$\leq 3c\left[\frac{|S|}{|B_\varrho|} + \left(\frac{|S|}{|B_\varrho|}\right)^{\frac{p+1-q}{p}}\left(\fint_{B_\varrho}|Du|^p\,dx\right)^{\frac{q-1}{p}}\right]$$
$$\leq 3c\left[\eta(\mu)^{-p} + \eta(\mu)^{q-p-1}\right]\fint_{B_\varrho}|Du|^p\,dx.$$

Collecting the estimates and recalling the condition $\eta(\mu) \leq 1$, which we imposed in Remark 5.8, we arrive at the claim. □

As demonstrated in [DM2], approximate p-harmonicity allows to obtain an excess estimate in the truly degenerate case. However, in the case $p > 2$ this excess estimate [DM2, Lemma 12] can be refined a bit leading to Remark 2.3. We give a formulation and a partial proof of this estimate.

Proposition 5.10. *We assume* (2.1), (2.4), (2.8)*, and Assumption 2.1 with* $q < p + 1$. *Furthermore, we consider arbitrary* $\chi > 0$ *and* $\alpha \in]0, \tilde{\alpha}[$, *where* $\tilde{\alpha}$ *is the constant from Theorem 5.7. Then, there are constants* $\varepsilon_1 \in]0, 1]$ *and* $\tau \in]0, 1[$ *such that for every ball* $B_\varrho(x_0) \subset \Omega$ *the conditions*

$$\chi|(Du)^V_{x_0,\varrho}|^p \leq \Phi(\varrho), \qquad \chi|(Du)^V_{x_0,\tau\varrho}|^p \leq \Phi(\tau\varrho), \tag{5.22}$$

and

$$\Phi(\varrho) \leq \varepsilon_1 \tag{5.23}$$

imply

$$\Phi(\tau\varrho) \leq \tau^{2\alpha}\Phi(\varrho).$$

Here τ depends only on n, N, p, q, Γ, $\lambda_{\tilde{M}}$, $\Lambda_{\tilde{M}}$, α, and χ, and ε_1 depends additionally on η, where $\tilde{M} > 0$ is yet another constant depending only on n, N, p, and χ. Once more our constants may depend on γ in the case (III). Note that in the case $p < 2$ we need only the first condition in (5.22)*, but not the second.*

Proof. We consider the case $p > 2$: Again we assume $x_0 = 0$, $\Phi(\varrho) \neq 0$ and set $\xi := (Du)_\varrho$. We follow the proof of [DM2, Lemma 12] first noting that by (5.22) we have

$$\fint_{B_\varrho} |Du|^p \, dx \leq 2^{p-1}[\Phi(\varrho) + |\xi|^p] \leq C_4\Phi(\varrho). \tag{5.24}$$

with $C_4 := 2^{p-1}(1 + \chi^{-1})$. Introducing the scaled function $w := \big(C_4\Phi(\varrho)\big)^{-\frac{1}{p}} u$ this means $\fint_{B_\varrho} |Dw|^p \, dx \leq 1$. For $\tau \in]0, \frac{1}{4}]$ to be fixed later we apply p-harmonic approximation based on Lemma 5.9 with $\varepsilon := \min\{\tau^{\frac{p}{2}(n+2+2\tilde{\alpha})}, \tau^{n+p+2\tilde{\alpha}}\}$ as for [DM2, (5.30)] finding a p-harmonic h such that we have

$$\fint_{B_\varrho} |Dh|^p \, dx \leq 1 \qquad \text{and} \qquad \fint_{B_\varrho} \left|\frac{w-h}{\varrho}\right|^p dx \leq \varepsilon.$$

Note that by Theorem 5.7, (5.23), and $\varepsilon_1 \leq 1$ this implies (see [DM2, p. 756])

$$\Phi(h, 0, \varrho) \leq c \tag{5.25}$$

$$|(Dh)_{2\tau\varrho}| \leq c, \tag{5.26}$$

$$|DP| = (C_4\Phi(\varrho))^{\frac{1}{p}} |(Dh)_{2\tau\varrho}| \leq \tilde{M}\Phi(\varrho)^{\frac{1}{p}} \leq \tilde{M}. \tag{5.27}$$

Here, $\tilde{M} > 0$ is a constant depending only on n, N, p, and χ and the affine function P is defined by $P(x) := \big(C_4\Phi(\varrho)\big)^{\frac{1}{p}}\big(h_{2\tau\varrho} + (Dh)_{2\tau\varrho}x\big)$. By (5.26), the inequalities of Poincaré and Hölder, Theorem 5.7, and (5.25) we conclude

$$\begin{aligned}
&|(Dh)_{2\tau\varrho}|^{p-2} \fint_{B_{2\tau\varrho}} \left|\frac{w(x) - h_{2\tau\varrho} - (Dh)_{2\tau\varrho}x}{2\tau\varrho}\right|^2 dx \\
&\leq c\left[\tau^{-2}\fint_{B_{2\tau\varrho}} \left|\frac{w-h}{\varrho}\right|^2 dx + |(Dh)_{2\tau\varrho}|^{p-2}\fint_{B_{2\tau\varrho}} \left|\frac{h(x) - h_{2\tau\varrho} - (Dh)_{2\tau\varrho}x}{\tau\varrho}\right|^2 dx\right] \\
&\leq c\left[\tau^{-n-2}\fint_{B_\varrho} \left|\frac{w-h}{\varrho}\right|^2 dx + |(Dh)_{2\tau\varrho}|^{p-2}\fint_{B_{2\tau\varrho}} |Dh - (Dh)_{2\tau\varrho}|^2 \, dx\right] \\
&\leq c\left[\tau^{-n-2}\varepsilon^{\frac{2}{p}} + \Phi(h, 0, 2\tau\varrho)\right] \\
&\leq c\tau^{2\tilde{\alpha}}[1 + \Phi(h, 0, \varrho)] \\
&\leq c\tau^{2\tilde{\alpha}}.
\end{aligned}$$

Rescaling we find

$$|(Dh)_{2\tau\varrho}|^{p-2} \fint_{B_{2\tau\varrho}} \left|\frac{u-P}{2\tau\varrho}\right|^2 dx \le c\tau^{2\tilde{\alpha}}\Phi(\varrho)^{\frac{2}{p}}.$$

Essentially the same argument with the exponent p instead of 2 is contained in [DM2, p. 756] and shows

$$\fint_{B_{2\tau\varrho}} \left|\frac{u-P}{2\tau\varrho}\right|^p dx \le c\tau^{2\tilde{\alpha}}\Phi(\varrho).$$

The last two inequalities together with (5.27) give

$$\fint_{B_{2\tau\varrho}} \left|V_{DP}\left(\frac{u-P}{2\tau\varrho}\right)\right|^2 dx \le c\tau^{2\tilde{\alpha}}\Phi(\varrho) \tag{5.28}$$

In addition, from (4.1), (5.24), and (5.27) we have

$$\Phi(2\tau\varrho, DP) \le c\left[\tau^{-n}\fint_{B_\varrho} |Du|^p\, dx + |DP|^p\right] \le c\tau^{-n}\Phi(\varrho). \tag{5.29}$$

In view of (5.27) we can apply Lemma 5.1 (with $\tilde{M}$ in place of M) deriving from (5.28) and (5.29)

$$\Phi(\tau\varrho, DP) \le c\left[\tau^{2\tilde{\alpha}}\Phi(\varrho) + (\tau^{2\tilde{\alpha}}\Phi(\varrho))^{\frac{q}{p}} + (\tau^{-n}\Phi(\varrho))^{\frac{q}{p}}\right].$$

Imposing as usual a smallness condition the right-hand side of the preceding inequality can be controlled by $c\tau^{2\tilde{\alpha}}\Phi(\varrho)$ exactly as in the proof of Proposition 5.6 leaving us with the analog of [DM2, (5.33)]. Now the conclusion is the same as in [DM2, Lemma 12].

For $p < 2$ the proof is quite close to [DM2, Lemma 15] and we omit the details. □

5.6 Conclusion

It has been demonstrated in [DM2] that the excess estimates for the nondegenerate and the degenerate case can be combined in a subtle iteration procedure. This procedure works in the same manner for our Propositions 5.6 and 5.10 and yields:

Lemma 5.11. *We assume* (2.1), (2.4), (2.8), (2.9) *and Assumption 2.1 with* $q < p + 1$. *Then, for every* $M > 0$ *and every* $\alpha \in]0, \tilde{\alpha}[$ *there is a constant* $\varepsilon_1 > 0$ *such that for every ball* $B_\varrho(x_0) \subset \Omega$ *with*

$$\Phi(\varrho) \le \varepsilon_1 \qquad \textit{and} \qquad |(Du)^V_{x_0,\varrho}| \le \frac{1}{2}M$$

we have

$$\Phi(r) \leq c \left(\frac{r}{\varrho}\right)^{2\alpha} \Phi(\varrho) \qquad \textit{for all } r \in]0, \varrho].$$

Here, c depends only on n, N, p, q, Γ, M, λ_M, Λ_M, $\lambda_{\tilde{M}}$, $\Lambda_{\tilde{M}}$, α, and ν_M, ε_1 depends additionally on η, and $\tilde{M}$ is the constant from Proposition 5.10 corresponding to the choice $\chi = \varepsilon_0$, where ε_0 appears in Proposition 5.6. As usual we have an additional dependence on γ in the case (III) of Assumption 2.1.

Actually, proving our statement of Lemma 5.11 there is a slight difference to the arguments from [DM2]. Namely, we need to verify the second part of (5.12) for all balls occurring in the iteration procedure, which can be accomplished by well-known arguments (see [E, pp. 246-247] and [CFM, p. 163]).

By Campanato's integral characterization of Hölder continuity, Theorem 2.2 is a routine consequence of Lemma 5.11. Hence, the proof of the theorem is now complete.

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