Partial regularity of strong local minimizers of quasiconvex integrals with (p,q)-growth

Sabine Schemm
Mathematisches Institut, Friedrich-Alexander-Universität Erlangen-Nürnberg, Bismarckstr. 1 1/2, 91054 Erlangen, Germany
(schemm@mi.uni-erlangen.de)

Thomas Schmidt
Mathematisches Institut, Heinrich-Heine-Universität Düsseldorf, Universitätsstr. 1, 40225 Düsseldorf, Germany
(schmidt.th@uni-duesseldorf.de)

We consider strictly quasiconvex integrals

$$F[u] := \int_\Omega f(Du)\,dx \qquad \text{for } u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$$

in the multi-dimensional calculus of variations. For the C^2-integrand $f : \mathbb{R}^{Nn} \to \mathbb{R}$ we impose (p,q)-growth conditions

$$\gamma|\xi|^p \le f(\xi) \le \Gamma(1+|\xi|^q) \qquad \text{for all } \xi \in \mathbb{R}^{Nn}$$

with γ, $\Gamma > 0$ and $1 < p \le q < \min\left\{p + \frac{1}{n}, \frac{2n-1}{2n-2}p\right\}$. Under these assumptions we prove partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity for strong local minimizers of F and the associated relaxed functional $\mathcal{F}$.

1. Introduction

In this paper we investigate the regularity of strong local minimizers of autonomous variational integrals

$$F[u] := \int_\Omega f(Du)\,dx \tag{1.1}$$

defined on vector-valued maps $u : \Omega \to \mathbb{R}^N$, $N \ge 1$. Here Ω denotes a bounded open set in $\mathbb{R}^n$, $n \ge 2$, and $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is a C^2-function satisfying suitable assumptions described below.

The existence and the partial regularity of minimizers of F are classical issues in the modern calculus of variations, and they have been extensively studied, especially over the last twenty years. Specifically, in the present paper we will focus on gradient regularity under the basic assumption that f is quasiconvex; that is

$$\fint_B f(\xi + D\varphi)\,dx \ge f(\xi)$$

holds for all $\xi \in \mathbb{R}^{Nn}$ and all $\varphi \in C_c^\infty(B;\mathbb{R}^N)$, where B denotes the unit ball in $\mathbb{R}^n$. Quasiconvexity, introduced by Morrey in his seminal paper [39], generalizes the classical convexity assumption in the calculus of variations and has turned out to be a key concept for both, the existence and the partial regularity of minimizers. In addition, the central role of quasiconvexity in nonlinear elasticity has been pointed out in the fundamental work of Ball [5].

Before presenting our theorems let us briefly describe some previous existence and regularity results. Primarily, imposing the standard growth conditions

$$\gamma|\xi|^p \leq f(\xi) \leq \Gamma(1+|\xi|^p) \tag{1.2}$$

for some $p>1$, Morrey [39] proved that quasiconvexity is a necessary and sufficient condition for lower semicontinuity of F with respect to weak $W^{1,p}$-convergence (see also [37, 1, 33, 30, 38]). This semicontinuity property is in turn, via the direct method of the calculus of variations, intimately linked to the existence of minimizers of F.

As for regularity, classical examples of minimizers with singularities of the gradient can be constructed [15, 41, 49, 50], even for smooth convex functionals and $n=3$, showing that in the vectorial case everywhere regularity of minimizers in the interior of Ω does not hold. Therefore, one is led to consider partial regularity, i.e. regularity outside a negligible closed subset of Ω, called the singular set. For quasiconvex functionals and $p \geq 2$ partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity of minimizers has first been shown by Evans [21]; see [23, 25, 2, 4, 17, 31] for extensions and variants. Partial regularity in the subquadratic case $1<p<2$ has been eventually proved by Carozza & Fusco & Mingione [13]; see also [48, 16].

Contrary to convex functionals, quasiconvex functionals may — in general — admit non-trivial critical points, i.e. weak solutions of the Euler equation which are not (absolutely) minimizing. Actually, Müller & Sverák [40] have even constructed examples of critical points which are nowhere C^1. This result is in sharp contrast to the partial regularity of minimizers and leads to the investigation of an intermediate notion, namely local minimizers of F in the sense of the following definition.

Definition 1.1 ($W^{1,\bar{q}}$ local minimizer, [32]). *Let $1 \leq p \leq \infty$, $1 \leq \bar{q} \leq \infty$. A map $\bar{u} \in W^{1,p}(\Omega,\mathbb{R})$ with $F[\bar{u}]<\infty$ is called a $W^{1,\bar{q}}$ local minimizer of F if there exists some $\delta>0$ such that*

$$F[\bar{u}] \leq F[\bar{u}+\varphi] \qquad \text{holds for all } \varphi \in W_0^{1,\bar{q}}(\Omega,\mathbb{R}^N)$$

with $\|D\varphi\|_{L^{\bar{q}}(\Omega,\mathbb{R}^{Nn})} \leq \delta$. In particular, for $1 \leq \bar{q} < \infty$ we call $\bar{u}$ a strong local minimizer and for $\bar{q}=\infty$ a weak local minimizer.

Having introduced this notion it is natural to ask whether non-trivial local minimizers of F exist and if they are still regular or not. Actually, the investigation of the existence (and non-existence) of local minimizers has followed previous developments [26, 27] for critical points, and has, up to now, focussed on the case of L^1 local minimizers with affine boundary data. For instance, if the underlying domain Ω is an annulus in $\mathbb{R}^2$ there exist non-trivial L^1 local minimizers [44, 32], while for a starshaped Ω every L^1 local minimizer is already absolutely minimizing [51]. In fact, generalizing these ideas, Taheri [52, 53] has provided multiplicity bounds for local minimizers in terms of topological invariants of Ω.

Clearly, the examples of L^1 local minimizers are also $W^{1,\bar{q}}$ local minimizers for every $1 \leq \bar{q} \leq \infty$. However, in the light of [32, Section 2] it would be interesting to discuss whether non-trivial examples of $W^{1,\bar{q}}$ local minimizers still exist in the simple case that Ω is a ball. Moreover, in view of Remark 2.5 below they should ideally possess the additional feature that they are not $W^{1,p}$ local minimizers. Indeed, for $\bar{q} > p$ we are not aware of any theoretical obstruction but no such examples seem to be present in the literature.

Assuming standard growth (1.2), the regularity theory for $W^{1,\bar{q}}$ local minimizers has been started in a recent interesting work of Kristensen & Taheri [32]. Let us restate their result:

Theorem 1.2 ([32])**.** *Let $2 \leq p < \infty$, $1 \leq \bar{q} < \infty$. Assume that $f \in C^2(\mathbb{R}^{Nn})$ is uniformly strictly quasiconvex with* (1.2) *and that $\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of F. Then there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \backslash \Omega_0| = 0$ such that $\bar{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $0 < \alpha < 1$.*

A similar result for weak local minimizers is contained in [32] and an analogous statement in the subquadratic case $1 < p < 2$ has been established in [12]. We stress that in the light of the counterexamples from [40] these theorems treat a borderline case of regularity. Finally, we mention that it would be desirable to remove the technical assumption $\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ in Theorem 1.2. However, at present it seems quite hard to achieve this.

Next we turn to a generalization of (1.2). Actually, starting with a series of papers by Marcellini (see e. g. [35, 36]) an increasing interest in more flexible growth conditions than (1.2) has emerged. In this paper we concentrate on the so-called (p, q)-growth conditions

$$\gamma|\xi|^p \leq f(\xi) \leq \Gamma(1 + |\xi|^q) \tag{1.3}$$

with two growth exponents $1 < p \leq q < \infty$. In the general vectorial setting the regularity theory for (1.3) was started in [42]. Assuming that f is strictly convex and $q < \min\{p + 1, \frac{pn}{n-1}\}$ the authors showed partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity of minimizers of F. Subsequently, considering less restrictive conditions on the growth exponents, higher integrability results for the gradient of minimizers have been given in [18, 19], and partial regularity has been established in [9]. Here, the most general condition on the exponents, appearing in [18, 9], reads $q < \frac{n+2}{n}p$. For results concerning non-autonomous functionals we refer to [20, 10, 14, 8].

The quasiconvex case is more recent. In [7, 34, 22, 28, 29] the semicontinuity properties in $W^{1,p}(\Omega, \mathbb{R}^N)$ of quasiconvex functionals satisfying (1.3) have been investigated. For our approach the following notion from [7] has turned out to be crucial:

Definition 1.3 ($W^{1,p}$-Quasiconvexity, [7])**.** *We say that f is $W^{1,p}$-quasiconvex iff*

$$\fint_B f(\xi + D\varphi)\, dx \geq f(\xi)$$

holds for all $\xi \in \mathbb{R}^{Nn}$ and all $\varphi \in W^{1,p}_0(B; \mathbb{R}^N)$

Relying on [7, 22] it has been shown in [46] that (strict) $W^{1,p}$-quasiconvexity together with (1.3) and some restrictions on the exponents p and q allows to establish both, existence and partial regularity, of absolute minimizers of F. Precisely, the restriction on the exponents reads $q < \frac{np}{n-1}$ for the existence and

$$1 < p \leq q < p + \frac{\min\{2, p\}}{2n} \tag{1.4}$$

for the regularity. We refer to the forthcoming paper [45] for similar results in the higher order case.

For $n = N$ an important class of examples is given by the polyconvex integrands

$$f(\xi) = (1 + |\xi|^2)^{\frac{p}{2}} + h(\det \xi), \tag{1.5}$$

where h is a convex function of growth rate $\frac{q}{n}$. These integrands are of some interest in non-linear elasticity as pointed out in [5, 6, 7, 34]. Moreover, we recall from [7] that f from (1.5) is $W^{1,p}$-quasiconvex if and only if $p \geq n$ holds. Thus, in this case the above existence and regularity results apply. Let us mention, at this stage, that polyconvex integrands with a structure related to the one in (1.5) and $p > n - 1$, but with a completely different growth behaviour, have previously been treated in [24] by means of more specific methods taking into account the peculiar nature of the functional.

In the case $p < n$ the integrands (1.5) are not $W^{1,p}$-quasiconvex and the above mentioned results do not apply. However, this case, in which F can potentially admit discontinuous minimizers, is of particular physical interest. To extend the existence and regularity results, a relaxation method, which is closely related to the classical idea of the Lebesgue-Serrin extension, has been introduced in [34, 22, 11, 47]. Precisely, one considers the relaxed functional

$$\mathcal{F}[u] := \inf \left\{ \liminf_{k \to \infty} F[u_k] : W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \ni u_k \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^N) \right\} \tag{1.6}$$

for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. It is not hard to see from this definition that the minimum of $\mathcal{F}$ is attained on every Dirichlet class; see [47]. Furthermore, invoking representation results from [22, 11], the approach of [46] has been carried over to minimizers of $\mathcal{F}$. Precisely, assuming that f is strictly quasiconvex with (1.3) and (1.4) partial regularity of minimizers of $\mathcal{F}$ has been established in [47].

The aim of the present paper is now to examine the regularity properties of local minimizers of quasiconvex functionals satisfying (p, q)-growth conditions. Clearly, our main interest remains in the model case (1.5), where examples of L^1 local minimizers have been provided in [53, Section 4]. Precisely, our results are the following: If f is strictly $W^{1,\bar{q}}$-quasiconvex, we prove partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity for $W^{1,\bar{q}}$ local minimizers $\bar{u}$ of F; see Theorem 2.1. In addition, if f is only strictly quasiconvex, following the approach of [47] we prove a similar regularity theorem for $W^{1,\bar{q}}$ local minimizers $\bar{u}$ of the relaxed functional $\mathcal{F}$; see Theorem 2.2. In both cases we are mainly interested in the case $\overline{q} > p$ where, as in [32], we need to impose the technical integrability assumption $\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Note that, as a byproduct, this assumption allows to work with $W^{1,\bar{q}}$-quasiconvexity instead of $W^{1,p}$-quasiconvexity in Theorem 2.1. Our results generalize those obtained in [32, 12, 46, 47].

Finally, let us briefly comment on some technical issues: In the subquadratic case $1 < p < 2$ we improve the condition (1.4) replacing it by

$$1 < p \le q < \min\Big\{\frac{2n-1}{2n-2}p, p + \frac{1}{n}\Big\}. \tag{1.7}$$

Note that in the model case (1.5) with $q = n = N = 2$ the bound (1.7) allows to replace the condition $p > \frac{8}{5}$ from (1.4) by $p > \frac{3}{2}$. However, the reason for this improvement is mainly a technical one. Moreover, we mention that following [32, 12] we use a blow-up argument based on the excess

$$E(x, r) = \fint_{B_r(x)} (1 + |D\bar{u} - (D\bar{u})_{x,r}|^2)^{\frac{p-2}{2}} |D\bar{u} - (D\bar{u})_{x,r}|^2 \, dx \tag{1.8}$$

to prove the partial regularity. In particular, even in the case of absolute minimizers, we provide an alternative proof of the results in [46, 47], where the $\mathcal{A}$-harmonic approximation method has been used.

2. Statement of the results

In this section we state our main results concerning partial regularity of strong local minimizers. Starting with a growth and a coercivity condition we will now supply precise statements of our assumptions.

(H1) ***q*-Growth:** There exists a bound $\Gamma > 0$ such that we have

$$0 \le f(\xi) \le \Gamma(1 + |\xi|^q) \quad \text{for every } \xi \in \mathbb{R}^{Nn}.$$

(H2) ***p*-Coercivity:** There is a coercivity constant $\gamma > 0$ such that we have

$$f(\xi) \ge \gamma |\xi|^p \quad \text{for every } \xi \in \mathbb{R}^{Nn}.$$

Next we state two quasiconvexity conditions, which will be imposed in Theorem 2.1 and Theorem 2.2, respectively.

(H3) **Strict $W^{1,\bar{q}}$-Quasiconvexity:** For each $L > 0$ there is a convexity constant $\nu_L > 0$ such that we have

$$\fint_{B_r(x_0)} \big(f(\xi + D\varphi) - f(\xi)\big) \ge \nu_L \fint_{B_r(x_0)} \big(1 + |D\varphi|^2\big)^{\frac{p-2}{2}} |D\varphi|^2 \, dx$$

for all balls $B_r(x_0) \subset \mathbb{R}^n$, for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \le L + 1$ and for all $\varphi \in W_0^{1,\bar{q}}(B_r(x_0), \mathbb{R}^N)$.

(H4) **Strict Quasiconvexity:** For each $L > 0$ there is a convexity constant $\nu_L > 0$ such that we have

$$\fint_{B_r(x_0)} \big(f(\xi + D\varphi) - f(\xi)\big) \ge \nu_L \fint_{B_r(x_0)} \big(1 + |D\varphi|^2\big)^{\frac{p-2}{2}} |D\varphi|^2 \, dx$$

for all balls $B_r(x_0) \subset \mathbb{R}^n$, for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \le L + 1$ and for all $\varphi \in C_c^\infty(B_r(x_0), \mathbb{R}^N)$.

Now we present our first main result, a regularity result for strong local minimizers of the functional F defined in (1.1).

Theorem 2.1. *Let $\bar{q} \in [1, \infty)$ and*

$$1 < p \leq q < \min\left\{p + \frac{1}{n}, \frac{2n-1}{2n-2}p\right\}. \tag{2.1}$$

Assume that $f \in C^2(\mathbb{R}^{Nn})$ satisfies (H1) and (H3) and that $\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of F in the sense of Definition 1.1. Then there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \backslash \Omega_0| = 0$ such that $\bar{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $0 < \alpha < 1$.

Our second main result concerns strong local minimizers of $\mathcal{F}$ from (1.6), where we define local minimizers of $\mathcal{F}$ along the lines of Definition 1.1 with F replaced by $\mathcal{F}$. Before stating the result we recall some properties of the functional $\mathcal{F}$: Assuming (H1) with $1 < p \leq q < \frac{np}{n-1}$ one has from [22, 47]

$$\mathcal{F}[u] \geq \int_\Omega Qf(Du)\,dx \qquad \text{for } u \in W^{1,p}(\Omega, \mathbb{R}^N), \tag{2.2}$$

$$\mathcal{F}[u] = \int_\Omega Qf(Du)\,dx \qquad \text{for } u \in W^{1,q}(\Omega, \mathbb{R}^N), \tag{2.3}$$

where Qf denotes the quasiconvex envelope of f. Furthermore, it has been shown in [22, 11] that $\mathcal{F}$ depends on the domain Ω like a Radon measure, whose absolutely continuous part has density $Qf(Du)$. These facts will be crucial in the proof of the following result. In particular, we mention that by an argument of [47] they can be used to prove the validity of Euler's equation for minimizers of $\mathcal{F}$. This is an important observation for the proof of the following theorem.

Theorem 2.2. *Let $\bar{q} \in [1, \infty)$ and*

$$1 < p \leq q < \min\left\{p + \frac{1}{n}, \frac{2n-1}{2n-2}p\right\}. \tag{2.4}$$

Assume that $f \in C^2(\mathbb{R}^{Nn})$ satisfies (H1), (H2) and (H4) and that $\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of $\mathcal{F}$ on Ω. Then there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \backslash \Omega_0| = 0$ such that $\bar{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $0 < \alpha < 1$.

We highlight some features of the previous theorems.

Remark 2.3. *In some sense $W^{1,\bar{q}}$-quasiconvexity is necessary for the existence of $W^{1,\bar{q}}$ local minimizers. Precisely, adapting the proof of [53, Proposition 4.1] one finds: If $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of F then for every $\varphi \in W^{1,\bar{q}}(B, \mathbb{R}^N)$ there holds*

$$\fint_B f(D\bar{u}(y) + D\varphi(x))\,dx \geq f(D\bar{u}(y)) \qquad \text{for a.e. } y \in \Omega.$$

Remark 2.4. *If $\bar{q} \geq q$ then (strict) $W^{1,\bar{q}}$-quasiconvexity is equivalent to (strict) quasiconvexity. Combining this with* (2.3)*, Theorem 2.1 and Theorem 2.2 turn out to be equivalent in this case.*

Remark 2.5. *In the case $\bar{q} \leq p$ it follows from [32, Section 2] that Theorem 2.1 can be reduced to the case of absolute minimizers.*

Remark 2.6. *Under the additional assumption*

$$\limsup_{r \to 0^+} \|Du - (Du)_{x,r}\|_{L^\infty(B_r(x), \mathbb{R}^{Nn})} < \delta$$

Theorem 2.1 and Theorem 2.2 hold also in the case of weak local minimizers, i e. for $\bar{q} = \infty$. This generalization is straightforward along the lines of [32, 12].

Remark 2.7. *The proofs of the theorems will show that we can choose Ω_0 such that*

$$\Omega \setminus \Omega_0 \subset \left\{ x \in \Omega \,:\, \liminf_{r \to 0+} E(x, r) > 0 \ \textit{or} \ \limsup_{r \to 0+} |(Du)_{x,r}| = \infty \right\}$$

holds, where $E(x, r)$ is defined in (1.8)*.*

Remark 2.8. *Under the assumptions of Theorem 2.1 or Theorem 2.2, if $f \in C^\infty(\mathbb{R}^{Nn})$ then we have $u \in C^\infty(\Omega_0, \mathbb{R}^N)$. Once $C^{1,\alpha}_{\mathrm{loc}}$-regularity is proved, this higher regularity result follows from the application of linear theory to the Euler equation.*

3. Preliminaries

Throughout this paper we denote by a c a positive constant possibly varying from line to line. The dependences of such constants will only occasionally be highlighted. We write $B_r(x)$ for the open ball with center x and radius r in $\mathbb{R}^n$ and abbreviate $B_r := B_r(0)$ and $B := B_1$. In addition, we will use the common abbreviations $u_{x,r} := \fint_{B_r(x)} u\,dx := \frac{1}{|B_r(x)|} \int_{B_r(x)} u\,dx$ and $u_r := u_{0,r}$ for mean values, where $|\cdot|$ denotes the n-dimensional Lebesgue measure. Moreover, for $\beta > 0$ we define the functions $V_\beta : \mathbb{R}^k \mapsto \mathbb{R}^k$ and $W_\beta : \mathbb{R}^k \mapsto \mathbb{R}^k$ by

$$V_\beta(\xi) = \left(1 + |\xi|^2\right)^{\frac{\beta-1}{2}} \xi, \qquad W_\beta(\xi) = \left(1 + |\xi|\right)^{\beta-1} \xi \tag{3.1}$$

for $\xi \in \mathbb{R}^k$, $k \in \mathbb{N}$. Since we are mostly dealing with $\beta = \frac{p}{2}$, where p is a fixed exponent, we abbreviate $V = V_{\frac{p}{2}}$ and $W = W_{\frac{p}{2}}$. Next, we will collect some useful properties of V and W: Clearly, we have $|V_\beta(\xi)| = V_\beta(|\xi|)$, $|W_\beta(\xi)| = W_\beta(|\xi|)$ and

$$c^{-1}|W_\beta(\xi)| \leq |V_\beta(\xi)| \leq c|W_\beta(\xi)|, \tag{3.2}$$

where c depends only on β. Furthermore, the functions $V_\beta(t)$ and $W_\beta(t)$ are both non-decreasing in $t \geq 0$ and some elementary calculations show that $|W|^2$ is convex for $1 \leq p < \infty$ and $|W|^{\frac{2}{p}}$ is convex for $1 \leq p \leq 2$ (in contrast to $|V|^2$ and $|V|^{\frac{2}{p}}$). Some additional properties are summarized in the following lemma.

Lemma 3.1. *Let $\beta > 0$, $1 < p < \infty$, and $M > 0$. Then, for all $\xi, \eta \in \mathbb{R}^k$ and $t > 0$ we have*

(i) $|V(t\xi)| \le \max\{t, t^{\frac{p}{2}}\}|V(\xi)|$;

(ii) $|V_\beta(\xi+\eta)| \le c(|V_\beta(\xi)| + |V_\beta(\eta)|)$;

(iii) $(1+|\xi|^2+|\eta|^2)^{\frac{p}{2}} \le c(1+|V(\xi)|^2+|V(\eta)|^2)$;

(iv) $|V_{p-1}(\xi)||\eta| \le |V(\xi)|^2 + |V(\eta)|^2$.

Here, c depends only on β and p, respectively.

Proof. The assertions (i) and (iii) are easy to check. (ii) has been proved for $\frac{1}{2} < \beta < 1$ in [13, Lemma 2.1] and is easily seen to hold for all $\beta > 0$. (iv) follows from the fact that V_{p-1} is non-decreasing. $\square$

Next we restate an integral inequality for V; see for instance [46].

Lemma 3.2. *Let $1 < p < \infty$ and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Then we have*

$$\int_\Omega |V(Du - (Du)_\Omega)|^2\,dx \le c \int_\Omega |V(Du)|^2\,dx$$

with a constant c depending only on p. $\square$

Furthermore, we recall a Poincaré type inequality and a Sobolev-Poincaré type inequality for V.

Lemma 3.3. *We consider $1 < p < \infty$, a ball $B_r(x_0)$ in $\mathbb{R}^n$ and a function $u \in W^{1,p}(B_r(x), \mathbb{R}^N)$. Then, we have*

$$\int_{B_r(x_0)} \Big|V\Big(\frac{u - u_{x_0,r}}{r}\Big)\Big|^2\,dx \le c \int_{B_r(x_0)} \big|V(Du)\big|^2\,dx. \tag{3.3}$$

In addition, setting $p^\# := \frac{2n}{n-p} > 2$ for $1 < p < 2$ we have

$$\Bigg(\fint_{B_r(x_0)} \Big|V\Big(\frac{u - u_{x_0,r}}{r}\Big)\Big|^{p^\#}\Bigg)^{\frac{1}{p^\#}} dx \le c\Bigg(\fint_{B_r(x_0)} \big|V(Du)\big|^2\,dx\Bigg)^{\frac{1}{2}}. \tag{3.4}$$

The constant c depends only on n, N and p in both inequalities. $\square$

Here, (3.4) has been proved in [16, Theorem 2] and (3.3) follows easily from the standard Poincaré inequality for $p \ge 2$ and from (3.4) for $1 < p < 2$. The reader should note that a weaker version of (3.4) has previously been established in [13].

Next we restate some estimates for smoothing operators, which will be crucial for our approach. These estimates, introduced first in [22, Lemma 2.2], have already been used in the regularity theory of integrals with (p,q)-growth; see [42, 46, 47]. We state them in the form of [46, Lemma 6.3].

Lemma 3.4. *Let $0<r<s$ and $B_s\subset\Omega$. We define a linear smoothing operator*

$$T_{r,s}:W^{1,1}(\Omega;\mathbb{R}^N)\to W^{1,1}(\Omega;\mathbb{R}^N)$$

for $u\in W^{1,1}(\Omega;\mathbb{R}^N)$ and $x\in\Omega$ by

$$T_{r,s}u(x):=\fint_{B_1}u(x+\vartheta(x)y)\,dy,\quad \text{where } \vartheta(x):=\frac{1}{2}\max\Big\{\min\{|x|-r,s-|x|\},0\Big\}. \tag{3.5}$$

With this definition, for all $1\le p\le q<\frac{np}{n-1}$ and all $u\in W^{1,p}(\Omega;\mathbb{R}^N)$ the following assertions are true:

$$T_{r,s}u\in W^{1,p}(\Omega;\mathbb{R}^N),$$

$$u=T_{r,s}u \qquad \text{almost everywhere on } (\Omega\setminus B_s)\cup B_r, \tag{3.6}$$

$$T_{r,s}u\in u+W_0^{1,p}(B_s\setminus\overline{B_r};\mathbb{R}^N), \tag{3.7}$$

$$|DT_{r,s}u|\le c(n)T_{r,s}|Du| \qquad \text{almost everywhere on } \Omega, \tag{3.8}$$

$$\|T_{r,s}u\|_{p;B_s\setminus B_r}\le c(n,p)\|u\|_{p;B_s\setminus B_r}, \tag{3.9}$$

$$\|DT_{r,s}u\|_{p;B_s\setminus B_r}\le c(n,p)\|Du\|_{p;B_s\setminus B_r}, \tag{3.10}$$

$$\|T_{r,s}u\|_{q;B_s\setminus B_r}\le c(n,p,q)(s-r)^{\frac{n}{q}-\frac{n-1}{p}}\left[\sup_{t\in]r,s[}\frac{\tilde\Xi(t)-\tilde\Xi(r)}{t-r}+\sup_{t\in]r,s[}\frac{\tilde\Xi(s)-\tilde\Xi(t)}{s-t}\right]^{\frac{1}{p}}, \tag{3.11}$$

$$\|DT_{r,s}u\|_{q;B_s\setminus B_r}\le c(n,p,q)(s-r)^{\frac{n}{q}-\frac{n-1}{p}}\left[\sup_{t\in]r,s[}\frac{\Xi(t)-\Xi(r)}{t-r}+\sup_{t\in]r,s[}\frac{\Xi(s)-\Xi(t)}{s-t}\right]^{\frac{1}{p}}. \tag{3.12}$$

Here we have used the abbreviations

$$\tilde\Xi(t):=\|u\|^p_{p;B_t}\quad\text{and}\quad \Xi(t):=\|Du\|^p_{p;B_t}.$$

□

Further estimates of the terms on the right-hand side of (3.11) and (3.12) can be obtained by means of the following simple lemma.

Lemma 3.5. *Let $-\infty<r<s<\infty$ and a continuous non-decreasing function $\Xi:[r,s]\to\mathbb{R}$ be given. Then, there are $\tilde r\in]r,\frac{2r+s}{3}[$ and $\tilde s\in]\frac{r+2s}{3},s[$ for which hold:*

$$\begin{aligned}\frac{\Xi(t)-\Xi(\tilde r)}{t-\tilde r}&\le 4\frac{\Xi(s)-\Xi(r)}{s-r}\\ \frac{\Xi(\tilde s)-\Xi(t)}{\tilde s-t}&\le 4\frac{\Xi(s)-\Xi(r)}{s-r}\end{aligned}\qquad \text{for every } t\in]\tilde r,\tilde s[. \tag{3.13}$$

In particular we have

$$\frac{s-r}{3}\le\tilde s-\tilde r\le s-r. \tag{3.14}$$

Proof. An elementary proof is given in [22]. □

Remark 3.6. *Assume that Ξ is absolutely continuous and non-decreasing and a set $N \subset \mathbb{R}$ of Lebesgue measure zero is given. Then, we can choose $\tilde{r}$ and $\tilde{s}$ as in Lemma 3.5 even with the additional property $\tilde{r}, \tilde{s} \notin N$; see [47, Lemma 4.6].*

Finally, we state another useful lemma concerning the smoothing operator $T_{r,s}$:

Lemma 3.7. *Let $1 < p < \infty$, $0 < r < s$ and $B_s(x_0) \subset \Omega$. Then, for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ we have*

$$\big|V(DT_{r,s}u)\big|^2 \le cT_{r,s}\big[\big|V(Du)\big|^2\big] \qquad \text{a. e. on } \Omega, \tag{3.15}$$

where c depends only on n and p.

Proof. Due to (3.2) it suffices to show the claim with V replaced by W. Since $|W|^2$ is a non-decreasing and convex function we obtain using (3.5), (3.8) and Jensen's inequality

$$\big|W(DT_{r,s}u)\big|^2 \le c\big|W(T_{r,s}|Du|)\big|^2 \le cT_{r,s}\big[\big|W(Du)\big|^2\big] \quad \text{a. e. on } \Omega.$$

This proves the claim. □

4. Proof of Theorem 2.1

First we note that the definition (1.8) of the excess reads

$$E(x,r) = \fint_{B_r(x)} |V(D\bar{u} - (D\bar{u})_{x,r})|^2 \, dx$$

in the terminology of Section 3. We will establish a decay estimate for this excess in the following proposition, which we prove by an indirect blow-up argument.

Proposition 4.1. *Under the assumptions of Theorem 2.1 for every $L > 0$ there is a constant $C > 0$ with the following property: For each $0 < \tau \le \frac{1}{2}$ there exists a number $\varepsilon > 0$ such that the conditions*

$$|(D\bar{u})_{x,r}| \le L, \quad r < \varepsilon \quad \text{and} \quad E(x,r) < \varepsilon$$

for a ball $B_r(x) \subset\subset \Omega$ imply

$$E(x, \tau r) \le C\tau^2 E(x,r).$$

Proof. We argue by contradiction. Assuming the proposition to be false there are $L > 0$ and $0 < \tau \le \frac{1}{2}$, corresponding to a constant C which will be chosen later, such that the following holds: There is a sequence of balls $B_{r_j}(x_j) \subset\subset \Omega$ with $r_j \to 0$ such that

$$|(D\bar{u})_{x_j,r_j}| \le L \quad \text{and} \quad 0 < \lambda_j := \sqrt{E(x_j,r_j)} \to 0 \text{ as } j \to \infty, \tag{4.1}$$

but

$$E(x_j, \tau r_j) > C\tau^2 E(x_j, r_j). \tag{4.2}$$

Step 1: Blow-up. We define $\xi_j := (D\bar{u})_{x_j,r_j}$ and

$$u_j(y) := \frac{1}{\lambda_j r_j}\big[\bar{u}(x_j + r_j y) - (\bar{u})_{x_j,r_j} - \xi_j r_j y\big] \quad \text{for } y \in B.$$

Then we have

$$\lambda_j Du_j(y) = D\bar{u}(x_j + r_j y) - \xi_j \quad \text{for } y \in B, \quad (u_j)_{0,1} = 0, \quad (Du_j)_{0,1} = 0$$

and

$$\fint_B \Big|\frac{V(\lambda_j Du_j)}{\lambda_j}\Big|^2 dx = 1. \tag{4.3}$$

Since $\lambda_j (Du_j)_{0,\tau} = (D\bar{u})_{x_j,\tau r_j} - \xi_j$ we get from (4.2)

$$\lambda_j^{-2} \fint_{B_\tau} |V(\lambda_j (Du_j - (Du_j)_{0,\tau})|^2 \, dx > C\tau^2. \tag{4.4}$$

Further we define

$$f_j(\xi) := \frac{f(\xi_j + \lambda_j \xi) - f(\xi_j) - Df(\xi_j)\lambda_j \xi}{\lambda_j^2} \quad \text{for } \xi \in \mathbb{R}^{Nn}.$$

Noting $|\xi_j| \le L$ we get from [2, Lemma II.3] that there is a positive constant $k(L)$ with

$$\begin{aligned} |f_j(\xi)| &\le k\lambda_j^{-2}|V_{\frac{q}{2}}(\lambda_j \xi)|^2, \\ |Df_j(\xi)| &\le k\lambda_j^{-1}|V_{q-1}(\lambda_j \xi)| \end{aligned} \tag{4.5}$$

for all $\xi \in \mathbb{R}^{Nn}$. Moreover, we rewrite the quasiconvexity hypothesis $(H3)$ in the following form:

$$\nu_L \int_B \Big|\frac{V(\lambda_j D\varphi)}{\lambda_j}\Big|^2 dx \le \int_B \big(f_j(\xi + D\varphi) - f_j(\xi)\big)\, dx \tag{4.6}$$

for all $\xi \in \mathbb{R}^{Nn}$ with $|\lambda_j \xi| \le 1$ and for all $\varphi \in W_0^{1,\bar{q}}(B, \mathbb{R}^N)$. In addition, setting

$$F_j[u] = \int_B f_j(Du)\, dx$$

the minimizing property of $\bar{u}$ can be rephrased as follows: For all $\varphi \in W_0^{1,\bar{q}}(B, \mathbb{R}^N)$ with

$$\|D\varphi\|_{L^{\bar{q}}(B,\mathbb{R}^{Nn})} \le \frac{\delta}{\lambda_j r_j^{\frac{n}{\bar{q}}}} \tag{4.7}$$

we have

$$F_j[u_j] \le F_j[u_j + \varphi]. \tag{4.8}$$

Next we claim

$$\fint_B |Du_j|^{\min\{2,p\}}\,dx \le c. \tag{4.9}$$

Actually, (4.9) follows immediately from (4.3) for $p \ge 2$. In contrast, for $p \le 2$ we first deduce $\fint_B |V(Du_j)|^2\,dx \le c$ from (4.3) by virtue of Lemma 3.1 (*i*) and then get (4.9) by Lemma 3.1 (*iii*). Thus, passing to subsequences we may assume that for some $u \in W^{1,\min\{2,p\}}(B,\mathbb{R}^N)$ and some $\xi_\infty \in \mathbb{R}^{Nn}$ we have

$$\begin{array}{rl} u_j \to u & \text{weakly in } W^{1,\min\{2,p\}}(B,\mathbb{R}^N);\\ u_j \to u & \text{strongly in } L^{\min\{2,p\}}(B,\mathbb{R}^N);\\ \lambda_j Du_j \to 0 & \text{a.e. on } B;\\ \xi_j \to \xi_\infty & \text{in } \mathbb{R}^{Nn}. \end{array} \tag{4.10}$$

Step 2: Linearization. In this step we will show that u is a weak solution of a linear system. Precisely, we claim

$$\int_B D^2 f(\xi_\infty)(Du, D\varphi)\,dx = 0 \qquad \text{for all } \varphi \in C^1_{\mathrm{c}}(B,\mathbb{R}^N). \tag{4.11}$$

Actually, the derivation of the limit equation (4.11) is well-known (see for instance [21, 2, 42, 9, 32]) and we will only sketch it. From the minimality property of u_j in (4.8) we get the following Euler-Lagrange equation:

$$\int_B Df_j(Du_j)D\varphi\,dx = 0 \qquad \text{for all } \varphi \in C^1_{\mathrm{c}}(B,\mathbb{R}^N).$$

We will show that the preceding equation converges to (4.11) as $j \to \infty$. Setting $B_j^+ := \{x \in B : |\lambda_j Du_j(x)| > 1\}$ and using (4.5) and $q \le p+1$ we obtain

$$\begin{aligned} \Big|\int_{B_j^+} Df_j(Du_j)D\varphi\,dx\Big| &\le \frac{c}{\lambda_j}\int_{B_j^+} |V_{q-1}(\lambda_j Du_j)|\,dx\ \sup_B |D\varphi| \\ &\le c\sup_B |D\varphi|\lambda_j \fint_B \frac{|V(\lambda_j Du_j)|^2}{\lambda_j^2}\,dx. \end{aligned}$$

By (4.3) we infer that this term vanishes as $j \to \infty$ and it remains to treat the integral over $B_j^- := \{x \in B : |\lambda_j Du_j(x)| \le 1\}$. Here, noting $|B_j^+| \le c\lambda_j^2 \to 0$ as in [32] we have

$$\begin{aligned} \int_{B_j^-} Df_j(Du_j)D\varphi\,dx &= \frac{1}{\lambda_j}\int_{B_j^-} \big(Df(\xi_j + \lambda_j Du_j) - Df(\xi_j)\big)D\varphi\,dx \\ &= \int_{B_j^-}\int_0^1 D^2 f(\xi_j + t\lambda_j Du_j)\,dt(Du_j, D\varphi)\,dx \\ &\to \int_B D^2 f(\xi_\infty)(Du, D\varphi) \end{aligned}$$

and (4.11) follows. The condition (H3) implies that $D^2f(\xi_\infty)$ is elliptic in the sense of Legendre-Hadamard with ellipticity constant $2\nu_L$ and upper bound $K_L := \sup_{|\xi|\le L} D^2f(\xi)$. Thus, we can apply linear theory to deduce that u is C^1 on B and

$$\fint_{B_\tau} |Du-(Du)_{0,\tau}|^2\,dx \le c\tau^2 \tag{4.12}$$

is valid. Here, c depends only on n, N, ν_L and K_L. The remainder of the proof is now mostly devoted to showing

$$\lambda_j^{-2}\int_{B_\tau} |V(\lambda_j(Du_j-Du))|^2\,dx \to 0 \quad \text{as } j\to\infty. \tag{4.13}$$

Once we have proved (4.13) we will see that (4.12) contradicts (4.4).

Step 3: Construction of test functions and preliminary estimates. We fix $\tau<\sigma<1$, $0<\alpha<1$ and consider $B_r(x_0)\subset\subset B_\sigma$. We define affine functions $a_j(x) := (u_j)_{x_0,r} + (Du_j)_{x_0,r}(x-x_0)$ and set

$$v_j(x) := u_j(x)-a_j(x).$$

Moreover, we introduce the abbreviation

$$\Xi_j(t) := \lambda_j^{-2}\int_{B_t(x_0)} \Big(\Big|V\Big(\frac{\lambda_j v_j}{(1-\alpha)r}\Big)\Big|^2 + \big|V(\lambda_j Dv_j)\big|^2\Big)\,dx$$

and choose for this function $\alpha r\le \tilde r_j<\tilde s_j\le r$ as in Lemma 3.5. In particular we have

$$\frac13(1-\alpha)r \le \tilde s_j-\tilde r_j \le (1-\alpha)r. \tag{4.14}$$

Now we consider smooth cut-off functions $\eta_j:\mathbb{R}^n\to[0,1]$ which satisfy $\eta_j\equiv 1$ in a neighborhood of $\overline{B_{\tilde r_j}(x_0)}$, $\eta_j=0$ in a neighborhood of $\mathbb{R}^n\setminus B_{\tilde s_j}(x_0)$ and $|\nabla\eta_j|\le\frac{2}{\tilde s_j-\tilde r_j}$ on $B_r(x_0)$. We define

$$\chi_j = \big[(1-\eta_j)v_j\big],\quad \psi_j := T_{\tilde r_j,\tilde s_j}\chi_j \quad\text{and}\quad \varphi_j := v_j-\psi_j,$$

where the smoothing operator T is defined in Lemma 3.4. According to (3.6) and (3.7) of Lemma 3.4 we have $\varphi_j\in W_0^{1,p}(B_{\tilde s_j}(x_0),\mathbb{R}^N)$, $\varphi_j=v_j$, $\psi_j=0$ on $B_{\tilde r_j}(x_0)$ and

$$Du_j-Da_j = Dv_j = D\varphi_j + D\psi_j \quad \text{on } B. \tag{4.15}$$

In addition, the product rule and (4.14) give

$$|D\chi_j| \le |Dv_j| + \left|\frac{v_j}{(1-\alpha)r}\right|. \tag{4.16}$$

Next, we will derive two preparatory estimates for χ_j, namely (4.17) and (4.18): Setting

$$Y_j := \lambda_j^{-2}\int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \Big(\Big|V\Big(\frac{\lambda_j v_j}{(1-\alpha)r}\Big)\Big|^2 + \big|V(\lambda_j Dv_j)\big|^2\Big)\,dx.$$

we apply in turn (3.8), Lemma 3.7, (3.9) (with $p=1$), (4.16) and Lemma 3.1 (*ii*) to get the estimate

$$\begin{aligned}\lambda_j^{-2}\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}|V(\lambda_j D\psi_j)|^2\,dx \;&\le\; c\lambda_j^{-2}\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}\big|T_{\tilde r_j,\tilde s_j}[|V(\lambda_j D\chi_j)|^2]\big|\,dx\\ &\le\; c\lambda_j^{-2}\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}|V(\lambda_j D\chi_j)|^2\,dx \le c\,Y_j. \end{aligned}\tag{4.17}$$

Arguing in a similar way, but using (3.11) (with $p=1$, $q=\kappa$) instead of (3.9) we find for $1\le\kappa<\frac{n}{n-1}$:

$$\begin{aligned}&\lambda_j^{-2}\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}|V(\lambda_j D\psi_j)|^{2\kappa}\le c\lambda_j^{-2}\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}\big|T_{\tilde r_j,\tilde s_j}[|V(\lambda_j D\chi_j)|^2]\big|^{\kappa}\,dx\\ &\quad\le\; c\lambda_j^{-2}(\tilde s_j-\tilde r_j)^{n-(n-1)\kappa}\Bigg(\sup_{t\in]\tilde r_j,\tilde s_j[}\frac{1}{t-\tilde r_j}\int\limits_{B_t(x_0)\setminus B_{\tilde r_j}(x_0)}|V(\lambda_j D\chi_j)|^2\,dx\\ &\qquad\qquad + \sup_{t\in]\tilde r_j,\tilde s_j[}\frac{1}{\tilde s_j-t}\int\limits_{B_{\tilde s_j}(x_0)\setminus B_t(x_0)}|V(\lambda_j D\chi_j)|^2\,dx\Bigg)^{\kappa}\\ &\quad\le\; c\lambda_j^{2\kappa-2}(\tilde s_j-\tilde r_j)^{n(1-\kappa)+\kappa}\Bigg(\sup_{t\in]\tilde r_j,\tilde s_j[}\frac{\Xi_j(t)-\Xi_j(\tilde r_j)}{t-\tilde r_j}+\sup_{t\in]\tilde r_j,\tilde s_j[}\frac{\Xi_j(\tilde s_j)-\Xi_j(t)}{\tilde s_j-t}\Bigg)^{\kappa}. \end{aligned}$$

Combining the last inequality with the estimates of Lemma 3.5 we obtain

$$\lambda_j^{-2}\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}|V(\lambda_j D\psi_j)|^{2\kappa}\;\le\; c\Bigg(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Bigg)^{\kappa-1}Y_j. \tag{4.18}$$

Step 4: The main estimate. In this step we will combine ideas of [32] and [46] to establish a key estimate. This estimate will lead to (4.13) later in the proof. Here, our first aim is to verify (4.7) for φ_j with j large, which will enable us to use (4.8). We start with the following computation and use for this purpose (4.14), formula (3.9) from Lemma 3.4 and the Poincaré inequality:

$$\begin{aligned}\int\limits_B|D\varphi_j|^{\bar q}\,dx &\le \int\limits_{B_{\tilde r_j}(x_0)}|Dv_j|^{\bar q}\,dx+\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}|D\psi_j|^{\bar q}\,dx\\ &\le \int\limits_{B_r(x_0)}|Dv_j|^{\bar q}\,dx + c\int\limits_{B_{\tilde s_j(x_0)}\setminus B_{\tilde r_j}(x_0)}|D\chi_j|^{\bar q}\,dx\\ &\le \int\limits_{B_r(x_0)}|Dv_j|^{\bar q}\,dx + c\Bigg(\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}|Dv_j|^{\bar q}\,dx+\int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)}\Big(\frac{|v_j|}{(1-\alpha)r}\Big)^{\bar q}\,dx\Bigg)\end{aligned}$$

$$\leq \quad c\Big(1+\frac{1}{1-\alpha}\Big)^{\bar q} \int\limits_{B_r(x_0)} |Dv_j|^{\bar q}\,dx.$$

Changing coordinates in view of $B_{rr_j}(x_j + r_j x_0) \subset B_{r_j}(x_j)$ we obtain

$$\|D\varphi_j\|_{L^{\bar q}(B,\mathbb{R}^{Nn})} \leq \frac{c}{\lambda_j r_j^{\frac{n}{\bar q}}}\Big(1+\frac{1}{1-\alpha}\Big)\Big(\int\limits_{B_{r_j}(x_j)} |D\bar u|^{\bar q}\,dx\Big)^{\frac{1}{\bar q}}.$$

Therefore the condition (4.7) is fulfilled if

$$c\Big(1+\frac{1}{1-\alpha}\Big)\Big(\int\limits_{B_{r_j}(x_j)} |D\bar u|^{\bar q}\,dx\Big)^{\frac{1}{\bar q}} \leq \delta$$

holds and this is satisfied for sufficiently large j, say for $j \geq j_1(\alpha)$. Furthermore, from the definition of a_j and (4.9) we see $|Da_j| \leq cr^{-n}$. Hence, there exists a $j_2(r)$ such that $|\lambda_j Da_j| \leq 1$ holds for all $j \geq j_2(r)$. We define $j_0(\alpha,r) := \max\{j_1(\alpha), j_2(r)\}$. Then, for $j \geq j_0(\alpha,r)$ we use (4.6), (4.8) and (4.15) to get

$$\begin{aligned}
\nu_L \int\limits_{B_{\tilde r_j}(x_0)} \Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 dx &\leq \nu_L \int\limits_{B_{\tilde s_j}(x_0)} \Big|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\Big|^2 dx \\
&\leq \int\limits_{B_{\tilde s_j}(x_0)} \big(f_j(Da_j + D\varphi_j) - f_j(Da_j)\big)\,dx \\
&= \int\limits_{B_{\tilde s_j}(x_0)} \big(f_j(Du_j - D\psi_j) - f_j(Du_j)\big)\,dx + \int\limits_{B} \big(f_j(Du_j) - f_j(Du_j - D\varphi_j)\big)\,dx \\
&\qquad + \int\limits_{B_{\tilde s_j}(x_0)} \big(f_j(Da_j + D\psi_j) - f_j(Da_j)\big)\,dx \\
&\leq \int\limits_{B_{\tilde s_j}(x_0)} \big(f_j(Du_j - D\psi_j) - f_j(Du_j)\big)\,dx + \int\limits_{B_{\tilde s_j}(x_0)} \big(f_j(Da_j + D\psi_j) - f_j(Da_j)\big)\,dx.
\end{aligned} \tag{4.19}$$

Recalling $\psi_j = 0$ on $B_{\tilde r_j}(x_0)$ we estimate the right-hand side by inequality (4.5) and Lemma 3.1 (ii):

$$\begin{aligned}
\nu_L \int\limits_{B_{\tilde r_j}(x_0)} \Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 dx &\leq \int\limits_{B_{\tilde s_j}(x_0)} \int_0^1 \big(Df_j(Da_j + tD\psi_j) - Df_j(Du_j - tD\psi_j)\big) D\psi_j\,dt\,dx \\
&\leq \quad c \int\limits_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} \Big(\Big|\frac{V_{q-1}(\lambda_j Da_j)}{\lambda_j}\Big||D\psi_j| + \Big|\frac{V_{q-1}(\lambda_j Dv_j)}{\lambda_j}\Big||D\psi_j| + \Big|\frac{V_{\frac{q}{2}}(\lambda_j D\psi_j)}{\lambda_j}\Big|^2\Big)\,dx \\
&=: \quad c\,(I + II + III)
\end{aligned}$$

with the obvious labeling.

Estimation of III. We estimate the last integral by Lemma 3.1 (iii), (4.17) and (4.18) with $\kappa = \frac{q}{p} < \frac{n}{n-1} = 1 + \frac{1}{n-1}$:

$$\begin{aligned} III &= \int\limits_{B_{\tilde{s}_j(x_0)}\setminus B_{\tilde{r}_j}(x_0)} \Big|\frac{V_{\frac{q}{2}}(\lambda_j D\psi_j)}{\lambda_j}\Big|^2\,dx \\ &\le \lambda_j^{-2} \int\limits_{B_{\tilde{s}_j(x_0)}\setminus B_{\tilde{r}_j}(x_0)} \big(1+|\lambda_j D\psi_j|^2\big)^{\frac{p}{2}\frac{q-p}{p}} |V(\lambda_j D\psi_j)|^2\,dx \\ &\le c\lambda_j^{-2} \int\limits_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \big(1+\big|V(\lambda_j D\psi_j)\big|^2\big)^{\frac{q-p}{p}} \big|V(\lambda_j D\psi_j)\big|^2\,dx \\ &\le c\,\lambda_j^{-2} \int\limits_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \big(\big|V(\lambda_j D\psi_j)\big|^2 + \big|V(\lambda_j D\psi_j)\big|^{\frac{2q}{p}}\big)\,dx \\ &\le c\bigg(Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{q}{p}-1} Y_j\bigg) \le c\bigg(Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{1}{n-1}} Y_j\bigg). \end{aligned}$$

Estimation of II. We estimate the integral II distinguishing the cases $p > 2\frac{n-1}{n}$ and $p \le 2\frac{n-1}{n}$.

Case $p > 2\frac{n-1}{n}$: In this case (2.1) reads $q < p + \frac{1}{n}$ and we have $\frac{p}{2} + 1 < p + \frac{1}{n}$. Hence, enlarging q if necessary we may assume $q \ge \frac{p}{2} + 1$ (without destroying $q < p + \frac{1}{n}$). Next we give a pointwise estimation of the integrand in II. For $|\lambda_j Dv_j| \le 1$ we obtain with Young's inequality

$$\begin{aligned} \big(1+|\lambda_j Dv_j|^2\big)^{\frac{q-2}{2}} |Dv_j||D\psi_j| &= \lambda_j^{-2}\big(1+|\lambda_j Dv_j|^2\big)^{\frac{p-2}{4}+\frac{2q-p-2}{4}} |\lambda_j Dv_j||\lambda_j D\psi_j| \\ &\le c\lambda_j^{-2}\big(1+|\lambda_j Dv_j|^2\big)^{\frac{p-2}{4}} |\lambda_j Dv_j||\lambda_j D\psi_j| \\ &\le c\bigg(\Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 + \lambda_j^{-2}|\lambda_j D\psi_j|^2\bigg), \end{aligned}$$

while for $|\lambda_j Dv_j| > 1$ a similar computation yields

$$\begin{aligned} \big(1+|\lambda_j Dv_j|^2\big)^{\frac{q-2}{2}} |Dv_j||D\psi_j| &= \lambda_j^{-2}\big(1+|\lambda_j Dv_j|^2\big)^{\frac{p-2}{2}\frac{q-1}{p}+\frac{2q-p-2}{2p}} |\lambda_j Dv_j||\lambda_j D\psi_j| \\ &\le c\lambda_j^{-2}\big(1+|\lambda_j Dv_j|^2\big)^{\frac{p-2}{2}\frac{q-1}{p}} |\lambda_j Dv_j|^{2\frac{q-1}{p}}\Big)|\lambda_j D\psi_j| \\ &\le c\bigg(\Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 + \lambda_j^{-2}|\lambda_j D\psi_j|^{\frac{p}{p+1-q}}\bigg). \end{aligned}$$

Thus, using $\frac{p}{p+1-q} \ge 2$, $q < p + \frac{1}{n}$, (4.17) and (4.18) (with $\kappa = \frac{1}{p+1-q} < \frac{n}{n-1}$) we argue essentially as for III:

$$II \le c \int\limits_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2\,dx + c\lambda_j^{-2} \int\limits_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \big(|\lambda_j D\psi_j|^2 + |\lambda_j D\psi_j|^{\frac{p}{p+1-q}}\big)\,dx$$

$$\begin{aligned}
&\leq c \int\limits_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \Big(\Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 + \Big|\frac{V(\lambda_j D\psi_j)}{\lambda_j}\Big|^2 + \lambda_j^{-2}\big|V(\lambda_j D\psi_j)\big|^{\frac{2}{p+1-q}}\Big)\,dx \\
&\leq c\bigg(Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{1}{p+1-q}-1} Y_j\bigg) \leq c\bigg(Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{1}{n-1}} Y_j\bigg).
\end{aligned}$$

Case $p \leq 2\frac{n-1}{n}$: In this case (2.1) reads $q < \frac{2n-1}{2n-2}p$ and we have, in particular, $q \leq \frac{p}{2}+1$. Again we will give an estimate for the integrand in II. In the case $|Dv_j| \leq |D\psi_j|$ we find since V_{q-1} is non-decreasing

$$\Big|\frac{V_{q-1}(\lambda_j Dv_j)}{\lambda_j}\Big||D\psi_j| \leq \Big|\frac{V_{\frac{q}{2}}(\lambda_j D\psi_j)}{\lambda_j}\Big|^2$$

and in the case $|Dv_j| > |D\psi_j|$ we get with Young's inequality and $q \leq \frac{p}{2}+1$

$$\begin{aligned}
\Big|\frac{V_{q-1}(\lambda_j Dv_j)}{\lambda_j}\Big||D\psi_j| &= \lambda_j^{-2}\big(1+|\lambda_j Dv_j|^2\big)^{\frac{p-2}{4}+\frac{2q-p-2}{4}}|\lambda_j Dv_j||\lambda_j D\psi_j| \\
&\leq \lambda_j^{-2}\big|V_{\frac{2q-p}{2}}(\lambda_j D\psi_j)\big|\big|V(\lambda_j Dv_j)\big| \\
&\leq \lambda_j^{-2}\Big(\big|V_{\frac{2q-p}{2}}(\lambda_j D\psi_j)\big|^2 + \big|V(\lambda_j Dv_j)\big|^2\Big).
\end{aligned}$$

Arguing essentially as supplied above and using $\frac{q}{p} \leq \frac{2q-p}{p} < \frac{n}{n-1}$ we derive the following estimate for II in this case:

$$\begin{aligned}
II &\leq c\bigg(Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{2q-p}{p}-1} Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{q}{p}-1} Y_j\bigg) \\
&\leq c\bigg(Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{1}{n-1}} Y_j\bigg).
\end{aligned}$$

Estimation of I. It remains to control I. Here, employing $|\lambda_j Da_j| \leq 1$ for $j \geq j_0(\alpha, r)$, Lemma 3.1 (iv) and (4.17) we get

$$\begin{aligned}
I &\leq c \int\limits_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \Big|\frac{V_{p-1}(\lambda_j Da_j)}{\lambda_j}\Big||D\psi_j|\,dx \leq c \int\limits_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \Big(\Big|\frac{V(\lambda_j Da_j)}{\lambda_j}\Big|^2 + \Big|\frac{V(\lambda_j D\psi_j)}{\lambda_j}\Big|^2\Big)\,dx \\
&\leq c\bigg(\int\limits_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j Da_j)}{\lambda_j}\Big|^2\,dx + Y_j\bigg).
\end{aligned}$$

Collecting the estimates for I, II and III we have proved

$$\int\limits_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 dx \leq c\bigg(Y_j + \Big(\frac{\lambda_j^2 Y_j}{\big((1-\alpha)r\big)^n}\Big)^{\frac{1}{n-1}} Y_j\bigg) + c\int\limits_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j Da_j)}{\lambda_j}\Big|^2 dx.$$

By the Poincaré-type inequality (3.3), Lemma 3.1 (i), Lemma 3.2 and (4.3) we have

$$\begin{aligned} Y_j &\leq \lambda_j^{-2} \int_{B_r(x_0)} \Big(\big|V(\lambda_j Dv_j)\big|^2 + \Big|V\Big(\frac{\lambda_j v_j}{(1-\alpha)r}\Big)\Big|^2 \Big)\, dx \\ &\leq \Big(1 + \frac{c}{(1-\alpha)^{\max\{2,p\}}}\Big) \int_{B_r(x_0)} \Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 dx \leq \frac{c}{(1-\alpha)^{\max\{2,p\}}}. \end{aligned}$$

Combining the last two inequalities we find

$$\int_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 dx \leq c\Big(Y_j + \int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j Da_j)}{\lambda_j}\Big|^2 dx\Big) + c_{\alpha,r}\lambda_j^{\frac{2}{n-1}},$$

where $c_{\alpha,r} > 0$ is a fixed constant depending, in particular, on α and r. The reader should note that, contrarily, the constants c in the preceding estimates do not depend on α or r. Applying Lemma 3.1 (ii) we deduce

$$\begin{aligned} &\int_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j(Du_j - Du)}{\lambda_j}\Big|^2 dx \\ &\leq c\Big[\int_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j Dv_j)}{\lambda_j}\Big|^2 dx + \int_{B_r(x_0)} \Big|\frac{V(\lambda_j(Du - Da_j)}{\lambda_j}\Big|^2 dx\Big] \\ &\leq c\Big[\int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j(Du_j - Du)}{\lambda_j}\Big|^2 dx + \int_{B_r(x_0)} \Big|\frac{V(\lambda_j(Du - Da_j)}{\lambda_j}\Big|^2 dx \\ &\quad + \int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j(Da_j)}{\lambda_j}\Big|^2 dx + \lambda_j^{-2}\int_{B_r(x_0)} \Big|V\Big(\frac{\lambda_j v_j}{(1-\alpha)r}\Big)\Big|^2 dx\Big] + c_{\alpha,r}\lambda_j^{\frac{2}{n-1}}. \end{aligned}$$

By Widman's hole filling trick, that is adding $c\int_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j(Du_j - Du))}{\lambda_j}\Big|^2 dx$ on both sides, we finally arrive at the main estimate

$$\begin{aligned} &\int_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j(Du_j - Du)}{\lambda_j}\Big|^2 dx \\ &\leq \theta \int_{B_r(x_0)} \Big|\frac{V(\lambda_j(Du_j - Du)}{\lambda_j}\Big|^2 dx + \int_{B_r(x_0)} \Big|\frac{V(\lambda_j(Du - Da_j)}{\lambda_j}\Big|^2 dx \\ &\quad + \int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j(Da_j)}{\lambda_j}\Big|^2 dx + \lambda_j^{-2}\int_{B_r(x_0)} \Big|V\Big(\frac{\lambda_j v_j}{(1-\alpha)r}\Big)\Big|^2 dx + c_{\alpha,r}\lambda_j^{\frac{2}{n-1}} \end{aligned} \tag{4.20}$$

for $j \geq j_0(\alpha, r)$ with $\theta = \frac{c}{1+c} < 1$. We stress that θ does not depend on α or r.

Step 5: Strong convergence. Recalling that u is C^1 on B it follows from (4.3) that $\lambda_j^{-2}\int_{B_\sigma} |V(\lambda_j(Du_j - Du))|^2\, dx$ remains bounded as $j \to \infty$. Thus, there exists a non-negative Radon measure μ on $\overline{B_\sigma}$ such that, passing to subsequences again, we have

$$\lambda_j^{-2}|V(\lambda_j(Du_j - Du))|^2\mathcal{L}^n \overset{*}{\rightharpoonup} \mu \qquad \text{weakly in the sense of measures on } \overline{B_\sigma}.$$

Introducing the affine function $a(x) := (u)_{x_0,r} + (Du)_{x_0,r}(x - x_0)$ we obviously have $a_j \to a$ and $\lambda_j^{-1}V(\lambda_j Da_j) \to Da$. Furthermore, setting $v := u - a$ we claim

$$\int_{B_r(x_0)} \lambda_j^{-2}\Big|V\Big(\frac{\lambda_j(v_j - v)}{(1-\alpha)r}\Big)\Big|^2\,dx \to 0. \tag{4.21}$$

For $1 < p < 2$ we will now prove (4.21) following an argument of [13]: We choose $t \in (0,1)$ such that $\frac{1}{2} = t + \frac{1-t}{p^\#}$ holds (recall $p^\# = \frac{2n}{n-p} > 2$) and apply the interpolation inequality and the Sobolev type inequality (3.4) to get

$$\begin{aligned}\int_{B_r(x_0)} \Big|\frac{V(\lambda_j(v_j - v))}{\lambda_j}\Big|^2 dx &\le \lambda_j^{2(t-1)}\Big(\int_{B_r(x_0)} |v_j - v|\,dx\Big)^{2t}\Big(\int_{B_r(x_0)} |V(\lambda_j(v_j - v))|^{p^\#}\,dx\Big)^{\frac{2(1-t)}{p^\#}} \\ &\le \Big(\int_{B_r(x_0)} |v_j - v|\,dx\Big)^{2t}\Big(\int_{B_r(x_0)} \Big|\frac{V(\lambda_j(Dv_j - Dv))}{\lambda_j}\Big|^2\,dx\Big)^{(1-t)}.\end{aligned}$$

By (4.3) and (4.10) the right-hand side converges to 0 for $j \to \infty$ and (4.21) is verified for $p < 2$. For $p \ge 2$ (4.21) follows from (4.10) and (4.3) by a simpler argument and we omit further details.

Returning to the general case, for every measurable subset A of $\overline{B_\sigma}$ we have

$$\begin{aligned}\mu(\mathrm{int}A) &\le \liminf_{j\to\infty}\int_A \lambda_j^{-2}|V(\lambda_j(Du_j - Du))|^2\,dx \\ &\le \limsup_{j\to\infty}\int_A \lambda_j^{-2}|V(\lambda_j(Du_j - Du))|^2\,dx \le \mu(\overline{A}).\end{aligned}$$

Keeping this in mind and passing to the limit in (4.20) we obtain

$$\begin{aligned}\mu(B_{\alpha r}(x_0)) &\le \theta\mu(\overline{B_r(x_0)}) \\ &\quad + \int_{B_r(x_0)} |Dv|^2\,dx + (1-\alpha^n)r^n|Da|^2 + c\int_{B_r(x_0)} \Big|\frac{v}{(1-\alpha)r}\Big|^2\,dx.\end{aligned}$$

Here, for the treatment of the fourth term on the right-hand side we have used Lemma 3.1 (*ii*) and (4.21). Since $0 < \alpha < 1$ is arbitrary we can, by virtue of a continuity argument, replace $\mu(B_{\alpha r}(x_0))$ by $\mu(\overline{B_{\alpha r}(x_0)})$ on the left-hand side of the previous inequality. Hence, dividing by r^n we have established

$$\alpha^n\frac{\mu(\overline{B_{\alpha r}(x_0)})}{\alpha^n r^n} \le \theta\frac{\mu(\overline{B_r(x_0)})}{r^n} + \varepsilon_1(r) + |Da|^2(1-\alpha^n) + \frac{\varepsilon_2(r)}{(1-\alpha)^2}, \tag{4.22}$$

where we have abbreviated

$$\varepsilon_1(r) = \frac{1}{r^n} \int\limits_{B_r(x_0)} |Dv|^2 \qquad \text{and} \qquad \varepsilon_2(r) = \frac{c}{r^{n+2}} \int\limits_{B_r(x_0)} |v|^2 \, dx.$$

Since u is C^1, we have $\varepsilon_1(r) + \varepsilon_2(r) \to 0$ and $Da \to Du(x_0)$ as $r \to 0^+$. Next we claim

$$\liminf_{r \to 0^+} \frac{\mu(\overline{B_r(x_0)})}{r^n} = 0. \tag{4.23}$$

To prove (4.23), following [32], we first suppose $\limsup_{r\to 0^+} \frac{\mu(\overline{B_r(x_0)})}{r^n} > 0$. Then, by an argument of [32, p. 78-79] we can pass $r \to 0^+$ in (4.22) arriving at

$$\alpha^n \le \theta + |Du(x_0)|^2 (1 - \alpha^n) \limsup_{r\to 0^+} \frac{r^n}{\mu(\overline{B_r(x_0)})}.$$

for all $0 < \alpha < 1$. Thus, passing $\alpha \to 1^-$ (recall that $\theta < 1$ is independent of α) we get (4.23) in any case and for all $x_0 \in B_\sigma$. Hence, following [32] again, by Vitali's covering theorem we deduce

$$\mu(\overline{B_\tau}) = 0,$$

which, in turn, implies the strong convergence stated in (4.13).

Step 6: Conclusion. Noting $(Du_j)_{0,\tau} \to (Du)_{0,\tau}$ we deduce from Lemma 3.1 (ii), (4.12) and (4.13)

$$\begin{aligned}
&\lim_{j\to\infty} \frac{1}{\lambda_j^2} \fint_{B_\tau} |V(\lambda_j(Du_j - (Du_j)_{0,\tau}))|^2 \, dx \\
&\quad \le \lim_{j\to\infty} \frac{c}{\lambda_j^2} \fint_{B_\tau} \big[|V(\lambda_j(Du_j - Du))|^2 + |V(\lambda_j(Du - (Du)_{0,\tau}))|^2 \\
&\qquad\qquad\qquad + |V(\lambda_j((Du)_{0,\tau} - (Du_j)_{0,\tau}))|^2 \big] dx \\
&\quad = c \fint_{B_\tau} |Du - (Du)_{0,\tau}|^2 \, dx \le C^* \tau^2
\end{aligned}$$

for some constant $C^* > 0$. Finally, the last inequality contradicts (4.4) if we choose $C = C^* + 1$ and the proof is finished. The reader should note that C and C^* depend only on n, N, p, ν_L and K_L. $\square$

Once Proposition 4.1 is established, Theorem 2.1 follows by a well-known iteration argument and Campanato's integral characterization of Hölder continuity. For further details see for instance [21, 13].

5. Proof of Theorem 2.2

In this section we present the proof of Theorem 2.2, modifying the proof of Theorem 2.1 along the lines of [47].

First we recall some simple estimates for the non-degenerate p-energy

$$e_p(\xi) := (1 + |\xi|^2)^{\frac{p}{2}}; \tag{5.1}$$

see e. g. [47] for a proof.

Lemma 5.1. *For $1 < p < \infty$, $L > 0$, $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \le L + 1$, a ball $B_r(x_0)$ in $\mathbb{R}^n$ and $\varphi \in W_0^{1,p}(B_\rho(x_0), \mathbb{R}^N)$ we have*

$$C_1^{-1} \int\limits_{B_r(x_0)} |V(D\varphi)|^2 \, dx \le \int\limits_{B_r(x_0)} \big[e_p(\xi + D\varphi) - e_p(\xi)\big] \, dx \le C_1 \int\limits_{B_r(x_0)} |V(D\varphi)|^2 \, dx$$

for some constant $C_1 > 0$ depending only on p and L. □

In the following lemmas we collect several properties of the relaxed functional $\mathcal{F}$. These lemmas have been proposed in [47] and rely heavily on (2.2), (2.3) and the measure and integral representation results obtained in [22, 11]. We mention that later in this section we will also apply (2.3) and the measure representation result [22, Theorem 3.1] explicitly. Next we give a reformulation of [47, Lemma 7.1]. Note that the growth condition imposed on Df in [47] follows from (H1) and the quasiconvexity of f.

Lemma 5.2. *We suppose that $f \in C^1$ is quasiconvex with (H1) and $1 < p \le q < \min\{p + 1, \frac{np}{n-1}\}$. Then, for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathcal{F}[u] < \infty$ and $\psi \in W^{1,\frac{p}{p+1-q}}(\Omega, \mathbb{R}^N)$ we have*

$$\mathcal{F}[u + \psi] - \mathcal{F}[u] = F[u + \psi] - F[u].$$

As in [47, Lemma 7.3] we see that Lemma 5.2 implies the validity of Euler's equation for $W^{1,\bar{q}}$ local minimizers of $\mathcal{F}$:

Lemma 5.3 (Euler's equation)**.** *Let $1 \le \bar{q} \le \infty$. We suppose that $f \in C^1$ is quasiconvex with (H1) and $1 < p \le q < \min\{p + 1, \frac{np}{n-1}\}$. Then, every $W^{1,\bar{q}}$ local minimizer $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$ of $\mathcal{F}$ is a weak solution of the Euler equation of F, i. e.*

$$\int\limits_\Omega Df(D\bar{u}) D\varphi \, dx = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^N).$$

Now we introduce the additional notation

$$\mathcal{F}[u; O] := \inf\Big\{ \liminf_{k\to\infty} \int\limits_O f(Du_k) \, dx : W^{1,q}_{\mathrm{loc}}(O, \mathbb{R}^N) \ni u_k \rightharpoonup u \text{ weakly in } W^{1,p}(O, \mathbb{R}^N) \Big\}$$

for open subsets O of Ω. We will need the next two lemmas, which can also be found in [47].

Lemma 5.4 ($W^{1,p}$-quasiconvexity)**.** *Assume (H1) and (H2) with $1 < p \le q < \frac{np}{n-1}$. Then, the following $W^{1,p}$-quasiconvexity condition holds for $\mathcal{F}$: For every ball $B_r(x_0)$ in $\mathbb{R}^n$, every $\xi \in \mathbb{R}^{Nn}$ and every $\varphi \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$ with compact support in $B_r(x_0)$ we have*

$$\mathcal{F}[l_\xi + \varphi; B_r(x_0)] \ge \mathcal{F}[l_\xi; B_r(x_0)], \tag{5.2}$$

where we have set $l_\xi(x) := \xi x$.

Lemma 5.5 (Additivity property.)**.** *Assume (H1) and (H2) with $1 < p \le q < \frac{np}{n-1}$. We consider a ball $B_s(x_0) \subset\subset \Omega$ and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that the boundary regularity condition*

$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{B_{s+\varepsilon}(x_0) \setminus B_{s-\varepsilon}(x_0)} |Du|^p \, dx < \infty \tag{5.3}$$

holds. Then we have

$$\mathcal{F}[u; \Omega] = \mathcal{F}[u; B_s(x_0)] + \mathcal{F}[u; \Omega \setminus \overline{B_s(x_0)}].$$

After these preparations we turn to the proof of Theorem 2.2: As for Theorem 2.1 it suffices to establish the following proposition, whose statement is completely analogous to Proposition 4.1.

Proposition 5.6. *Under the assumptions of Theorem 2.2 for every $L > 0$ there is a constant $C > 0$ with the following property: For each $0 < \tau \le \frac{1}{2}$ there exists a number $\varepsilon > 0$ such that the conditions*

$$|(D\bar{u})_{x,r}| \le L, \quad r < \varepsilon \quad \textit{and} \quad E(x, r) < \varepsilon$$

for a ball $B_r(x) \subset\subset \Omega$ imply

$$E(x, \tau r) < C\tau^2 E(x, r).$$

Sketch of proof. We argue by contradiction. Assuming the proposition to be wrong we proceed by blow-up as for Proposition 4.1 and we will highlight only the necessary modifications in the proof. First we note that by Lemma 5.3 the Euler equation used in Step 2 is available. Thus, the remaining modifications, which will be outlined now, concern only the handling of the quasiconvexity hypothesis and the minimizing property. We use the nomenclature of the proof of Proposition 4.1 up to the following difference: We choose $\tilde{r}_j$, $\tilde{s}_j$ as in Remark 3.6 avoiding the set

$$N_j := \left\{ t \in]\alpha r, r[: \ t \mapsto \int_{B_t(x_0)} |Du_j|^p \, dx \text{ is not differentiable at } t \right\}.$$

Thus, u_j satisfies the condition (5.3) near $\partial B_{\tilde{s}_j}(x_0)$. As explained in the proof of [47, Lemma 7.13] it is easy to see that the same condition holds also for $a_j + \varphi_j$ and $u_j - \varphi_j$. We will use this fact later when applying Lemma 5.5. Next we will rewrite the quasiconvexity hypothesis (H4) in an adequate form for our purposes. To this aim we introduce the auxiliary integrand

$$g(\xi) := f(\xi) - \frac{\nu_L}{C_1} e_p(\xi) \quad \text{for } \xi \in \mathbb{R}^{Nn}.$$

where e_p is defined in (5.1) and C_1, γ and ν_L denote the constants from Lemma 5.1, (H2) and (H4). Moreover, for $W^{1,p}$-functions w we set

$$\begin{aligned}
G[w] &:= \int_\Omega g(Dw) \, dx, \\
\mathcal{G}[w] &:= \inf \left\{ \liminf_{k \to \infty} G[w_k] : W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \ni w_k \rightharpoonup w \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^N) \right\}, \\
\mathcal{F}_j[w] &:= \inf \left\{ \liminf_{k \to \infty} F_j[w_k] : W^{1,q}_{\mathrm{loc}}(B, \mathbb{R}^N) \ni w_k \rightharpoonup w \text{ weakly in } W^{1,p}(B, \mathbb{R}^N) \right\},
\end{aligned}$$

and we will also use the obvious modifications of this notations for open subsets O of Ω and B, respectively. Following an argument from the proof of [47, Lemma 7.13] it is not hard to see from the definitions of $\mathcal{F}$ and $\mathcal{G}$ that we have

$$\mathcal{G}[w;O] \le \mathcal{F}[w;O] - \frac{\nu_L}{C_1}\int_O e_p(Dw)\,dx$$

for all open subsets O of Ω and all $w \in W^{1,p}(O,\mathbb{R}^N)$. In addition, from (H1) and (H2) we see that g fulfills the growth conditions

$$\Big(\gamma - \frac{2^{\frac{p}{2}}\nu_L}{C_1}\Big)|\xi|^p - \frac{2^{\frac{p}{2}}\nu_L}{C_1} \le g(\xi) \le \Gamma(1+|\xi|^q)$$

for all $\xi \in \mathbb{R}^{Nn}$. Imposing the condition $2^{\frac{p}{2}}\nu_L < C_1\gamma$, which is clearly not restrictive, we infer that g satisfies (H1) and (H2) up to an additive constant. Obviously, this is sufficient to allow the application of Lemma 5.4 to $\mathcal{G}$. Furthermore, we deduce from (H4) and Lemma 5.1 that g is quasiconvex at all ξ with $|\xi| \le L+1$. In particular, recalling (2.3) this gives $\mathcal{G}[l_\xi; B_{r_j}(x_j)] = |B_{r_j}(x_j)|Qg(\xi) = |B_{r_j}(x_j)|g(\xi)$ for these ξ, where we have used the notation l_ξ from Lemma 5.4. Consequently, applying Lemma 5.4 to $\mathcal{G}$ we get

$$\begin{aligned} 0 &\le \mathcal{G}[l_\xi+\varphi; B_{r_j}(x_j)] - \mathcal{G}[l_\xi; B_{r_j}(x_j)] \\ &\le \mathcal{F}[l_\xi+\varphi; B_{r_j}(x_j)] - |B_{r_j}(x_j)|f(\xi) - \frac{\nu_L}{C_1}\int_{B_{r_j}(x_j)} \big[e_p(\xi+D\varphi) - e_p(\xi)\big]\,dx \end{aligned}$$

for $|\xi| \le L+1$ and all $\varphi \in W^{1,p}(B_{r_j}(x_j),\mathbb{R}^N)$ with compact support in $B_{r_j}(x_j)$. By Lemma 5.1 we conclude

$$\int_{B_{r_j}(x_j)} |V(D\varphi)|^2\,dx \le c\,\big(\mathcal{F}[l_\xi+\varphi; B_{r_j}(x_j)] - |B_{r_j}(x_j)|f(\xi)\big)$$

for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \le L+1$ and all $\varphi \in W^{1,p}(B_{r_j}(x_j),\mathbb{R}^N)$ with compact support in $B_{r_j}(x_j)$. Rescaling gives us[1]

$$\int_B \Big|\frac{V(\lambda_j D\varphi)}{\lambda_j}\Big|^2\,dx \le c(\mathcal{F}_j[l_\xi+\varphi;B] - |B|f_j(\xi)),$$

for all $\xi \in \mathbb{R}^{Nn}$ with $|\lambda_j\xi| \le 1$ and for all $\varphi \in W^{1,p}(B,\mathbb{R}^N)$ with compact support in B. Next, we recall $\varphi_j \in W^{1,p}_0(B_{\tilde{s}_j}(x_0),\mathbb{R}^N)$ and $\mathcal{F}_j[a_j;O] = |O|f_j(Da_j)$ for all open subsets O of Ω; see Step 3 in the proof of Proposition 4.1 and (2.3). Using these facts, Lemma 5.5 and the choice of $\tilde{s}_j$ we deduce from the previous inequality

$$\int_{B_{\tilde{s}_j}(x_0)} \Big|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\Big|^2\,dx \le c\,\big(\mathcal{F}_j[a_j+\varphi_j; B_{\tilde{s}_j}(x_0)] - |B_{\tilde{s}_j}(x_0)|f_j(Da_j)\big) \tag{5.4}$$

[1] Actually, this can be verified by a straightforward computation using the definitions of $\mathcal{F}_j$, F_j, f_j $\mathcal{F}$, F and the fact that the integral of the linear term in the definition of f_j is weakly continuous.

for $j \geq j_2(r)$. In the following we will use the quasiconvexity hypothesis in the form (5.4). Next we turn to a reformulation of the minimizing property: We have assumed that $\bar{u}$ is a $W^{1,\bar{q}}$ local minimizer of $\mathcal{F}$ on Ω. By the measure representation theorem [22, Theorem 3.1] this is easily seen to imply

$$\mathcal{F}[\bar{u}; B_{r_j}(x_j)] \leq \mathcal{F}[\bar{u} - \varphi; B_{r_j}(x_j)]$$

for all functions $\varphi \in W^{1,\bar{q}}(B_{r_j}(x_j), \mathbb{R}^N)$ with compact support in $B_{r_j}(x_j)$ and $\|D\varphi\|_{L^{\bar{q}}(B_{r_j}(x_j), \mathbb{R}^{Nn})} \leq \delta$. Rescaling as before we get

$$\mathcal{F}_j[u_j; B] \leq \mathcal{F}_j[u_j - \varphi; B]$$

for all $\varphi \in W^{1,\bar{q}}(B, \mathbb{R}^N)$ with compact support in B and $\|D\varphi\|_{L^{\bar{q}}(B,\mathbb{R}^{Nn})} \leq \frac{\delta}{\lambda_j r_j^{\frac{n}{\bar{q}}}}$. Recalling $\|D\varphi_j\|_{L^{\bar{q}}(B,\mathbb{R}^{Nn})} \leq \frac{\delta}{\lambda_j r_j^{\frac{n}{\bar{q}}}}$ for $j \geq j_1(\alpha)$ (see Step 3 in the proof of Proposition 4.1) we get from Lemma 5.5 and the choice of $\tilde{s}_j$

$$\mathcal{F}_j[u_j; B_{\tilde{s}_j}(x_0)] \leq \mathcal{F}_j[u_j - \varphi_j; B_{\tilde{s}_j}(x_0)]. \tag{5.5}$$

Finally, recalling (4.15) and putting together (5.4) and (5.5) we find

$$\begin{aligned}\int_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\Big|^2 dx \quad \leq \quad & c\Big(\mathcal{F}_j[u_j - \psi_j; B_{\tilde{s}_j}(x_0)] - \mathcal{F}_j[u_j; B_{\tilde{s}_j}(x_0)] \\ & + \mathcal{F}_j[a_j + \psi_j; B_{\tilde{s}_j}(x_0)] - |B_{\tilde{s}_j}(x_0)| f_j(Da_j)\Big).\end{aligned}$$

Since $q \leq \frac{p}{p+1-q} < \frac{np}{n-1}$ we see $\psi_j \in W^{1,\frac{p}{p+1-q}}(B_{\tilde{s}_j}(x_0), \mathbb{R}^N) \subset W^{1,q}(B_{\tilde{s}_j}(x_0), \mathbb{R}^N)$ from the estimates of Lemma 3.4. Thus, we can apply Lemma 5.2 and (2.3) to simplify the right-hand side of the preceding formula deriving

$$\begin{aligned}\int_{B_{\alpha r}(x_0)} \Big|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\Big|^2 dx \quad \leq \quad & c\Big(\int_{B_{\tilde{s}_j}(x_0)} \big(f_j(Du_j - D\psi_j) - f_j(Du_j)\big)\, dx \\ & + \int_{B_{\tilde{s}_j}(x_0)} \big(f_j(Da_j + D\psi_j) - f_j(Da_j)\big)\, dx\Big)\end{aligned}$$

for $j \geq j_0(\alpha, r)$. Since the last inequality coincides with the estimate in (4.19) we can now argue exactly as in the proof of Proposition 4.1. □

Acknowledgment

The second author would like to express his gratitude for the kind hospitality of the Mathematical Institute in Erlangen-Nürnberg, where parts of this paper were written in summer 2007.

References

1 E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Ration. Mech. Anal. **86** (1984), 125–145.

2 E. Acerbi and N. Fusco, *A regularity theorem for minimizers of quasiconvex integrals*, Arch. Ration. Mech. Anal. **99** (1987), 261–281.

3 E. Acerbi and N. Fusco, *Regularity for minimizers of non-quadratic functionals: the case $1 < p < 2$*, J. Math. Anal. Appl. **140** (1989), 115–135.

4 E. Acerbi and G. Mingione, *Regularity results for a class of quasiconvex functionals with nonstandard growth*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **30** (2001), 311–339.

5 J.M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal. **63** (1977), 337–403.

6 J.M. Ball, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Philos. Trans. R. Soc. Lond., A **306** (1982), 557–611.

7 J.M. Ball and F. Murat, *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal. **58** (1984), 225–253.

8 I. Benedetti and E. Mascolo, *Regularity of minimizers for nonconvex vectorial integrals with p-q growth via relaxation methods*, Abstr. Appl. Anal. **2004** (2004), 27–44.

9 M. Bildhauer and M. Fuchs, *Partial regularity for variational integrals with (s, μ, q)-growth*, Calc. Var. Partial Differ. Equ. **13** (2001), 537–560.

10 M. Bildhauer and M. Fuchs, *$C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems*, Calc. Var. Partial Differ. Equ. **24** (2005), 309–340.

11 G. Bouchitté, I. Fonseca and J. Malý, *The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent*, Proc. R. Soc. Edinb., Sect. A, Math. **128** (1998), 463–479.

12 M. Carozza and A. Passarelli di Napoli, *Partial regularity of local minimizers of quasiconvex integrals with sub-quadratic growth*, Proc. R. Soc. Edinb., Sect. A, Math. **133** (2003), 1249–1262.

13 M. Carozza, N. Fusco and G. Mingione, *Partial regularity of minimizers of quasiconvex integrals with subquadratic growth*, Ann. Mat. Pura Appl., IV. Ser. **175** (1998), 141–164.

14 G. Cupini, M. Guidorzi and E. Mascolo, *Regularity of minimizers of vectorial integrals with p-q growth*, Nonlinear Anal., Theory Methods Appl. **54** (2003), 591–616.

15 E. De Giorgi, *Un essempio di estremali discontinue per un problema variazionale di tipo ellitico*, Boll. Unione Mat. Ital., IV. Ser. **1** (1968), 135–137.

16 F. Duzaar, J.F. Grotowski and M. Kronz, *Regularity of almost minimizers of quasi-convex variational integrals with subquadratic growth*, Ann. Mat. Pura Appl., IV. Ser. **184** (2005), 421–448.

17 F. Duzaar and M. Kronz, *Regularity of ω-minimizers of quasi-convex variational integrals with polynomial growth*, Differ. Geom. Appl. **17** (2002), 139–152.

18 L. Esposito, F. Leonetti and G. Mingione, *Higher integrability for minimizers of integral functionals with (p, q) growth*, J. Differ. Equations **157** (1999), 414–438.

19 L. Esposito, F. Leonetti and G. Mingione, *Regularity results for minimizers of irregular integrals with (p, q) growth*, Forum Math. **14** (2002), 245–272.

20 L. Esposito, F. Leonetti and G. Mingione, *Sharp regularity for functionals with (p, q) growth*, J. Differ. Equations **204** (2004), 5–55.

21 L.C. Evans, *Quasiconvexity and partial regularity in the calculus of variations*, Arch. Ration. Mech. Anal. **95** (1986), 227–252.

22 I. Fonseca and J. Malý, *Relaxation of multiple integrals below the growth exponent*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **14** (1997), 309–338.

23 N. Fusco and J.E. Hutchinson, *$C^{1,\alpha}$ partial regularity of functions minimising quasiconvex integrals*, Manuscr. Math. **54** (1985), 121–143.

24 N. Fusco and J.E. Hutchinson, *Partial regularity in problems motivated by nonlinear elasticity*, SIAM J. Math. Anal. **22** (1991), 1516–1551.

25 M. Giaquinta and G. Modica, *Partial regularity of minimizers of quasiconvex integrals*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **3** (1986), 185–208.

26 F. John, *Uniqueness of non-linear elastic equilibrium for prescribed boundary displacements and sufficiently small strains*, Commun. Pure Appl. Math. **25** (1972), 617–634.

27 R.J. Knops and C.A. Stuart, *Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity*, Arch. Ration. Mech. Anal. **86** (1984), 233–249.

28 J. Kristensen, *Lower semicontinuity in Sobolev spaces below the growth exponent of the integrand*, Proc. R. Soc. Edinb., Sect. A **127** (1997), 797–817.

29 J. Kristensen, *Lower semicontinuity of quasi-convex integrals in BV*, Calc. Var. Partial Differ. Equ. **7** (1998), 249–261.

30 J. Kristensen, *Lower semicontinuity in spaces of weakly differentiable functions*, Math. Ann. **313** (1999), 653–710.

31 J. Kristensen and G. Mingione, *The singular set of Lipschitzian minima of multiple integrals*, Arch. Ration. Mech. Anal. **184** (2007), 341–369.

32 J. Kristensen and A. Taheri, *Partial regularity of strong local minimizers in the multi-dimensional calculus of variations*, Arch. Ration. Mech. Anal. **170** (2003), 63–89.

33 P. Marcellini, *Approximation of quasiconvex functions and lower semicontinuity of multiple integrals*, Manuscr. Math. **51** (1985), 1–28.

34 P. Marcellini, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **3** (1986), 391–409.

35 P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions*, Arch. Ration. Mech. Anal. **105** (1989), 267–284.

36 P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differ. Equations **90** (1991), 1–30.

37 N.G. Meyers, *Quasi-convexity and lower semi-continuity of multiple variational integrals of any order*, Trans. Am. Math. Soc. **119** (1965), 125–149.

38 G. Mingione and D. Mucci, *Integral functionals and the gap problem: sharp bounds for relaxation and energy concentration*, SIAM J. Math. Anal. **36** (2005), 1540–1579.

39 C.B. Morrey, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

40 S. Müller and V. Sverák, *Convex integration for Lipschitz mappings and counterexamples to regularity*, Ann. Math. (2) **157** (2003), 715–742.

41 J. Necas, *Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity*, Theor. Nonlin. Oper., Constr. Aspects, Proc. int. Summer Sch., Berlin 1975 (1977), 197–206.

42 A. Passarelli di Napoli and F. Siepe, *A regularity result for a class of anisotropic systems*, Rend. Ist. Mat. Univ. Trieste **28** (1996), 13–31.

43 P. Pedregal, *Jensen's inequality in the calculus of variations*, Differ. Integral Equ. **7** (1994), 57–72.

44 K.D.E. Post and J. Sivaloganathan, *On homotopy conditions and the existence of multiple equilibria in finite elasticity*, Proc. R. Soc. Edinb., Sect. A **127** (1997), 595–614.

45 S. Schemm, *Partial regularity of minimizers of higher order integrals with (p,q)-growth*, submitted.

46 T. Schmidt, *Regularity of minimizers of $W^{1,p}$-quasiconvex variational integrals with (p,q)-growth*, Calc. Var. Partial Differ. Equ. **32** (2008), 1–24.

47 T. Schmidt, *Regularity of relaxed minimizers of quasiconvex variational integrals with (p,q)-growth*, Arch. Ration. Mech. Anal., to appear.

48 V. Sverak, *Quasiconvex functions with subquadratic growth*, Proc. R. Soc. Lond., Ser. A **433** (1991), 723–725.

49 V. Sverak and X. Yan, *A singular minimizer of a smooth strongly convex functional in three dimensions*, Calc. Var. Partial Differ. Equ. **10** (2000), 213–221.

50 V. Sverák and X. Yan, *Non-Lipschitz minimizers of smooth uniformly convex functionals*, Proc. Natl. Acad. Sci. USA **99** (2002), 15269–15276.

51 A. Taheri, *Quasiconvexity and uniqueness of stationary points in the multi-dimensional calculus of variations*, Proc. Am. Math. Soc. **131** (2003), 3101–3107.

52 A. Taheri, *On Artin's braid group and polyconvexity in the calculus of variations*, J. Lond. Math. Soc., II. Ser. **67** (2003), 752–768.

53 A. Taheri, *Local minimizers and quasiconvexity – the impact of topology*, Arch. Ration. Mech. Anal. **176** (2005), 363–414.

(Version July 2008)