INTRINSIC LIPSCHITZ GRAPHS IN HEISENBERG GROUPS AND CONTINUOUS SOLUTIONS OF BALANCE EQUATIONS

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ABSTRACT. In this paper we provide a characterization of intrinsic Lipschitz graphs in the sub-Riemannian Heisenberg groups in terms of their distributional gradients. Moreover, we prove the equivalence of different notions of continuous weak solutions to the equation $\phi_y + [\phi^2/2]_t = w$, where w is a bounded function.

CONTENTS

1. INTRODUCTION

In the last years it has been largely developed the study of intrinsic submanifolds inside the Heisenberg groups $\mathbb{H}^n$ or more general Carnot groups, endowed with their Carnot-Carathéodory metric structure, also named sub-Riemannian. By an intrinsic regular (or intrinsic Lipschitz) hypersurfaces we mean a submanifold which has, in the intrinsic geometry of $\mathbb{H}^n$, the same role like a C^1 (or Lipschitz) regular graph has in the Euclidean geometry. Intrinsic regular graphs had several applications

within the theory of rectifiable sets and minimal surfaces in CC geometry, in theoretical computer science, geometry of Banach spaces and mathematical models in neurosciences.

We postpone complete definitions of $\mathbb{H}^n$ to Section 2. We only remind that the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ is the simplest example of Carnot group, endowed with a left-invariant metric d_∞ (equivalent to its Carnot-Carathéodory metric), not equivalent to the Euclidean metric. $\mathbb{H}^n$ is a (connected, simply connected and stratified) Lie group and has a sufficiently rich compatible underlying structure, due to the existence of intrinsic families of left translations and dilations and depending to the horizontal fields $X_1, ..., Y_n$. We call intrinsic any notion depending directly by the structure and geometry of $\mathbb{H}^n$. For a complete description of Carnot groups [6, 16, 21, 22, 23] are recommended.

As we said, we will study intrinsic submanifolds in $\mathbb{H}^n$. An intrinsic regular hypersurface $S \subset \mathbb{H}^n$ is locally defined as the non critical level set of an horizontal differentiable function, more precisely there exists locally a continuous function $f : \mathbb{H}^n \to \mathbb{R}$ such that $f(P) = 0$, there exists in the sense of distributions $\nabla_{\mathbb{H}} f = (X_1 f, \ldots, Y_n f)$, it is continuous and non-vanishing for $P \in S$. Intrinsic regular hypersurfaces can be locally be represented as X_1-graph by a function $\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{R}$, where $\mathbb{W} = \{x_1 = 0\}$, through an implicit function theorem (see [12]). In [2, 4, 5] the parametrization ϕ has been characterized as weak solution of a system of non linear first order PDEs $\nabla^\phi \phi = w$, where $w : \omega \to \mathbb{R}^{2n-1}$ and $\nabla^\phi = (X_2, \ldots, X_n, \partial_y + \phi \partial_t, Y_2, \ldots, Y_n)$, see Theorem 2.7. By an intrinsic point of view, the operator $\nabla^\phi \phi$ is the intrinsic gradient of the function $\phi : \mathbb{W} \to \mathbb{R}$. In particular, [5] shows that ϕ is a continuous distributional solution of the problem $\nabla^\phi \phi = w$ with $w \in C^0(\omega, \mathbb{R}^{2n-1})$ if and only if ϕ induces an intrinsic regular graph.

Let us point out that an intrinsic regular graph can be very irregular from the Euclidean point of view: indeed, there are examples of intrinsic regular graphs in $\mathbb{H}^1$ which are fractal in the Euclidean sense ([19]).

The aim of our work is to characterize Intrinsic Lipschitz graphs in terms of the intrinsic distributional gradient. In the Euclidean setting, Lipschitz graphs can be either defined

- by means of cones: there exists a cone which translated to any point of the surface locally intersects it only in the vertex;
- in a metric way: there exists $L > 0$ such that $|\phi(x) - \phi(y)| \le L|y - x|$ for every $x, y \in \omega$;
- by the distributional derivatives, which must be absolutely continuous measures with L^∞ density.

Intrinsic Lipschitz graphs in $\mathbb{H}^n$ have been introduced and studied in [15]. In particular the equivalence of the first two points for intrinsic Lipschitz graphs has been established. A subset $S \subset \mathbb{H}^n$ is intrinsic Lipschitz if at each point $P \in S$ there is an intrinsic cone with vertex P and fixed opening, intersecting S only in P. As consequence, the metric definition (see Definition 2.8) is given with respect to the the graph quasidistance d_ϕ, see (2.5), i.e the function $\phi : (\omega, d_\phi) \to \mathbb{R}$ is meant Lipschitz in classical metric sense. This notion turned out to be the right one in the setting of the intrinsic rectifiability in $\mathbb{H}^n$. Indeed it was proved in [15] that the notion of rectifiable set in terms of an intrinsic regular hypersurfaces is equivalent to the one in terms of intrinsic Lipschitz graphs.

We will denote by $\text{Lip}_{\mathbb{W}}(\omega)$ the class of all intrinsic Lipschitz function $\phi : \omega \to \mathbb{R}$ and by $\text{Lip}_{\mathbb{W},\text{loc}}(\omega)$ the one of locally intrinsic Lipschitz function. Notice that $\text{Lip}_{\mathbb{W}}(\omega)$ is not a vector space and that

$$\text{Lip}(\omega) \subsetneq \text{Lip}_{\mathbb{W},\text{loc}}(\omega) \subsetneq C^{0,1/2}_{\text{loc}}(\omega),$$

where $\text{Lip}(\omega)$ and $C^{0,1/2}_{\text{loc}}(\omega)$ denote respectively the classes of Euclidean Lipschitz and $1/2$-Hölder functions in ω. For a complete presentation of intrinsic Lipschitz graphs and functions [8, 15, 20] are recommended.

The first main result of this paper is the characterization of a parametrization $\phi : \omega \to \mathbb{R}$ of an intrinsic Lipschitz graph as a continuous distributional solution of $\nabla^\phi \phi = w$, where $w \in L^\infty(\omega, \mathbb{R}^{2n-1})$.

Theorem 1.1. *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open set, $\phi : \omega \to \mathbb{R}$ be a continuous function and $w \in \mathrm{L}^\infty(\omega; \mathbb{R})$. $\phi \in \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega; \mathbb{R})$ if and only if there exists $w \in \mathrm{L}^\infty(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a distributional solution of the system $\nabla^\phi \phi = w$.*

We stress that this is indeed different from proving a Rademacher theorem, which is more related to a pointwise rather than distributional characterization for the derivative, see [15]. Nevertheless, we find that the density of the (intrinsic) distributional derivative is indeed given by the function one finds by Rademacher theorem. We also stress that there are a priori different notions of *continuous* solutions $\phi : \omega \to \mathbb{R}$ to $\nabla^\phi \phi = w$, which express the Lagrangian and Eulerian viewpoints. They will turn out to be equivalent descriptions of intrinsic Lipschitz graphs, when the source w belongs to $\mathrm{L}^\infty(\omega; \mathbb{R}^{2n-1})$. This is proved in Section 6 and it is summarized as follows.

Theorem 1.2. *The following conditions are equivalent*

- (i) *ϕ is a distributional solution of the system $\nabla^\phi \phi = w$ with $w \in \mathrm{L}^\infty(\omega; \mathbb{R}^{2n-1})$;*
- (ii) *ϕ is a broad solution of $\nabla^\phi \phi = w$, i.e. there exists a Borel function $\hat{w} \in \mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ s.t.*
  - (B.1): *$w(A) = \hat{w}(A)$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$;*
  - (B.2): *for every continuous vector field ∇^ϕ_i having an integral curve $\Gamma \in \mathrm{C}^1((-\delta, \delta); \omega)$, ϕ satisfies*

$$\phi\left(\Gamma(s)\right) - \phi\left(\Gamma(0)\right) = \int_0^s \hat{w}\left(\Gamma(r)\right) \, dr \quad \forall s \in [-\delta, \delta].$$

In the statement, $\mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ denotes the set of functions from ω to $\mathbb{R}^{2n-1}$, while $\mathrm{L}^\infty(\omega; \mathbb{R}^{2n-1})$ denotes the equivalence classes of Lebesgue measurable functions in $\mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ which are identified when differing on a Lebesgue negligible set. We will keep this notation throughout the paper: its relevance is remarked by Examples 1.3, 1.4 below.

Outline of the proofs. With the intention of focusing on the nonlinear field, we fix the attention on the case $n = 1$. The variables will be denoted by t and y, and the subscripts $[\cdot]_t$, $[\cdot]_y$ will denote the distributional derivatives $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial y}$ in the Euclidean sense w.r.t. this variables.

Given a continuous distributional solution ϕ of the PDE

$$\nabla^\phi \phi = \phi_y + \left[\frac{\phi^2}{2}\right]_t = w$$

we first prove that it is Lipschitz when restricted along any characteristic curve $\Gamma(y) = (y, \gamma(y))$, $\dot{\gamma} = \phi \circ \Gamma$, adapting an argument by Dafermos. On the other hand, by a construction based on the classical existence theory of ODEs with continuous coefficients, we can define a change of variable $(y, \chi(y, \tau))$ which straightens characteristics. This change of variables does not enjoy BV or Lipschitz regularity, it fails injectivity in an essential way, though it is continuous and we impose an important monotonicity property. This monotonicity, relying on the fact that we basically work in dimension 2, is the regularity property which allows us the change of variables. As we exemplify below, we indeed have an approach different from providing a regular Lagrangian flow of Ambrosio-Di Perna's theory, and it is essentially two dimensional. After the change of variables, the PDE is, roughly, linear, and we indeed find a family of ODEs for ϕ on the family of characteristics composing χ, with coefficients which now are not anymore continuous, but which are however bounded. By generalizing a lemma on ODEs already present in [4], we prove the 1/2-Hölder continuity on the vertical direction (y constant), and a posteriori in the whole domain. This are the main ingredients for establishing that ϕ defines indeed a Lipschitz graph: given two points, we connect them by a curve made first by a characteristic curve which joins the two vertical lines through the points, then by the remaining vertical segment. We manage this way to control the variation of ϕ between the two points with their graph distance d_ϕ, checking therefore the metric definition of intrinsic Lipschitz graphs.
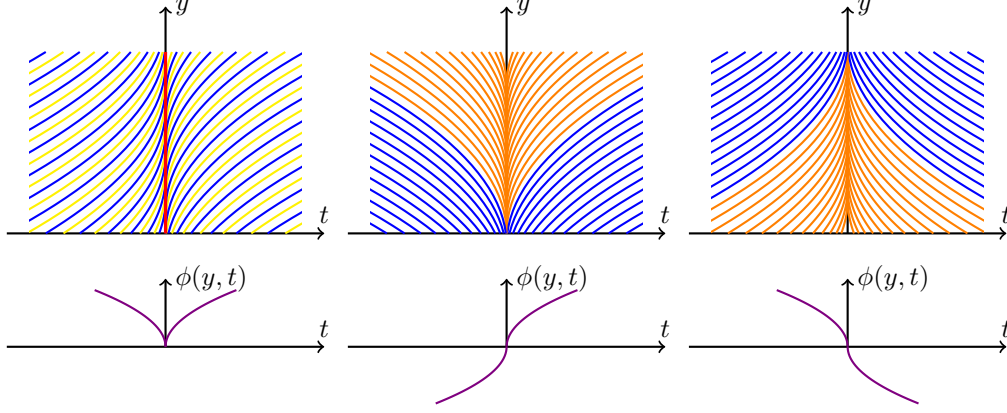
FIGURE 1. Illustration of Examples 1.3, 1.4. Characteristics are drown for three particular continuous distributional solution to the equation $\phi_y + [\phi^2/2]_t = \operatorname{sgn}(t)/2$. Below the graphs of the corresponding functions ϕ are depicted.

The other side of Theorem 1.1 is based on the possibility of suitably approximating an intrinsic Lipschitz graph with intrinsic regular graphs. A geometric approximation is provided by [8]. We also provide a more analytic, and weaker, approximation as a byproduct of the change of variable χ which straightens characteristics, by mollification.

We stop now for a while in order to clarify the features of the statement in Theorem 1.2, and why it is so important to differentiate between $\mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ and $L^\infty(\omega; \mathbb{R}^{2n-1})$. Lagrangian formulations are affected by altering the representative, as the following example stresses.

Example 1.3 (Figure 1). *Consider the continuous function $\phi(y,t) = \sqrt{|t|}$ in the domain $\omega = (0,1) \times (-1,1)$. For simplicity, it does not depend on y. It is easy to calculate*

$$\nabla^\phi \phi = \frac{\partial}{\partial y}\phi + \frac{\partial}{\partial t}\frac{\phi^2}{2} = \begin{cases} 1/2 & \text{if } t \geq 0 \\ -1/2 & \text{if } t < 0 \end{cases} =: w(y,t).$$

Consider the specific characteristic curve $(y, \gamma(y)) := (y, 0)$. Even if $\dot{\gamma}(y) = \phi(y, \gamma(y)) = 0$, the derivative of ϕ along this characteristic curve is not the right one:

$$\frac{\partial}{\partial y}\phi(y,0) = 0 \neq \frac{1}{2} = w(y,0).$$

Equation (3.5) holds however on every characteristic curves provided we choose correctly an L^∞-representative $\hat{w} \in \mathfrak{L}^\infty(\omega; \mathbb{R})$ of the source w: it is enough to consider

$$\hat{w}(y,t) := \begin{cases} 1/2 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1/2 & \text{if } t < 0 \end{cases}$$

Notice that $w(y,t) = \hat{w}(y,t)$ for $\mathcal{L}^2$-a.e. $(y,t) \in \omega$. □

Before outlining Theorem 1.2 we exemplify other features mentioned above by similar examples.

Example 1.4 (Figure 1). *Let $\omega = (0,1) \times (-1,1)$, and choose*

$$\phi(y,t) = -\operatorname{sgn} t \sqrt{|t|}.$$

Again

$$\nabla^\phi \phi = \frac{\partial}{\partial y}\phi + \frac{\partial}{\partial t}\frac{\phi^2}{2} = \begin{cases} 1/2 & \text{if } t > 0 \\ -1/2 & \text{if } t \leq 0 \end{cases} =: w(y,t).$$

One easily sees that characteristics do collapse in an essential way. Considering instead

$$\phi(y,t) = \operatorname{sgn} t \sqrt{|t|}$$

characteristics do split in an unavidable way. Therefore, while it is proved in [10] that in Example 1.3 one can choose for changing variable a flux which is better than a generic other—the regular Lagrangian flow—we are not always in this case. This is the main reason why we refer in our first change of variables to a monotonicity property.

We have now motivated the further study for the stronger statement of Theorem 1.2. In order to prove it, we consider also the weaker concept of *Lagrangian solution*: the idea is that the reduction on characteristics is not required on *any* characteristic, but on a set of characteristic composing the change of variables χ that one has chosen. Indeed, exhibiting a suitable set of characteristics for the change of variables χ is part of the proof. Therefore, a $\mathfrak{L}^\infty$-representative w_χ for the source of the ODEs related to χ is provided by taking the y-derivative of $\phi(y, \chi(y,\tau))$, which by construction is also the second y-derivative of $\chi(y,\tau)$. Certainly, there is an additional further technicality in coming back from (y,τ) to (y,t), which can be overcome. However, if one changes the set of characteristics in general one arrives to a different function $w_{\chi'} \in \mathfrak{L}^\infty(\omega; \mathbb{R})$.

We have called *broad solution* a function which satisfies the reduction on every characteristic curve. In order to have this stronger characterization, we give a different argument borrowed from [1]. We define a universal source term $\hat{w}$ in an abstract way, by a selection theorem, at each point where there exists a characteristic curve with second derivative. After showing that this is well defined, we have provided a universal representative of the intrinsic gradient of ϕ. In cases as Examples 1.3, 1.4 it extends the one, defined only almost everywhere, provided by Rademacher theorem.

Outline of the paper. The paper is organized as follows. In Section 2 we recall basic notions about the Heinsenberg groups. In Section 3 we fix instead notations relative to the PDE, mainly specifying the different notions of solutions we will consider. One of them will involve a change of variables, for passing to the Lagrangian formulation, which is mainly matter of Section 4 and it is basically concerned with classical theory on ODEs. Then Appendix A also explains how to extend a partial change of variables of that kind to become surjective, and provides a counterexample to its local Lipschitz regularity. In Section 5 we prove the equivalence among the facts that either a continuous function ϕ describes a Lipschitz graph or it is a distributional solution to the PDE $\nabla^\phi \phi = w$, $w \in \mathrm{L}^\infty$. The further equivalencies are finally matter of Section 6.

With some simplification, we can illustrate the main connections by the following papillon. As mentioned above, there is also a connection with the existence of smooth approximations, which is not emphasized. Here it is noticed in Corollary 6.4 and applied relatively to the equivalence with the distributional formulation. See [8] for a different smoothing.
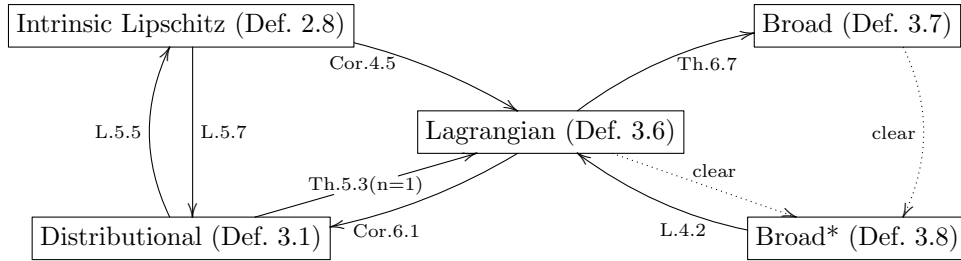
2. Sub-Riemannian Heisenberg Group

Definition: a noncommutative Lie group. We denote the points of $\mathbb{H}^n \equiv \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ by

$$P = [z,t] = [x+iy,t] = (x,y,t), \qquad z \in \mathbb{C}^n, \ x,y \in \mathbb{R}^n, \ t \in \mathbb{R}.$$

If $P = [z,t]$, $Q = [z',t'] \in \mathbb{H}^n$ and $r > 0$, the group operation reads as

$$(2.1) \qquad\qquad P \cdot Q := \left[z+z', t+t' - \frac{1}{2}\Im m(\langle z, \bar{z'} \rangle) \right].$$

The group identity is the origin 0 and one has $[z,t]^{-1} = [-z,-t]$. In $\mathbb{H}^n$ there is a natural one parameter group of non isotropic dilations $\delta_r(P) := [rz, r^2 t]$, $r > 0$.

The group $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\}$$

and with the left-invariant and homogeneous distance

$$d_\infty(P,Q) := \|P^{-1} \cdot Q\|_\infty.$$

The metric d_∞ is equivalent to the standard Carnot-Carathéodory distance. It follows that the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $2n+2$, whereas its topological dimension is $2n+1$.

The Lie algebra $\mathfrak{h}_n$ of left invariant vector fields is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}, \quad j = 1, \ldots, n, \quad T = \frac{\partial}{\partial t}$$

and the only nonvanishing commutators are

$$[X_j, Y_j] = T, \qquad j = 1, \ldots n.$$

We also use the notation $X_j := Y_{j-n}$ for $j = n+1, \ldots, 2n$.

Horizontal fields and differential calculus. We shall identify vector fields and associated first order differential operators; thus the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ generate a vector bundle on $\mathbb{H}^n$, the so called *horizontal* vector bundle $H\mathbb{H}^n$ according to the notation of Gromov (see [16]), that is a vector subbundle of $T\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$. Since each fiber of $H\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section φ of $H\mathbb{H}^n$ can be identified with a map $\varphi : \mathbb{H}^n \to \mathbb{R}^{2n+1}$. At each point $P \in \mathbb{H}^n$ the horizontal fiber is indicated as $H\mathbb{H}^n_P$ and each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_P$ and the associated norm $\| \cdot \|_P$ that make the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ orthonormal.

Definition 2.1. *A real valued function f, defined on an open set $\Omega \subset \mathbb{H}^n$, is said to be of class $C^1_{\mathbb{H}}(\Omega)$ if $f \in C^0(\Omega)$ and the distribution*

$$\nabla_{\mathbb{H}} f := (X_1 f, \ldots, Y_n f)$$

is represented by a continuous function.

Definition 2.2. *We shall say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular hypersurface if for every $P \in S$ there exist an open ball $U_\infty(P,r)$ and a function $f \in C^1_{\mathbb{H}}(U_\infty(P,r))$ such that*

    i: $S \cap U_\infty(P,r) = \{Q \in U_\infty(P,r) : f(Q) = 0\}$;
    ii: $\nabla_{\mathbb{H}} f(P) \neq 0$.

The *horizontal normal* to S at P is $\nu_S(P) := -\dfrac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|}$.

In the spirit of [2, 3, 4, 5] we set $\mathbb{W} := \{(x,y,t) \in \mathbb{H}^n : x_1 = 0\} \equiv \mathbb{R}^{2n}$, $\mathbb{V} := \{(x,y,t) \in \mathbb{H}^n : x_2 = 0, \ldots, y_1 = 0, \ldots, y_n = 0\} \equiv \mathbb{R}$. Therefore if $A \in \mathbb{W}$ then $A = (0, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t)$, we will write $A = (y_1, v, t)$, where $v = (x_2, \ldots, x_n, y_2, \ldots, y_n)$ if $n \geq 2$ and $A = (y_1, t) = (y, t)$ if $n = 1$.

Definition 2.3. *A set $S \subset \mathbb{H}^n$ is an X_1-graph if there is a function $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$ such that $S = G^1_{\mathbb{H},\phi}(\omega) := \{A \cdot \phi(A)e_1 : A \in \omega\}$.*

Let us recall the following results proved in [12].

Theorem 2.4 (Implicit Function Theorem). *Let Ω be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in \mathrm{C}^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f > 0$. Let $S := \{(x,y,t) \in \Omega : f(x,y,t) = 0\}$; then there exist a connected open neighborhood $\mathcal{U}$ of 0 and a unique continuous function $\phi : \omega \subset \mathbb{W} \to [-h, h]$ such that $S \cap \overline{\mathcal{U}} = \Phi(\omega)$, where $h > 0$ and Φ is the map defined as*

$$\omega \ni (y_1, v, t) \mapsto \Phi(y_1, v, t) = (y_1, v, t) \cdot \phi(y_1, v, t)e_1$$

and given explicitly by

$$\Phi(y_1, v, t) = \left(\phi(y_1, v, t), x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t - \frac{y_1}{2}\phi(y_1, v, t)\right) \quad \text{if } n \geq 2$$
$$\Phi(y_1, \tau) = \left(\phi(y_1, t), y_1, t - \frac{y_1}{2}\phi(y_1, t)\right) \quad \text{if } n = 1.$$

Let $n \geq 2$, $A_0 = (x_2^0, \ldots, x_n^0, y_1^0, \ldots, y_n^0, t^0) \in \mathbb{R}^{2n}$ and define

$$I_r(A_0) \quad := \left\{(x_2, \ldots, x_n, y_1, \ldots, y_n, t) \in \mathbb{R}^{2n} : |y_1 - y_1^0| < r, \right.$$
$$\left. \textstyle\sum_{i=2}^n [(x_i - x_i^0)^2 + (y_i - y_i^0)^2] < r^2, |t - t^0| < r\right\}.$$

When $n = 1$ and $A_0 = (y^0, t^0) \in \mathbb{R}^2$ let

$$I_r(A_0) := \left\{(y, t) \in \mathbb{R}^2 : |y - y^0| < r, |t - t^0| < r\right\}.$$

Following [2, 3, 25] we define the graph quasidistance d_ϕ on ω. We set $\mathbb{O}_1 := \{(x, y, t) \in \mathbb{H}^n : x_1 = 0, t = 0\}$, $\mathbb{T} := \{(x, y, t) \in \mathbb{H}^n : x_1 = 0, \ldots, y_n = 0\}$

Definition 2.5. *For $A = (y_1, v, t)$, $B = (y_1', v', t') \in \omega$ we define*

$$(2.2) \qquad d_\phi(A, B) := \|\pi_{\mathbb{O}_1}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty + \|\pi_{\mathbb{T}}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty$$

If $n \geq 2$ we have explicitly

$$d_\phi(A, B) = |(y_1', v') - (y_1, v)| + \left|t' - t - \frac{1}{2}(\phi(A) + \phi(B))(y_1' - y_1) + \sigma(v, v')\right|^{1/2};$$

where $\sigma(v, v') = \frac{1}{2}\sum_{j=2}^n (v_{n+j}v_j' - v_j v_{n+j}')$. If $n = 1$ and $A = (y_1, t), B = (y_1', t') \in \omega$ we have

$$d_\phi(A, B) = |y_1' - y_1| + \left|t' - t - \frac{1}{2}(\phi(A) + \phi(B))(y_1' - y_1)\right|^{1/2}.$$

An intrinsic differentiable structure can be induced on $\mathbb{W}$ by means of d_ϕ, see [2, 3, 25]. We remind that a map $L : \mathbb{W} \to \mathbb{R}$ is $\mathbb{W}$-linear if it is a group homeomorphism and $L(ry_1, rv, r^2 t) = rL(y_1, v, t)$ for all $r > 0$ and $(y_1, v, t) \in \mathbb{W}$. We remind then the notion of ∇^ϕ-differentiablility.

Definition 2.6. *Let $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$ be a fixed continuous function, and let $A_0 \in \omega$ and $\psi : \omega \to \mathbb{R}$ be given.*

- *We say that ψ is ∇^ϕ-differentiable at A_0 if there is an $\mathbb{W}$-linear functional $L : \mathbb{W} \to \mathbb{R}$ such that*

$$(2.3) \qquad \lim_{A \to A_0} \frac{\psi(A) - \psi(A_0) - L(A_0^{-1} \cdot A)}{d_\phi(A_0, A)} = 0.$$

- *We say that ψ is uniformly ∇^ϕ-differentiable at A_0 if there is an $\mathbb{H}$-linear functional $L : \mathbb{W} \to \mathbb{R}$ such that*

$$(2.4) \qquad \lim_{r \to 0} \sup_{\substack{A, B \in I_r(A_0) \\ A \neq B}} \left\{\frac{|\psi(B) - \psi(A) - L(B^{-1} \cdot A)|}{d_\phi(A, B)}\right\} = 0$$

If ϕ is uniformly ∇^ϕ-differentiable at A_0, then ϕ is ∇^ϕ-differentiable at A_0.

In [2] it has been proved that each $\mathbb{H}$- regular graph $\Phi(\omega)$ admits an intrinsic gradient $\nabla^\phi\phi \in C^0(\omega; \mathbb{R}^{2n})$, in sense of distributions, which shares a lot of properties with the Euclidean gradient. Indeed, since $\mathbb{W} = \exp(\text{span}\{X_2, \ldots, X_n, Y_1, \ldots, Y_n, T\})$, it is possible to define the differential operators given, in sense of distributions, by

$$
(2.5) \quad
\begin{aligned}
W^\phi\phi &:= Y_1\phi + \frac{1}{2}T(\phi^2) = \frac{\partial}{\partial y_1}\phi + x_1\frac{\partial}{\partial t}\phi + \frac{1}{2}\frac{\partial}{\partial t}(\phi^2) = \frac{\partial}{\partial y_1}\phi + \frac{1}{2}\frac{\partial}{\partial t}(\phi^2), \\
\nabla^\phi\phi &:= \begin{cases} (X_2\phi, \ldots, X_n\phi, W^\phi\phi, Y_2\phi, \ldots, Y_n\phi) & \text{if } n \geq 2 \\ W^\phi\phi & \text{if } n = 1 \end{cases}.
\end{aligned}
$$

We also denote by $\nabla^\phi := (\nabla^\phi_2, \ldots, \nabla^\phi_{2n})$ the family of vector fields on $\mathbb{R}^{2n}$, $\nabla^\phi_j := X_j$ for $j = 2, \ldots, n$, $\nabla^\phi_{n+1} = W^\phi := Y_1 + \phi T$ and $\nabla^\phi_j := Y_{j-n}$ for $j = n+2, \ldots, 2n$.

The following characterizations were proved in [2, 3, 5]. The definitions of broad* and distributional solution of the system $\nabla^\phi\phi = w$ are recalled in Section 3.

Theorem 2.7. *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. Then*

(i) *The set $S := \Phi(\omega)$ is an $\mathbb{H}$-regular surface and $\nu^1_S(P) < 0$ for all $P \in S$, where $\nu_S(P) = (\nu^1_S(P), \ldots, \nu^{2n}_S(P))$ is the horizontal normal to S at P.*

is equivalent to each one of the following conditions:

(ii) *There exists $w \in C^0(\omega; \mathbb{R}^{2n-1})$ and a family $(\phi_\epsilon)_{\epsilon > 0} \subset C^1(\omega)$ such that, as $\epsilon \to 0^+$,*

$$\phi_\epsilon \to \phi \quad \text{and} \quad \nabla^{\phi_\epsilon}\phi_\epsilon \to w \quad \text{in } L^\infty_{\text{loc}}(\omega),$$

and $\nabla^\phi\phi = w$ in ω, in sense of distributions.

(i) *There exists $w \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a broad* solution of the system $\nabla^\phi\phi = w$.*
(ii) *There exists $w \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a distributional solution of $\nabla^\phi\phi = w$.*
(iii) *ϕ is uniformly ∇^ϕ-differentiable at A for all $A \in \omega$.*

Introduction to the concern of this paper.

Let us now introduce the concept of intrinsic Lipschitz function and intrinsic Lipschitz graph.

Definition 2.8. *Let $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$. We say that ϕ is an intrinsic Lipschitz continuous function in ω and write $\phi \in \text{Lip}_{\mathbb{W}}(\omega)$, if there is a constant $L > 0$ such that*

$$(2.6) \qquad\qquad |\phi(A) - \phi(B)| \leq Ld_\phi(A, B) \qquad \forall A, B \in \omega$$

Moreover we say that ϕ is a locally intrinsic Lipschitz function in ω and we write $\phi \in \text{Lip}_{\mathbb{W},\text{loc}}(\omega)$ if $\phi \in \text{Lip}_{\mathbb{W}}(\omega)$ for every $\omega' \Subset \omega$.

We remark that when ϕ is intrinsic Lipschitz, then there exists $C > 0$ such that

$$\frac{1}{C}d_\phi(A, B) \leq d_\infty(\Phi(A), \Phi(B)) \leq Cd_\phi(A, B) \qquad \forall A, B \in \omega.$$

In particular, the graph distance d_ϕ is also equivalent to the Carnot-Carathéodory distance restricted to the corresponding points on the graph of the Lipschitz intrinsic hypersurface. This means that ϕ is Lipschitz continuous also in the classical sense when evaluated on any fixed integral curve of the vector field W^ϕ, while it is 1/2-Hölder on the lines where t is fixed.

In [8] is proved the following characterization for intrinsic Lipschitz functions.

Theorem 2.9. *Let $\omega \subset \mathbb{W}$ be open and bounded, let $\phi : \omega \to \mathbb{R}$. Then the following are equivalent:*

(i) *$\phi \in \text{Lip}_{\mathbb{W},\text{loc}}(\omega)$*

(ii) *there exist* $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathrm{C}^\infty(\omega)$ *and* $w \in (\mathrm{L}^\infty_{\mathrm{loc}}(\omega))^{2n-1}$ *such that* $\forall \omega' \Subset \omega$ *there exists* $C = C(\omega') > 0$ *such that*

    (ii1) $\{\phi_k\}_{k\in\mathbb{N}}$ *uniformly converges to* ϕ *on the compact sets of* ω;

    (ii2) $|\nabla^{\phi_k}\phi_k(A)| \le C \quad \mathcal{L}^{2n}$-*a.e.* $x \in \omega'$, $k \in \mathbb{N}$;

    (ii3) $\nabla^{\phi_k}\phi_k(A) \to w(A) \quad \mathcal{L}^{2n}$-*a.e.* $A \in \omega$.

Moreover if (ii) holds, then $\nabla^\phi\phi(A) = w(A)$ $\mathcal{L}^{2n}$-*a.e.* $A \in \omega$.

Let us finally recall the following Rademacher type Theorem, proved in [15].

Theorem 2.10. *If* $\phi \in \mathrm{Lip}_{\mathbb{W}}(\omega)$ *then* ϕ *is* ∇^ϕ-*differentiable for* $\mathcal{L}^{2n}$-*a.e* $A \in \omega$.

3. Different Solutions of the Intrinsic Gradient Differential Equation

Even when $w \in \mathrm{C}^0(\omega)$, where ω is an open subset of $\mathbb{W} \equiv \mathbb{R}^2$, the equation

$$(3.1) \qquad \phi_y(y,t) + \left[\frac{\phi^2(y,t)}{2}\right]_t = w(y,t) \qquad \text{in } \omega$$

allows in general for discontinuous solution. However, it is the case $n = 1$ of the system (2.5)

$$(3.2) \qquad \nabla^\phi\phi = w \qquad \Leftrightarrow \qquad \begin{cases} X_j\phi = w_j \\ W^\phi(\phi) = w_{n+1} \quad j = 2,\dots,n \\ Y_j\phi = w_{n+j} \end{cases}$$

where $w \in \mathrm{C}^0(\omega, \mathbb{R}^{2n-1})$ and this system, by Theorem 2.7, describes an $\mathbb{H}$-regular surface $S := \Phi(\omega)$ which is an X_1-graph. Since we want to study in the present paper intrinsic Lipschitz graphs, then we do not require anymore the continuity of w but we allow $w \in \mathrm{L}^\infty(\omega; \mathbb{R}^{2n-1})$. Notwithstanding that, the continuity of ϕ remains natural.

There are a priori different notions of *continuous* solutions $\phi : \omega \to \mathbb{R}$. We recall some of them in this section: distributional, Lagrangian, broad, broad*. All of them will finally coincide.

After giving in the present sections the definitions for all n, we will focus in the next one the analysis on the non-linear equation in the case $n = 1$, which conveys the attention on the planar case (3.1). We will remind this reduction by adopting often the variables (s, τ) instead of (y, t). The generalization to other cases $n \ge 2$ of most of the lemmas is straightforward, because the fields X_j and Y_j are linear. It is not basically in Lemma 4.4, where we prefer taking advantage of the continuity of χ; however, we have no reason to prove it in full generality.

We recall that in general solutions are not smooth, even if we assume the continuity—see e.g. Example A.2 below. The equation is then interpreted in a distributional way.

Definition 3.1 (Distributional solution)**.** *A continuous function* $\phi : \omega \to \mathbb{R}$ *is a* distributional solution *to* (3.2) *if for each* $\varphi \in \mathrm{C}^\infty_c(\omega)$

$$(3.3) \qquad \int_\omega \phi \, X_j\varphi \, d\mathcal{L}^{2n} = -\int_\omega w_j\varphi \, d\mathcal{L}^{2n}, \quad \int_\omega \phi \, Y_j\varphi \, d\mathcal{L}^{2n} = -\int_\omega w_{j+n}\varphi \, d\mathcal{L}^{2n} \qquad j = 2,\dots,n$$

and

$$\int_\omega \left(\phi\frac{\partial\varphi}{\partial y} + \frac{1}{2}\phi^2\frac{\partial\varphi}{\partial t}\right) d\mathcal{L}^{2n} = -\int_\omega w_{n+1}\varphi \, d\mathcal{L}^{2n}.$$

We consider now different versions for the Lagrangian formulation of the PDE. The first one somehow englobes a choice of trajectories for passing from Lagrangian to Eulerian variables, and imposes the evolution equation on these trajectories. If B is a subset $\mathbb{R}^{2n}$, we will denote

$$B_{v,t} := \{y_1 \in \mathbb{R} : (y_1, v, t) \in B\}, \quad B_{y_1,v} := \{t \in \mathbb{R} : (y_1, v, t) \in B\}.$$

Definition 3.2 (Lagrangian parameterization)**.** *A partial Lagrangian parameterization associated to a continuous function $\phi : \omega \to \mathbb{R}$ and to the balance law (3.2) is any couple $(\tilde{\omega}, \chi)$ with $\tilde{\omega} \subset \mathbb{R}^{2n}$, usually open, and $\chi : \tilde{\omega} \ni (y_1, v, \tau) \mapsto \mathbb{R}$ Borel, such that*

    (L.1): *the function $\Upsilon(y_1, v, \tau) = (y_1, v, \chi(y_1, v, \tau))$ is valued in ω:*
    (L.2): *χ is nondecreasing in the τ variable for each fixed y_1, v;*
    (L.3): *for each v, τ fixed, $\chi(y_1, v, \tau)$ is absolutely continuous in y_1 and almost everywhere*

$$(3.4) \qquad\qquad\qquad \frac{d}{dy_1} \chi(y_1, v, \tau) = \phi(\Upsilon(y_1, v, \tau)).$$

We call it (full) *Lagrangian parameterization if $\chi(y_1, v, \tau)$ is onto the section $\omega_{y_1, v}$ for all y_1, v.*

We remark again that we emphasized in this definition the nonlinear PDE of the system: a Lagrangian parameterizaiton provides a covering of ω by characteristic lines for that equation. Indeed, a covering by characteristic lines of the other equations is immediately given by an expression like

$$\chi_i(x_2, \dots, x_n, y_1, \dots, y_n, t) = \begin{cases} t - \frac{y_i}{2} x_i & i = 2, \dots, n \\ t + \frac{x_i}{2} y_i & i = n+1, \dots, 2n \end{cases}, \qquad \Psi_i(y_1, v, t) = (y_1, v, \chi_i(y_1, v, t)).$$

Moreover, the reduction along characteristics for the linear equations, and thus the equivalence between Lagrangian and distributional solution, holds with less technicality.

Definition 3.3. *A (partial) parameterization $(\tilde{\omega}, \chi)$ extends the (partial) parameterization $(\tilde{\omega}', \tilde{\chi}')$, we denote $(\tilde{\omega}', \tilde{\chi}') \preceq (\tilde{\omega}, \chi)$, if there exists a Borel injective map*

$$J : \tilde{\omega}' \ni (y_1, v, \tau) \mapsto (y_1, v, j(y_1, v, \tau)) \in \tilde{\omega} \qquad such\ that \qquad \chi \circ J = \tilde{\chi}'.$$

When $(\tilde{\omega}', \tilde{\chi}') \preceq (\tilde{\omega}, \chi)$ and $(\tilde{\omega}, \chi) \preceq (\tilde{\omega}', \tilde{\chi}')$ they are called equivalent.

Remark 3.4. *The notion of Lagrangian parameterization given above does not consist in a different formulation for the notion of regular Lagrangian flow in the sense by Ambrosio-Di Perna (see [9] for an effective presentation). Particles are really allowed both to split and to join, therefore in particular the compressibility condition here is not required, while instead we have a monotonicity property.*

Notation 3.5. When we need to distinguish letters, we denote with $\bar{\cdot}$ functions defined on ω but possibly related to a parameterization, with $\tilde{\cdot}$ functions defined on $\tilde{\omega}$, and with $\hat{\cdot}$ functions defined on ω not related to specific parameterizations.

Notice that a full Lagrangian parameterization is continuous in the two variables for free: indeed, e.g. for $n = 1$, by monotonicity one has that for each s

$$\sup_{\tau' < \tau} \chi(s, \tau') \le \chi(s, \tau) \le \inf_{\tau' > \tau} \chi(s, \tau').$$

By the surjectivity then equality must hold. Considering the Lipschitz continuity in the other variables one gains the joint continuity in (s, τ). The same holds for $n > 1$ provided that χ is continuous on the hyperplane $y_1 = 0$, since the argument above gives the continuity only on the planes where v is constant. We do not mind about continuity in v.

Before giving the notion of Lagrangian solution, we recall that a set $A \subset \mathbb{R}^n$ is *universally measurable* if it is measurable w.r.t. every Borel measure. Universally measurable sets constitute a σ-algebra, which includes analytic sets. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said universally measurable if it is measurable w.r.t. this σ-algebra. In particular, it will be measurable w.r.t. any Borel measure.
Notice that Borel counterimages of universally measurable sets are universally measurable. Then the composition $\varphi \circ \psi$ of any universally measurable function φ with a Borel function ψ is universally measurable. This composition would be nasty if φ were just Lebesgue measurable.
Since restrictions of Borel functions on Borel sets are Borel, all the terms in the following definition are thus meaningful.

Definition 3.6. *[Lagrangian solution] A continuous function ϕ is a* Lagrangian solution *of* (3.2) *if there exists a Lagrangian parameterization $(\tilde{\omega}, \chi)$, associated to ϕ, and a universally measurable function $\bar{w}_\chi : \omega \to \mathbb{R}$, for which*

$$(3.5) \qquad \phi(\Upsilon(s_2, v, t)) - \phi(\Upsilon(s_1, v, t)) = \int_{s_1}^{s_2} \bar{w}_\chi(\Upsilon(r, v, t)) \, dr \quad \forall (s_1, s_2) \subset \tilde{\omega}_{v,\tau}, \ \forall \tau.$$

We are going to prove in Section 6.1 that ϕ above is a distributional solution to (3.2) provided that the function $\bar{w}_\chi$ is a pointwise representative of the source term w, in $\mathrm{L}^\infty(\omega, \mathcal{L}^{2n})$. The fact that we do not require in the definition that $\bar{w}_\chi$ and w represent the same function is just a notational convenience for the proofs below.

We give now the strongest notion of solution: the evolution equation is imposed on every trajectories.

Definition 3.7. *[Broad solution] A continuous function $\phi : \omega \to \mathbb{R}$ is a* broad solution *of* (3.2) *if there exists a Souslin function $\hat{w} \in \mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ such that*

(B.1): *it is $\mathcal{L}^{2n}$-a.e. equal to w;*
(B.2): *if for every continuous vector field ∇_i^ϕ having an integral curve $\Gamma \in \mathrm{C}^1((-\delta, \delta); \omega)$, satisfies*

$$\phi\left(\Gamma(s)\right) - \phi\left(\Gamma(0)\right) = \int_0^s \hat{w}\left(\Gamma(r)\right) \, dr \quad \forall s \in [-\delta, \delta].$$

We also remind the intermediate notion of broad* solution introduced in [4].

Definition 3.8. *[Broad* solution] A continuous function ϕ is a* broad* solution *of* (3.2) *if for every $A_0 \in \omega$ there exists a map, that we will call exponential map,*

$$(3.6) \qquad \exp_{A_0}(\nabla_i^\phi)(\cdot) : [-\delta, \delta] \times I_\delta(A_0) \to I_{\delta_1}(A_0) \Subset \omega,$$

where $0 < \delta < \delta_1$ such that, if $\Gamma^A(s) = \exp_{A_0}(s\nabla_i^\phi)(A)$ for some $i = 2, \ldots, 2n$, then

(E.1): $\Gamma^A \in \mathrm{C}^1([-\delta, \delta]; \mathbb{R}^{2n})$,
(E.2): $\begin{cases} \dot{\Gamma}^A = \nabla_i^\phi \circ \Gamma^A \\ \Gamma^A(0) = A \end{cases}$,
(E.3): *there exists an $\mathrm{L}^\infty(\omega; \mathbb{R})$-representative $\bar{w}$ of w such that*

$$\phi\left(\Gamma^A(s)\right) - \phi\left(\Gamma^A(0)\right) = \int_0^s \bar{w}\left(\Gamma^A(r)\right) \, dr \quad \forall s \in [-\delta, \delta] \qquad \forall A \in I_\delta(A_0).$$

Being a distributional solution to the PDE, a Lagrangian solution will be in particular a broad* solution with $w := \bar{w}$. Viceversa, if the exponential map related to a broad solution satisfies the relative monotonicity property, then one can prove that the broad* solution is also a Lagrangian solution, with the same $\bar{w}$. One can moreover derive a procedure for constructing a Lagrangian parameterization in Lemma 4.2, where the curves $\gamma(s; \bar{s}, \bar{t})$ should be replaced by the ones of the exponential map.

4. EXISTENCE OF LAGRANGIAN PARAMETERIZATIONS

As mentioned, we can focus in this section on the case $n = 1$. We will give information about the generalization to the case $n \geq 2$ in Remark 4.3.

Given a continuous function ϕ on $\mathrm{clos}(\omega) \subset \mathbb{W}$, we highlight here that one can directly construct a partial Lagrangian parameterization. Extending it to a full one will be matter of the next section. However, we immediately give an example of a global one (Lemma 4.2).

Notices that the definition of Lagrangian parameterization concerns only classical theory on ODEs with continuous coefficients, as therefore Lemmas 4.1, 4.4, A.1, 5.4, 4.2.

Lemma 4.1. *Let $\phi : \mathrm{clos}(\omega) \to \mathbb{R}$ be a continuous function, $(0,0) \in \omega$. Then there are domains $\tilde{\omega}_m$, $\tilde{\omega}_M$ associated to the functions*

$$\chi_m(s,\tau) := \min\Big\{\gamma(s) : \ (r,\gamma(r)) \in \mathrm{clos}(\omega), \ \dot{\gamma}(r) = \phi(r,\gamma(r)), \ \gamma(0) = \tau\Big\}$$

$$\chi_M(s,\tau) := \max\Big\{\gamma(s) : \ (r,\gamma(r)) \in \mathrm{clos}(\omega), \ \dot{\gamma}(r) = \phi(r,\gamma(r)), \ \gamma(0) = \tau\Big\}$$

for which $(\tilde{\omega}_m, \chi_m)$, $(\tilde{\omega}_M, \chi_M)$ are partial Lagrangian parameterizaiton relative to ϕ.

Proof. For every $(\bar{s}, \bar{\tau}) \in \omega$ one could consider the minimal and maximal curves satisfying the ODE for characteristics (3.4) and passing through that point: indeed the functions

(4.1a) $\qquad \gamma_{(\bar{s},\bar{\tau})}(s) := \min\Big\{\gamma(s) : \ (r,\gamma(r)) \in \mathrm{clos}(\omega), \ \dot{\gamma}(r) = \phi(r,\gamma(r)), \ \gamma(\bar{s}) = \bar{\tau}\Big\}$

(4.1b) $\qquad \gamma^{(\bar{s},\bar{\tau})}(s) := \max\Big\{\gamma(s) : \ (r,\gamma(r)) \in \mathrm{clos}(\omega), \ \dot{\gamma}(r) = \phi(r,\gamma(r)), \ \gamma(\bar{s}) = \bar{\tau}\Big\}$

are well defined, Lipschitz, and because of the continuity of ϕ they are still integral curves. This shows that the functions χ_m, χ_M in the statement are C^1 in the s variable for every τ fixed.

Denoting by r. i. the relative interior of a set, we define the domain

$$\tilde{\omega}_m = \mathrm{r.\,i.}\,\big\{(s,\tau): \ (s,\chi_m(s,\tau)) \in \omega\big\}.$$

Since we are assuming that ω contains the origin, $\tilde{\omega}_m$ is nonempty. The domain $\tilde{\omega}_M$ is analogous.

χ_m, χ^M are jointly Borel in (s,τ) by continuity in s and monotonicity in τ, that now we show.

Notice first the semigroup property: for $t, h > 0$, for example for (4.1a),

$$\gamma_{(0,\tau)}(s) =: \bar{\tau}, \quad \gamma_{(s,\bar{\tau})}(h) =: \bar{\bar{\tau}}, \quad \Longrightarrow \quad \gamma_{(0,\tau)}(s+h) = \bar{\bar{\tau}}.$$

This yields the monotonicity of $\chi_m(s,\tau) = \gamma_{(0,\tau)}(s)$ for each s fixed. Indeed, if $\tau_1 < \tau_2$ and $\gamma_{(0,\tau_1)}(\hat{s}) \geq \gamma_{(0,\tau_2)}(\hat{s})$ at a certain $\hat{s} > 0$, by the continuity of the curves there exists $0 < \bar{s} \leq \hat{s}$ when $\gamma_{(0,\tau_1)}(\bar{s}) = \gamma_{(0,\tau_2)}(\bar{s})$. But then the curve

$$\gamma(s) = \begin{cases} \gamma_{(0,\tau_1)}(s) & \text{for } s \leq \bar{s} \\ \gamma_{(0,\tau_2)}(s) & \text{for } s \geq \bar{s} \end{cases}$$

is a good competitor for the definition of $\gamma_{(0,\tau_1)}$, which implies

$$\gamma_{(0,\tau_2)}(\hat{s}) = \gamma(\hat{s}) \geq \gamma_{(0,\tau_1)}(\hat{s}),$$

and therefore equality. For $\gamma^{(0,\tau)}$ the argument is similar. $\qquad \square$

In Lemma A.1 we show how to make a partial parameterization χ surjective: thus we cover ω by a family of characteristic curves which includes the ones of χ. In the following lemma, w.l.o.g. in a simpler setting, we provide instead a full Lagrangian parameterization, defined at once instead of extending a given one.

Lemma 4.2. *There exists a Lagrangian parameterization associated to a continuous function $\phi : [0,1]^2 \to \mathbb{R}$.*

Proof. Let ϕ be a continuous function on $[0,1]^2$. Then we associate to each point $(\bar{s}, \bar{t}) \in [0,1]^2$ a curve $\gamma(s; \bar{s}, \bar{t})$ which is minimal forward in s, maximal backward:

$$\gamma(s; \bar{s}, \bar{t}) = \begin{cases} \gamma_{(\bar{s},\bar{t})}(s) & s \geq \bar{s}, \\ \gamma^{(\bar{s},\bar{t})}(s) & s < \bar{s}. \end{cases}$$

where $\gamma_{(\bar{s},\bar{t})}(s)$, $\gamma^{(\bar{s},\bar{t})}(s)$ were defined in (4.1). One can verify that the set of these curves is totally ordered by the relation

$$\big\{\gamma(s; s_1, yt_1)\big\}_{s \in [0,1]} \preceq \big\{\gamma(s; s_2, t_2)\big\}_{s \in [0,1]} \qquad \Leftrightarrow \qquad \gamma(s; s_1, t_1) \leq \gamma(s; s_2, t_2) \ \forall s.$$

Then one can consider the strictly order-preserving map from these curves to $[0, 2]$

$$\theta : \ \{\gamma(s; \bar{s}, \bar{t})\}_{s \in [0,1]} \mapsto \sum_{n \in \mathbb{N}_0} 2^{-n} \gamma(r_n; \bar{s}, \bar{t}),$$

where $\{r_n\}_{n \in \mathbb{N}}$ is an enumeration of the rational numbers in $[0, 1]$. One can verify that the image is an interval $[0, T]$. Therefore we can consider the continuous map $\chi : [0, 1] \times [0, T] \to [0, 1]$

$$\chi(s, \tau) := \gamma(s; \bar{s}, \bar{t}) \qquad \text{with } \tau = \theta(\bar{s}, \bar{t}).$$

It is an easy verification that it provides a full Lagrangian parameterization. $\qquad \square$

Remark 4.3. Notice that the same construction works with more variables, considering analogous characteristic curves $\gamma(s; \bar{s}, \bar{v}, \bar{t})$—having the same order relation at $\bar{v}$ frozen—and the relative parameterization given by $\chi(s, v, \tau) := \gamma(s; \bar{s}, v, \bar{t})$ with $\tau = \theta(\gamma(s; \bar{s}, v, \bar{t}))$ defined as above. The continuity in v however is not guaranteed.

The parameterization that we introduced in the above lemma, as well as the one of Lemma A.1 below, satisfies

$\bigstar$ whenever $\chi(\bar{s}, \tau_1) = \chi(\bar{s}, \tau_2)$, then $\chi(s, \tau_1) = \chi(s, \tau_2)$ either for $s > \bar{s}$ or for $s < \bar{s}$.

In Lemma 5.1 we follow a computation in [11] and we show that continuous distributional solutions ϕ to the balance law (3.1) are Lipschitz continuous along characteristics. Therefore, the following lemma will yield that continuous distributional solutions to the balance law (3.1) are also Lagrangian solutions. We prove it here because it is simple and on the ODE front. Moreover, in order to avoid technicalities we give a proof valid for $n = 2$, since there will be a stronger (and independent) proof for all n in Section 6.2, consisting in the construction of the universal source term $\hat{w}$.

Lemma 4.4. *Let ϕ be a continuous function. Consider a Lagrangian parameterization $(\tilde{\omega}, \chi(s, \tau))$ satisfying $\bigstar$ and assume that $\phi(s, \chi(s, \tau))$ is Lipschitz in s for all τ. Then there exists a Borel function $\bar{w} : \omega \to \mathbb{R}$ such that for all τ*

$$\partial_s \phi(s, \chi(s, \tau)) = \partial_{ss} \chi(s, \tau) = \bar{w}(s, \chi(s, \tau)) \qquad \text{for } \mathcal{L}^1\text{-a.e. } s.$$

Before giving the proof of the lemma, we just remind that, as recalled just after the Definition 2.8, an intrinsic Lipschitz continuous function is Lipschitz along characteristics. We immediately emphasize therefore the following corollary as a particular case.

Corollary 4.5. *If $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$ is an intrinsic Lipschitz continuous function, then it is a Lagrangian solution to the equation $W^\phi \phi = w$.*

Proof. By assumption $\partial_s \chi(s, \tau) = \phi(s, \chi(s, \tau))$ is Lipschitz in s. Being ϕ continuous, one can see that the subset $B \subset \tilde{\omega}$ of those (s, τ) where $\chi(s, \tau)$ is twice s-differentiable is $F_{\sigma\delta}$, and $\partial_{ss} \chi(s, \tau)$ is a Borel function on it. Moreover, by Rademacher's theorem the τ-sections of B have full measure. Clearly, by Tonelli theorem also B has full measure. However, we have still to work because we are looking for a function defined on ω, not on $\tilde{\omega}$.

Consider the map

$$\Upsilon : \tilde{\omega} \to \omega \qquad \Upsilon(s, \tau) := (s, \chi(s, \tau)).$$

In order to check that Υ lifts $\partial_{ss} \chi$ to a map $\bar{w}$ a.e. defined on ω, which would provide our thesis, it would be natural to show that

- $\Upsilon(B)$ is a Lebesgue measurable subset of ω with full measure;
- $\partial_{ss} \chi(s, \tau)$ is constant on the level set of Υ intersected with B.

Since χ is not Lipschitz, we do not prove at this point that $\Upsilon(B)$ has full measure. We instead assign a specific value, 0, to the function out of $\Upsilon(B)$. This choice of the extension does not affect our claim. It could be however noticed for completeness that $\Upsilon(B)$ has full measure because of the approximation given in Corollary 6.4.

`Exact study of` $\Upsilon(B)$`.` The proof of the Borel measurability of $\Upsilon(B)$ requires some technicality: we apply a theorem due to Srivastava ([24], Theorem 5.9.2) deriving that there exists a Borel restriction χ which is one-to-one to $\Upsilon(B)$; then Theorem 4.12.4 in [24], due to Lusin, would provide the thesis. In order to apply the first theorem, we partition $\tilde{\omega}$ into the level sets of Υ, which are G_δ. We need also to observe that $\Upsilon^{-1}(\Upsilon)(O)$ is Borel for each open set O. For simplicity, consider the case when χ is already a full parameterization and thus it is continuous. Every open set O is σ-compact: thus by continuity $\Upsilon(O)$ is σ-compact, and finally $\Upsilon^{-1}(\Upsilon)(O)$ is σ-compact. Therefore by Srivastava's theorem there is a Borel cross section S for the partition: Υ restricted to $S \cap B$ is Borel, injective and onto $\Upsilon(B)$. Being a Borel image by a one-to-one map, Lusin's theorem asserts on one hand that that $\Upsilon(B)$ is Borel, and moreover that this restriction has a Borel inverse

$$\Xi : \omega \to \tilde{\omega}.$$

Maybe there is a more elementary argument which allows to approximate B with a σ-compact subset B_K whose τ-section have full measure. Then one could as well work with B_K instead of B avoiding measurability difficulties, even without investigating the size of its image.

`Analysis of` $\partial_{ss}\chi(s,\tau)$`.` Whenever Υ maps two points $(\bar{s}, \tau_1)$, $(\bar{s}, \tau_2)$ to a single point $(\bar{s}, \bar{t})$, then by construction the curves $\chi(s, \tau)$ coincide for $s \geq \bar{s}$ and $\tau \in [\tau_1, \tau_2]$. In particular, they will have precisely the same second s-derivative at points of second s-differentiability. This means that the set

$$\left\{ \partial_{ss}\chi(s,\tau) : \ t = \chi(s,\tau), \ (s,\tau) \in B \right\}$$

is at most a singleton. We can thus define $\bar{w}$ as

$$\bar{w}(s,t) = \begin{cases} \partial_{ss}\chi(s,\tau) & t = \chi(s,\tau), \ (s,\tau) \in B, \\ 0 & \text{otherwise,} \end{cases}$$

and this map will be exactly $\tilde{w} \circ \Xi$ on $\Upsilon(B)$. $\square$

We remark that one could prove the same statement for any Lagrangian parameterization χ, with an argument similar to the proof of Theorem 6.8. We preferred above this simpler statement. Indeed, since we prove below that Lagrangian solutions are broad solution, the lemma will hold for free of any Lagrangian parameterization.

Notice finally that for the moment the function $\bar{w}$ in the proof of Lemma 4.4 has no relation with the RHS of (3.1), which will come from Section 6.1. As well ϕ has not been yet related to the PDE.

5. Equivalence among distributional PDE and intrinsic Lipschitz condition.

In this section we prove Theorem 1.1, without dimensional restrictions.

5.1. **Some properties of distributional solutions.** Preliminary we highlight here two properties of continuous distributional solutions $\phi(y_1, v, t)$ to the problem $\nabla^\phi \phi = w$. In particular we need regularity results of the solution along the characteristics lines γ of the fields ∇_j^ϕ. In the case of the non linear field W^ϕ, the integral curves of $\dot{\gamma}(s) = \phi(s, v, \gamma(s))$ exist by the continuity and boundedness of ϕ.

Lemma 5.1. *Let* $\omega \subset \mathbb{W}$ *be an open set. A continuous distributional solution* $\phi : \omega \to \mathbb{R}$ *to* $\nabla^\phi \phi = w$ *is* $\|w_{n+1}\|_\infty$*-Lipschitz along any characteristic line* $\gamma : [-\delta, \delta] \to \mathbb{R}$ *satisfying*

$$\dot{\gamma}(s) = \phi(s, v, \gamma(s)) \qquad s \in [-\delta, \delta], \ v \text{ fixed}.$$

Proof. In the same way in Dafermos [11] we obtain for $a, b \in (-\delta, \delta)$ and v fixed

$$(5.1) \qquad \int_{\gamma(b)}^{\gamma(b)+\epsilon} \phi(b, v, t)dt - \int_{\gamma(a)}^{\gamma(a)+\epsilon} \phi(a, v, t)dt - \int_a^b \int_{\gamma(s)}^{\gamma(s)+\epsilon} w(s, v, t) \, dt \, ds =$$

$$= -\int_a^b [\phi(s, v, \gamma(s) + \epsilon) - \phi(s, v, \gamma(s))]^2 \, ds \le 0$$

and then

$$\int_{\gamma(b)}^{\gamma(b)+\epsilon} \phi(b, v, t)dt - \int_{\gamma(a)}^{\gamma(a)+\epsilon} \phi(a, v, t)dt \le \int_a^b \int_{\gamma(s)}^{\gamma(s)+\epsilon} w(s, v, t) \, dt \, ds \le \|w_{n+1}\|_{\mathrm{L}^\infty(\omega)}(b - a)\epsilon$$

Dividing by ϵ and getting to the limit to $\epsilon \to 0$ we obtain

$$\phi(b, v, \gamma(b)) - \phi(a, v, \gamma(a)) \le \|w_{n+1}\|_{\mathrm{L}^\infty(\omega)}(b - a).$$

The opposite inequality is obtained in a similar way integrating on the left of the characteristic. $\square$

We obtain the same result of Lemma 5.1 for the linear fileds X_j, Y_j following the same proof.

Lemma 5.2. *Let $\omega \subset \mathbb{W}$ be an open set. A continuous distributional solution $\phi : \omega \to \mathbb{R}$ to $\nabla^\phi \phi = w$ is $\|w_j\|_{\mathrm{L}^\infty(\omega)}$-Lipschitz along any characteristic line $\Gamma : [-\delta, \delta] \to \omega$ satisfying*

$$\dot{\Gamma}(s) = \nabla_j^\phi \circ \Gamma(s) \qquad j = 2, \ldots, n, n+2, \ldots, 2n.$$

As announced before Lemma 4.4[1], where we assumed the regularity of the above lemma, we can now state that a continuous distributional solution ϕ to the balance law (3.2) always solves a Lagrangian formulation.

Corollary 5.3. *A continuous distributional solution ϕ to (3.2) is also a Lagrangian solution.*

The fact that the function $\bar{w}$ introduced in the Lagrangian formulation is a representative of the source term w can be deduce by the next section (Theorem 6.1). The fact that there exists a universal $\hat{w}$ independent of χ is proved instead in Theorem 6.8.

We pass now to the Hölder continuity in the other variable.

Lemma 5.4. *Let $f \in \mathrm{C}^0(\omega)$ be such that of all $\tau \in \mathbb{R}$ there are $\gamma : [-\delta, \delta] \to \mathbb{R}$ satisfying*

$$\begin{cases} \dot{\gamma}(s) = f(s, \gamma(s)) & s \in [-\delta, \delta] \\ \gamma(0) = \tau \end{cases}$$

and that

$$|f(s, \gamma(s)) - f(s', \gamma(s'))| \le L|s - s'|.$$

Then we have

$$|f(0, \tau_1) - f(0, \tau_2)| \le 2\sqrt{2L}\sqrt{|\tau_1 - \tau_2|} \qquad for \ |\tau_1 - \tau_2| \le r_0, \quad 0 < r_0 < \frac{\delta^4}{16}.$$

Proof. Let us denote

$$\beta := L, \quad \alpha := \max\{2\sqrt{2L}, r_0^{1/4}\}, \quad f_0(\tau) = f(0, \tau).$$

Let us observe that $\dfrac{\beta}{\alpha^2} \le \dfrac{1}{8}$.

By contradiction, let us assume there exist $-\delta \le \tau_2 < \tau_1 \le \delta$, $0 < \bar{r} < r_0 < \delta$ such that

$$(5.2) \qquad\qquad\qquad 0 < |\tau_1 - \tau_2| \le \bar{r}$$

---

[1]As mentioned, Lemma 4.4 is proved there for simplicity with $n = 2$. It has an independent generalization in Section 6.2

$$(5.3) \qquad \frac{|f_0(\tau_1) - f_0(\tau_2)|}{\sqrt{\tau_1 - \tau_2}} > \alpha.$$

The idea of the proof is the following: the Lipschitz condition in the hypothesis is an upper bound on the second derivative of the mentioned curves γ. Therefore, if their first derivative wants to change it takes some time in s. If we assume that at $s = 0$ the first derivative differs at two points τ_1, τ_2 at least of the ratio (5.3), then the the relative curves γ_1, γ_2 starting from those points must meet soon. However, at the time they meet the first derivative did not have the time to change enough to become equal, providing an absurd.

Let us introduce curves γ_1, γ_2 such that for $i = 1, 2$, $s \in (-\delta, \delta)$

$$\dot{\gamma}_i(s) = f(s, \gamma_i(s)),$$

$$\gamma_i(0) := \tau_i.$$

We observe that, by our Lipschitz assumption, $\dfrac{d}{ds} f(s, \gamma_i(s)) := h_i(s)$ is a function bounded by L. Therefore we can represent each $\gamma_i(s)$ for each $s \in [-\delta, \delta]$ as

$$(5.4) \qquad \gamma_i(s) = \tau_i + \int_0^s f(s, \gamma_i(\sigma)) \, d\sigma$$

$$= \tau_i + g(\tau_i) \, s + \int_0^s \int_0^\sigma h_i(z) \, dz \, d\sigma \quad \forall s \in [-\delta, \delta].$$

In particular by the second equality in (5.4), for $0 \le s \le \delta$,

$$(5.5) \qquad \gamma_1(s) - \gamma_2(s) \le \tau_1 - \tau_2 + (f_0(\tau_1) - f_0(\tau_2))s + 2\beta s^2$$

for $s \in [-\delta, \delta]$. By (5.3) we get

$$(5.6) \qquad f_0(\tau_1) - f_0(\tau_2) < -\alpha \sqrt{\tau_1 - \tau_2}$$

or

$$(5.7) \qquad f_0(\tau_1) - f_0(\tau_2) > \alpha \sqrt{\tau_1 - \tau_2}$$

Let us prove now that if (5.6) holds then there exists $0 < s^* < \delta$ such that

$$(5.8) \qquad \gamma_1(s^*) = \gamma_2(s^*).$$

Let $\bar{s} := 2 \dfrac{\sqrt{\tau_1 - \tau_2}}{\alpha}$ then

$$(5.9) \qquad \bar{s} \in [0, \delta], \quad \gamma_1(s^*) \le \gamma_2(s^*).$$

Indeed by (5.2) and the definition of α, $\bar{s} = 2 \frac{\sqrt{\tau_1 - \tau_2}}{\alpha} \le 2(\tau_1 - \tau_2)^{1/4} \le 2\bar{r}^{1/4} \le \delta$. On the other hand by (5.5) (with $s = \bar{s}$), (5.6) gives

$$\gamma_1(\bar{s}) - \gamma_2(\bar{s}) \quad \le \tau_1 - \tau_2 - 2(\tau_1 - \tau_2) + \frac{8\beta}{\alpha^2}(\tau_1 - \tau_2)$$

$$= (\tau_1 - \tau_2)\left(-1 + \frac{8\beta}{\alpha^2}\right) \le 0$$

Then (5.9) follows. Let

$$s^* := \sup\{s \in [0, \delta_2] : \gamma_1(s) > \gamma_2(s)\}$$

then by (5.8) $0 < s^* < \bar{s} \le \delta$ and it satisfies (5.8).

If (5.7) holds we can repeat the argument reversing the s-axis getting that there exist $-\delta < s* < 0$ such that (5.8) still holds.

Let us prove now that

$$(5.10) \qquad f(s*, \gamma_1(s*)) \neq f(s*, \gamma_2(s*)),$$

then a contradiction and the thesis will follow. Indeed, for instance, let us assume (5.6). Then by (5.4) and the bound on h_i

$$f(s*, \gamma_1(s*)) - f(s*, \gamma_2(s*)) = f_0(\tau_1) - f_0(\tau_2) + \int_0^{s*} h_1(\sigma) - h_2(\sigma)d\sigma \leq$$

$$\leq f_0(\tau_1) - f_0(\tau_2) + 2\beta\, s* \leq f_0(\tau_1) - f_0(\tau_2) + 2\beta\, \bar{s}$$

$$\leq -\alpha\, \sqrt{\tau_1 - \tau_2} + 2\frac{2\beta}{\alpha}\, \sqrt{\tau_1 - \tau_2} \leq$$

$$\leq 2\,\alpha\, \sqrt{\tau_1 - \tau_2}\left[-\frac{1}{2} + \frac{2\beta}{\alpha^2}\right].$$

Therefore we get that

$$f(s*, \gamma_1(s*)) - f(s*, \gamma_2(s*))) < 0$$

and (5.10) follows. □

5.2. Proof of the equivalence.

We are now able to give the proof of Theorem 1.1. We distinguish the two different implications in the following two lemmas.

Lemma 5.5. Let $\omega \subset \mathbb{W}$ be an open set. If ϕ is a distributional solution of $\nabla^\phi \phi = w$, then $\phi \in \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega)$.

Proof. Let $B, B' \in I_{\delta_0}(A)$ for a sufficiently small δ_0. For $n \geq 2$ let $\overline{X}, W^\phi$ be the vector fields given by

$$\overline{X} := \sum_{\substack{j=2 \\ j \neq n+1}}^{2n} (v'_j - v_j)\widetilde{X}_j, \qquad W^\phi := \frac{\partial}{\partial y_1} + \phi\frac{\partial}{\partial t}.$$

Define

$$\begin{aligned}
B^* &:= \exp(\overline{X})(B) \\
&= B \star (0, (v'_2 - v_2, \ldots, v'_n - v_n, v'_{n+2} - v_{n+2}, \ldots, v'_{2n} - v_{2n}), 0) \\
&= (y_1, v', t - \sigma(v', v)) \\
B'' &:= \exp((y'_1 - y_1)W^\phi)(B^*) = (y'_1, v', t'') \quad \text{(for a certain } t''\text{)};
\end{aligned}$$

observe that B^* and B'' are well defined. For $n = 1$, $\overline{X}$ is not defined and we set $B^* = B$ and $B'' := \exp((y'_1 - y_1)W^\phi)(B) = (y'_1, t'')$.

We have to show that there exists $L > 0$ such that

$$(5.11) \qquad |\phi(B) - \phi(B')| \leq L d_\phi(B, B').$$

We have

$$(5.12) \qquad |\phi(B) - \phi(B')| \leq |\phi(B) - \phi(B^*)| + |\phi(B^*) - \phi(B'')| + |\phi(B'') - \phi(B')|$$

Notice that from Lemma 5.2

$$(5.13) \qquad |\phi(B) - \phi(B^*)| \leq \|(w_2, \ldots, w_n, w_{n+2}, \ldots, w_{2n})\|_{\mathrm{L}^\infty(\omega, \mathbb{R}^{2n-2})}\, |v' - v|$$

and then $|\phi(B) - \phi(B^*)| \leq d_\phi(B, B')$, from Lemma 5.1

$$(5.14) \qquad |\phi(B^*) - \phi(B'')| \leq \|w_{n+1}\|_{\mathrm{L}^\infty(\omega)}\, |y'_1 - y_1|$$

and then $|\phi(B^*) - \phi(B'')| \leq d_\phi(B, B')$.

By (5.14) we can apply Lemma 5.4 and obtain

$$(5.15) \qquad |\phi(B') - \phi(B'')| \leq C\sqrt{|t' - t''|}$$

Let's observe that

$$|t' - t''|$$

$$= \left| t' - t + \sigma(v', v) - \int_0^{y_1' - y_1} \phi(\exp(sW^\phi)(B^*)) \, ds \right|$$

$$\leq \left| t' - t - \frac{1}{2}(\phi(B') + \phi(B))(y_1' - y_1) + \sigma(v', v) \right| +$$

(5.16)

$$+ \frac{1}{2} \left| (\phi(B') + \phi(B))(y_1' - y_1) - 2 \int_0^{y_1' - y_1} \phi(\exp(sW^\phi)(B^*)) \, ds \right|$$

$$\leq d_\phi(B', B)^2 + \frac{1}{2}|\phi(B') - \phi(B'')| \, |y_1' - y_1| + \frac{1}{2}|\phi(B^*) - \phi(B)| \, |y_1' - y_1| +$$

$$+ \frac{1}{2} \left| [\phi(B'') + \phi(B^*)](y_1' - y_1) - 2 \int_0^{y_1' - y_1} \phi(\exp(sW^\phi)(B^*) \, ds \right|$$

$$=: \quad d_\phi(B', B)^2 + R_1(B', B) + R_2(B', B) + R_3(B', B).$$

For the case $n = 1$ we arrive to (5.16) with the same line (it is sufficient to follow the same steps "erasing" the term $\sigma(v', v)$).

Now we want to prove that for all $\epsilon > 0$ there is a $\delta_\epsilon \in]0, \delta_0]$ such that, for $\delta \in]0, \delta_\epsilon[$,

(5.17) $$R_1(B', B) \leq |y_1' - y_1|^2 + \epsilon|t' - t''|$$

for all $B', B \in I_\delta(A)$ and that there exist $C_1, C_2 > 0$ such that

(5.18) $$R_2(B', B) \leq C_2 d_\phi(B', B)^2$$

(5.19) $$R_3(B', B) \leq C_1 |y_1' - y_1|^2$$

for all $B', B \in I_{\delta_0}(A)$,

These estimates are sufficient to conclude: in fact, choosing $\epsilon := 1/2$ and using (5.16), (5.19), (5.17) and (5.18), we get

$$|t' - t''| \leq d_\phi(B', B)^2 + C_1|y_1' - y_1|^2 + |y_1' - y_1|^2 + |t' - t''|/2 + C_2 d_\phi(B', B)^2$$

whence

$$|t' - t''|^{1/2} \leq C_3 d_\phi(B, B')$$

which is $|\phi(B') - \phi(B'')| \leq d_\phi(B, B')$ and then the thesis (5.11).

By (5.15) we obtain

$$R_1(B', B) \leq 2C\sqrt{|t' - t''|} \, |y_1' - y_1| \leq \epsilon|t' - t''| + \frac{1}{\epsilon}|y_1' - y_1|^2,$$

whence (5.17) follows.

Observe that (5.18) follows from $R_2(B, B') = 0$ if $n = 1$, and from

$$R_2(B', B) = |y_1' - y_1||\phi(B) - \phi(B^*)|$$
$$\leq 2C_2|y_1' - y_1||v' - v| \leq C_2|(y_1' - y_1, v' - v)|^2 \leq C_2 d_\phi(B', B)^2$$

if $n \geq 2$. Finally, for $s \in [-\delta_0, \delta_0]$ we can define

(5.20) $$g(s) := 2 \int_0^s \phi(\exp(rW^\phi)(B^*)) \, dr - \left[\phi\big(\exp(sW^\phi)(B^*)\big) + \phi(B^*) \right] s;$$

We have

$$g(s) = 2 \int_0^s \left[\phi(\exp(rW^\phi)(B^*)) - \phi(B^*) \right] dr - \left[\phi\big(\exp(sW^\phi)(B^*)\big) - \phi(B^*) \right] s = o(s^2)$$

because $(-\delta_0, \delta_0) \ni s \mapsto \phi(\exp(sW^\phi)(B^*))$ is Lipschitz. Therefore (5.19) follows with $s = y_1' - y_1$. $\square$

Corollary 5.6. *Let $\omega \subset \mathbb{W}$ be an open set. If $\phi \in \mathrm{C}^0(\omega)$ is a distributional solution of $\nabla^\phi \phi = w$ in ω, then*

$$\text{(5.21)} \qquad |\phi(A) - \phi(B)| \leq C\sqrt{|A - B|} \qquad \forall\, A, B \in \omega$$

Lemma 5.7. *If $\phi \in \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega)$, then ϕ is a distributional solution of $\nabla^\phi \phi = w$ in ω.*

Proof. By Theorem 2.9 there exist $\{\phi_k\}_{k \in \mathbb{N}} \subset \mathrm{C}^\infty(\omega)$, such that ϕ_k uniformly converges to ϕ on the compact sets of ω, $|\nabla^{\phi_k}\phi_k(A)| \leq C$ $\mathcal{L}^n$-a.e. $A \in \omega$ for every $k \in \mathbb{N}$ and $\nabla^{\phi_k}\phi_k(A) \to w(A)$ $\mathcal{L}^n$-a.e. $A \in \omega$. Therefore, denoting $w_k := \nabla^{\phi_k}\phi_k$, we have for every $k \in \mathbb{N}$ and for every $\varphi \in \mathrm{C}_c^\infty(\omega)$

$$\int_\omega \phi_k X_j \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_{j,k}\varphi \, d\mathcal{L}^{2n}, \qquad j = 2, \ldots, n$$

$$\int_\omega \left(\phi_k \frac{\partial \varphi}{\partial y} + \frac{1}{2}\phi_k^2 \frac{\partial \varphi}{\partial t} \right) d\mathcal{L}^{2n} = -\int_\omega w_{n+1,k}\varphi \, d\mathcal{L}^{2n}.$$

$$\int_\omega \phi_k Y_j \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_{j+n,k}\varphi \, d\mathcal{L}^{2n}, \qquad j = 2, \ldots, n$$

Getting to the limit for $k \to \infty$ we obtain

$$\int_\omega \phi X_j \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_j \varphi \, d\mathcal{L}^{2n}, \qquad j = 2, \ldots, n$$

$$\int_\omega \left(\phi \frac{\partial \varphi}{\partial y} + \frac{1}{2}\phi^2 \frac{\partial \varphi}{\partial t} \right) d\mathcal{L}^{2n} = -\int_\omega w\varphi \, d\mathcal{L}^{2n}$$

$$\int_\omega \phi Y_j \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_{j+n}\varphi \, d\mathcal{L}^{2n}, \qquad j = 2, \ldots, n$$

i.e. ϕ is a distributional solution of the problem $\nabla^\phi \phi = w$. $\qquad\square$

6. Further Equivalences

In the previous section we established the equivalence between

$$\text{`}\phi : \omega \subset \mathbb{W} \to \mathbb{R} \text{ is intrinsic Lipschitz continuous'}$$

and

$$\text{`}\phi \in \mathrm{C}^0(\omega; \mathbb{R}) \text{ and there exists } w \in \mathrm{L}^\infty(\omega; \mathbb{R}^{2n-1}) \text{ such that } \nabla^\phi \phi = w \text{ in } \mathcal{D}'(\omega)\text{'.}$$

We establish now a characterization more related to the Lagrangian formulation: we prove that one can the reduce the PDE

$$\nabla^\phi \phi = w$$

along any integral line of the vector fields ∇_i^ϕ, $i = 2, \ldots, n$, provided that one chooses suitably the $\mathrm{L}^\infty(\omega; \mathbb{R}^{2n-1})$ representative $\hat{w}$ of the distribution identified by w. As well, if the characteristic equation is satisfied one has a distributional solution to the PDE.

We already motivated why focusing on the case $n = 1$, whereas the equation reduces to

$$\text{(3.1)} \qquad \phi_y(y,t) + \left[\frac{\phi^2(y,t)}{2} \right]_t = w(y,t).$$

We give generalizations to the case $n \geq 2$ in Remark 6.5

6.1. **Lagrangian solutions are distributional solutions.** In this section we prove, without passing through the implicit function theorem, that if a continuous function ϕ satisfies the Lagrangian formulation 3.6 of the balance law (3.1), then in the Eulerian variables ϕ solves indeed the balance law in distributional sense. See also [8], where a different, pointwise approximation of the distributional solution is provided, starting from a broad* solution. This basically shows the converse of Dafermos' statement in [11].

Theorem 6.1. *Every Lagrangian solution to* (3.1) *is also a distributional solution.*

Corollary 6.2. *Any continuous broad solution ϕ to* (3.1) *is also a distributional solution.*

Proof. Let $\chi(s,\tau)$ be a Lagrangian parameterization associated to ϕ, and $w : \omega \to \mathbb{R}$ such that

$$\phi(s,\chi(s,\tau)) - \phi(0,\chi(0,\tau)) = \int_0^s w(r,\chi(r,\tau))dr.$$

We prove then that ϕ is a distributional solution of the balance equation

$$\phi_y(y,t) + \left[\frac{\phi^2(y,t)}{2}\right]_t = w(y,t).$$

`Smoothing of` ϕ,χ `in the` τ`-variable.` Consider a suitable convolution kernel $\rho_\varepsilon(\tau)$, so that the τ-regularized function $\chi^\varepsilon(s,\tau)$ given by

$$\chi^\varepsilon(s,\tau) = \chi(s,\tau) * \rho_\varepsilon(\tau)$$

is strictly monotone in the τ variable. Define then the approximation $\phi^\varepsilon(s,t)$ by

$$\phi^\varepsilon(s,\chi^\varepsilon(s,\tau)) = \partial_s \chi^\varepsilon(s,\tau) = \partial_s \chi(s,\tau) * \rho_\varepsilon(\tau) = \phi(s,\chi(s,\tau)) * \rho_\varepsilon(\tau).$$

Since both χ and ϕ are continuous, the above relations immediately imply the uniform convergence of the regularized functions:

$$\chi^\varepsilon(s,\tau) \rightrightarrows \chi(\tau) \qquad\qquad \phi^\varepsilon(s,\chi^\varepsilon(s,\tau)) \rightrightarrows \phi(s,\chi(s,\tau)).$$

`Convergence in the` (s,t)`-variables of` ϕ^ε`.` We have chosen ρ in order that $\partial_\tau \chi^\varepsilon$ is strictly positive and $\chi^\varepsilon : (s,\tau) \mapsto y$ is invertible in the τ variable, for each s. Notice that $\partial_\tau \chi^\varepsilon(s,\tau)dsd\tau$ is converging (as a measure) to $\partial_\tau \chi(s,\tau)$. The above procedure then not only defines correctly the continuous functions $\phi^\varepsilon(s,t)$, but it allows to establish their convergence in L^1: indeed

$$\int |\phi^\varepsilon(s,t) - \phi(s,t)|dy = \int |\phi^\varepsilon(s,\chi^\varepsilon(s,\tau)) - \phi(s,\chi^\varepsilon(s,\tau))|\partial_\tau \chi^\varepsilon(s,\tau)d\tau$$

$$\leq \int |\phi^\varepsilon(s,\chi^\varepsilon(s,\tau)) - \phi(s,\chi(s,\tau))|\partial_\tau \chi^\varepsilon(s,\tau)d\tau$$

$$+ \int |\phi(s,\chi(s,\tau)) - \phi(s,\chi^\varepsilon(s,\tau))|\partial_\tau \chi^\varepsilon(s,\tau)d\tau.$$

The first factor in both the integrals is uniformly convergent to zero, while $\partial_\tau \chi^\varepsilon(s,\tau)dy$ converges to the measure $\partial_\tau \chi(s,\tau)$. The convergence of $(\phi^\varepsilon)^2/2$ is then straightforward.

`Approximation of the source.` Define now an approximate source $w^\varepsilon(s,t)$ by

$$w^\varepsilon(s,\chi^\varepsilon(s,\tau)) = \partial_{ss}\chi^\varepsilon(s,\tau) = \frac{d}{ds}\phi^\varepsilon(s,\chi^\varepsilon(s,\tau)).$$

Being a smooth function, the above relation is immediately equivalent to

(6.1) $$\phi_y^\varepsilon(y,t) + \left[\frac{(\phi^\varepsilon)^2}{2}\right]_t(y,t) = w^\varepsilon(y,t) \qquad \partial_s \chi^\varepsilon(s,\tau) = \phi^\varepsilon(s,\tau).$$

Since we started from a Lagrangian parameterization, the further regularity in the t variable

$$\partial_{ss}\chi(s,\tau) = \frac{d}{ds}\phi(s,\chi(s,\tau)) = \bar{w}(s,\chi(s,\tau))$$

for the relative pointwise representative $\bar{w}$ implies the relation

$$w^\varepsilon(s, \chi^\varepsilon(s, \tau)) = \partial_{ss}\chi^\varepsilon(s, \tau) = \partial_{ss}\chi(s, \tau) * \rho_\varepsilon(\tau) = \bar{w}(s, \chi(s, \tau)) * \rho_\varepsilon(\tau).$$

In particular, the sources w^ε are uniformly bounded by the L^∞ bound for $\bar{w}$. Moreover, for each t fixed $w^\varepsilon(s, \chi^\varepsilon(s, \tau))$ converges in all $\mathrm{L}^p(dt)$ to $\bar{w}(s, \chi(s, \tau))$, and thus in $\mathrm{L}^p(dydt)$; the convergence is clearly uniform when $\bar{w}$ is continuous.

The LHS of equation (6.1) passes to the weak limit by the L^1-convergence of $\phi^\varepsilon(y, t)$ to $\phi(y, t)$ established above. The same holds as well for the RHS, since $w^\varepsilon(y, t)dydt$ converge in $w^* - L^\infty$ to $\bar{w}(y, t)dydt$. Thus

$$\phi_y(y, t) + \left[\frac{\phi^2}{2}\right]_t (y, t) = \bar{w}(y, t)$$

holds, with $\phi, \bar{w}(y, t)$ precisely the functions given in the Lagrangian formulation. $\qquad\square$

We recall that the converse of this theorem is provided by Corollary 5.3: now we know also that $\bar{w}$ is a representative of the $\mathcal{L}^\infty(dydt)$-function w. We emphasize it with the following statement.

Corollary 6.3. *Suppose ϕ is a distributional solution to $\nabla^\phi \phi = w$. Then there exists a Borel pointwise representative $\bar{w}$ of w, associated to a Lagrangian parameterization, such that ϕ is a Lagrangian solution satisfying (3.5) precisely with $\bar{w}$.*

We state separately the approximation result in the proof of Theorem 6.1. It is useful for example in order to prove that if $\Upsilon^{-1}(B)$ is Lebesgue negligible, for some Borel set $B \subset \omega$, then also B must be Lebesgue negligible, even if Υ is not Lipschitz. We remind that $\Upsilon(s, \tau) = (s, \chi(s, \tau))$.

Corollary 6.4. *Under the hypothesis of Theorem 6.1, the τ-regularized functions*

$$\chi^\varepsilon(s, \tau) = \chi(s, \tau) * \rho_\varepsilon(\tau)$$

provide a strictly monotone in space flux whose associated vector field

$$\phi^\varepsilon(y, t) = \partial_s \chi^\varepsilon(s, (\chi^\varepsilon)^{-1}(t))$$

satisfies the Burgers equation with source term $w^\varepsilon(y, t) = \bar{w}(s, \chi(s, \tau)) * \rho_\varepsilon(\tau) \upharpoonright_{t=\chi(s,\tau),\ y=s}$,

$$\partial_y \phi^\varepsilon(y, t) + \partial_t \frac{\phi^{\varepsilon 2}(y, t)}{2} = w^\varepsilon(y, t),$$

and converge in $\mathrm{L}^1(dydt)$ to the vector field ϕ. The function w^ε converges to $\bar{w}$ in W_∞.

Remark 6.5. We finally remark that Theorem 6.1 works immediately also in higher dimensions, because for (almost every) $v \in \mathbb{R}^{2(n-1)}$ the restriction to the plane ω_v of ϕ is still a Lagrangian solution, with source term the restriction $\tilde{w} \upharpoonright_{\omega_v}$. For every test function $\varphi \in \mathrm{C}_c^\infty(\omega, \mathbb{R})$ we have then by Fubini-Tonelli Theorem

$$\iint_\omega \left[\varphi_{y_1}\phi + \varphi_t \frac{\phi^2}{2}\right] = \iint_{\mathbb{R}^{2(n-1)}} dv \int_{\omega_v} dy_1 dt \left[\varphi_{y_1}\phi + \varphi_t \frac{\phi^2}{2}\right] \upharpoonright_{\omega_v}$$

$$\stackrel{\text{Th. 6.1}}{=} -\iint_{\mathbb{R}^{2(n-1)}} dv \int_{\omega_v} dy_1 dt \, [\tilde{w} \upharpoonright_{\omega_v}] = -\iint_\omega \tilde{w}.$$

Considering also the linear fields we gain the implication (ii) to (i) in Theorem 1.2, and more generally that a Lagrangian solution is also a distributional solution.

6.2. **Distributional solutions are broad solutions.** Here we show that there exists a Borel function $\hat{w}(y,t)$ such that every curve $\gamma(s)$ satisfying the ODE $\dot{\gamma}(s) = \phi(s, \gamma(s))$ has time derivative Lipschitz, and it has second derivative precisely $\hat{w}(s, \gamma(s))$ for a.e. s. The remarkable fact is that $\hat{w}(y,t)$ could be defined independently of any set of characteristic curves. This is thus different, and stronger, from what we already proved, which is that there exists a Lagrangian parameterization χ and an associated function w_χ satisfying $\partial_s^2 \chi(s, \tau) = w_\chi(s, \chi(s, \tau))$. The new point is indeed that $\hat{w}$ is a universal representative for the source term.

Moreover, due to its nature this proof works for any n with no further complication. The only difference is that ω will be a subset of $\mathbb{R}^{2n}$ instead of $\mathbb{R}^2$. We write it with $n = 1$ only for notational convenience. Thus in particular we generalize here the previous Lemma 4.4 where we opted for a shortcut taking advantage from the continuity of χ, immediate only for $n = 1$.

Since the argument is more intuitive, we mention first how to construct such a Souslin function $\hat{w}(y,t)$. We proceed then with the Borel construction because it gives a better result.

6.2.1. *Souslin selection.* The first idea is to define pointwise, but in a measurable way, a function $\hat{w}(y,t)$ such that t is a Lebesgue point for the second derivative of a curve $s \mapsto \gamma(s)$ with $\gamma(y) = t$ and satisfying the ODE, whenever there exists one. Therefore one applies a selection theorem to the subset of

$$\omega \times \mathrm{C}^1 \times \mathbb{R} \supset \mathcal{G} \ni (y, t, \gamma, \zeta)$$

defined by

$$\mathcal{G} = \left\{ \gamma(y) = t, \ \dot{\gamma} = \phi \circ (\mathbb{I} \otimes \gamma), \ |\dot{\gamma}(r) - \dot{\gamma}(s)| \leq \|w\|_\infty |r - s|, \ \zeta = \lim_{\sigma \downarrow 0} \frac{1}{\pm\sigma} \int_y^{y \pm \sigma} \ddot{\gamma}(s) ds \right\}.$$

Lemma 6.6. *$\mathcal{G}$ is Borel. It has full measure projection on $(y,t) \in \omega$.*

Proof. `Components` (y, t, γ). The subset

(6.2) $$(y, t, \gamma, \zeta) \ni C \subset \omega \times \mathrm{C}^1(\mathbb{R}) \times \mathbb{R}$$

identified by the constraints

$$\gamma(s) = x \qquad \dot{\gamma} = u \circ (\mathbb{I} \otimes \gamma) \qquad |\dot{\gamma}(\tau) - \dot{\gamma}(\sigma)| \leq \|w\|_\infty |\tau - \sigma|$$

is closed. By Lemma 5.1, moreover, its projection on (y,t) is all $\mathbb{R}^+ \times \mathbb{R}$. By Rademacher theorem, we have moreover seen in the same lemma also that the projection of $\mathcal{G}$ on (y,t) has full measure.

`Component` ζ, `discretization.` In order to establish the existence (and the value) of the limit for the second derivative of γ at y, it suffices to consider e.g. the sequence

$$h_{n+1} = h_n - h_n^2, \qquad h_1 = 1/2.$$

Indeed, then for $h \in (h_{n+1}, h_n]$, for example at $y = 0$

$$\left| \frac{1}{h} \int_0^h \ddot{\gamma} - \frac{1}{h_n} \int_0^{h_n} \ddot{\gamma} \right| = \left| \left(\frac{1}{h} - \frac{1}{h_n} \right) \int_0^h \ddot{\gamma} - \frac{1}{h_n} \int_h^{h_n} \ddot{\gamma} \right| \leq 2\|w\|_\infty \frac{h_n - h}{h_n}.$$

By construction however

$$|h_n - h| \leq |h_n - h_{n+1}| = h_n^2,$$

yielding that the existence of the limit along $\{h_n\}_n$ implies the existence of the limit for any $h \downarrow 0$. Notice that this would not hold choosing a generic $\tilde{h}_n \downarrow 0$ instead of $\{h_n\}_n$.

`Measurability of` $\mathcal{G}$. The further constraint in γ can be written as

$$\forall k \ \exists n \ \forall \bar{n} \geq n : \qquad \left| \zeta - \frac{1}{\pm h_{\bar{n}}} \int_y^{y \pm h_{\bar{n}}} \ddot{\gamma}(s) ds \right| \leq 2^{-k}.$$

Therefore, we are considering the following subset of C:

$$\mathcal{G} = C \cap \bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \bigcap_{\bar{n} \geq n} \left\{ \left| \zeta - \frac{\dot{\gamma}(y \pm h_{\bar{n}}) - \dot{\gamma}(y)}{\pm h_{\bar{n}}} \right| \leq 2^{-k} \right\}.$$

Since the set within brackets is closed, $\mathcal{G}$ is Borel. $\qquad\square$

This allows to define an $\mathcal{A}$-selection $(y,t) \mapsto \gamma_{(y,t)}$, which by construction is a measurable function assigning to every point (y,t) an integral curve $\gamma_{(y,t)}$ for the ODE $\dot{\gamma}_{(y,t)}(s) = \phi(s, \gamma_{(y,t)}(s))$, whenever there exists a Lipschitz one having y as a Lebesgue point for the right, left and total second derivative. As well, one can define the Souslin function

$$(y,t) \mapsto \bar{w}(y,t) := \ddot{\gamma}_{(y,t)}(t) = w_{(y,t)}.$$

The importance of this selection is due to the following theorem.

Theorem 6.7. *For every curve γ satisfying the ODE $\dot{\gamma}(s) = \phi(s, \gamma(s))$, one has*

$$\phi(s, \gamma(s)) - \phi(0, \gamma(0)) = \int_0^s \hat{w}(r, \gamma(r)) dr.$$

Corollary 6.8. *A continuous distributional solution ϕ to $\nabla^\phi \phi = w$ is also a broad solution.*

6.2.2. *Borel selection.* Before proving the theorem, for the sake of completeness we show that one can define as well a Borel fucntion, that we still denote as $\hat{w}(y,t) = w_{y,t}$, for which Theorem 6.7 still holds. This requires a bit more work than the previous argument, and it is conceptually a little more involved: we do not associate immediately to each point (where it is possible) an eligible curve and its second derivative, but something which must be close to it. We will find then with the proof of Theorem 6.7 that we end up basically with the same selection.

Lemma 6.9. *For every $\varepsilon > 0$, there is a Borel function defined on the (y,t)-projection $\mathcal{D}$ of $\mathcal{G}$*

$$(y,t) \mapsto (\gamma_{\varepsilon,y,t}, w_{\varepsilon,y,t})$$

such that $(y, t, \gamma_{\varepsilon,y,t}, w_{\varepsilon,y,t}) \in C$ of (6.2) and that for $|h|$ sufficiently small

$$\left| w_{\varepsilon,y,t} - \frac{1}{h} \int_y^{y+h} \ddot{\gamma}_{\varepsilon,y,t} \right| < \varepsilon.$$

Definition 6.10. *We define the Borel representative of w as the function*

$$\hat{w}(y,t) = w_{y,t} = \chi_{\{(y,t) \in \mathcal{D}\}} \liminf_{\varepsilon \downarrow 0} w_{\varepsilon,y,t}.$$

Proof of Lemma 6.9. We apply Arsenin-Kunugui selection theorem ([18], or Th. 5.12.1 of [24]) to the set

$$(6.3) \qquad \bigcup_{n \in \mathbb{N}} \bigcap_{m > n} \left\{ (y, t, \gamma, w) \in C : \left| w - \frac{\dot{\gamma}(y + h_m) - \dot{\gamma}(y)}{h_m} \right| \leq \varepsilon \right\},$$

where C was defined in (6.2) and $\{h_n\}_{n \in \mathbb{N}}$ immediately below that, in the same proof. More precisely, C is closed and

$$\left\{ (\gamma, \zeta) : \left| w - \frac{\dot{\gamma}(y + h) - \dot{\gamma}(y)}{h} \right| \leq \varepsilon. \right\}$$

is also closed: therefore each (y,t)-section of (6.3) is σ-compact. Then the hypothesis of the theorem are satisfied: it assures that the projection of (6.3) on the first factor $\omega \ni (y,t)$ is Borel and there exists a Borel section of (6.3) defied on it, which is the function in our statement.

Notice that the domain fo this function, containing $\mathcal{D}$, has full Lebesgue measure. $\qquad\square$

6.2.3. *Proof of Theorem 6.7.* We provide now the proof of Theorem 6.7 with $w_{y,t}$ either the Borel or the Souslin one: we consider any characteristic $\bar\gamma$ for the balance law and we prove that for almost every y its second derivative is precisely $w_{y,\bar\gamma(y)}$. We remind that $\dot{\bar\gamma}(y)$ is Lipschitz (Lemma 5.1).

Step 1, countable decomposition. The set of y where the second derivative of $\bar\gamma(y)$ exists and it is different from $w_{t,\bar\gamma(y)}$ can be reduced to

$$\bigcup_{\varepsilon\downarrow 0}\left\{s:\ \left|w_{y,\bar\gamma(y)}-\ddot{\bar\gamma}(y)\right|\geq\varepsilon\right\}$$

$$\subset\bigcup_{\varepsilon\downarrow 0}\bigcup_{n\in\mathbb{N}}\left\{y:\ \forall\sigma<2^{-n}\ \ \left|w_{y,\bar\gamma(y)}-\frac{1}{\sigma}\int_y^{y+\sigma}\ddot{\bar\gamma}\right|\geq\varepsilon,\ \ \left|w_{y,\bar\gamma(y)}-\frac{1}{\sigma}\int_{y-\sigma}^y\ddot{\bar\gamma}\right|\geq\varepsilon\right\}$$

$$=\bigcup_{\tilde\varepsilon<\varepsilon\downarrow 0}\bigcup_{n\in\mathbb{N}}\left\{y:\ \forall\sigma<2^{-n}\ \ \left|w_{y,\bar\gamma(y)}-\frac{1}{\pm\sigma}\int_y^{y\pm\sigma}\ddot{\bar\gamma}\right|\geq 3\varepsilon,\ \ \left|w_{t,\bar\gamma(s)}-\frac{1}{\pm\sigma}\int_y^{y\pm\sigma}\ddot\gamma_{\tilde\varepsilon,t,\bar\gamma(s)}\right|<\varepsilon\right\}.$$

If one is considering the Souslin selection, clearly there is the simplification $\gamma_{\tilde\varepsilon,y,t}=\gamma_{y,t}$.

Step 2, reduction argument. We prove that the set

$$(6.4)\qquad\left\{y:\ \forall\sigma<2^{-n}\ \ w_{y,\bar\gamma(y)}>\frac{1}{\pm\sigma}\int_y^{y\pm\sigma}\ddot{\bar\gamma}+3\varepsilon,\ \ \left|w_{y,\bar\gamma(y)}-\frac{1}{\sigma}\int_y^{y+\sigma}\ddot\gamma_{\tilde\varepsilon,y,\bar\gamma(y)}\right|<\varepsilon\right\}$$

cannot contain points y_1,y_2 with $|y_1-y_2|\leq 2^{-n}$. Then the thesis will follow: by the previous step the set of y where the second derivative of $\bar\gamma(y)$ exists and it is different from $w_{t,\bar\gamma(y)}$ will be countable. Therefore the second derivative of $\bar\gamma(y)$ will be almost everywhere precisely $w_{t,\bar\gamma(y)}$.

Step 3, analysis of the single sets. By contradiction, assume that (6.4) contains two such points, for example $y_1=0$, $y_2=\sigma$. By definition of the set of points we are considering, the two selected curves through $(0,\bar\gamma(0))$, $(\sigma,\bar\gamma(\sigma))$,

$$\gamma_0:=\gamma_{\tilde\varepsilon,0,\bar\gamma(0)},\qquad\gamma_\sigma:=\gamma_{\tilde\varepsilon,\sigma,\bar\gamma(\sigma)}$$

must intersect in the time interval $[0,\sigma]$, say at time σ'. Since they satisfy the ODE for characteristics, where they intersect they have the same derivative. Being all of them Lipschitz, we have then

$$\dot{\bar\gamma}(0)+\int_0^{\sigma'}\ddot\gamma_0=\dot\gamma_\sigma(\sigma')=\dot{\bar\gamma}(\sigma)-\int_{\sigma'}^\sigma\ddot\gamma_\sigma.$$

Comparing the LHS and the RHS, one arrives to

$$\dot{\bar\gamma}(\sigma)-\dot{\bar\gamma}(0)=\int_0^{\sigma'}\ddot\gamma_0+\int_{\sigma'}^\sigma\ddot\gamma_\sigma.$$

However, since the times $0,\sigma$ belong by construction to the set (6.4) one has

$$\int_0^{\sigma'}\ddot\gamma_0+\int_{\sigma'}^\sigma\ddot\gamma_\sigma>w_{0,\bar\gamma(0)}\sigma'+w_{\sigma,\bar\gamma(\sigma)}(\sigma-\sigma')-2\varepsilon>\int_0^\sigma\ddot{\bar\gamma}+\varepsilon=\dot{\bar\gamma}(\sigma)-\dot{\bar\gamma}(0)+\varepsilon$$

reaching a contradiction. $\qquad\square$

Corollary 6.11. *The various notions of continuous solutions we have considered are thus equivalent.*

APPENDIX A. FROM PARTIAL TO FULL LAGRANGIAN PARAMETERIZATIONS

In the present section we deal with the issue of extending a partial Lagrangian parameterization to a 'full' one. We construct a function $\chi(s,\tau)$ satisfying the ODE (3.4), both monotone and surjective in the τ variable, which extends a given one $\tilde\chi$. This is the matter of Lemma A.1: we recall below how to extend a solution to an ODE with continuous coefficients, whose existence is a classical result.

The procedure can be first understood considering Example A.2 below, illustrated in Figure 2. This deals with the simpler case of an s-independent ϕ, but it has all the ingredients of the construction below.

Moreover, Example A.2 provides a counterexample for the following fact: even if characteristics are C^1 in s with Lipschitz derivative, *it is not possible in general to extend a partial, monotone Lagrangian parameterization to a full one which is locally Lipschitz continuous.*

The reduction of the balance law along characteristics, which is Equation (3.5), will be instead studied more deeply in the following. Here we just notice that it holds, with some $\bar{w}_\chi$ pointwise defined in ω, for a particular Lagrangian parameterization $(\tilde{\omega}, \chi)$. The argument is 2-dimensional.

Lemma A.1. *Any partial Lagrangian parameterization can be extended to a full one.*

Proof. Let $\tilde{\chi}(s, \tau)$ be a partial Lagrangian parameterization. Focus e.g. the attention on $s, \tau \in [0, 1]$ and $\tilde{\chi}(s, \tau)$ valued in $[0, 1]$ and right continuous, the general case being similar.

We construct an extension $\tilde{\chi}'$ by a recursive procedure. For convenience, the induction index is given by couples (h, n) with $n \in \mathbb{N}$ and $h = 0, \ldots, 2^{n-1} - 1$. The ordering is lexicographic, starting from the second variable: $(h_1, n_1) \le (h_2, n_2)$ iff either $n_1 < n_2$ or $n_1 = n_2$ and $h_1 \le h_2$.

The starting point is $\chi^0 = \tilde{\chi}$ defined for $s \in [0, 1]$, $\tau \in T_0 = [0, 1]$. Consider the dichotomous points $s^{h,n} = 2^{-n} + 2^{-n+1} h$, which go from 2^{-n} to $1 - 2^{-n}$ at step 2^{-n+1}, associated to the indexes (h, n) with $n \in \mathbb{N}$ and $h = 0, \ldots, 2^{n-1} - 1$.

`Induction step` (h, n), $n \ge 1$: Assume you have been given χ defined on $(s, \tau) \in [0, 1] \times T$ by a previous step. If at $s = s^{h,n}$ the map $\tau \mapsto \chi(s^{h,n}, \tau)$ is not onto $[\chi(s^{h,n}, 0), \chi(s^{h,n}, 1)]$ we construct an extension $\chi^{h,n}$ such that

- $\tau \mapsto \chi^{h,n}(s^{h,n}, \tau)$ is onto $[\chi(s^{h,n}, 0), \chi(s^{h,n}, 1)]$ for $h = 0, \ldots, 2^{n-1} - 1$;
- there exists a strictly increasing map $j^{h,n}$, with $\mathcal{L}^1(j^{h,n}(T)) - \mathcal{L}^1(T) \le 2^{1-2n}$, such that

$$\chi^{h,n}(t, j^{h,n}(\tau)) = \chi(s, \tau).$$

These properties of the new partial Lagrangian parameterization will determine that we get at the end a limit which is a full parameterization extending $\tilde{\chi}$.

Because of monotonicity the complementary of the image of $\tau \mapsto \chi(s^{h,n}, \tau)$ is the at most countable union of disjoint intervals $\{I_k\}_k$. Let τ_k be those points corresponding to characteristics in χ which bifurcate, and open to an interval I_k at time $s^{h,n}$:

$$\min I_k = \chi(s^{h,k}, (\tau_k)^-) \qquad \sup I_k = \chi(s^{h,k}, \tau_k).$$

Define consequently the strictly increasing map opening each of those points τ_k into an interval proportional (with factor $1/2^{2n-1}$) to the hole I_k that the relative characteristics leave at $s^{h,n}$:

$$j^{h,n} : \qquad \begin{matrix} T & \to & T_{h,n} = [0, j^{h,n}(1)] \\ \tau & \mapsto & \tau + \mathcal{L}^1(\cup_{\tau_k \le \tau} I_k)/2^{2n-1}. \end{matrix}$$

The inequality $|T_{h,n}| - |T| \le 2^{1-2n}$ holds because we are assuming that χ is valued in $[0, 1]$, thus $\mathcal{L}^1(\cup_k I_k) \le 1$.

For each $z \in I_k$, consider the C^1 maximal curve through $(s^{h,n}, z)$ until it touches $s \mapsto \chi(s, \tau_k^\pm)$: recalling the curve $\gamma^{(s,z)}$ defined at (4.1), and setting

$$s_+^\pm(s^{h,n}, z) = \inf\{s > s^{h,n} : \gamma^{(s^{h,n}, z)}(s) = \chi(s, \tau_k^\pm)\}$$

$$s_-^\pm(s^{h,n}, z) = \sup\{s > s^{h,n} : \gamma^{(s^{h,n}, z)}(s) = \chi(s, \tau_k^\pm)\},$$

then a possible right continuous extension is give by

$$\gamma(s; s^{h,n}, z) := \begin{cases} \gamma^{(s^{h,n},z)}(s) & \text{for } \max s^{\pm}_{-}(s^{h,n}, z) \leq s \leq \max s^{\pm}_{+}(s^{h,n}, z), \\ \chi(s, \tau_k) & \text{otherwise.} \end{cases}$$

Then define on $[0,1] \times T_{h,n}$ the Lagrangian parameterization

$$\chi^{h,n}(s,w) = \begin{cases} \chi(s,\tau) & \text{if } w = j^{h,n}(\tau), \\ \gamma(s; s^{h,n}, z) & \text{if } w \in \left[j^{h,n}(\tau_k^-), j^{h,n}(\tau_k) \right), \; z = \chi(s^{h,k}, \tau_k) - \left(j^{h,n}(\tau_k) - w \right). \end{cases}$$

Notice that also the surjectivity property at $s = s^{h,n}$ is satisfied.

Conclusion. Let us first look at how much the domain T_0 grows in the extension process. Since 2^{n-1} couples of indices have second variable n, then the total size of the intervals added by those couples alltogether is at most 2^{-n}: thus, setting $T_n = \cup_h T_{h,n}$,

$$T_0 = [0,1], \quad T_1 \subset [0, 3/2], \quad \dots, \quad T_n \subset [0, 2 - 2^{-n}], \quad \dots.$$

The maps $j^{\bar{h},\bar{n}} \circ \cdots \circ j^{0,1}$ are strictly increasing, valued in $[0, 2 - 2^{-\bar{n}}]$ with 1-Lipschitz inverses. By Ascoli-Arzelà theorem the inverses converge uniformly, to a monotone map which is the inverse of a right continuous function $j : [0,1] \to [0,2]$. By construction j is strictly increasing.

Moreover, by construction each $\chi^{\bar{h},\bar{n}}$ is surjective at each $s = s^{h,n}$ with $(h,n) \leq (\bar{h}, \bar{n})$.

Being $\|\chi^{\bar{h},\bar{n}} - \chi^{h,n}\|_\infty \leq 2^{-n}$ for $(h,n) \geq (\bar{h}, \bar{n})$, the sequence $\chi^{h,n}$ converges uniformly. One immediately verifies that the limit is a monotone, Lagrangian parameterization χ which extends $\tilde{\chi}$, with injection map j.

Being surjective and monotone, each $\tau \to \chi(s^{h,m}, \tau)$ is continuous. By the continuity in s we deduce then surjectivity also at the remaining times: indeed if by absurd we had $\chi(\bar{s}, \bar{\tau}^-) \neq \chi(\bar{s}, \bar{\tau}^+)$, we could not have $\chi(s^{h,m}, \bar{\tau}^-) = \chi(s^{h,m}, \bar{\tau}^+)$ at $s^{h,m}$ arbitrarily close to $\bar{s}$. $\qquad \square$

The following example introduces the extension of a Lagrangian parameterization. It shows moreover that it is not possible in general to get a full one which is Lipschitz continuous, even though characteristics below are twice continuously differentiable.

Example A.2. *We consider χ_m for the very simple case of an equation $(\phi^2/2)_t = w$, $\phi_y = 0$. The function ϕ will be constructed below, and its integral curves $\dot{\gamma}(s) = \phi(\gamma(s))$ are as in Figure 2, that now we describe.*

Define first a smooth function $\gamma(s)$, for $s \in [0,1]$, which increases continuously from 0 to 1. Let $\dot{\gamma}(s - 1/2)$ be even, strictly increasing in the first half interval from 0 to its maximum. Let $\ddot{\gamma}$ vanish at $0; 1/2; 1$ and be positive in $[0, 1/2]$. For instance, consider

$$\gamma(s) = s + 1/(2\pi)\sin(2\pi s - \pi), \qquad \gamma'(s) = 1 + \cos(2\pi s - \pi).$$

Define then the following points where $\phi(t)$ and $w(t)$ will vanish: for $i \in \mathbb{N}$, $h = 0, \dots, 2^i$

$$z_0 := 0, \qquad z_\infty := -\sum_{j=1}^{\infty} \frac{2^{-j}}{\ln(j+2)},$$

$$z_i = -\sum_{j=1}^{i} \frac{2^{-j}}{\ln(j+2)} = z_{2^i, i} = z_{0, i+1}, \qquad z_{h,i} := z_{i-1} - \frac{2^{-2i}h}{\ln(i+2)}.$$

The first $\{z_i\}_{i \in \mathbb{N}}$ are approximatively $0, -0.455, -0.635, -0.713, -0.748, -0.764, -0.772, \dots$, and each interval $[z_{i+1}, z_i]$ is divided into 2^i equal subintervals.
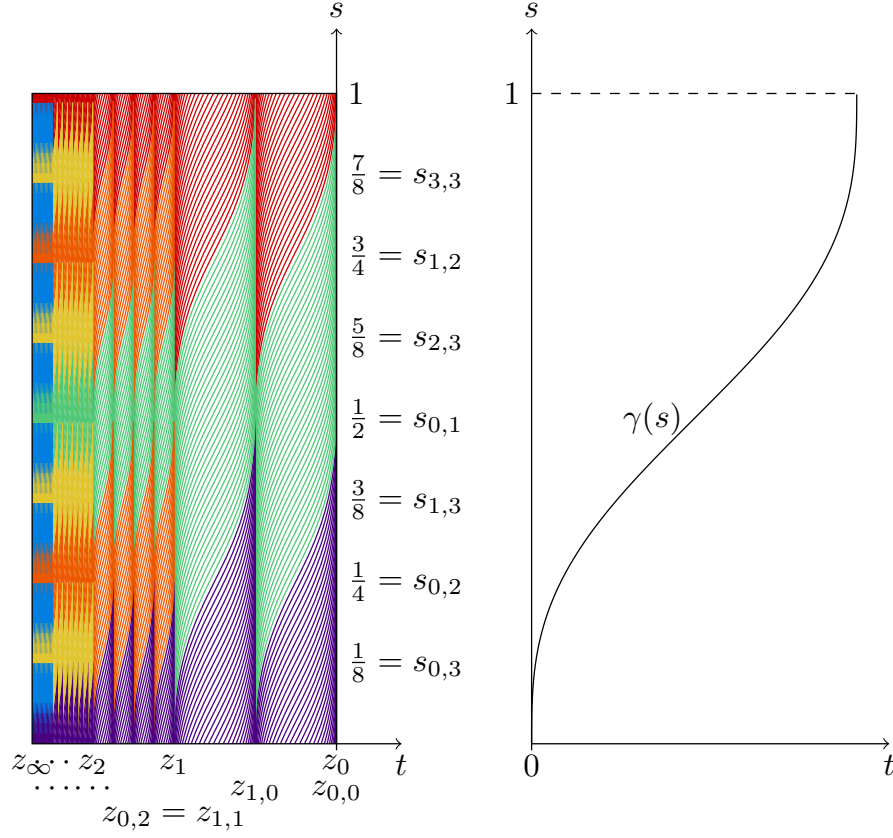
FIGURE 2. A partial monotone Lagrangian parameterization is extended to a 'full' one. Different colors denote different steps in the extension, which are countably many. Each step corresponds to a dichotomous value of s: the relative extension of the parameterization must cover the relative s-section. In this example the full parameterization can not be locally Lipschitz.

Now we define the functions $\phi(t)$, $w(t)$ on each subinterval $[z_{h+1,i}, z_{h,i}]$. As a preliminary step consider the rescaled smooth functions $\gamma_{h,i}(s) = \chi_m(s, z_{h+1,i}^+)$ given by

$$\gamma_{h,i}(s) = z_{h+1,i} + \frac{\gamma(2^i s)}{2^{2i} \ln(i+2)}, \qquad s \in [0, 2^{-i}],$$

Notice that each increases monotonically from $z_{h+1,i}$ to $z_{h,i}$. Define then

$$\phi(t) = \gamma'_{h,i}(s), \quad w(t) = \gamma''_{h,i}(s) \quad : \quad \gamma_{h,i}(s) = t.$$

Since $|\dot{\gamma}_i| \leq C2^{-i}/\ln(i+2)$ and $|\ddot{\gamma}_i| \leq C/\log(i+2)$, therefore w and ϕ are continuous up to z_∞, where they vanish.

In this case there is no Lipschitz extension of the parameterization χ_m to a surjective one. Indeed, minimal characteristics starting from $s = 0$ do not cover almost all the interval at $s = 1$. Adding those starting at $s = 1$, it remains to cover open intervals of total length

$$\sum_{j=1}^{\infty} 2^j \frac{2^{-2j}}{\ln(j+2)} = z_0 - z_\infty$$

at $s = 1/2$. Including all those characteristics which intersect the line $s = 1/2$, similarly, one do not cover the whole line $s = 1/4; 3/4$: a length

$$\sum_{j=2}^{\infty} 2^{-j} \ln(j+2) = z_1 - z_\infty$$

remains to cover at both those two values of s. At the subsequent i-th step, one has to cover a length $\sum_{j=i}^{\infty} 2^{-j} \ln(j+2)$ at $s_{h,i} = 2^{-i} + h2^{-i+1}$ for $h = 0, \ldots, 2^{i-1} - 1$. In the whole process, it must be covered a total length equal to

$$\sum_{j=1}^{\infty} (1 + \cdots + 2^{j-1}) \frac{2^{-j}}{\ln(j+2)} = \sum_{j=1}^{\infty} (2^{j+1} - 2) \frac{2^{-j}}{\ln(j+2)} \geq \sum_{j=1}^{\infty} \frac{1}{\ln(j+2)} = +\infty.$$

Any monotone, Lagrangian parameterization χ must map a disjoint family of real intervals $\{I_{h,i}\}_{h,i}$ with $\sum_{h,i} |I_{h,i}| < 1$ to the intervals $\{[x_\infty, x_{i+1}]\}_i$, respectively at the above $s_{h,i}$. However, we have just computed that for all constant C

$$\infty = \sum_{h=0,\ldots,2^{i-1}-1, \ i \in \mathbb{N}} |x_{i+1} - x_\infty|$$

$$= \sum_{h=0,\ldots,2^{i-1}-1, \ i \in \mathbb{N}} \chi(s_{h,i}, [I_{h,i}])$$

$$\geq C \sum_{h=0,\ldots,2^{i-1}-1, \ i \in \mathbb{N}} |I_{h,i}|,$$

preventing any Lipschitz regularity. Indeed, the map $\tau \mapsto \chi(s_{h,i}, \tau)$ can be Lipschitz with some constant C_i, but C_i must blow up as $i \to \infty$. At other values of s, this map is just $\mathrm{W}^{1,1}(\mathbb{R})$.

One can as well construct examples where χ has a Cantor part.

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