

CAHN-HILLIARD AND THIN FILM EQUATIONS WITH NONLINEAR MOBILITY AS GRADIENT FLOWS IN WEIGHTED-WASSERSTEIN METRICS

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ABSTRACT. In this paper, we establish a novel approach to proving existence of non-negative weak solutions for degenerate parabolic equations of fourth order, like the Cahn-Hilliard and certain thin film equations. The considered evolution equations are in the form of a gradient flow for a perturbed Dirichlet energy with respect to a Wasserstein-like transport metric, weak solutions are obtained as curves of maximal slope. Our main assumption is that the mobility of the particles is a concave function of their spatial density. A qualitative difference of our approach to previous ones is that essential properties of the solution — non-negativity, conservation of the total mass and dissipation of the energy — are automatically guaranteed by the construction from minimizing movements in the energy landscape.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

This paper is concerned with the following class of initial-boundary value problems for non-linear fourth order parabolic equations,

$$\partial_t u = -\operatorname{div}(\mathbf{m}(u) \operatorname{D}(\Delta u - G'(u))) \quad \text{in } (0, \infty) \times \Omega, \quad (1)$$

$$\mathbf{n} \cdot \operatorname{D} u = 0, \quad \mathbf{n} \cdot (\mathbf{m}(u) \operatorname{D}(\Delta u - G'(u))) = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \quad (2)$$

$$u(0, x) = u_0(x), \quad \text{in } \Omega. \quad (3)$$

The problem (1)–(3) is posed on a bounded, smooth convex domain $\Omega \subset \mathbb{R}^d$. \mathbf{n} denotes the normal vector field to the boundary $\partial\Omega$. The sought solution $u : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ is subject to the constraint $0 \leq u(t, x) \leq M$, where either $M > 0$ is a given number, or $M = +\infty$. The *mobility* \mathbf{m} is a non-negative concave function $\mathbf{m} : (0, M) \rightarrow \mathbb{R}_+$ that vanishes at 0, and also at M if $M < \infty$. The mobility \mathbf{m} and the *free energy* $G : (0, M) \rightarrow \mathbb{R}$ are subject to certain regularity assumptions, specified below. Introducing the *pressure* P satisfying

$$P'(s) = \mathbf{m}(s)G''(s), \quad (4)$$

equation (1) can be rewritten in the more familiar form

$$\partial_t u = -\operatorname{div}(\mathbf{m}(u) \operatorname{D} \Delta u) + \Delta P(u). \quad (5)$$

Equations of the form (1) or (5) arise, for instance, as hydrodynamic approximation to models for many-particle systems in gas dynamics, and also in lubrication theory. In particular, the classical Cahn-Hilliard equation for phase separation in a binary alloy as well as the (de)stabilized thin film equation are of the shape (1); we comment on these special cases further below. The value of the solution $u(t, x)$ is typically related to the particle density, or the fraction of one component of a binary alloy (in the case of the Cahn-Hilliard equation) or the height of the film (in the case of the thin film equation) at time $t \geq 0$ and location $x \in \Omega$.

There is a rich literature on the mathematical structure of Cahn-Hilliard, thin film and related equations. In particular, the techniques developed in the seminal papers by Elliott and Garcke [12] and by Bernis and Friedman [2] have been proven extremely powerful to carry out existence analysis, and have been extended by many other authors afterwards. As a core feature, these techniques allow to replace (1) by a family of regularized problems with smooth solutions u_δ that satisfy certain bounds which produce the desired constraint $0 \leq u \leq M$ in the limit $\delta \downarrow 0$. We emphasize that this bound does not come for free since solutions to fourth order equations do not obey comparison principles in general.

More specifically, in the existence proof for the Cahn-Hilliard equation [12], the degenerate mobility \mathbf{m} is replaced by a strictly positive approximation \mathbf{m}_δ defined on all \mathbb{R} . The resulting parabolic problems are non-degenerate and possess global and smooth solutions u_δ , which, however, may attain arbitrary real values. Using additional a priori estimates, it is then shown that the integral of u_δ in the region where $u < 0$ or $u > M$ converges to zero as \mathbf{m}_δ approaches \mathbf{m} , which yields $0 \leq u(t) \leq M$ in the limit. In fact, a corollary of this method of proof is that solutions to (1) with a sufficiently degenerate mobility function \mathbf{m} preserve the *strict* inequalities $0 < u(t) < M$ for all times $t \geq 0$. This property has been used in the existence proofs for thin film equations [2, 9], where the original mobility is approximated by very degenerate \mathbf{m}_δ .

The techniques from [2, 12] rely on the dissipation of certain Lyapunov functionals by solutions to (1). One distinguished Lyapunov functional is a perturbed Dirichlet energy, which is defined on functions $u \in H^1(\Omega)$ with $0 \leq u \leq M$ by

$$\mathbf{E}[u] = \frac{1}{2} \int_{\Omega} |\mathbf{D}u(x)|^2 dx + \int_{\Omega} G(u(x)) dx. \quad (6)$$

This energy and its dissipation provide regularity estimates. Another Lyapunov functional introduced in [2, 12] and since then widely used in the literature is

$$\mathbf{U}[u] := \int_{\Omega} U(u(x)) dx \quad \text{with} \quad U''(s) = \frac{1}{\mathbf{m}(s)}. \quad (7)$$

If the mobility $\mathbf{m}(s)$ degenerates sufficiently strongly for $s \downarrow 0$ and $s \uparrow M$, then \mathbf{U} allows to control the solution u in the zones where u is close to 0 or M . The functional \mathbf{U} has thus become a key tool for proving the bounds $0 \leq u \leq M$.

Here we develop an alternative approach to existence which avoids the cumbersome discussion of the propagation of the bound $0 \leq u \leq M$ and shows a new interesting variational structure behind equations of the form (1). Our starting point is the classical observation that (1) is in the shape of a gradient flow for \mathbf{E} on the space of non-negative density functions of fixed mass. On a purely formal level, the corresponding metric tensor is readily determined: to a tangential vector $\mathbf{v} := \partial_s \rho(0)$ to a smooth curve $\rho : (-\varepsilon, \varepsilon) \rightarrow L^1(\Omega)$ of strictly positive densities $\rho(s)$ at $\rho_0 = \rho(0)$, it assigns the length

$$\|\mathbf{v}\|^2 = \int_{\Omega} |\mathbf{D}\varphi(x)|^2 \mathbf{m}(\rho_0(x)) dx, \quad \text{with } \varphi \text{ satisfying } -\operatorname{div}(\mathbf{m}(\rho_0(x)) \mathbf{D}\varphi(x)) = \mathbf{v} \quad \text{in } \Omega \quad (8)$$

and variational boundary conditions on $\partial\Omega$.

In the particular case of a constant mobility $\mathbf{m} \equiv 1$, this is simply the dual of the Sobolev seminorm in $W^{1,2}(\Omega)$, and one can work in the well known setting of gradient flows in Hilbert spaces, see e.g. [3]. For the linear mobility $\mathbf{m}(s) = s$, the tensor (8) is induced by a non-Hilbertian metric, namely the celebrated L^2 -Wasserstein distance, see [23]. In this framework, weak solutions to specific cases of (1) have been obtained as curves of steepest descent in the energy landscape of \mathbf{E} ; see [15, 21] for respective results on the Hele-Shaw flow.

For more general mobilities, the existence proof presented here seems to be the first based on the gradient flow structure of (1) with respect to a metric that is not the L^2 -Wasserstein distance nor a flat Hilbertian one. It has been proven only recently by Dolbeault, Nazaret and the third author [11] that even for certain *non-linear* mobilities \mathbf{m} , the formal metric structure indicated in (8) still leads to a genuine metric $\mathbf{W}_\mathbf{m}$ on the space of positive measures. One needs to assume, however, that \mathbf{m} is a concave function to get nice analytic and geometric properties of $\mathbf{W}_\mathbf{m}$: they have been studied in [7] and [20] and we review selected results in Section 2 below.

The goal of this paper is to prove rigorously that weak solutions to (1)–(3) can be obtained by the variational *minimizing movement/JKO scheme* under suitable conditions on the nonlinear concave function \mathbf{m} . The terminology *minimizing movement scheme* is due to De Giorgi [10], whereas *JKO scheme* enters in common use after the paper [17]. Preservation of the total mass, dissipation of the energy and, most notably, non-negativity of the density along the solution are direct consequences of the applied construction: the solution is a weak limit of time-discrete energy minimizing curves that lie in the convex cone of non-negative densities. The difficulty of

this approach consists in proving a posteriori that the curve of maximal slope indeed corresponds to a weak solution. For this, a priori estimates resulting from the dissipation of \mathbf{U} are employed.

1.1. **Hypotheses.** We recall that

either $M > 0$ is a given number, or $M = +\infty$.

All appearing measures are assumed to be absolutely continuous with respect to the Lebesgue measure \mathcal{L}^d , and we identify them with their Lebesgue densities on Ω . The densities have fixed total mass $\mathbf{m} > 0$, and are bounded from above by M if the latter is finite. Thus our ambient space will be the metric space $(X(\Omega), \mathbf{W}_{\mathbf{m}})$ where

$$X(\Omega) := \left\{ u \in L^1(\Omega) : 0 \leq u \leq M \text{ a.e. in } \Omega, \int_{\Omega} u \, dx = \mathbf{m} \right\}. \quad (9)$$

The possible choices for mobility functions $\mathbf{m} : (0, M) \rightarrow \mathbb{R}_+ := (0, \infty)$ are subject to the following conditions:

$$\begin{aligned} \mathbf{m} \text{ is concave, } \mathbf{m} \in C^\infty(0, M), \quad \mathbf{m} > 0 \text{ in } (0, M), \\ \mathbf{m}(0) := \lim_{s \downarrow 0} \mathbf{m}(s) = 0, \quad \mathbf{m}(M) := \lim_{s \uparrow M} \mathbf{m}(s) = 0 \quad \text{if } M < +\infty. \end{aligned} \quad (\text{M})$$

Moreover, we say that the mobility \mathbf{m} satisfies a LSC condition (i.e. \mathbf{m} is Lipschitz and \mathbf{m}^2 is Semi-Convex) if

$$\sup_{s \in (0, M)} |\mathbf{m}'(s)| < +\infty \quad \text{and} \quad \sup_{s \in (0, M)} (-\mathbf{m}''(s)\mathbf{m}(s)) < +\infty. \quad (\text{M-LSC})$$

The restriction to concave mobilities in (M) is necessary, since only for those, the corresponding metric $\mathbf{W}_{\mathbf{m}}$ is well-defined. Notice that this hypothesis is somewhat opposite to the one made in [12], where an asymptotic behaviour $\mathbf{m}(s) \sim s^\alpha$ for $s \rightarrow 0$ with $\alpha \geq 1$ has been assumed. Typical examples for mobility functions with finite $M > 0$ are $\mathbf{m}(r) = r(M - r)$, or, more generally, $\mathbf{m}(r) = r^{\alpha_0}(M - r)^{\alpha_1}$ with exponents $\alpha_0, \alpha_1 \in (0, 1]$. These mobilities satisfies (M-LSC) iff $\alpha_0 = \alpha_1 = 1$. In the case that $M = +\infty$, the mobility \mathbf{m} is nondecreasing because it is concave and strictly positive in $(0, +\infty)$. Typical examples are $\mathbf{m}(r) = r^\alpha$ with $\alpha \in (0, 1]$; such a mobility is LSC only in the Wasserstein case $\alpha = 1$.

Concerning the free energy $G \in C^2(0, M)$ and the associated pressure P with (4), we assume that there exist a constant $C \geq 0$ and an exponent $q > 2$ with $q < 2d(d - 4)$ if $d > 4$ such that

$$\begin{aligned} \mathbf{m}G'' \geq -C \text{ in } (0, M), \quad P \in C^0([0, M]) & \quad \text{if } M < \infty, \\ \mathbf{m}G'' \geq -C(1 + \mathbf{m}) \text{ in } (0, \infty), \quad P \in C^0([0, \infty)), \quad \lim_{s \rightarrow \infty} \frac{P(s)}{s^q + |G(s)|} = 0 & \quad \text{if } M = +\infty. \end{aligned} \quad (\text{G})$$

The condition (G) yields in particular (see §2.4)

$$G \in C^0([0, M]) \quad \text{if } M < \infty; \quad G(s) \geq -C(1 + s^2) \quad \text{for every } s > 0 \text{ if } M = +\infty. \quad (10)$$

Examples for sensible choices of G (and P) fulfilling these assumptions are given after the statements of our main results.

1.2. **The minimizing movement approximation and the existence result for LSC mobilities.** The *minimizing movement/JKO scheme* is a variational algorithm to obtain a time-discrete approximation (of given step size $\tau > 0$) to a curve of steepest descent, see [1]. In the situation at hand, we start from the initial condition $u_0 \in X(\Omega)$ with $\mathbf{E}[u_0] < +\infty$ and define inductively

$$u_\tau^0 := u_0, \quad u_\tau^{n+1} := \operatorname{argmin} \Psi_\tau^n \in X(\Omega) \quad \text{where} \quad \Psi_\tau^n(v) := \frac{1}{2\tau} \mathbf{W}_{\mathbf{m}}(u_\tau^n, v)^2 + \mathbf{E}[v], \quad (11)$$

and we set $\mathbf{E}[u] := +\infty$ if $u \notin H^1(\Omega)$. The approximation $\bar{u}_\tau : [0, \infty) \rightarrow X(\Omega)$ is defined by constant interpolation, using $\bar{u}_\tau(t) = u_\tau^n$ for $(n - 1)\tau < t \leq n\tau$.

Theorem 1. *Assume that Ω is a smooth bounded convex open subset of \mathbb{R}^d , the mobility function \mathbf{m} satisfies (M) and (M-LSC), and the free energy G satisfies (G).*

Then, for any initial condition $u_0 \in X(\Omega)$ of finite energy $\mathbf{E}[u_0] < +\infty$, the scheme (11) admits time-discrete solutions \bar{u}_τ for all $\tau > 0$. For every sequence $\tau_n \downarrow 0$ there exists a subsequence, still denoted by τ_n , and a function u satisfying:

$$u \in L^2_{\text{loc}}([0, \infty); H^2(\Omega)) \cap C_w^0([0, \infty); H^1(\Omega)) \cap \text{AC}^2_{\text{loc}}([0, \infty); X(\Omega)) \quad (12)$$

$$\bar{u}_{\tau_n}(t) \rightarrow u(t) \text{ strongly in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ for all } t \in [0, +\infty), \quad (13)$$

$$\bar{u}_{\tau_n} \rightarrow u \text{ strongly in } L^2(0, T; H^1(\Omega)) \text{ and weakly in } L^2(0, T; H^2(\Omega)) \text{ for all } T > 0. \quad (14)$$

The energy satisfies the bound

$$\mathbf{E}[u(t)] \leq \mathbf{E}[u_0] \quad \text{for all } t \geq 0, \quad (15)$$

there exists a decreasing function $\varphi : [0, +\infty) \rightarrow \mathbb{R}$ such that

$$\varphi(t) \geq \mathbf{E}[u(t)] \quad \text{for all } t \geq 0, \quad (16)$$

and

$$\mathbf{E}[\bar{u}_{\tau_n}(t)] \rightarrow \mathbf{E}[u(t)] \quad \text{for a.e. } t \geq 0. \quad (17)$$

Finally u satisfies the equation (1) and the boundary conditions (2) in the following weak sense:

$$\int_0^{+\infty} \int_\Omega \partial_t \zeta u dx dt = \int_0^{+\infty} \int_\Omega \Delta u \operatorname{div}(\mathbf{m}(u) \mathbf{D} \zeta) dx dt - \int_0^{+\infty} \int_\Omega P(u) \Delta \zeta dx dt \quad (18)$$

for every test function $\zeta \in C_c^\infty((0, +\infty) \times \bar{\Omega})$ such that $\mathbf{D} \zeta \cdot \mathbf{n} = 0$ on $\partial\Omega$.

Remark 1.1. We add a few comments on the previous result:

- $C_w^0([0, \infty); H^1(\Omega))$ denotes the space of weakly continuous curves $u : [0, \infty) \rightarrow H^1(\Omega)$.
- A curve $u : [0, \infty) \rightarrow X(\Omega)$ belongs to $\text{AC}^2_{\text{loc}}(0, \infty; X(\Omega))$ if there exists a function $g \in L^2_{\text{loc}}([0, \infty))$ such that

$$\mathbf{W}_{\mathbf{m}}(u(s), u(t)) \leq \int_s^t g(r) dr \quad \text{for all } 0 \leq s \leq t < \infty. \quad (19)$$

- The condition $u \in \text{AC}^2_{\text{loc}}(0, \infty; X(\Omega))$ implies that

$$0 \leq u \leq M, \quad \int_\Omega u(t, x) dx = \mathbf{m} \quad \text{for all } t \geq 0. \quad (20)$$

- The condition (12) implies that the initial datum in (3) is attained in the sense that $u(t)$ converges to u_0 in $X(\Omega)$ with respect to the distance $\mathbf{W}_{\mathbf{m}}$ and weakly in $H^1(\Omega)$ as $t \downarrow 0$.
- Since $u \in L^2_{\text{loc}}([0, \infty); H^2(\Omega))$ and \mathbf{m} is LSC, $\operatorname{div}(\mathbf{m}(u) \mathbf{D} \zeta) \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$.
- Even in the case $M = +\infty$, (15), the asymptotic behaviour (G), the lower bound (10) and the Sobolev embedding of $H^2(\Omega)$ in $L^q(\Omega)$ imply that $P(u) \in L^2_{\text{loc}}([0, +\infty); L^1(\Omega))$.

The main examples that fits into the framework of Theorem 1 are the classical Cahn-Hilliard equations [4]: For the mobility, one chooses $\mathbf{m}(r) = r(1-r)$, so $M = 1$. Typical choices for the free energy G are the double well potential,

$$G(r) = \theta r^2(1-r)^2, \quad \theta > 0,$$

yielding a model for the phase separation for a binary alloy,

$$\partial_t u = -\operatorname{div}(u(1-u) \mathbf{D} \Delta u) + \theta \Delta u^2(1-u)^2; \quad (21)$$

or the function

$$G(r) = \theta(r \log r + (1-r) \log(1-r)), \quad \theta \in \mathbb{R},$$

that lead to an equation for the volume fraction of one component in binary gas mixture,

$$\partial_t u = -\operatorname{div}(u(1-u) \mathbf{D} \Delta u) + \theta \Delta u. \quad (22)$$

See [4, 5] for a derivation of (21) and (22), and [12] for a related existence analysis.

1.3. The existence result for more general mobilities. The assumption (M-LSC) about mobility can be weakened to condition $(M_{1/2})$ below, at the price that the weak solution to (1)–(3) is no longer obtained as a curve of steepest descent in \mathbf{W}_m , but appears as the weak limit of such curves in metrics satisfying (M-LSC).

Theorem 2. *Assume that Ω is a smooth bounded convex open subset of \mathbb{R}^d , the mobility function \mathbf{m} satisfies (M), and the free energy G satisfies (G). In addition, assume that \mathbf{m} satisfies*

$$\lim_{s \downarrow 0} s^{1/2} \mathbf{m}'(s) = 0, \quad \text{and, if } M < \infty, \text{ also } \lim_{s \uparrow M} (M - s)^{1/2} \mathbf{m}'(s) = 0. \quad (M_{1/2})$$

Then, for any initial condition $u_0 \in X(\Omega)$ of finite energy $\mathbf{E}[u_0] < \infty$, there exists a function $u \in L^2_{\text{loc}}(0, \infty; H^2(\Omega)) \cap C_w^0([0, \infty); H^1(\Omega))$ satisfying the constant mass and maximum estimate (20), the energy bound (15) and the weak formulation (18) of equation (1) with the boundary conditions (2). The initial condition (3) is met in the sense that $u(t)$ weakly converges to u_0 in $H^1(\Omega)$ as $t \downarrow 0$.

The first condition in $(M_{1/2})$ is needed to give a meaning to the gradient of $\mathbf{m}(u)$ in the weak formulation (18), particularly on the set $Z = \{(t, x) \in (0, +\infty) \times \Omega : u(t, x) = 0\}$. We briefly indicate the problem: Since $u(t) \in H^2(\Omega)$ for a.e. $t \geq 0$, and $u(t)$ satisfies homogeneous Neumann boundary conditions, a known estimate on the square root (see Lemma A.1) states that

$$\|D \sqrt{u(t)}\|_{L^4(\Omega)}^2 \leq C \|D^2 u(t)\|_{L^2(\Omega)}. \quad (23)$$

Thus $u \in L^2_{\text{loc}}(0, \infty; H^2(\Omega))$ in combination with $(M_{1/2})$ implies that $\mathbf{m}(u) \in L^2_{\text{loc}}(0, \infty; H^1(\Omega))$. In fact, in the proof it turns out that $D \mathbf{m}(u) = 0$ a.e. on the set Z . A similar reasoning applies to the zero set of $D \sqrt{M - u}$ if $M < \infty$. Therefore, it suffices to evaluate the second spatial integral in (18) only on the subset $\{0 < u(t) < M\} \subset \Omega$, thus avoiding to discuss the singularity of $\mathbf{m}'(s)$ at $s = 0$ or $s = M$. Unfortunately, (23) cannot be extended to obtain L^{2p} estimates on roots $u(t)^{1/p}$ with $p > 2$, as is easily seen. Without further a priori estimates, there is apparently no way to remove condition $(M_{1/2})$.

The main example in the framework of Theorem 2 is the – (de)stabilized – lubrication or thin film equation, where one chooses $M = +\infty$ and $\mathbf{m}(r) = r^\alpha$ with $1/2 < \alpha \leq 1$. The equation is

$$\partial_t u = -\operatorname{div}(u^\alpha D \Delta u) + \kappa \Delta(u^\beta), \quad (24)$$

where $u : \Omega \times (0, \infty) \rightarrow [0, +\infty)$ describes the height of a thin viscous liquid film on a substrate, moving under the influence of surface tension; the lower order perturbation is typically attributed to van der Waals forces or similar intermolecular interactions. The destabilized case corresponds to $\kappa < 0$ while for $\kappa > 0$, the contribution has a stabilizing effect.

The existence theory of (24) for the unperturbed flow $\kappa = 0$ is fairly well understood [9]. In particular, the Hele-Shaw equation obtained for $\alpha = 1$ has been analyzed thoroughly as a gradient flow of the Dirichlet functional in the L^2 -Wasserstein metric, see e.g. [6, 15, 21]. The perturbed flow has essentially been treated in $d = 1$ dimensions only, see e.g. [19, 25], but some results (e.g. on the blow-up behavior of solutions) are available also in multiple dimensions [13]. For non-linear mobilities \mathbf{m} , the equation's gradient flow structure has apparently not been exploited for rigorous analytical treatment before.

In order to obtain (24) from (1), one would like to choose

$$G(r) = \kappa \frac{\beta}{(\beta - \alpha)(\beta - \alpha + 1)} r^{\beta - \alpha + 1}$$

in the definition of the energy (6). This is, however, only possible for certain regimes of β and κ :

- If $1 \leq \beta \leq \alpha + 1$, then G satisfies (G) for all $\kappa \in \mathbb{R}$.
- If $\alpha < 1$ and $\alpha + 1 < \beta$ or $\alpha = 1$ and $\beta > 2$ with $\beta < 2d/(d - 4)$ if $d > 4$ then G satisfies (G) provided that $\kappa \geq 0$, i.e., the perturbation must be stabilizing.
- If $\beta < 1$, then there is no way to accommodate the perturbation into our framework.

1.4. Key ideas of the existence proof. The discrete approximation scheme in (11) provides a family of piecewise constant approximate solutions $\bar{u}_\tau : [0, \infty) \rightarrow H^1(\Omega)$. Weak convergence towards a limit curve $\bar{u} : [0, \infty) \rightarrow H^1(\Omega)$ along a sequence $\tau_n \downarrow 0$ is easily obtained, using the machinery developed in [1]. The difficulty lies in identifying the weak limit \bar{u} as a weak solution to (1)–(3).

For mobilities satisfying (M-LSC), a semi-discrete version of the weak formulation (18) is derived by variational methods, i.e., we use suitable perturbations of the minimizers u_τ^n in each step of the scheme (11). Our variations of the u_τ^n are obtained by applying an auxiliary gradient flow to them. Specifically, in order to arrive at (18), we would like to use variations in the direction of the flow generated by the functional

$$\mathbf{V}[u] := \int_{\Omega} V u \, dx$$

with a given test function $V \in C^\infty(\bar{\Omega})$ satisfying homogeneous Neumann boundary conditions on $\partial\Omega$. To motivate this particular choice, assume for the moment that the test function ζ factors as $\zeta(t, x) = \psi(t)V(x)$. Then the left-hand side of (18) can formally be read as

$$\int_0^T \int_{\Omega} u(t, x) \psi'(t) V(x) \, dx \, dt = - \int_0^T \psi(t) \frac{d}{dt} \mathbf{V}[u(t)] \, dt,$$

i.e., as the temporal derivative of the functional \mathbf{V} along the sought gradient flow for \mathbf{E} . Further, the “flow interchange” Lemma 3.2, taken from [21], says that this expression can equally be understood as the temporal derivative of the functional \mathbf{E} along the gradient flow of \mathbf{V} . Thus, variations of the minimizers for (11) along the flow of \mathbf{V} are expected to provide a form of (18).

Unfortunately, \mathbf{V} itself is not a suitable choice for carrying out estimates, since the gradient flow generated by \mathbf{V} is not regular enough to apply the flow interchange lemma. In particular, the functional \mathbf{V} is not geodesically λ -convex in the metric \mathbf{W}_m for any $\lambda \in \mathbb{R}$ (see [7]). As a matter of fact, the trajectories of the gradient flow of \mathbf{V} with respect to \mathbf{W}_m are formally given by the solutions of the non-linear conservation law

$$\partial_t v_t = \operatorname{div}(\mathbf{m}(v_t) D V). \quad (25)$$

These solutions are expected to develop shocks in finite time. To circumvent this technical problem, we consider a modification of \mathbf{V} ,

$$\mathbf{V}_\varepsilon[u] := \mathbf{V}[u] + \varepsilon \mathbf{U}[u],$$

where \mathbf{U} is defined in (7), that induces the following viscous regularization in (25)

$$\partial_t v_t = \operatorname{div}(\mathbf{m}(v_t) D V) + \varepsilon \Delta v_t.$$

For LSC mobilities, the viscous regularized flow generated by \mathbf{V}_ε with respect to \mathbf{W}_m is λ_ε -convex and our strategy goes through. For more general mobilities, even the viscous flow lacks convexity. This makes it necessary to perform further approximations: we replace the mobility function by LSC-ones, obtain a weak formulation (18) for the corresponding flows, and then pass to the non-LSC limit.

Even with the discrete version of (18) at hand, we still need to facilitate sufficiently strong compactness to pass to the time-continuous limit $\tau \downarrow 0$. Our key estimate is obtained from the dissipation of the functional \mathbf{U} in (7) along solutions of (1). A direct calculation shows that the dissipation term provides a bound in $L^2_{\text{loc}}(0, \infty; H^2(\Omega))$. The rigorous proof of this H^2 -bound is obtained by another application of the strategy above: we interchange flows and calculate the variations of \mathbf{E} with respect to perturbations of the minimizers in the direction of the gradient flow generated by \mathbf{U} . This time, the strategy goes through smoothly since the auxiliary functional \mathbf{U} , which generates the heat flow with homogeneous Neumann boundary condition, is geodesically convex with respect to the considered metric \mathbf{W}_m .

The paper is organized as follows. Section 2 below provides the essential definitions for the measure-theoretic formulation of the problem. In Section 3 we have collected a variety of technical results that are applied in subsequent sections to obtain a priori estimates on the semi-discrete

approximation \bar{u}_τ . Sections 4 and 5 are devoted to the proofs of Theorems 1 and 2, respectively. There, we follow the strategy outlined above.

2. PRELIMINARIES

2.1. Basic assumptions. Here and in the rest of this paper, we will always assume that

$$\Omega \text{ is a convex, smooth and bounded open set of } \mathbb{R}^d, \quad \mathbf{m}_\Omega := \frac{\mathbf{m}}{|\Omega|} \in (0, M), \quad (\Omega\text{-conv})$$

where $M \in (0, +\infty]$ characterizes the domain of the mobility function \mathbf{m} as in (M). We will always assume that \mathbf{m} is a mobility function satisfying (M) and G is a free energy density satisfying (G).

2.2. Notation: admissible and regular densities. As in (9), for a given mobility \mathbf{m} and a mass \mathbf{m} satisfying $(\Omega\text{-conv})$ we introduce the sets of *admissible* and *regular densities* on Ω

$$X(\Omega) := \left\{ \rho \in L^1(\Omega) \mid 0 \leq \rho \leq M, \int_{\Omega} \rho \, dx = \mathbf{m} \right\}, \quad (26)$$

$$X^r(\Omega) := \left\{ \rho \in C^\infty(\Omega) \mid 0 < \inf \rho \leq \sup \rho < M, \int_{\Omega} \rho \, dx = \mathbf{m} \right\}. \quad (27)$$

Since we will keep fixed the mobility \mathbf{m} (and its domain of definition $(0, M)$) and the total mass \mathbf{m} , we will omit to indicate the explicit dependence of the above spaces from these two parameters.

We often identify an element $u \in X(\Omega)$ with the nonnegative measure $u = u \mathcal{L}^d$ in \mathbb{R}^d supported in $\bar{\Omega}$ and we will consider weak convergence of sequences in $X(\Omega)$ in the sense of distributions of $\mathcal{D}'(\mathbb{R}^d)$:

$$u_n \rightharpoonup u \text{ in } \mathcal{D}'(\mathbb{R}^d) \iff \lim_{n \rightarrow \infty} \int_{\Omega} u_n \varphi \, dx = \int_{\Omega} u \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^d). \quad (28)$$

Remark 2.1. Since u_n are nonnegative with fixed total mass, we could also equivalently consider test functions $\varphi \in C_c^0(\mathbb{R}^d)$ in (28); when $M < \infty$, $X(\Omega)$ is a convex and bounded subset of $L^\infty(\Omega)$ and (28) also coincides with the weak- \star convergence in the latter space.

For every extended-valued real functional $\mathcal{F} : X(\Omega) \rightarrow (-\infty, +\infty]$ we denote by $\text{Dom}(\mathcal{F})$ its proper domain $\text{Dom}(\mathcal{F}) := \{u \in X(\Omega) : \mathcal{F}(u) < \infty\}$. \mathcal{F} is called proper if $\text{Dom}(\mathcal{F})$ is not empty.

We will consider curves in $X^r(\Omega)$: they are maps $\gamma : [0, 1] \rightarrow X(\Omega)$ which we will also identify with functions $\gamma : [0, 1] \times \Omega \rightarrow [0, M]$ such that $\gamma(t, \cdot) \in X^r(\Omega)$ for every $t \in [0, 1]$. We say that

$$\gamma : [0, 1] \rightarrow X^r(\Omega) \text{ is regular if } t \mapsto \gamma(t, x) \in C^1([0, 1]) \text{ for all } x \in \Omega. \quad (29)$$

In a similar way, a functional

$$\mathcal{F} : X^r(\Omega) \rightarrow \mathbb{R} \text{ is regular if } t \mapsto \mathcal{F}[\gamma(t)] \in C^1([0, 1]) \text{ for every regular curve } \gamma, \quad (30)$$

and a map $\mathbf{S} : [0, \infty) \times X^r(\Omega) \rightarrow X^r(\Omega)$ is regular if the curves

$$\mathbf{S}(\cdot, u), \mathbf{S}(t, \gamma(\cdot)) \text{ are regular for every } u \in X^r(\Omega), t \geq 0, \text{ and for every regular curve } \gamma. \quad (31)$$

2.3. Survey: weighted transport distances. We shall now review the weighted transport distances $\mathbf{W}_\mathbf{m}$ introduced in [11] (see also [7] and [20]) without going into details about their formal definition. When $M = \infty$ they could in fact be pseudo-metrics, i.e. they satisfy all the axioms of the usual notion of distance except for the fact that the value $+\infty$ may be attained; nevertheless, even in the case $M = \infty$ the next proposition shows that the restriction of $\mathbf{W}_\mathbf{m}$ to the sublevels of the convex functional (recall (7))

$$\mathbf{U}[u] := \int_{\Omega} U(u) \, dx \quad \text{where } U''(s) = \frac{1}{\mathbf{m}(s)}, \quad U(\mathbf{m}_\Omega) = U'(\mathbf{m}_\Omega) = 0, \quad (32)$$

is a finite distance. Notice moreover that, besides \mathbf{m} , $\mathbf{W}_\mathbf{m}$ also depends on the domain Ω : we will denote it by $\mathbf{W}_{\mathbf{m}, \Omega}$ when we want to stress this dependence. In particular, for every $\delta > 0$ we will also sometimes consider the δ -neighborhood $\Omega_{[\delta]}$ of Ω

$$\Omega_{[\delta]} := \Omega + \delta \mathbb{B}^d = \{x \in \mathbb{R}^d \mid \text{dist}(x, \Omega) < \delta\}. \quad (33)$$

Proposition 2.2. *In the setting of §2.1, the pseudo-metric \mathbf{W}_m on the space $X(\Omega)$ has the following properties:*

(a) For every $u_n, u \in X(\Omega)$,

$$\lim_{n \rightarrow \infty} \mathbf{W}_m(u_n, u) = 0 \quad \Leftrightarrow \quad u_n \rightharpoonup u \quad \text{in } \mathcal{D}'(\mathbb{R}^d) \quad (34)$$

according to (28) (but see also remark 2.1).

(b) For every $c \geq 0$ the sublevels of \mathbf{U}

$$\left\{ u \in X(\Omega) \mid \mathbf{U}[u] \leq c \right\} \quad \text{are compact metric spaces w.r.t. } \mathbf{W}_m. \quad (35)$$

(c) For every decreasing sequence of convex sets Ω^n converging to Ω , if two sequences $u_0^n, u_1^n \in X(\Omega^n)$ converge to u_0 and u_1 in the sense of distributions respectively, i.e.

$$\lim_{n \rightarrow \infty} \int_{\Omega^n} u_i^n \varphi \, dx = \int_{\Omega} u_i \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d), \quad i = 0, 1, \quad (36)$$

then

$$\mathbf{W}_{m, \Omega}(u_0, u_1) \leq \liminf_{n \rightarrow +\infty} \mathbf{W}_{m, \Omega^n}(u_0^n, u_1^n). \quad (37)$$

(d) If $\gamma : [0, 1] \rightarrow X^r(\Omega)$ is a regular curve according to (29) and $\varphi : [0, 1] \rightarrow H^1(\Omega)$ is the corresponding curve of weak solutions to

$$-\operatorname{div}(\mathbf{m}(\gamma) \mathbf{D} \varphi) = \partial_s \gamma \quad \text{in } \Omega, \quad \mathbf{n} \cdot \mathbf{D} \varphi = 0 \quad \text{on } \partial \Omega, \quad (38)$$

then the \mathbf{W}_m -distance between $u_0 = \gamma(0)$ and $u_1 = \gamma(1)$ is bounded as follows:

$$\mathbf{W}_m(u_0, u_1)^2 \leq \int_0^1 \int_{\Omega} \mathbf{m}(\gamma) |\mathbf{D} \varphi|^2 \, dx \, ds. \quad (39)$$

(e) Let $u_i \in X(\Omega)$, $i = 0, 1$, be given with $\mathbf{W}_m(u_0, u_1) < \infty$. For every decreasing sequence of smooth convex sets Ω_n converging to Ω as $n \rightarrow \infty$, such that $\Omega_n \supset \Omega_{[\delta_n]}$ for a vanishing sequence δ_n , there exists a sequence of regular curves (“approximate geodesics”) $\gamma_n : [0, 1] \rightarrow X^r(\Omega_n)$, such that:

- $\gamma_n(0)$ and $\gamma_n(1)$ converge to u_0 and u_1 , respectively, in $L^1(\mathbb{R}^d)$ as $n \rightarrow \infty$ and for every proper and lower semicontinuous convex integrand $F : [0, M] \rightarrow [0, \infty]$

$$\lim_{n \rightarrow \infty} \int_{\Omega_n} F(\gamma_n(i, x)) \, dx = \int_{\Omega} F(u_i) \, dx \quad i = 0, 1. \quad (40)$$

- if $\varphi_n : [0, 1] \rightarrow H^1(\Omega_n)$ are the corresponding curves of weak solutions to (38) on Ω_n , then

$$\mathbf{W}_m(u_0, u_1)^2 = \lim_{n \rightarrow \infty} \int_0^1 \int_{\Omega_n} \mathbf{m}(\gamma_n) |\mathbf{D} \varphi_n|^2 \, dx \, ds. \quad (41)$$

No properties of the metric \mathbf{W}_m other than those listed above will be used in the sequel. Notice that (39)&(41) establish the connection between the metric \mathbf{W}_m and the formal definition of the metric tensor given in (8).

2.4. The entropy and energy functionals. Let us first study a few property of the entropy and the energy densities U, G introduced in (7) and (6).

For definiteness, we make the following specific choice for the function U in (7):

$$U(s) := \int_{s_0}^s \frac{s-r}{\mathbf{m}(r)} \, dr, \quad s_0 := \mathbf{m}_\Omega = \frac{\mathbf{m}}{|\Omega|}. \quad (42)$$

Lemma 2.3. *The entropy functional \mathbf{U} is lower semi-continuous with respect to the weak convergence (28), and satisfies the following bounds*

$$0 \leq \mathbf{U}[u] \leq C(1 + \|u\|_{L^2(\Omega)}^2), \quad \text{for all } u \in \operatorname{Dom}(\mathbf{U}). \quad (43)$$

The constant C above only depends on Ω , \mathbf{m} , and $\mathbf{m}(s_0) = \mathbf{m}(\mathbf{m}_\Omega)$.

Proof. Lower semi-continuity is a consequence of the convexity of U , which, in turn, follows from $U''(s) = 1/\mathbf{m}(s) > 0$ for every $s \in (0, M)$. The lower bound in (43) follows from non-negativity of U , indeed convexity of U and (32) yield that \mathbf{m}_Ω is a minimum for U and $U(\mathbf{m}_\Omega) = 0$. For showing the upper bound in (43), first note that

$$\mathbf{m}(s) \geq \begin{cases} \frac{\mathbf{m}(s_0)}{s_0} s & \text{if } s \leq s_0 \\ \frac{\mathbf{m}(s_0)}{M-s_0} (M-s) & \text{if } s > s_0, \quad M < +\infty \\ \mathbf{m}(s_0) & \text{if } s > s_0, \quad M = +\infty, \end{cases} \quad (44)$$

by concavity of \mathbf{m} . Thus for $C_0 := \frac{s_0^2}{\mathbf{m}(s_0)}$ and $C_M := \frac{(M-s_0)^2}{\mathbf{m}(s_0)}$

$$U(s) \leq \begin{cases} \frac{s_0}{\mathbf{m}(s_0)} (s(\log s - \log(s_0)) - 1) + s_0 \leq C_0 & \text{if } 0 \leq s \leq s_0, \\ \frac{M-s_0}{\mathbf{m}(s_0)} ((M-s)(\log(M-s)) - \log(M-s_0) - 1) + M - s_0 \leq C_M & \text{if } s > s_0, M < \infty, \\ \frac{1}{2\mathbf{m}(s_0)} (s - s_0)^2 \leq \frac{1}{2\mathbf{m}(s_0)} s^2 & \text{if } s > s_0, M = \infty. \end{cases}$$

Now (43) follows by the boundedness of Ω . \square

Concerning the function G , we decompose its second derivative G'' into the difference of its positive and negative part

$$L := G'' = L^+ - L^-, \quad L^- := -\min(G'', 0), \quad L^+ := \max(G'', 0). \quad (45)$$

Fixing $s_0 \in (0, M)$ (e.g. $s_0 = \mathbf{m}_\Omega$ as before) and assuming without loss of generality that $G(s_0) = G'(s_0) = 0$ (recall that the integral of elements in $X(\Omega)$ is fixed to be \mathbf{m}) we have the decomposition

$$G = G_{\text{conv}} + G_{\text{conc}}, \quad G_{\text{conv}}(s) = \int_{s_0}^s L^+(r)(s-r) dr, \quad G_{\text{conc}}(s) = - \int_{s_0}^s L^-(r)(s-r) dr, \quad (46)$$

and the corresponding one

$$P = P_{\text{incr}} + P_{\text{decr}}, \quad P_{\text{incr}}(s) = \int_{s_0}^s L^+(r)\mathbf{m}(r) dr, \quad P_{\text{decr}}(s) = - \int_{s_0}^s L^-(r)\mathbf{m}(r) dr. \quad (47)$$

(46) and the upper bound $L^- \leq C(1 + 1/\mathbf{m}) = C(1 + U'')$ of (G) yield

$$G_{\text{conv}}(s) \geq G_{\text{conv}}(s_0) = 0 = G_{\text{conc}}(s_0) \geq G_{\text{conc}}(s) \geq -C(1 + s^2) \quad \text{for all } s \in (0, M), \quad (48)$$

proving the lower bound in (10). It follows immediately from the lower bounds in (G) that P_{decr} is Lipschitz continuous and G_{conc} is continuous in $[0, M)$ (and also in M if $M < \infty$) since it is concave and bounded from below.

In order to check the continuity of G_{conv} in 0 (the same argument applies to M when $M < \infty$), let us first observe that $P'_{\text{incr}} = L^+\mathbf{m}$ is integrable around 0 since $P_{\text{incr}} = P - P_{\text{decr}}$ is locally bounded around 0 by (G). Recalling (44) we easily get for $0 < s < s_0$

$$G_{\text{conv}}(s) = \int_s^{s_0} \frac{P'_{\text{incr}}(r)}{\mathbf{m}(r)} (r-s) dr \leq \frac{s_0}{\mathbf{m}(s_0)} \int_s^{s_0} P'_{\text{incr}}(r) dr \leq -P_{\text{incr}}(0).$$

Since G_{conv} is convex we conclude that it has a right limit at 0.

With (10) and the above remarks at our disposal, we can obtain simple lower bounds on the the energy functional \mathbf{E} defined in (6).

Lemma 2.4 (Basic properties of \mathbf{E}). *The functional \mathbf{E} is bounded from below in the space $X(\Omega)$ and lower semi-continuous with respect to the distributional convergence (28) in the space $X(\Omega)$. Finally the following estimate holds*

$$\frac{1}{8} \|u\|_{H^1(\Omega)}^2 + \int_{\Omega} G_{\text{conv}}(u) dx \leq \mathbf{E}[u] + \mathbf{E}_0 \quad \text{for all } u \in X(\Omega), \quad (49)$$

where G_{conv} has been defined by (46)-(45) and the constant \mathbf{E}_0 only depend on Ω , the mass \mathbf{m} , the dimension d and the function G .

Proof. To begin with, we prove the estimate (49). We recall the following Gagliardo-Nirenberg ([14], [22]) interpolation inequality

$$\|u\|_{L^2(\Omega)} \leq C_1 \|Du\|_{L^2(\Omega)}^\theta \|u\|_{L^1(\Omega)}^{1-\theta} + C_2 \|u\|_{L^1(\Omega)}, \quad u \in H^1(\Omega) \quad (50)$$

where $\theta = d/(d+2)$ and the constants C_1, C_2 only depend on Ω and d . In our specific case of $u \in X(\Omega) \cap H^1(\Omega)$ we have, for every $\varepsilon > 0$,

$$\|u\|_{L^2(\Omega)} \leq C_1 \|Du\|_{L^2(\Omega)}^\theta \mathbf{m}^{1-\theta} + C_2 \mathbf{m} \leq \varepsilon \|Du\|_{L^2(\Omega)} + C_3(\varepsilon) \mathbf{m} \quad (51)$$

where $C_3(\varepsilon) := (2C_1/\varepsilon)^{1/(1-\theta)} + C_2$. In particular

$$\|u\|_{L^2(\Omega)}^2 \leq 2\varepsilon^2 \|Du\|_{L^2(\Omega)}^2 + 2C_3(\varepsilon)^2 \mathbf{m}^2. \quad (52)$$

Using the decomposition (46), the lower bound (48), and (52), for the constant C in (48) we have

$$\begin{aligned} \mathbf{E}[u] &\geq \frac{1}{2} \|Du\|_{L^2}^2 - C(|\Omega| + \|u\|_{L^2}^2) + \int_{\Omega} G_{\text{conv}}(u) dx \\ &\geq \left(\frac{1}{2} - 2\varepsilon^2 C\right) \|Du\|_{L^2}^2 - C(|\Omega| + 2C_3(\varepsilon)^2 \mathbf{m}^2) + \int_{\Omega} G_{\text{conv}}(u) dx. \end{aligned} \quad (53)$$

Choosing $\varepsilon^2 = 1/(8C)$ in (53) and using again (52) with $\varepsilon^2 = 1/2$ we obtain (49) with the constant $\mathbf{E}_0 := C(|\Omega| + 2C_3(1/(2\sqrt{2C}))^2 \mathbf{m}^2) + 1/4 C_3(1/\sqrt{2})^2 \mathbf{m}^2$. Boundedness of \mathbf{E} from below is an immediate consequence of (49), recalling that G_{conv} is non-negative.

In order to prove lower semi-continuity, assume that a sequence $u_k \in X(\Omega)$ converges to a limit $u \in X(\Omega)$ according to (28). It is not restrictive to assume that $u_k \in H^1(\Omega)$ and that $\sup_{k \rightarrow \infty} \mathbf{E}[u_k] < +\infty$. By estimate (49) the sequence u_k is bounded in $H^1(\Omega)$. Hence, up to subsequences, u_k converges weakly in $H^1(\Omega)$, converges strongly in $L^2(\Omega)$, and converges pointwise \mathcal{L}^d -a.e. to u .

By (48) and Fatou's Lemma we have that

$$\liminf_{k \rightarrow +\infty} \int_{\Omega} G_{\text{conc}}(u_k) + C(1 + u_k^2) dx \geq \int_{\Omega} G_{\text{conc}}(u) + C(1 + u^2) dx. \quad (54)$$

The $L^2(\Omega)$ strong convergence of u_k and concavity of G_{conc} yield

$$\limsup_{k \rightarrow +\infty} \int_{\Omega} G_{\text{conc}}(u_k) + C(1 + u_k^2) dx \leq \int_{\Omega} G_{\text{conc}}(u) + C(1 + u^2) dx. \quad (55)$$

From (54) and (55) it follows

$$\lim_{k \rightarrow +\infty} \int_{\Omega} G_{\text{conc}}(u_k) dx = \int_{\Omega} G_{\text{conc}}(u) dx. \quad (56)$$

Second, by Fatou's Lemma, it follows that

$$\liminf_{k \rightarrow +\infty} \int_{\Omega} G_{\text{conv}}(u_k) dx \geq \int_{\Omega} G_{\text{conv}}(u) dx. \quad (57)$$

Finally, since Du_k converges weakly in $L^2(\Omega)$ to Du ,

$$\liminf_{k \rightarrow +\infty} \int_{\Omega} |Du_k|^2 dx \geq \int_{\Omega} |Du|^2 dx. \quad (58)$$

The lower semi-continuity of \mathbf{E} follows from (56), (57) and (58). \square

We will denote by \mathbf{E}_{\min} the minimum value (depending on Ω, \mathbf{m}, G) of \mathbf{E} on $X(\Omega)$. Notice that estimate (43) in combination with (49) yields

$$0 \leq \mathbf{U}[u] \leq C(\mathbf{E}[u] + \mathbf{E}_0), \quad \text{for all } u \in \text{Dom}(\mathbf{E}), \quad (59)$$

with some constant C only depending on $\mathbf{m}(s_0), \Omega, \mathbf{m}$ and G .

3. A PRIORI ESTIMATES

3.1. Semi-discrete approximation. We begin by invoking a result from [1] that guarantees the well-posedness of the minimizing movement scheme (11), i.e. the existence of the semi-discrete curves \bar{u}_τ and their compactness for vanishing step size $\tau \downarrow 0$.

Proposition 3.1. *In the setting of §2.1, for every $u_0 \in X(\Omega) \cap \text{Dom}(\mathbf{E})$ and $\tau > 0$ there exists a sequence $\{u_\tau^n\}_{n \geq 0}$ satisfying (11) and the following energy estimate:*

$$\mathbf{E}[u_\tau^N] + \frac{1}{2\tau} \sum_{n=1}^N \mathbf{W}_m(u_\tau^n, u_\tau^{n-1})^2 \leq \mathbf{E}[u_0] \quad \text{for all } N \in \mathbb{N}. \quad (60)$$

Moreover, for every sequence $\tau_n \downarrow 0$ there exists a subsequence, still denoted by τ_n , and a continuous limit curve $u : [0, +\infty) \rightarrow X(\Omega)$ such that $\bar{u}_{\tau_n}(t)$ converges weakly to u_t in $H^1(\Omega)$ for every $t \geq 0$. The curve u is globally $1/2$ -Hölder continuous

$$\mathbf{W}_m(u_t, u_s) \leq (2(\mathbf{E}[u_0] - \mathbf{E}_{\min}))^{1/2} |s - t|^{1/2} \quad \text{for all } s, t \in [0, +\infty). \quad (61)$$

The curve $t \mapsto u_t$ satisfies (15), (16), (17) and (20).

Proposition 3.1 is obtained by combining the results from Chapters 2 and 3 (see in particular Sections 2.1, 2.2 and Corollary 3.3.4) of [1]. The properties of \mathbf{E} proven in Lemma 2.4 are sufficient to apply this general theory and the uniform upper bound on $\mathbf{E}[\bar{u}_\tau]$ given by (60) improves the pointwise convergence of \bar{u}_{τ_n} with respect to \mathbf{W}_m to the weak convergence in $H^1(\Omega)$. It should be remarked that we do not claim uniqueness of solutions, even on this discrete level, except in the case when \mathbf{E} is a convex functional.

3.2. Flow interchange lemma. For the derivation of τ -independent a priori estimates on the interpolations \bar{u}_τ , we employ the device of the *flow interchange lemma*, which has been proven in [21]. Before reviewing the lemma and its proof, we recall the definition of λ -flow in the metric space $X(\Omega)$ given in [8].

Definition 1. *Let $\mathcal{F} : X(\Omega) \rightarrow (-\infty, +\infty]$ be a proper lower semi-continuous functional and $\lambda \in \mathbb{R}$. A continuous semi-group $\mathbf{S}^t : \text{Dom}(\mathcal{F}) \rightarrow \text{Dom}(\mathcal{F})$, $t \geq 0$, is a λ -flow for \mathcal{F} if it satisfies the Evolution Variational Inequality (EVI)*

$$\frac{1}{2} \limsup_{h \downarrow 0} \left[\frac{\mathbf{W}_m(\mathbf{S}^h(u), v)^2 - \mathbf{W}_m(u, v)^2}{h} \right] + \frac{\lambda}{2} \mathbf{W}_m(u, v)^2 + \mathcal{F}[u] \leq \mathcal{F}[v], \quad (62)$$

for all measures $u, v \in \text{Dom}(\mathcal{F})$ with $\mathbf{W}_m(u, v) < +\infty$.

Recall that a continuous semigroup \mathbf{S} on a set $D \subset X(\Omega)$ is a family of maps $\mathbf{S}^t : D \rightarrow D$, $t \geq 0$, satisfying

$$\mathbf{S}^{t+s}(u) = \mathbf{S}^t(\mathbf{S}^s(u)), \quad \lim_{t \downarrow 0} \mathbf{W}_m(\mathbf{S}^t(u), u) = 0 \quad \text{for all } u \in D. \quad (63)$$

Notice that the continuity of \mathbf{S} is already coded in (62): it is sufficient to choose $v := u$ in (62).

(63) and the triangle inequality yields

$$\mathbf{W}_m(\mathbf{S}^t(u), u) < +\infty \quad \text{for all } u \in D, t \geq 0; \quad (64)$$

in particular the “lim sup” in (62) is well defined.

Lemma 3.2 (Flow interchange Lemma [21]). *Assume that $\mathbf{S}_\mathcal{F}$ is a λ -flow for the proper, lower semi-continuous functional \mathcal{F} in $X(\Omega)$ and let u_τ^n be a n -th step approximation constructed by the minimizing movement scheme (11). If $u_\tau^n \in \text{Dom}(\mathcal{F})$ then*

$$\mathcal{F}[u_\tau^n] - \mathcal{F}[u_\tau^{n-1}] \leq \tau \liminf_{h \downarrow 0} \left(\frac{\mathbf{E}[\mathbf{S}_\mathcal{F}^h(u_\tau^n)] - \mathbf{E}[u_\tau^n]}{h} \right) - \frac{\lambda}{2} \mathbf{W}_m(u_\tau^n, u_\tau^{n-1})^2. \quad (65)$$

Proof. (C.f. [21]) By definition of u_τ^n as a minimizer in (11),

$$\frac{1}{2\tau} \mathbf{W}_m(u_\tau^n, u_\tau^{n-1})^2 + \mathbf{E}[u_\tau^n] \leq \frac{1}{2\tau} \mathbf{W}_m(\mathbf{S}_{\mathcal{F}}^h(u_\tau^n), u_\tau^{n-1})^2 + \mathbf{E}[\mathbf{S}_{\mathcal{F}}^h(u_\tau^n)]$$

holds for every $h > 0$. This implies

$$-\tau \liminf_{h \downarrow 0} [h^{-1}(\mathbf{E}[\mathbf{S}_{\mathcal{F}}^h(u_\tau^n)] - \mathbf{E}[u_\tau^n])] \leq \frac{1}{2} \limsup_{h \downarrow 0} [h^{-1} \mathbf{W}_m(\mathbf{S}_{\mathcal{F}}^h(u_\tau^n), u_\tau^{n-1})^2 - \mathbf{W}_m(u_\tau^n, u_\tau^{n-1})^2].$$

To conclude (65) from here, apply (62) with the choices $u = u_\tau^n$ and $v = u_\tau^{n-1}$. \square

3.3. Eulerian calculus. In order to apply the flow interchange Lemma 3.2 with a particular auxiliary functional \mathcal{F} , we need to exhibit the associated semigroup \mathbf{S} (usually given implicitly as the solution to a nonlinear evolution equation) and to verify that it is indeed a λ -flow, i.e., it satisfies the EVI (62) with a finite constant λ . A very general strategy to attack this problem is the Eulerian calculus for transportation metrics, that has been developed by the third author in [8], based on earlier work by Otto and Westdickenberg [24]. Similar to the flow interchange estimate, the basic idea is to simplify estimates by exchanging two time-like derivatives. We also need that \mathbf{S} can be suitably approximated by semigroups on smooth densities: here is the relevant definition.

Definition 2. *Let us fix a nonnegative vanishing sequence δ_n , let $\Omega_n \supset \Omega_{[\delta_n]}$, a decreasing sequence of smooth convex sets converging to Ω , let $\mathcal{F} : X(\Omega) \rightarrow (-\infty, +\infty]$ be proper and l.s.c. functionals, and let \mathbf{S} be a semi-group on $\text{Dom}(\mathcal{F}) \subset X(\Omega)$.*

We say that $\{\mathcal{F}_n, \mathbf{S}_n\}_{n \in \mathbb{N}}$ is a family of mollifications for \mathcal{F}, \mathbf{S} if

- (a) $\mathcal{F}_n : X^r(\Omega_n) \rightarrow \mathbb{R}$ is a regular functional according to (30).
- (b) Each \mathbf{S}_n is a regular semi-group on $X^r(\Omega_n)$ according to (31).
- (c) For every $\rho_0, \rho_1 \in \text{Dom}(\mathcal{F})$ with $\mathbf{W}_m(\rho_0, \rho_1) < \infty$ the regular densities $\gamma_n(i)$ given as in (e) of Proposition 2.2 satisfy

$$\lim_{n \rightarrow \infty} \mathcal{F}_n[\gamma_n(i)] = \mathcal{F}[\rho_i] \quad i = 0, 1. \quad (66)$$

- (d) For every sequence $\rho_n \in \text{Dom}(\mathcal{F}_n)$ converging to $\rho \in \text{Dom}(\mathcal{F})$ in $L^1(\mathbb{R}^d)$ as $n \rightarrow \infty$ with $\mathcal{F}_n[\rho_n] \rightarrow \mathcal{F}[\rho]$ we have $\mathbf{S}_n^t \rho_n \rightarrow \mathbf{S}^t \rho$ in $\mathcal{D}'(\mathbb{R}^d)$ and $\liminf_{n \rightarrow \infty} \mathcal{F}_n[\mathbf{S}_n^t \rho_n] \geq \mathcal{F}[\mathbf{S}^t \rho]$ for all $t \geq 0$.

Let $\mathcal{F}, \mathcal{F}_n$ and \mathbf{S}, \mathbf{S}_n as in the definition above. For a given $n \in \mathbb{N}$ consider a regular curve $\gamma_n : [0, 1] \rightarrow X^r(\Omega_n)$ and for every $h \geq 0$, introduce $\gamma_n^h : [0, 1] \rightarrow X^r(\Omega_n)$ — the h -perturbation of γ_n — by

$$\gamma_n^h(s) = \mathbf{S}_n^{hs} \gamma_n(s).$$

Notice that the $(h, s) \mapsto \gamma_n^h(s, x)$ is of class C^1 in $[0, \infty) \times [0, 1]$ thanks to the regularity of \mathbf{S}_n . Also, introduce the *action* of the perturbed curves

$$\mathbf{A}_n^h(s) = \int_{\Omega_n} |D \varphi_n^h(s)|^2 \mathbf{m}(\gamma_n^h(s)) \, dx, \quad (67)$$

where the $\varphi_n^h(s) \in H^1(\Omega_n)$ form a s -differentiable family of solutions to the associated Neumann problems

$$-\text{div}(\mathbf{m}(\gamma_n^h(s, \cdot)) D \varphi_n^h(s, \cdot)) = \partial_s \gamma_n^h(s, \cdot) \quad \text{in } \Omega_n, \quad \mathbf{n} \cdot D \varphi_n^h(s, \cdot) = 0 \quad \text{on } \partial \Omega_n. \quad (68)$$

These Neumann problems are solvable because γ_n^h is a regular curve of densities in $X^r(\Omega_n)$; in particular, the mass is constant, and thus $\partial_s \gamma_n^h(s, \cdot)$ has vanishing average on Ω_n . The following result is essentially an adaptation of Theorem 2.2 in [8] to the situation at hand.

Lemma 3.3. *Under the hypotheses and with the definitions above, assume that $\inf_n \mathcal{F}_n \geq F > -\infty$ and $h \mapsto \mathcal{F}_n[\mathbf{S}_n^h \rho]$ are non-increasing for every $\rho \in X^r(\Omega_n)$, and there exists $\lambda \leq 0$ (independent of n and of the considered curves γ_n) such that the inequality*

$$\frac{1}{2} \partial_h \mathbf{A}_n^h(s) + s \lambda \mathbf{A}_n^h(s) \leq -\partial_s \mathcal{F}_n[\gamma_n^h(s)] \quad (69)$$

holds for all $s \in [0, 1]$ and all $h \geq 0$. Then \mathbf{S} is a λ -flow for \mathcal{F} .

Proof. The core idea is to prove an integrated form of (62) by estimating the perturbed action (67) starting from a family of approximating geodesics – provided by (e) in Proposition 2.2 – connecting two given admissible measures $u, v \in \text{Dom}(\mathcal{F}) \subset X(\Omega)$. Without loss of generality, we assume that \mathcal{F}_n are non-negative and $\lambda < 0$; the case $\lambda = 0$ follows by obvious modifications.

Given $u, v \in \text{Dom}(\mathcal{F}) \subset X(\Omega)$ at finite distance, and a family of approximating geodesic γ_n on $X^r(\Omega_n)$ between v and u in the sense of (e) in Proposition 2.2, define $u_n^0 = \gamma_n(0)$ and $u_n^1 = \gamma_n(1)$. Multiply (69) by $e^{2\lambda hs}$ and integrate with respect to $s \in [0, 1]$; this gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dh} \int_0^1 e^{2\lambda hs} \mathbf{A}_n^h(s) ds &\leq - \int_0^1 e^{2\lambda hs} \partial_s \mathcal{F}_n[\gamma_n^h(s)] ds \\ &= \mathcal{F}_n[u_n^0] - e^{2\lambda h} \mathcal{F}_n[\mathbf{S}_n^h u_n^1] + \int_0^1 2\lambda h e^{2\lambda hs} \mathcal{F}_n[\gamma_n^h(s)] ds \\ &\leq \mathcal{F}_n[u_n^0] - e^{2\lambda h} \mathcal{F}_n[\mathbf{S}_n^h u_n^1], \end{aligned}$$

since $\lambda < 0$ while \mathcal{F}_n is non-negative. Next, integrate with respect to $h \in [0, H]$, which yields

$$\frac{1}{2} \int_0^1 e^{2\lambda Hs} \mathbf{A}_n^H(s) ds + \frac{1 - e^{2\lambda H}}{-2\lambda} \mathcal{F}_n[\mathbf{S}_n^H u_n^1] \leq \frac{1}{2} \int_0^1 \mathbf{A}_n^0(s) ds + H \mathcal{F}_n[u_n^0],$$

where we also used the fact that $h \mapsto \mathcal{F}_n[\mathbf{S}_n^h u_n^1]$ is non-increasing. Further, a reparametrization of $s \mapsto \gamma_n^h(s)$ similar to that used in [8, Lemma 5.2] yields

$$\frac{e^{-2\lambda H} - 1}{-2\lambda H} \mathbf{W}_m(u_n^0, \mathbf{S}_n^H u_n^1)^2 \leq \int_0^1 e^{2\lambda Hs} \mathbf{A}_n^H(s) ds.$$

In summary, we have

$$\frac{e^{-2\lambda H} - 1}{-2\lambda H} \mathbf{W}_m(u_n^0, \mathbf{S}_n^H u_n^1)^2 + \frac{1 - e^{2\lambda H}}{-2\lambda} \mathcal{F}_n[\mathbf{S}_n^H u_n^1] \leq \frac{1}{2} \int_0^1 \mathbf{A}_n^0(s) ds + H \mathcal{F}_n[u_n^0].$$

By our choice of γ_n , (e) of Proposition 2.2 and (66) yield on one hand that

$$\lim_{n \rightarrow \infty} \int_0^1 \mathbf{A}_n^0(s) ds = \mathbf{W}_m(u, v)^2, \quad \lim_{n \rightarrow \infty} \mathcal{F}_n[u_n^0] = \mathcal{F}[v].$$

On the other hand, we know by Proposition 2.2 and the properties listed in Definition 2 that u_n^1 converges to the density u in $L^1(\mathbb{R}^d)$ and

$$\mathcal{F}[\mathbf{S}^H u] \leq \liminf_{n \rightarrow \infty} \mathcal{F}_n[\mathbf{S}_n^H u_n^1], \quad \mathbf{W}_m(\mathbf{S}^H u, v)^2 \leq \liminf_{n \rightarrow \infty} \mathbf{W}_m(\mathbf{S}_n^H u_n^1, u_n^0)^2.$$

Altogether, this yields the inequality

$$\frac{e^{-2\lambda H} - 1}{-2\lambda H} \mathbf{W}_m(\mathbf{S}^H u, v)^2 + \frac{1 - e^{2\lambda H}}{-2\lambda} \mathcal{F}[\mathbf{S}^H u] \leq \frac{1}{2} \mathbf{W}_m(u, v)^2 + H \mathcal{F}[v],$$

from which the EVI property (62) is deduced after division by $H > 0$ in the limit $H \downarrow 0$. \square

4. PROOF OF THEOREM 1

Throughout this section we use the notation introduced in §1.2.

4.1. **H^2 -regularity and strong convergence.** The goal of the following is to prove:

Proposition 4.1. *In the setting of §2.1 each solution u_τ^n of the minimizing movement scheme (11) satisfies*

$$u_\tau^n \in H^2(\Omega) \quad \text{for all } n \in \mathbb{N}, \tau > 0, \quad (70)$$

and the following bound holds for the piecewise constant interpolant \bar{u}_τ ,

$$\int_0^T \|\bar{u}_\tau(t)\|_{H^2(\Omega)}^2 dt \leq CT(\mathbf{E}_0 + \mathbf{E}[u_0]) < +\infty \quad \text{for all } T > 0, \quad (71)$$

with a constant C independent of $\tau > 0$. Moreover, every sequence $\tau_k \downarrow 0$ contains a subsequence (still denoted by τ_k) such that

$$\bar{u}_{\tau_k} \rightarrow u \text{ strongly in } L^2(0, T; H^1(\Omega)) \quad \text{for all } T > 0, \quad (72)$$

$$\bar{u}_{\tau_k} \rightarrow u \text{ weakly in } L^2(0, T; H^2(\Omega)) \quad \text{for all } T > 0. \quad (73)$$

The proof of Proposition 4.1 rests on the fact that the densities u_τ are τ -uniformly bounded in $L^2(0, T; H^2(\Omega))$ for arbitrary $T > 0$. As motivation for the arguments below, we provide the relevant formal calculations in the case $G \equiv 0$: assuming that u is a smooth solution to (1) satisfying (2), differentiation of the entropy functional $\mathbf{U}[u(t)]$ introduced in (7) yields

$$\begin{aligned} \frac{d}{dt} \mathbf{U}[u(t)] &= \int_{\Omega} U'(u(t)) \partial_t u(t) dx = - \int_{\Omega} U'(u(t)) \operatorname{div}(\mathbf{m}(u(t)) \mathbf{D} \Delta u(t)) dx \\ &= \int_{\Omega} U''(u(t)) \mathbf{m}(u(t)) \mathbf{D} u(t) \cdot \mathbf{D} \Delta u(t) dx = \int_{\Omega} \mathbf{D} u(t) \cdot \mathbf{D} \Delta u(t) dx \\ &= - \int_{\Omega} (\Delta u(t))^2 dx \end{aligned}$$

because of the identity $U''(s) = 1/\mathbf{m}(s)$. Consequently, $\mathbf{U}[u(t)]$ is decreasing with respect to t , and (still formally),

$$\mathbf{U}[u(T)] + \int_0^T \int_{\Omega} (\Delta u)^2 dx dt \leq \mathbf{U}[u^0].$$

Taking also into account the contribution of G and the convexity of Ω , one ends up with an estimate of the form (71). The goal for the rest of this section is the rigorous proof of this estimate.

We wish to apply the flow interchange Lemma 3.2 with $\mathcal{F} = \mathbf{U}$. To this end, we need to identify the associated semi-group \mathbf{S} , with $\mathbf{S}^t v$ given by the smooth solution v_t to the Neumann problem

$$\partial_t v_t = \Delta v_t \quad \text{in } \Omega, \quad \mathbf{n} \cdot \mathbf{D} v_t = 0 \quad \text{on } \partial\Omega, \quad v_0 = v. \quad (74)$$

Lemma 4.2. *The semi-group \mathbf{S} induced by solutions v_t of the problem (74) on $X^r(\Omega)$ extends to a 0-flow \mathbf{S} for \mathbf{U} .*

This fact is a special case of a more general result proven in [7, Theorem 6.1]. We provide the relevant calculations for the specific situation of Lemma 4.2 as we shall refer to it later.

Proof. We wish to apply Lemma 3.3. In order to define a family of mollifiers $\{\mathcal{F}_n, \mathbf{S}_n\}_{n \in \mathbb{N}}$ for \mathbf{U}, \mathbf{S} , we consider a sequence of domains $\Omega_n := \Omega_{[\delta_n]}$ for some vanishing sequence $\delta_n > 0$ and we define $\mathcal{F}_n(u) := \mathbf{U}_n[u] := \int_{\Omega_n} U(u(x)) dx$ and the heat-semigroup \mathbf{S}_n on Ω_n with homogeneous Neumann boundary conditions. The contraction properties of the heat flow imply that convergence of the initial conditions in L^1 imply the same convergence of the solution at any time $h > 0$ and it is easy to verify all properties required in Definition 2.

Let regular curves $\gamma_n : [0, 1] \rightarrow X^r(\Omega_n)$ be given. By classical parabolic theory, the h -perturbed curves $\gamma_n^h : [0, 1] \rightarrow X^r(\Omega_n)$ are well-defined for any $h \geq 0$, such that, for every $s \in [0, 1]$, the function $(h, x) \mapsto \gamma_n^h(s, x)$ is a classical solution to

$$\partial_h \gamma_n^h = s \Delta \gamma_n^h \quad \text{in } \Omega_n, \quad \mathbf{D} \gamma_n^h \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega_n, \quad \gamma_n^0(s) = \gamma_n(s). \quad (75)$$

We need to verify the principal estimate (69) of Eulerian calculus, which reads in the situation at hand (we will omit to indicate n in the following) as follows:

$$\frac{1}{2} \int_{\Omega} \partial_h [\mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2] dx \leq - \int_{\Omega} \partial_s [U(\gamma^h)] dx. \quad (76)$$

Using the definition of φ^h in (68) and its boundary conditions, the right-hand side evaluates after integration by parts to

$$- \int_{\Omega} U'(\gamma^h) \partial_s \gamma^h dx = - \int_{\Omega} U''(\gamma^h) \mathbf{D} \gamma^h \cdot (\mathbf{m}(\gamma^h) \mathbf{D} \varphi^h) dx = - \int_{\Omega} \mathbf{D} \gamma^h \cdot \mathbf{D} \varphi^h dx = \int_{\Omega} \gamma^h \Delta \varphi^h dx$$

since $U''(s)\mathbf{m}(s) = 1$ by definition of U . For the h -derivative of the action, we find

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \partial_h [\mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2] dx &= \frac{1}{2} \int_{\Omega} \partial_h \mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2 dx + \int_{\Omega} \mathbf{m}(\gamma^h) \partial_h \mathbf{D} \varphi^h \cdot \mathbf{D} \varphi^h dx \\ &= - \frac{1}{2} \int_{\Omega} \partial_h \mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2 dx + \int_{\Omega} (\partial_h \mathbf{m}(\gamma^h) \mathbf{D} \varphi^h + \mathbf{m}(\gamma^h) \partial_h \mathbf{D} \varphi^h) \cdot \mathbf{D} \varphi^h dx. \end{aligned} \quad (77)$$

To simplify the second integral above, first observe that for every smooth function $\theta \in C^\infty(\bar{\Omega})$ it follows from (68) that

$$\int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} \varphi^h \cdot \mathbf{D} \theta dx = \int_{\Omega} \partial_s \gamma^h \theta dx. \quad (78)$$

Taking the h -derivative yields

$$\int_{\Omega} (\partial_h \mathbf{m}(\gamma^h) \mathbf{D} \theta + \mathbf{m}(\gamma^h) \partial_h \mathbf{D} \varphi^h) \cdot \mathbf{D} \theta dx = \int_{\Omega} \partial_h \partial_s \gamma^h \theta dx. \quad (79)$$

If θ satisfies homogeneous Neumann boundary conditions, $\mathbf{n} \cdot \mathbf{D} \theta = 0$ on $\partial\Omega$, we obtain from (75)

$$\int_{\Omega} \partial_h \gamma^h \vartheta dx = s \int_{\Omega} \gamma^h \Delta \vartheta dx, \quad (80)$$

and the s -derivative amounts, in view of (78), to

$$\int_{\Omega} \partial_s \partial_h \gamma^h \vartheta dx = \int_{\Omega} \gamma^h \Delta \vartheta dx + s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} \varphi^h \cdot \mathbf{D} \Delta \vartheta dx, \quad (81)$$

and thus allows to express the mixed derivative $\partial_h \partial_s \gamma^h$ in (79). Using as test function $\theta = \varphi^h$ in (79) and (81), the integrals in (77) become

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \partial_h [\mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2] dx &= - \frac{1}{2} \int_{\Omega} \partial_h \mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2 dx \\ &\quad + \int_{\Omega} \gamma^h \Delta \varphi^h dx + s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} \varphi^h \cdot \mathbf{D} \Delta \varphi^h dx. \end{aligned} \quad (82)$$

We evaluate the first integral on the right-hand side,

$$\begin{aligned} - \frac{1}{2} \int_{\Omega} \partial_h \mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2 dx &= - \frac{1}{2} \int_{\Omega} \mathbf{m}'(\gamma^h) s \Delta \gamma^h |\mathbf{D} \varphi^h|^2 dx \\ &= \frac{s}{2} \int_{\Omega} \mathbf{D}(\mathbf{m}'(\gamma^h) |\mathbf{D} \varphi^h|^2) \cdot \mathbf{D} \gamma^h dx \\ &= \frac{s}{2} \int_{\Omega} \mathbf{D}(\mathbf{m}'(\gamma^h)) \cdot \mathbf{D} \gamma^h |\mathbf{D} \varphi^h|^2 dx + \frac{s}{2} \int_{\Omega} \mathbf{m}'(\gamma^h) \mathbf{D} \gamma^h \cdot \mathbf{D} (|\mathbf{D} \varphi^h|^2) dx \\ &= \frac{s}{2} \int_{\Omega} \mathbf{m}''(\gamma^h) |\mathbf{D} \gamma^h|^2 |\mathbf{D} \varphi^h|^2 dx - \frac{s}{2} \int_{\Omega} \mathbf{m}(\gamma^h) \Delta (|\mathbf{D} \varphi^h|^2) dx \\ &\quad + \frac{s}{2} \int_{\partial\Omega} \mathbf{m}(\gamma^h) \mathbf{D} (|\mathbf{D} \varphi^h|^2) \cdot \mathbf{n} d\mathcal{H}^{d-1}. \end{aligned}$$

Using the last identity in (82), taking into account the Bochner formula

$$- \frac{1}{2} \Delta (|\mathbf{D} \varphi^h|^2) + \mathbf{D} \Delta \varphi^h \cdot \mathbf{D} \varphi^h = - \|\mathbf{D}^2 \varphi^h\|^2 \leq 0,$$

and that $\mathbf{D}(|\mathbf{D}\varphi^h|^2) \cdot \mathbf{n} \leq 0$ on $\partial\Omega$ since Ω is convex, see (124), we find that

$$\frac{1}{2} \int_{\Omega} \partial_h [\mathbf{m}(\gamma^h) |\mathbf{D}\varphi^h|^2] dx \leq - \int_{\Omega} \partial_s [U(\gamma^h)] dx + \frac{s}{2} \int_{\Omega} \mathbf{m}''(\gamma^h) |\mathbf{D}\gamma^h|^2 |\mathbf{D}\varphi^h|^2 dx. \quad (83)$$

By concavity of \mathbf{m} , this proves (76). \square

The following Lemma provides the last missing piece for proving (71) by means of the flow interchange Lemma 3.2, namely the dissipation of the energy \mathbf{E} along the heat flow (74).

Lemma 4.3. *Let $v : [0, \infty) \rightarrow H^1(\Omega)$ be a solution to (74). If*

$$\liminf_{s \downarrow 0} \frac{1}{s} (\mathbf{E}[v_s] - \mathbf{E}[v_0]) > -\infty, \quad (84)$$

then $v_0 \in H^2(\Omega)$ and

$$- \liminf_{s \downarrow 0} \frac{1}{s} (\mathbf{E}[v_s] - \mathbf{E}[v_0]) \geq \frac{1}{2} \int_{\Omega} (\Delta v_0)^2 dx - C(\mathbf{E}_0 + \mathbf{E}[v_0]), \quad (85)$$

where the constant C depends only on $\mathbf{m}(s_0)$, $|\Omega|$ and G .

Proof. By classical parabolic theory, the solution v to (74) is smooth and for every $0 < s_0 < s_1 < \infty$ it satisfies $0 < \inf_{x \in \Omega} v_s(x) \leq \sup_{x \in \Omega} v_s(x) < M$ for $(x, s) \in \Omega \times (s_0, s_1)$. Thus $\mathbf{E}[v_s]$ is continuously differentiable with respect to s in $[s_0, s_1]$ with

$$\frac{d}{ds} \mathbf{E}[v_s] = \int_{\Omega} \mathbf{D}v_s \cdot \mathbf{D}\Delta v_s dx + \int_{\Omega} G'(v_s) \Delta v_s dx = - \int_{\Omega} (\Delta v_s)^2 dx - \int_{\Omega} G''(v_s) |\mathbf{D}v_s|^2 dx, \quad (86)$$

where the last equality follows after integration by parts, using that the boundary condition $\mathbf{n} \cdot \mathbf{D}v_s = 0$ is satisfied for any $s > 0$. Taking into account of (G) the second integral can be estimated as follows,

$$- \int_{\Omega} G''(v_s) |\mathbf{D}v_s|^2 dx \leq - \int_{\Omega} G''_1(v_s) |\mathbf{D}v_s|^2 dx \leq C \left(\int_{\Omega} |\mathbf{D}v_s|^2 dx + \int_{\Omega} \frac{|\mathbf{D}v_s|^2}{\mathbf{m}(v_s)} dx \right).$$

Recall (44) and the identity

$$\frac{|\mathbf{D}v_s|^2}{v_s} = 4 |\mathbf{D}\sqrt{v_s}|^2.$$

In case that $M = +\infty$, we obtain

$$\begin{aligned} \int_{\Omega} \frac{|\mathbf{D}v_s|^2}{\mathbf{m}(v_s)} dx &\leq \frac{s_0}{\mathbf{m}(s_0)} \int_{\Omega} \frac{|\mathbf{D}v_s|^2}{v_s} dx + \frac{1}{\mathbf{m}(s_0)} \int_{\Omega} |\mathbf{D}v_s|^2 dx \\ &\leq C \left(\int_{\Omega} |\mathbf{D}\sqrt{v_s}|^2 dx + \int_{\Omega} |\mathbf{D}v_s|^2 dx \right). \end{aligned}$$

Moreover, by Hölder's inequality and estimate (123) from the Appendix,

$$\int_{\Omega} |\mathbf{D}\sqrt{v_s}|^2 dx \leq \left(|\Omega| \int_{\Omega} |\mathbf{D}\sqrt{v_s}|^4 dx \right)^{1/2} \leq \left(\frac{(d+8)|\Omega|}{16} \int_{\Omega} (\Delta v_s)^2 dx \right)^{1/2} \leq \varepsilon \int_{\Omega} (\Delta v_s)^2 dx + K_{\varepsilon}.$$

with $K_{\varepsilon} = (d+8)|\Omega|/(64\varepsilon)$. And analogously, if $M < \infty$,

$$\begin{aligned} \int_{\Omega} \frac{|\mathbf{D}v_s|^2}{\mathbf{m}(v_s)} dx &\leq \frac{s_0}{\mathbf{m}(s_0)} \int_{\Omega} \frac{|\mathbf{D}v_s|^2}{v_s} dx + \frac{M-s_0}{\mathbf{m}(s_0)} \int_{\Omega} \frac{|\mathbf{D}(M-v_s)|^2}{M-v_s} dx \\ &\leq C \left(\int_{\Omega} |\mathbf{D}\sqrt{v_s}|^2 dx + \int_{\Omega} |\mathbf{D}\sqrt{M-v_s}|^2 dx \right), \end{aligned}$$

where we use that

$$\int_{\Omega} |\mathbf{D}\sqrt{M-v_s}|^2 dx \leq \varepsilon \int_{\Omega} (\Delta v_s)^2 dx + K_{\varepsilon}.$$

Choosing ε above sufficiently small, and observing that

$$\int_{\Omega} |\mathbf{D}v_s|^2 dx \leq \int_{\Omega} |\mathbf{D}v_0|^2 dx,$$

it thus can be achieved that

$$\frac{d}{ds} \mathbf{E}[v_s] \leq -\frac{1}{2} \int_{\Omega} (\Delta v_s)^2 dx + C(1 + \|v_0\|_{H^1(\Omega)}^2),$$

for a suitable constant C .

Recall that the curve $s \mapsto v_s$ is continuous in $H^1(\Omega)$ and that G can be decomposed as in (46). The continuity of G_{conc} and the lower bound (48) yield that

$$\lim_{s \downarrow 0} \int_{\Omega} G_{\text{conc}}(v_s) dx = \int_{\Omega} G_{\text{conc}}(v_0) dx;$$

on the other hand, since G_{conv} is convex, we have

$$\int_{\Omega} G_{\text{conc}}(v_s) dx \leq \int_{\Omega} G_{\text{conv}}(v_0) dx \quad \text{for all } s > 0$$

so that Fatou's Lemma and the continuity of G yields

$$\lim_{s \downarrow 0} \int_{\Omega} G_{\text{conv}}(v_s) dx = \int_{\Omega} G_{\text{conv}}(v_0) dx.$$

Consequently, the function $s \mapsto \mathbf{E}[v_s]$ is continuous at $s = 0$ and we have that

$$\frac{1}{s} (\mathbf{E}[v_s] - \mathbf{E}[v_0]) \leq -\frac{1}{2} \frac{1}{s} \int_{\Omega} (\Delta v_{\theta(s)})^2 dx + C(1 + \|v_0\|_{H^1(\Omega)}^2),$$

with $0 < \theta(s) < s$. By (84) it follows that the family $\{\Delta v_{\theta(s)}\}_{s \in (0, s_0)}$ for $s_0 > 0$ is weakly compact in $L^2(\Omega)$. Since v_s converges to v_0 strongly in $H^1(\Omega)$ as $s \downarrow 0$, we have that $v_0 \in H^2(\Omega)$ and

$$\begin{aligned} -\liminf_{s \downarrow 0} \frac{1}{s} (\mathbf{E}[v_s] - \mathbf{E}[v_0]) &\geq \frac{1}{2} \liminf_{s \downarrow 0} \int_{\Omega} (\Delta v_{\theta(s)})^2 dx - C(1 + \|v_0\|_{H^1(\Omega)}^2) \\ &\geq \frac{1}{2} \int_{\Omega} (\Delta v_0)^2 dx - C(1 + \|v_0\|_{H^1(\Omega)}^2). \end{aligned}$$

Another application of the estimate (49) finally provides (85). \square

Proof of Proposition 4.1. By Lemma 2.3 we can apply the flow interchange Lemma 3.2 with $\mathcal{F} = \mathbf{U}$. By Lemma 4.3 applied to $v_0 = u_{\tau}^n$ we have that u_{τ}^n lies in $H^2(\Omega)$, and by (85) and (65), for any $n \in \mathbb{N}$, it follows that

$$\frac{\tau}{2} \int_{\Omega} (\Delta u_{\tau}^n)^2 dx \leq \mathbf{U}[u_{\tau}^{n-1}] - \mathbf{U}[u_{\tau}^n] + C(\mathbf{E}_0 + \mathbf{E}[u_{\tau}^n])\tau. \quad (87)$$

Here the constant C is the same as in (85), and does not depend on τ , on n or on the solution u_{τ} .

Let $T > 0$ and $\tau \in (0, 1)$ be given, and define $N \in \mathbb{N}$ such that $(N-1)\tau < T \leq N\tau$. In view of (59) and $\mathbf{E}[u_{\tau}^n] \leq \mathbf{E}[u_0]$, summing (87) from $n = 1$ to $n = N$, we find that the interpolating function \bar{u}_{τ} satisfies

$$\|\Delta \bar{u}_{\tau}\|_{L^2(0, T; L^2(\Omega))}^2 \leq \tau \sum_{n=1}^N \int_{\Omega} (\Delta u_{\tau}^n)^2 dx \leq 2CT(\mathbf{E}_0 + \mathbf{E}[u_0]),$$

which is obviously independent of $\tau \in (0, 1)$. Combining this with (49) and applying again (122), we conclude that \bar{u}_{τ} remains uniformly bounded in $L^2(0, T; H^2(\Omega))$ as $\tau \downarrow 0$, for any $T > 0$:

$$\int_0^T \|\bar{u}_{\tau}(t)\|_{H^2(\Omega)}^2 dt \leq CT(\mathbf{E}_0 + \mathbf{E}[u_0]) < +\infty. \quad (88)$$

By (88) we have that, up to subsequences, \bar{u}_{τ_n} converge weakly to u in $L^2(0, T; H^2(\Omega))$ for every $T > 0$. Since we have already seen in Proposition 3.1 that \bar{u}_{τ} pointwise converge weakly in $H^1(\Omega)$ and thus strongly in $L^2(\Omega)$ by Rellich Theorem, the dominated convergence theorem shows that \bar{u}_{τ} converges strongly in $L^2(0, T; L^2(\Omega))$. From here, strong convergence $\bar{u}_{\tau_n} \rightarrow u$ in $L^2(0, T; H^1(\Omega))$ follows by standard interpolation between the uniform bound (88). \square

Corollary 4.4. *In the setting of §2.1 we have for all $T > 0$*

$$P(u_{\tau_n}) \rightarrow P(u) \quad \text{strongly in } L^1(0, T; L^1(\Omega)), \quad (89)$$

and, if \mathbf{m} is also Lipschitz (as for (M-LSC))

$$\mathbf{m}(\bar{u}_{\tau_n}) \rightarrow \mathbf{m}(u) \quad \text{strongly in } L^2(0, T; H^1(\Omega)). \quad (90)$$

Proof. (89) is trivial when $M < \infty$. When $M = +\infty$, by Sobolev imbedding Theorem and the uniform estimates (71) and (60) we know that

$$\int_0^T \int_{\Omega} \left((\bar{u}_{\tau})^q + |\bar{u}_{\tau}| \right) dx dt \leq C_T \quad (91)$$

uniformly with respect to τ . Since \bar{u}_{τ} (up to subsequence) converges strongly to u in $L^1((0, T) \times \Omega)$, we deduce the same property for $P(\bar{u})$ thanks to (G).

(90) is a standard consequence of the fact that \mathbf{m} is Lipschitz and C^1 . \square

4.2. Weak formulation. The remaining section is devoted to prove the following Proposition stating that the time-continuous limit u obtained before is a weak solution in the sense of (18).

Proposition 4.5. *Under the assumptions of Theorem 1, let V be a spatial test function satisfying*

$$V \in C^\infty(\bar{\Omega}), \quad \text{D}V \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \quad (92)$$

and a temporal test function $\psi \in C_c^\infty(0, +\infty)$ be given. Then

$$-\int_0^{+\infty} \psi'(t) \mathbf{V}[u(t)] dt = \int_0^{+\infty} \psi(t) \mathbf{N}[u(t), V] dt, \quad (93)$$

where the nonlinear functional \mathbf{N} is given by

$$\mathbf{N}[u, V] := - \int_{\Omega} \Delta u \operatorname{div}(\mathbf{m}(u) \text{D}V) dx + \int_{\Omega} P(u) \Delta V dx.$$

In the spirit of the ideas developed in [17], we would like to use the flow interchange Lemma 3.2 with $\mathcal{F} := \mathbf{V}$ the potential energy functional $\mathbf{V} : X(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mathbf{V}[u] := \int_{\Omega} V(x) u(x) dx,$$

with a test function V satisfying (92). As already mentioned in the introduction, the functionals \mathbf{V} are — unfortunately — never λ -convex (for any $\lambda \in \mathbb{R}$) along geodesics of the space $(X(\Omega), \mathbf{W}_{\mathbf{m}})$, unless the mobility \mathbf{m} is a linear function [7, Section 2.3]. To cure this problem, we shall construct a λ_ε -flow for the regularized functional

$$\mathbf{V}_\varepsilon[u] := \mathbf{V}[u] + \varepsilon \mathbf{U}[u], \quad (94)$$

with $\varepsilon > 0$ instead, which amounts to solutions of the classical viscous approximation of (25),

$$\partial_s v_s - \operatorname{div}(\mathbf{m}(v_s) \text{D}V) - \varepsilon \Delta v_s = 0 \quad \text{in } (0, +\infty) \times \Omega \text{ with } \text{D}v_s \cdot \mathbf{n} = 0 \text{ on } (0, +\infty) \times \partial\Omega. \quad (95)$$

Proposition 4.6. *Under the assumptions (M), (M-LSC) on \mathbf{m} , suppose that V satisfies (92). Define the semigroup \mathbf{S}_ε by taking $\mathbf{S}_\varepsilon^s v_0 = v_s$, the unique solution to (95) with initial condition v_0 . Then \mathbf{S}_ε extends to a λ_ε -flow \mathbf{S}_ε for \mathbf{V}_ε with respect to $\mathbf{W}_{\mathbf{m}}$, with some $\lambda_\varepsilon \geq -K/\varepsilon$ where $K > 0$ only depends on V and \mathbf{m} .*

Proof. As in the proof of Lemma 4.2, we need to verify (69) for the flow \mathbf{S}_ε and the functional $\mathcal{F} = \mathbf{V}_\varepsilon$. The calculations are similar to the proof there, but more terms need to be controlled.

Below, we shall implicitly use various properties of the solution semi-group \mathbf{S}_ε for (95). A summary of these relevant properties are given in Lemma A.2 in the Appendix. In particular, note that \mathbf{S}_ε is well-defined and L^1 -continuous on the admissible densities $X(\Omega)$, and that it leaves the regular densities $X^r(\Omega)$ invariant.

In order to define a family of mollifications $\{\mathbf{V}_{\varepsilon, n}, \mathbf{S}_{\varepsilon, n}\}$ for $\mathbf{V}_\varepsilon, \mathbf{S}_\varepsilon$, we assume without restriction that $0 \in \Omega$, we take a monotone sequence $\eta_n \downarrow 1$ and we define $\Omega_n := \eta_n \Omega = \{\eta_n x : x \in \Omega\}$,

choosing $\delta_n \downarrow 0$ so that $\Omega_{[\delta_n]} \subset \Omega_n$. We define $V_n(x) = V(x/\eta_n)$ and $\mathbf{V}_{\varepsilon,n}[u] = \int_{\Omega_n} V_n(x) dx + \varepsilon \mathbf{U}[u]$. It is easy to check that V_n satisfies (92) in Ω_n . We define for every n the solution semi-group $\mathbf{S}_{\varepsilon,n}$ of the problem (95) for V_n on the domain Ω_n . It is not difficult to check that all the conditions of Definition 2 are satisfied.

We turn to prove (69), writing for simplicity Ω in place of Ω_n everywhere. The s -derivative of \mathbf{V}_ε amounts to

$$-\int_{\Omega} \partial_s [\varepsilon U(\gamma^h) + V\gamma^h] dx = \varepsilon \int_{\Omega} \gamma^h \Delta \varphi^h dx + \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} V \cdot \mathbf{D} \varphi^h dx.$$

Moreover, the weak formulation (80) is modified as follows,

$$\int_{\Omega} \partial_h \gamma^h \vartheta dx = s\varepsilon \int_{\Omega} \gamma^h \Delta \vartheta dx - s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} V \cdot \mathbf{D} \vartheta dx, \quad (96)$$

and, consequently, (81) is replaced by

$$\begin{aligned} \int_{\Omega} \partial_s \partial_h \gamma^h \vartheta dx &= \varepsilon \int_{\Omega} \gamma^h \Delta \vartheta dx + s\varepsilon \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} \varphi^h \cdot \mathbf{D} \Delta \vartheta dx \\ &\quad - \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} V \cdot \mathbf{D} \vartheta dx - s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{D} \varphi^h \cdot \mathbf{D} (\mathbf{m}'(\gamma^h) \mathbf{D} V \cdot \mathbf{D} \vartheta). \end{aligned}$$

Performing the same manipulations as in the proof of Lemma 4.2, one obtains

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \partial_h \mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2 dx &= \frac{\varepsilon s}{2} \int_{\Omega} \mathbf{m}''(\gamma^h) |\mathbf{D} \gamma^h|^2 |\mathbf{D} \varphi^h|^2 dx - s\varepsilon \int_{\Omega} \mathbf{m}(\gamma^h) \Delta (|\mathbf{D} \gamma^h|^2) dx \\ &\quad - s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{m}'(\gamma^h) \mathbf{D} V \cdot \mathbf{D}^2 \varphi^h \cdot \mathbf{D} \varphi^h dx - s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{m}''(\gamma^h) \mathbf{D} \gamma^h \cdot \mathbf{D} V |\mathbf{D} \varphi^h|^2 dx. \end{aligned}$$

Summing up everything provides

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \partial_h [\mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2] dx &= -\partial_s \left(\int_{\Omega} V \gamma^h dx + \varepsilon \int_{\Omega} U(\gamma^h) dx \right) \\ &\quad + \frac{s\varepsilon}{2} \int_{\Omega} \mathbf{m}''(\gamma^h) |\mathbf{D} \gamma^h|^2 |\mathbf{D} \varphi^h|^2 dx \end{aligned} \quad (97)$$

$$+ \frac{s}{2} \int_{\partial\Omega} \mathbf{m}(\gamma^h) \mathbf{D} (|\mathbf{D} \varphi^h|^2) \cdot \mathbf{n} d\mathcal{H}^{d-1} \quad (98)$$

$$- s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{m}''(\gamma^h) (\mathbf{D} \gamma^h \cdot \mathbf{D} \varphi^h \mathbf{D} \varphi^h \cdot \mathbf{D} V - \mathbf{D} \gamma^h \cdot \mathbf{D} V |\mathbf{D} \varphi^h|^2) dx \quad (99)$$

$$- s \int_{\Omega} \mathbf{m}(\gamma^h) \mathbf{m}'(\gamma^h) \mathbf{D} \varphi^h \mathbf{D}^2 V \mathbf{D} \varphi^h dx. \quad (100)$$

We need to show that the sum of the terms from (97) to (100) are less than $-s\lambda_\varepsilon \mathbf{A}$ for a sufficiently small (negative) constant λ_ε . The integral in (100) is readily controlled by a multiple of \mathbf{A} , recalling that \mathbf{m} has the Lipschitz property (M-LSC) and observing that

$$|\mathbf{m}(\gamma^h) \mathbf{m}'(\gamma^h) \mathbf{D} \varphi^h \mathbf{D}^2 V \mathbf{D} \varphi^h| \leq \sup_s |\mathbf{m}'(s)| \|V\|_{C^2(\Omega)} \mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2.$$

In order to absorb the integral in (99) into the (non-positive) integral in (97) and a multiple of \mathbf{A} , we apply Young's inequality to the integrand and estimate

$$\begin{aligned} |\mathbf{m}(\gamma^h) \mathbf{m}''(\gamma^h) (\mathbf{D} \gamma^h \cdot \mathbf{D} \varphi^h \mathbf{D} \varphi^h \cdot \mathbf{D} V - \mathbf{D} \gamma^h \cdot \mathbf{D} V |\mathbf{D} \varphi^h|^2)| &\leq 2\mathbf{m}(\gamma^h) |\mathbf{m}''(\gamma^h)| |\mathbf{D} \gamma^h| |\mathbf{D} \varphi^h|^2 |\mathbf{D} V| \\ &\leq \frac{\varepsilon}{2} (-\mathbf{m}''(\gamma^h)) |\mathbf{D} \gamma^h|^2 |\mathbf{D} \varphi^h|^2 + \frac{2}{\varepsilon} (-\mathbf{m}''(\gamma^h) \mathbf{m}(\gamma^h)^2) |\mathbf{D} \varphi^h|^2 |\mathbf{D} V|^2 \\ &\leq \frac{\varepsilon}{2} (-\mathbf{m}''(\gamma^h)) |\mathbf{D} \gamma^h|^2 |\mathbf{D} \varphi^h|^2 + \frac{2}{\varepsilon} \|V\|_{C^1(\Omega)}^2 \sup_s (-\mathbf{m}''(s) \mathbf{m}(s)) \mathbf{m}(\gamma^h) |\mathbf{D} \varphi^h|^2. \end{aligned}$$

Thus, defining, for every n

$$\lambda_{\varepsilon,n} := -\sup_s |\mathbf{m}'(s)| \|V_n\|_{C^2(\Omega_n)} - \frac{2}{\varepsilon} \|V_n\|_{C^1(\Omega_n)}^2 \sup_s (-\mathbf{m}''(s) \mathbf{m}(s)),$$

and recalling that (98) is non-positive for convexity of Ω we obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega_n} \partial_h [\mathbf{m}(\gamma^h) |D \varphi^h|^2] dx + s \lambda_{\varepsilon, n} \int_{\Omega_n} [\mathbf{m}(\gamma^h) |D \varphi^h|^2] dx \\ \leq -\partial_s \left(\int_{\Omega_n} V_n \gamma^h dx + \varepsilon \int_{\Omega_n} U(\gamma^h) dx \right). \end{aligned} \quad (101)$$

Defining $\lambda_\varepsilon := \inf_n \lambda_{\varepsilon, n} > -\infty$ (thanks to the uniform boundedness of all the derivatives of V_n) the principal estimate (69) follows from (101). \square

The flow interchange estimate (65) is applicable. To obtain a sensible a priori estimate, we still need to express the dissipation term in (65).

Lemma 4.7. *Let v_s be as in Proposition 4.6, and assume that $v_0 \in \text{Dom}(\mathbf{E}) \cap H^2(\Omega)$. Then*

$$-\liminf_{s \downarrow 0} \frac{1}{s} (\mathbf{E}[v_s] - \mathbf{E}[v_0]) \geq -\mathbf{N}[v_0, V] + \varepsilon \left(\frac{1}{2} \int_{\Omega} (\Delta v_0)^2 dx - C(\mathbf{E}_0 + \mathbf{E}[v_0]) \right). \quad (102)$$

Proof. For $\delta > 0$ sufficiently small, define approximations of G by

$$G_\delta(s) := \begin{cases} G\left(\delta + \frac{M-2\delta}{M}s\right) & \text{if } M < \infty, \\ G(\delta + s) & \text{if } M = +\infty. \end{cases}$$

This regularizes the possible singularities of $G'(s)$ for $s \downarrow 0$ and $s \uparrow M$. Denote by \mathbf{E}_δ the energy functional associated to G_δ instead of G .

For the following calculations, we need some properties of solutions to the problem (95), which are summarized in Lemma A.2 in the Appendix. By (125) we have that $s \mapsto \mathbf{E}_\delta[v_s]$ is absolutely continuous and we can calculate for almost every $s > 0$ its derivative

$$\begin{aligned} \frac{d}{ds} \mathbf{E}_\delta[v_s] &= \varepsilon \left(- \int_{\Omega} \Delta v_s \Delta v_s dx + \int_{\Omega} G'_\delta(v_s) \Delta v_s dx \right) \\ &\quad - \int_{\Omega} \Delta v_s \operatorname{div}(\mathbf{m}(v_s) D V) dx + \int_{\Omega} G'_\delta(v_s) \operatorname{div}(\mathbf{m}(v_s) D V) dx \\ &\leq -\varepsilon \left(\frac{1}{2} \int_{\Omega} (\Delta v_s)^2 dx - C(\mathbf{E}_0 + \mathbf{E}_\delta[v_0]) \right) + \mathbf{N}_\delta[v_s, V]. \end{aligned}$$

The last estimate is obtained by treating the term multiplied by ε exactly as in the proof of Lemma 4.3, and integrating by parts in the last two integrals (which is allowed for the smooth approximation G_δ and does not produce boundary terms since V satisfies homogeneous Neumann conditions (92)). Moreover, following the proof of Lemma 4.3, it is easy to check that the constant C in the last integral can be chosen uniformly with respect to δ . Then we have

$$\frac{\mathbf{E}_\delta[v_s] - \mathbf{E}_\delta[v_0]}{s} \leq \frac{1}{s} \int_0^s \left(-\frac{\varepsilon}{2} \int_{\Omega} (\Delta v_t)^2 dx + \mathbf{N}_\delta[v_t, V] \right) dt + \varepsilon C(1 + \mathbf{E}_\delta[v_0]).$$

By (10) it is easy to check that $\int_{\Omega} G_\delta(v) dx \rightarrow \int_{\Omega} G(v) dx$ and $\int_{\Omega} P_\delta(v) \Delta V dx \rightarrow \int_{\Omega} P(v) \Delta V dx$ as $\delta \downarrow 0$. Passing to the limit as $\delta \downarrow 0$ we obtain

$$\frac{\mathbf{E}[v_s] - \mathbf{E}[v_0]}{s} \leq \frac{1}{s} \int_0^s \left(-\frac{\varepsilon}{2} \int_{\Omega} (\Delta v_t)^2 dx + \mathbf{N}[v_t, V] \right) dt + \varepsilon C(1 + \mathbf{E}[v_0]).$$

By the right continuity property (126) we can pass to the limit by $s \downarrow 0$ obtaining (102). \square

The flow interchange estimate (65) provides the following.

Lemma 4.8. *Let V be a given test function satisfying (92) and $\psi \in C_c^\infty(0, +\infty)$ be a given temporal test function satisfying $\psi \geq 0$. Then,*

$$-\int_0^\infty \psi'(t) \mathbf{V}_\varepsilon[\bar{u}_\tau(t)] dt \leq \int_0^\infty \bar{\psi}_\tau(t) \mathbf{N}[\bar{u}_\tau(t), V] dt + C \frac{\tau}{\varepsilon}, \quad (103)$$

where the simple function $\bar{\psi}_\tau : (0, +\infty) \rightarrow [0, +\infty)$ is defined by $\bar{\psi}_\tau(t) = \psi((n-1)\tau)$ for $(n-1)\tau < t \leq n\tau$ for all $n \in \mathbb{N}$. The constant C in (103) is independent of τ and ε and depends only on the test functions V and ψ , and on the initial energy $\mathbf{E}[u_0]$.

Proof. Since \bar{u}_τ is a simple function with respect to $t \geq 0$, which is constant on intervals $((n-1)\tau, n\tau]$, and ψ is smooth with compact support, it follows that, for some sufficiently large $N \in \mathbb{N}$,

$$\begin{aligned} - \int_0^{+\infty} \psi'(t) \mathbf{V}_\varepsilon[\bar{u}_\tau(t)] dt &= - \sum_{n=1}^N \int_{(n-1)\tau}^{n\tau} \psi'(t) \mathbf{V}_\varepsilon[u_\tau^n] dt \\ &= \sum_{n=1}^N (\psi((n-1)\tau) - \psi(n\tau)) \mathbf{V}_\varepsilon[u_\tau^n] \\ &= \sum_{n=1}^N \psi((n-1)\tau) (\mathbf{V}_\varepsilon[u_\tau^n] - \mathbf{V}_\varepsilon[u_\tau^{n-1}]). \end{aligned} \quad (104)$$

By Proposition 4.6 we can apply the flow interchange Lemma 3.2 with $\mathcal{F} = \mathbf{V}_\varepsilon$. By (70) we can apply Lemma 4.7 with $v_0 = u_\tau^n$. Combining the flow interchange estimate (65) and inequality (102) we find

$$\mathbf{V}_\varepsilon[u_\tau^n] - \mathbf{V}_\varepsilon[u_\tau^{n-1}] \leq \tau \mathbf{N}[u_\tau^n, V] + \varepsilon \tau \left(C(\mathbf{E}_0 + \mathbf{E}[u_0]) - \frac{1}{2} \int_\Omega (\Delta u_\tau^n)^2 dx \right) + \frac{K}{2\varepsilon} \mathbf{W}_m(u_\tau^n, u_\tau^{n-1})^2. \quad (105)$$

Combining (104) with (105) and recalling that $\psi \geq 0$ we obtain

$$\begin{aligned} - \int_0^{+\infty} \psi'(t) \mathbf{V}_\varepsilon[\bar{u}_\tau(t)] dt &\leq \tau \sum_{n=1}^N \psi((n-1)\tau) \mathbf{N}[u_\tau^n, V] + \varepsilon C(\mathbf{E}_0 + \mathbf{E}[u_0]) \tau \sum_{n=0}^{N-1} \psi(n\tau) \\ &\quad + \frac{K\tau}{2\varepsilon} \sup_{t>0} \psi(t) \tau \sum_{n=1}^{+\infty} \left(\frac{\mathbf{W}_m(u_\tau^n, u_\tau^{n-1})}{\tau} \right)^2 \\ &\leq \int_0^{+\infty} \bar{\psi}_\tau(t) \mathbf{N}[\bar{u}_\tau(t), V] dt + C(\mathbf{E}_0 + \mathbf{E}[u_0]) \varepsilon \int_0^{+\infty} \bar{\psi}_\tau(t) dt \\ &\quad + \frac{K\tau}{\varepsilon} \sup_{t>0} \psi(t) (\mathbf{E}[u_0] + \mathbf{E}_0), \end{aligned}$$

where the energy inequality (60) has been used to obtain the last line. The claim (103) follows. \square

In order to finish the proof of (93), we pass to the time-continuous limit $\tau \downarrow 0$ and the limit as $\varepsilon \downarrow 0$ simultaneously, in such a way that the remainder term in (103) goes to zero.

Proof of Proposition 4.5. For definiteness, let $(\tau_n)_{n \in \mathbb{N}}$ be a vanishing sequence for which $\bar{u}_{\tau_n} \rightarrow u$ strongly in $L^2(0, T, H^1(\Omega))$ according with Proposition 4.1. Without loss of generality, for (72) we may further assume that $\bar{u}_{\tau_n} \rightarrow u$ and $D\bar{u}_{\tau_n} \rightarrow Du$ almost everywhere on $(0, +\infty) \times \Omega$. It is sufficient to choose the vanishing sequence $\varepsilon_n := \sqrt{\tau_n}$ in order to have that $C \frac{\tau_n}{\varepsilon_n} \downarrow 0$ in (103).

We start by proving convergence of the left-hand side in (103). By the bounds from (59) and the the monotonicity of the energy (60), one finds that

$$|\mathbf{V}_\varepsilon[\bar{u}_\tau(t)] - \mathbf{V}[\bar{u}_\tau(t)]| \leq \varepsilon C(\mathbf{E}_0 + \mathbf{E}[u_0]) \quad (106)$$

for every $\tau > 0$ and $t \geq 0$. Choosing $T > 0$ such that $\text{supp}(\psi) \subset [0, T]$, using (106), we have

$$\begin{aligned} &\left| \int_0^T \psi'(t) \mathbf{V}_{\varepsilon_n}[\bar{u}_{\tau_n}(t)] dt - \int_0^T \psi'(t) \mathbf{V}[u(t)] dt \right| \\ &\leq \sup_{t \in (0, T)} |\psi'(t)| \left(\int_0^T |\mathbf{V}_{\varepsilon_n}[\bar{u}_{\tau_n}(t)] - \mathbf{V}[\bar{u}_{\tau_n}(t)]| dt + \int_0^T |\mathbf{V}[\bar{u}_{\tau_n}(t)] - \mathbf{V}[u(t)]| dt \right) \\ &\leq \sup_{t \in (0, T)} |\psi'(t)| \left(T \varepsilon_n C(\mathbf{E}_0 + \mathbf{E}[u_0]) + \sup_{x \in \Omega} |V(x)| \int_0^T \int_\Omega |\bar{u}_{\tau_n} - u| dx dt \right) \end{aligned}$$

which shows that

$$\lim_{n \rightarrow +\infty} \int_0^{+\infty} \psi'(t) \mathbf{V}_{\varepsilon_n}[\bar{u}_{\tau_n}(t)] dt = \int_0^{+\infty} \psi'(t) \mathbf{V}[\bar{u}(t)] dt. \quad (107)$$

From (103) and (107) one concludes that

$$-\int_0^{+\infty} \psi'(t) \mathbf{V}[\bar{u}(t)] dt \leq \liminf_{n \rightarrow \infty} \int_0^{+\infty} \bar{\psi}_{\tau_n}(t) \mathbf{N}[\bar{u}_{\tau_n}(t), V] dt. \quad (108)$$

Next, we claim that the minimum limit in (108) is actually a limit, and that

$$\lim_{n \rightarrow +\infty} \int_{\Omega_T} \bar{\psi}_{\tau_n} \Delta \bar{u}_{\tau_n} \operatorname{div}(\mathbf{m}(u_{\tau_n}) \mathbf{D}V) dx dt = \int_{\Omega_T} \psi \Delta \bar{u} \operatorname{div}(\mathbf{m}(\bar{u}) \mathbf{D}V) dx dt, \quad (109)$$

$$\lim_{n \rightarrow \infty} \int_{\Omega_T} \bar{\psi}_{\tau_n} P_i(\bar{u}_{\tau_n}) \Delta V dx dt = \int_{\Omega_T} \psi P_i(u) \Delta V dx dt, \quad (110)$$

for $i = 1, 2$. In fact, (109) and (110) follow almost immediately from Corollary 4.4: Combining (90) with the weak convergence (73) and the uniform convergence of $\bar{\psi}_{\tau_n}$ to ψ in Ω_T , one obtains (109). And recalling that $\bar{\psi}_{\tau_n}$ uniformly converges to ψ in Ω_T , we obtain (110) from (89).

Inserting (109) and (110) into (108) we obtain that

$$-\int_0^{+\infty} \psi'(t) \mathbf{V}[u(t)] dt \leq \int_0^{+\infty} \psi(t) \mathbf{N}[u(t), V] dt \quad (111)$$

for all $V \in C^\infty(\bar{\Omega})$ satisfying (92), and all non-negative $\psi \in C_c^\infty(0, +\infty)$. Exchanging V with $-V$ in (111) yields the respective equality (93). Trivially, (93) extends from non-negative test functions ψ to all $\psi \in C_c^\infty(0, +\infty)$, thus finishing the proof. \square

Since any space-temporal test function in (18) $\zeta \in C^\infty((0, +\infty) \times \bar{\Omega})$, with $\mathbf{D}\zeta \cdot \mathbf{n} = 0$ on $\partial\Omega$, can be approximated in $C^\infty((0, +\infty) \times \bar{\Omega})$ by sums of functions of the type $\zeta(t, x) = \psi(t)V(x)$ with $V \in C^\infty(\bar{\Omega})$ satisfying (92) and $\psi \in C_c^\infty(0, +\infty)$, Theorem 1 follows.

5. PROOF OF THEOREM 2

5.1. Approximation. Theorem 2 is now proven by approximation of the more general mobility function \mathbf{m} satisfying (M_{1/2}) by mobilities \mathbf{m}_δ that have the Lipschitz property (M-LSC). To this end, define for all $\delta > 0$ sufficiently small:

- if $M < +\infty$:

$$\mathbf{m}_\delta(s) := \mathbf{m}\left(\frac{s_\delta^2 - s_\delta^1}{M}s + s_\delta^1\right) - \delta,$$

where $s_\delta^1 < s_\delta^2$ are the two solutions of $\mathbf{m}(s) = \delta$.

- if $M = +\infty$:

$$\mathbf{m}_\delta(s) := \mathbf{m}(s + s_\delta) - \delta.$$

where $s_\delta > 0$ is the unique solution of $\mathbf{m}(s) = \delta$.

Introduce accordingly P_δ by

$$P_\delta(s) = \int_0^s \mathbf{m}_\delta(r) G''(r) dr. \quad (112)$$

Lemma 5.1. *For all $\delta > 0$ sufficiently small, the \mathbf{m}_δ are smooth functions that have the Lipschitz property (M-LSC) and satisfy the pointwise bounds $0 \leq \mathbf{m}_\delta \leq \mathbf{m}$. In particular, we have*

$$\mathbf{W}_\mathbf{m}(u, v) \leq \mathbf{W}_{\mathbf{m}_\delta}(u, v) \quad \text{for all } u, v \in X(\Omega). \quad (113)$$

For $\delta \downarrow 0$, the \mathbf{m}_δ converge monotonically and globally uniformly to \mathbf{m} . Moreover, if G satisfies (G) with respect to \mathbf{m} , then it also satisfies (G) with respect to each \mathbf{m}_δ . Finally, the P_δ are continuous functions, and there is a constant K such that

$$-K(1 + s^2) \leq P_\delta(s) \leq P(s) + K(1 + s) \quad (114)$$

for all $s \in (0, M)$ and all $\delta > 0$ sufficiently small, and P_δ converges to P as $\delta \downarrow 0$, uniformly on $[0, M]$ if $M < \infty$, or uniformly on each $[0, \bar{s}]$ if $M = +\infty$.

Proof. Smoothness, non-negativity and the Lipschitz property of \mathbf{m}_δ are evident from its definition, and the concavity and smoothness of \mathbf{m} . In the case $M = +\infty$, also the upper bound $\mathbf{m}_\delta \leq \mathbf{m}$ is a trivial consequence of concavity, as is the uniform convergence for $\delta \downarrow 0$:

$$0 \leq \mathbf{m}(s) - \mathbf{m}_\delta(s) \leq \delta - \mathbf{m}'(s + s_\delta)s_\delta \leq \delta. \quad (115)$$

In the case $M < +\infty$, the upper bound can be proven as follows: assume that \mathbf{m} attains its maximal value at $\sigma \in (0, M)$; then \mathbf{m}_δ attains its maximum at $\sigma_\delta = (\sigma - s_\delta^1)M/(s_\delta^2 - s_\delta^1)$. Without loss of generality, assume $\sigma_\delta \leq \sigma$. For all $s \in [0, \sigma_\delta]$, we have

$$\mathbf{m}'_\delta(s) = \underbrace{\frac{s_\delta^2 - s_\delta^1}{M}}_{\leq 1} \mathbf{m}'\left(\underbrace{\frac{s_\delta^2 - s_\delta^1}{M}s + s_\delta^1}_{\geq s}\right) \leq \mathbf{m}'(s)$$

and thus also $\mathbf{m}_\delta(s) \leq \mathbf{m}(s)$. A similar argument provides $\mathbf{m}_\delta(s) \leq \mathbf{m}(s)$ for all $s \in [\sigma, M]$. For $s \in [\sigma_\delta, \sigma]$, the inequality follows since \mathbf{m}_δ is non-increasing and \mathbf{m} is non-decreasing on that interval. The argument for uniform convergence of \mathbf{m}_δ to \mathbf{m} is established essentially with the same argument as in (115), making again a case distinction whether $s \in [0, \sigma_\delta]$, $s \in [\sigma_\delta, \sigma]$, or $s \in [\sigma, M]$.

Condition (G) on G is less stringent for the approximations \mathbf{m}_δ since $\mathbf{m}_\delta \leq \mathbf{m}$ in case $M = +\infty$, and $\mathbf{m}_\delta/(1 + \mathbf{m}_\delta) \leq \mathbf{m}/(1 + \mathbf{m})$ in case $M < \infty$. Concerning the continuity of P_δ , we remark that, in view of $\mathbf{m}_\delta \leq \mathbf{m}$, the integrability of $\mathbf{m}_\delta(s)G''(s)$ near $s = 0$ (and near $s = M$ if $M < \infty$) implies the respective integrability of $\mathbf{m}(s)G''(s)$. Moreover,

$$\sup_{0 < s < \bar{s}} |P(s) - P_\delta(s)| \leq \int_0^{\bar{s}} (\mathbf{m}(s) - \mathbf{m}_\delta(s)) |G''(s)| ds$$

in combination with the pointwise convergence of \mathbf{m}_δ to \mathbf{m} implies uniform convergence of P_δ to P on all intervals $[0, \bar{s}]$; notice that the dominated convergence theorem is applicable since $0 \leq (\mathbf{m} - \mathbf{m}_\delta)|G''| \leq \mathbf{m}|G''|$, and the latter is integrable by assumption. Finally, if $M = +\infty$, then (114) is another consequence of (G). Indeed, on one hand,

$$P_\delta(s) = \int_0^s \mathbf{m}_\delta(r)G''(r)dr \geq -C \int_0^s (1 + \mathbf{m}(r))dr \geq -K(1 + s^2),$$

and on the other hand, using also (115),

$$\begin{aligned} P_\delta(s) &= P(s) - \int_0^s (\mathbf{m}(r) - \mathbf{m}_\delta(r))G''(r)dr \leq P(s) + \delta \int_0^s (G''(r))_- dr \\ &\leq P(s) + C\delta \int_0^s \left(1 + \frac{1}{\mathbf{m}(r)}\right)dr \leq P(s) + K(1 + s). \end{aligned}$$

For $M < \infty$, (114) simply amounts to δ -uniform boundedness of P_δ , which is clear from the uniform convergence to P . \square

5.2. Weak and strong convergence. Lemma 5.1 implies that Theorem 1 is applicable to the approximate mobilities \mathbf{m}_δ for each δ sufficiently small: there exist respective solutions $u_\delta : [0, +\infty) \rightarrow X(\Omega)$ of problem (1)–(3), i.e.,

$$\int_0^{+\infty} \int_\Omega \partial_t \zeta u_\delta dx dt = \int_0^{+\infty} \int_\Omega \Delta u_\delta D(\mathbf{m}_\delta(u_\delta) D\zeta) dx dt + \int_0^{+\infty} \int_\Omega P_\delta(u_\delta) \Delta \zeta dx dt \quad (116)$$

for all test functions $\zeta \in C_c^\infty((0, +\infty) \times \bar{\Omega})$ such that $D\zeta \cdot \mathbf{n} = 0$ on $\partial\Omega$. We wish to pass to the limit as $\delta \downarrow 0$ in (116).

Lemma 5.2. *There exists an absolutely continuous curve $u : [0, \infty) \rightarrow X(\Omega)$ and a vanishing sequence δ_k such that the u_{δ_k} converge to u weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ pointwise in time, as well as strongly in $L^2(0, T; L^2(\Omega))$ for every $T > 0$ and pointwise a.e. on $(0, +\infty) \times \Omega$.*

Proof. The curves u_δ satisfy estimate (61) in the respective metric $\mathbf{W}_{\mathbf{m}_\delta}$ with the global Hölder constant determined by $\mathbf{E}[u_0]$. In view of (113), the family $(u_\delta)_{\delta>0}$ is equi-continuous with respect to $\mathbf{W}_{\mathbf{m}}$. Moreover, the sublevel sets of the energy \mathbf{E} are compact. The claim on convergence now follows by the variant of the Arzelá-Ascoli theorem given in [1, Proposition 3.3.1]. An application of the dominated convergence theorem with respect to time provides strong convergence of u_δ to u also in sense of $L^2(0, T; L^2(\Omega))$, and thus (without loss of generality) also pointwise a.e. convergence. \square

In the following we write $\delta \downarrow 0$ to indicate “along a suitable vanishing sequence δ_k for $k \rightarrow \infty$ ”.

Lemma 5.3. *For every $T > 0$,*

$$\limsup_{\delta \downarrow 0} \int_0^T \|u_\delta(t)\|_{H^2(\Omega)}^2 dt < +\infty. \quad (117)$$

Consequently, u_δ converges to u weakly in $L^2(0, T; H^2(\Omega))$ and strongly $L^2(0, T; H^1(\Omega))$ as $\delta \downarrow 0$. Moreover, $P_\delta(u_\delta)$ converges to $P(u)$ in $L^1(0, T; L^1(\Omega))$.

Proof. Define the δ -approximations of the entropy functional \mathbf{U}_δ as in (42) with \mathbf{m}_δ instead of \mathbf{m} . Then estimate (59) holds with a constant C independent of δ for every \mathbf{U}_δ , at least for all $\delta > 0$ sufficiently small. Indeed, observe that $\mathbf{m}_\delta(s_0) \geq \frac{1}{2}\mathbf{m}(s_0)$ if δ is small enough, and hence inequality (44) follows. In the same way, inequalities (85) and (88) can be obtained with δ -independent constants C . In combination, (117) follows.

The stated weak convergence of u_δ is now a consequence of Alaoglu’s theorem and the uniqueness of the weak limit. The strong convergence is obtained by interpolation of the strong convergence in $L^2(0, T; L^2(\Omega))$ with the bound (117). To prove convergence of $P_\delta(u_\delta)$, we argue as in Corollary 4.4. \square

5.3. Convergence of the mobility gradient. The next goal is to establish convergence of $\mathbf{m}_\delta(u_\delta)$ in $L^2(0, T; H^1(\Omega))$.

Lemma 5.4. *$\mathbf{m}_\delta(u_\delta)$ converges to $\mathbf{m}(u)$ strongly in $L^2(\Omega_T)$ for every $T > 0$ as $\delta \downarrow 0$.*

Proof. The uniform convergence of the mobility functions \mathbf{m}_δ to \mathbf{m} , and the pointwise a.e. convergence of u_δ to u suffice to conclude pointwise a.e. convergence of $\mathbf{m}_\delta(u_\delta)$ to $\mathbf{m}(u)$ on Ω_T . Moreover, if $M < \infty$, then $\mathbf{m}_\delta(u_\delta)$ is δ -independently bounded, and by dominated convergence it follows that $\mathbf{m}_\delta(u_\delta)$ converges strongly to $\mathbf{m}(u)$ in $L^2(\Omega_T)$. In the case $M = +\infty$, the δ -uniform bound $\mathbf{m}_\delta(s) \leq \mathbf{m}(s) \leq C(1+s)$ and the strong convergence of u_δ in $L^2(\Omega_T)$ imply equi-integrability of $|\mathbf{m}_\delta(u_\delta)|^2$ in Ω_T . We invoke Vitali’s theorem to conclude the proof. \square

For the proof of convergence of the gradients $\mathbf{D}\mathbf{m}_\delta(u_\delta)$ in $L^2(\Omega_T)$, we distinguish the cases $M < \infty$ and $M = \infty$.

Lemma 5.5. *Assume $M < \infty$. Define $g : [0, M] \rightarrow \mathbb{R}$ by $g_\delta(s) := \sqrt{s(M-s)}\mathbf{m}'_\delta(s)$ for all $\delta > 0$ sufficiently small. For $\delta \downarrow 0$, the g_δ converge uniformly to the continuous function $g_0 : [0, M] \rightarrow \mathbb{R}$ given by $g_0(s) = \sqrt{s(M-s)}\mathbf{m}'(s)$ for all $s \in (0, M)$, and $g_0(0) = g_0(M) = 0$.*

Proof. Let $\epsilon > 0$ be given. Since $\sqrt{s(M-s)}\mathbf{m}'(s) \rightarrow 0$ for $s \downarrow 0$ and for $s \uparrow M$, respectively, by hypothesis $(M_{1/2})$, and since $\mathbf{m}'(s)$ is a non-increasing function on $(0, M)$, there are $\sigma_\epsilon^1 < \sigma_\epsilon^2$ such that $|g_\delta(s)| \leq \sqrt{s(M-s)}|\mathbf{m}'(s)| < \epsilon$ for all $s \in [0, \sigma_\epsilon^1] \cup [\sigma_\epsilon^2, M]$, and all $\delta > 0$ is sufficiently small. For $\delta \downarrow 0$, the points s_δ^1 and s_δ^2 in the definition of \mathbf{m}_δ converge to 0 and to M , respectively. By smoothness of \mathbf{m} , one thus has local uniform convergence of \mathbf{m}'_δ to \mathbf{m}' , and consequently $|g_\delta(s) - g_0(s)| < \epsilon$ for all $s \in [\sigma_\epsilon^1, \sigma_\epsilon^2]$, for sufficiently small $\delta > 0$. The uniform convergence of the g_δ also proves continuity of g_0 . \square

Lemma 5.6. *Assume $M < \infty$. Then $\mathbf{D}\mathbf{m}_\delta(u_\delta)$ converges to $\mathbf{D}\mathbf{m}(u)$ strongly in $L^2(\Omega_T)$ for every $T > 0$ as $\delta \downarrow 0$.*

Proof. Observe that

$$|\mathbf{D} \mathbf{m}_\delta(u_\delta(t))|^2 = (g_\delta(u_\delta))^2 \frac{1}{M} (|\mathbf{D} \sqrt{u_\delta(t)}|^2 + |\mathbf{D} \sqrt{M - u_\delta(t)}|^2) \quad (118)$$

for every $t \geq 0$ at which $u_\delta(t) \in H^2(\Omega)$. Pointwise convergence of u_δ to u almost everywhere on Ω_T and uniform convergence of g_δ to g_0 imply pointwise convergence of the compositions $g_\delta(u_\delta)$ to $g_0(u)$ almost everywhere. In combination with the δ -uniform boundedness of g_δ it follows in particular that $g_\delta(u_\delta) \rightarrow g_0(u)$ in $L^4(\Omega_T)$.

Now let $Q \subset \Omega_T$ be a measurable subset. From (118) we obtain

$$\begin{aligned} \int_Q |\mathbf{D} \mathbf{m}_\delta(u_\delta)|^2 dx dt &= \frac{1}{M} \int_Q |g_\delta(u_\delta)|^2 |\mathbf{D} \sqrt{u_\delta}|^2 dx dt + \frac{1}{M} \int_Q |g_\delta(u_\delta)|^2 |\mathbf{D} \sqrt{M - u_\delta}|^2 dx dt \\ &\leq \frac{1}{M} \left(\int_0^T \|\mathbf{D} \sqrt{u_\delta(t)}\|_{L^4}^4 dt \right)^{1/2} \left(\int_Q g_\delta(u_\delta)^4 dx dt \right)^{1/2} \\ &\quad + \frac{1}{M} \left(\int_0^T \|\mathbf{D} \sqrt{M - u_\delta(t)}\|_{L^4}^4 dt \right)^{1/2} \left(\int_Q g_\delta(u_\delta)^4 dx dt \right)^{1/2}. \end{aligned} \quad (119)$$

Choose $Q = Z_T := \{(t, x) \in \Omega_T : u(t, x) = 0 \text{ or } u(t, x) = M\}$. Since $g_\delta(u_\delta) \rightarrow g_0(u) = 0$ in $L^4(Z_T)$, the right-hand side of (119) vanishes as $\delta \downarrow 0$, and $\mathbf{D} \mathbf{m}_\delta(u_\delta) \rightarrow 0$ in $L^2(Z_T)$.

Further, since the Q in inequality (119) can be chosen as any arbitrary subset of $\Omega_T \setminus Z_T$, the equi-integrability of $|g_\delta(u_\delta)|^4$ on Ω_T is inherited to $|\mathbf{D} \mathbf{m}_\delta(u_\delta)|^2$ on $\Omega_T \setminus Z_T$. On $\Omega_T \setminus Z_T$, the composition $\mathbf{m}'(u)$ is everywhere finite, and $\mathbf{m}'_\delta(u_\delta) \rightarrow \mathbf{m}'(u)$ pointwise a.e. In combination with the pointwise a.e. convergence of $\mathbf{D} u_\delta$, it follows that

$$\mathbf{D} \mathbf{m}_\delta(u_\delta) = \mathbf{m}'_\delta(u_\delta) \mathbf{D} u_\delta \rightarrow \mathbf{m}'(u) \mathbf{D} u = \mathbf{D} \mathbf{m}(u)$$

strongly in $L^2(\Omega_T \setminus Z_T)$, invoking Vitali's theorem once again.

The proof of strong convergence $\mathbf{m}_\delta(u_\delta) \rightarrow \mathbf{m}(u)$ in $L^2(0, T; H^1(\Omega))$ is now concluded by observing the following. Since $\mathbf{m}_\delta(u_\delta) \rightarrow \mathbf{m}(u)$ in $L^2(\Omega_T)$, the strong $L^2(\Omega_T)$ -limit of $\mathbf{D} \mathbf{m}_\delta(u_\delta)$ coincides with the (uniquely determined) distributional derivative $\mathbf{D} \mathbf{m}(u)$. In particular, $\mathbf{D} \mathbf{m}(u)$ vanishes a.e. on the set Z_T . \square

Lemma 5.7. *Assume $M = +\infty$. Then $\mathbf{D} \mathbf{m}_\delta(u_\delta)$ converges to $\mathbf{D} \mathbf{m}(u)$ strongly in $L^2(\Omega_T)$ for every $T > 0$ as $\delta \downarrow 0$.*

Proof. Similar as in the proof of Lemma 5.6, define $g_\delta : [0, \infty) \rightarrow \mathbb{R}$ by $g_\delta(s) := \sqrt{s} \mathbf{m}'_\delta(s)$ for all $\delta > 0$ small enough. Analogously to Lemma 5.5, one obtains convergence — uniformly on every interval $[0; \bar{s}]$ — of g_δ to the continuous function $g_0 : [0, \infty) \rightarrow \mathbb{R}$ with $g_0(s) := \sqrt{s} \mathbf{m}'(s)$ for $s \in (0, +\infty)$, and $g_0(0) = 0$. Like in (118), we observe that

$$|\mathbf{D} \mathbf{m}_\delta(u_\delta(t))|^2 = 4(g_\delta(u_\delta))^2 |\mathbf{D} \sqrt{u_\delta(t)}|^2 \quad (120)$$

for every $t \geq 0$ at which $u_\delta(t) \in H^2(\Omega)$. As before, we conclude that $g_\delta(u_\delta)$ converges to $g_0(u)$ almost everywhere on $(0, +\infty) \times \Omega$. Moreover, by construction of the \mathbf{m}_δ and $(M_{1/2})$, one has $0 \leq g_\delta(s) \leq C(1 + s^{1/2})$ for all $s \geq 0$. Since $|u_\delta|^2$ is equi-integrable in Ω_T for arbitrary $T > 0$, the compositions $|g_\delta(u_\delta)|^4$ are also equi-integrable in Ω_T . By Vitali's Theorem, it follows that $g_\delta(u_\delta) \rightarrow g_0(u)$ in $L^4(\Omega_T)$.

From (120), we conclude that

$$\begin{aligned} \int_Q |\mathbf{D} \mathbf{m}_\delta(u_\delta)|^2 dx dt &\leq 4 \int_Q |g_\delta(u_\delta)|^2 |\mathbf{D} \sqrt{u_\delta}|^2 dx dt \\ &\leq 4 \left(\int_0^T \|\mathbf{D} \sqrt{u_\delta(t)}\|_{L^4}^4 dt \right)^{1/2} \left(\int_Q g_\delta(u_\delta)^4 dx dt \right)^{1/2} \end{aligned} \quad (121)$$

holds for any measurable set $Q \subset \Omega_T$. From this point on, the proof is identical to the one for Lemma 5.6, with the only change that $Z_T := \{(t, x) \in \Omega_T : u(t, x) = 0\}$. \square

5.4. End of the proof. We are now able to pass to the limit in the weak formulation (116). By Lemma 5.2, Δu_δ converges to Δu weakly in $L^2(\Omega_T)$ as $\delta \downarrow 0$, and by Lemma 5.6 or 5.7, respectively, $\mathbf{D} \mathbf{m}_\delta(u_\delta)$ converges to $\mathbf{D} \mathbf{m}(u)$ strongly in that space. Hence, the term inside the first integral on the right-hand side of (116) converges weakly to the desired limit in $L^1(\Omega_T)$. Convergence of the second integral follows from the strong convergence of $P_\delta(u_\delta)$ to $P(u)$ in $L^1(\Omega_T)$ stated in Lemma 5.3. This finishes the proof of Theorem 2.

APPENDIX A. APPENDIX

We recall a Sobolev like inequality, that will be useful in order to estimate the rate of dissipation of the energy \mathbf{E} along the heat flow.

Lemma A.1. *Assume that Ω is a smooth convex open set and that $u \in H^2(\Omega)$ satisfies homogeneous Neumann boundary conditions, $\mathbf{n} \cdot \mathbf{D} u = 0$ on $\partial\Omega$. Then*

$$\int_{\Omega} \|\mathbf{D}^2 u\|^2 dx \leq \int_{\Omega} (\Delta u)^2 dx \leq d \int_{\Omega} \|\mathbf{D}^2 u\|^2 dx. \quad (122)$$

If, in addition, u is non-negative, then $\sqrt{u} \in W^{1,4}(\Omega)$ and

$$16 \int_{\Omega} |\mathbf{D} \sqrt{u}|^4 dx \leq (d+8) \int_{\Omega} (\Delta u)^2 dx. \quad (123)$$

Proof. By density, it obviously suffices to prove (122) for $u \in C^\infty(\bar{\Omega})$ satisfying $\mathbf{n} \cdot \mathbf{D} u = 0$ on $\partial\Omega$. Integrating by parts, it follows

$$\begin{aligned} \int_{\Omega} \|\mathbf{D}^2 u\|^2 dx &= \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{D}^2 u \cdot \mathbf{D} u d\sigma - \int_{\Omega} \mathbf{D} \Delta u \cdot \mathbf{D} u dx \\ &= \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{D}^2 u \cdot \mathbf{D} u d\sigma - \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{D} u \Delta u d\sigma + \int_{\Omega} (\Delta u)^2 dx. \end{aligned}$$

Now observe that the second boundary integral vanishes because of the no-flux boundary conditions, whereas the integrand in the first is pointwise non-negative,

$$\mathbf{n} \cdot \mathbf{D}^2 u \cdot \mathbf{D} u \leq 0. \quad (124)$$

(For a proof of this classical fact, see e.g. [16, Lemma 5.2].) This shows the first inequality in (122). The second inequality follows by observing the pointwise relation

$$(\Delta u)^2 = \sum_{i,j=1}^d \partial_{ii} u \partial_{jj} u \leq \frac{1}{2} \sum_{i,j=1}^d (\partial_{ii} u)^2 + (\partial_{jj} u)^2 = d \sum_{i=1}^d (\partial_{ii} u)^2 \leq d \|\mathbf{D}^2 u\|^2.$$

The estimate (123) follows by combination of [16, Lemma 3.1] with (122) above. \square

The next lemma summarizes selected properties of solutions to the viscous conservation law (95). These are used in the calculation of the right derivative of the energy \mathbf{E} along the corresponding flow.

Lemma A.2. *Assume that Ω is a smooth bounded domain, \mathbf{m} satisfies (M), (M-LSC) and V satisfies (92). If $v_0 \in X^r(\Omega)$, then there exists a unique smooth classical solution $v(s, x)$ of problem (95) and $v(s, \cdot) \in X^r(\Omega)$. If v_0 and \bar{v}_0 are two initial conditions in $X^r(\Omega)$, then the L^1 -contractivity estimate*

$$\|v_s - \bar{v}_s\|_{L^1(\Omega)} \leq \|v_0 - \bar{v}_0\|_{L^1(\Omega)}$$

holds at any time $s \geq 0$ for the associated solutions v, \bar{v} . If $v_0 \in H^2(\Omega)$, then $v_s \in H^2(\Omega)$ for every $s \geq 0$,

$$\Delta v, \operatorname{div}(\mathbf{m}(v) \mathbf{D} V) \in L^\infty(0, +\infty; L^2(\Omega)), \quad (125)$$

and the maps

$$s \mapsto \Delta v_s, \quad s \mapsto \operatorname{div}(\mathbf{m}(v_s) \mathbf{D} V) \quad \text{are right continuous with values in } L^2(\Omega). \quad (126)$$

Proof. This lemma is deduced with standard methods for parabolic equations, see e.g. [18], and we leave most of its proof to the interested reader. Here, we shall only comment on the well-definiteness of the flow on $X^r(\Omega)$. From the classical theory, it follows that $v \in C^\infty(\mathbb{R}_+ \times \Omega)$, i.e., the generalized solution is smooth and classical for positive times, and is continuous initially. One then easily verifies, using (M-LSC), that v satisfies the comparison principle. Next, observe that (95) admits a family of stationary solutions $(\tilde{v}^c)_{c \in \mathbb{R}}$ of the form

$$\tilde{v}^c(x) = F^{-1}(c - \varepsilon^{-1}V(x)) \quad \text{with} \quad F(s) = \int_{s_0}^s \frac{d\sigma}{\mathbf{m}(\sigma)}.$$

It is easily seen from the properties of \mathbf{m} and by smoothness of V , that $\tilde{v}^c \in X^r(\Omega)$ for every $c \in \mathbb{R}$, and that $\tilde{v}^c \downarrow 0$ or $\tilde{v}^c \uparrow M$ uniformly on Ω , for $c \downarrow -\infty$ or $c \uparrow \infty$, respectively. Hence, any solution v can be sandwiched between two stationary solutions. Since $v_0 \in X^r(\Omega)$, one has $v(s) \in X^r(\Omega)$ for all $s > 0$. □

REFERENCES

- [1] Ambrosio, L.; Gigli, N.; Savaré, G. Gradient flows in metric spaces and in the space of probability measures. Second edition. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008.
- [2] Bernis, F.; Friedman, A. *Higher order nonlinear degenerate parabolic equations*. J. Differential Equations 83 (1990), no. 1, 179–206.
- [3] Brézis, H. Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Publishing Co., Amsterdam-London, 1973.
- [4] Cahn, J. W.; Hilliard, J. E. *Spinodal decomposition: A reprise*. Acta Metallurgica 19 (1961) 151–161.
- [5] Taylor, J.E.; Cahn, J.W. *Linking anisotropic sharp and diffuse surface motion laws via gradient flows*. J. Statist. Phys. 77 (1994), no. 1-2, 183–197.
- [6] Carlen, E. A.; Ulusoy, S. *Asymptotic equipartition and long time behavior of solutions of a thin-film equation*. J. Differential Equations 241 (2007), no. 2, 279–292.
- [7] Carrillo J.A.; Lisini, S.; Savaré, G.; Slepcev, D. *Nonlinear mobility continuity equations and generalized displacement convexity*. J. Funct. Anal. 258 (2010) 1273-1309.
- [8] S. Daneri and G. Savaré, Eulerian calculus for the displacement convexity in the Wasserstein distance, SIAM J. Math. Anal., 3 (2008), pp. 1104-1122.
- [9] Dal Passo, R.; Garcke, H.; Grün, G. *On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions*. SIAM J. Math. Anal. 29 (1998), no. 2, 321–342.
- [10] De Giorgi, E. *New problems on minimizing movements*. Boundary value problems for partial differential equations and applications, 81-98, RMA Res. Notes Appl. Math., 29, Masson, Paris, 1993.
- [11] Dolbeault, J.; Nazaret, B.; Savaré, G. *A new class of transport distances between measures*. Calc. Var. Partial Differential Equations 34 (2009), no. 2, 193–231.
- [12] Elliott, C. M.; Garcke, H. *On the Cahn-Hilliard equation with degenerate mobility*. SIAM J. Math. Anal. 27 (1996), no. 2, 404–423.
- [13] Evans, J. D.; Galaktionov, V. A.; King, J. R. *Blow-up similarity solutions of the fourth-order unstable thin film equation*. European J. Appl. Math. 18 (2007), no. 2, 195–231.
- [14] Gagliardo, E. *Ulteriori proprietà di alcune classi di funzioni in pi variabili*. Ricerche Mat. 8 (1959) 24-51.
- [15] Giacomelli, L.; Otto, F. *Variational formulation for the lubrication approximation of the Hele-Shaw flow*. Calc. Var. Partial Differential Equations 13 (2001), no. 3, 377–403.
- [16] Gianazza, U.; Savaré, G.; Toscani, G. *The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation*. Arch. Ration. Mech. Anal. 194 (2009), 133–220.
- [17] Jordan, R.; Kinderlehrer, D.; Otto, F. *The variational formulation of the Fokker-Planck equation*. SIAM J. Math. Anal. 29 (1998), no. 1, 1–17.
- [18] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva, Linear and quasilinear equations of parabolic type. American Mathematical Society, Providence, R.I. 1967.
- [19] Laugesen, R. S.; Pugh, M. C. *Linear stability of steady states for thin film and Cahn-Hilliard type equations*. Arch. Ration. Mech. Anal. 154 (2000), no. 1, 3–51.
- [20] Lisini, S.; Marigonda, A. *On a class of modified Wasserstein distance induced by concave mobility functions defined on bounded intervals* Manuscripta math. 133 (2010), 197–224.
- [21] Matthes, D.; McCann, R. J.; Savaré, G. *A family of fourth order equations of gradient flow type*. Comm. Partial Differential Equations 34 (2009), no. 10–12, 1352–1397.
- [22] Nirenberg, L. *On elliptic partial differential equations*. Ann. Scuola Norm. Sup. Pisa 13 (1959) 115-162.
- [23] Otto, F. *The geometry of dissipative evolution equation: the porous medium equation*. Comm. Partial Differential Equations 26 (2001), 101–174.
- [24] Otto, F.; Westdickenberg, M. *Eulerian calculus for the contraction in the Wasserstein distance*. SIAM J. Math. Anal. 37 (2005), 1227–1255.

- [25] Slepcev, D.; Pugh, M. C. *Selfsimilar blowup of unstable thin-film equations*. Indiana Univ. Math. J. **54** (2005), no. 6, 1697–1738.

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