SMOOTH APPROXIMATION FOR INTRINSIC LIPSCHITZ FUNCTIONS IN THE HEISENBERG GROUP

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Abstract. We characterize intrinsic Lipschitz functions as maps which can be approximated by a sequence of smooth maps, with pointwise convergent intrinsic gradient. We also provide an estimate of the Lipschitz constant of an intrinsic Lipschitz function in terms of the $L^\infty-$norm of its intrinsic gradient.

1. Introduction

In the last few years, it has been carried out an intensive study of submanifolds inside the Heisenberg groups $\mathbb{H}^n$ or more general Carnot groups, endowed with a sub-Riemannian structure. A notable class of intrinsic C^1 and Lipschitz surfaces has been identified by means of their invariance with respect to the sub-Riemannian structure. They have been defined in the framework of rectifiable sets (see, for instance, [2, 3, 15, 23, 24, 26, 29, 32, 35, 40]), with several applications to geometry of Banach spaces, theoretical computer science, mathematical models in neurosciences (see, for instance, [12, 17, 19, 30, 42]). We refer the reader to the monograph [11] and the references therein for a more detailed introduction to the Heisenberg group and the afore-mentioned arguments.

Heisenberg groups $\mathbb{H}^n$ provide the simplest non-trivial examples of stratified, connected and simply connected Lie groups. Their Lie algebra $\mathfrak{h}_n$ can be identified with $\mathbb{R}^{2n+1}$ with the stratification

$$\mathfrak{h}_n = \mathfrak{h}_1 \oplus \mathfrak{h}_2, \quad [\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_2 \text{ and } [\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}, \tag{1}$$

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where $\mathfrak{h}_1$ has dimension $2n$ and $\mathfrak{h}_2$ has dimension 1. We will denote by

$$\nabla_1^{\mathbb{H}}, \dots, \nabla_{2n}^{\mathbb{H}} \text{ a basis of } \mathfrak{h}_1, \quad \nabla_{2n+1}^{\mathbb{H}} \text{ a non zero element of } \mathfrak{h}_2$$

satisfying only the following non trivial commutations

$$[\nabla_i^{\mathbb{H}}, \nabla_{i+n}^{\mathbb{H}}] = 2\,\nabla_{2n+1}^{\mathbb{H}} \quad i = 1, \dots, n\,. \tag{2}$$

The subspace $\mathfrak{h}_1$ canonically generates the so-called horizontal subbundle $H\mathbb{H}^n$. The Heisenberg group $(\mathbb{H}^n, \cdot)$ can be conveniently represented by its Lie algebra equipped with the group operation arising from the Baker-Campbell-Hausdorff formula, hence making the exponential mapping equal to the identity. We will endow $\mathbb{H}^n$ with a homogeneous norm $\|\cdot\|_*$, which canonically induces a left-invariant homogeneous distance d on $\mathbb{H}^n$, see (19). It is well known (see for example [38]) that d is equivalent to the standard Carnot-Carathéodory metric associated to the horizontal subbundle $H\mathbb{H}^n$ defined in a classical way by means of $\mathfrak{h}_1$. Moreover the metric d and the Euclidean norm $|\cdot|_{\mathbb{R}^{2n+1}}$ on $\mathbb{R}^{2n+1}$ are not equivalent, even though $(\mathbb{R}^{2n+1}, \mathrm{d})$ and $(\mathbb{R}^{2n+1}, |\cdot|_{\mathbb{R}^{2n+1}})$ are homeomorphic. Intrinsic s-dimensional spherical Hausdorff measure $\mathcal{S}^s$ on $\mathbb{H}^n$, $s \geq 0$, is obtained from d, following Carathéodory construction (see, for instance, [34]). The intrinsic metric (or Hausdorff) dimension $\dim_{\mathbb{H}}(S)$ of a set S is the number

$$\dim_{\mathbb{H}}(S) \stackrel{\text{def}}{=} \inf\{s \geq 0 \ : \ \mathcal{S}^s(S) = 0\}.$$

Here we focus only on intrinsic C^1 surfaces in $\mathbb{H}^n$ with codimension one, defined in [23] and extended in general dimensions and codimensions, in [25]. This notion was also extended, for codimension one, to general Carnot-Carathéodory spaces in [16] and in [33] for general codimensions. If we choose a basis $\nabla^{\mathbb{H}}$ of the horizontal tangent space at every point, and an open set $\mathcal{U}$ in $\mathbb{H}^n$, we will say that a function $f : \mathcal{U} \to \mathbb{R}$ is of class C^1 if its horizontal gradient $\nabla^{\mathbb{H}} f : \mathcal{U} \longrightarrow \mathbb{R}^{2n}$ is continuous. An $\mathbb{H}$-regular surface is the level set of a C^1 function if its gradient never vanishes. Note that $\mathbb{H}$-regular surfaces can be very irregular from an Euclidean point of view and in general these surfaces are not Euclidean C^1 submanifolds, not even locally (see [31]). Nevertheless, it can be proved that they have metric dimension $2n+1$, topological dimension $2n$, as well as they locally have finite $\mathcal{S}^{2n+1}$ measure. Moreover, at each point there is a continuously varying, intrinsic tangent $2n$-plane that is a coset of a maximal subgroup of $\mathbb{H}^n$.

Implicit function theorems have been proved in this context in [23] and yield that any $\mathbb{H}$-regular surface S can be considered locally a graph in a suitable intrinsic sense. Precisely, identifying $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ by a suitable choice of exponential coordinates (we address the reader for more details to Section 2), then there are two homogeneous subgroups

$$\begin{aligned} \mathbb{W} &:= \{p \in \mathbb{H}^n \mid p = (0, x),\ x \in \mathbb{R}^{2n}\} \\ \mathbb{V} &:= \{p \in \mathbb{H}^n \mid p = (s, 0),\ s \in \mathbb{R}\}, \end{aligned} \tag{3}$$

such that

$$\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}\,.$$

Moreover, for every $U \subset \mathbb{H}^n$ open there exists a relative open set ω in $\mathbb{W}$ and a continuous function $\phi : \omega \longrightarrow \mathbb{V}$ such that

$$S \cap U = \Phi(\omega)$$

where Φ is the *graph map* of ϕ defined by (see Remark 2.1):

$$\Phi : \omega \longrightarrow \mathbb{H}^n, \quad \Phi(x) := (\phi(x), x). \tag{4}$$

Further regularity properties for the function ϕ have been studied in [4, 6, 7] and [16] In particular it has been proved that if a surface is of class C^1, it locally defines an implicit function ϕ, which is of class C^1 with respect to suitable nonlinear vector fields ∇^ϕ expressed in terms of the function ϕ itself. This is not surprising, since non linear vector fields often arise when studying geometric surfaces and equations in the subriemannian context (we refer to the papers of [44] and [13, 14], for nonlinear vector fields introduced in connection with a Monge Ampère subelliptic equation or a Levi equation). We also refer to the paper [18], where the nonlinear vector fields ∇^ϕ have been proposed for studying a problem of mathematical finance. In these papers, it has been observed that, since the Heisenberg space has a natural stratification, a stratification is also induced on the domain $\mathbb{W}$ of ϕ, and, if ϕ was regular in the Euclidean sense, we could use the standard notion of exponential distance, proposed and exploited in [38]. On the other side, if ϕ is not regular, their definition does not apply. The notion of distance in this nonlinear context has been introduced for the first time in [13], and exploited with minimal regularity [14] and [18] using a freezing method, based on the exponential mapping. The definition of distance first introduced in these papers, has been extended in large generality in [16] and turned out to be equivalent to the notion of distance introduced in the Heisenberg case by [4].

The notion of Lipschitz continuous function introduced in [14] is equivalent to the one recently studied in [5], [26] [37] and [46]. However the point of view in these papers is different from the previous ones, since inspired by the metric structure of the ambient space $(\mathbb{H}^n, \mathrm{d})$.

Precisely while dealing with intrinsic graphs, the distance and Lipschitz continuity are defined as follows:

1.1. **Definition.** Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open and bounded set and let $\phi : \omega \longrightarrow \mathbb{R}$ be a continuous function. The graph distance between $x, y \in \omega$ is defined by

$$\mathrm{d}_\phi(x,y) := \frac{1}{2}\Big(\|\pi_{\mathbb{W}}(\Phi(x)^{-1}\cdot\Phi(y))\| + \|\pi_{\mathbb{W}}(\Phi(y)^{-1}\cdot\Phi(x))\|\Big) \tag{5}$$

where Φ is defined in (4), $\|\cdot\|$ in (18) and

$$\pi_{\mathbb{W}} : \mathbb{H}^n \longrightarrow \mathbb{W} \quad \pi_{\mathbb{W}}((s,x)) := x \tag{6}$$

1.2. **Definition.** We say that $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ is an (intrinsic) Lipschitz continuous function in ω and we write $\phi \in Lip_{\mathbb{W}}(\omega)$, if there is a constant $L > 0$ such that

$$|\phi(x) - \phi(y)| \leq L\mathrm{d}_\phi(x,y) \qquad \forall x, y \in \omega. \tag{7}$$

The Lipschitz constant of ϕ in ω is the infimum of the numbers L such that (7) holds and we write $Lip(\phi)$ to denote it. Moreover, we say that $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ is a locally (intrinsic) Lipschitz function in ω and we write $\phi \in Lip_{loc,\mathbb{W}}(\omega)$, if $\phi \in Lip_{\mathbb{W}}(\omega')$ for every $\omega' \Subset \omega$. We denote by $Lip(\phi, \omega')$ the Lipschitz constant of $\phi_{|\omega'}$.

Note that $Lip_{\mathbb{W}}(\omega)$ does not turn to be a vector space (see [43, Remark 4.2]). Nevertheless, the intrinsic Lipschitz functions amount to a thick class of functions. Indeed, we have that ([26, Propositions 4.8 and 4.11])

$$Lip(\omega) \subsetneq Lip_{\mathbb{W},loc}(\omega) \subsetneq C^{1/2}_{loc}(\omega)\,, \tag{8}$$

where, respectively, $Lip(\omega)$ and $C^{1/2}_{loc}(\omega)$ denote the classes of real valued Euclidean Lipschitz and locally 1/2-Hölder functions on ω. The main properties of Lipschitz functions have been proved in [26]:

1.3. **Theorem.** *If $\phi \in Lip_{\mathbb{W}}(\omega)$ then it can be extended to a function $\overline{\phi} : \mathbb{W} \longrightarrow \mathbb{R}$ with $\overline{\phi} \in Lip_{\mathbb{W}}(\mathbb{W})$. Moreover, if $\omega = \mathbb{W}$ and ψ, ϕ are intrinsic Lipschitz functions with Lipschitz constant L, then there exists $\bar{L} = \bar{L}(L) \geq L$ such that* $\max\{\phi,\psi\}$ *and* $\min\{\phi,\psi\}$ *are intrinsic Lipschitz with Lipschitz constant $\bar{L}$.*
Furthermore, the subgraph

$$E_\phi := \{(s,x) \in \mathbb{R}\times\omega :\ s < \phi(x)\,\}$$

is a set of locally finite perimeter in $\mathbb{H}^n$. Besides ϕ is ∇^ϕ-differentiable for $\mathcal{L}^{2n}$-a.e $x \in \omega$, in the sense defined by [4].

Precisely

1.4. **Definition.** We say that ϕ is $\nabla^\phi-$differentiable at $w_0 \in \omega$ if there exists a homogeneous homomorphism $L : \mathbb{W} \longrightarrow \mathbb{V}$ such that

$$\lim_{w \to w_0} \frac{|\phi(w) - \phi(w_0) - L(\pi_{\mathbb{W}}(\Phi(w_0)^{-1} \cdot \Phi(w)))|}{\mathrm{d}_\phi(w_0, w)} = 0.$$

The map L is called the ∇^ϕ-*differential* of ϕ at w_0.

The main properties of differentiable functions are collected in Proposition 4.1.

We are now in position to state our results and describe the organization of the paper. In Section 2, we recall some standard definitions and results of geometric measure theory in $\mathbb{H}^n$. In Section 3, we study the distance d_ϕ. This section is organized in two parts. First, we exploit the notion of distance induced on the graph by the metric and the group law of the Heisenberg group. Then we compare this distance with the more classical Carnot-Carathéodory distance defined on the domain of ϕ, in terms of the family of nonlinear vector fields ∇^ϕ. We obtain for non regular vector fields the same result true for regular ones (see [38]), using a technique similar to the one used in [16]. Precisely we prove that

1.5. **Proposition** (Equivalence). *If $\phi \in Lip_{\mathbb{W}}(\omega)$ and $n \geq 2$ then the distance d_ϕ is locally equivalent to the Carnot Carathéodory distance induced by the family ∇^ϕ, with equivalence constants only dependent on $Lip(\phi, \omega)$ (see Proposition 3.8 below).*

Section 4 is devoted to the notion of intrinsic differentiability. In particular we prove that the pointwise intrinsic gradient $\nabla^\phi\phi : \omega \longrightarrow \mathbb{R}^{2n-1}$ of a given $\phi \in Lip_{\mathbb{W}}(\omega)$, is the distributional one. The main difficulty in this assertion is that the notion of formal adjoint of the ∇^ϕ is not well defined in $Lip_{\mathbb{W}}(\omega)$. We also extend the area formula for intrinsic C^1 graphs obtained in [4] to intrinsic Lipschitz functions. Precisely, we prove the following representation formula for the $\mathbb{H}-$perimeter and for the spherical Hausdorff $(2n+1)-$measure of the intrinsic graph of a Lipschitz function ϕ in terms of its $\nabla^\phi-$gradient.

1.6. **Theorem.** *If $\phi \in Lip_{\mathbb{W}}(\omega)$ with $\omega \subset \mathbb{W}$ open and bounded, then there exists a dimensional constant $c_n > 0$ such that the following area formula hold*

$$|\partial E_\phi|_{\mathbb{H}}(\mathbb{R} \times \omega) = c_n \ \mathcal{S}^{2n+1}(\Phi(\omega)) = \int_\omega \sqrt{1 + |\nabla^\phi\phi|^2} \ \mathrm{d}\mathcal{L}^{2n}.$$

In Section 5, we prove the main result of this paper. It deals with the approximation of a given intrinsic Lipschtiz function $\phi \in Lip_{\mathbb{W}}(\omega)$

as well its gradient $\nabla^\phi\phi$, by means of regular functions. The classical approximation by convolution does not apply because of the nonlinearity of $\nabla^\phi\phi$. Here we refine the technique used in [37], which relies on an extension to $\mathbb{H}^n$ of the classical Euclidean technique which goes back to De Giorgi ([20]). The strategy is somehow indirect, indeed we approximate in $\mathbb{H}^n$ the intrinsic subgraph $E_\phi := \{(s,x) \in \mathbb{H}^n \mid s \le \phi(x)\}$ of a given $\phi \in Lip_{\mathbb{W}}(\omega)$ (see (32)), rather than the function itself. The key point is that E_ϕ is a set of locally finite $\mathbb{H}$- perimeter (see Definition 2.6 and Theorem 1.3) and hence the recent results of geometric measure theory obtained in this setting permit an extension of De Giorgi's results for sets of finite Euclidean perimeter. Precisely, we prove the following Theorem

1.7. **Theorem** (Approximation of intrinsic Lipschitz functions)**.** *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be a bounded open set and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then there exists a sequence $\{\phi_k\} \subset C^\infty(\omega)$,*

(i) *$\phi_k \to \phi$ uniformly in ω;*
(ii) *$|\nabla^{\phi_k}\phi_k(x)| \le ||\nabla^\phi\phi||_{L^\infty(\omega)}$ $\forall x \in \omega$;*
(iii) *$\nabla^{\phi_k}\phi_k(x) \to \nabla^\phi\phi(x)$ $\mathcal{L}^{2n}-a.e$ $x \in \omega$.*

Section 6 is devoted to an application of the approximation Theorem. In particular, as in the Euclidean framework, we prove that the above approximation result is sharp for intrinsic Lipschitz functions, see Theorem 6.1. Moreover, as a corollary of Theorem 1.7, we get a local comparison, for a given $\phi \in Lip_{\mathbb{W}}(\omega)$, between the Lipschitz constant of ϕ and the $L^\infty-$norm of its intrinsic gradient. Precisely, we prove the following

1.8. **Proposition.** *Let $\omega \subset \mathbb{W}$ be open and bounded, $\phi \in Lip_{\mathbb{W}}(\omega)$. Then there exists a postive constant $c_1 = c_1(Lip(\phi,\omega))$ depending only on $Lip(\phi,\omega)$ such that*

$$\|\nabla^\phi\phi\|_{L^\infty(\omega)} \le\ c_1\,. \tag{9}$$

If $n \ge 2$, there exists a positive constant $c_2 = c_2(\|\nabla^\phi\phi\|_{L^\infty(\omega)})$ depending only on $\|\nabla^\phi\phi\|_{L^\infty(\omega)}$ such that for each $\bar{x} \in \omega$ and each $r > 0$ sufficiently small

$$Lip(\phi, U_\phi(\bar{x},r)) \le c_2\, \|\nabla^\phi\phi\|_{L^\infty(\omega)} \tag{10}$$

whereas if $n = 1$

$$Lip(\phi, U_\phi(\bar{x},r)) \le c_3 \sqrt{1 + \|\nabla^\phi\phi\|^2_{L^\infty(\omega)}} \tag{11}$$

for a suitable geometric positive constant c_3.

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2. Preliminaries

In this section we briefly recall some useful facts about the sub-Riemannian Heisenberg group, for a more detailed treatment see [8], [11], [29] and [36]. The definition of the Heisenberg group and Heisenberg algebra has been given in the introduction see (1). For our purposes, it is convenient to use the following exponential coordinates. Precisely, if a point, called 0, is fixed in $\mathbb{H}^n$, the exponential map (see [21])

$$Exp_0 : \mathfrak{h}_n \equiv \mathbb{R}^{2n+1} \longrightarrow \mathbb{H}^n$$

$$Exp_0(\xi) := \exp\Big(s\nabla_1^{\mathbb{H}}\Big)\Big(\exp\Big(\sum_{i=1}^{2n} x_i \nabla_{i+1}^{\mathbb{H}}\Big)(0)\Big) \tag{12}$$

is a global diffeomorphism, where $(s, x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n+1} \equiv \mathbb{R} \times \mathbb{R}^{2n}$ are such that $\xi = s\nabla_1^{\mathbb{H}} + \sum_{i=1}^{2n} x_i \nabla_{i+1}^{\mathbb{H}}$. Because of (2) and the Baker-Campbell- Hausdorff formula (see, for instance, [8, Theorem 15.1.1]), the map Exp_0 can be represented as

$$Exp_0(\xi) = \exp\Big(s\nabla_1^{\mathbb{H}} + \sum_{i=1}^{2n-1} x_i \nabla_{i+1}^{\mathbb{H}} + (x_{2n} - s\, x_n)\nabla_{2n}^{\mathbb{H}}\Big)(0)\,.$$

Using this map, $\mathbb{H}^n \equiv \mathbb{R} \times \mathbb{R}^n$ by identifying each point $p \in \mathbb{H}$ with $p \equiv (s, x) := Exp_0^{-1}(p)$. The group law can be obtained simply projecting the Baker-Campbell-Hausdorff operation defined on $\mathfrak{h}_n$. In coordinates, if $p = (s, x), q = (t, y) \in \mathbb{R} \times \mathbb{R}^{2n}$ we obtain

$$q \cdot p = (t + s, y_1 + x_1, \ldots, y_{2n} + x_{2n} + 2\, t\, x_n + \sigma(x, y)) \tag{13}$$

where

$$\sigma(x, y) := \sum_{i=1}^{n-1}(y_i x_{n+i} - y_{n+i} x_i) \text{ if } n \geq 2 \text{ and } \sigma(x, y) = 0 \text{ if } n = 1. \tag{14}$$

2.1. *Remark.* Notice that formula (13) immediately gives

$$(0, x) \cdot (s, 0) = (s, x) \quad \forall x \in \mathbb{R}^{2n},\ s \in \mathbb{R}\,.$$

Thus the graph map assumes the simple form (4).

2.2. *Remark.* From (13), taking into account that the origin $0 = (0, \ldots, 0) \in \mathbb{R}^{2n+1}$ is the unit element of $\mathbb{H}^n$, we easily get that $p^{-1} \neq -p$. Namely, with the identification $p = (s, x)$, then

$$(s, x)^{-1} = (-s, -x_1, \ldots, -x_{2n-1}, -x_{2n} + 2\, s\, x_n) \neq (-s, -x).$$

In particular

(15)
$$q^{-1} \cdot p = (s-t, x_1 - y_1, \dots, x_{2n-1} - y_{2n-1}, x_{2n} - y_{2n} - 2\,t\,(x_n - y_n) - \sigma(x,y)).$$

In these coordinates the canonical basis of $\mathfrak{h}_n$, is expressed as

$$(16)\quad \begin{cases} \nabla_1^{\mathbb{H}} = \partial_s \\ \nabla_{i+1}^{\mathbb{H}} = \partial_{x_i} - x_{i+n}\partial_{x_{2n}} \quad \text{if } n>1 \text{ and } i=1,\dots,n-1 \\ \nabla_{n+1}^{\mathbb{H}} = \partial_{x_n} + 2s\partial_{x_{2n}} \\ \nabla_{i+1}^{\mathbb{H}} = \partial_{x_i} + x_{i-n}\partial_{x_{2n}} \quad \text{if } n>1 \text{ and } i=n+1,\dots,2n-1 \\ \nabla_{2n+1}^{\mathbb{H}} = \partial_{x_{2n}} \end{cases}.$$

2.3. **Definition.** For each $p \in \mathbb{H}^n$ with canonical coordinates $(s,x) \in \mathbb{R} \times \mathbb{R}^{2n}$ and for any $\lambda > 0$ we define the dilations

$$\delta_\lambda(p) := (\lambda s, \lambda x_1, \dots, \lambda x_{2n-1}, \lambda^2 x_{2n}),$$

and translations:

$$p \mapsto \tau_q(p) := q \cdot p$$

for any fixed $q \in \mathbb{H}^n$.

Every fiber $H_p\mathbb{H}^n$ is the linear span of $\nabla_1^{\mathbb{H}}(p), \dots, \nabla_{2n}^{\mathbb{H}}(p)$. We define the following homogeneous norm $\|p\|_*$

$$(17)\qquad \|p\|_* := \frac{1}{2}\,\left[\|p\| + \|p^{-1}\|\right],$$

where

$$(18)\qquad \|p\| := \max\left\{|(s, x_1, \dots, x_{2n-1})|_{\mathbb{R}^{2n}}, |x_{2n}|^{\frac{1}{2}}\right\},$$

and $|\cdot|_{\mathbb{R}^{2n}}$ denotes the Euclidean norm on $\mathbb{R}^{2n}$.

Note that the functions defined in (17) and in (18) are equivalent homogeneous norm on $\mathbb{H}^n$. In addition, the norm $\|\cdot\|_*$ is also symmetric.

Let $\mathrm{d} : \mathbb{H}^n \times \mathbb{H}^n \to [0, +\infty)$

$$(19)\qquad \mathrm{d}(p,q) = \|q^{-1} \cdot q\|_* \quad \text{if } p, q \in \mathbb{H}^n.$$

By definition, it immediately follows that d is a left-invariant homogenous distance on $\mathbb{H}^n$ and d is equivalent to the quasi-distance induced from the norm $\|\cdot\|$, that is

$$(20)\qquad \frac{1}{c}\,\|q^{-1} \cdot p\| \le \mathrm{d}(p,q) \le c\,\|q^{-1} \cdot p\| \quad \forall p, q \in \mathbb{H}^n.$$

Finally, it is easy to see that d is locally equivalent to the Carnot-Carathéodory (or sub-Riemannian) distance associated to the horizontal subbundle $H\mathbb{H}^n$ in a standard way, see [29]. We denote by $U(p,r)$

and by $B(p,r)$ the open and the closed ball associated with d. Moreover the bounded sets in $(\mathbb{H}^n, \mathrm{d})$ coincide with the ones of $\mathbb{R}^{2n+1}$ endowed with Euclidean metric $|\cdot|_{\mathbb{R}^{2n+1}}$ and for each bounded set $A \subseteq \mathbb{H}^n$ there exists a positive constant $c = c(A) > 0$ such that

$$\frac{1}{c}|p-q|_{\mathbb{R}^{2n+1}} \leq \mathrm{d}(p,q) \leq c|p-q|^{\frac{1}{2}}_{\mathbb{R}^{2n+1}} \quad \forall p,q \in A. \tag{21}$$

Denoting by $\mathcal{L}^{2n+1}$ the Lebesgue measure on $\mathbb{R}^{2n+1}$ we have that for all $(2n+1)-$ dimensional measurable $E \subset \mathbb{H}^n$ it holds

$$\mathcal{L}^{2n+1}(\tau_q(E)) = \mathcal{L}^{2n+1}(E) \quad \forall q \in \mathbb{H}^n,$$
$$\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E) \quad \forall \lambda > 0,$$

hence $\mathcal{L}^{2n+1}$ is the Haar measure of $\mathbb{H}^n$ and $\mathbb{H}^n$ turns out to be a homogeneous group with homogeneous dimension $Q := 2n+2$ (see [22]). We also recall that the homogeneous dimension Q coincides with the Hausdorff dimension of $(\mathbb{H}^n, \mathrm{d})$. Finally, we denote by $\mathcal{S}^m$ the $m-$dimensional spherical measure obtained from the distance d in $\mathbb{H}^n$.

From now on we will identify a section $F : \mathbb{H}^n \longrightarrow H\mathbb{H}^n$ with its canonical coordinates with respect to the moving frame $\nabla_1^{\mathbb{H}}, \dots, \nabla_{2n}^{\mathbb{H}}$, in other words each section F will be identified with a function

$$F = (F_1, \dots, F_{2n}) : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}.$$

With this convention we can give the following definition

2.4. **Definition.** Let $\Omega \subseteq \mathbb{H}^n$ be open. If $f \in L^1_{loc}(\Omega;\mathbb{R})$ and if $F = (F_1, \dots, F_{2n}) \in (L^1_{loc}(\Omega,\mathbb{R}))^{2n}$ is a section, we define the $\mathbb{H}-$divergence of F and the $\mathbb{H}-$gradient of f as the following distributions

$$\mathrm{div}_{\mathbb{H}} F := \sum_{j=1}^{2n} \nabla_j^{\mathbb{H}} F_j, \qquad \nabla_{\mathbb{H}} f := (\nabla_1^{\mathbb{H}} f, \dots, \nabla_{2n}^{\mathbb{H}} f).$$

2.1. Sets of intrinsic finite perimeter in $\mathbb{H}^n$.

In this subsection we will recall some useful concepts of geometric measure theory in $\mathbb{H}^n$, see [23], [24], [27], [11] and reference therein.

From now on we will denote by $C_c^\infty(\Omega, H\mathbb{H}^n)$ the set of C^∞ section of $H\mathbb{H}^n$ with compact support in Ω where, of course, the C^∞ regularity is understood as regularity between manifolds and $\Omega \subset \mathbb{H}^n$ is an open set.

2.5. **Definition.** ([10]) We say that $f : \Omega \longrightarrow \mathbb{R}$ is of bounded $\mathbb{H}$-variation in an open set $\Omega \subset \mathbb{H}^n$ and we write $f \in BV_{\mathbb{H}}(\Omega)$, if $f \in L^1(\Omega)$

and the total variation

$$|Df|_{\mathbb{H}}(\Omega) := \sup\left\{ \int_\Omega f \operatorname{div}_{\mathbb{H}}\varphi \mathrm{d}\mathcal{L}^{2n+1} \mid \varphi \in C_c^\infty(\Omega, H\mathbb{H}^n),\ |\varphi|_{\mathbb{R}^{2n}} \leq 1 \right\}$$

is finite. Moreover we say that f is of locally finite $\mathbb{H}-$variation in Ω ($f \in BV_{\mathbb{H},loc}(\Omega)$) if $f \in L^1_{loc}(\Omega)$ and $f \in BV_{\mathbb{H}}(\Omega')$ for every $\Omega' \Subset \Omega$.

2.6. **Definition.** A set $E \subset \mathbb{H}^n$ is said to be of *finite* $\mathbb{H}-$*perimeter* in Ω if $\chi_E \in BV_{\mathbb{H}}(\Omega)$. Analogously, a set $E \subset \mathbb{H}^n$ is of locally finite $\mathbb{H}-$perimeter in Ω if $\chi_E \in BV_{loc,\mathbb{H}}(\Omega)$.

2.7. *Remark.* It is well-known that if $E \subseteq \mathbb{H}^n$ is a set of locally finite $\mathbb{H}-$perimeter in Ω, then $|\partial E|_{\mathbb{H}}$ is a Radon measure on Ω and its support is such that $\operatorname{spt}(|\partial E|_{\mathbb{H}}) \subseteq (\partial E \cap \Omega)$, where ∂E denotes the topological boundary of E.

2.8. **Proposition.** *Let $f, f_n \in L^1(\Omega)$, $n \in \mathbb{N}$, be such that $f_n \to f$ in $L^1(\Omega)$. Then*

$$|Df|_{\mathbb{H}}(\Omega) \leq \liminf_{n\to\infty} |Df_n|_{\mathbb{H}}(\Omega).$$

In analogy with the Euclidean case, by Riesz's representation Theorem, the following formula holds, see [23].

2.9. **Theorem.** *Let $E \subset \Omega$ be a set with locally finite $\mathbb{H}-$perimeter. Then there exists a $|\partial E|_{\mathbb{H}}-$ measurable section ν_E of $H\mathbb{H}^n$, called* generalized inward normal, *such that*

$$|\nu_E(p)|_{\mathbb{R}^{2n}} = 1 \ \textit{for}\ |\partial E|_{\mathbb{H}} - a.e\ p \in \Omega;$$

$$\int_E \operatorname{div}_{\mathbb{H}}\varphi\ \mathrm{d}\mathcal{L}^{2n+1} = -\int_{\mathbb{H}^n} \langle \nu_E, \varphi\rangle\ \mathrm{d}|\partial E|_{\mathbb{H}} \quad \forall\ \varphi \in C_c^\infty(\mathbb{H}^n, H\mathbb{H}^n),$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{2n}$.

2.10. **Definition** ([23])**.**

(i) Let $E \subset \mathbb{H}^n$ be a set of locally finite $\mathbb{H}-$perimeter; we say that $p \in \partial^*_{\mathbb{H}} E$ (the $\mathbb{H}-$*reduced boundary of* E) if

$|\partial E|_{\mathbb{H}}(U(p,r)) > 0 \quad \forall\ r > 0;$

$\exists\ \lim_{r\to 0} \ \fint_{U(p,r)} \nu_E\ \mathrm{d}|\partial E|_{\mathbb{H}} = \nu_E(p);$

$|\nu_E(p)|_p = 1.$

(ii) Let $E \subset \mathbb{H}^n$ be a measurable set, we say that $p \in \partial_{*,\mathbb{H}} E$ (*the measure theoretic boundary of* E) if

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^{2n+1}(E \cap U(p,r))}{\mathcal{L}^{2n+1}(U(p,r))} > 0$$

and

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^{2n+1}(E^c \cap U(p,r))}{\mathcal{L}^{2n+1}(U(p,r))} > 0.$$

2.11. **Lemma** ([23])**.** *The $\mathbb{H}-$reduced boundary of a set of finite $\mathbb{H}$-perimeter is invariant under left translations, that is if $q \in \partial^*_{\mathbb{H}} E$ if and only if $\tau_p(q) \in \partial^*_{\mathbb{H}}(\tau_p(E))$, moreover, $\nu_E(q) = \nu_{\tau_p(E)}(\tau_p(q))$.*

2.12. **Lemma** ([1])**.** *Assume E is a set of locally finite $\mathbb{H}-$perimeter in $\mathbb{H}^n$, then*

$$\lim_{r\to 0} \fint_{U(p,r)} \nu_E \, \mathrm{d}|\partial E|_{\mathbb{H}} = \nu_E(p) \quad \textit{for } |\partial E|_{\mathbb{H}} - a.e\ p \in \mathbb{H}^n.$$

2.13. *Remark.* From Definition 2.10 and Lemma 2.12 we immediately deduce that $|\partial E|_{\mathbb{H}}-$a.e $p \in \mathbb{H}^n$ belongs to the reduced boundary $\partial^*_{\mathbb{H}} E$.

We end this section with a collection of results that are the Heisenberg counterpart of the BV function theory in the Euclidean space, see [11], [23] and [24].

2.14. **Theorem** ([27])**.** *There is a constant $c > 0$ independent of $r > 0$ such that for any set $E \subset \mathbb{H}^n$ of locally finite $\mathbb{H}-$perimeter, $\forall p \in \mathbb{H}^n$, $\forall r > 0$*

(22)
$$\min\{\mathcal{L}^{2n+1}(E \cap U(p,r)), \mathcal{L}^{2n+1}(E^c \cap U(p,r))\}^{\frac{Q-1}{Q}} \le c|\partial E|_{\mathbb{H}}(U(p,r))$$

and

$$\min\{\mathcal{L}^{2n+1}(E), \mathcal{L}^{2n+1}(E^c)\}^{\frac{Q-1}{Q}} \le c|\partial E|_{\mathbb{H}}(\mathbb{H}^n). \tag{23}$$

2.15. **Definition.** For each $q \in \mathbb{H}^n$, we define the map $\pi_q : \mathbb{H}^n \longrightarrow H_q\mathbb{H}^n$

$$\pi_q(p) = \pi_q((s,x)) := s\nabla^{\mathbb{H}}_1(q) + \sum_{j=2}^{2n} x_j \nabla^{\mathbb{H}}_j(q).$$

2.16. **Theorem** ([23])**.** *If $E \subseteq \mathbb{H}^n$ is a set with locally finite $\mathbb{H}$-perimeter then there exist $c = c(n) > 0$ such that*

$$|\partial E|_{\mathbb{H}} = c\, \mathcal{S}^{2n+1} \llcorner \partial^*_{\mathbb{H}} E. \tag{24}$$

2.17. **Theorem** ([23])**.** *If E is a locally finite $\mathbb{H}-$perimeter set, $p \in \partial^*_{\mathbb{H}} E$ and $\nu_E(p) \in H\mathbb{H}^n_p$ is the generalized inward normal to E in p, then*

$$\lim_{r\to 0} \boldsymbol{1}_{E_{r,p}} = \boldsymbol{1}_{S^+_{\mathbb{H}}(\nu_E(p))} \quad \textit{in } L^1_{loc}(\mathbb{H}^n), \tag{25}$$

where $E_{r,p} := \delta_{1/r}(\tau_{p^{-1}} E)$ and

$$S^+_{\mathbb{H}}(\nu_E(p)) := \{q \in \mathbb{H}^n \mid \langle \pi_p(q), \nu_E(p)\rangle_p \ge 0\}.$$

Moreover, for all $R > 0$ it holds

$$\lim_{r\to 0} |\partial E_{r,p}|_{\mathbb{H}}(U(0,R)) = |S^+_{\mathbb{H}}(\nu_E(p))|_{\mathbb{H}}(U(p,R)) = 2\omega_{2n-1}R^{2n+1},$$

where, as usual, ω_{2n-1} denotes the $(2n-1)-$dimensional Lebesgue measure of the unit Euclidean ball in $\mathbb{R}^{2n-1}$.

3. Intrinsic Lipschitz functions

In this section we analyze the notion of Lipschitz functions introduced in Definition 1.2. As usual, we have two approaches to study the geometric properties of an embedded graph. One is to use directly the geometry of the ambient space. A second way is to induce vector fields on the domain and using them to define a geometric structure on the surface. We will see that the two approaches are equivalent for intrinsic Lipschitz function.

3.1. **Lipschitz functions: properties inherited by the ambient space.** If we denote $\mathbb{W}, \mathbb{V}$ the homogeneous subgroups introduced in (3), then $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ and $\mathbb{W}\cap\mathbb{V} = \{0\}$. From now on, when no confusion can arise, we denote a point $(0, x) \in \mathbb{W}$ by $x \in \mathbb{R}^{2n}$ and $(s, 0) \in \mathbb{V}$ by $s \in \mathbb{R}$. We also assume to have a fixed continuous function

$$\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V},$$

and we explicitly write in coordinates the distance d_ϕ defined in Definition 1.1.

If $x = (x_1, \dots, x_{2n}) \in \mathbb{W} \equiv \mathbb{R}^{2n}$, we denote by $\hat{x} := (x_1, \dots, x_{2n-1}) \in \mathbb{R}^{2n-1}$ and, if $x, y \in \omega$

$$\sigma_\phi(x, y) := |y_{2n} - x_{2n} - 2\phi(x)(y_n - x_n) + \sigma(x, y)|^{1/2} \tag{26}$$

where σ is defined in(14). Let $x, y \in \omega$, we get:

$$\mathrm{d}_\phi(x, y) = \frac{1}{2}\max\Big\{|\hat{x}-\hat{y}|_{\mathbb{R}^{2n-1}}, \sigma_\phi(x, y)\Big\} + \frac{1}{2}\max\Big\{|\hat{x}-\hat{y}|_{\mathbb{R}^{2n-1}}, \sigma_\phi(y, x)\Big\}. \tag{27}$$

3.1. *Remark.* It immediately follows from the explicit expression of the distance (see also [14]) that, if $\phi \in Lip_{\mathbb{W}}(\omega)$ (see Definition 1.2) then d_ϕ is a quasi-distance on ω. Indeed for each $x, y, z \in \omega$:

$$\mathrm{d}_\phi(x, y) \leq \tag{28}$$

$$\leq \mathrm{d}_\phi(x, z) + \mathrm{d}_\phi(y, z) + |\phi(x) - \phi(z)|^{1/2}|x_n - z_n|^{1/2} + |\phi(y) - \phi(z)|^{1/2}|y_n - z_n|^{1/2}$$

so that

$$\mathrm{d}_\phi(x, y) \leq (1 + \; Lip(\phi))^{1/2}\Big(\mathrm{d}_\phi(x, z) + \mathrm{d}_\phi(y, z)\Big).$$

3.2. *Remark.* It is easy to see that, if $\phi \in Lip_{\mathbb{W}}(\omega)$, then

$$\sigma_\phi(y, x) \leq \; \sigma_\phi(x, y) + |\phi(x) - \phi(y)|^{1/2}|x_n - y_n|^{1/2} \quad \forall x, y \in \omega$$

whence, by (27), $\forall x, y \in \omega$

$$\text{(29)}\quad \mathrm{d}_\phi(x,y) \le\ |\hat{x} -\ \hat{y}|_{\mathbb{R}^{2n-1}} + \sigma_\phi(x,y) + |\phi(x)-\phi(y)|^{1/2}|x_n - y_n|^{1/2}$$

3.3. *Remark.* Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open set and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then, arguing like as before, it is easy to see that, for each $\omega' \Subset \omega$ there exists a constant $\alpha = \alpha(\omega', \|\phi\|_{L^\infty(\omega')}, Lip(\phi)) > 0$ such that

$$\frac{1}{\alpha}\,|x-y|_{\mathbb{R}^{2n}} \le\ \mathrm{d}_\phi(x,y) \le\ \alpha\,\sqrt{|x-y|_{\mathbb{R}^{2n}}} \quad \forall x, y \in \omega'\,.$$

As a consequence, the identity map $Id:\ (\omega, \mathrm{d}_\phi) \to (\omega, |\cdot|_{\mathbb{R}^{2n}})$ is an homemorphism.

If $\phi \in Lip_{\mathbb{W}}(\omega)$, we will denote

$$U_\phi(x,r) := \{y \in \omega \mid \mathrm{d}_\phi(x,y) < r\}\,. \tag{30}$$

Let us now recall that the map $\pi_{\mathbb{W}}$ defined in (6) and the map $\pi_{\mathbb{V}}$ defined as

$$\pi_{\mathbb{V}} : p = (s,x) \in \mathbb{H}^n \to (s,0) \in \mathbb{V}$$

are continuous and

$$\frac{1}{\sqrt{2}}(\|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\|) \le \|p\| \le \|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\|, \tag{31}$$

(see also [5] for a generalization of this statement in Carnot groups of any step).
We will call *intrinsic (left) graph* of a function $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V}$ the set

$$\mathrm{graph}(\phi) := \{(\phi(x), x) \mid x \in \omega\} \tag{32}$$

and *intrinsic subgraph of* ϕ the set

$$E_\phi := \{(s,x) \in \mathbb{R}\times\omega \mid s \le \phi(x)\}. \tag{33}$$

With these notations the notion of intrinsic Lipschitz function, can be rephrased in terms of a notion of closed cones, as introduced in the interesting paper [26].

3.4. **Definition.** Let $q \in \mathbb{H}^n$ and $\alpha > 0$. The intrinsic (closed) cone $C_{\mathbb{W},\mathbb{V}}(q,\alpha)$ with base $\mathbb{W}$, axis $\mathbb{V}$, vertex q and opening α is

$$C_{\mathbb{W},\mathbb{V}}(q,\alpha) := \{p = (s,x) \in \mathbb{H}^n \mid \|\pi_{\mathbb{W}}(q^{-1}\cdot p)\| \le \alpha\|\pi_{\mathbb{V}}(q^{-1}\cdot p)\|\}.$$

In our simplified context if $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ is a continuous map and $p, q \in C_{\mathbb{W},\mathbb{V}}(q,\alpha) \cap \mathrm{graph}(\phi)$, then there exist x, $y \in \omega$ such that $p = (\phi(x), x)$, $q = (\phi(y), y)$,

$$\|\pi_{\mathbb{V}}(q^{-1}\cdot p)\| = |\phi(x) - \phi(y)|,$$

so that it is clear that ϕ is an intrinsic Lipschitz function if and only if there is $L > 0$ such that, for all $q \in \mathrm{graph}(\phi)$,

$$C_{\mathbb{W},\mathbb{V}}(q, 1/L) \cap \mathrm{graph}(\phi) = \{q\}. \tag{34}$$

Using this notation, in [37] a general theorem was proved, which in our context reduces to the following result.

3.5. **Theorem.** *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}-$perimeter in $U(0,r)$, $r > 0$, ν_E be the measure theoretic inward normal of E, and $\nu \in S^{2n-1}$. Assume there exists $k \in (0,1]$ such that $\pi_{\mathbb{V}}(\nu_E(p)) \leq -k$ for $|\partial E|_{\mathbb{H}}-a.e$ $p \in U_r$. Then there exists $\alpha > 0$ such that possibly modifying E in a negligible set, for all $p \in \partial E \cap U_r$*

$$\{q \in U_r \mid \|\pi_{\mathbb{W}}(p^{-1} \cdot q)\| < -\alpha \pi_{\mathbb{V}}(p^{-1} \cdot q)\} \subset E;$$
$$\{q \in U_r \mid \|\pi_{\mathbb{W}}(p^{-1} \cdot q)\| < \alpha \pi_{\mathbb{V}}(p^{-1} \cdot q)\} \subset \mathbb{H}^n \setminus E.$$

If in particular $n = 1$, then

$$\alpha^2 + 2\alpha \leq \frac{h}{2}, \quad \textit{with } h := \sqrt{\frac{k^2}{2-k^2}}. \tag{35}$$

We conclude this section comparing the distance d_ϕ with the distance of points of the graph:

3.6. **Proposition.** *If $\phi \in Lip_{\mathbb{W}}(\omega)$ then there is $C_1 = C_1(Lip(\phi)) > 0$ such that*

$$U_\phi(x, C_1 r) \subseteq \pi_{\mathbb{W}}(U(\Phi(x), r) \cap \mathit{graph}(\phi)) \subseteq U_\phi(x, r) \tag{36}$$

for all $x \in \omega$ and $r > 0$,where $U_\phi(x,r)$ is defined in (30).

Proof. Let $z \in \pi_{\mathbb{W}}(U(\Phi(x), r) \cap \mathrm{graph}(\phi))$ then $\mathrm{d}(\Phi(x), \Phi(z)) < r$. Since the intrinsic projection $\pi_{\mathbb{W}} : \mathbb{H}^n \longrightarrow \mathbb{W}$ is such that $\forall p \in \mathbb{H}^n$

$$\begin{aligned}\|\pi_{\mathbb{W}}(p)\| &= \max\{|(0, x_1, \dots, x_{2n-1})|, |x_{2n}|^{\frac{1}{2}}\} \leq \\ &\leq \max\{|(s, x_1, \dots, x_{2n-1})|, |x_{2n}|^{\frac{1}{2}}\} = \|p\|.\end{aligned}$$

We have

$$\begin{aligned}\mathrm{d}_\phi(x,z) &= \frac{1}{2}\Big(\|\pi_{\mathbb{W}}(\Phi(x)^{-1} \cdot \Phi(z))\| + \|\pi_{\mathbb{W}}(\Phi(z)^{-1} \cdot \Phi(x))\|\Big) \\ &\leq \frac{1}{2}\Big(\|(\Phi(x)^{-1} \cdot \Phi(z))\| + \|(\Phi(z)^{-1} \cdot \Phi(x))\|\Big) = \|\Phi(x)^{-1} \cdot \Phi(z)\|_* \\ &= \mathrm{d}(\Phi(x), \Phi(z)) < r.\end{aligned}$$

Hence the second inclusion follows.
For the first inclusion, let us note that for all $x, z \in \omega$

$$\mathrm{d}(\Phi(x), \Phi(z)) \leq |\phi(z) - \phi(x)| + \mathrm{d}_\phi(x,z)$$

therefore, since $\phi \in Lip_{\mathbb{W}}(\omega)$, for every $z \in U_\phi(x, Cr)$ with $C > 0$ to be determinated we obtain

$$\begin{aligned} d(\Phi(x), \Phi(z)) &\le Lip(\phi) d_\phi(x, z) + d_\phi(x, z) \\ &\le C(Lip(\phi) + 1)r \end{aligned}$$

and the first inclusion follows choosing $0 < C < 1/(Lip(\phi) + 1)$. $\square$

3.2. **Equivalence between d_ϕ and the cc-distance.** In this section we give an equivalent approach to the distance d_ϕ, more similar to the one initially proposed in [16]. Indeed, the presence of the function $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V}$ induces a family of nonlinear vector fields on the domain ω, and a distance is directly defined in terms of these new vector fields.

3.7. **Definition.** Let $\phi : \omega \longrightarrow \mathbb{R}$ be a continuous function defined on an open and bounded set $\omega \subset \mathbb{W}$. We introduce the family $\nabla^\phi = (\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi)$ of nonlinear vector fields, namely of first order differential operators, on ω by

$$\begin{aligned} &\nabla_i^\phi(x) = \partial_{x_i} - x_{i+n}\partial_{x_{2n}}, \quad i = 1, \dots, n-1, \\ &\nabla_n^\phi(x) = \partial_{x_n} + 2\phi(x)\partial_{x_{2n}}, \\ &\nabla_i^\phi(x) = \partial_{x_i} + x_{i-n}\partial_{x_{2n}}, \quad i = n+1 \dots, 2n-1, \end{aligned} \tag{37}$$

if $n \ge 2$, and by

$$\nabla_1^\phi(x) = \partial_{x_1} + 2\phi(x)\partial_{x_2} \quad \text{if } n = 1. \tag{38}$$

We will call horizontal tangent bundle $H\mathbb{W}$ of $\mathbb{W}$ the linear span of the family $(\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi)$. Note that, if ϕ is continuous, and $n \ge 2$, $\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi$ together with $[\nabla_1^\phi, \nabla_{n+1}^\phi] = 2\,\partial_{x_{2n}}$ span the whole tangent space to $\mathbb{W}$. If the vector fields were smooth this condition would be enough to ensure that it is possible to connect each couple of points $\bar{x}$ and y in Ω with a piecewise continuous integral curve of the horizontal vector fields. This means that there exists an absolutely continuous (a.c.) curve $\gamma_h : [0,1] \to \omega$ with extrema $\bar{x}$ an y such that

$$\dot{\gamma_h}(t) = \sum_{j=1}^{2n-1} h_j(t) \nabla_j^\phi(\gamma_h(t)) \quad a.e \;\; t \in (0,1) \tag{39}$$

with h_j a piecewise continuous function $h : [0,1] \to \mathbb{R}^{2n-1}$. In this case, the vector fields are only continuous, but is it immediate to recognize that the local connectivity property is still satisfied. In order to prove this, we remark that the Euclidean topology on $\mathbb{R}^{2n}$ coincides with the one induced by d_ϕ (see Remark 3.3) then we need only to prove the local connectivity in ω. We choose a point $\bar{x}$ in an open set ω and consider a rectangle

$$Q_{Eb} = \{y : |(y - \bar{x})_i| \le b\} \Subset \omega.$$

From standard ODE theory it follows that, if we choose any point $y \in Q_{E\delta}$, with $\delta < 1/(1 + \max_{Q_{Eb}} |\phi|)$ the curve $\gamma_h : [0,1] \to \omega$, with $h_j = (y - \bar{x})_j$ lies in ω and connects $\bar{x}$ with a point $\tilde{y}$ with the same first $2n-1$ components as y. Due to the Chow theorem, we can now connect $\tilde{y}$ and y with a piecewise integral curve of the vector fields ∇_1^ϕ and ∇_{n+1}^ϕ, again lying in ω. Consequently, $\bar{x}$ can be connected with any point of Q_δ. Since the set of points which can be connected with $\bar{x}$ is also closed in ω, we can deduce that any open connect set ω satisfies the connectivity property. Hence the Carnot-Carathéodory distance $\mathrm{d}_{cc,\phi}$ in the set ω will be defined as in [38]:

$$\begin{aligned} \mathrm{d}_{cc,\phi}(x,y) = \inf\{ &\|h\|_{L^\infty((0,1),\,\mathbb{R}^{2n-1})} \mid \exists \text{ an curve } \gamma_h : [0,1] \to \omega \\ &\text{satisfying (39)}\,, \gamma_h(0) = x, \gamma_h(1) = y\}. \end{aligned}$$

From now on we will denote by

$$U_{cc,\phi}(x,r) := \{y \in \omega \mid \mathrm{d}_{cc,\phi}(x,y) < r\}\,.$$

We will denote respectively $Lip_{cc,\phi}$ the set of Lipschitz continuous functions with respect to the $\mathrm{d}_{cc,\phi}$ distance and $Lip_{cc}(\phi,\omega)$ the corresponding Lipschitz constant on a set ω. We will also denote it $Lip_{cc}(\phi)$, if it is clear which is ω.

3.8. **Proposition.** *Let $\omega \subset \mathbb{W}$ be an open set. Let $n \ge 2$ and $\phi \in Lip_{\mathbb{W}}(\omega)$. For each $\bar{x} \in \omega$ let*

$$\bar{r} := \min\Big\{\mathrm{d}_\phi(\bar{x},\partial\omega), \mathrm{d}_{cc,\phi}(\bar{x},\partial\omega)\Big\} > 0\,.$$

If $L := \min\{Lip(\phi), Lip_{cc}(\phi)\}$, there exist $C_i = C_i(L)$, (continuous function of L) $i \in \{1,2\}$ such that:

$$U_{cc,\phi}(\bar{x},r) \subset U_\phi(\bar{x},C_2 r) \Subset \omega \quad \forall\, 0 < r < \frac{\bar{r}}{2C_2}; \tag{40}$$

$$U_\phi(\bar{x},r) \subset U_{cc,\phi}(\bar{x},C_1 r) \Subset \omega \quad \forall\, 0 < r < \frac{\bar{r}}{2C_1}\,. \tag{41}$$

In particular, $\mathrm{d}_{cc,\phi}$ and d_ϕ are locally equivalent.

Proof. The proof is similar to the one contained in [16, Proposition 4.2] for general vector fields. In order to establish inclusion (40) we fix $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{2n}) \in \omega$ and $0 < r < \bar{r}/2C_2$, and C_2 is a constant only dependent on L to be chosen later. For each $y \in U_{cc,\phi}(\bar{x},r)$ let $\gamma : [0,1] \to \omega$

be an absolutely continuous curve satisfying (39) with a piecewise continuous function $h = (h_1, \dots, h_{2n-1}) \in L^\infty((0,1), \mathbb{R}^{2n-1}) \equiv L^\infty(0,1)$ and $\gamma(0) = \bar{x}$, $\gamma(1) = y$. We first note that

$$\gamma_i(t) - \bar{x}_i = \int_0^t h_i(s)\mathrm{d}s \quad \forall t \in [0,1],\ i \in \{1, \dots, 2n-1\} \tag{42}$$

and

(43)

$$\gamma_{2n}(t) - \bar{x}_{2n} = \int_0^t (2h_n(s)\phi(\gamma(s))\mathrm{d}s + \sigma(h(s), \gamma(s))\mathrm{d}s \quad \forall t \in [0,1]\,.$$

Since $\phi \in Lip_{\mathbb{W}}(\omega)$, then $L := \min\{Lip(\phi), Lip_{cc}(\phi)\} < +\infty$. Therefore, if $L = Lip(\phi)$ then

$$|\phi(x) - \phi(y)| \le L\, \mathrm{d}_\phi(x, y) \quad \forall\, x, y \in \omega\,, \tag{44}$$

whereas, if $L = Lip_{cc}(\phi)$, then

$$|\phi(x) - \phi(y)| \le L\, \mathrm{d}_{cc,\phi}(x, y) \quad \forall\, x, y \in \omega\,, \tag{45}$$

hence, by (44) and (45) we get,

$$|\phi(x) - \phi(y)| \le L\, (\mathrm{d}_{cc,\phi}(x, y) + \mathrm{d}_\phi(x, y)) \quad \forall\, x, y \in \omega\,. \tag{46}$$

By (29) and (46), $\forall t \in [0,1]$,

$$\begin{aligned} \mathrm{d}_\phi(\gamma(t), \bar{x}) \le |\widehat{\gamma(t) - \bar{x}}|_{\mathbb{R}^{2n-1}} + \sigma_\phi(\bar{x}, \gamma(t)) + \\ + (L/2)^{1/2} |\gamma_n(t) - \bar{x}_n|^{1/2} (\mathrm{d}_\phi(\gamma(t), \bar{x})^{1/2} + \mathrm{d}_{cc,\phi}(\gamma(t), \bar{x})^{1/2}) \end{aligned} \tag{47}$$

Setting $M_1 := 1 + 2\,L$ and recalling that $|\gamma_n(t) - \bar{x}_n| \le |\widehat{\gamma(t) - \bar{x}}|_{\mathbb{R}^{2n-1}}$, from (47) we obtain

$$\mathrm{d}_\phi(\gamma(t), \bar{x}) \le M_1 |\widehat{\gamma(t) - \bar{x}}|_{\mathbb{R}^{2n-1}} + 2\,\sigma_\phi(\bar{x}, \gamma(t)) + \mathrm{d}_{cc,\phi}(\gamma(t), \bar{x})\,. \tag{48}$$

Let us now observe that, from the definition of $\mathrm{d}_{cc,\phi}$,

$$\mathrm{d}_{cc,\phi}(\gamma(t), \bar{x}) \le \|h\|_{L^\infty(0,1)} \quad \forall t \in [0,1]\,. \tag{49}$$

Meanwhile

$$\begin{aligned} \sigma_\phi(\bar{x}, \gamma(t))^2 = \Big| \int_0^t \Big(2h_n(s)\phi(\gamma(s))\mathrm{d}s + \sigma(h(s), \gamma(s)\Big)\mathrm{d}s - \\ -2\phi(\bar{x}) \int_0^t h_n(s)\mathrm{d}s + \sigma(\bar{x}, \int_0^t 2h(s)\mathrm{d}s)\Big| \end{aligned}$$

then, by (42) and (43), for each $t \in [0,1]$, we get

(50)

$$\sigma_\phi(\bar{x},\gamma(t))^2 \le 2\int_0^t h_n(s)\Big|\phi(\gamma(s))-\phi(\bar{x})\Big|\mathrm{d}s + 2\int_0^t \Big|\sigma(h(s),\gamma(s)-\bar{x})\Big|\mathrm{d}s$$
$$\le M_2\,\|h\|_{L^\infty(0,1)}\Big(\max_{s\in[0,1]} \mathrm{d}_\phi(\gamma(s),\bar{x}) + \|h\|_{L^\infty(0,1)}\Big),$$

where $M_2 = 2(L+1) > M_1 \ge 1$. In the last inequality we used the fact that $\sigma(v,w) \le |\hat{v}|_{\mathbb{R}^{2n-1}}|\hat{w}|_{\mathbb{R}^{2n-1}}$ $\forall v,w \in \mathbb{R}^{2n}$ together with (42) and (43). From (48), (49) and (50), it now follows that

(51)

$$\max_{s\in[0,1]} \mathrm{d}_\phi(\gamma(s),\bar{x}) \le (M_1(2n-1)+1)\,\|h\|_{L^\infty(0,1)} +$$
$$+ 2\,M_2^{1/2}\|h\|_{L^\infty(0,1)}^{1/2}\Big(\max_{s\in[0,1]} \mathrm{d}_\phi(\gamma(s),\bar{x}) + \|h\|_{L^\infty(0,1)}\Big)^{1/2}$$
$$\le 2nM_2\|h\|_{L^\infty(0,1)} + \frac{1}{2}\max_{s\in[0,1]} \mathrm{d}_\phi(\gamma(s),\bar{x})$$

and hence, for $t=1$

$$\mathrm{d}_\phi(y,\bar{x}) = \mathrm{d}_\phi(\gamma(1),\bar{x}) \le 4n\,M_2\,\|h\|_{L^\infty(0,1)}\,. \tag{52}$$

Taking the infimum in (52) on the piecewise continuous function $h = (h_1,\dots,h_{2n-1}) \in L^\infty((0,1),\mathbb{R}^{2n-1})$ satisfying (39) for a suitable absolutely continuous curve $\gamma : [0,1] \to \omega$ with $\gamma(0) = \bar{x}$ and $\gamma(1) = y$, we get

$$\mathrm{d}_\phi(y,\bar{x}) \le M_3 \mathrm{d}_{cc,\phi}(y,\bar{x}), \tag{53}$$

where $C_2 = M_3 := 8n(L+1)$.

In order to prove inclusion (41), we fix $0 < r < \min\{\bar{r}/2C_1, \bar{r}/2C_2\}$ where C_1 is a constant only dependent on L to be chosen later and $y \in U_\phi(\bar{x},r)$. Thanks to the choice of r and inequality (53) the curve

$$\gamma(s) := \exp(s(y_n-\bar{x}_n)\nabla_n^\phi)(\bar{x})\,. \tag{54}$$

exists for s small and it is defined for $t \in [0,s]$, with $s<1$. Moreover, by (51), it satisfies

$$\max_{t\le s} \mathrm{d}_\phi(\gamma(t),\bar{x}) \le M_3\,|\bar{x}_n - y_n| \le M_3\,\mathrm{d}_\phi(y,\bar{x}) \tag{55}$$

and hence it is further prolungable on the set $t \in [0,1]$. In coordinates it can be expressed as

(56)

$$\gamma(s) = \Big(\bar{x}_1,\dots,\bar{x}_n + s(y_n-\bar{x}_n),\dots,\bar{x}_{2n-1},\bar{x}_{2n} + 2(y_n-\bar{x}_n)\int_0^s \phi(\gamma(\tau))\mathrm{d}\tau\Big).$$

By definition of $\mathrm{d}_{cc,\phi}$ and d_ϕ

$$\mathrm{d}_{cc,\phi}(\gamma(1),\bar{x}) \leq |y_n - \bar{x}_n| \leq \mathrm{d}_\phi(y,\bar{x})\,. \tag{57}$$

Hence in order to get the thesis we only need to estimate $\mathrm{d}_{cc,\phi}(\gamma(1),y)$. Note that the points $\gamma(1)$ and y have the same n-th component, and the metric d_ϕ on

$$\{(z_1,\dots,z_{n-1},y_n,z_n,\dots,z_{2n-1}) \in \mathbb{W} :\, (z_1,\dots,z_{2n-1}) \in \mathbb{R}^{2n-1}\} \equiv \mathbb{H}^{n-1}$$

reduces to the distance d in (19) on $\mathbb{H}^{n-1}$. On the other hand, when we discard the vector field ∇^ϕ_n, the family $\{\nabla^\phi\}$ reduces to the standard Heisenberg vector fields in $\mathbb{H}^{n-1}$. Hence , if we denote by $\hat{\mathrm{d}}_{cc}$ the CC metric induced on $\mathbb{H}^{n-1}$, it is well-known that d and $\hat{\mathrm{d}}_{cc}$ are equivalent on $\mathbb{H}^{n-1}$. Then there exists a geometric constant $M_4 >\, 0$ such that

$$\frac{1}{M_4}\,\hat{\mathrm{d}}_{cc}(\gamma(1),y) \leq\, \mathrm{d}_\phi(\gamma(1),y) \leq\, M_4\,\hat{\mathrm{d}}_{cc}(\gamma(1),y)\,. \tag{58}$$

Because, by definition, $\mathrm{d}_{cc,\phi}(\gamma(1),y) \leq\, \hat{\mathrm{d}}_{cc}(\gamma(1),y)$, by (58), we have

$$\mathrm{d}_{cc,\phi}(\gamma(1),y) \leq\, M_4\,\mathrm{d}_\phi(\gamma(1),y)\,. \tag{59}$$

By the triangle inequality, (57) and (59) we have $\forall y \in U_\phi(\bar{x},r)$.

$$\begin{aligned}\mathrm{d}_{cc,\phi}(\bar{x},y) &\leq \mathrm{d}_{cc,\phi}(\gamma(1),\bar{x}) + \mathrm{d}_{cc,\phi}(\gamma(1),y)\\ &\leq \mathrm{d}_\phi(\bar{x},y) + M_4\,\mathrm{d}_\phi(\gamma(1),y).\end{aligned} \tag{60}$$

Then, by (28) and (46), we get

$$\begin{aligned}\mathrm{d}_\phi(\gamma(1),y) &\leq \mathrm{d}_\phi(\gamma(1),\bar{x}) + \mathrm{d}_\phi(y,\bar{x})\\ &+ L^{1/2}\mathrm{d}_\phi^{1/2}(\gamma(1),\bar{x})\Big[\mathrm{d}_\phi^{1/2}(\gamma(1),\bar{x}) + \mathrm{d}_{cc,\phi}^{1/2}(\gamma(1),\bar{x})\Big]+\\ &+ L^{1/2}\mathrm{d}_\phi^{1/2}(y,\bar{x})\Big[\mathrm{d}_\phi^{1/2}(y,\bar{x}) + \mathrm{d}_{cc,\phi}^{1/2}(y,\bar{x})\Big]\\ &= \Big(L^{1/2}+1\Big)(\mathrm{d}_\phi(\gamma(1),\bar{x}) + \mathrm{d}_\phi(y,\bar{x}))\\ &+ L^{1/2}\mathrm{d}_\phi^{1/2}(\gamma(1),\bar{x})\mathrm{d}_{cc,\phi}^{1/2}(\gamma(1),\bar{x})\\ &+ L^{1/2}\mathrm{d}_\phi^{1/2}(y,\bar{x})\mathrm{d}_{cc,\phi}^{1/2}(y,\bar{x})\end{aligned} \tag{61}$$

so that, by (57) and (60)

$$\mathrm{d}_\phi(\gamma(1),y) \leq (2L^{1/2}+2+LM_4/2)\mathrm{d}_\phi(y,\bar{x})+(L^{1/2}+1+L)\mathrm{d}_\phi(\gamma(1),\bar{x})+\frac{1}{2}\mathrm{d}_\phi(\gamma(1),y)$$

which implies

$$\mathrm{d}_\phi(\gamma(1),y) \leq 2(2L^{1/2}+2+LM_4/2)\mathrm{d}_\phi(y,\bar{x})+2(L^{1/2}+1+L)\mathrm{d}_\phi(\gamma(1),\bar{x}).$$

Inserting this in (60) and using (55) we have

$$\begin{aligned}\mathrm{d}_{cc,\phi}(\bar{x},y) &\le \mathrm{d}_\phi(\bar{x},y) + 2M_4(2L^{1/2}+2+LM_4/2)\mathrm{d}_\phi(y,\bar{x}) + \\ &+ 2M_4(L^{1/2}+1+L)\mathrm{d}_\phi(\gamma(1),\bar{x}) \\ &= M_5 \mathrm{d}_\phi(\bar{x},y)\end{aligned}$$

for a suitable positive constant $M_5 = M_5(L)$. □

4. Intrinsic differentiability of intrinsic Lipschitz functions

In this section we study the notion of intrinsic differentiability introduced for the first time in [4, 16] and stated in Definition 1.4. This concept is particularly interesting since as recalled in the Introduction (see Theorem 1.3), an intrinsic Lipschitz function is differentiable in the intrinsic pointwise sense almost everywhere with respect to $\mathcal{L}^{2n}$. This result, as in the Euclidean case, open the possibility of proving much finer results on intrinsic Lipschitz functions. Indeed, we are able to prove an area formula beside the spherical Hausdorff measure for the graph of an intrinsic Lipschitz function (see Theorem 1.6) and that the pointwise gradient coincides with the weak one (see Proposition 4.7). We point out that this fact is not elementary at all in our situation, since the formal adjoint of the vector fields ∇^ϕ is not well defined when $\phi \in Lip_{\mathbb{W}}(\omega)$.

Throughout this section we refer to Definition 1.4 for the concept of intrinsic differentiability.

We start our analysis pointing out an important property of the $\nabla^\phi-$differential, see [4] and [16].

4.1. **Proposition.** *Let $\phi : \omega \longrightarrow \mathbb{R}$ be such that ϕ is ∇^ϕ differentiable at $x \in \omega$ then the $\nabla^\phi-$differential of ϕ at x is unique. Moreover, there is a unique vector $\nabla^\phi\phi(x) \in \mathbb{R}^{2n-1}$ such that*

$$L(y) = \left\langle \nabla^\phi \phi(x), \tilde{\pi}(y) \right\rangle \qquad \forall y \in \mathbb{W} \tag{62}$$

where $\langle \cdot,\cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{2n-1}$ and

$$\tilde{\pi}(x_1,\dots,x_{2n-1},x_{2n}) := (x_1,\dots,x_{2n-1}) \qquad \forall x \in \mathbb{W}.$$

We call the vector $\nabla^\phi\phi(x)$ the $\nabla^\phi-$gradient of ϕ at $x \in \omega$.

The following Corollaries are easy consequences of Theorem 1.3.

4.2. **Corollary.** *Let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then the intrinsic generalized inward normal ν_{E_ϕ} to the subgraph E_ϕ in $\mathbb{H}^n$ has the following representation*

$$\nu_{E_\phi}(\Phi(x)) = \Big(\frac{-1}{\sqrt{1+|\nabla^\phi\phi(x)|^2}}, \frac{\nabla^\phi\phi(x)}{\sqrt{1+|\nabla^\phi\phi(x)|^2}}\Big) \tag{63}$$

for $\mathcal{L}^{2n}-$a.e $x \in \omega$.

Proof. By Theorem 4.15, Lemma 4.28 and Theorem 4.29 in [26], for each $x_0 \in \omega$ point of intrinsic differentiability of ϕ, we get

$$\text{graph}(L) = \{p \in \mathbb{H}^n \mid \langle \nu_{E_\phi}(\Phi(x_0)), \bar{\pi}(p)\rangle = 0\}$$

where $\bar{\pi} : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}$ is given by

$$\bar{\pi}(p) = \bar{\pi}(s, x_1, \dots, x_{2n-1}, x_{2n}) := (s, x_1, \dots, x_{2n-1})$$

and L is the intrinsic differential of ϕ at x_0. Therefore, by Proposition 4.1, we obtain

$$\Big\{\Big(\big\langle \nabla^\phi\phi(x_0), \tilde{\pi}(y)\big\rangle, y\Big),\ y \in \mathbb{R}^{2n}\Big\} = \{p \in \mathbb{H}^n \mid \langle \nu_{E_\phi}(\Phi(x_0)), \bar{\pi}(p)\rangle = 0\}.$$

Hence, denoting by $\nu_{E_\phi}^{(1)}, \dots, \nu_{E_\phi}^{(2n)}$ the components of ν_{E_ϕ} we conclude that

$$\nu_{E_\phi}^{(1)}(\Phi(x_0))\,\big\langle \nabla^\phi\phi(x_0), \tilde{\pi}(y)\big\rangle + \sum_{i=2}^{2n} \nu_{E_\phi}^{(i)}(\Phi(x_0))y_i \quad \forall y \in \mathbb{R}^{2n}. \tag{64}$$

The thesis follows using (64) with $y = e_i$, $i = 1, \dots, 2n$, where $(e_1, \dots, e_{2n})$ is the canonical basis of $\mathbb{R}^{2n}$ and recalling that $|\nu_{E_\phi}(\Phi(x))| = 1$ $\mathcal{L}^{2n}-$a.e. in ω. □

4.3. **Corollary.** *Let $\phi : \mathbb{W} \longrightarrow \mathbb{R}$ be an intrinsic Lipschitz function with parametric map $\Phi : \mathbb{W} \longrightarrow \mathbb{H}^n$. Then*

$$(\Phi)_\#(\mathcal{L}^{2n} \llcorner \mathbb{W}) = -\nu_1 |\partial E_\phi|_{\mathbb{H}}$$

where $(\Phi)_\#(\mathcal{L}^{2n} \llcorner \mathbb{W})$ and ν_1 denote respectively the image measure $\mathcal{L}^{2n} \llcorner \mathbb{W}$ through the map Φ and the first component of the intrinsic generalized inward normal to E_ϕ.

Proof. In order to get the thesis it suffices to prove that, in our setting, the constant $c(\mathbb{W}, \mathbb{V}) > 0$ provided by Lemma 4.30 in [26] is equal to 1. This follow easily observing that when $\mathbb{W}$ and $\mathbb{V}$ are as in (3), then the map $\Psi : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1}$ defined in Lemma 4.30 of [26] becames the identity map. □

Using the fact that every intrinsic Lipschitz function is differentiable almost everywhere, we have the following proposition.

4.4. **Proposition.** *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be a bounded open set, and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then the intrinsic gradient $\nabla^\phi \phi$, which is defined $\mathcal{L}^{2n}$-a.e. in ω, satisfies*

$$\|\nabla^\phi \phi(x)\|_{L^\infty(\omega)} \le C\, Lip(\phi)\,(Lip(\phi)+1) \quad \mathcal{L}^{2n} - a.e.\ x \in \omega\,, \tag{65}$$

where $C = C(n) > 0$ depends only on n. As a consequence $\nabla^\phi \phi \in (L^\infty(\omega))^{2n-1}$.

Proof. Firstly, notice that, for each $i = 1, \ldots, 2n-1$ and $x \in \omega$, the exponential map of the vector field ∇_i^ϕ, $[-\delta, \delta] \ni s \to \exp(s\,\nabla_i^\phi)(x) \in \omega$ is well-defined for $\delta > 0$ small enough and

$$\mathrm{d}_\phi(\exp(s\,\nabla_i^\phi)(x), x) \le\ 8n\,(Lip(\phi)+1)\,|s| \quad \forall s \in [-\delta, \delta]\,, \tag{66}$$

for each $i = 1, \ldots, 2n-1$. Indeed, the existence of the exponential map is well-knwon for $i \ne n$ because the vector field ∇_i^ϕ has regular coefficients, and for $i = n$ it is consequence of (55). Let us now fix $s \in [-\delta, \delta]$ and $i = 1, \ldots, 2n-1$, denote $\bar{x} :=\ x$, $\gamma(t) :=\ \exp(st\,\nabla_i^\phi)(x)$ and $h(t) :=\ s\,e_i$ if $0 \le\ t \le 1$, . Then from (51), (66) follows . Now, by (66) and repeating verbatim the argument contained in the proof of [4, Proposition 3.7] , it can be proved that, at each point $x \in \omega$ where ϕ is ∇^ϕ-differentiable (see Definition 1.4)

$$\nabla_i^\phi \phi(x) = \lim_{s \to 0} \frac{\phi\left(\exp(s\,\nabla_i^\phi)(x)\right) - \phi(x)}{s} \quad \forall i = 1, \ldots, 2n-1\,. \tag{67}$$

Eventually, from Theorem 1.3, (66) and (67),(65) follows. $\square$

4.5. **Lemma** ([26]). *Let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then*

$$\partial_{*,\mathbb{H}} E_\phi \cap (\mathbb{R} \times \omega) = \partial E_\phi \cap (\mathbb{R} \times \omega) = graph(\phi) \tag{68}$$

and

$$\mathcal{S}^{2n+1}(\partial_{*,\mathbb{H}} E_\phi \setminus \partial_{\mathbb{H}}^* E_\phi) = 0. \tag{69}$$

4.6. **Proposition.** *Let $\omega \subset \mathbb{W}$ be open and bounded and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then for each $\varphi = (\varphi_1, \ldots, \varphi_{2n}) \in C_c^1(\mathbb{R} \times \omega, \mathbb{R}^{2n})$*

$$\int_{E_\phi} \mathrm{div}_{\mathbb{H}} \varphi \ \mathrm{d}\mathcal{L}^{2n+1} = \int_\omega \varphi_1 \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi} \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n} \tag{70}$$

where $\hat{\varphi} := (\varphi_2, \ldots, \varphi_{2n})$ and $\Phi : \omega \longrightarrow \mathbb{H}^n$ is as in (4).

Proof. We define $E := E_\phi$ the subgraph of ϕ and $\Omega := \omega \cdot \mathbb{R}e_1 = \mathbb{R} \times \omega$. By Theorem 1.3 E is a set of locally finite perimeter in $\mathbb{H}^n$, then there exists a unique $|\partial E|_{\mathbb{H}}$-measurable function $\nu_E : \Omega \longrightarrow \mathbb{R}^{2n}$ such that $|\nu_E|_{\mathbb{R}^{2n}} = 1$ $|\partial E|_{\mathbb{H}}$-a.e in Ω and

$$\int_E \mathrm{div}_{\mathbb{H}}\varphi \,\mathrm{d}\mathcal{L}^{2n+1} = -\int_\Omega \langle \varphi, \nu_E\rangle \,\mathrm{d}|\partial E|_{\mathbb{H}} \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^{2n}),\ |\varphi|_{\mathbb{R}^{2n}} \le 1.$$

By using Corollaries 4.2 and 4.3, we have that the first component $\nu_E^{(1)}$ of ν_E is such that $\nu_E^{(1)} < 0$ $|\partial E|_{\mathbb{H}}-$a.e in Ω.

Hence

$$\int_\Omega \langle \varphi, \nu_E\rangle \,\mathrm{d}|\partial E|_{\mathbb{H}} = \int_\Omega \frac{\langle \varphi, \nu_E\rangle}{\nu_E^{(1)}} \nu_E^{(1)} \,\mathrm{d}|\partial E|_{\mathbb{H}}$$

and by Corollary 4.3 we obtain

$$\int_\Omega \langle \varphi, \nu_E\rangle \,\mathrm{d}|\partial E|_{\mathbb{H}} = -\int_\Omega \frac{\langle \varphi, \nu_E\rangle}{\nu_E^{(1)}} \mathrm{d}\Phi_\#(\mathcal{L}^{2n} \llcorner \mathbb{W})$$

finally by a change of variables

$$\int_\Omega \frac{\langle \varphi, \nu_E\rangle}{\nu_E^{(1)}} \mathrm{d}\Phi_\#(\mathcal{L}^{2n} \llcorner \mathbb{W}) = \int_\omega \frac{\langle \nu_E \circ \Phi, \varphi \circ \Phi\rangle}{\nu_E^{(1)} \circ \Phi} \mathrm{d}\mathcal{L}^{2n}.$$

Now, by the characterization of the inward normal provided in Theorem 1.3 we have for every $\varphi \in C^1_c(\Omega, \mathbb{R}^{2n})$ with $|\varphi|_{\mathbb{R}^{2n}} \le 1$,

$$\begin{aligned}
\int_\Omega \langle \varphi, \nu_E\rangle \,\mathrm{d}|\partial E|_{\mathbb{H}} &= -\int_\omega \frac{\langle \nu_E \circ \Phi, \varphi \circ \Phi\rangle}{\nu_E^{(1)} \circ \Phi} \mathrm{d}\mathcal{L}^{2n} \\
&= -\int_\omega \varphi_1 \circ \Phi + \sum_{i=2}^{2n} \frac{(\nu_E \circ \Phi)_i (\varphi \circ \Phi)_i}{\nu_E^{(1)} \circ \Phi} \mathrm{d}\mathcal{L}^{2n} \\
&= -\int_\omega \varphi_1 \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi} \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n},
\end{aligned}$$

where $\hat{\varphi} = (\varphi_2, \dots, \varphi_{2n})$. Hence

$$-\int_\Omega \langle \varphi, \nu_E\rangle \,\mathrm{d}|\partial E|_{\mathbb{H}} = \int_\omega \varphi_1 \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi} \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n} \tag{71}$$

as desired. $\square$

Now we are going to prove that the gradient $\nabla^\phi \phi$ of a Lipschitz continuous function $\phi \in Lip_{\mathbb{W}}(\omega)$ also agrees with the distributional gradient. We emphasize the fact that the gradient $\nabla^\phi \phi$ exists almost everywhere thanks to Proposition 4.1.

4.7. **Proposition.** *Let $\omega \subset \mathbb{R}^{2n}$ be open and bounded and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then the pointwise intrinsic gradient $\nabla^\phi \phi$ is also distributional, that is, for each $\psi \in C^1_c(\omega)$*

(i) $\int_\omega \phi \ \nabla^\phi_i \psi \,\mathrm{d}\mathcal{L}^{2n} = -\int_\omega \nabla^\phi_i \phi \ \psi \ ,\mathrm{d}\mathcal{L}^{2n} \quad \forall i \ne n;$

(ii) $\int_\omega(\phi\,\partial_{n+1}\psi+\phi^2\,\partial_{2n+1}\psi)\,\mathrm{d}\mathcal{L}^{2n}=-\int_\omega\nabla^\phi_n\phi\ \psi\,\mathrm{d}\mathcal{L}^{2n}$.

Proof. Let us denote by $M:=\|\phi\|_{L^\infty(\omega)}<+\infty$. By standard considerations, there is a sequence $\{\phi_j\}_{j\in\mathbb{N}}\subset C^\infty_c(\omega)$ converging uniformly to ϕ on every $\omega'\Subset\omega$. Let us now prove that the sequence $(\nabla^{\phi_j}\phi_j)_j$ converges to $\nabla^\phi\phi$, in sense of distributions, that is

(72)
$$\int_\omega\big\langle\nabla^\phi\phi(x),\psi(x)\big\rangle\,\mathrm{d}\mathcal{L}^{2n}=\lim_{j\to\infty}\int_\omega\big\langle\nabla^{\phi_j}\phi_j(x),\psi(x)\big\rangle\,\mathrm{d}\mathcal{L}^{2n}\quad\forall\psi\in C^1_c(\omega,\mathbb{R}^{2n-1}).$$

We denote by $\Phi_j:\omega\longrightarrow\mathbb{H}^n$ the graph map of ϕ_j and by E_j the subgraph of ϕ_j. Therefore, by Proposition 4.6, we obtain that for every $\varphi=(\varphi_1,\dots,\varphi_{2n})\in C^1_c(\mathbb{R}\times\omega,\mathbb{R}^{2n})$
$$\int_\omega\varphi_1\circ\Phi-\big\langle\nabla^\phi\phi,\hat\varphi\circ\Phi\big\rangle\,\mathrm{d}\mathcal{L}^{2n}=\int_E\mathrm{div}_{\mathbb{H}}\varphi\ \mathrm{d}\mathcal{L}^{2n}.$$
By the uniform convergence of ϕ_j to ϕ we get
$$\int_E\mathrm{div}_{\mathbb{H}}\varphi\ \mathrm{d}\mathcal{L}^{2n}=\lim_{j\to\infty}\int_{E_j}\mathrm{div}_{\mathbb{H}}\varphi\ \mathrm{d}\mathcal{L}^{2n}$$
and applying again Proposition 4.6 to each ϕ_j, we obtain
$$\int_{E_j}\mathrm{div}_{\mathbb{H}}\varphi\ \mathrm{d}\mathcal{L}^{2n}=\int_\omega\varphi_1\circ\Phi_j-\big\langle\nabla^{\phi_j}\phi_j,\hat\varphi\circ\Phi_j\big\rangle\,\mathrm{d}\mathcal{L}^{2n},$$
where $\hat\varphi:=(\varphi_2,\dots,\varphi_{2n})$. Putting together the last three equalities we deduce
$$\int_\omega\varphi_1\circ\Phi-\big\langle\nabla^\phi\phi,\hat\varphi\circ\Phi\big\rangle\,\mathrm{d}\mathcal{L}^{2n}=\lim_{j\to+\infty}\int_\omega\varphi_1\circ\Phi_j-\big\langle\nabla^{\phi_j}\phi_j,\hat\varphi\circ\Phi_j\big\rangle\,\mathrm{d}\mathcal{L}^{2n}.$$
Clearly, if $\varphi_1=0$, this implies
$$\int_\omega\big\langle\nabla^\phi\phi,\hat\varphi\circ\Phi\big\rangle\,\mathrm{d}\mathcal{L}^{2n}=\lim_{j\to+\infty}\int_\omega\big\langle\nabla^{\phi_j}\phi_j,\hat\varphi\circ\Phi_j\big\rangle\,\mathrm{d}\mathcal{L}^{2n}.$$
We explicitly point out that we will deduce the weak converge of $\nabla^{\phi_j}\phi_j$ to $\nabla^\phi\phi$ from the uniform convergence of ϕ_j to ϕ and from Proposition 4.6, which is an integration by part type formula. This will be done with a suitable choice of φ. Indeed, if $\varphi((s,x)):=\psi(x)\xi(s)$ with $\psi=(\psi_2,\dots,\psi_{2n-1})\in C^1_c(\omega,\mathbb{R}^{2n-1})$ and $\xi\in C^1_c(\mathbb{R})$ such that $\xi(s)=1$ for all $s\in\mathbb{R}$ with $-M-1\le s\le M+1$, then $\varphi\in C^1_c(\mathbb{R}\times\omega,\mathbb{R}^{2n})$. Hence

(73)
$$\int_\omega\big\langle\nabla^\phi\phi(x),\psi(x)\xi(\phi(x))\big\rangle\,\mathrm{d}\mathcal{L}^{2n}=\lim_{j\to\infty}\int_\omega\big\langle\nabla^{\phi_j}\phi_j(x),\psi(x)\xi(\phi_j(x))\big\rangle\,\mathrm{d}\mathcal{L}^{2n}$$

and since ϕ_j converges uniformly to ϕ, there exist $\bar{j} \in \mathbb{N}$ such that for all $j \geq \bar{j}$ and for all x in the support of ψ,

$$-M - 1 \leq \phi_j(x) \leq M + 1$$

and hence $\xi(\phi_j(x)) = 1$ for all $j \geq \bar{j}$ and for all x in the support of ψ. This implies (72).

If $\psi(x) := (0, \dots, \psi_i(x), \dots, 0) \in C_c^1(\omega, \mathbb{R}^{2n-1})$ and $i \neq n$ then by (72) we obtain

$$\int_\omega \nabla_i^\phi \phi \ \psi_i \, \mathrm{d}\mathcal{L}^{2n} = \lim_{j\to\infty} \int_\omega \nabla_i^{\phi_j} \phi_j \ \psi_i \, \mathrm{d}\mathcal{L}^{2n} =$$
$$= - \lim_{j\to\infty} \int_\omega \phi_j \nabla_i^{\phi_j} \psi_i \mathrm{d}\mathcal{L}^{2n} = \int_\omega \phi \nabla_i^\phi \psi_i \mathrm{d}\mathcal{L}^{2n}$$

where we used the fact that, if $i \neq n$, then $\nabla_i^{\phi_j} \phi_j = \nabla_i^\phi \phi_j$. On the other hand if $i = n$ we obtain

$$\int_\omega \nabla_n^\phi \phi \ \psi_n \, \mathrm{d}\mathcal{L}^{2n} = \lim_{j\to\infty} \int_\omega \nabla_n^{\phi_j} \phi_j \ \psi_n \, \mathrm{d}\mathcal{L}^{2n} =$$
$$= - \lim_{j\to\infty} \int_\omega (\phi_j \partial_n \psi_n + \phi_j^2 \partial_{2n} \psi) \, \mathrm{d}\mathcal{L}^{2n} = - \int_\omega (\phi \partial_n \psi + \phi^2 \partial_{2n} \psi_n) \, \mathrm{d}\mathcal{L}^{2n}.$$

□

We end this Section proving the area formula stated in Theorem 1.6:

Proof of Theorem 1.6. Denoting by E the subgraph of ϕ and by Ω the cylinder $\mathbb{R} \times \omega$, being $|\partial E|_{\mathbb{H}}$ a Radon measure, a classical approximation result ensure the existence of a sequence

$$(\varphi_j)_{j\in\mathbb{N}} = ((\varphi_{j,1}, \dots, \varphi_{j,2n}))_{j\in\mathbb{N}} \subset C_c^1(\Omega, \mathbb{R}^{2n})$$

with $|\varphi_j|_{\mathbb{R}^{2n}} \leq 1$ such that

$$\varphi_j \to \nu_E \qquad |\partial E|_{\mathbb{H}} - \text{a.e in } \Omega$$

moreover by Corollaries 4.2 and 4.3 it is easy to see that

$$\varphi_j \circ \Phi \to \nu_E \circ \Phi \quad \mathcal{L}^{2n} - \text{a.e in } \omega.$$

Inserting this sequence in (71) of Prosposition 4.6 we obtain that for all $j \in \mathbb{N}$,

$$- \int_\Omega \langle \varphi_j, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} = \int_\omega \varphi_{j,1} \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi}_j \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n} \tag{74}$$

and the first part of the thesis follows taking the limit as $j \to \infty$ in (74). The fact that $|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = c_n \mathcal{S}^{2n+1}(\text{graph}(\phi))$ for some dimensional constant $c_n > 0$ is a direct consequence of Theorem 2.16 and Lemma 4.5. □

5. Approximation result

In this section we are going to prove Theorem 1.7. We will strictly follow here the approximation techniques contained in [37] and [43], which are extensions to Heisenberg setting of the classical De Giorgi's techniques for the Euclidean one [20].

Before stating the approximation Theorem we need to recall two results that will be fundamental in the proof.

5.1. **Theorem** ([45]). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a strictly convex function and let $(g_j)_j$ and g be in $(L^1(\Omega))^n$. If*

(1) *$g_j \to g$ weakly in $(L^1(\Omega))^n$;*
(2) *$\int_\Omega f \circ g_j \; \mathrm{d}\mathcal{L}^n \to \int_\Omega f \circ g \; \mathrm{d}\mathcal{L}^n$*

then $g_j \to g$ strongly in $(L^1(\Omega))^n$.

5.2. **Lemma.** *([43]) Suppose that $M > 0$ $c > 0$ and $u \in C^1((-M, M) \times \omega, \mathbb{R}) \cap C^0([-M, M] \times \omega)$ are such that $\nabla_1^{\mathbb{H}} u \leq 0$ and*

$$u(x, M) > c, \quad u(x, -M) \leq 0 \quad \forall x \in \omega.$$

Assume also that $\nabla_1^{\mathbb{H}} u(p) < 0$ on the set $A = \{p \in (-M, M) \times \omega : u(p) = c$. Then there exists $\phi : \omega \longrightarrow (-M, M)$ such that ϕ is $\nabla^\phi-$differentiable in ω and

$$\{u > c\} \cap (-M, M) \times \omega = E_\phi \cap (-M, M) \times \omega.$$

We are now in position to prove Theorem 1.7.

Proof of Theorem 1.7. Let us assume firstly that $\phi : \mathbb{W} \longrightarrow \mathbb{R}$. Let $M := \|\phi\|_{L^\infty(\mathbb{W})} < +\infty$. For each $\alpha > 0$ we define $u_\alpha : \mathbb{H}^n \longrightarrow \mathbb{R}$ by

$$\begin{aligned} u_\alpha(p) := (\rho_\alpha * \chi_{E_\phi})(p) &= \int_{\mathbb{H}^n} \rho_\alpha(p \cdot q^{-1}) \chi_{E_\phi}(q) \; \mathrm{d}\mathcal{L}^{2n+1}(q) \\ &= \int_{\mathbb{H}^n} \rho_\alpha(q) \chi_{E_\phi}(q^{-1} \cdot p) \; \mathrm{d}\mathcal{L}^{2n+1}(q) \end{aligned} \tag{75}$$

where $\rho_\alpha(x) := \alpha^{2n+2} \rho(\delta_{1/\alpha}(x))$ and $\rho \in C_c^\infty(U(0,1))$ is a smooth mollifier with $\rho(p^{-1}) = \rho(-p) = \rho(p)$ $\forall p \in \mathbb{H}^n$. Namely let us exploit the classical technique of approximation by convolution in $\mathbb{H}^n$ introduced in [22] of which the main properties are collected in [22] and [43, Lemma 2.4].

Claim 0. *Let us first show that u_α is constant far from the graph of ϕ, so that the integral (75) is indeed extended only in a neighborhood of the graphs itself.*

To this end, for each $\alpha > 0$ it follows that $u_\alpha \in C_c^\infty(\mathbb{H}^n)$ and

$$spt(u_\alpha) \subset U(0,\alpha) \cdot spt(\chi_{E_\phi}).$$

Moreover, let us observe that for each $\alpha > 0$

$$0 \le u_\alpha(p) \le 1 \ \forall p \in \mathbb{H}^n;$$

and for all sufficiently small $\alpha > 0$

$$u_\alpha(p) = 1 \quad \forall p \in (-\infty, -2M] \times \mathbb{W}. \tag{76}$$

Notice also that E_ϕ is open in $\mathbb{H}^n$ and

$$\begin{aligned} spt(\chi_{E_\phi}) = \overline{E}_\phi &\subseteq \{(s,x) \mid x \in \mathbb{R}^{2n},\ s \le \phi(x)\} \\ &\subseteq (-\infty, M] \times \mathbb{W}. \end{aligned} \tag{77}$$

Hence

$$spt(u_\alpha) \subseteq \bar{U}_\alpha \cdot spt(\chi_{E_\phi}) \subseteq (-\infty, 2M) \times \mathbb{W} \tag{78}$$

for $\alpha < M$. In particular, (78) implies

$$u_\alpha(p) = 0 \quad \forall\ p \in [2M, +\infty) \times \mathbb{W}. \tag{79}$$

Claim 1. *Let us compute explicitly* $\nabla_1^{\mathbb{H}} u_\alpha$.

Let $\varphi \in C_c^\infty((-3M, 3M) \times \omega)$, then

(80)

$$\begin{aligned} \left\langle \nabla_1^{\mathbb{H}} u_\alpha, \varphi \right\rangle &= -\int_{(-3M,3M)\times\omega} u_\alpha(p') \nabla_1^{\mathbb{H}} \varphi(p') \mathrm{d}\mathcal{L}^{2n+1}(p') \\ &= -\int_{\bar{U}_\alpha} \rho_\alpha(p) \mathrm{d}\mathcal{L}^{2n+1}(p) \int_{(-3M,3M)\times\omega} \chi_{E_\phi}(p^{-1} \cdot p') \nabla_1^{\mathbb{H}} \varphi(p') \mathrm{d}\mathcal{L}^{2n+1}(p') \\ &= -\int_{\bar{U}_\alpha} \rho_\alpha(p) \mathrm{d}\mathcal{L}^{2n+1}(p) \int_{p^{-1}\cdot((-3M,3M)\times\omega)} \chi_{E_\phi}(q) \nabla_1^{\mathbb{H}} \varphi(p \cdot q) \mathrm{d}\mathcal{L}^{2n+1}(q). \end{aligned}$$

With the notation $\varphi_p(q) = \varphi(p \cdot q)$ we have $\nabla_1^{\mathbb{H}}(\varphi(p \cdot q)) = \nabla_1^{\mathbb{H}} \varphi_p(q)$, because $\nabla_1^{\mathbb{H}}$ is left-invariant; moreover $\varphi_p \in C_c^\infty(p^{-1} \cdot ((-3M, 3M) \times \omega))$, then

(81)

$$\begin{aligned} &\left\langle \nabla_1^{\mathbb{H}} u_\alpha, \varphi \right\rangle = \\ &= -\int_{\bar{U}_\alpha} \rho_\alpha(p) \mathrm{d}\mathcal{L}^{2n+1}(p) \int_{p^{-1}\cdot((-3M,3M)\times\omega)} \chi_{E_\phi}(q) \nabla_1^{\mathbb{H}} \varphi_p(q) \mathrm{d}\mathcal{L}^{2n+1}(q). \end{aligned}$$

Put $\mathcal{C}(p,3M) := p^{-1}\cdot((-3M,3M)\times\omega)$ then by an integration by parts, we have

(82)
$$\int_{\mathcal{C}(p,3M)} \chi_{E_\phi}(q)\nabla_1^{\mathbb{H}}\varphi_p(q)\mathrm{d}\mathcal{L}^{2n+1}(q) = -\int_{\mathcal{C}(p,3M)} \nu^1_{E_\phi}(q)\varphi_p(q)\mathrm{d}|\partial E_\phi|(q)$$

where $\nu^1_{E_\phi}$ is the first component of the horizontal inward normal $\nu_{E_\phi} = (\nu^1_{E_\phi},\dots,\nu^{2n}_{E_\phi})$ to E_ϕ.

Because $spt(\varphi_p) \Subset \mathcal{C}(p,3M)$ and $p \in \overline{U}_\alpha$ if α is small enough, we can replace $\mathcal{C}(p,3M)$ by $\mathcal{C}(0,3M)$. Thus, by Fubini-Tonelli Theorem and a change of variable, we obtain

$$\left\langle \nabla_1^{\mathbb{H}}u_\alpha,\varphi\right\rangle = \int_{\mathcal{C}(0,3M)} \nu^1_{E_\phi}(q)\mathrm{d}|\partial E_\phi|(q)\Big(\int_{\mathbb{H}^n}\rho_\alpha(p)\varphi(p\cdot q)\mathrm{d}\mathcal{L}^{2n+1}(p)\Big).$$

Then for each $p \in \mathcal{C}(0,3M) = (-3M,3M)\times\omega$ and for all $\alpha > 0$ small enough

$$\begin{aligned}(83)\qquad \nabla_1^{\mathbb{H}}u_\alpha(p) &= \int_{\mathcal{C}(0,3M)} \rho_\alpha(p\cdot q^{-1})\nu_1(q)\mathrm{d}|\partial E_\phi|(q) = \\ &= \int_{U^R(p,\alpha)} \rho_\alpha(p\cdot q^{-1})\nu_1(q)\mathrm{d}|\partial E_\phi|(q)\end{aligned}$$

where $U^R(p,\alpha) := U(0,\alpha)\cdot p$.

In particular we immediately deduce from (83) the following assertion. *For each couple (ω,ω_0) of open and bounded subset of $\mathbb{W}$ with $\omega_0 \Supset \omega$ there exists $\bar\alpha = \bar\alpha(\omega_0) > 0$ such that for all $0<\alpha<\bar\alpha$*

$$(84)\qquad \int_{(-2M,2M)\times\omega} |\nabla_{\mathbb{H}}u_\alpha|\mathrm{d}\mathcal{L}^{2n+1} \le |\partial E_\phi|((-2M,2M)\times\omega_0).$$

Claim 2. *For every fixed α and $c\in(0,1)$ the set*

$$A = \{p \in (-2M,2M)\times\omega : u_\alpha(p) = c\}$$

implicitly defines a function ϕ_α. This family has a subsequence $\{\phi_k\}_k$ such that $|\nabla^{\phi_k}\phi_k| \le ||\nabla^\phi\phi||_{L^\infty(\omega)}$ $\forall k\in\mathbb{N}$ on $\omega\subset\mathbb{W}$.

From Claim 1 we will first deduce that

$$(85)\qquad \nabla_1^{\mathbb{H}}u_\alpha(p) < 0 \quad \forall p \in A.$$

Indeed, recalling that (see Corollary 4.2)

$$\nu_1\circ\Phi = -\frac{1}{\sqrt{1+|\nabla^\phi\phi|^2}} \quad \text{in } \omega$$

and denoting by

$$I_\alpha(p) := \int_{U^R(p,\alpha)} \rho_\alpha(p \cdot q^{-1}) \mathrm{d}|\partial E_\phi|(q) \tag{86}$$

we obtain

$$\nabla_1^{\mathbb{H}} u_\alpha(p) \le -\frac{1}{\sqrt{1+\|\nabla^\phi \phi\|^2_{L^\infty(\omega)}}} I_\alpha(p) \quad \forall p \in (-3M, 3M) \times \omega. \tag{87}$$

In order to prove (85) for every $c \in (0,1)$ let us define

$$E_\alpha = E_{\alpha,c} := \{p \in \mathbb{R} \times \omega \mid u_\alpha(p) > c\}$$

and notice that for each $p \in (-2M, 2M) \times \omega$ with $u_\alpha(p) = c$

$$\mathcal{L}^{2n+1}(U^R(p,\alpha) \cap E_\phi) > 0 \qquad \mathcal{L}^{2n+1}(U^R(p,\alpha) \cap E_\phi^c) > 0. \tag{88}$$

Otherwise, by contradiction, assume, for instance, that $\mathcal{L}^{2n+1}(U^R(p,\alpha)\cap E_\phi) = 0$. Then, since E_ϕ is open, we can assume $U^R(p,\alpha)\cap E_\phi = \emptyset$. By definition of convolution, it follows that $u_\alpha(p) = 0$ and then a contradiction. Analogously, it follows that $u_\alpha(p) = 1$ if $\mathcal{L}^{2n+1}(U^R(p,\alpha)\cap E_\phi^c) = 0$.

By (88) and Theorem 2.14, we have $|\partial E_\phi|(U^R(p,\alpha)) > 0$ $\forall p \in (-2M, 2M) \times \omega$ with $u_\alpha(p) = c$, then

$$I_\alpha(p) > 0 \quad p \in (-2M, 2M) \times \omega \text{ with } u_\alpha(p) = c. \tag{89}$$

From (87) and (89), (85) follows. Applying Lemma 5.2 we deduce that there is a function $\phi_\alpha : \omega \longrightarrow [-2M, 2M]$ such that

$$E_\alpha \cap ([-2M, 2M] \times \omega) = E_{\phi_\alpha} \cap ([-2M, 2M] \times \omega), \tag{90}$$

From (79), (76), it follows that

$$\partial E_\alpha \cap (\mathbb{R} \times \omega) = \{p \in [-2M, 2M] \times \omega \mid u_\alpha(p) = c\} = \Phi_\alpha(\omega) \tag{91}$$

where $\Phi_\alpha : \omega \longrightarrow \mathbb{H}^n$ is the graph map defined as in (4).

We can now estimate from above the gradient of ϕ_α. Letting $\hat{\nabla}_{\mathbb{H}} u_\alpha := (\nabla_2^{\mathbb{H}} u_\alpha, \dots, \nabla_{2n}^{\mathbb{H}} u_\alpha)$, $\hat{\nu}_{E_\phi} = (\nu^2_{E_\phi}, \dots, \nu^{2n}_{E_\phi})$ and arguing as in Claim 1 we get,

$$\begin{aligned} |\nabla^{\phi_\alpha}\phi_\alpha| = & \frac{|\hat{\nabla}_{\mathbb{H}} u_\alpha(p)|}{|\nabla_1^{\mathbb{H}} u_\alpha(p)|} \le \\ & \frac{1}{|\nabla_1^{\mathbb{H}} u_\alpha(p)|} \int_{U^R(p,\alpha)} |\hat{\nu}_{E_\phi}(q)||\rho_\alpha(p \cdot q^{-1})| \, \mathrm{d}|\partial E_\phi|(q) \\ & \le I_\alpha(p) \frac{\|\nabla^\phi \phi\|_{L^\infty(U^R(p,\alpha))}}{|\nabla_1^{\mathbb{H}} u_\alpha(p)| \sqrt{1+\|\nabla^\phi \phi\|^2_{L^\infty(U^R(p,\alpha))}}} \\ & \le \|\nabla^\phi \phi\|_{L^\infty(U^R(p,\alpha))}, \end{aligned} \tag{92}$$

the last inequality being a consequence of (87). It follows that for all $\alpha > 0$

$$|\nabla^{\phi_\alpha}\phi_\alpha| \leq \|\nabla^{\phi}\phi\|_{L^\infty(\omega)} \text{ in } \omega. \tag{93}$$

Claim 3. *There is a constant L only dependent on $\|\nabla^{\phi}\phi\|_{L^\infty(\omega)}$ such that for each $\alpha > 0$ sufficiently small and each $x \in \omega$ it holds $|\phi_\alpha(x) - \phi(x)| \leq L\alpha$. Hence $\{\phi_\alpha\}$ coverges uniformly on ω.*

This is a direct consequence of Theorem 3.5. Indeed for β fixed as in Theorem 3.5, there exists $L > 0$ only dependent on β such that

$$U((-L,0),1) \subset \{q : \|\pi_{\mathbb{W}}(q)\| < -\beta\pi_{\mathbb{V}}(q)\}$$

$$U((L,0),1) \subset \{q : \|\pi_{\mathbb{W}}(q)\| < \beta\pi_{\mathbb{V}}(q)\}.$$

Let $p(x) := (\phi(x), x)$, $p'(x) = (\phi(x) - \alpha L, x)$ and $p''(x) = (\phi(x) + \alpha L, x)$ with $x \in \omega$ and and $\alpha \in (0,1]$. Observe that it holds:

$$\|p'(x)\| \leq M + L + (2n-1)C + \sqrt{C} \quad \forall x \in \omega \tag{94}$$

where $C := \max_{1\leq i\leq n}(\max_{x\in\omega}|x_i|) < +\infty$. Moreover, by standard considerations (see [29, 38]), we know that for each $p \in U(0, r_0)$ there exists $c = c(r_0) > 0$ such that

$$U^R(p,r) \subset U(p, c(r_0)\sqrt{r}) \quad \forall r \in (0,1). \tag{95}$$

Hence, if $x \in \omega$ and $\alpha \in (0,1]$, by (94) and (95), we conclude:

(96)

$$U^R(p'(x),\alpha) \subset U(p'(x), c(r_0)\sqrt{\alpha}) \subset \{q : \|\pi_{\mathbb{W}}(p(x)^{-1}\cdot q)\| < -\beta\pi_{\mathbb{V}}(p^{-1}\cdot q)\} \subset E$$

$$U^R(p''(x),\alpha) \subset U(p''(x), c(r_0)\sqrt{\alpha}) \subset \{q : \|\pi_{\mathbb{W}}(p(x)^{-1}\cdot q)\| < \beta\pi_{\mathbb{V}}(p^{-1}\cdot q)\} \subset \mathbb{H}^n - E,$$

where $r_0 := M + L + (2n-1)C + \sqrt{C} > 0$. In particular, by definition of u_α

$$u_\alpha(\phi(x) - \alpha L, x) = 1, \quad u_\alpha(\phi(x) + \alpha L, x) = 0,$$

and by definition of ϕ_α and (85) we conclude that:

$$\phi(x) - \alpha L \leq \phi_\alpha(x) \leq \phi(x) + \alpha L \quad \forall \alpha \in (0,1], \forall x \in \omega.$$

Claim 4. *There exists a positive sequence $(\alpha_h)_h$ such that, if $\phi_h \equiv \phi_{\alpha_h}$ then*

$$\nabla^{\phi_h}\phi_h(x) \to \nabla^{\phi}\phi(x) \quad \mathcal{L}^{2n} - a.e\ x \in \omega. \tag{97}$$

In order to get (97), we need only to prove that there exists a positive sequence $(\alpha_h)_h$ converging to 0 such that there exists

$$\lim_{h\to\infty}\int_\omega \sqrt{1+|\nabla^{\phi_h}\phi_h|^2}\mathrm{d}\mathcal{L}^{2n} = \int_\omega \sqrt{1+|\nabla^{\phi}\phi|^2}\mathrm{d}\mathcal{L}^{2n} \tag{98}$$

where $\phi_h \equiv \phi_{\alpha_h}$. Indeed, up to subsequence, by (93) and Proposition 4.7 we can assume , that the sequence in (98) also satisfies

$$\nabla^{\phi_h}\phi_h \to \nabla^\phi\phi \text{ weakly in } (L^1(\omega))^{2n-1}. \tag{99}$$

Then, by Theorem 5.1, it follows that

$$\nabla^{\phi_h}\phi_h \to \nabla^\phi\phi \text{ strongly in } (L^1(\omega))^{2n-1}. \tag{100}$$

Therefore, up to a subsequence, (97) follows. Let us now prove (98). It is sufficient to show that there exists $\bar{c} \in (0,1)$ and $(\alpha_h)_h \subset (0,+\infty)$ converging to 0 such that

$$\exists \lim_{h\to\infty} |\partial E_{\alpha_h,\bar{c}}|_{\mathbb{H}}((-2M,2M)\times\omega) = |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega). \tag{101}$$

In fact, by Proposition 1.6 and well-known $\mathbb{H}-$perimeter properties

$$\begin{aligned}
&\int_\omega \sqrt{1+|\nabla^{\phi}\phi|^2}\mathrm{d}\mathcal{L}^{2n} = |\partial E_\phi|_{\mathbb{H}}(\mathbb{R}\times\omega) = \\
&= |\partial E_\phi|_{\mathbb{H}}((-\infty,2M]\times\omega) + |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega) + \\
&+ |\partial E_\phi|_{\mathbb{H}}([2M,+\infty)\times\omega) = \\
&= |\partial E_\phi|_{\mathbb{H}}((-\infty,-2M]\times\omega\cap\partial E_\phi) + |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega) + \\
&+ |\partial E_\phi|_{\mathbb{H}}([2M,+\infty)\times\omega\cap\partial E_\phi) = \\
&= |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega),
\end{aligned} \tag{102}$$

where in the last equality we have used the inequality $|\phi| \le M$ which implies $(-\infty,-2M]\times\omega\cap\partial E_\phi = [2M,+\infty)\times\omega\cap\partial E_\phi = \emptyset$. Analogously, by (90),(76) and (79)

$$|\partial E_{\alpha_h,c}|_{\mathbb{H}}((-2M,2M)\times\omega) = |\partial E_{\phi_h}|_{\mathbb{H}}((-2M,2M)\times\omega) \tag{103}$$

and

$$\int_\omega \sqrt{1+|\nabla^{\phi_h}\phi_h|^2}\mathrm{d}\mathcal{L}^{2n} = |\partial E_{\phi_h}|(\mathbb{R}\times\omega) \tag{104}$$

$$= |\partial E_{\phi_h}|_{\mathbb{H}}((-2M,2M)\times\omega), \tag{105}$$

where $\phi_h = \phi_{\alpha_h}$. Therefore (102),(103) and (104) imply (98). Finally let us prove (101). We will follow the technique exploited in [43]. Notice that, by the semicontinuity of $\mathbb{H}-$perimeter measure and Claim 3, we have

$$|\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega) \le \liminf_{\alpha\to 0^+} |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M,2M)\times\omega) \tag{106}$$

for each $c\in(0,1)$. On the other hand, by (106) and the coarea formula it follows that

$$\begin{aligned}|\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega) &\le \int_0^1 \liminf_{\alpha\to 0^+} |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M,2M)\times\omega)\mathrm{d}c \le \\ &\le \liminf_{\alpha\to 0^+}\int_0^1 |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M,2M)\times\omega)\mathrm{d}c = \\ &= \liminf_{\alpha\to 0^+}\int_{(-2M,2M)\times\omega} |\nabla_{\mathbb{H}} u_\alpha|\mathrm{d}\mathcal{L}^{2n+1} =: I(\omega,c).\end{aligned}$$

Now, for each $\omega_0 \Supset \omega$ open and bounded, by Claim 2, it holds

$$I(\omega,c) \le |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega_0). \tag{107}$$

Indeed, by Claim 2, for each $\omega_0 \Supset \omega$ open and bounded there exists a sequence $\{\alpha_h\}_h \subset (0,+\infty)$ which converges to 0 and $\bar h = \bar h(\omega_0) > 0$ such that for each $h \le \bar h$

$$\int_{(-2M,2M)\times\omega} |\nabla u_{\alpha_h}|\mathrm{d}\mathcal{L}^{2n+1} \le |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega_0). \tag{108}$$

Hence

$$I(\omega,c) \le |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega_0) \tag{109}$$

for each $c\in(0,1)$ and each $\omega_0 \Supset \omega$ open and bounded. Moreover, since $|\partial E_\phi|_{\mathbb{H}}$ is a Radon measure then by a standard approximation argument we can rewrite (107) with ω instead of ω_0. Using again (106), we obtain that $\mathcal{L}^1-$a.e $c\in(0,1)$

$$\liminf_{\alpha\to 0} |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M,2M)\times\omega) = |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega).$$

In particular there exists $\bar c\in(0,1)$ and a positive sequence $(\alpha_h)_h$ converging to 0 such that (101) holds.

We conclude the proof proving that the assumption $\phi:\mathbb{W}\longrightarrow\mathbb{R}$ can be relaxed to $\phi:\omega\longrightarrow\mathbb{R}$ where $\omega\subset\mathbb{W}$ is open and bounded. Indeed, by (8) ϕ is locally uniformly continuous on ω. Thus ϕ can be extended to a continuous function $\phi:\overline{\omega}\to\mathbb{V}\equiv\mathbb{R}$ and let $M:=\sup_\omega|\phi| < +\infty$. By Theorem 1.3, there exists a Lipschitz extension $\overline{\phi}:\mathbb{W}\equiv\mathbb{R}^{2n}\to\mathbb{V}\equiv\mathbb{R}$ of ϕ. Define $\phi^*:\mathbb{W}\to\mathbb{V}\equiv\mathbb{R}$ by

$$\phi^*(x) = \max\{\min\{\overline{\phi}(x),\, M\},\, -M\} \quad x\in\mathbb{W}\,.$$

Theorem 1.3 yields that ϕ^* is a bounded Lipschitz function, which still extends ϕ. Applying the previous part of the proof to ϕ^* we get the thesis. $\square$

6. Application

In this section we provide a characterization of $Lip_{\mathbb{W}}(\omega)$ in terms of approximating sequences. We first give the proof of Proposition 1.8.

Proof of Proposition 1.8. Estimate (9) follows from (66). Let us fix $\phi \in Lip_{\mathbb{W}}(\omega)$, $\bar{x} \in \omega$ and $0 < r < \bar{r}/(2C_2)$, here $\bar{r}$ and C_2 are as in Proposition 3.8. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a sequnce of smooth functions as in Theorem 1.7. For every i sufficiently big $U_{cc,\phi_i}(\bar{x}, r/2) \subset \omega$. Moreover, by [28, Theorem 2.7], $\forall x, y \in U_{cc,\phi_i}(\bar{x}, r/2)$

$$|\phi_i(x) - \phi_i(y)| \le \|\nabla^{\phi_i}\phi_i\|_{L^\infty(U_{cc,\phi_i}(\bar{x},r/2)))}\mathrm{d}_{cc,\phi_i}(x, y) \tag{110}$$

so that

$$Lip_{cc}(\phi_i, U_{cc,\phi_i}(\bar{x}, r/2)) \le \|\nabla^{\phi_i}\phi_i\|_{L^\infty(U_{cc,\phi_i}(\bar{x},r/2))} \quad \forall i \in \mathbb{N}. \tag{111}$$

Viceversa, by definition of differential it follows that

$$\|\nabla^{\phi_i}\phi_i\|_{L^\infty(U_{cc,\phi_i}(\bar{x},r/2)))} \le Lip_{cc}(\phi_i, U_{cc,\phi_i}(\bar{x}, r/2)) \quad \forall i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$ let $\bar{r}_i, C_1^i$ and C_2^i be as in Proposition 3.8. Since every ϕ_i is Lipschitz continuous on $U_{cc,\phi_i}(\bar{x}, r/2)$ with respect to the distance d_{cc,ϕ_i} then it is clear, from the proof of Proposition 3.8 and (111), that each $\bar{r}_i, C_1^i, C_2^i$ depend only on $\|\nabla^{\phi_i}\phi_i\|_{L^\infty(\omega)}$. By Theorem 1.7, the sequence $\{\|\nabla^{\phi_i}\phi_i\|_{L^\infty(\omega)}\}_{i\in\mathbb{N}}$ is bounded, hence we can as well choose $\bar{r}, C_1$ and C_2 independent of i and dependent only on $\|\nabla^\phi\phi\|_{L^\infty(\omega)}$ such that

$$\begin{aligned} |\phi_i(x) - \phi_i(y)| &\le \|\nabla^\phi\phi\|_{L^\infty(\omega)}\mathrm{d}_{cc,\phi_i}(x, y) \quad \forall x, y \in U_{\phi_i}(\bar{x}, r/2C_1) \\ &\le C_2\|\nabla^\phi\phi\|_{L^\infty(\omega)}\mathrm{d}_{\phi_i}(x, y) \end{aligned}$$

letting $i \to +\infty$ and using Theorem 1.7 and the explicit expression of d_{ϕ_i} (see (27)) we conclude that $\forall x, y \in U_\phi(\bar{x}, r/2C_1)$

$$|\phi(x) - \phi(y)| \le C_2\|\nabla^\phi\phi\|_{L^\infty(\omega)}\mathrm{d}_\phi(x, y) \tag{112}$$

which implies (10).
To prove (11), we use the fact, recalled in in Definition 3.4, that the cone opening is the inverse of the Lipschitz constant, and the estimate of the cone opening provided in Theorem 3.5, with $k = -\frac{1}{\sqrt{1+\|\nabla^\phi\phi\|^2_{L^\infty(\omega)}}}$ $\square$

Using this fact, from proposition 3.8 we immediately get:

6.1. **Theorem** (Characterization of locally intrinsic Lipschitz functions)**.** *Let $\omega \Subset \mathbb{W}$ be open and bounded, and let $\phi : \omega \to \mathbb{R}$. Then the following are equivalent:*

(i) $\phi \in Lip_{\mathbb{W},loc}(\omega)$*;*

(ii) *there exist* $\{\phi_k\}_{k\in\mathbb{N}} \subset C^\infty(\omega)$ *and* $w \in (L^\infty_{loc}(\omega))^{2n-1}$ *such that*

(ii$_1$) $\{\phi_k\}_{k\in\mathbb{N}}$ *uniformly converges to* ϕ *on the compact sets of* ω*;*

(ii$_2$) *for each* $\omega' \Subset \omega$ *there exists* $C = C(\omega')$ *such that* $|\nabla^{\phi_k}\phi_k(x)| \le C \quad \mathcal{L}^{2n}$*-a.e.* $x \in \omega'$, $k \in \mathbb{N}$*;*

(ii$_3$) $\nabla^{\phi_k}\phi_k(x) \longrightarrow w(x)\ \mathcal{L}^{2n}-a.e\ x \in \omega$*.*

Moreover if (ii) holds, then $w \equiv \nabla^\phi\phi\ \mathcal{L}^{2n}-a.e$ *in* ω*.*

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