POINCARÉ INEQUALITIES IN QUASIHYPERBOLIC BOUNDARY CONDITION DOMAINS

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ABSTRACT. We study the validity of (q, p)-Poincaré inequalities, $q < p$, on domains in $\mathbb{R}^n$ which satisfy a quasihyperbolic boundary condition, i.e. domains whose quasihyperbolic metric satisfies a logarithmic growth condition. In the present paper, we show that the quasihyperbolic boundary condition domains support a (q, p)-Poincaré inequality whenever $p > p_0$, where p_0 is an explicit constant depending on q, on the logarithmic growth condition, and on the boundary of the domain.

1. INTRODUCTION

A bounded domain G in $\mathbb{R}^n$, $n \geq 2$, is said to support a (q, p)-Poincaré inequality if there exists a finite constant c such that the inequality

$$\left(\int_G |u(x) - u_G|^q \, dx\right)^{1/q} \leq c \left(\int_G |\nabla u(x)|^p \, dx\right)^{1/p} \tag{1.1}$$

holds for all functions u in the Sobolev space $W^{1,p}(G)$; here $1 \leq p, q < \infty$ and u_G is the integral average of u over G. If G is a John domain (see Definition 4.1), then it is well known that (1.1) is valid for all (q, p) where $1 \leq p \leq q \leq np/(n-p)$ [1, Theorem 5.1]. Property (4.2) of John domains implies that a Poincaré inequality supported by balls is valid also in John domains. In this paper we consider a larger class of domains which do not inherit the inequalities which balls support; we study bounded domains satisfying the quasihyperbolic boundary condition, see Definition 2.2.

A proper subclass of quasihyperbolic boundary condition domains is formed by John domains, but domains in the former class allow narrow gaps which can destroy the John condition (4.2), [4, Example 2.26]. This kind of effect implies that a (p, p)-Poincaré inequality fails

2010 *Mathematics Subject Classification.* 46E35, 26D10, 35A23.

Key words and phrases. Irregular domain, John domain, Minkowski dimension, Poincaré inequality, Quasihyperbolic boundary condition, Quasihyperbolic distance, Sobolev space, Whitney decomposition.

A. V. V. was supported by the Academy of Finland, grants 75166001 and 1134757, and by the Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation.

to hold for small values of p, whereas the domain does support the (p,p)-Poincare inequality for large enough p.

A (q,p)-Poincaré inequality is valid in a β-quasihyperbolic boundary condition domain, if $n-n\beta<q=p<\infty$, see [10, Theorem 1.4], and also [7, Remark 7.11]; and if $n-n\beta<p\le q<\beta np/(n-p)$, whenever $p<n$, [10, Theorem 1.5], [8, Theorem 1.4]. It is shown in [10, Example 5.5] that if $1\le p<n-n\beta$, there exists β-quasihyperbolic boundary condition domains which do not support the (p,p)-Poincaré inequality. We remark that β-quasihyperbolic boundary condition domains support $(1,p)$-Poincaré inequality for all $p>n-n\beta$ by Hölder's inequality while John domains support $(1,p)$-Poincaré inequality for all $1\le p<\infty$. The question one may ask is, what can be said about the validity of (q,p)-Poincaré inequalities in the case $1\le q<\min\{n-n\beta,p\}$.

Poincaré inequalities, (1.1), in the case $1\le q<p$ have been considered on general domains, e.g., in [13, Section 6.4], see also [5] and the $(1,p)$-case in [6]. Maz'ya [13], Theorem 6.4.3/2 on p. 344, gives a characterization for domains which support (1.1) when $q<p$. In addition, this class of domains characterizes certain compact embeddings, see Theorem 6.8.2/2 on p. 376 [13] for more details. Maz'ya presents also applications to the Neumann problems for strongly elliptic operators in domains which characterize (1.1) with $p=2$ and $1\le q<2$, cf. Section 6.10.1. We shall discuss applications in Section 6.

In the present paper, we answer the question about (q,p)-Poincaré inequality for quasihyperbolic boundary condition domains in the case $1\le q<\min\{n-n\beta,p\}$. We use the upper Minkowski dimension of the boundary. Roughly speaking, an issue is the counting of the number of those Whitney cubes, of a given size, whose shadows are comparable in measure. The shadow of a fixed Whitney cube is the union of those cubes to which one goes through the fixed cube when approaching the boundary of the domain from inside. The use of the upper Minkowski dimension enables us to count the aforementioned cubes in an efficient manner. Previously the upper Minkowski dimension of the boundary has been used in studying weighted Poincaré inequalities in [2] and [3], but maybe not to its full potential. On the other hand, the upper Minkowski dimension seems to be a right tool for the (q,p)-Poincaré inequality with $q<p$, see Lemma 3.4 and Lemma 3.9 in Section 3.

More precisely, we show that a β-quasihyperbolic boundary condition domain with the upper Minkowski dimension of the boundary being less than or equal to $\lambda\in[n-1,n)$ supports the (q,p)-Poincaré inequality (1.1) with $1\le q<p<\infty$ if

$$p>\frac{q(n-\lambda\beta)}{q+\beta(n-\lambda)}\,, \tag{1.2}$$

see Theorem 3.1. Here, the right hand side bound is, as it should be, an increasing function of λ, when $q<n-n\beta$. Namely a quasihyperbolic

boundary condition domain is more irregular and Poincaré inequality (1.1) fails to hold more easily when the upper Minkowski dimension of the boundary is larger. We also show that the bound in (1.2) is essentially sharp in essentially all the possible cases in the plane, see Theorem 5.1 and Remark 5.9; and we discuss sharpness of the bound in higher dimensions, see Theorem 4.7, Theorem 4.15, and Remark 5.2.

To show that our results are sharp in the plane we introduce a method for modifying any given John domain in a controlled manner so that the resulting domain is no more a John domain but it satisfies a quasihyperbolic boundary condition.

The structure of this paper is as follows. In Section 2 we recall the quasihyperbolic boundary condition and some basic facts related to this condition and the geometry of Whitney cubes; we also recall the shadow of a Whitney cube. Lemma 3.4, Lemma 3.6 and Lemma 3.9 in Section 3 are the key ingredients in the proof of the main result of the paper, Theorem 3.1. In Section 4 we modify a given John domain in order to revoke its John properties and to obtain a quasihyperbolic boundary condition domain. We use such a modification in Section 5 where we consider sharpness of our main result in the plane. We close the paper by giving an application to the solvability of the Neumann problem on quasihyperbolic boundary condition domains in Section 6.

2. Notation and preliminaries

Throughout the paper G is a bounded domain (an open connected set) in $\mathbb{R}^n$, $n \geq 2$. The closure, the interior, and the boundary of a set $E \subset \mathbb{R}^n$ are denoted by $\overline{E}$, $\mathrm{int}(E)$, and ∂E, respectively. We write χ_E for the characteristic function of E, and the Lebesgue n-measure of a measurable set E is written as $|E|$. The Hausdorff dimension is denoted by $\dim_{\mathcal{H}}(E)$. The upper Minkowski dimension of a set E is

$$\dim_{\mathcal{M}}(E) = \sup\{\lambda \geq 0 \,:\, \limsup_{r\to 0+} \mathcal{M}_\lambda(E,r) = \infty\},$$

where for each $r > 0$

$$\mathcal{M}_\lambda(E,r) = \frac{|\bigcup_{x\in E} B^n(x,r)|}{r^{n-\lambda}}$$

is the λ-dimensional Minkowski precontent.

The family of closed dyadic cubes is denoted by $\mathcal{D}$. The side length of a cube $Q \subset \mathbb{R}^n$ is $\ell(Q)$ and its centre is x_Q. We let $\mathcal{D}_j$ be the family of those dyadic cubes whose side length is 2^{-j}, $j \in \mathbb{Z}$. For a domain G we fix a Whitney decomposition $\mathcal{W} = \mathcal{W}_G \subset \mathcal{D}$. We write $\mathcal{W}_j = \mathcal{W} \cap \mathcal{D}_j$, $j \in \mathbb{Z}$, and $\sharp\mathcal{W}_j$ is the number of those cubes in $\mathcal{W}$ whose side length is 2^{-j}. For a Whitney cube $Q \in \mathcal{W}$ let us write $Q^* = \frac{9}{8}Q$. Then

$$\mathrm{diam}(Q) \leq \mathrm{dist}(Q, \partial G) \leq 4\,\mathrm{diam}(Q),$$

and $\sum \chi_{Q^*} \leq 12^n$. For the construction of Whitney cubes we refer to Stein [15].

Let us fix a cube $Q_0 \in \mathcal{W}$. Then for each $Q \in \mathcal{W}$ there exists a chain of cubes, $\mathcal{C}(Q) := (Q_0^*, Q_1^*, \dots, Q_k^*)$, joining Q_0^* to $Q_k^* = Q^*$ such that $Q_i^* \cap Q_j^* \neq \emptyset$ if and only if $|i-j| \leq 1$. The length of this chain is $\ell(\mathcal{C}(Q)) = k$. Moreover, the shadow $S(Q)$ of a cube $Q \in \mathcal{W}$ is defined as follows

$$S(Q) = \bigcup_{\substack{R \in \mathcal{W} \\ Q^* \in \mathcal{C}(R)}} R. \tag{2.1}$$

The quasihyperbolic distance between points x and y in G is defined as

$$k_G(x,y) = \inf_\gamma \int_\gamma \frac{ds}{\operatorname{dist}(z,\partial G)},$$

where the infimum is taken over all rectifiable curves γ joining x to y in G. In this paper we study bounded domains satisfying the following quasihyperbolic boundary condition. Other equivalent definitions can be found, e.g., in [7, p. 25].

2.2. **Definition.** A bounded domain $G \subset \mathbb{R}^n$ is said to satisfy a β-*quasihyperbolic boundary condition*, $\beta \in (0,1]$, if there exist a point $x_0 \in G$ and a constant $c < \infty$ such that

$$k_G(x,x_0) \leq \frac{1}{\beta} \log \frac{1}{\operatorname{dist}(x,\partial G)} + c$$

holds for every $x \in G$.

The following theorem is from [11, Theorem 5.1].

2.3. **Theorem.** *Suppose G satisfies a β-quasihyperbolic boundary condition. Then $\dim_{\mathcal{M}}(\partial G) \leq n - c_n\beta^{n-1}$ with a constant $c_n > 0$ depending only on the dimension n.*

The notation $a \lesssim b$ means that an inequality $a \leq cb$ holds for some constant $c > 0$ whose exact value is not important. We use subscripts to indicate the dependence on parameters, for example, c_λ means that the constant depends only on the parameter λ.

3. Poincaré inequalities

The following theorem is our main result.

3.1. **Theorem.** *Suppose G satisfies a β-quasihyperbolic boundary condition, $\beta \in (0,1]$, and $\dim_{\mathcal{M}}(\partial G) \leq \lambda \in [n-1,n)$. If $1 \leq q < p < \infty$ are real numbers such that*

$$p > \frac{q(n-\lambda\beta)}{q+\beta(n-\lambda)}, \tag{3.2}$$

then G supports the (q,p)-Poincaré inequality (1.1).

3.3. *Remark.* Theorem 3.1 is concerned with the case when the upper Minkowski dimension is bounded by λ. Observe that the right hand side of (3.2) is an increasing function of λ, when $q < n - n\beta$; recall from the introduction that this is the interesting case. This reflects the fact that a quasihyperbolic boundary condition domain is more irregular and the Poincaré inequality fails to hold more easily when the upper Minkowski dimension of the boundary is larger.

Preparations for the proof of Theorem 3.1. Let us choose $\lambda' \in (\lambda, n)$ such that inequality (3.2) holds if λ is replaced by λ'. Then $\dim_{\mathcal{M}}(\partial G) < \lambda'$ and we may assume that $\dim_{\mathcal{M}}(\partial G)$ is strictly less than $\lambda \in [n-1, n)$. The following lemma from [6, Lemma 4.4] relies on this strict inequality.

3.4. **Lemma.** *Let $K \subset \mathbb{R}^n$ be a compact set such that*

$$\dim_{\mathcal{M}}(K) < \lambda$$

where $\lambda \in [n-1, n)$. Assume that $\{B_1, B_2, \ldots, B_N\}$ is a family of N disjoint balls in $\mathbb{R}^n$, each of which is centered in K and whose radius is $r \in (0, 1]$. Then

$$N \le c r^{-\lambda},$$

where the constant c is independent of the disjoint balls.

Let $Q_0 \in \mathcal{W}$ and $x_0 \in Q_0$ be fixed. Choose any $Q \in \mathcal{W}$ and join x_0 to x_Q by a quasihyperbolic geodesic. By using those Whitney cubes that intersect the quasihyperbolic geodesic we find, as in [7, Proposition 6.1], a chain $C(Q)$ connecting Q_0 to Q such that

$$\ell(C(Q)) \le c_n k_G(x_0, x) + 1 \le 5c_n(\ell(C(Q)) + 1), \tag{3.5}$$

for every $x \in Q$.

3.6. **Lemma.** *Suppose G satisfies a β-quasihyperbolic boundary condition, $\beta \in (0, 1]$. Let $\varepsilon \in (0, 1)$ and $1 \le q < \infty$. Then*

$$\sum_{\substack{Q \in \mathcal{W} \\ Q \subset S(A)}} \ell(C(Q))^{q-1} |Q| \le c |\, S(A)\,|^{1-\varepsilon}, \tag{3.7}$$

where c is a positive constant, independent of $A \in \mathcal{W}$.

Proof. By inequality (3.5)

$$\Sigma := \sum_{\substack{Q \in \mathcal{W} \\ Q \subset S(A)}} \ell(C(Q))^{q-1} |Q| \lesssim \sum_{\substack{Q \in \mathcal{W} \\ Q \subset S(A)}} (k_G(x_0, x_Q) + 1)^{q-1} |Q|. \tag{3.8}$$

We employ Hölder's inequality with $r \in (1,\infty)$ and $r' = r/(r-1)$ and estimate as follows

$$\begin{aligned}\Sigma &\lesssim \Bigg(\sum_{\substack{Q\in\mathcal{W}\\ Q\subset S(A)}} (k_G(x_0,x_Q)+1)^{r'(q-1)} |Q|^{(1-1/r)r'} \Bigg)^{1/r'} \Bigg(\sum_{\substack{Q\in\mathcal{W}\\ Q\subset S(A)}} |Q| \Bigg)^{1/r} \\ &\le \Bigg(\sum_{Q\in\mathcal{W}} (k_G(x_0,x_Q)+1)^{r'(q-1)} |Q| \Bigg)^{1/r'} |\,S(A)\,|^{1/r}.\end{aligned}$$

By inequality (3.5) and [7, Theorem 7.7] the last series is finite, and its least upper bound depends only on n, q, r, x_0, and G. Indeed, by [14, Corollary 1], domain G satisfies the required Whitney-$\sharp$ condition.

The above estimates give

$$\Sigma \le c|\,S(A)\,|^{1/r},$$

where c depends only on q, n, r, x_0, and G. Choosing $r = 1/(1-\varepsilon) > 1$ gives inequality (3.7). □

We write $[s]$ for the integer part of $s \in \mathbb{R}$.

3.9. **Lemma.** *Suppose G satisfies a β-quasihyperbolic boundary condition, $\beta \in (0,1]$. Suppose further that $\dim_{\mathcal{M}}(\partial G) < \lambda$, where $\lambda \in [n-1,n)$. Then there is a number $\sigma \ge 1$ such that*

$$\mathcal{W}_j = \bigcup_{k=0}^{[j-j\beta]} \mathcal{W}_{j,k,\sigma} \tag{3.10}$$

for every $j \in \mathbb{N}$, where

$$\mathcal{W}_{j,k,\sigma} := \{Q \in \mathcal{W}_j \,:\, 2^{-(j-k)n} \le |S(Q)| \le \sigma \cdot 2^{-(j-k-1)n}\}.$$

Also, if $k \in \{0,1,\dots,[j-j\beta]\}$, then

$$\sharp\mathcal{W}_{j,k,\sigma} \le c\,j\,2^{n(j-k)+j\beta(\lambda-n)}. \tag{3.11}$$

Here c is a positive constant independent of j and k.

Proof. Let us fix $j \in \mathbb{N}$. The $5r$-covering theorem, [12, p. 23], implies that there is a finite family

$$\mathcal{F} \subset \{B^n(x, 2^{-j\beta}) \,:\, x \in \partial G\}$$

of disjoint balls such that

$$\partial G \subset \bigcup_{B\in\mathcal{F}} 5B. \tag{3.12}$$

We claim that, if $Q \in \mathcal{W}_j$, there exists a ball $B \in \mathcal{F}$ such that

$$Q \subset c_1 B. \tag{3.13}$$

Here c_1 is a constant depending on n only. To verify this let $y \in \partial G$ be a nearest point in ∂G to the centre x_Q of Q. By inclusion (3.12) there is a point x in ∂G such that $B^n(x, 2^{-j\beta}) \in \mathcal{F}$ and $y \in B^n(x, 5 \cdot 2^{-j\beta})$. If $z \in Q$ the triangle inequality implies

$$|z - x| \le |z - x_Q| + |x_Q - y| + |y - x| \le c2^{-j} + c2^{-j} + 5 \cdot 2^{-j\beta} < c_1 2^{-j\beta}.$$

Inclusion (3.13) follows because $Q \subset B^n(x, c_1\, 2^{-j\beta}) = c_1\, B^n(x, 2^{-j\beta})$.

Let us fix $Q \in \mathcal{W}_j$ and a ball $B := B^n(x, 2^{-j\beta})$ in $\mathcal{F}$ such that $Q \subset c_1 B$. We claim that

$$S(Q) \subset B^n(x, c_2\, 2^{-j\beta}), \tag{3.14}$$

where $c_2 > c_1$ is a constant which depends on β, n, x_0, and G only. To prove inclusion (3.14), let $R \in \mathcal{W}$ be a cube such that $Q^* \in C(R)$. Then $R \subset S(Q)$. By [10, Lemma 2.8]

$$\operatorname{diam}(R) \le \operatorname{diam}(S(Q)) \le c_n \operatorname{dist}(x_0, \partial G)^{1-\beta} \operatorname{diam}(Q)^\beta = c2^{-j\beta}.$$

Hence, if $y \in R$, the triangle inequality gives

$$\begin{aligned} |y - x| &\le |y - x_R| + |x_R - x_Q| + |x_Q - x| \\ &\le c2^{-j\beta} + c2^{-j\beta} + c_1 2^{-j\beta} < c_2\, 2^{-j\beta}. \end{aligned}$$

Inclusion (3.14) follows.

As a consequence of (3.14), we obtain

$$2^{-jn} = |Q| \le |S(Q)| \le \sigma \cdot 2^{-jn\beta}$$

with a constant $\sigma \ge 1$, depending on β, n, x_0, and G only. Identity (3.10) is valid with this constant.

To prove estimate (3.11) we use the following inequality

$$\sharp\{Q \in \mathcal{W}_j \,:\, Q^* \in C(R)\} \le c_n j, \qquad R \in \mathcal{W}, \tag{3.15}$$

from [10, Lemma 2.5].

Let us fix $k \in \{0, 1, \dots, [j - j\beta]\}$ and let us consider an arbitrary ball $B := B^n(x, 2^{-j\beta})$ in $\mathcal{F}$. We estimate the number of cubes that are included in $c_1 B$. By inclusion (3.14)

$$\begin{aligned} &\sharp\{Q \in \mathcal{W}_{j,k,\sigma} \,:\, Q \subset c_1 B\} \\ &\le \sum_{\substack{Q \in \mathcal{W}_{j,k,\sigma} \\ Q \subset c_1 B}} 2^{(j-k)n} |S(Q)| \le 2^{(j-k)n} \sum_{Q \in \mathcal{W}_{j,k,\sigma}} |S(Q) \cap c_2 B| \\ &\le 2^{(j-k)n} \sum_{Q \in \mathcal{W}_{j,k,\sigma}} \sum_{\substack{R \in \mathcal{W} \\ Q^* \in C(R)}} |R \cap c_2 B| = 2^{(j-k)n} \sum_{R \in \mathcal{W}} \sum_{\substack{Q \in \mathcal{W}_{j,k,\sigma} \\ Q^* \in C(R)}} |R \cap c_2 B|. \end{aligned}$$

Inequality (3.15) shows that the last sum is bounded by

$$c_n\, j\, 2^{(j-k)n} |c_2 B| \le c_3\, j\, 2^{-kn} 2^{j(n - n\beta)}$$

with a constant $c_3 > 0$ which depends on β, n, x_0, and G only.

Inclusion (3.13) implies that

$$\sharp \mathcal{W}_{j,k,\sigma} \le \sum_{B \in \mathcal{F}} \sharp\{Q \in \mathcal{W}_{j,k,\sigma} \,:\, Q \subset c_1 B\} \le c_3 \sum_{B \in \mathcal{F}} j\, 2^{-kn} 2^{j(n - n\beta)}. \tag{3.16}$$

Recall that $\mathcal{F}$ is a family of disjoint balls, each of which are centered in ∂G with radius $2^{-j\beta} \in (0,1]$. Therefore, Lemma 3.4 yields

$$\sharp\mathcal{F} \le c2^{j\lambda\beta}.$$

Combining this estimate with inequalities (3.16) yields

$$\sharp\mathcal{W}_{j,k,\sigma} \le c\,j\,2^{j\lambda\beta}2^{-kn}2^{j(n-n\beta)},$$

which is estimate (3.11). □

Proof of Theorem 3.1. We may assume, by scaling, that $\operatorname{diam}(G) < 1$. Hence $\mathcal{W} = \bigcup_{j=0}^{\infty} \mathcal{W}_j$. Using Hölder's inequality, and inequalities $|a + b|^q \le 2^{q-1}(|a|^q + |b|^q)$ and $|a+b|^{1/q} \le |a|^{1/q} + |b|^{1/q}$, $a, b \in \mathbb{R}$, we obtain

$$
\begin{aligned}
\Big(\int_G |u(x) - u_G|^q\,dx\Big)^{1/q} &\le 2\Big(\int_G |u(x) - u_{Q_0^*}|^q\,dx\Big)^{1/q} \\
&\lesssim \Big(\sum_{Q\in\mathcal{W}} \int_{Q^*} |u(x) - u_{Q^*}|^q\,dx\Big)^{1/q} + \Big(\sum_{Q\in\mathcal{W}} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q\,dx\Big)^{1/q}. \qquad (3.17)
\end{aligned}
$$

The first term on the right hand side of (3.17) is estimated by the (q,p)-Poincaré inequality in cubes and by Hölder's inequality,

$$\Big(\sum_{Q\in\mathcal{W}} \int_{Q^*} |u(x) - u_{Q^*}|^q\,dx\Big)^{1/q} \lesssim \Big(\int_G |\nabla u(x)|^p\,dx\Big)^{1/p}.$$

For the second term on the right hand side of (3.17) let us connect Q_0 to every cube $Q \in \mathcal{W}$ by a chain $C(Q) = (Q_0^*, Q_1^*, \ldots, Q_k^*)$, $Q_k^* = Q^*$, that is constructed by using quasihyperbolic geodesics as in connection with (3.5). By the triangle and Hölder's inequalities, by the properties of Whitney cubes and by the validity of (q,p)-Poincaré inequality in cubes we obtain

$$
\begin{aligned}
\sum_{Q\in\mathcal{W}} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q\,dx &\lesssim \sum_{Q\in\mathcal{W}} \int_{Q^*} \ell(C(Q))^{q-1} \sum_{j=0}^{k-1} |u_{Q_j^*} - u_{Q_{j+1}^*}|^q\,dx \\
&\lesssim \sum_{Q\in\mathcal{W}} \int_{Q^*} \ell(C(Q))^{q-1} \sum_{j=0}^{k} |Q_j^*|^{-1} \int_{Q_j^*} |u(y) - u_{Q_j^*}|^q\,dy\,dx \\
&\lesssim \sum_{Q\in\mathcal{W}} \int_{Q^*} \ell(C(Q))^{q-1} \sum_{j=0}^{k} |Q_j^*|^{q/n-q/p}\Big(\int_{Q_j^*} |\nabla u(y)|^p\,dy\Big)^{q/p}\,dx\,.
\end{aligned}
$$

By rearranging the double sum and by using Hölder's inequality with $(p/q, p/(p-q))$ we obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{W}} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q \, dx \\
&\lesssim \sum_{A\in\mathcal{W}} \sum_{\substack{Q\in\mathcal{W}\\ Q\subset S(A)}} \ell(C(Q))^{q-1}|Q||A|^{q/n-q/p}\Big(\int_{A^*} |\nabla u(x)|^p \, dx\Big)^{q/p} \\
&\lesssim \Bigg(\sum_{A\in\mathcal{W}}\Big(\sum_{\substack{Q\in\mathcal{W}\\ Q\subset S(A)}} \ell(C(Q))^{q-1}|Q||A|^{q/n-q/p}\Big)^{p/(p-q)}\Bigg)^{(p-q)/p}\Big(\int_G |\nabla u(x)|^p \, dx\Big)^{q/p}.
\end{aligned}$$

We write

$$\Sigma := \sum_{A\in\mathcal{W}}\Big(\sum_{\substack{Q\in\mathcal{W}\\ Q\subset S(A)}} \ell(C(Q))^{q-1}|Q||A|^{q/n-q/p}\Big)^{p/(p-q)}.$$

Hence it is enough to show that the quantity Σ is finite. The preceding part of the proof followed the chaining argument in [6, Theorem 3.2], which is nowadays a standard approach dating back to [9].

Fix $\varepsilon \in (0, q/p)$. By Lemma 3.6,

$$\Sigma \lesssim \sum_{A\in\mathcal{W}} (|\, S(A)\,|^{1-\varepsilon}\, |A|^{q/n-q/p})^{p/(p-q)}.$$

By (3.10) we obtain

$$\Sigma \lesssim \sum_{j=0}^{\infty} \sum_{k=0}^{[j-j\beta]} \sum_{A\in\mathcal{W}_{j,k,\sigma}} (|S(A)|^{1-\varepsilon}\, |A|^{q/n-q/p})^{p/(p-q)}.$$

Definition of $\mathcal{W}_{j,k,\sigma}$ and inequality (3.11) imply

$$\begin{aligned}
\Sigma &\lesssim \sum_{j=0}^{\infty} \sum_{k=0}^{[j-j\beta]} j\cdot 2^{n(j-k)+j\beta(\lambda-n)} \cdot (2^{-n(j-k)(1-\varepsilon)} \cdot 2^{-jnq(1/n-1/p)})^{p/(p-q)} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{[j-j\beta]} j\cdot 2^{kn(p(1-\varepsilon)/(p-q)-1)} 2^{j(n+\beta(\lambda-n)-n(1-\varepsilon)p/(p-q)-qp/(p-q)+nq/(p-q))}.
\end{aligned}$$

Let j and k be as in the summation. Then,

$$kn\Big(\frac{p(1-\varepsilon)}{p-q}-1\Big) \le n(j-j\beta)\Big(\frac{p(1-\varepsilon)}{p-q}-1\Big) = \frac{jn(1-\beta)(q-\varepsilon p)}{p-q}.$$

By the estimate $[j-j\beta] \le j$, where $\beta \in (0,1]$,

$$\begin{aligned}
\Sigma &\lesssim \sum_{j=0}^{\infty} j^2 \cdot 2^{j(n(1-\beta)(q-\varepsilon p)/(p-q)+n+\beta(\lambda-n)-n(1-\varepsilon)p/(p-q)-qp/(p-q)+nq/(p-q))} \\
&\lesssim \sum_{j=0}^{\infty} j^2 \cdot 2^{\varepsilon j(np/(p-q)-n(1-\beta)p/(p-q))} 2^{j(\beta(\lambda-n)-q((\beta-1)n+p)/(p-q))}.
\end{aligned}$$

Inequality (3.2) allows us to choose $\varepsilon > 0$ so small that the last series converges. □

We have the following corollary of Theorem 3.1.

3.18. **Corollary.** *Let $1 \le q < n - n\beta$, where $\beta \in (0,1)$. If G satisfies a β-quasihyperbolic boundary condition, then G supports the $(q, n-n\beta)$-Poincaré inequality* (1.1)*.*

Proof. By Theorem 2.3, $\lambda = \dim_{\mathcal{M}}(\partial G) < n$. Observe that $r = n - n\beta$ satisfies the identity

$$p(r) = \frac{r(n-\beta\lambda)}{r+\beta(n-\lambda)} = r.$$

On the other hand, since $1 \le q < r$ and $p'(t) > 0$ if $t > 0$, we obtain that $p(q) < p(r) = r = n - n\beta$. The claim follows from Theorem 3.1. □

3.19. *Remark.* A conjecture [10, p. 813] states that a domain satisfying a β-quasihyperbolic boundary condition supports the (p,p)-Poincaré inequality, where $p = n - n\beta \ge 1$. To our knowledge, this boundary case is still an open question.

4. Modification of a John domain

In this section, we introduce a method how to modify a given John domain G in a controlled way such that the resulting domain, denoted by G_β and called a β-version of G, is no more a John domain but it satisfies a $\beta/4$-quasihyperbolic boundary condition if $\beta \le \varkappa c_J$. Here $\varkappa$ is a constant depending on n only, and c_J is the John constant of G (see Definition 4.1). By studying the validity of Poincaré inequalities (1.1) on these β-versions of John domains we shall show that Theorem 3.1 is essentially sharp in the plane. Theorems 4.7 and 4.15 are the main results of this section.

We assume that G is a bounded domain such that $\operatorname{diam}(G) \le 4$ in Section 4 and Section 5. We recall the following definition.

4.1. **Definition.** A bounded domain $G \subset \mathbb{R}^n$ is a *John domain*, if there exist a point x_0 in G and a constant $c_J \in (0,1]$ such that every point $x \in G$ can be joined to x_0 by a rectifiable curve $\gamma : [0, \ell(\gamma)] \to G$ which is parametrized by its arc length, $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, and

$$\operatorname{dist}(\gamma(t), \partial G) \ge c_J t \tag{4.2}$$

for each $t \in [0, \ell(\gamma)]$. The point x_0 is called a *John center* of G, and the largest $c_J \in (0,1]$ is called the *John constant* of G. The curve γ is called a *John curve.*

Let us fix $\beta \in (0,1]$ and let $Q \subset \mathbb{R}^n$ be a closed cube that is centered at $x_Q = (x_1, \ldots, x_n)$, and whose side length is $\ell(Q) = \ell \le 4$, i.e.

$$Q := \prod_{i=1}^{n} [x_i - \ell/2,\ x_i + \ell/2].$$

The *room* in Q is the open cube

$$R(Q) := \operatorname{int}\left(\frac{1}{4}Q\right) = \prod_{i=1}^{n}(x_i - \ell/8,\ x_i + \ell/8)$$

whose center is x_Q and side length is $\ell/4$. The *β-passage* in Q is the open set

$$P_\beta(Q) := \Big(\prod_{i=1}^{n-1}(x_i - (\ell/8)^{1/\beta},\ x_i + (\ell/8)^{1/\beta})\Big) \times (x_n + \ell/8,\ x_n + \ell/8 + (\ell/8)^{1/\beta}).$$

Note that $\ell/8 < 1$ and $1/\beta \geq 1$, hence we have $(\ell/8)^{1/\beta} < \ell/8$. Thus, $P_\beta(Q) \subset \frac{1}{2}Q$. The open cube

$$E(Q) := \operatorname{int}\left(\frac{3}{4}Q\right) = \prod_{i=1}^{n}(x_i - 3\ell/8,\ x_i + 3\ell/8) \subset Q$$

contains the room and β-passage in Q. The *long β-passage* in Q is the open set

$$L_\beta(Q) := \Big(\prod_{i=1}^{n-1}(x_i - (\ell/8)^{1/\beta},\ x_i + (\ell/8)^{1/\beta})\Big) \times (x_n,\ x_n + \ell/2) \subset Q.$$

The *β-apartment* in Q is the set

$$A_\beta(Q) := L_\beta(Q) \cup Q \setminus (\partial R(Q) \cup \partial P_\beta(Q)) \subset Q.$$

Figure 1 depicts these geometric objects in a cube Q when $\beta = 1/2$.

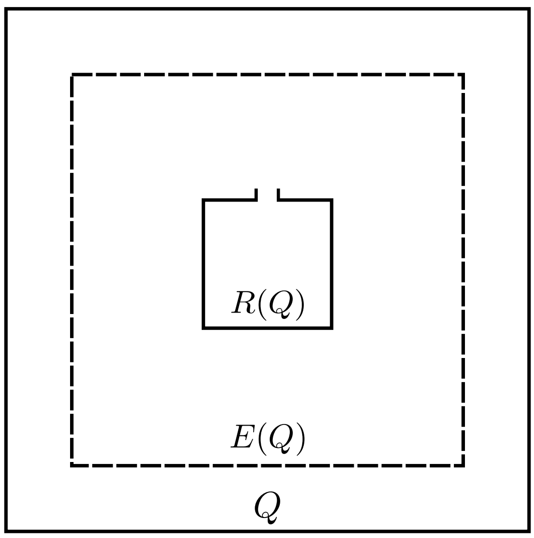

FIGURE 1. A modified Whitney cube with $\beta = 1/2$.

4.3. **Definition.** Let G be a John domain. A *β-version of G* is the domain

$$G_\beta := \bigcup_{Q \in \mathcal{W}_G} A_\beta(Q).$$

The following proposition is a modification of [6, Proposition 5.11].

4.4. **Proposition.** *Let $G \subset \mathbb{R}^n$ be a John domain. Then*

$$\dim_{\mathcal{M}}(\partial G) = \dim_{\mathcal{M}}(\partial G_\beta)$$

for every $\beta \in (0,1]$.

Let us next study the validity of Poincaré inequalities on a β-version of a given John domain. Let $Q \subset \mathbb{R}^n$ be a closed cube, $\ell(Q) \le 4$, and define a continuous function

$$u^{A_\beta(Q)} \colon G_\beta \to \mathbb{R}$$

which has linear decay along the n^{th} variable in $P_\beta(Q)$ and satisfies

$$u^{A_\beta(Q)}(x) = \begin{cases} \ell(Q)^{(\lambda-n)/q}, & \text{if } x \in R(Q); \\ 0, & \text{if } x \in G_\beta \setminus (R(Q) \cup \overline{P_\beta(Q)}). \end{cases} \tag{4.5}$$

Moreover, in the sense of distributions in G_β the following holds

$$\nabla u^{A_\beta(Q)} = (0, \dots, 0, -8^{1/\beta}\ell(Q)^{(\lambda-n)/q-1/\beta}\chi_{P_\beta(Q)}). \tag{4.6}$$

The following is the first main result in this section.

4.7. **Theorem.** *Let $G \subset \mathbb{R}^n$ be a John domain such that*

$$\limsup_{k\to\infty} 2^{-\lambda k} \cdot \sharp\mathcal{W}_k > 0, \tag{4.8}$$

where $\lambda = \dim_{\mathcal{M}}(\partial G)$. Suppose that

$$p \le \frac{q(n-\lambda\beta)}{q+\beta(n-\lambda)}, \tag{4.9}$$

where $1 \le q < p < \infty$ and $\beta \in (0,1]$. Then the β-version of G does not satisfy the (q,p)-Poincaré inequality (1.1).

By [6, Proposition 5.2] for every $\lambda \in [n-1, n)$, $n \ge 2$, there exists a John domain $G \subset \mathbb{R}^n$ such that $\dim_{\mathcal{M}}(\partial G) = \lambda$ and hypothesis (4.8) in Theorem 4.7 is satisfied.

Proof of Theorem 4.7. Choose $k_0 \in \mathbb{N}$ such that $\limsup_{k\to\infty} 2^{-\lambda(k-k_0)} \cdot \sharp\mathcal{W}_k > 2$. This allows us to inductively choose indices $j(k)$, $k \in \mathbb{N}$, such that

$$k_0 \le j(1) < j(2) < \cdots$$

and $\sharp\mathcal{W}_{j(k)} \ge 2 \cdot 2^{\lambda(j(k)-k_0)}$ for every $k \in \mathbb{N}$. Let us write $M_j := 2^{[\lambda(j-k_0)]}$, where $[\lambda(j-k_0)]$ is the integer part of $\lambda(j-k_0)$, and let us choose cubes $Q^1_{j(k)}, \dots, Q^{2M_{j(k)}}_{j(k)} \in \mathcal{W}_{j(k)}$. For every $m \in \mathbb{N}$ let us write

$$v_m := \sum_{k=1}^{m} \Bigg(\sum_{i=1}^{M_{j(k)}} u^{A_\beta(Q^i_{j(k)})} - \sum_{i=M_{j(k)}+1}^{2M_{j(k)}} u^{A_\beta(Q^i_{j(k)})} \Bigg) \in W^{1,p}(G_\beta).$$

Note that $(v_m)_{G_\beta} = 0$ and

$$A_m := \Big(\int_{G_\beta} |v_m - (v_m)_{G_\beta}|^q\Big)^{1/q} = \Big(\sum_{k=1}^{m}\sum_{i=1}^{2M_{j(k)}} \int_{G_\beta} |u^{A_\beta(Q^i_{j(k)})}(x)|^q\,dx\Big)^{1/q}$$
$$\geq \Big(\sum_{k=1}^{m} 2\cdot 2^{\lambda(j(k)-k_0)-1}\cdot 2^{-j(k)(\lambda-n)}\cdot 4^{-n}\cdot 2^{-j(k)n}\Big)^{1/q} = c_{n,q,\lambda,k_0} m^{1/q}.$$

On the other hand, by inequality (4.9),

$$B_m := \Big(\int_{G_\beta} |\nabla v_m(x)|^p\,dx\Big)^{1/p}$$
$$= \Big(\sum_{k=1}^{m}\sum_{i=1}^{2M_{j(k)}} \int_{G_\beta} |\nabla u^{A_\beta(Q^i_{j(k)})}(x)|^p\,dx\Big)^{1/p}$$
$$\leq c_\beta\Big(\sum_{k=1}^{m} 2\cdot 2^{\lambda(j(k)-k_0)}\cdot 2^{-pj(k)((\lambda-n)/q-1/\beta)}\cdot 2^{n-1}\cdot 2^{-nj(k)/\beta}\Big)^{1/p}$$
$$\leq c_{n,\beta,p,\lambda,k_0} m^{1/p}.$$

Hence, we obtain

$$\frac{A_m}{B_m} \geq c_{n,\beta,p,q,k_0,\lambda} m^{1/q-1/p} \xrightarrow{m\to\infty} \infty,$$

where $1 \leq q < p$. It follows that G_β does not satisfy the (q,p)-Poincaré inequality (1.1). □

We shall show that a β-version of a given John domain G satisfies a $\beta/4$-quasihyperbolic boundary condition if $\beta \leq \varkappa c_J$, where c_j is the John constant of G and $\varkappa$ is a constant depending on n only, Theorem 4.15. Let us begin with the following auxiliary result.

4.10. **Lemma.** *Let $G \subset \mathbb{R}^n$ be a John domain with the John constant c_J. Let further $\beta \in (0,1)$ and G_β be the β-version of G. If $x \in E(Q)\cap G_\beta$, $Q \in \mathcal{W}_G$, then there is a point $z \in \partial E(Q)$ such that*

$$k_{G_\beta}(x,z) \leq \Big(\frac{2}{\beta} + \frac{1}{\varkappa}\Big)\log\frac{1}{\operatorname{dist}(x,\partial G_\beta)} + c. \tag{4.11}$$

If $x \in Q \setminus E(Q)$, then

$$k_{G_\beta}(x,x_0) \leq \frac{1}{\varkappa c_J}\log\frac{1}{\operatorname{dist}(x,\partial G_\beta)} + c. \tag{4.12}$$

The constant $\varkappa \in (0,1)$ appearing in both inequalities (4.11) and (4.12) depends on n only.

Proof. Let us consider the case $x \in R(Q) \subset E(Q)$. We will join x to the point $z := x_Q + (3\ell/8)e_n \in \partial E(Q)$, where $e_n = (0,\ldots,0,1)$, by a rectifiable curve that is to be constructed next. For this purpose, we record the following inequality

$$\operatorname{dist}(x,\partial G_\beta) \leq \ell/8. \tag{4.13}$$

Notice that there is a constant $\varkappa$ such that

$$k_{G_\beta}(x, x_Q) \le \frac{1}{\varkappa} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c.$$

Next we connect x_Q to the point z by a curve $\gamma : [0, 3\ell/8] \to G_\beta$ which parametrizes the line segment $[x_Q, z]$. Write $\xi = \ell/8 - (\ell/8)^{1/\beta}$ and $\eta = \ell/8 + 2(\ell/8)^{1/\beta}$. Observe that

$$\operatorname{dist}(\gamma(t), \partial G_\beta) \ge \ell/8 - t$$

for all $t \in [0, \ell/8]$. By this inequality and (4.13), we obtain

$$\int_0^\xi \frac{dt}{\operatorname{dist}(\gamma(t), \partial G_\beta)} \le \int_0^\xi \frac{dt}{\ell/8 - t} \le \frac{1}{\beta} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)}.$$

In the following step, we pass through the β-passage in Q. If $t \in [\xi, \eta]$ then $\operatorname{dist}(\gamma(t), \partial G_\beta) \ge (\ell/8)^{1/\beta}$. Hence

$$\int_\xi^\eta \frac{dt}{\operatorname{dist}(\gamma(t), \partial G_\beta)} \le \frac{3(\ell/8)^{1/\beta}}{(\ell/8)^{1/\beta}} = 3.$$

For $t \in [\eta, 3\ell/8]$,

$$\operatorname{dist}(\gamma(t), \partial G_\beta) \ge \min\{\ell/8, (\ell/8)^{1/\beta} + t - \eta\}.$$

By this inequality and the fact that $(3\ell/8 - \eta)/(\ell/8) \le 3$,

$$\begin{aligned}\int_\eta^{3\ell/8} \frac{dt}{\operatorname{dist}(\gamma(t), \partial G_\beta)} &\le 3 + \int_\eta^{3\ell/8} \frac{dt}{(\ell/8)^{1/\beta} + t - \eta} \\ &\le \frac{1}{\beta} \log \frac{1}{\ell/8} + c \le \frac{1}{\beta} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c.\end{aligned}$$

By these estimates we have

$$k_{G_\beta}(x, z) \le k_{G_\beta}(x, x_Q) + k_{G_\beta}(x_Q, z) \le \left(\frac{2}{\beta} + \frac{1}{\varkappa}\right) \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c$$

for every $x \in R(Q)$. This gives inequality (4.11).

Let us consider the case $x \in G_\beta \cap \overline{P_\beta(Q)}$. There is a point ω on the line segment from x_Q to z such that

$$k_{G_\beta}(x, \omega) \le \frac{1}{\varkappa} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c.$$

Joining ω to z by the line segment $[\omega, z] \subset [x_Q, z]$ gives the inequality

$$k_{G_\beta}(\omega, z) \le \frac{1}{\beta} \log \frac{1}{\ell/8} + c.$$

Since $\operatorname{dist}(x, \partial G_\beta) \le (\ell/8)^{1/\beta} \le \ell/8$, we have

$$k_{G_\beta}(x, z) \le k_{G_\beta}(x, \omega) + k_{G_\beta}(\omega, z) \le \left(\frac{1}{\beta} + \frac{1}{\varkappa}\right) \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c.$$

This gives inequality (4.11).

If $x \in E(Q) \setminus (R(Q) \cup \overline{P_\beta(Q)})$, then clearly there is a point $z \in \partial E(Q)$ such that

$$k_{G_\beta}(x, z) \le \frac{1}{\varkappa} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c.$$

But this is inequality (4.11).

Finally, let us consider the case $x \in Q \setminus E(Q)$. The idea is to construct a curve $\gamma : [0, \ell(\gamma)] \to G_\beta$, parametrized by arc length such that $\gamma(0) = x$ and $\gamma(\ell(\gamma)) = x_0$, so that

$$\operatorname{dist}(\gamma(t), \partial G_\beta) \ge \varkappa c_J t \tag{4.14}$$

for every $t \in [0, \ell(\gamma)]$. This is done by taking a John curve from x to the John center x_0, and modifying it whenever it intersects with $E(Q)$, $Q \in \mathcal{W}_G$. This is illustrated in Figure 2. For further details, we refer to the proof of [6, Proposition 5.16].

Let us write $\delta = \operatorname{dist}(x, \partial G_\beta)$. If $\ell(\gamma) \le \delta/2$, then $k_{G_\beta}(x, x_0) \le 1$. Hence, we may assume that $\ell(\gamma) > \delta/2$ and therefore

$$\begin{aligned} k_{G_\beta}(x, x_0) &\le \int_\gamma \frac{ds}{\operatorname{dist}(z, \partial G_\beta)} \\ &= \int_0^{\delta/2} \frac{dt}{\operatorname{dist}(\gamma(t), \partial G_\beta)} + \int_{\delta/2}^{\ell(\gamma)} \frac{dt}{\operatorname{dist}(\gamma(t), \partial G_\beta)}. \end{aligned}$$

If $t \in [0, \delta/2]$ then $\operatorname{dist}(\gamma(t), \partial G_\beta) \ge \delta/2$. Inequality (4.14) implies that $\ell(\gamma) \le \operatorname{diam}(G_\beta)/\varkappa c_J =: T$. Hence,

$$\begin{aligned} k_{G_\beta}(x, x_0) \le 1 + \frac{1}{\varkappa c_J} \int_{\delta/2}^{T} \frac{dt}{t} &\le 1 + \frac{\log(T) - \log(\delta/2)}{\varkappa c_J} \\ &= \frac{1}{\varkappa c_J} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c. \end{aligned}$$

This gives inequality (4.12). □

The following is the second main result in this section.

4.15. **Theorem.** *Suppose $G \subset \mathbb{R}^n$ is a John domain with a John constant c_J. Then the β-version of G, G_β, satisfies a $\beta/4$-quasihyperbolic boundary condition if $0 < \beta \le \varkappa c_J$, where $\varkappa \in (0, 1)$ is the same constant as in Lemma 4.10.*

Proof. Fix a point x in $G_\beta \subset G$. We shall show that the inequality

$$k_{G_\beta}(x, x_0) \le \frac{4}{\beta} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c \tag{4.16}$$

is valid with some constant c; the point x belongs to some Whitney cube $Q \in \mathcal{W}_G$.

Let us consider the case $x \in E(Q) \cap G_\beta$. By Lemma 4.10, there is a point $z \in \partial E(Q)$ such that inequality (4.11) holds. Applying

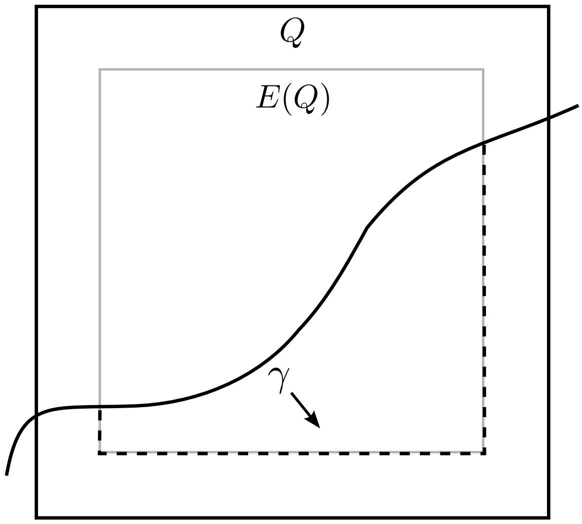

FIGURE 2. John curve γ and its modification (dotted line) on $E(Q)$.

Lemma 4.10 again, now with the point $z \in Q \setminus E(Q)$, yields

$$\begin{aligned} k_{G_\beta}(x, x_0) &\le k_{G_\beta}(x, z) + k_{G_\beta}(z, x_0) \\ &\le \left(\frac{2}{\beta} + \frac{1}{\varkappa}\right) \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + \frac{1}{\varkappa c_J} \log \frac{1}{\operatorname{dist}(z, \partial G_\beta)} + c. \end{aligned}$$

By inequality $\operatorname{dist}(z, \partial G_\beta) \ge \ell/8 \ge \operatorname{dist}(x, \partial G_\beta)/c_n$,

$$k_{G_\beta}(x, x_0) \le \frac{4}{\beta} \log \frac{1}{\operatorname{dist}(x, \partial G_\beta)} + c.$$

This gives inequality (4.16).

If $x \in Q \setminus E(Q)$, then inequality (4.16) follows from Lemma 4.10. □

5. SHARPNESS OF THEOREM 3.1 IN THE PLANE

We consider sharpness of our main result, Theorem 3.1, in the plane. We introduce new constants τ and $\bar{c}_2$ whose values will be clear later. It will become apparent that we would like to have the constant $\bar{c}_2$ as close as possible to the constant c_2 in Theorem 2.3. The following theorem is a delicate extension of Theorem 4.7 in the plane.

5.1. **Theorem.** *Let $\lambda \in [1, 2)$ and let $\beta \in (0, 1)$ be such that $\lambda \le 2 - \bar{c}_2\beta$. Then there is a domain $G_\beta \subset \mathbb{R}^2$ which satisfies a $\beta/4$-quasihyperbolic boundary condition, $\dim_{\mathcal{M}}(\partial G_\beta) = \lambda$, and which does not satisfy the (q, p)-Poincaré inequality (1.1) if $1 \le q < p < \infty$, and*

$$p \le \frac{q(2 - \lambda\beta)}{q + \beta(2 - \lambda)}.$$

5.2. *Remark.* A counterpart of Theorem 5.1 for $n \ge 3$ is not known to us. It seems that the construction behind Theorem 5.1 can be generalized only if $\lambda \le n - \bar{c}_n\beta$. However, Theorem 2.3 suggests that it is natural to consider the less restrictive condition, $\lambda \le n - \bar{c}_n\beta^{n-1}$, which allows larger values of β, i.e. more cases to be covered by a counterexample, for a a given λ. An obstacle is that it seems not to be known whether Theorem 2.3 is sharp in case of $n \ge 3$.

We need the following construction for the proof of Theorem 5.1.

5.3. **Proposition.** *Let $\lambda \in [1,2)$ and $\beta \in (0,1)$ be such that $\lambda \le 2 - \tau\beta$. Then there exists a John domain $G \subset \mathbb{R}^2$ with a John constant $c_J \ge \beta$ such that $\dim_{\mathcal{M}}(\partial G) = \lambda$ and*

$$\limsup_{k\to\infty} 2^{-\lambda k} \cdot \sharp\mathcal{W}_k > 0. \tag{5.4}$$

Proof. Let $Q := [-1,1] \times [-1,1] \subset \mathbb{R}^2$ and, for $\kappa \in (0,1)$, $r(\kappa) := (1-\kappa)/2 \in (0,1/2)$. Let us write $z_1 := (\kappa + r(\kappa), \kappa + r(\kappa))$, and let z_2, z_3, z_4 be the corresponding symmetric points in the three remaining quadrants in any order. Let S_1, S_2, S_3, S_4 be similitudes that are defined by $S_i(x) := r(\kappa)x + z_i$, $i = 1,2,3,4$. By reasoning as in [12, pp. 66–67] we have a non-empty compact set K in Q such that

$$K = S_1(K) \cup S_2(K) \cup S_3(K) \cup S_4(K). \tag{5.5}$$

The similitudes S_1, S_2, S_3, S_4 satisfy an open set condition [12, p. 67]. Hence, by [12, Corollary 5.8, Theorem 4.14],

$$\dim_{\mathcal{M}}(K) = \dim_{\mathcal{H}}(K) = -\frac{\log 4}{\log r(\kappa)}.$$

If we let κ vary between $(0,1/2]$, then $\dim_{\mathcal{M}}(K)$ reaches all values in $[1,2)$. There exists, in particular, $\kappa = \kappa(\lambda) \in (0,1/2]$ for which the upper Minkowski dimension of $K_\lambda := K$ is λ. We define G to be the open set

$$G := B^2(0,2) \setminus K_\lambda.$$

Since $\partial G = \partial B^2(0,2) \cup K_\lambda$, we obtain $\dim_{\mathcal{M}}(\partial G) = \lambda$.

Estimate (5.4) has been verified in [6, Proposition 5.2].

We shall estimate the John constant of the domain G. We show that there is a constant $c > 0$ such that for every $x \in G$, there is a rectifiable curve $\gamma : [0, \ell(\gamma)] \to G$, parametrized by its arc length so that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = 0$, and

$$\operatorname{dist}(\gamma(t), \partial G) \ge c\kappa t \tag{5.6}$$

for all $t \in [0, \ell(\gamma)]$. Before the construction of γ, let us explain why the property above implies that the John constant c_J of G is larger than β if $\lambda \le 2 - \tau\beta$, where $\tau := 4/(c \log 2)$. By the mean value theorem (recall that $d\log_2(t)/dt = (t\log 2)^{-1}$ for $t > 0$) and the fact that $\kappa \in (0, 1/2]$,

$$\begin{aligned}\tau\beta \le 2 - \lambda &= 2\Big(1 + \frac{1}{\log_2[(1-\kappa)/2]}\Big) = 2\Big(\frac{\log_2(1-\kappa)}{\log_2(1-\kappa) - 1}\Big)\\ &= 2\Big(\frac{\log_2(1-\kappa) - \log_2(1)}{\log_2(1-\kappa) - \log_2(1) - 1}\Big) < \frac{2\kappa}{(\log 2)(1-\kappa)} \le \frac{4\kappa}{\log 2}.\end{aligned}$$

It follows that $\beta < c\kappa \le c_J$, where c_J is the John constant of G.

We construct the curve γ. If x lies in $G \setminus Q$ then the construction is clear. Hence, we may assume without loss of generality that x lies in $G \cap Q = Q \setminus K_\lambda$. Let $m \ge 0$ the smallest positive integer for which $x \in Q \setminus K_\lambda^{m+1}$, where for each $j \ge 1$

$$K_\lambda^j := \bigcup_{i_1=1}^{4} \cdots \bigcup_{i_j=1}^{4} S_{i_1} \circ \cdots \circ S_{i_j}(Q). \tag{5.7}$$

To see that such an m exists, we use the fact that iterations K_λ^j converge to K_λ in the Hausdorff metric ρ as $j \to \infty$, [12, pp. 66–67]. Hence, there is $M \in \mathbb{N}$ such that $\rho(K_\lambda^M, K_\lambda) < \mathrm{dist}(x, K_\lambda)/2$. Especially,

$$K_\lambda^M \subset \{y \in \mathbb{R}^2 \,:\, \mathrm{dist}(y, K_\lambda) \le \mathrm{dist}(x, K_\lambda)/2\},$$

and it follows that $x \in Q \setminus K_\lambda^M$. Hence, the smallest m exists.

We write $X^0 = \{y \in \mathbb{R}^2 \,:\, y_1 = 0 \text{ or } y_2 = 0\}$ for the coordinate axes in $\mathbb{R}^2$, and

$$X^j := \bigcup_{i_1=1}^{4} \cdots \bigcup_{i_j=1}^{4} S_{i_1} \circ \cdots \circ S_{i_j}(X^0)$$

for all $j \ge 1$. We connect x to the set $X^m \cap Q$ by a curve $\gamma : [0, t_m] \to G$ such that inequality (5.6) is valid and

$$t_m = \mathrm{dist}(x, X^m) \le \kappa r(\kappa)^m < r(\kappa)^m.$$

We proceed inductively: We connect sets X^j and X^{j-1} to each other, when $j \ge 1$; we connect X^0 to 0, when $j = 0$. Figure 3 depicts one of the intermediate construction steps. Let us first consider the case $1 \le j \le m$: inequality (5.6) is valid for all $t \in [0, t_j]$, point $\gamma(t_j)$ lies in $X^j \cap Q$, and

$$t_j \le \sum_{i=j}^{\infty} r(\kappa)^i. \tag{5.8}$$

Let us connect $\gamma(t_j)$ to X^{j-1} as follows: we define γ in $[t_j, t_{j-1}]$ by tracing along set $X^j \cap Q$ until we reach $X^j \cap X^{j-1} \cap Q$. This can be done in a way that $t_{j-1} - t_j \le r(\kappa)^{j-1}$. Hence,

$$t_{j-1} = t_{j-1} - t_j + t_j \le r(\kappa)^{j-1} + \sum_{i=j}^{\infty} r(\kappa)^i \le \sum_{i=j-1}^{\infty} r(\kappa)^i \le 2r(\kappa)^{j-1},$$

and inequality (5.8) is true for $j-1$. Since $K_\lambda \subset K_\lambda^{m+1}$, we have for all $t \in [t_j, t_{j-1}]$

$$\begin{aligned}\operatorname{dist}(\gamma(t), \partial G) = \operatorname{dist}(\gamma(t), K_\lambda) &\geq \operatorname{dist}(\gamma(t), K_\lambda^{m+1}) \geq \kappa r(\kappa)^j \\ &\geq 4^{-1}\kappa r(\kappa)^{j-1} \geq 8^{-1}\kappa t_{j-1} \geq 8^{-1}\kappa t.\end{aligned}$$

Hence, inequality (5.6) is valid for these values of t.

We consider the case $j = 0$. Now $\gamma(t_0)$ lies in $X^0 \cap Q$, the curve γ satisfies inequality (5.6) for all $t \in [0, t_0]$, and $t_0 \leq \sum_{i=0}^{\infty} r(\kappa)^i$. Define γ in $[t_0, t_{-1}]$ by tracing along $X^0 \cap Q$ during time $t_{-1} - t_0 \leq 1$. This yields a curve $\gamma : [0, t_{-1}] \to G$ satisfying estimate (5.6) for all $t \in [0, t_1]$. □

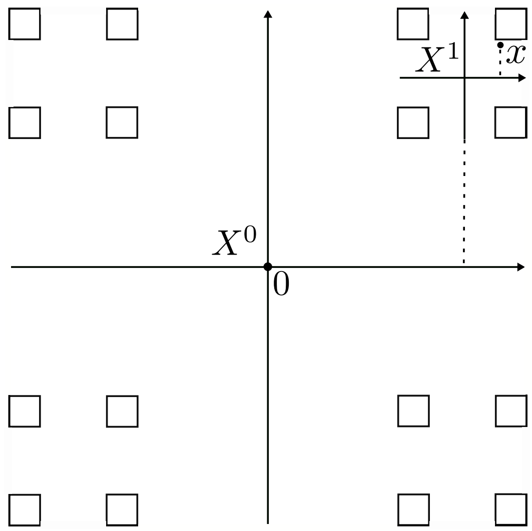

FIGURE 3. $Q \setminus K_\lambda^2$ with coordinate axes X^0 and X^1 ($\kappa = 1/2$).

We are now ready for the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $\varkappa \in (0,1)$ be as in Lemma 4.10 and let $\tau > 0$ be as in Proposition 5.3. Choose $\bar{c}_2 := \max\{2/\varkappa, \tau/\varkappa\}$. Fix $\lambda \in [1,2)$ and suppose that $\beta \in (0,1)$ is such that $\lambda \leq 2 - \bar{c}_2\beta$. Then $\lambda \leq 2 - \tau\beta/\varkappa$, and hence by Proposition 5.3 there exists a John domain G in $\mathbb{R}^2$ whose John constant is $c_J \geq \beta/\varkappa$, i.e. $\beta \leq \varkappa c_J$. By Theorem 4.15, G_β satisfies a $\beta/4$-quasihyperbolic boundary condition. Proposition 4.4 gives that $\dim_{\mathcal{M}}(\partial G_\beta) = \dim_{\mathcal{M}}(\partial G) = \lambda$. By Theorem 4.7 the domain G_β does not satisfy the (q,p)-Poincaré inequality (1.1). □

5.9. *Remark.* Theorem 3.1 states that a domain $G \subset \mathbb{R}^2$ supports the (q,p)-Poincaré inequality (1.1) with $1 \leq q < p < \infty$ if the following conditions are met: G satisfies a β-quasihyperbolic boundary condition,

$\dim_{\mathcal{M}}(\partial G) \le \lambda \in [1, 2)$, and

$$p > \frac{q(2 - \lambda\beta)}{q + \beta(2 - \lambda)}. \tag{5.10}$$

Recall a natural restriction among these parameters; by Theorem 2.3 there is a constant $c_2 > 0$ such that $\dim_{\mathcal{M}}(\partial G) \le 2 - c_2\beta$. Taking this into account, Theorem 5.1 shows that our main result is essentially sharp in essentially all the possible cases when $n = 2$. More precisely, if $\lambda \le 2 - \bar{c}_2\beta$, then there is a domain G_β which satisfies a $\beta/4$-quasihyperbolic boundary condition, $\dim_{\mathcal{M}}(\partial G_\beta) = \lambda$, and G_β does not support the (q, p)-Poincaré inequality (1.1) if inequality (5.10) fails to hold and $1 \le q < p < \infty$.

6. Applications to the Neumann problem

We discuss an application of our main result to the study of the solvability of the elliptic second order Neumann problem in quasihyperbolic boundary condition domains. We hence supplement Corollary 4.5 in [10].

Let G in $\mathbb{R}^n$ be a bounded domain which satisfies a β-quasihyperbolic boundary condition. We consider the following second order strongly elliptic partial differential operator

$$\mathcal{L}_A u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right), \tag{6.1}$$

where $A = \{a_{ij}(x)\}_{i,j=1}^n$, with a_{ij} being real-valued measurable functions in G, $a_{ij} = a_{ji}$, and there exists a constant $c \ge 1$ such that

$$c^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le c|\xi|^2$$

for almost every $x \in G$ and for all $\xi \in \mathbb{R}^n$. We deal with the equation in divergence form

$$\mathcal{L}_A u = f \tag{6.2}$$

subject to the Neumann boundary condition.

Let us fix $1 \le q < \infty$. We say that u is in the domain $\mathcal{D}(\mathcal{L}_A)$ of $\mathcal{L}_A$ if u satisfies (6.2) in the weak sense, i.e. $u \in W^{1,2}(G) \cap L^q(G)$, $f \in L^{q'}(G)$ with $q' = q/(q-1)$, and

$$\int_G \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_j}\,dx = \int_G \phi(x) f(x)\,dx$$

for all $\phi \in W^{1,2}(G) \cap L^q(G)$. Then the Neumann problem is said to be q-solvable if for each $f \in L^{q'}(G)$ with

$$\int_G f\,dx = 0$$

there exists $u \in \mathcal{D}(\mathcal{L}_A)$ such that (6.2) holds in the weak sense.

We have the following corollary of Theorem 3.1.

6.3. **Corollary.** *Suppose $G \subset \mathbb{R}^n$, $n \geq 2$, satisfies a β-quasihyperbolic boundary condition for some $1-\frac{2}{n} \leq \beta \leq 1$. Then the Neumann problem (6.2) on G is q-solvable for each $1 \leq q < 2$.*

Proof. By [13, 6.10.1/Lemma, p. 381], the Neumann problem (6.2) on G is q-solvable if and only if G supports a $(q, 2)$-Poincaré inequality. Therefore, the claim follows from Corollary 3.18. □

We note that in the plane the Neumann problem (6.2) is q-solvable in every β-quasihyperbolic boundary condition domain G, $0 < \beta \leq 1$, whenever $1 \leq q < 2$.

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