Elements of Geometric Measure Theory on sub-Riemannian groups

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A mia madre,
a cui sarò sempre debitore

Contents

Chapter 1

Introduction

The main purpose of this thesis is to extend methods and results of Geometric Measure Theory to the geometries of sub-Riemannian groups.

A detailed description of all the material of the thesis is given in the introductions to individual chapters and in the more concise overview at the end of this chapter. Here we want to outline how the research on this topic historically grew, trying to reach the more recent developments. In this way, we hope to provide for the reader the mathematical landscape where this thesis should be fitted in and the reasons that motivate this study.

Typical features of “sub-Riemannian structures” historically appeared in several fields of Mathematics. Perhaps, the first seeds can already be found in the 1909 work, [34], by Constantin Carathéodory, on the second principle of Thermodynamics. Here a thermodynamic process can be represented by a curve in $\mathbb{R}^q$ and the heat exchanged during the process by the integral of a suitable one-form θ along the curve. The work of the physicist Joules Carnot led to the existence of two thermodynamic states that cannot be connected by adiabatic processes, namely, curves where θ vanishes at every point. These are the so-called *horizontal curves*, whose velocities belong to the distribution of nullspaces of θ. Carathéodory proved that if there exist two points that cannot be connected by horizontal curves, then θ is integrable. From both this theorem and the result by Carnot, we can conclude that θ is integrable, namely, there exist two functions T and S such that $\theta = T\,dS$. When T and S are interpreted as the Temperature and the Entropy, respectively, the last equation becomes the mathematical formulation of second principle of Thermodynamics. The geometric content of Carathéodory Theorem becomes clearer when it is stated in a different way: if θ is a nonintegrable one-form, then any two points can be connected by an horizontal curve. Here we want to mention that this result was probably already known to Hertz, although without proof. If we consider a frame of vector fields that span the distribution of nullspaces of the nonintegrable one-form θ, then Frobenius Theorem tells us that there exist two vector fields of the distribution whose Lie bracket is not contained in the distribution itself. In this case the distribution of subspaces associated to θ has codimension one, hence the Lie algebra generated by the vector fields at any point has the same dimension of the tangent space. This version of Carathéodory Theorem can be generalized to distributions of any codimension, whose Lie algebra generates the tangent space at every point. The condition on the distributions above is known in Nonholonomic Mechanics, subelliptic PDE's and Optimal Control Theory under different names, as "total nonholonomicity", "Hörmander condition", "bracket generating condition" or "Chow condition". The reason is the following fundamental theorem which extends the previous Carathéodory result: a connected manifold with a bracket generating distribution is connected by curves which are tangent to the distribution itself. This result was independently proved in the 1938-1939 papers by Rashevsky and Chow, [38], [160] and it is the foremost basic result of sub-Riemannian Geometry. A complete characterization of system of vector fields which give a distribution of subspaces such that the manifold is connected by curves tangent to the distribution was proved by Sussmann in the 1973 paper [176].

A distribution of subspaces seen as a fiber bundle will be called *horizontal subbundle* and its tangent curves will be referred to as *horizontal curves*. Throughout the thesis we will frequently use the adjective "horizontal" to indicate objects related to the horizontal subbundle and we will often use the prefix H. The Chow-Rashevsky Theorem allows us to introduce a distance that takes into account the geometry induced by the horizontal subbundle on the manifold. The distance between two points is the infimum of lengths of all horizontal curves connecting them. This is the so-called *Carnot-Carathéodory distance*, as it was named in [87], [154] (CC-distance). The manifold together with its horizontal subbundle is the so-called *Carnot-Carathéodory space* (CC-space). A first general formulation of the method to find geodesics in CC-spaces dates back to the 1973 paper by Hermann, [97]. Here the author notes that the classical formulation of geodesic equations using the Hamilton-Jacobi Theory allows of cometrics that may vanish on a subspace of one-forms. The Riemannian metrics which are dual to these cometrics have formally infinite value on the subspace of directions (the vertical ones) which is dual to that of one-forms. This method amounts to seek geodesics subject to constraints imposed on their directions. This type of question is a feature of CC-geometries which has no counterpart in Riemannian Geometry. It is worth to mention that smoothness of singular geodesics in CC-spaces is still an open issue. In the 1994 paper by Montgomery, [136], it has been shown an example of length minimizer which does not satisfy the geodesic equations. We quote the paper by Liu and Sussmann, [121], and the recent book by Montgomery, [137], for a detailed exposition and further studies on this question.

In the field of PDE's the importance of both the bracket generating condition on the horizontal subbundle and the CC-distance first appeared in the 1967 paper by Hörmander, [99]. He proved the hypoellipticity of the second order degenerate elliptic operator

$$\mathcal{L}_X = -\sum_{i=1}^{m} X_i^2 \,, \tag{1.1}$$

under the condition that the first order operators X_i, which form a frame for the horizontal subbundle, satisfy the crucial bracket generating condition. A surprising relation between CC-distance and hypoellipticity was shown in the 1981 paper by Fefferman and Phong, [56]. Here it is established that an Hölder estimate of the following type

$$\rho(x, y) \le C\,|x-y|^{\varepsilon} \,,$$

where ρ is the CC-distance, is equivalent to the subelliptic estimate

$$\|u\|_{H^{\varepsilon}} \le C\Big(\|u\|^2 + \sum_{i=1}^{m} \|X_i u\|^2\Big) \,,$$

which gives in turn the hypoellipticity. In the previous formula we have denoted by $\|\cdot\|$ the L^2 norm and by $\|\cdot\|_{H^{\varepsilon}}$ the fractional Sobolev norm. Precisely, the previous result was proved for a distance associated with a degenerate elliptic operator. Furthermore, when the degenerate elliptic operator is of type (1.1), its associated distance coincides with the CC-distance.

A large number of important works has appeared in this area, following the points of view of Sobolev Space Theory, Harmonic Analysis, Regularity Theory for PDE's, Spectral Theory, fundamental solutions for subelliptic operators and other aspects. Among these ones, we mention the papers by Bony, [20], Capogna [28], [29], Capogna, Danielli and Garofalo, [30], [32], Citti, Garofalo and Lanconelli, [39], Fabes, Kenig and Serapioni, [54], Folland, [57], [58], Franchi and Lanconelli, [62], [63], Franchi and Serapioni, [68], Garofalo and Lanconelli, [78], Gaveau [81], Métivier, [132], Nagel, Ricci and Stein, [147], [148], Sánchez-Calle [166], Xu and Zuily, [189].

At the same time, the notion of Sobolev space was extended to CC-spaces, requiring that only distributional derivatives along the vector fields of the subbundle are p-summable. This naturally occurred in order to fit the corresponding PDE's theory. The crucial role played by Sobolev inequality and Poincaré inequality in the regularity theory of elliptic PDE's was clear since the celebrated results by De Giorgi, Nash and Moser, [48], [144], [150]. In the context of CC-spaces, the first use of the so-called Moser iteration technique dates back to the 1983 paper by Franchi and Lanconelli, [62], where it was shown that solutions to degenerate second order operators of "Grushin type" are Hölder continuous with respect to the CC-distance induced by the operator itself. A few years later Jerison, [100], generalized the Poincaré inequality for vector fields X_i under the bracket generating condition

$$\int_{U_{x,r}} |w(z) - w_{U_{x,r}}|^2\, dz \le C\, r^2 \int_{U_{x,r}} \sum_{j=1}^{m} |X_j w(z)|^2\, dz. \tag{1.2}$$

Here, the ball $U_{x,r}$ of center x and radius r is considered with respect to the CC-distance, $w \in C^\infty(U_{x,r})$ and $w_{U_{x,r}}$ is the average integral of w on $U_{x,r}$. The proof of this result is first accomplished in the model case of *stratified groups* (CC-spaces with a Lie group structure). Then the Lifting Theorem of Rothschild and Stein, [163], according to which CC-spaces can be seen as submanifolds of suitable stratified groups, allows of the extension to general CC-spaces. Incidentally, this paper arose in connection with the solution to the Yamabe problem on CR manifolds, [101], [102], [103]. Recall that CR structures can be pointwise approximated by the Heisenberg group, which is the simplest paradigm of nonabelian stratified group. We also mention that a new proof of the Poincaré inequality for vector fields has been recently given by Lanconelli and Morbidelli, [118].

These results confirmed the possibility to extend much of the classical theory of Sobolev spaces to the setting of CC-spaces, together with various other connected issues. We mention questions as Sobolev embeddings, isoperimetric inequalities, traces theorems, representation formulae, mapping with finite distortion, quasiconformal mappings, monotone maps and other more. It is really difficult to give an exhaustive account of all literature in this field. We refer the reader to the works by Buckley, Koskela and Lu, [24], Capogna, Danielli and Garofalo, [30], [31], Chernikov and Vodop'yanov, [36], [37], Danielli, Garofalo and Nhieu, [42], [43], Franchi, Gallot and Wheeden, [60], Franchi, Gutiérrez and Wheeden, [61], Franchi, Lu and Wheeden, [66], Franchi Serapioni and Serra Cassano, [69], [70], Garofalo and Nhieu, [79], Greshnov and Vodop'yanov, [82], [83] Heinonen and Holopainen, [94], Korányi and Reimann [113], [114], Lanconelli and Morbidelli, [118], Lu, [122], Marchi, [128], Margulis and Mostow, [129], Monti and Morbidelli, [140], Morbidelli, [142], Pansu, [154], Sawyer and Wheeden, [167], Vodop'yanov [182], [183]. Some of the previous papers consider stratified groups, which form a privileged class of CC-spaces, as we will discuss later.

In the last few years there has been an impressive development of these theories in general metric spaces. The beginning of this research line can be dated to around 1995, with papers by Biroli and Mosco, [17], [18], [19], Hajlasz, [89], and Hailasz and Koskela, [90]. In the paper [89] Sobolev spaces have been defined by means of a generalized Lipschitz condition

$$|u(x) - u(y)| \leq \Big(g(x) + g(y)\Big)\, d(x, y)\,,$$

where u is a Borel map on a metric measure space X and $g \in L^p(X)$. Some years before this paper, in connection with a conjecture by Yau on the space of harmonic functions with polynomial growth in a Riemannian manifold, Grigor'yan and Saloff-Coste independently extended the Yau result, [190], to Riemannian manifolds with a doubling volume measure and where the Poincaré inequality holds, [84], [165]. We recall that a doubling measure μ on a metric space X has the property

$$\mu(B_{x,2r}) \leq C\, \mu(B_{x,r})$$

for some constant $C > 0$ and every ball $B_{x,r}$ of center $x \in X$ and radius $r > 0$. The brief note by Hajlasz and Koskela, [90], shows that a doubling condition on the measure and the validity of Poincaré inequality are enough to obtain the Sobolev inequality in an arbitrary metric measure space. In the abstract setting of Dirichlet forms on metric spaces the same kind of implication was proved in [18], [19]. The abstract approach of considering a doubling measure, the validity of Poincaré inequality and Sobolev inequality in metric spaces was already made in [17].

This is a good occasion to remark that CC-spaces endowed with the Lebesgue measure are doubling, as Nagel, Stein and Wainger proved, [149], and the Poincaré inequality (1.2) holds. We also mention that other types of geometries that have a Poincaré inequality has been found, [21], [22], [116], [172]. Hence it is not surprising that even other settings are suitable to develop the Sobolev Space Theory, as graphs, fractals and metric spaces with Dirichlet forms. Clearly, the metric theory of Sobolev spaces provides a unified picture where these geometries fit in. A survey on these themes and references on the previously mentioned research lines can be found in the monograph by Hajlasz and Koskela [91], where a particular attention is devoted to the case of CC-spaces. Concerning relations between heat equation and Sobolev Theory on groups we quote the book by Varopoulos, Saloff-Coste and Coulhon, [180], where a section is specialized on groups with a CC-structure (stratified groups). Another approach to metric Sobolev spaces is given in the paper by Shanmugalingam, [173]. Concerning functions of bounded variation in metric spaces see [134], by Miranda.

These and other observations have lead several authors to tackle various geometrical questions in a pure metric setting. On the other hand, this approach has also other motivations and it can be seen as a part of modern developments of Anaysis and Geometry; see the works by Ambrosio, [5], Ambrosio and Kirchheim, [7], [8], Assouad, [11], Biroli and Mosco, [17], [18], [19], Cheeger, [35], David and Semmes [45], De Giorgi [49], Franchi, Hajlasz and Koskela, [64], Franchi, Lu and Wheeden, [67], Gromov, [87], [88], Heinonen and Koskela [95], Kirchheim, [110], Kirchheim and Magnani, [111], Korevaar and Schoen, [115], Lang and Schroeder, [119], Preiss and Tisier, [158], Semmes, [170], [171], Weaver, [186], but surely this list could be enlarged. About an overview of metric geometry we mention the recent textbook [26], by D.Burago, Y.Burago and Ivanov. Concerning methods of Analysis in metric spaces we mention the book [93], by Heinonen, where there is an account of several recent results and open questions in this field. New types of geometries with “good calculus properties” are studied in [172], by Semmes.

We have seen that results in CC-geometries also served as a model of inspiration for further generalizations to a pure metric setting. This process can be explained because CC-geometries contain a wider class of metric spaces than the Riemannian one. In fact, a CC-space, also called “nonriemannian space”, is far from being Euclidean from a metric point of view in that the CC-distance is not bilipschitz equivalent to the Euclidean distance in a coordinate chart. This is the standard situation corresponding to a bracket generating horizontal subbundle whose dimension is less than the topological dimension of the manifold. In the case when these dimensions coincide we obtain the well known Riemannian manifold, hence CC-spaces encompass Riemannian manifolds. The study of the geometry of the CC-distance fits into the area of "sub-Riemannian Geometry". One of the leading themes of a thorough paper by Gromov, [86], is the study of the possibility to obtain all information about a CC-space using only its CC-distance. The same paper and the book [88] provide more information on this research stream. These works clearly show how CC-spaces constitute a new terrain for Analysis and Geometry, where classical theories can be extended and developed, keeping only the fundamental principles.

Now it is time to introduce the ambient where our investigations will take place, i.e. *sub-Riemannian groups*. These are stratified groups endowed with a Riemannian metric restricted to the horizontal subbundle. The term "sub-Riemannian" is taken from [175] in order to emphasize the particular metric structure that characterizes these groups, which is strictly related to the horizontal subbundle. To ensure that the horizontal subbundle yields a homogeneous structure compatible with the algebraic structure of the group, we require that translations of the group preserve the distribution of subspaces which forms the horizontal subbundle. Precisely, taking a subspace H of the tangent space $T_e\mathbb{G}$ of the group at the unit element $e \in \mathbb{G}$, one moves it to any point of $\mathbb{G}$ by means of the differential of left translations $l_p x = p \cdot x$ and call the collection of all these subspaces the horizontal subbundle generated by the subspace H. A *stratified group* is a simply connected nilpotent Lie group, whose Lie algebra $\mathcal{G}$ admits a decomposition into the direct sum $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$, where the relations

$$V_{j+1} = [V_j, V_1] \tag{1.3}$$

hold for every $j \in \mathbb{N}$ and $V_k = \{0\}$ whenever $k \geq \iota$. This last condition tells us that the algebra is *nilpotent*. Groups whose Lie algebra is nilpotent are called nilpotent groups and the integer ι is called the *step* of the group. Recall also that if $\mathfrak{a}$ and $\mathfrak{b}$ are subspaces of a Lie algebra, the expression $[\mathfrak{a}, \mathfrak{b}]$ represents the vector space spanned by the Lie brackets $[X, Y]$, where $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$. The canonical choice of the horizontal subbundle is given defining

$$H = \{X(e) \mid X \in V_1\} \subset T_e\mathbb{G} \tag{1.4}$$

The terminology "stratified group" is taken from [59]. In the literature the name "Carnot group" is also used, following the terminology of [154].

As we have previously mentioned, stratified groups form a particular class of CC-spaces. In fact, the horizontal subbundle of the group is spanned by the left invariant vector fields belonging to V_1, therefore relations (1.3) yield the bracket generating condition. In order to obtain a metric structure compatible with the algebraic structure of the group, the scalar product on the horizontal subbundle is taken to be left invariant, hence the associated CC-distance is also left invariant.

Much of the previously quoted works on CC-spaces were exactly on stratified groups, that constitute a simplified model of these spaces. A significant way to see the interconnection between stratified groups and CC-spaces is given by a result of Mitchell, [135], according to which a stratified group can be seen as the tangent cone to a CC-space at some point. Precisely, looking at the couple formed by the CC-space M and one of its points p as a pointed metric space (M, p, d), it is studied the limit of metric spaces $(M, p, \lambda\, d)$, as $\lambda \to \infty$. This enlargement of the distance amounts to dilate the space around the point p. When M is a Riemannian manifold the limit space coincides with the classical tangent space T_pM. In the general case of a CC-space the limit is exactly a stratified group, that generalizes the Euclidean space. Refined versions of this result can be found in [15], [130]. The way of interpreting stratified groups as tangent cones of CC-spaces offers us an enlightening comparison between sub-Riemannian Geometry and Riemannian Geometry: stratified groups are to CC-spaces what Euclidean spaces are to Riemannian manifolds.

Another context where these groups naturally arise is that of infinite discrete groups. An element of a discrete group can have different representations of the form $g = g_1^{i_1} \cdots g_2^{i_k}$ and the length of the representation is the positive integer $\sum_{j=1}^{k} i_j$. The distance $d(g, 1)$ of g from the unit element 1 is the minimum length of all representations of g. One can check that $d(g^{-1}, 1) = d(g, 1)$. As a result, defining $d(g, h) = d(g^{-1}h, 1)$ we obtain a natural left invariant distance on the group and we can consider the number N_r of elements contained in a ball of radius r. If N_r is less than or equal to a function of type Cr^d for some positive numbers C and d, we say that the group has polynomial growth. A result by Bass, Milnor and Wolf, [14], [133], [188] establishes that every discrete finitely generated nilpotent group has polynomial growth. A deep result due to Gromov provides the striking "viceversa" to the previous result, [85], answering a 1968 Milnor conjecture: every discrete finitely generated group with polynomial growth contains a nilpotent subgroup of finite index. It is striking that the only information on the growth of the group yields a nilpotent structure. The geometric idea is to look at the discrete group from infinity, i.e. moving the observation point far away from the group and then obtaining a continuous structure that corresponds to the limit Lie group. This process corresponds to consider the limit of metric spaces $(\Gamma, \varepsilon d)$ as $\varepsilon \to 0^+$, where (Γ, d) is the discrete group with its left invariant distance. We mention that the notion of convergence of metric spaces was introduced by Gromov exactly in connection with this problem, [85]. Further studies on this notion can be found in [87], [88] and [157].

More can be said about the limit space. In fact, it is not only nilpotent, but even stratified and it carries a family of dilations $\delta_r : \mathbb{G} \longrightarrow \mathbb{G}$, see [153]. These maps, also called self-similarities, are one of the most important features of the group. They are compatible with the CC-distance in the sense that $\rho(\delta_r p, \delta_r q) = r\, \rho(p, q)$, where

ρ is the CC-distance, $r > 0$, $p, q \in \mathbb{G}$, and well behave with respect to the group operation, $\delta_r(p \cdot q) = \delta_r p \cdot \delta_r q$. We can always extend the Riemannian metric on the horizontal subbundle to the whole tangent bundle of the group, obtaining a left invariant Riemannian metric. This yields a Riemannian volume measure v_g that is also left invariant and then it coincides with the Haar measure of the group. Dilations scale well with the volume measure, as the formula

$$v_g(\delta_r E) = r^Q v_g(E) \tag{1.5}$$

shows, where E is a measurable subset of $\mathbb{G}$ and Q is the Hausdorff dimension of the group with respect to the CC-distance. The Hausdorff dimension of $\mathbb{G}$ with respect to the CC-distance is related to the dimension of subspaces V_i by the formula

$$Q = \sum_{j=1}^{\iota} j \dim V_j \,. \tag{1.6}$$

Stratified groups are in particular nilpotent and simply connected, hence in view of Theorem 2.3.10 they are linearly isomorphic to a finite dimensional vector space.

All these features could remind us of the familiar Euclidean structure, but as soon as we consider nonabelian groups, in many respects we are dealing with a geometry closer to the fractal one. For instance, formula (1.6) tells us that in the nonabelian case $\iota > 1$ the Hausdorff dimension of the group is always greater than its topological dimension and this is a typical feature of fractal objects. Throughout the thesis we will refer to the non-Euclidean case $\iota > 1$, that contains the new features of these geometries. Nevertheless, all our results hold in particular for Euclidean spaces, which seen as particular stratified groups are recovered in the case $\iota = 1$.

The fractal nature of these groups also appears in other respects. We can have purely unrectifiable sub-Riemannian groups, as it was first shown by Ambrosio and Kirchheim, [7], for Heisenberg groups, that constitute the simplest class of nonabelian sub-Riemannian groups (see Subsection 2.3.1). Recall that the notion of pure unrectifiability can be stated in metric spaces, see 3.2.14 of [55]. A full characterization of all purely unrectifiable sub-Riemannian groups is given in Section 4.4 as an original contribution of the thesis. Also "regular surfaces", in the sense of sub-Riemannian groups, possess a fractal nature, that will be explained later. All these features are certainly a source of difficulties in dealing with these geometries and often prevent us from utilizing the "Euclidean intuition".

Returning to the properties of sub-Riemannian groups, formula (1.5) and homogeneity of dilations with respect to the CC-distance imply that the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ constructed by the CC-distance is finite. Hence, from left invariance of CC-distance we conclude that $\mathcal{H}^Q$ is proportional to the volume measure. The advantage of $\mathcal{H}^Q$ is its "intrinsic nature", which requires only the distance of the group. It is also meaningful the study of surface measures with any codimension, although this study in sub-Riemannian groups for surfaces of codimension greater than one is still to be explored. We will touch on this issue in Section 3.5, giving a novel perspective to be followed in the investigation of these geometries. In fact, much of the recent studies on the geometry of CC-spaces are essentially devoted to surfaces of codimension one.

The rich structure of sub-Riemannian groups, despite their fractal nature, is undoubtedly sufficient to start the study of Geometric Measure Theory on these groups. Classical Analysis tells us the importance of Geometric Measure Theory in connection with Calculus of Variations, elliptic PDE's, isoperimetric inequalities, fine properties of functions and so forth. It is surprising that although a bulk of work has been done both in CC-spaces and stratified groups from different points of view, the study of Geometric Measure Theory in these spaces is still in its very beginning. Only recently some papers have begun this project, as the works by Ambrosio and Magnani, [9], Balogh, [12], Danielli, Garofalo and Nhieu, [43], Franchi, Serapioni and Serra Cassano, [71], [72], [73], Garofalo and Nhieu, [79], Leonardi and Rigot, [120], Magnani, [124], [125], [126], Monti and Serra Cassano, [141], Pauls, [155], [156], Ukhlov and Vodop'yanov, [177], Vodop'yanov, [184] and many others.

We can take these recent works back to some important starting points: the isoperimetric inequality in CC-spaces, BV functions on CC-spaces and an intrinsic notion of differentiability on sub-Riemannian groups. The first isoperimetric inequality proved in a sub-Riemannian context is due to Pansu, [152], for the Heisenberg group, but only recently it was extended to CC-spaces by Capogna, Danielli and Garofalo, [31], Franchi, Gallot and Wheeden, [60], and Garofalo and Nhieu, [79]. Isoperimetric inequality is an important tool in order to obtain the rectifiability of finite perimeter sets, [47]. Recently, this famous result due to De Giorgi has been extended by Franchi, Serapioni and Serra Cassano to the case when the ambient space is the Heisenberg group, or more generally a sub-Riemannian group of step two, [71], [73]. A set of finite perimeter is those set whose characteristic map is a function of bounded variation. This is a well known notion in the Euclidean context. It has been introduced in CC-geometries by Capogna, Danielli and Garofalo in the 1994 paper [31], concerning a general Sobolev embedding on CC-spaces, and it was subsequently studied with various characterizations and applications to Calculus of Variations by Franchi, Serapioni and Serra Cassano, [69]. A locally summable function $u : \Omega \longrightarrow \mathbb{R}$ is of X-*bounded variation* if the supremum of integrals

$$\int_\Omega u(y) \sum_{j=1}^m X_j^* \varphi(y)\, dx \tag{1.7}$$

over all maps $\varphi \in C_c^\infty(\Omega)$ is finite. As usual, vector fields X_j span the horizontal subbundle and satisfy the bracket generating condition. The symbol X_j^* represents the formal adjoint operator. A measurable subset whose characteristic function has bounded X-variation is said to be a set of *finite X-perimeter*. An important aspect related to the notion of X-perimeter measure is that it does not require a fixed Hausdorff dimension of the space. In fact, the Hausdorff dimension of a CC-space can vary on different parts of the space, hence the use of the Hausdorff measure in CC-spaces is meaningless. Other notions of surface measure make sense in CC-spaces, as the Minkowski content that was proved to be equal to the X-perimeter measure, for sufficiently regular domains, by Monti and Serra Cassano, [141]. However, the use of Lebesgue measure in (1.7) is not really intrinsic. A canonical notion of volume measure actually lacks in CC-spaces and only in some special cases a first answer has been given, [137]. This and other reasons make hard a complete development of Geometric Measure Theory in CC-spaces, mainly on the study of surfaces with higher codimension.

Now, it remains the notion of intrinsic differentiability on sub-Riemannian groups. Here we meet the first important theme of the thesis that will be studied in Chapter 3. Differentiability on sub-Riemannian groups was introduced in the 1989 paper by Pansu, [154], where it was used to extend Mostow rigidity, [146]. This notion is intrinsic since it employs the group operation, dilations and a natural family of "linear maps" of the group, called H-*linear maps*. Let $\mathbb{G}$ and $\mathbb{M}$ be two sub-Riemannian groups, $\Omega \subset \mathbb{G}$ be an open subset and $f : \mathbb{G} \longrightarrow \mathbb{M}$. We say that f is H-differentiable at $p \in \Omega$ if there exists an H-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ such that

$$\frac{\rho\left(f(p)^{-1}f(x), L(p^{-1}x)\right)}{d(p,x)} \longrightarrow 0 \quad \text{as} \quad x \to p\,, \tag{1.8}$$

where d and ρ are the CC-distances of $\mathbb{G}$ and $\mathbb{M}$, respectively. The map L is denoted by $d_H f(p)$ and called the H-differential. Recall that an H-linear map is a group homomorphism that is 1-homogeneous with respect to dilations. When $\mathbb{G}$ and $\mathbb{M}$ are Euclidean spaces these definitions give the classical notion of differentiability. Perhaps the core of many ideas that allow us to employ several methods of Geometric Measure Theory in sub-Riemannian groups is the following fundamental result due to Pansu, [154]. A Lipschitz map $f : \Omega \longrightarrow \mathbb{M}$ is H-differentiable $\mathcal{H}^Q$-a.e. on Ω. This is an extension of the classical Rademacher Theorem to sub-Riemannian groups. A manageable version of this theorem in view of applications to Geometric Measure Theory has to encompass the case when the domain of the map f is only measurable. This was the beginning of the author's research on this topic. Due to the lack of Lipschitz extension theorems for maps between sub-Riemannian groups the task of extending the previous result to the case of measurable domains becomes technically nontrivial, [124]. The first paper dealing with this question among others is due to Ukhlov and Vodop'yanov, [177]. We refer the reader to Chapter 3 for the proof of this theorem (Theorem 3.4.11) and more detailed comments on this argument.

Differentiability of Lipschitz maps also allows of the extension of the classical area formula to Lipschitz maps between sub-Riemannian groups,

$$\int_A J_Q(d_H f(x))\; d\mathcal{H}^Q_d(x) = \int_{\mathbb{M}} N(f,A,y)\; d\mathcal{H}^Q_\rho(y)\,, \tag{1.9}$$

provided that a suitable notion of jacobian is used, [124]. The map $N(f,A,y)$ is the multiplicity function (Definition 2.1.11) and $J_Q(d_Hf(x))$ is the H-*jacobian* of $d_Hf(x)$, introduced in Definition 4.2.1. An original contribution of this thesis concerns a general formulation of the area formula in metric spaces, that provides a unified way to obtain the area formula in several contexts. This general approach is developed in Section 4.1. A classical proof of (1.9) will be given in Chapter 4. Area formula also gives an easy way to prove a "rigidity result" for sub-Riemannian groups. This question was already considered by Pansu, [154], and Semmes, [168]. Basically, it is a direct consequence of the a.e. differentiability of Lipschitz maps. Consider two sub-Riemannian groups $\mathbb{G}$ and $\mathbb{M}$ and call them "equivalent" if there exist two measurable subsets $A \subset \mathbb{G}$ and $B \subset \mathbb{M}$ both with positive measure, such that there exists a bilipschitz map $f: A \longrightarrow B$. Then every equivalence class contains only one sub-Riemannian group up to H-linear isomorphisms, see Theorem 4.4.6. Our novel contribution in this result is the extension of the rigidity result to measurable subsets, instead of open subsets, and the simple use of the area formula. As a consequence, a measurable subset of a nonabelian sub-Riemannian group cannot be parametrized by a bilipschitz map on a subset of an Euclidean space. This tells us immediately that nonabelian sub-Riemannian groups are substantially different from Riemannian metric spaces. Furthermore, each of these groups has an own geometry, that is essentially different from that of any other nonisomorphic group. An unpleasant consequence is that it is not possible to adopt "Euclidean methods" directly by means of bilipschitz parametrizations with pieces of Euclidean spaces. The good one is that results valid for sub-Riemannian groups encompass a wide class of different geometries, where the Euclidean one is an example among the others. The above rigidity theorem emphasizes also another aspect which comes up from this thesis. This is the interconnection between metric and algebraic properties of sub-Riemannian groups. There are different situations where this principle occurs. In the terminology of [45], we say that $\mathbb{M}$ looks down on $\mathbb{G}$ if there exists a closed set $A \subset \mathbb{M}$ and a Lipschitz map $f: A \longrightarrow \mathbb{G}$ such that $\mathcal{H}^Q(f(A)) > 0$, where $\mathbb{G}$ and $\mathbb{M}$ are sub-Riemannian groups. This means that $\mathbb{R}^k$ does not look down on $\mathbb{G}$ if and only if $\mathbb{G}$ is purely k-unrectifiable. This observation offers us the possibility to study whether a given sub-Riemannian group looks down on another one using the same technique of Theorem 4.4.4, where purely k-unrectifiable groups are characterized by checking algebraic conditions on the groups and exploiting the area formula. Another situation related to the above mentioned principle occurs when one establishes whether two sub-Riemannian groups are bilipschitz equivalent by checking H-linear maps between the groups, as it is done in Theorem 4.4.6. In Theorem 6.3.4 we have a different case where this principle applies. Here it is proved the nonexistence of nontrivial coarea formulae between different Heisenberg groups, using the fact that every H-linear between these groups cannot be surjective, due to algebraic constraints.

The notion of differentiability on sub-Riemannian groups also provides a natural way to introduce "intrinsic regular surfaces". This concept was first introduced by Franchi, Serapioni and Serra Cassano in [71], [72], [73], in order to obtain a natural notion of rectifiability that fits the geometry of sub-Riemannian groups. A subset $\Sigma \subset \Omega$ is $\mathbb{G}$-regular if there exists an H-differentiable map $f : \Omega \longrightarrow \mathbb{R}$ with continuous differential $p \longrightarrow d_H f(p)$ such that for every $p \in \Sigma$ the H-linear map $d_H f(p)$ is nonvanishing. In the case when $\mathbb{G}$ is an Euclidean space the above definition yields classical C^1 regular submanifolds. But things can change tremendously as soon as the group is nonabelian. In fact, it seems that it is possible to construct an example of $\mathbb{H}^3$-regular surface with Hausdorff dimension $5/2$ with respect to the Euclidean distance, where $\mathbb{H}^3$ is the three dimensional Heisenberg group, [112]. This interesting surface cannot be 2-rectifiable in the sense of 3.2.14 of [55], even though in view of an Implicit Function Theorem proved in [72], its topological dimension is still two. Hence $\mathbb{G}$-regularity is clearly an intrinsic notion, since what is regular with eyes of the sub-Riemannian groups has a definitive fractal nature from the Euclidean viewpoint. This is another confirmation of the fact that the study of intrinsic regular surfaces has to be accomplished employing more general tools and methods. The notion of $\mathbb{G}$-regularity can be extended to subsets of higher codimension and modeled on the geometry of another sub-Riemannian group $\mathbb{M}$. This is precisely explained by the notion of $(\mathbb{G}, \mathbb{M})$-*regularity*, a novel notion introduced in the thesis, which will be discussed in Section 3.5. This study opens many new questions together with a good perspective to introduce a theory of currents according to the geometry of sub-Riemannian groups.

The first motivation for the above notions of $\mathbb{G}$-regular surfaces is the validity of the De Giorgi Rectifiability Theorem on the class of sub-Riemannian groups of step two, [71], [73], as we will explain later. We point out that the definition of functions with bounded X-variation can be specialized to sub-Riemannian groups adopting only the left invariant Riemannian metric restricted to the horizontal subbundle (Definition 2.4.3). In this way we obtain a notion independent from the choice of vector fields utilized in (1.7). Following the general terminology adopted in this thesis, we will speak of *functions* of H-*bounded variation* and of *sets* of H-*finite perimeter* on sub-Riemannian groups. Sets of H-finite perimeter also naturally possess the notion of H-*reduced boundary*, along with the Euclidean notion (Definition 2.4.10). The recent version of the Rectifiability Theorem proved by Franchi, Serapioni and Serra Cassano establishes that the H-reduced boundary of an H-finite perimeter set in a step two sub-Riemannian group is a countable union of $\mathbb{G}$-regular surfaces up to $\mathcal{H}^{Q-1}$-negligible sets, where Q is the Hausdorff dimension of the group. The validity of this result for groups of higher step is an open problem and already in step three a counterexample to the classical method is possible, [73].

One of the crucial points in the Rectifiability Theorem is the blow-up method of enlarging the subset around a point up to obtain its "generalized" tangent space. This method was introduced by De Giorgi, [47]. Here we meet the second important theme of the thesis, e.g. the “blow-up principle”. The blow-up technique applied by rescaling a Lipschitz map $f : A \longrightarrow \mathbb{M}$ on its differentiability points yields a general coarea inequality

$$\int_{\mathbb{M}} \mathcal{H}^{Q-P}\left(A \cap f^{-1}(\xi)\right) d\mathcal{H}^P(\xi) \leq \int_A C_P(d_H f(\xi))\, d\mathcal{H}^Q(\xi)\,, \tag{1.10}$$

proved in [125]. The symbol $C_P(d_H f(\xi))$ denotes the H-*coarea factor* of $d_H f(\xi)$, according to Definition 6.1.3. This formula will be proved in Section 6.2. An important application is the $\mathcal{H}^{Q-1}$-negligibility of characteristic points of C^1 hypersurfaces, proved in Theorem 6.6.2. This completes and extends some previous results in the literature, [12], [73] and it is one of the foremost contributions of this thesis. The blow-up technique which relies on the rectifiability of the perimeter measure stated in Theorem 6.4.7 yields the coarea formula for real-valued Lipschitz maps $u : \mathbb{G} \longrightarrow \mathbb{R}$,

$$\int_{\mathbb{G}} h(w)\, |\nabla_H u|(w)\, dv_g(w) = \int_{\mathbb{R}} \int_{u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_{E_t}(w))}{\omega_{Q-1}}\, h(w)\, d\mathcal{S}^{Q-1}(w)\, dt\,,$$

where $\mathcal{S}^{Q-1}$ is the spherical Hausdorff measure, θ^g_{Q-1} is the metric factor first introduced in [126] and studied in Chapter 5, and ν_{E_t} is the generalized inward normal to the set $E_t = \{x \in \mathbb{G} \mid u(x) > t\}$, defined in Definition 2.4.9. In Section 6.5 we give a novel proof of this formula, which was first obtained for the Heisenberg group in [125], by means of the coarea inequality (1.10).

If we consider C^1 sets instead of general sets with H-finite perimeter the Blow-up Theorem is possible in every sub-Riemannian group, see Theorem 7.4.2. This basic observation and the results on characteristic points above mentioned give the following representation of the perimeter measure

$$|\partial E|_H = \frac{\theta^g_{Q-1}(\nu_H)}{\omega_{Q-1}}\, \mathcal{S}^{Q-1} \llcorner \partial E\,. \tag{1.11}$$

This last formula is another original result of the thesis and it generalizes the previous one in the Heisenberg group [71]. Its main consequence is the answer to a conjecture raised by Danielli, Garofalo and Nhieu in [42]. These facts are discussed and proved in Chapter 7.

It is well known that BV functions on Euclidean spaces are a.e. approximately differentiable. This result can be extended to H-BV functions defined on open subsets of sub-Riemannian groups. In Chapter 8 higher order approximate differentiability is also proved for functions of H-bounded higher order variation, see [9].

In conclusion, although sub-Riemannian groups have an homogeneous structure given by dilations, a notion of differentiability, a precise Hausdorff dimension, intrinsic regular surfaces and so forth, still many questions are to be answered. We mention for instance the validity of a general coarea formula, corresponding to the equality in

(1.10), a characterization of $\mathbb{G}$-regular surfaces in terms of Lipschitz parametrizations from suitable subgroups and an intrinsic theory of currents. The understanding of these and other questions on the class of sub-Riemannian geometries is undoubtedly a good starting point in order to grasp what are the fundamental principles in some of the already known theories of Geometric Measure Theory. As a result, this path is also useful within the project of extending Analysis and Geometry in metric spaces.

References

Part of the materials used for this introduction is taken from [16] by Berger, [91] by Hajlasz and Koskela, [137] by Montgomery and [181] by Vershik and Gershkovich. In these works further references on the above mentioned topics can be found.

1.1 A concise overview of the thesis

Chapter 2 is devoted to the main notions utilized throughout the thesis, with an essentially self-contained exposition. After a brief introduction to some general concepts on metric spaces, our study specializes in CC-spaces and finally in sub-Riemannian groups. In particular, Theorem 2.2.24 shows that the distance associated to a sub-elliptic operator by subunit curves equals the one associated to a horizontal subbundle by horizontal curves. We introduce graded coordinates (Definition 2.3.43), that represent an important tool in many proofs of the thesis. We give a novel presentation of H-BV functions on sub-Riemannian showing that the associated variational measure only depends on the left invariant Riemannian metric restricted to the horizontal subbundle. Moreover, once a system of graded coordinates is fixed this definition coincides with the already known notion in the literature, where the Lebesgue measure is commonly used. This definition has been first used in [9].

Chapter 3 extends several tools of classical Calculus to sub-Riemannian groups. After a detailed description of H-linear maps, the notion of H-differentiability is introduced and the chain rule on sub-Riemannian groups is proved. A section is devoted to the proof of the Inverse Mapping Theorem for H-differentiable maps. This is our first novel application of H-differentiability to "sub-Riemannian Calculus". The main result of the chapter is the a.e. H-differentiability of Lipschitz maps defined on measurable subsets of sub-Riemannian groups. This part is taken from a recent paper of the author, [124]. A section of the chapter is devoted to a brief survey of all recent notions of rectifiability on sub-Riemannian groups, presenting and discussing the novel and general notion of $(\mathbb{G}, \mathbb{M})$-rectifiability. In the last part of the chapter we present a counterexample to the H-differentiability of Lipschitz maps as soon as we replace a homogeneous distance of the target with a left invariant distance which is not homogeneous. This example was obtained in collaboration with Bernd Kirchheim and it is essentially taken from [111].

Chapter 4 deals with the area formula in different contexts. An original contribution of the thesis is the proof of the area formula for Lipschitz maps in a purely metric setting, once a suitable notion of "metric jacobian" is adopted. Besides a unified approach to area formula in several spaces, as Riemannian manifolds and stratified groups, this result also emphasizes the key role played by the notion of jacobian. Subsequently we introduce the H-jacobian for H-linear maps and we present two proofs of the area formula for Lipschitz maps between sub-Riemannian groups. The first one is derived from the general metric area formula. The second one, of more classical fashion, utilizes the H-jacobian. The notion of H-jacobian was first introduced in [124] and it was inspired by the metric definition of [7]. The second proof of the area formula is also taken from [124]. Finally, we present two new applications of the sub-Riemannian area formula. We characterize all purely k-unrectifiable sub-Riemannian groups for every $k \geq 1$ and we prove the following rigidity theorem. Let $\mathbb{G}$ and $\mathbb{M}$ be sub-Riemannian groups with two subsets $A \subset \mathbb{G}$ and $B \subset \mathbb{M}$ with positive measure such that there exists a bilipschitz map $f : A \longrightarrow B$. Then $\mathbb{G}$ and $\mathbb{M}$ are isomorphic.

Chapter 5 presents the notion of isometry in sub-Riemannian groups and the class of sub-Riemannian groups that are "symmetric" with respect to these maps. An horizontal isometry $T : \mathbb{G} \longrightarrow \mathbb{G}$ must respect both the metric structure and the algebraic structure of the group, i.e. it is both an H-linear map and an isometry with respect to the sub-Riemannian metrics of the groups. A group which has a family $\mathcal{R}$ of horizontal isometries that acts transitively on vertical hyperplanes of the group is said to be $\mathcal{R}$-invariant (see Definition 5.1.4). A metric notion associated to a homogeneous distance (Definition 2.3.35) is that of metric factor $\theta_{Q-1}(\nu)$, where ν is a direction of the Lie algebra. This function plays the same role of ω_{n-1} introduced in (2.5) about the representation of the Euclidean Hausdorff measure $\mathcal{H}^{n-1}$ in $\mathbb{R}^n$. The dependence of θ_{Q-1} on the direction ν takes into account the anisotropy of the homogeneous distance. We prove that $\mathcal{R}$-invariant groups possess a constant metric factor. All this notions have been first introduced in [126] in connection with the representation of the Q-1 dimensional spherical Hausdorff measure of hypersurfaces with respect to an arbitrary homogeneous distance.

Chapter 6 contains various coarea formulae on sub-Riemannian groups together with some applications. We first prove a general coarea inequality for Lipschitz maps between sub-Riemannian groups (6.1). A first application is a Sard-type theorem for Lipschitz maps of sub-Riemannian groups (Theorem 6.3.1) and the nonexistence of nontrivial coarea formulae between different Heisenberg groups (Theorem 6.3.4). These results are taken from [125]. We prove a general representation formula for the perimeter measure (6.31) on generating groups (Definition 6.4.8). As a consequence, we obtain the coarea formula for real-valued Lipschitz maps on generating groups (6.42). The technique used for the representation formula of the perimeter measure is a refined version of that used in [125] in the case of the Heisenberg group. The method to prove the coarea formula differs from the one used in [125], based on the coarea inequality. We adopt a new and simpler proof relying on Theorem 6.5.1. Here it is proved that the H-reduced boundary (Definition 2.4.10) of a.e. upper level set of a Lipschitz map coincides with the corresponding level set up to an $\mathcal{H}^{Q-1}$-negligible set and the generalized inward normal of upper level sets is proportional to the horizontal gradient of the map restricted to the level set. In the end, we prove that the characteristic set of a C^1 hypersurface (Definition 2.2.8) has $\mathcal{H}^{Q-1}$-negligible measure (Theorem 6.6.2). The proof of this fact relies on the Sard-type Theorem and it is a new contribution of the thesis that extends previous results relative to the case of two step groups, [12], [73].

Chapter 7 analyzes the blow-up procedure in two main cases relative to C^1 subsets. In the first one, it is studied the limit of the measure of a dilated and rescaled C^1 hypersurface around one of its noncharacteristic points (7.11). The expression of the limit contains the metric factor studied in Chapter 5 and yields relations between the Q-1 dimensional spherical Hausdorff measure and the Riemannian surface measure of the hypersurface (7.16), (7.17). The validity of these formulae for C^1 hypersurfaces with $\mathcal{H}^{Q-1}$-negligible characteristic set was already proved in [126]. Due to the $\mathcal{H}^{Q-1}$-negligibility of characteristic points, proved in Chapter 6, these formulae always hold without any additional assumption. The same formulae are also used to prove the coarea formula for real-valued Lipschitz maps with respect to the Riemannian distance of the group (7.19). The validity of this formula in every sub-Riemannian group is due to the assumption of the Lipschitz property with respect to the Riemannian distance. This is a stronger request than the natural one of considering the Lipschitz property with respect to the CC-distance. The same blow-up technique applied at characteristic points of $C^{1,1}$ surfaces of two step groups yields an estimate of the $\mathcal{S}^{Q-2}$-measure of the characteristic set. As a consequence, we obtain a sharp upper estimate of the Hausdorff dimension of the characteristic set (7.35). This result is taken from [126]. In the second case the blow-up technique is applied to subsets with C^1 boundary, i.e. C^1 subsets, obtaining an explicit formula for the perimeter measure of these sets in terms of the Q-1 dimensional spherical Hausdorff measure, (7.51). This formula was first obtained in [71] in the case of the Heisenberg group. The validity of (7.51) in any sub-Riemannian group was an open question raised in [72] and [73]. Its proof is another contribution of this thesis. The same formula immediately yields a reciprocal estimate between the perimeter measure of a C^1 subset and the Q-1 dimensional Hausdorff measure of its boundary (7.52). The general validity of this formula was conjectured in [42]. Finally, some intrinsic divergence theorems for C^1 subsets are proved, (7.54), (7.55), (7.56).

Chapter 8 is devoted to the study of approximate differentiability of H-BV functions on sub-Riemannian groups. Its content essentially stems from a recent collaboration with Luigi Ambrosio, [9]. The concept of approximate differentiability easily extends from Euclidean spaces to sub-Riemannian groups, considering the corresponding notion of H-differentiability. We prove that an H-BV functions is $\mathcal{H}^Q$-a.e. approximately differentiable and that the H-differential is given by the density of the absolutely continuous part of the vector measure associated to the H-BV map. Part of the chapter deals with the structure of the approximately discontinuity set of an H-BV function. We prove that this set is a countable union of essential boundaries (Definition 2.1.16) of sets with H-finite perimeter. Thus, whenever one is able to prove that these boundaries are $\mathbb{G}$-rectifiable, the same property holds for the approximate discontinuity set. Actually, this rectifiability result is true in all two step sub-Riemannian groups, [73]. A section of the chapter recalls the representation formula on sub-Riemannian groups (8.19). Our proof of this formula is taken from [66], where the general case of spaces of homogeneous type is considered. This formula is an important tool in order to obtain higher order differentiability. In Theorem 8.5.7 we prove that functions with H-bounded k-variation are $\mathcal{H}^Q$-a.e. k-approximately differentiable. The case $k = 2$ fits into a weak version of an Alexandrov type differentiability on sub-Riemannian groups. In the last section we present some nontrivial example of functions with H-bounded 2-variation, arising from inf-convolution of a suitable cost function.

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Chapter 2

Main notions

In this chapter we present a self-contained exposition of all basic materials we will use throughout the thesis. In this way we provide also for the reader that is not familiar with these notions by giving all necessary information required to enter safely into the topic of the thesis. In order to clarify the generality of several notions, we have divided the chapter into different sections that go from general metric spaces to the richer structure of sub-Riemannian groups. Next, we present a brief overview of the chapter.

In Section 2.1 we recall some elementary facts about measures in metric spaces. We show a simple change of variable formula for Borel maps and we introduce the general notion of doubling space. We present a standard covering theorem and an estimate between measures by means of their reciprocal spherical density.

In Section 2.2 we present the so called Carnot-Carathéodory spaces, in short CC-spaces. After some basic definitions of Differential Geometry we introduce the notions of horizontal curve, horizontal gradient, horizontal vector field and characteristic point, which come directly from the geometry induced by the "horizontal subbundle". We state the important theorem of Chow-Rashevsky, which says that connected manifolds, where horizontal vector fields and their iterated commutators generate the tangent bundle, are H-connected. Finally, in Subsection 2.2.1 we introduce sub-Riemannian metrics on a CC-space, obtaining the notion of "sub-Riemannian manifold". We define the Carnot-Carathéodory distance, in short CC-distance, and we provide some characterizations that connect different notions used in the literature.

In Section 2.3 we recall some general facts on nilpotent groups. A particular attention is devoted to the Heisenberg group, that is the simplest nonabelian sub-Riemannian group and represents a precious source of manageable examples. Subsequently, we present the class of nilpotent groups that constitute the privileged ambient on which we can extend most of the classical Geometric Measure Theory, namely "sub-Riemannian groups". The connection between sub-Riemannian manifolds and sub-Riemannian groups is given by the following result: the "tangent space" to a sub-Riemannian manifold is a sub-Riemannian group. Here the notion of tangent space has to be considered in appropriate way, e.g. it is the limit of a sequence formed by pointed metric spaces that correspond to the sub-Riemannian manifold with dilated distance at the given point, see [15], [130], [135]. Sub-Riemannian groups can be regarded as stratified groups with a left invariant metric. We will always consider the class of left invariant metrics that respect the grading, namely "graded metrics". The assumption on the stratification guarantees that a sub-Riemannian group is a particular example of CC-space. Via the graded metric we have a natural CC-distance, that turns out to be a homogeneous distance (Proposition 2.3.39). Then any sub-Riemannian group has a privileged homogeneous distance, that is also "geodesic" in the sense that the infimum of all lengths of rectifiable curves that connect two points is equal to their distance. Another important subsection is devoted to "graded coordinates". Throughout the thesis we will see their important role in the proof of many theorems. Basically, they can be thought of as privileged charts to look at the group, where several objects introduced in the abstract group $\mathbb{G}$ can be translated into $\mathbb{R}^q$ with manageable computations. By means of graded coordinates we can define polynomials on groups with an intrinsic notion of polynomial degree. Homogeneous polynomials will be useful to obtain an explicit formula for left invariant vector fields when translated into $\mathbb{R}^q$ via graded coordinates, (2.42).

In Section 2.4 we present functions of bounded variation in sub-Riemannian groups, namely H-BV functions. This notion can be stated in the general framework of Carnot-Carathéodory spaces, [31], [69], and metric spaces, [134]. In our presentation we use the horizontal divergence and the Riemannian volume, so one easily recognizes that the variational measure associated to an H-BV function depends only on the graded metric fixed on the group. We point out that the horizontal divergence (Definition 2.4.1) is a differential operator independent of the graded metric. However, graded metrics have the following compatibility: the horizontal divergence is equal to the Riemannian divergence when the last one is referred to a graded metric, (Proposition 2.4.7). It turns out that the Riemannian divergence with respect to a graded metric depends only on the horizontal subbundle and it is indeed independent of the choice of the graded metric itself. This phenomenon occurs analogously in Euclidean spaces with the canonical associated metric. An H-BV function that is the characteristic map of some measurable subset yields a set of H-finite perimeter. We introduce this class of subsets and the related concepts of generalized inward normal and of H-reduced boundary. Due to a general result of L. Ambrosio, [5], the H-reduced boundary is the set where the perimeter measure is concentrated.

In Section 2.5 we state some important results, as the coarea formula for H-BV functions, the Poincaré inequality and the isoperimetric estimate. All these known facts hold in general CC-spaces. Throughout the thesis these results will be applied to sub-Riemannian groups.

2.1 Some facts in metric spaces

In this section (X, d) and (Y, ρ) will denote two metric spaces. The set of extended real numbers will be denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ and the family of subsets of X by $\mathcal{P}(X)$.

Definition 2.1.1 A nonnegative function $\mu : \mathcal{P}(X) \longrightarrow \overline{\mathbb{R}}$ is a *measure* over X if for any sequence of subsets $(E_j) \subset \mathcal{P}(X)$ and any $E \in \mathcal{P}(X)$ such that $E \subset \bigcup E_j$ we have

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(E_j)\,.$$

It is well known that there exists a σ-algebra of μ-measurable sets $\mathcal{A}_\mu(X) \subset \mathcal{P}(X)$ where the measure μ is countably additive.

Definition 2.1.2 (Borel measures) We denote by $\mathcal{B}(X)$ the smallest σ-algebra containing all the open sets of X. Elements of $\mathcal{B}(X)$ are called *Borel sets*. A measure μ on X is called a *Borel measure* if $\mathcal{B}(X) \subset \mathcal{A}_\mu(X)$. A Borel measure μ such that for every $A \subset X$ there exists $B \in \mathcal{B}(X)$ with $A \subset B$ and $\mu(A) = \mu(B)$, is said to be a *Borel regular measure*.

Definition 2.1.3 Let μ be a measure on X and N be a topological space. A map $f : X \longrightarrow N$ is *measurable* if for any open subset $O \subset N$ we have $f^{-1}(O) \in \mathcal{A}_\mu(X)$. We say that f is *Borel map* if $f^{-1}(O) \in \mathcal{B}(X)$.

Remark 2.1.4 It is not difficult to recognize that if $f : X \longrightarrow Y$ is Borel, then $f^{-1}(E) \in \mathcal{B}(X)$ whenever $E \in \mathcal{B}(Y)$. It follows easily that compositions of Borel maps are still Borel.

We recall that a measurable map $F : X \longrightarrow N$, where N is either $\overline{\mathbb{R}}$ or a normed space, is called *p-summable*, with $p \geq 1$, if we have

$$\int_X \|F(x)\|^p \, d\mu(x) < +\infty\,.$$

If $p = 1$ we simply say that F is summable. The space of all p-summable maps is denoted by $L^p_\mu(X, N)$, sometimes we will omit either the symbols N or μ when $N = \overline{\mathbb{R}}$ or μ is the Haar measure of the locally compact group X.

Definition 2.1.5 Let μ be a measure over X. The *image measure* of μ under the map $F : X \longrightarrow Y$ is defined as follows

$$F_\sharp\mu(A) = \mu(F^{-1}(A)) \quad \text{for any } A \subset Y\,.$$

By previous definitions we can prove the following theorem.

Theorem 2.1.6 *Let $F : X \longrightarrow Y$ and $u : Y \longrightarrow N$ be Borel maps, μ be a Borel measure over X and $u \circ F$ be either μ-summable or nonnegative. Assume that N is either $\overline{\mathbb{R}}$ or a finite dimensional space. Then for any $B \in \mathcal{B}(X)$ we have*

$$\int_{F^{-1}(B)} u \circ F \, d\mu = \int_B u \, dF_\sharp \mu \,. \tag{2.1}$$

Proof. First of all, we note that $u \circ F$ is a Borel map and that the image measure $F_\sharp \mu$ is a Borel measure over Y. The latter assertion follows by the Carathéodory's criterion (see for instance 2.3.2(9) of [55]). Now, following a standard argument, we check formula (2.1) on the class of finite linear combinations of characteristic maps of Borel sets. Thus, considering the decomposition $u = u^+ - u^-$, where $u^+, u^- \geq 0$ and approximating u^+ and u^- with maps of this class our claim follows by the Beppo Levi Monotone Convergence Theorem. □

Definition 2.1.7 Let μ be a measure on X and let $f : X \longrightarrow Y$ be a μ-summable map, where Y is either $\overline{\mathbb{R}}$ or a finite dimensional space. We denote by $f\,\mu$ the measure, or vector measure, defined on any set $A \in \mathcal{A}_\mu(X)$ as follows

$$f\,\mu(A) = \int_A f \, d\mu \,.$$

Notice that up to this point we have used only the topology of X, without referring to the distance.

Definition 2.1.8 (Metric ball) We denote by $B_{x,r} = \{y \in X \mid d(y,x) < r\}$ the open ball with center x and radius r and we simply write $B_r = B_{e,r}$, if some particular element e of the space is understood. We will also write $B^d_{x,r}$ to emphasize the distance. For the closed ball $D_{x,r} = \{y \in X \mid d(y,x) \leq r\}$ of center x and radius r we follow the same conventions adopted for open balls.

Definition 2.1.9 (Lipschitz functions) Let $f : X \longrightarrow Y$ be a map of metric spaces. We say that f is *L-Lipschitz* and if there exists a constant $L \geq 0$ such that

$$\rho(f(u), f(v)) \leq L\, d(u,v) \qquad \text{for any } u, v \in X \,.$$

The number L is a *Lipschitz constant* of f and $\mathrm{Lip}(f)$ is the infimum among all Lipschitz constants of f.

Definition 2.1.10 (Rectifiable curves) Let I be an interval of $\mathbb{R}$. We say that a curve $\gamma : I \longrightarrow X$ is rectifiable if the following number is finite

$$l_d(\gamma) = \sup\Big\{ \sum_{j=1}^n d\left(\gamma(t_{i-1}), \gamma\left(t_i\right)\right) \Big| \text{where } t_{i-1} < t_i \text{ for any } i = 1, \dots, n \text{ and } n \in \mathbb{N} \Big\}.$$

The number $l_d(\gamma)$ is the *length* of γ with respect to the distance d of X.

Definition 2.1.11 (Multiplicity function) Let $f : A \subset X \longrightarrow Y$ and $B \subset A$. We define the *multiplicity function* of f relatively to B as $y \longrightarrow N(f, B, y) = \#(\{f^{-1}(y) \cap B\}) \in \mathbb{N} \cup \{+\infty\}$, where $\#$ indicates the cardinality of the set.

In the sequel we denote a metric space with a measure by the triplet (X, d, μ) and call it a *metric measure space*.

Definition 2.1.12 (Doubling spaces) Let (X, d, μ) be a metric measure space. We say that μ is *doubling* if it is finite and positive on some open set and there exists a constant $C > 0$ such that for any ball $B_{x,2r} \subset X$ we have

$$\mu(B_{x,2r}) \leq C\, \mu(B_{x,r}). \tag{2.2}$$

In this case we say that (X, d, μ) is a *doubling space*.

Remark 2.1.13 Notice that if μ is positive and finite on some open set, the doubling property (2.2) implies that it is finite on bounded sets and positive on all open sets of X. Furthermore, it is standard to notice that iterating (2.2) one obtains constant $C', s > 0$ such that

$$\mu(B_{x,tr}) \leq C'\, t^s\, \mu(B_{x,r}) \tag{2.3}$$

for any $x \in \mathbb{G}$, $r > 0$ and $t > 1$.

Throughout the thesis we will follow the standard convention to denote the averaged integral

$$\fint_E u\, d\mu = \frac{1}{\mu(E)} \int_E u\, d\mu\,,$$

where $E \subset X$ is μ-measurable and $u : E \longrightarrow \overline{\mathbb{R}}$ is either a μ-summable or nonnegative measurable map.

Definition 2.1.14 (Density points) Let (X, d, μ) be a metric measure space and consider a μ-measurable set $A \subset X$. We define $\mathcal{I}(A)$ as the set of points $x \in X$ such that

$$\fint_{B_{x,r}} \mathbf{1}_A\, d\mu \longrightarrow 1 \quad \text{as} \quad r \to 0^+\,.$$

We call every element of $\mathcal{I}(A)$ a *density point*.

Note that in a doubling space μ-measurable sets have the property $\mu\,(A \setminus \mathcal{I}(A)) = 0$. This follows by Theorem 2.1.22 stated in this section and Theorem 2.9.8 of [55].

Lemma 2.1.15 *Let (X, d, μ) be a doubling space and $A \subset X$. Then for any $x \in \mathcal{I}(A)$ we have* $\operatorname{dist}(y, A) = o\,(d(y, x))$ *as $y \to x$.*

PROOF. We define $t_y = \mathrm{dist}(y, A)$. If $t_y > 0$ we have

$$B_{y,t_y} \subset B_{x,t_y+d(x,y)} \setminus A$$

and the property $x \in \mathcal{I}(A)$ together with (2.3) yield

$$\frac{1}{C'}\left(\frac{t_y}{t_y + d(x,y)}\right)^s = \frac{1}{C'\,\mu(B_{y,t_y})}\left(\frac{t_y}{t_y + d(x,y)}\right)^s \mu(B_{y,t_y}) \le \frac{\mu(B_{y,t_y})}{\mu(B_{x,t_y+d(x,y)})}$$
$$\le \frac{\mu(B_{x,t_y+d(x,y)} \setminus A)}{\mu(B_{x,t_y+d(x,y)})} \longrightarrow 0^+ \quad \text{as} \quad r \to 0^+. \quad \square$$

The notion of density point allows us to introduce the measure theoretic boundary of a set in a metric measure space.

Definition 2.1.16 (Essential boundary) Let (X, d, μ) be a metric measure space and let $E \subset X$. The *essential boundary* of E is the set

$$\partial^* E = \{p \in X \mid \ p \text{ is a density point neither of } E \text{ nor of } X \setminus E\}\ .$$

We use the following notation to indicate the diameter of a set A in a metric space

$$\mathrm{diam}(A) = \sup_{x,y \in A} d(x,y)\,.$$

Now we recall the Carathéodory's construction (see [55] for the general definition).

Definition 2.1.17 (Carathéodory measure) Let (X, d) be a metric space and let $\mathcal{F}$ be a family of subsets of X. We fix $a \ge 0$ and define for every $t > 0$ the measures

$$\Phi_t^a(E) = \beta_a \inf\left\{\sum_{i=1}^{\infty} \mathrm{diam}(D_i)^a \mid E \subset \bigcup_{i=1}^{\infty} D_i,\ \mathrm{diam}(D_i) \le t,\ D_i \in \mathcal{F}\right\}\ ,$$
$$\Phi^a(E) = \lim_{t \to 0} \Phi_t^a(E)\,,$$

with $E \subset X$ and $\beta_a > 0$. We assume that the family $\mathcal{F}$ has the following property

$$\Theta_a^{-1}\,\mathcal{H}^a \le \Phi^a \le \Theta_a\,\mathcal{H}^a\,, \tag{2.4}$$

where $\Theta_a > 0$ and $\mathcal{H}^a$ is the Hausdorff measure built with $\mathcal{F} = \mathcal{P}(X)$, $\beta_a = \omega_a/2^a$,

$$\omega_a = \frac{\pi^{a/2}}{\Gamma(1 + a/2)} \quad \text{and} \quad \Gamma(s) = \int_0^{\infty} r^{s-1} e^{-r}\, dr\,. \tag{2.5}$$

For instance, if $\mathcal{F}$ is the family of closed (or open) balls and $\beta_a = \omega_a/2^a$, the corresponding measure Φ^a satisfies the latter estimate with $\Theta_a = 2^a$. Indeed, in this case Φ^a is the well known *spherical Hausdorff measure*, denoted by $\mathcal{S}^a$. Sometimes we will also write both $\mathcal{H}^a_d$ or $\mathcal{S}^a_d$ to emphasize the dependence on the distance d we have used to build the measure.

Definition 2.1.18 (Hausdorff dimension) Let E be a subset of a metric space (X, d). We define the *Hausdorff dimension* of E as the following number

$$\mathcal{H}\text{–}\dim(E) = \inf \{\alpha > 0 \mid \mathcal{H}^\alpha(E) = 0\} \,.$$

Now we state an important coarea estimate that holds for Lipschitz maps between metric spaces. In fact, after a work of Davies [46], the assumptions in paragraph 2.10.25 of [55] can be removed.

Theorem 2.1.19 (Coarea estimate) *Let $f : X \longrightarrow Y$ be a Lipschitz map of metric spaces and consider $A \subset X$, with $0 \le P \le Q$. Then the following estimate holds*

$$\int_Y^* \mathcal{H}^{Q-P}\left(A \cap f^{-1}(\xi)\right)\, d\mathcal{H}^P(\xi) \le Lip(f) \frac{\omega_{Q-P}\,\omega_P}{\omega_Q}\, \mathcal{H}^Q(A)\,. \tag{2.6}$$

The symbol $\int^*$ denotes the upper integral (see for instance [55]). We can easily transform (2.6) using our measures Φ^a from Definition 2.1.17, obtaining

$$\int_Y^* \Phi^{Q-P}\left(A \cap f^{-1}(\xi)\right)\, d\Phi^P(\xi) \le \mathrm{Lip}(f)\, \frac{\omega_P\,\omega_{Q-P}}{\omega_Q}\, \Theta_{Q-P}\,\Theta_P\,\Theta_Q\,\Phi^Q(A)\,. \tag{2.7}$$

The following definition is taken from 2.8.16 of [55].

Definition 2.1.20 (Vitali relation) Let μ be a measure on a metric space (X, d). We say that a family of Borel sets $\mathcal{V} \subset \mathcal{P}(X)$ is a μ-*Vitali relation* if for any $\mathcal{C} \subset \mathcal{V}$ and $A \subset X$ such that for any $x \in A$

$$\inf \{\operatorname{diam}(S) \mid S \in \mathcal{C},\, x \in S\} = 0\,,$$

then the family $\{S \mid S \in \mathcal{C},\, x \in S,\, x \in A\}$ has a countable disjoint subfamily $\mathcal{F}$ such that

$$\mu\Big(A \setminus \bigcup_{S \in \mathcal{F}} S\Big) = 0\,.$$

Definition 2.1.21 (Asymptotically doubling measures) Let μ be a Borel measure, that is finite on bounded sets of X. We say that μ is *asymptotically doubling* on X if for μ-a.e. $p \in X$ we have

$$\limsup_{r \to 0^+} \frac{\mu(B_{p,\lambda_0 r})}{\mu(B_{p,r})} < +\infty$$

for some $\lambda_0 > 1$ (and thus for any $\lambda > 1$).

In view of Theorem 2.8.17 of [55] we state the following result.

Theorem 2.1.22 *Let μ be an asymptotically doubling measure on X, which is finite on bounded sets and such that*

$$\limsup_{r\to 0^+} \frac{\mu(B_{x,\lambda r})}{\mu(B_{x,r})} < +\infty$$

for some $\lambda > 1$ and μ-a.e. $x \in X$. Then closed balls of X form a μ-Vitali relation.

Remark 2.1.23 By virtue of Theorem 2.1.22 the family of closed balls of X is a μ-Vitali relation whenever (X, d, μ) is a doubling space.

The next lemma is a simple variant of Lemma 2.9.3 in [55], where we replace the Borel regularity of ν with the absolute continuity with respect to μ.

Lemma 2.1.24 *Let ν and μ be measures that are finite on bounded sets of X, where ν is absolutely continuous with respect to μ. Assume that the family $\mathcal{V}$ of closed balls is a μ-Vitali relation and that μ is Borel regular. Then for any $\alpha > 0$ and any μ-measurable set*

$$A \subset \left\{ x \in X \mid \liminf_{r\to 0} \frac{\nu(D_{x,r})}{\mu(D_{x,r})} < \alpha \right\}$$

we have $\nu(A) \leq \alpha\, \mu(A)$.

PROOF. First of all, we fix $\varepsilon > 0$. By Theorem 2.2.2 of [55] and the fact that μ is Borel regular and finite on bounded sets it follows that there exists an open subset O such that $\mu(O \setminus A) \leq \varepsilon$. Let us consider the family of closed balls

$$\mathcal{C} = \left\{ D_{x,r} \subset O \,\Big|\, x \in A,\ \frac{\nu(D_{x,r})}{\mu(D_{x,r})} < \alpha \right\},$$

and notice that by our assumptions, defining $I_x = \{r \mid B_{x,r} \in \mathcal{C}\}$ we have $\inf I_x = 0$ for any $x \in A$. Thus, by the μ-Vitali property there exists a countable disjoint subfamily $\{D_{x_j,r_j}\} \subset \mathcal{C}$ such that

$$\mu\Big(A \setminus \bigcup_{j\in\mathbb{N}} D_{x_j,r_j}\Big) = 0\,.$$

Utilizing the absolutely continuity of ν and the previous equation we get

$$\nu(A) \leq \alpha \sum_{j=1} \mu(D_{x_j,r_j}) = \alpha\, \mu\Big(\bigcup_{j\in\mathbb{N}} D_{x_j,r_j}\Big) \leq \alpha\, \mu(O) \leq \alpha\, \mu(A) + \alpha\varepsilon$$

and letting $\varepsilon \to 0^+$ we achieve our claim. $\square$

2.2 Carnot-Carathéodory spaces

Throughout the section, we will denote by M a smooth manifold with topological dimension q. We start recalling some elementary notions of Differential Geometry.

Definition 2.2.1 Let M and N be smooth manifolds and let $\Omega \subset M$ be an open subset. We denote by $C^k(\Omega, N)$, $k \geq 1$, the set of k-times continuously differentiable maps $f : \Omega \longrightarrow N$ and we define $C^\infty(\Omega, N) = \bigcap_{k\in\mathbb{N}\setminus\{0\}} C^k(\Omega, N)$. If $N = \mathbb{R}$ we simply write $C^k(M)$.

Definition 2.2.2 (Vector fields) We denote by $\Gamma(TM)$ the linear space of smooth sections of TM, that is a module over $C^\infty(M)$.

The space $\Gamma(TM)$ can be identified with the space of all derivations $\mathfrak{D}^1(M)$, see Theorem 1.51 of [75], where we can define the differential operator

$$f \longrightarrow X(Yf) - Y(Xf) = [X, Y]f \tag{2.8}$$

for any $X, Y \in \mathfrak{D}^1(M)$ and $f \in C^\infty(M)$. This operator is indeed a derivation, so we have uniquely defined the corresponding vector field $[X, Y] \in \Gamma(TM)$, namely the *Lie bracket* of X and Y. This product has the following properties:

1. the map $\Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$, $(X, Y) \longrightarrow [X, Y]$ is bilinear
2. $[X, Y] + [Y, X] = 0 \quad$ (antisymmetric property)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad$ (Jacobi identity).

Then $\Gamma(TM)$ has a natural structure of infinite dimensional Lie algebra.

Definition 2.2.3 (Image of vector fields) Let $f : M \longrightarrow N$ be a C^1 diffeomorphism of differentiable manifolds and let $X \in \Gamma(TM)$. Then the *image* of X under f is the vector field of $\Gamma(TN)$ defined for any $n \in N$ as follows

$$f_*X(n) = df(f^{-1}(n)) \left(X \left(f^{-1}(n) \right) \right) .$$

Remark 2.2.4 If we read the vector fields in terms of derivations it is not difficult to recognize the following rule

$$f_*Xu = [X(u \circ f)] \circ f^{-1} \tag{2.9}$$

for any $u \in C^1(N)$.

An easy relation connects Lie bracket of vector fields with their image through a diffeomorphism:

$$[f_*X, f_*Y] = f_*[X, Y] \tag{2.10}$$

whenever $f : M \longrightarrow N$ is a diffeomorphism and $X, Y \in \Gamma(TM)$. This formula easily follows checking its validity for the corresponding derivations.

Definition 2.2.5 (Horizontal subbundle) A *horizontal subbundle* is a distribution of subspaces $H_pM \subset T_pM$, for any $p \in M$, that is locally generated by a set of Lipschitz vector fields. The collection of all subspaces is denoted by *HM*. We will say that *HM* is either a smooth, C^k or Lipschitz horizontal subbundle if the locally defining vector fields have the corresponding regularity.

We mention that in the terminology of Nonholonomic Mechanics, smooth horizontal subbundles are called "differential systems" or simply distributions, [181]. Notice that the dimension of H_pM may depend on the point p.

Definition 2.2.6 (Horizontal vector fields) Let HM be a horizontal subbundle. We denote by $\Gamma(HM)$ the space of sections of HM that possess the same regularity of HM. The space $\Gamma_c(HM)$ denotes all elements of $\Gamma(HM)$ with compact support. A section of $\Gamma(HM)$ is called *horizontal vector field*.

Definition 2.2.7 (Horizontal gradient) Let (M, g) be a Riemannian manifold with a C^1 horizontal subbundle HM and let $u \in C^1(M)$. We denote by $\mathfrak{p}_H$ the fiberwise orthogonal projection of TM onto HM. The horizontal vector field $\nabla_H u \in \Gamma(HM)$ defined by

$$du(p) \circ \mathfrak{p}_H(X) = g(p)(\nabla_H u, X)$$

for any $p \in M$ and $X \in T_pM$ is called the *horizontal gradient* of u.

Definition 2.2.8 (Characteristic points) Let M be a C^1 manifold with horizontal subbundle HM and let $\Sigma \subset M$ be a hypersurface of class C^1 with $p \in \Sigma$. We say that $p \in \Sigma$ is a *characteristic point* of Σ if $H_pM \subset T_p\Sigma$. The *characteristic set* of Σ, denoted by $C(\Sigma)$, is the subset of Σ which contains all characteristic points.

Definition 2.2.9 (Horizontal normal) Let (M, g) be a Riemannian manifold with a horizontal subbundle HM and let $\Sigma \subset M$ be a hypersurface of class C^1 with $p \in \Sigma$. Let $\nu(p)$ be a unit normal of Σ at p. We define $\nu_H(p) = \mathfrak{p}_H(\nu(p))$ to be the *horizontal normal* of Σ at p.

Proposition 2.2.10 *Let Σ be a C^1 hypersurface of (M, g) with horizontal subbundle HM. Then $p \in C(\Sigma)$ if and only if $\nu_H(p) = 0$.*

PROOF. Suppose that $\nu_H(p) = 0$ and assume by contradiction that there exists $w \in H_pM \setminus T_p\Sigma$. Then we can write $w = u + \lambda\nu(p)$, with $\lambda \neq 0$ and $u \in T_p\Sigma$. Let $\mathfrak{p}_H : T_pM \longrightarrow H_pM$ be the projection associated to the scalar product $g(p)$ on T_pM. Hence we have

$$w = \mathfrak{p}_H(w) = \mathfrak{p}_H(u) + \lambda\, \mathfrak{p}_H\,(\nu(p)) = \mathfrak{p}_H(u) + \lambda\, \nu_H(p) = \mathfrak{p}_H(u) = u + \lambda\, \nu(p)\,,$$

that yields the following contradiction

$$|\mathfrak{p}_H(u)|^2 = |u + \lambda\, \nu(p)|^2 = |u|^2 + \lambda^2\, |\nu(p)|^2 \geq |\mathfrak{p}_H(u)|^2 + \lambda^2\, |\nu(p)|^2\,.$$

Now suppose that $H_pM \subset T_p\Sigma$. We know that $\nu(p)$ is perpendicular to $T_p\Sigma$, then it is perpendicular to H_pM. □

Definition 2.2.11 (Horizontal curve) A *horizontal curve* is an absolutely continuous map $\gamma : [a, b] \longrightarrow M$, with $-\infty < a < b < +\infty$, such that $\gamma'(t) \in H_{\gamma(t)}M$ for a.e. $t \in [a, b]$.

Definition 2.2.12 (H-connectedness) A smooth connected manifold M with horizontal subbundle *HM* is *horizontally connected*, or in short *H-connected*, if any two points of M can be joined by a horizontal curve.

Definition 2.2.13 (CC-space) We say that a smooth H-connected manifold M is a *Carnot-Carathéodory space*, or in short a *CC-space*.

We have assumed only Lipschitz regularity on the vector fields of *HM* for different reasons. There are simple examples of CC-spaces where the system of vector fields is Lipschitz but it is not smooth. We mention the so called Grushin plane $(\mathbb{R}^2, H\mathbb{R}^2)$ where $H_{(x,y)}\mathbb{R}^2$ is generated by ∂_x and $\lambda(x)\partial_y$, where λ is a nonnegative and nonconstant Lipschitz map, that is C^1 outside the origin. Many important results such as Poincaré inequalities, Harnack inequalities and the De Giorgi-Nash regularity Theorem were obtained in general versions of the Grushin plane, see [62], [63].

Another reason comes from the theory of degenerate elliptic equations. In fact, if the matrix $A = (a^{ij})$ of the second order derivatives is smooth with rank less than or equal to m, with $m < q$, then the operator can be decomposed as a sum of m squares of Lipschitz vector fields. This form is particularly convenient when the vector fields are smooth, where if the condition of Definition 2.2.14 is satisfied, then hypoellipticity holds for such operators [99]. However, even if vector fields are only Lipschitz it is possible to define Sobolev spaces with respect to them and to obtain a wide variety of embedding theorems and Sobolev-Poincare' inequalities. This is a wide subject of increasing interest, with contributions of many authors. We address the reader to the recent monograph [91], where an exhaustive list of reference is given.

A well known condition that ensures the H-connectedness in the smooth case is the following.

Definition 2.2.14 Let *HM* be a smooth horizontal subbundle of M. Then we say that *HM* satisfies the *Chow condition* if for any $p \in M$ the Lie algebra generated by H_pM with respect to the Lie product of vector fields coincides with T_pM.

The proof of H-connectedness in the assumptions of Definition 2.2.14 is due to W.L. Chow and to P.K. Rashevsky independently, [38], [160]. See also [99], [117] and the recent approaches of [15], [86].

Theorem 2.2.15 (Chow-Rashevsky Theorem) *Let M be a smooth connected manifold, such that HM satisfies the Chow condition. Then M is H-connected.*

A complete characterization of systems of vector fields that yield H-connectedness is given in [176].

Remark 2.2.16 If the Chow condition holds and we assume in addition that at any point $p \in M$ the Lie algebra generated by *HM* at p is nilpotent of step less than or equal to ι, for some positive integer ι, we have the following estimate (in local coordinates)

$$|x - y| \leq d(x, y) \leq C\,|x - y|^{1/\iota} \text{ for any } x, y \in K \subset M\,, \tag{2.11}$$

where K is a compact and C is a dimensional constant depending on K, see [149].

2.2.1 CC-distance

In this subsection we characterize the CC-distance of a CC-space with respect to different points of view adopted in the literature.

Definition 2.2.17 (Sub-Riemannian manifold) Let (M, HM) be a smooth manifold with a horizontal subbundle. A quadratic form g on TM

$$TM \ni (p, W) \longrightarrow g(p, W) \in [0, +\infty]$$

such that the restriction $g_{|HM}$ is Lipschitz regular on HM is called a *sub-Riemannian metric* on M. We call the triplet (M, HM, g) a *sub-Riemannian manifold.*

The previous notion of sub-Riemannian metric is taken from [15].

Definition 2.2.18 Let $\gamma : [c, d] \longrightarrow M$ be a horizontal curve. The length of γ with respect to the sub-Riemannian metric g is defined as follows

$$l_g(\gamma) = \int_a^b \sqrt{g(\gamma(t), \gamma'(t))}\, dt\,.$$

Definition 2.2.19 (CC-distance) Let (M, HM, g) be a sub-Riemannian manifold and let $p, p' \in M$. We denote by $\mathcal{H}_{p,p'}$ the set of horizontal curves that connect p to p'. The *Carnot-Carathéodory distance*, in short *CC-distance*, between p and p' is defined as follows

$$\rho(p, p') = \inf \left\{ l_g(\gamma) \mid \gamma \in \mathcal{H}_{p,p'} \right\} ,$$

where we assume that $\inf \emptyset = +\infty$.

It is clear that any CC-space has finite CC-distance.

Now we want to emphasize the importance of the CC-distance in connection with sub-elliptic PDE's. This is another feature that illustrates how CC-distance naturally fits the intrinsic geometry induced by HM. Following [56] we will define the distance associated to a sub-elliptic operator $\mathcal{L}$ on M in local coordinates

$$\mathcal{L} = -\sum_{i,j=1}^{q} a^{ij}(x) \frac{\partial^2}{\partial_{x_i x_j}} + \sum_{j=1}^{q} b_j(x) \frac{\partial}{\partial_{x_j}} + c(x) \, , \tag{2.12}$$

where the controvariant matrix $(a^{ij}(x))$ is symmetric and nonnegative.

Definition 2.2.20 We say that $V \in T_pM$ is a *subunit vector* if

$$V^i V^j \leq a^{ij}(p) \, .$$

An absolutely continuous curve $\gamma : [c, c'] \longrightarrow M$, with $-\infty < c < c' < +\infty$, is a *subunit curve* if $\gamma'(t)$ is a subunit vector for a.e. $t \in [c, c']$.

The above definition does not depend on the coordinate system that we consider and can be expressed in a more intrinsic way considering $a(x) = a^{ij}(x) \, \partial_{x_i} \otimes \partial_{x_j}$ as a semidefinite metric on the cotangent bundle T^*M. So the condition of being a subunit vector is equivalently expressed as follows

$$\langle \eta, V \rangle^2 \leq a(p)(\eta, \eta) \tag{2.13}$$

for 1-form $\eta \in T_pM^*$.

Definition 2.2.21 Let a be a semidefinite metric on T^*M. For any couple of points $p, p' \in M$ the *a-distance* between p and p' is

$$d_a(p, p') = \inf \left\{ c' - c \mid \gamma : [c, c'] \longrightarrow M \text{ is a subunit curve which connects } p \text{ with } p' \right\}$$

where we assume that $\inf \emptyset = +\infty$.

To understand the role of d_a, we mention the following remarkable result due to C.Fefferman and D.H.Phong, [56], where it is proved that the condition

$$|x-y| \le C\ d_a(x,y)^\varepsilon$$

for some $\varepsilon > 0$ is equivalent to the sub-elliptic estimate

$$\|u\|^2_{H^\varepsilon} \le C\Big(\|u\|^2 + \int \sum_{i,j=1}^{q} a^{ij}(x) u_{x_i} u_{x_j}\, dx\Big)\,, \tag{2.14}$$

where a is the semidefinite metric on T^*M associated to the operator $\mathcal{L}$ as in (2.12). The subelliptic estimate (2.14) in turn implies the hypoellipticity of $\mathcal{L}$ (see also [117]). Now, via the Legendre transformation we define the sub-Riemannian metric associated to the controvariant semidefinite metric a, as it is done in [15].

Definition 2.2.22 Let a be a semidefinite metric on T^*M. The sub-Riemannian metric associated to a is defined as follows

$$g_a(p,V) = \sup_{\eta \in T_pM^*} \Big\{2\langle \eta, V\rangle - a(p)(\eta,\eta)\Big\}\,.$$

By definition of g_a one can verify that it is a sub-Riemannian metric on M. We check the homogeneity of degree 2. For each $\lambda > 0$ we have

$$g_a(p,\lambda V) = \sup_{\eta \in T_pM^*} \Big\{2\langle \eta\lambda^{-1}, \lambda^2 V\rangle - a(p)(\eta,\eta)\Big\}$$

$$= \sup_{\lambda\eta \in T_pM^*} \Big\{2\langle \eta, \lambda^2 V\rangle - \lambda^2 a(p)(\eta,\eta)\Big\} = \lambda^2\, g_a(p,V)\,.$$

Lemma 2.2.23 *A vector $V \in T_pM$ is subunit if and only if $g_a(p,V) \le 1$.*

Proof. Suppose that $V \in T_pM$ is a subunit vector, then we have

$$g_a(p,V) \le 2\langle \eta, V\rangle - a(p)(\eta,\eta) \le 2\langle \eta, V\rangle - \langle \eta, V\rangle^2 \le 1\,.$$

Viceversa, if we assume by contradiction that V is not subunit, then there exists a linear map η_0 such that

$$\langle \eta_0, V\rangle^2 > a(p)(\eta_0,\eta_0)\,.$$

Up to a multiplication by a positive constant we can suppose that $\langle \eta_0, V\rangle = 1$. It follows that

$$g_a(p,V) \ge 2\langle \eta_0, V\rangle - a(p)(\eta_0,\eta_0) > 2\langle \eta_0, V\rangle - \langle \eta_0, V\rangle^2 = 1\,,$$

so the proof is complete. □

Theorem 2.2.24 *Let ρ_a be the CC-distance associated to g_a. Then we have $d_a = \rho_a$.*

PROOF. Let $p, p' \in M$ be such that $d_a(p, p') < \infty$ and let $\varepsilon > 0$. There exists a subunit curve $\gamma : [c, c'] \longrightarrow M$ that connects p and p' such that $c' - c < d_a(p, p') + \varepsilon$. By Lemma 2.2.23 it follows

$$l_g(\gamma) = \int_c^{c'} \sqrt{g_a(\gamma(t), \gamma'(t))} \leq c' - c\,,$$

then $\rho_a(p, p') \leq d_a(p, p') + \varepsilon$ and letting $\varepsilon \to 0^+$ we have the first inequality. Now suppose that $\rho_a(p, p') < \infty$ and consider $\varepsilon > 0$. We can find a curve $\gamma : [0, 1] \longrightarrow M$ with $L = l_g(\gamma) < \rho_a(p, p') + \varepsilon$. We define the nondecreasing maps

$$\lambda(t) = \int_0^t \sqrt{g_a(\gamma(r), \gamma'(r))}\, dr\,, \qquad h(s) = \inf\{t \mid \lambda(t) = s\}\ ,$$

where $\lambda : [0, 1] \longrightarrow [0, L]$, $h : [0, L] \longrightarrow [0, 1]$ and $\lambda \circ h = \mathrm{Id}_{[0,L]}$. We define the set

$$F = \{s \in [0, L] \mid \text{ either } \lambda \text{ or } \gamma \text{ is not differentiable at } h(s)\}\,.$$

If we denote by G the set of points where either γ or λ are not differentiable we have $h(F) \subset G$ and $F \subset \lambda(G)$. By the absolutely continuity of λ and the fact that G is negligible we conclude that F has vanishing measure. Now, defining $\Gamma(s) = \gamma \circ h(s)$ it follows that for a.e. $s \in [0, L]$

$$1 = \lambda'(h(s))\, h'(s) = \sqrt{g_a(\gamma(h(s)), \gamma'(h(s)))}\, h'(s) = \sqrt{g_a(\Gamma(s), \Gamma'(s))}\,.$$

Again, by Lemma 2.2.23 the curve Γ is subunit, then $d_a(p, p') \leq \rho_a(p, p') + \varepsilon$. Letting $\varepsilon \to 0^+$ the thesis follows. □

As we have mentioned previously, a smooth symmetric nonnegative matrix (a^{ij}) of rank less than or equal to m, can be decomposed locally as a product of Lipschitz matrices $(a^{ij}) = CC^T$, where C is a $m \times q$ matrix, $m < q$, see [15]. If we define X_j, with $j = 1, \ldots, m$ as the columns of C, then it is easy to check that

$$a(p)(\eta, \eta) = \sum_{i=1}^m \langle \eta, X_j(p) \rangle^2\,. \tag{2.15}$$

Lemma 2.2.25 *Assume that vectors $\{X_j(p) \mid j = 1, \ldots, m\}$ in (2.15) are linearly independent. Then $v \in T_pM$ is a subunit vector if and only if $v = \sum_{i=1}^m a^j X_j(p)$, with $\sum_{i=1}^m (a^j)^2 \leq 1$.*

PROOF. Suppose that v satisfies (2.13). We prove first that v is a linear combination of $(X_j(p))$. Reasoning by contradiction, if $v \notin \mathrm{span}\{X_1(p), \ldots, X_m(p)\}$, then there exists a linear map η such that $\langle \eta, v \rangle \neq 0$ and $\langle \eta, X_i \rangle = 0$ for any $i = 1, \ldots, m$.

In view of (2.15) and (2.13) the previous conditions give a contradiction. Then $v = \sum_{i=1}^m a^j X_j(p)$ for some constants a^j, with $j = 1, \dots, m$. Again, conditions (2.15) and (2.13) imply

$$\left(\sum_{i=1}^m a^j \langle \eta, X_j \rangle \right)^2 \leq \sum_{j=1}^m \langle \eta, X_j \rangle^2$$

for any linear map η. By the fact that $\{X_j(p)\}$ are linearly independent we deduce that $\sum_{i=1}^m (a^j)^2 \leq 1$. The opposite implication follows easily from Cauchy-Schwarz inequality. □

Remark 2.2.26 In view of the previous lemma the distance d_a corresponds to the common notion of CC-distance defined with respect to a set of Lipschitz vector fields $\{X_j(p) \mid j = 1, \dots, m\}$, see for instance the definitions used in [79], [100], [118]. Notice that when a semidefinite controvariant metric a is given, the notion of d_a clearly does not depend on the particular choice of vector fields that we use to define locally the controvariant metric a itself.

Now we consider that case when the H_pM has a fixed dimension $m < \dim(M)$ for any $p \in M$. We assume to have a sub-Riemannian metric g on M. By the Gram-Schmidt procedure it is possible to construct locally a set of orthonormal vector fields $\{X_j(p) \mid j = 1, \dots, m\}$. By these vector fields we can define a semidefinite controvariant metric a by formula (2.15). Notice that this definition does not depend on the orthonormal basis we consider. It is easy to check that

$$g(p)\Big(\sum_{i=1}^m c^j X_j(p), \sum_{i=1}^m c^j X_j(p) \Big) = g_a\Big(p, \sum_{i=1}^m c^j X_j(p) \Big) = \sum_{i=1}^m (c^j)^2 \,.$$

So, in view of Theorem 2.2.24 and Lemma 2.2.25 we obtain the following result.

Theorem 2.2.27 *Let (M, HM, g) be a sub-Riemannian manifold and assume that $\dim(H_pM) = m < \dim(M)$ for any $p \in M$. Then for any $p, p' \in M$ we have*

$$\rho(p, p') = \inf\{c' - c \mid \gamma \in \mathcal{S}_{p,p'},\, \gamma : [c, c'] \longrightarrow M\} \,,$$

where the ρ is the CC-distance and $\mathcal{S}_{p,p'}$ is the family of absolutely continuous curves $\gamma : [c, c'] \longrightarrow M$ such that $\gamma(c) = p$, $\gamma(c') = p'$ and $\gamma'(t) = \sum_{i=1}^m c^j(t) X_j(\gamma(t))$, for some system of orthonormal vector fields under the condition $\sum_{i=1}^m (c^j)^2 \leq 1$.

In the previous theorem we have characterized the CC-distance between two points $p, p' \in M$ as either the infimum among "times" of subunit curves or the infimum among lengths of horizontal curves, where both of them connect p and p'.

2.3 Nilpotent groups

Let $\mathbb{G}$ be a second countable and locally compact Lie group, i.e. a differentiable manifold with a smooth group operation $\mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$, $(g, h) \longrightarrow g^{-1}h$. It is well known that there exists an analytic structure on $\mathbb{G}$ such that the above map is analytic, [178]. Classical examples of Lie groups are the following:

1. the Euclidean space $\mathbb{E}^n$ under the additive operation.

2. the unit circle $\mathbb{S}^1$ under the product operation of complex numbers.

3. the manifold $GL_n(\mathbb{R})$ of all non-singular real matrices under matrix product operation

4. the submanifold $O_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ of all orthogonal matrices.

Definition 2.3.1 (Left translations) Let $\mathbb{G}$ be a Lie group and let $p \in \mathbb{G}$. The *left translation* associated to p is the diffeomorphism $l_p : \mathbb{G} \longrightarrow \mathbb{G}$ defined as $l_p(w) = pw$.

Note that the group of left translations is a subgroup of Diffeo$(\mathbb{G})$ and it is isomorphic to $\mathbb{G}$. We will use the symbol e to denote the unit element of the group.

Definition 2.3.2 (Left invariance) We say that a vector field $X \in \Gamma(T\mathbb{G})$ is *left invariant* if for any $p \in \mathbb{G}$ we have $dl_p\left(X(e)\right) = X(p)$. The linear space of all left invariant vector fields of $\mathbb{G}$ will be denoted by $\mathcal{G}$.

Remark 2.3.3 Notice that Definition 2.3.2 provides also a way to construct left invariant vector fields starting from tangent vectors of $T_e\mathbb{G}$. It suffices to define $X_v(p) = dl_p(e)(v)$ for any $p \in \mathbb{G}$ when $v \in T_e\mathbb{G}$ and check that X_v is a left invariant vector field. Then the map $v \longrightarrow X_v$ is a isomorphism between $T_e\mathbb{G}$ and $\mathcal{G}$, so the dimension of $\mathcal{G}$ is equal to the topological dimension of $\mathbb{G}$.

We also observe that if we look at vector fields as differential operators, the left invariance can be stated requiring that for any $u \in C^\infty(\mathbb{G})$ and any $p \in \mathbb{G}$ we have

$$X\left(u \circ l_p\right) = (Xu) \circ l_p\,,$$

where $X \in \mathcal{G}$.

Definition 2.3.4 (Lie algebra) We say that a finite dimensional vector space $\mathfrak{g}$ is a *Lie algebra* if there exists an antisymmetric bilinear map

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}\,, \quad (X, Y) \longrightarrow [X, Y]\,,$$

such that the Jacobi identity holds (see properties of the Lie bracket in Section 2.2). A linear subspace $\mathfrak{a} \subset \mathfrak{g}$ is a Lie *subalgebra* of $\mathfrak{g}$ if $[X, Y] \in \mathfrak{a}$ for any $X, Y \in \mathfrak{a}$.

By formula (2.10) it follows that the Lie bracket of left invariant vector fields is still left invariant, hence $\mathcal{G}$ is a finite dimensional Lie subalgebra of $\Gamma(TM)$.

Example 2.3.5 Let us consider the space of $n \times n$ real or complex matrices $\mathfrak{gl}_n$ with the product operation

$$[A, B] = A \cdot B - B \cdot A\,,$$

for any $A, B \in \mathfrak{gl}_n$. The space $\mathfrak{gl}_n$ with this product operation is a Lie algebra.

Broadly speaking, the Jacobi identity replaces the associative property of the product operation on a ring. Indeed, given an associative algebra $\mathfrak{u}$ we can always build a Lie algebra structure on it, defining $[v, w] = v \cdot w - w \cdot v$ for any $v, w \in \mathfrak{u}$. We mention that due to a deep result of Ado any finite dimensional real (or complex) Lie algebra can be characterized as the Lie algebra of a subgroup of $GL_n(\mathbb{R})$ (or $GL_n(\mathbb{C})$), for some positive integer n, see [178].

Now, in order to introduce the exponential map in Lie groups we consider the following system of O.D.E.

$$\begin{cases} \partial_t \Phi(p, t) = V\left(\Phi(p, t)\right) \\ \Phi(p, 0) = p \end{cases} \tag{2.16}$$

where $V \in \mathcal{G}$. The flow Φ associated to this system is defined on all of $\mathbb{R}$. In fact, if we consider $\Phi(e, \cdot)$ defined on some interval $[0, b]$, we observe that $\Phi(p, \cdot) = p \cdot \Phi(e, t)$ is again defined on $[0, b]$, so $\Phi\left(\Phi(e, b/2), t\right) = \Phi(e, b/2)\Phi(e, t) = \Phi(e, t + b/2)$ can be extended $[0, b]$, then $\Phi(e, \cdot)$ can be defined on $[0, 3b/2]$, and so forth. It is clear that this argument can be repeated analogously on the left half line. Thus, we can give the following definition.

Definition 2.3.6 For any $V \in \mathcal{G}$ we define the map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ to be

$$\exp(V) = \Phi(e, 1)\,,$$

where Φ is the flow associated to the system (2.16).

Remark 2.3.7 Notice that this definition of exponential map involves only the differentiable structure of $\mathbb{G}$ and it does not refer to any metric on $\mathbb{G}$.

Definition 2.3.8 (Nilpotent group) Consider a Lie algebra $\mathfrak{g}$ and two subspaces $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$. We define $[\mathfrak{a}, \mathfrak{b}]$ to be the subspace of $\mathfrak{g}$ generated by all linear combinations of elements $[X, Y]$, where $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$. For each $k \in \mathbb{N} \setminus \{0\}$ we define by induction the following sequence of subspaces

$$\mathfrak{g}^{(1)} = \mathfrak{g}\,,$$

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}\,]\,.$$

The family $(\mathfrak{g}^{(k)})_{k\geq 1}$ is called the *descending central sequence* of $\mathfrak{g}$. If there exists a positive integer ι such that $\mathfrak{g}^{(\iota+1)} = 0$ we say that $\mathfrak{g}$ is a *nilpotent Lie algebra*, precisely $\mathfrak{g}$ is *ι-step nilpotent*. The integer ι is called the *step* of $\mathfrak{g}$, or the *degree of nilpotency* of $\mathfrak{g}$. We use the same terminology for Lie groups whose Lie algebra is nilpotent.

Remark 2.3.9 Notice that if $\mathfrak{g}$ is ι-step nilpotent, then for each $1 \leq j \leq \iota$ the subalgebra $\mathfrak{g}^{(j+1)}$ is strictly contained in $\mathfrak{g}^{(j)}$.

An important theorem for simply connected nilpotent Lie groups holds, see for instance Theorem 1.2.1 of [40].

Theorem 2.3.10 *Let $\mathbb{G}$ be a simply connected nilpotent Lie group and let $\mathcal{G}$ be its Lie algebra. Then the exponential map* $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ *is a diffeomorphism.*

Due to the preceding theorem we can define the inverse map $\ln = \exp^{-1}$ in simply connected nilpotent groups.

Remark 2.3.11 It is a standard fact that for any $A \in \mathfrak{gl}_n(\mathbb{R})$ the function

$$e^A = \sum_{i=0}^{\infty} \frac{A^k}{k!} \tag{2.17}$$

is the exponential map of $GL_n(\mathbb{R})$, according to Definition 2.3.6. If we restrict the exponential map to the orthogonal subalgebra $\mathfrak{o}_n(\mathbb{R})$, that corresponds to the Lie group of orthogonal matrices $O_n(\mathbb{R})$, i.e.

$$\exp : \mathfrak{o}_n(\mathbb{R}) \longrightarrow O_n(\mathbb{R}),$$

we have an example where the exponential map is not a diffeomorphism. This follows observing that $O_n(\mathbb{R})$ is a compact topological space.

It is possible to get an explicit representation of simply connected nilpotent Lie groups. Precisely, for any nilpotent Lie algebra $\mathfrak{g}$ of topological dimension q, there exists an isomorphic Lie algebra $\mathcal{G} \subset \Gamma(T\mathbb{R}^q)$ of polynomial vector fields that can be constructed explicitly from the Lie bracket relations, see Proposition 2.4 of [107]. These vector fields yield the polynomial group operation on $\mathbb{R}^q$, which makes it a Lie group with algebra $\mathcal{G}$.

To get this operation we proceed as follows: let $Y_1, Y_2, \ldots, Y_q$ be the vector fields which induce the nilpotent structure in $\mathbb{R}^q$ and consider the O.D.E.

$$\begin{cases} \partial_t \Phi(x,t) = \sum_{i=1}^q y^i Y_i\left(\Phi(x,t)\right) \\ \Phi(x,0) = x \end{cases}.$$

The flow $(x,y,t) \longrightarrow \Phi(x,y,t)$ defines the group element $\tilde{y} = \Phi(0,y,1) \in \mathbb{R}^q$, and the group operation in $\mathbb{R}^q$ is given by

$$\tilde{x} \cdot \tilde{y} = \Phi\left(\Phi(0,x,1), y, 1\right) .$$

The nilpotence of vector fields and their polynomial expression imply that the operation above has a polynomial form. It turns out that $\mathbb{R}^q$ endowed with this polynomial operation is a nilpotent group with Lie algebra isomorphic to $\mathfrak{g}$. So we can associate (in a noncanonical way) to any nilpotent algebra $\mathfrak{g}$ a simply connected Lie group $\mathbb{R}^q$ with the same nilpotent algebra. This means that we can identify a nilpotent group with $\mathbb{R}^q$ together with a polynomial operation. In Definition 2.3.13 we will see in detail how can be made precise this identification.

Next, we will state the remarkable Baker-Campbell-Hausdorff formula, where a relation between vectors of the algebra and the product of their corresponding exponentials is established. In the sequel we will say shortly BCH formula.

Theorem 2.3.12 (Baker-Campbell-Hausdorff formula) *Let $X, Y \in \mathcal{G}$, where $\mathcal{G}$ is the nilpotent Lie algebra of a simply connected group $\mathbb{G}$ of step ι and define*

$$\ln\Big(\exp X \exp Y\Big) = X \circledcirc Y .$$

Then we have

$$X \circledcirc Y = \sum_{n=1}^{\iota} \frac{(-1)^{n+1}}{n} \sum_{1 \le |\alpha|+|\beta| \le \iota} \frac{(\mathrm{Ad}X)^{\alpha_1}(\mathrm{Ad}Y)^{\beta_1} \cdots (\mathrm{Ad}X)^{\alpha_n}(\mathrm{Ad}Y)^{\beta_n - 1}(Y)}{\alpha!\beta!\,|\alpha+\beta|}, \tag{2.18}$$

where for any $Z \in \mathcal{G}$ the map $\mathrm{Ad}Z : \mathcal{G} \longrightarrow \mathcal{G}$ is the linear operator defined by $\mathrm{Ad}Z(W) = [Z, W]$ and for any $\alpha \in \mathbb{N}^n$ we have assumed the convention $\alpha! = \prod_{l=1}^n \alpha_l$ and $|\alpha| = \sum_{l=1}^n \alpha_l$.

Now, to better clarify the BCH formula we state it for groups of step 3. In this case for any $X, Y \in \mathcal{G}$ we have

$$X \circledcirc Y = X + Y + \frac{[X,Y]}{2} + \frac{[X,[X,Y]] - [Y,[X,Y]]}{12}. \tag{2.19}$$

Definition 2.3.13 (Exponential coordinates) Let $\mathbb{G}$ be simply connected nilpotent Lie group and let $(W_1, \ldots, W_q)$ be a basis of $\mathcal{G}$. We define the diffeomorphism $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ as

$$F(y) = \exp\Big(\sum_{i=1}^q y_i W_i\Big) .$$

We say that (F, W) is a system of *exponential coordinates* associated to the basis $W = (W_1, \ldots, W_q)$.

In the following theorem we recall the uniqueness of the simply connected Lie group associated to a Lie algebra, see [187].

Theorem 2.3.14 *Let $\mathbb{G}$ and $\mathbb{M}$ be simply connected Lie groups with isomorphic Lie algebras. Then there exists a group isomorphism between $\mathbb{G}$ and $\mathbb{M}$.*

Remark 2.3.15 Since the map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism for simply connected nilpotent Lie groups, the operation $\circledcirc$ defined in (2.18) makes exp a group isomorphism, which allows us to identify the algebra with the group. It is immediate to observe from formula (2.18) that

$$\ln(x^{-1}) = -\ln(x) \quad \text{for any} \quad x \in \mathbb{G}.$$

It is enough to observe that $\mathrm{Ad}Z(Z) = 0$ whenever $Z \in \mathcal{G}$.

Now we introduce a particular class of nilpotent Lie algebras, where it is possible to define a one parameter group of dilations.

Definition 2.3.16 (Graded algebra) We say that a Lie algebra $\mathcal{G}$ is *graded* if it can be decomposed as the following direct sum

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota \,, \quad \iota \in \mathbb{N}\,, \tag{2.20}$$

with $V_{i+1} \subset [V_i, V_1]$ for any $i \in \mathbb{N}\setminus\{0\}$ and $V_j = \{0\}$ for any $j > \iota$. A Lie group whose Lie algebra is graded is called *graded group*. The decomposition (2.20) is called the *grading* of the group. If $\mathbb{G}$ is the simply connected group associated to $\mathcal{G}$, we define for every $p \in \mathbb{G}$ the *subspace of degree* k at p as follows

$$H_p^j\mathbb{G} = \left\{ X(p) \,\middle|\, X \in V_j \right\} \subset T_p\mathbb{G}\,,$$

we also write $H_p\mathbb{G} = H_p^1\mathbb{G}$. We denote by $\mathbb{V}_k = \exp V_k \subset \mathbb{G}$ the space of elements in $\mathbb{G}$ of degree $k = 1, \ldots, \iota$.

Remark 2.3.17 Notice that any graded group is in particular nilpotent and the positive integer ι is the step of the group. This fact holds because the group is assumed finite dimensional.

The grading property guarantees the existence of a one parameter group of dilations.

Definition 2.3.18 (Dilations) Let $\mathcal{G}$ be a graded algebra. Then for any $r \geq 0$ we define the map $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$ as

$$\delta_r(v) = \sum_{i=1}^{\iota} r^i\, v_i\,,$$

where $v = \sum_{i=1}^{\iota} v_i$ and $v_i \in V_i$ for any $i = 1, \ldots, \iota$. We can extend dilations also for negative parameters $t < 0$

$$\delta_t(v) = -\delta_{|t|} v \,.$$

The *sign map* is defined as $\sigma_t(v) = \delta_{t/|t|} v$ whenever $t \neq 0$. Dilations and the sign map can be read on the group as $\exp \circ \delta_r \circ \ln$ and $\exp \circ \sigma_t \circ \ln$. For the sake of simplicity, we will denote them with the same symbol.

The above definition of dilation comes in a rather natural way. It suffices to consider the unique extension of the standard dilation $w \longrightarrow r\, w$ on V_1 to an algebra homomorphism of $\mathcal{G}$. This yields just Definition 2.3.18 and allows us to see that $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$ is an algebra homomorphism, i.e. δ_r is linear and satisfies $\delta_r(v \circledcirc w) = \delta_r v \circledcirc \delta_r w$ for any $v, w \in \mathcal{G}$. The one parameter group property $\delta_{rs} = \delta_r \circ \delta_s$ with $r, s > 0$, comes from the fact that compositions of algebra homomorphisms are still algebra homomorphisms. Notice that by Definition 2.3.18 and Remark 2.3.15 if $x \in \mathbb{G}$ and $t < 0$ we have $\delta_t x = \delta_{|t|} x^{-1}$.

In the terminology of [59] we give the following definition.

Definition 2.3.19 (Stratified algebra) A graded algebra $\mathcal{G}$ with grading

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota \,, \quad \iota \in \mathbb{N} \,,$$

is called *stratified* if for any $i \in \mathbb{N} \setminus \{0\}$ we have $V_{i+1} = [V_i, V_1]$, where $V_j = \{0\}$ for any $j > \iota$. A Lie group whose Lie algebra is stratified is called *stratified group.*

Definition 2.3.20 The horizontal subbundle associated to a graded group is defined as follows

$$H\mathbb{G} = \bigcup_{p \in \mathbb{G}} H^1_p \mathbb{G}. \tag{2.21}$$

Remark 2.3.21 With the previous definition it is easy to notice that the horizontal subbundle $H\mathbb{G}$ of a stratified group satisfy the Chow condition (Definitions 2.2.14), so all stratified groups are CC-spaces.

All notions of Section 2.2, as horizontal curve, horizontal vector field, characteristic point and so forth, are understood for all graded groups, regarded as smooth manifolds endowed with the horizontal subbundle $H\mathbb{G}$ defined in (2.21).

The H-connectedness of stratified groups can be stated in a more precise way using the group operation and dilations. This is done in the following proposition, whose proof is essentially taken from Lemma 1.40 of [59].

Proposition 2.3.22 (Generating property) *Let $\mathbb{G}$ be a stratified group and let $(v_1, v_2, \ldots, v_m)$ be a basis of V_1. Then there exists a positive integer γ and an open bounded neighbourhood of the origin $U \subset \mathbb{R}^\gamma$ such that the following set*

$$\left\{ \prod_{s=1}^{\gamma} \exp(a_s v_{i_s}) \mid (a_s) \subset U \right\} , \tag{2.22}$$

where $i_s = 1, \dots, m$ and $s = 1, \dots, \gamma$, is an open neighbourhood of $e \in \mathbb{G}$.

PROOF. By the fact that $\mathcal{G}$ is stratified, for any $j = 2, \dots, \iota$ there exists a set of multi-indices $\mathcal{A}_j \subset (I_m)^j$, with $I_m = \{1, 2, \dots, m\}$, such that for any $\alpha = (i_1, \dots, i_j) \in \mathcal{A}_j$ we have $i_1 \leq i_2 \leq \dots \leq i_j$ and $(v_\alpha)_{\alpha \in \mathcal{A}_j}$ is a basis of V_j, where we have denoted

$$v_\alpha = [\cdots[[v_{i_1}, v_{i_2}], v_{i_3}], \dots, v_{i_j}].$$

We write $[x, y] = xyx^{-1}y^{-1}$ for any $x, y \in \mathbb{G}$ to denote the commutator of group elements. Utilizing formula (2.19) that amounts to consider the first terms of (2.18) we can recognize that

$$\varphi_v(\exp Y) = [\exp Y, \exp v] = \exp\left([Y, v] + R(Y, v)\right),$$

where $R(Y, v)$ contains terms in both Y and v of order higher than 2, in the sense that these terms appear in the iterated Lie products more than twice. We identify the Lie algebra $\mathcal{G}$ with the tangent space $T_e\mathbb{G}$, obtaining $d\varphi_v(0)(Y) = [Y, v]$ for every $Y \in \mathcal{G}$. By the chain rule formula for composition of differentiable maps we obtain

$$d\left(\varphi_v \circ \varphi_w\right)(0)(Y) = [[Y, w], v],$$

where $\varphi_v \circ \varphi_w(\exp Y) = [[\exp Y, \exp w], \exp v]$ and $v, w \in \mathcal{G}$. Now, for every $j = 2, \dots, \iota$ and every $\alpha \in \mathcal{A}_j$ we define

$$\varphi^j_\alpha(\exp Y) = [\cdots[[\exp Y, \exp v_{i_2}], \exp v_{i_3}], \dots, \exp v_{i_j}]$$

where $\alpha = (i_1, i_2, \dots, i_j)$. By previous considerations we get

$$d\varphi^j_\alpha(0)(Y) = [\cdots[[Y, v_{i_2}], v_{i_3}], \dots, v_{i_j}].$$

We consider the map $\phi : \mathbb{R}^q \longrightarrow \mathbb{G}$ defined as

$$\phi\Big(\sum_{l=1}^m y^1_l e^1_l + \sum_{j=2}^{\iota} \sum_{\alpha \in \mathcal{A}_j} y^j_\alpha e^j_\alpha\Big) = \Big(\prod_{l=1}^m \exp(y^1_l v_l)\Big) \prod_{j=2}^{\iota} \prod_{\alpha \in \mathcal{A}_j} \varphi^j_\alpha(y^j_\alpha v_{\alpha^1})$$

where $\{e^1_l \mid l = 1, \dots, m\} \cup \{e^j_\alpha \mid j = 2, \dots, \iota,\ \alpha \in \mathcal{A}_j\}$ is a basis of $\mathbb{R}^q$ and α^1 is the first component of the integer vector α. Thus, for every $l = 1, \dots, m$, every $j = 2, \dots, \iota$ and every $\alpha \in \mathcal{A}_j$ we have $\partial_{y^1_l}\phi(0)(v_l) = v_l$ and $\partial_{y^j_\alpha}\phi(0)(v_{\alpha^1}) = v_\alpha$. It follows that the differential $d\phi(0)$ is invertible and the map ϕ maps an open neighbourhood of $\mathbb{R}^q$ onto an open neighbourhood of $e \in \mathbb{G}$. To reach the form (2.22), for every $j = 2, \dots, \iota$ and every $\alpha = (i_1, i_2, \dots, i_j) \in \mathcal{A}_j$, we develop the iterated commutators as an ordered product

$$[\cdots[[\exp v_{i_1}, \exp v_{i_2}], \exp v_{i_3}], \dots, \exp v_{i_j}] = \prod_{s=1}^{N_j} \exp(\sigma_s\, v_{k_s})$$

where $N_j = 2^{j+1}-2-2^{j-1}$, $\sigma_s \in \{-1,1\}$ and $k_s \in \{i_1, i_2, \dots, i_j\}$. Then we define

$$E^j_\alpha(b^\alpha) = \prod_{s=1}^{N_j} \exp(b^\alpha_s\, v_{k_s})$$

where $b^\alpha = (b^\alpha_1, b^\alpha_2, \dots, b^\alpha_{N_j}) \in \mathbb{R}^{N_j}$ and

$$E(b) = \Big(\prod_{l=1} \exp(b^1_l v_l)\Big) \prod_{j=2}^{\iota} \prod_{\alpha\in\mathcal{A}_j} E^j_\alpha(b^\alpha)\,,$$

where $b = \sum_{l=1}^m b^1_l e^1_l + \sum_{j=2}^{\iota} \sum_{\alpha\in\mathcal{A}_j} \sum_{s=1}^{N_j} b^\alpha_s\, e^\alpha_s \in \mathbb{R}^\gamma$ with $\gamma = m + \sum_{j=2}^{\iota} n_j\, N_j$ and $\dim V_j = n_j$. The map $E : \mathbb{R}^\gamma \longrightarrow \mathbb{G}$ takes in particular the values of ϕ, hence there exists a neighbourhood of the origin in $\mathbb{R}^\gamma$ that is mapped onto a neighbourhood of $e \in \mathbb{G}$ through the map E. $\Box$

2.3.1 The Heisenberg group

In this subsection we describe the most simple example of nonabelian stratified Lie group, namely the Heisenberg group.

Definition 2.3.23 A Lie algebra with a basis $(X_1, \dots, X_n, Y_1, \dots Y_n, T)$ that satisfies relations

$$[X_i, X_j] = 0\,, \quad [Y_i, Y_j] = 0\,, \quad [X_j, Y_j] = \alpha T\,, \tag{2.23}$$

for some $\alpha \in \mathbb{R}\setminus\{0\}$ and every $i, j = 1, \dots, n$, is called *Heisenberg algebra* and it is denoted by $\mathfrak{h}_{2n+1}$. The *Heisenberg group* $\mathbb{H}^{2n+1}$ is the simply connected nilpotent Lie group associated to $\mathfrak{h}_{2n+1}$.

In the following proposition we see that the algebraic structure of the Heisenberg group does not depend on α, up to group isomorphisms.

Proposition 2.3.24 *For any $\alpha \neq 0$ the Heisenberg algebra $\mathfrak{h}_{2n+1}$ yields a unique simply connected group $\mathbb{H}^{2n+1}$ up to group isomorphisms.*

PROOF. For any $\alpha \neq 0$ let $(X^\alpha_1, \dots, X^\alpha_n, Y^\alpha_1, \dots Y^\alpha_n, T^\alpha)$ be a basis of $\mathfrak{h}_{2n+1}$ that satisfies

$$[X^\alpha_i, X^\alpha_j] = 0\,, \quad [Y^\alpha_i, Y^\alpha_j] = 0\,, \quad [X^\alpha_j, Y^\alpha_j] = \alpha T^\alpha\,. \tag{2.24}$$

Let $\mathcal{A}_\alpha$ be the Heisenberg algebra associated to the basis defined above and let $\mathbb{G}_\alpha$ the simply connected nilpotent group associated to $\mathcal{A}_\alpha$. We define the Lie algebra homomorphism $L : \mathcal{A}_\alpha \longrightarrow \mathcal{A}_1$ such that

$$L(X^\alpha_i) = X^1_i\,, \quad L(Y^\alpha_i) = Y^1_i\,, \quad L(T^\alpha) = \frac{T^1}{\alpha}.$$

Then by Theorem 2.3.14 the Heisenberg group $\mathbb{G}_\alpha$ is isomorphic to $\mathbb{G}_1$. $\Box$

Remark 2.3.25 By relations (2.23) and the Jacobi identity we get immediately that $[X_i, T] = [Y_i, T] = 0$ for any $i = 1, \ldots, 2n$. Therefore $\mathfrak{h}_{2n+1}$ is a nilpotent Lie algebra. Defining

$$V_1 = \mathrm{span}\{X_1, \ldots, X_n, Y_1, \ldots Y_n\} \quad V_2 = \mathrm{span}\{T\}$$

we also see that $\mathfrak{h}_{2n+1}$ is a 2-step stratified Lie algebra. The group of dilations is easily described

$$\delta_r(X_i) = r\,X_i\,, \quad \delta_r(Y_i) = r\,Y_i\,, \quad \delta_r(Z) = r^2\,T$$

for any $i = 1, \ldots, n$.

The Heisenberg algebra can be realized in different ways. We start considering the subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ constituted by all the upper triangular $(n+2)\times(n+2)$-matrices of the following form

$$\begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_n & \zeta \\ 0 & 0 & \cdots & \cdots & 0 & y_1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & \vdots & y_n \\ 0 & 0 & \cdots & \cdots & \vdots & 0 \end{pmatrix}, \tag{2.25}$$

which correspond to vectors $\zeta T + \sum_{i=1}^n x_i X_i + y_i Y_i$. The Lie product of matrices restricted to that of the form (2.25) gives relations (2.23) with $\alpha = 1$. By formula (2.17) we get a realization of $\mathbb{H}^{2n+1}$ as a subgroup of $GL_{n+2}(\mathbb{R})$, where any element of $\mathbb{H}^{2n+1}$ can be represented as follows

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & \zeta \\ 0 & 1 & \cdots & \cdots & 0 & y_1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 & y_n \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{2.26}$$

and the group operation is given by the standard matrix product.

Another way to realize $\mathfrak{h}_{2n+1}$ is to consider $2n+1$ vector fields in $\mathbb{R}^{2n+1}$, satisfying the commutator relations (2.23). There is not a unique choice for such vector fields. We can consider for instance the following

$$X_j = \partial_{x_j} - \frac{\alpha}{2} y_j\,\partial_\zeta, \quad Y_j = \partial_{y_j} + \frac{\alpha}{2} x_j\,\partial_\zeta\,, \quad T = \partial_\zeta \tag{2.27}$$

$$\tilde{X}_j = \partial_{x_j}, \quad \tilde{Y}_j = \partial_{y_j} + \alpha\,x_j \partial_\zeta \quad \tilde{T} = \partial_\zeta\,. \tag{2.28}$$

Both systems of vector fields (2.27) and (2.28) satisfy relations (2.23). Now we want to obtain the explicit form of the group operation in $\mathbb{H}^{2n+1}$ with respect to the exponential coordinates corresponding to the system (2.27). We use directly the definition of exponential map in the same way we have done in Section 2.3. Let us consider the ordinary differential system

$$\gamma'(t) = \sum_{i=1}^{n} \xi_i \, X_i(\gamma(t)) + \xi_{n+i} \, Y_i(\gamma(t)) \,,$$

that can be written as follows

$$\begin{cases} \gamma_i'(t) = \xi_i, \quad i = 1, \dots, 2n \\ \gamma_{2n+1}'(t) = \xi_{2n+1} + \frac{\alpha}{2} \sum_{i=1}^{n} \xi_{n+i}\gamma_i - \xi_i\gamma_{n+i} \\ \gamma(0) = \eta \end{cases} .$$

When $\eta = 0$ it is straightforward that $\gamma(1, 0, \xi) = \xi$, then

$$\exp\Big(\sum_{j=1}^{n} \xi_j X_j + \xi_{n+j} Y_j\Big) = \xi \in \mathbb{R}^{2n+1} \,.$$

To compute the group operation we use the fact that for any $i = 1, \dots, 2n$ we have $\gamma_i(t, \eta, \xi) = \eta_i + t\xi_i$. It follows the equation

$$\gamma_{2n+1}'(t, \eta, \xi) = \xi_{2n+1} + \frac{\alpha}{2} \sum_{i=1}^{n} \xi_{n+i}\eta_i - \xi_i\eta_{n+i} \,,$$

so we have

$$\gamma_{2n+1}(1, \eta, \xi) = \eta_{2n+1} + \xi_{2n+1} + \frac{\alpha}{2} \sum_{i=1}^{n} \xi_{n+i}x_i - \xi_i\eta_{n+i} \,.$$

We have obtained the following group operation

$$\eta \cdot \xi = \left(\eta_1 + \xi_1, \dots, \eta_{2n} + \xi_{2n}, \eta_{2n+1} + \xi_{2n+1} + \frac{\alpha}{2} \sum_{i=1}^{n} \xi_{n+i} \, \eta_i - \xi_i \, \eta_{n+i}\right) . \tag{2.29}$$

Then the group operation of $\mathbb{H}^{2n+1}$ with respect to exponential coordinates of the basis (2.27) is given by (2.29).

Remark 2.3.26 The preceding calculation could have been accomplished also with respect to the basis (2.28). In this case we would have had a different expression of the group operation. However, Proposition 2.3.24 guarantees that these two different operations yield isomorphic groups, i.e. the same Heisenberg group.

Remark 2.3.27 If we choose the exponential coordinates corresponding to $\alpha = -4$ we can use the complex notation to write the group operation in $\mathbb{C}^n \times \mathbb{R}$, that is identified with $\mathbb{R}^{2n+1}$. We denote $\eta = (z,s)$ and $\xi = (w,t)$ where $z, w \in \mathbb{C}^n$ and $s, t \in \mathbb{R}$. Then formula (2.29) yields

$$(z,s) \cdot (w,t) = \Big(z + w, s + t + 2\mathrm{Im}\langle z, w \rangle \Big) , \tag{2.30}$$

where $\langle , \rangle$ denote the Hermitian product in $\mathbb{C}^n$.

To understand the "twisted structure" of nonabelian nilpotent groups we will show that the 3-dimensional Heisenberg group cannot be realized as a product of proper subgroups.

Proposition 2.3.28 *The Heisenberg group $\mathbb{H}^3$ is not isomorphic to any product of two nontrivial Lie groups.*

PROOF Let us assume by contradiction that $\mathbb{H}^3$ is isomorphic to $\mathbb{G}_1 \times \mathbb{G}_2$, where $\dim(\mathcal{G}_i) \geq 1$ and $\mathcal{G}_i$ is the Lie algebra of $\mathbb{G}_i$ for $i = 1, 2$. It follows that $\mathcal{G}_i$ is 2-step nilpotent and $\dim(\mathcal{G}_i) \leq 2$ for any $i = 1, 2$. These conditions imply that $\mathcal{G}_i$ is abelian, i.e. $[X, Y] = 0$ for any $X, Y \in \mathcal{G}_i$. This is trivial when $\dim(\mathcal{G}_i) = 1$. When $\dim(\mathcal{G}_i) = 2$, taking a basis (X, Y) of $\mathcal{G}_i$, we suppose by contradiction that $[X, Y] = \lambda X + \mu Y$ with $\lambda \neq 0$. By hypothesis we get $[Y, [X, Y]] = 0 = \lambda X$ that implies the contradiction $\lambda = 0$. We reason in the same way if $\mu \neq 0$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are abelian Lie algebras. It follows that $\mathcal{G}_1 \times \mathcal{G}_2$ is abelian and $\mathbb{G}_1 \times \mathbb{G}_2$ is also. This would imply that $\mathbb{H}^3$ is abelian, hence the isomorphism above cannot occur. □

2.3.2 Sub-Riemannian groups

In the sequel any Lie group will be assumed connected and simply connected.

Definition 2.3.29 Let $\mathbb{G}$ be a Lie group. A *left invariant metric* on $\mathbb{G}$ is a Riemannian metric such that all left translations of the group are isometries.

Throughout the thesis g will denote a left invariant metric on $\mathbb{G}$, if not otherwise stated. When it will be clear from the context we will also use the simpler notation

$$\langle X, Y \rangle_p = g(p)(X, Y) \quad \text{for any } X, Y \in T_p\mathbb{G} \,. \tag{2.31}$$

Definition 2.3.30 (Graded metric) Let $\mathbb{G}$ be a graded group. We say that a left invariant metric g on $\mathbb{G}$ is a *graded metric* if all subspaces $V_j \subset \mathcal{G}$ of the grading are orthogonal each other.

Throughout the thesis the Riemannian volume with respect to g, seen as a measure over $\mathbb{G}$, will be denoted by v_g. We point out that if d_g is the Riemannian distance associated to g we have $v_g = \mathcal{H}^q_{d_g}$, see 3.2.46 of [55]. When a left invariant metric is understood the norm of a vector $X \in T_pM$ with respect to the metric will be denoted simply by $|X| = \sqrt{g(p)(X,X)}$.

Definition 2.3.31 (Sub-Riemannian group) We say that stratified group $\mathbb{G}$ is a *sub-Riemannian group* if it is endowed with a graded metric.

Throughout the thesis we will use the term "sub-Riemannian group" when we are using its metric structure, otherwise we will use the term "stratified group".

Remark 2.3.32 Notice that the notion of horizontal gradient of Definition 2.2.7 can be written more explicitly in a sub-Riemannian group. In fact, if we consider a C^1 map $u : \Omega \longrightarrow \mathbb{R}$, where $\Omega \subset \mathbb{G}$ is an open set and we fix an orthonormal frame $(X_1, \ldots, X_m)$ of V_1, we have

$$\nabla_H u(x) = \sum_{i=1}^{m} X_i u(x)\, X_i$$

and the expression clearly does not depend of the frame.

Definition 2.3.33 (CC-distance) Let $\mathbb{G}$ be a sub-Riemannian group with a graded metric g. We define $\tilde{g}$ as the restriction of g on V_1 and as $+\infty$ otherwise. According to Definition 2.2.17 $\tilde{g}$ is a sub-Riemannian metric over $\mathbb{G}$. Regarding $\mathbb{G}$ as a particular CC-space we define the *CC-distance of* $\mathbb{G}$ referring to Definition 2.2.19, where the sub-Riemannian metric is given by $\tilde{g}$.

Remark 2.3.34 By virtue of Theorem 2.2.27 the CC-distance between two points $w, w' \in \mathbb{G}$ corresponds to the infimum of all $T > 0$ such that $\gamma : [0, T] \longrightarrow \mathbb{G}$ is horizontal, $\gamma(0) = w$, $\gamma(T) = w'$ and for a.e. $t \in [c, d]$ we have

$$\gamma'(t) = \sum_{j=1}^{m} c_j(t)\, X_j\left(\gamma(t)\right)\,,$$

where $\sum_{j=1}^{m} c_j(t)^2 \leq 1$ and (X_j) is an orthonormal basis of V_1. Notice that the latter notion of distance is currently used in general Carnot-Carathéodory spaces, see for instance [79], [100], [118].

In the following definition we single out a class of distances that are compatible with the geometry of graded groups.

Definition 2.3.35 (Homogeneous distance) Let $\mathbb{G}$ be a graded group. A *homogeneous distance* on $\mathbb{G}$ is a continuous map $d : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty[$ that makes $(\mathbb{G}, d)$ a metric space and has the following properties

1. $d(x, y) = d(ux, uy)$ for every $u, x, y \in \mathbb{G}$ (left invariance),
2. $d(\delta_r x, \delta_r y) = r\, d(x, y)$ for every $r > 0$ (homogeneity).

We simply write $d(x) = d(x, e)$, where e is the unit element of the group.

Throughout the thesis it will be always understood the use of a homogeneous distance, if not otherwise stated.

Remark 2.3.36 Note that the symmetry property $d(x, y) = d(y, x)$ and the left invariance imply that $d(x) = d(x^{-1})$ for any $x \in \mathbb{G}$. The homogeneity of homogenous distances yields for any $r > 0$ the relation

$$\delta_r B_1 = B_r \,, \tag{2.32}$$

where B_r is the metric ball with respect to an arbitrary homogeneous distance. Thus, we can write a metric ball $B_{p,r}$ with respect to a homogeneous distance as $p\, \delta_r B_1$.

Proposition 2.3.37 *Let d and δ be homogeneous distances on $\mathbb{G}$. Then there exist two positive constants C_1 and C_2 such that for any $x, y \in \mathbb{G}$ we have*

$$C_1\, \delta(x, y) \leq d(x, y) \leq C_2\, \delta(x, y)\,.$$

PROOF. We define the sphere $S = \{x \in \mathbb{G} \mid \delta(x) = 1\}$ and the numbers

$$C_1 = \min_{y \in S} d(e, y) \qquad C_2 = \max_{y \in S} d(e, y).$$

By the fact that $d(e, \cdot)$ is strictly positive and continuous on S the numbers C_1 and C_2 are positive constants. By property 2 of homogeneous distances we get

$$C_1\, \delta(e, y) \leq d(e, y) \leq C_2\, \delta(e, y)\,,$$

for any $y \in \mathbb{G}$. Now the left invariance (property 1) leads us to the conclusion. □

Example 2.3.38 We present an example of homogeneous distance that is used in [71] to obtain explicit calculations in the Heisenberg group.

Let us consider the Heisenberg group $\mathbb{H}^{2n+1}$ endowed with the exponential coordinates $(F, (X_i, Y_i, Z))$, where the only nontrivial bracket relations are $[X_i, Y_i] = -4Z$ for any $i = 1, \ldots, n$. For any element $p \in \mathbb{H}^{2n+1}$ we adopt the complex notation $F^{-1}(p) = (z, t) \in \mathbb{C}^n \times \mathbb{R}$. In these coordinates the group operation reads as in formula (2.30). Define the map $\tilde{N}(z, t) = \max\{|z|, |t|^{1/2}\}$ and $N = \tilde{N} \circ F : \mathbb{H}^3 \longrightarrow \mathbb{R}$. Now, for any $p, q \in \mathbb{H}^{2n+1}$ we consider the continuous map

$$d_\infty(p, q) = N(p^{-1}q)\,.$$

It is easy to check that d_∞ is left invariant. We also have

$$N(\delta_r p) = \tilde{N}(rz, r^2 t) = r\, \tilde{N}(z, t) = r\, N(p)\,,$$

that yields the homogeneity. We notice that $F(p^{-1}) = -(z,t)$, so $N(p) = N(p^{-1})$ and the symmetry property of d follows. Now, to prove the triangle inequality it suffices to prove that

$$N(pq) \le N(p) + N(q)\,.$$

Denoting $F^{-1}(p) = (z,t)$ and $F^{-1}(q) = (w,s)$ we obtain

$$N(pq) = \max\{|z+w|, |t+s+2\mathrm{Im}\langle z,w\rangle|^{1/2}\}.$$

If $N(pq) = |z+w|$, we have

$$N(pq) \le |z| + |w| \le N(p) + N(q)\,.$$

If $N(pq) = |t+s+2\mathrm{Im}\langle z,w\rangle|^{1/2}$, we have

$$N(pq)^2 \le |t| + |s| + 2|w|\,|z| \le N(p)^2 + N(q)^2 + 2N(p)\,N(q)\,,$$

so d_∞ is a homogeneous distance on $\mathbb{H}^{2n+1}$.

Proposition 2.3.39 *Let ρ be the CC-distance of a sub-Riemannian group $\mathbb{G}$. Then ρ is a homogeneous distance.*

PROOF. By Remark 2.2.16 the continuity of ρ follows. The left invariance of g implies that translations $l_p : \mathbb{G} \longrightarrow \mathbb{G}$, $p \in \mathbb{G}$, are isometries, so horizontal curves are moved into horizontal curves preserving the velocities. From this we get

$$\rho(l_p w, l_p w') = \rho(w,w') \quad \text{for any} \quad p \in \mathbb{G}\,,$$

that yields the left invariance of ρ. To prove the homogeneity let us consider a horizontal curve $\gamma : [c,d] \longrightarrow \mathbb{G}$ that connects w and w' and define $\Gamma = \delta_r \circ \gamma$. Since $\delta_r X = rX$ whenever $X \in V_1$ we see easily that $|\Gamma'(t)| = r\,|\gamma'(t)|$ for a.e. $t \in [c,d]$, then $l_g(\Gamma) = r\,l_g(\gamma)$. The last equality yields $\rho(\delta_r w, \delta_r w') = r\,\rho(w,w')$. $\square$

Throughout the thesis we will also utilize the classical notions of jacobian and coarea factor in finite dimensional Hilbert spaces, [6]. Note that these spaces formally correspond to abelian sub-Riemannian groups.

Definition 2.3.40 (Jacobian) Let $\mathcal{G}$ and $\mathcal{M}$ be Hilbert spaces with dimensions q and p, respectively. Let $L : \mathcal{G} \longrightarrow \mathcal{M}$ be a linear map and assume that $q \le p$. The *jacobian* of L is the following number

$$\mathcal{J}_q(L) = \sqrt{\det(L^* \circ L)}\,,$$

where $L^* : \mathcal{M} \longrightarrow \mathcal{G}$ is the adjoint map.

Definition 2.3.41 (Coarea factor) Let $\mathcal{G}$ and $\mathcal{M}$ be Hilbert spaces with dimensions q and p, respectively. Let $L : \mathcal{G} \longrightarrow \mathcal{M}$ be a linear map and assume that $q \ge p$. Then the *coarea factor* of L is the following number

$$\mathcal{C}_p(L) = \sqrt{\det(L \circ L^*)}\,,$$

where $L^* : \mathcal{M} \longrightarrow \mathcal{G}$ is the adjoint map.

2.3.3 Graded coordinates

Graded coordinates represent a privileged system of coordinates that fits the geometry of the group. Here we present their main properties.

Definition 2.3.42 (Adapted basis) We denote $n_j = \dim V_j$ for any $j = 1, \dots, \iota$, $m_0 = 0$ and $m_i = \sum_{j=1}^{i} n_j$ for any $i = 1, \dots \iota$. We say that a basis $(W_1, \dots, W_q)$ of $\mathcal{G}$ is an *adapted basis*, if

$$(W_{m_{j-1}+1}, W_{m_{j-1}+2}, \dots, W_{m_j})$$

is a basis of V_j for any $j = 1, \dots \iota$.

Definition 2.3.43 (Graded coordinates) Let $\mathbb{G}$ be a graded group. A system of exponential coordinates (F, W) associated to an adapted basis of $\mathcal{G}$ will be called a system of *graded coordinates*. Posing $F(y) = \exp\Big(\sum_{i=1}^{q} y_i W_i\Big)$, we define for any $i = 1, \dots, q$ the *degree* of the coordinate y_i as $d_i = j + 1$, if $m_j \leq i \leq m_{j+1}$.

Remark 2.3.44 We emphasize the attention on the fact that whenever a graded metric on the graded group is considered together with a system of graded coordinates, then it is understood that the adapted basis of the system is orthonormal with respect to the graded metric. It is also understood that any graded metric on a graded group admits a system of graded coordinates with respect to an orthonormal basis. So, whenever a graded metric is considered on the group the system of graded coordinates will be understood with respect to an orthonormal adapted basis. We also mention that Definition 2.3.43 has a natural generalization in Carnot-Carathéodory spaces, see [15], [130].

Definition 2.3.45 (Coordinate dilations) Let (F, W) be a system of graded coordinates. We say that the maps $\Lambda_r : \mathbb{R}^q \longrightarrow \mathbb{R}^q$, with $r > 0$, defined as

$$\Lambda_r(\xi) = \sum_{j=1}^{q} r^{d_j} \, \xi_j \, e_j \, , \tag{2.33}$$

where (e_j) is the canonical basis of $\mathbb{R}^q$, are *coordinate dilations* with respect to (F, W).

Notice that coordinate dilations constitute a one parameter group with the product $\Lambda_{rs} = \Lambda_r \circ \Lambda_s$ for any $r, s > 0$.

Remark 2.3.46 Note that coordinate dilations commute with F as follows

$$F \circ \Lambda_r = \delta_r \circ F \, . \tag{2.34}$$

In the following proposition we analyze the relation between the Lebesgue measure in graded coordinates and the Riemannian volume.

Proposition 2.3.47 *Let $\mathbb{G}$ be a graded group and let (F, W) be a system of graded coordinates. Then we have the formula $F_\sharp \mathcal{L}^q = v_g$.*

PROOF. We know that $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ is a smooth diffeomorphism. Let A be a measurable set of $\mathbb{R}^q$. By the classical area formula and taking into account the left invariance of both v_g and $F_\sharp \mathcal{L}^q$ we have

$$c\,\mathcal{L}^q(A) = v_g(F(A)) = \int_A \mathcal{J}_q\left(dF(\xi)\right)\,d\xi$$

for some constant $c > 0$. Then $⨍_A \mathcal{J}_q(dF) = c$ for any measurable A. By continuity of $\xi \longrightarrow \mathcal{J}_q(dF(\xi))$ we obtain that $\mathcal{J}_q(dF(\xi)) = c$ for any $\xi \in \mathbb{R}^q$. We know that $F = \exp \circ L$, where $L(\xi) = \sum_{i=1}^q \xi_j\, W_j$ and (W_j) is an orthonormal basis of $\mathcal{G}$. Since the map $dF(0) = d\exp(0) \circ L = L$ has jacobian equal to one, then $c = 1$ and the thesis follows. □

The previous proposition and the notion of coordinate dilation allow us to establish an explicit formula for the Hausdorff dimension of a graded group endowed with a homogeneous distance. We have

$$v_g(B_{p,r}) = v_g(B_r) = c\,\mathcal{L}^q(F^{-1}(B_r)) = c\,\mathcal{L}^q(F^{-1} \circ \delta_r(B_1)) \tag{2.35}$$

$$= c\,\mathcal{L}^q(\Lambda_r F^{-1}(B_1)) = c\,r^Q\,\mathcal{L}^q(F^{-1}(B_1)) = r^Q\,v_g(B_1)\,. \tag{2.36}$$

The first equality of (2.35) follows by the fact that left translations are isometries with respect to the left invariant Riemannian metric, the third equality of (2.35) is a consequence of (2.32), the first equality of (2.36) follows by (2.34) and the second equality of (2.36) is due to a simple computation of the jacobian of Λ_r. In fact, by Definition 2.33, a simple calculation shows that $\mathcal{J}_q(\Lambda_r) = r^Q$ and we have the formula

$$Q = \sum_{j=1}^{\iota} j \dim V_j\,. \tag{2.37}$$

Then we have prove that

$$v_g(B_{p,r}) = r^Q\,v_g(B_1) \tag{2.38}$$

for every $p \in \mathbb{G}$ and $r > 0$, where Q is given by formula (2.37). Applying the classical result of Theorem 2.56 in [6] we can conclude from formula (2.38) that $\mathcal{H}_d^Q$ is finite and positive on open subsets of $\mathbb{G}$, hence the Hausdorff dimension of $\mathbb{G}$ is equal to Q. This fact is true for an arbitrary homogeneous distance. Furthermore, by left invariance of homogeneous distances it follows that $\mathcal{H}_d^Q$ is proportional to v_g. Throughout the thesis, it will be always assumed that the Hausdorff measure on a graded group is built with respect to a homogeneous distance and we will omit the symbol d when it will be clear from the context.

Polynomials on groups

Via graded coordinates we review some basics about polynomials on groups, see Chapter 1.C of [59].

Definition 2.3.48 (Polynomials) Let (F, W) be a system of graded coordinates of $\mathbb{G}$. We say that a function $P : \mathbb{G} \longrightarrow \mathbb{R}$ is a *polynomial* on $\mathbb{G}$ if the composition $P \circ F$ is a polynomial on $\mathbb{R}^q$.

Notice that if $(\tilde{F}, \tilde{W})$ is another system of graded coordinates the map $F^{-1} \circ F : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is a linear. Thus, $P \circ F$ is a polynomial if and only if $P \circ \tilde{F}$ is also and the previous definition does not depend on the fixed graded coordinates.

Definition 2.3.49 Let $\mathfrak{p}_j : \mathbb{R}^q \longrightarrow \mathbb{R}$ be the canonical projection $x \longrightarrow x_j$. We define the *graded projections* associated to a system of graded coordinates (F, W) as $\eta_j(s) = \mathfrak{p}_j\left(F^{-1}(s)\right)$ for any $s \in \mathbb{G}$. We will also use the simpler notation $x_j = x_j(s)$.

Note that any polynomial of $\mathbb{G}$ can be expressed in the form

$$P(s) = \sum_{\alpha \in \mathcal{A}} c_\alpha \eta^\alpha(s), \tag{2.39}$$

where $\eta^\alpha = \prod_{j=1}^q \eta_j^{\alpha_j}$ is a *monomial of graded projections* and $\mathcal{A}$ is a finite subset of $\mathbb{N}^q$.

Definition 2.3.50 We associate to a monomial of graded projections η^α the following integer

$$\deg_H(\eta^\alpha) = \sum_{j=1}^q d_j\, \alpha_j\,.$$

The *homogeneous degree* of a polynomial P with expression (2.39) is defined as follows

$$\deg_H(P) = \max_{\alpha \in \mathcal{A}} \left\{\deg_H(\eta^\alpha)\right\}\,.$$

We denote by $\mathcal{P}_{H,k}(\mathbb{G})$ the space of polynomials of homogeneous degree less than or equal to k.

For instance, in the Heisenberg group $\mathbb{H}^3$ with graded coordinates (x, y, t) with respect to the basis (X, Y, T) with $[X, Y] = T$, the polynomial $P(x, y, t) = t^2 - x^3$ has homogeneous degree equal to 4.

Proposition 2.3.51 *The homogeneous degree of a polynomial does not depend on the choice of graded coordinates.*

Proof. Let (F, W) and $(\tilde{F}, \tilde{W})$ be two system of graded coordinates. We have the linear relations

$$\tilde{W}_j = \sum_{k=1}^{q} A_j^k W_k, \qquad F^{-1} \circ \tilde{F}(\tilde{x}) = \sum_{k=1}^{q} \sum_{j=1}^{q} A_j^k \tilde{x}^j e_k ,$$

where $A_j^k = 0$ for any $m_{d_{j-1}} < k \le m_{d_j}$, due to the fact that (W_i) and $(\tilde{W}_i)$ are adapted bases. Then the $q \times q$ matrix $A = (A_j^k)$ has ι diagonal blocks of dimensions n_i for any $i = 1, \ldots, \iota$, where ι is the step of the group. Let us consider a monomial of graded projections $\eta^\alpha = x^\alpha$ with respect to the system (F, W) and represent it with respect to $(\tilde{F}, \tilde{W})$

$$x^\alpha = \prod_{k=1}^{q} \left(\sum_{k=1}^{q} A_j^k \tilde{x}^j \right)^{\alpha_k} .$$

Since the matrix A is invertible with diagonal blocks for any $k = 1, \ldots, q$ there exists $A_{j_k}^k \neq 0$, with $d_{k-1} < j_k \le d_k$. Moreover, in the sum $\sum_{k=1}^{q} A_j^k \tilde{x}^j$ we have $A_j^k = 0$ whenever $d_j \neq d_k$. It follows that

$$\deg_H \left(\sum_{k=1}^{q} A_j^k \tilde{x}^j \right)^{\alpha_k} = d_k \alpha_k .$$

As a result, observing that the homogeneous degree is additive on products of polynomials we obtain that

$$\deg_H(x^\alpha) = \sum_{k=1}^{q} d_k \alpha_k = \sum_{k=1}^{q} \deg_H \left(\sum_{k=1}^{q} A_j^k \tilde{x}^j \right)^{\alpha_k} = \deg_H \left(\prod_{k=1}^{q} \left(\sum_{k=1}^{q} A_j^k \tilde{x}^j \right)^{\alpha_k} \right) .$$

By the general representation (2.39) the latter equality yields our claim. □

Remark 2.3.52 It might be misleading to try to determine the homogeneous degree of a polynomial expressed with respect to coordinates that are not graded, but only of exponential type (Definition 2.3.13).

Consider the simple polynomial $P \circ \tilde{F}(x, y, t) = t$ of $\mathbb{H}^3$, where $(\tilde{F}, (T, X, Y))$ are exponential coordinates and $[X, Y] = T$. It might naively seem that the homogeneous degree of P is two, if one does not look carefully to the order of the basis. But, if we represent P with respect to the graded coordinates $(F, (X, Y, T))$ we obtain

$$P \circ F(x, y, t) = P \circ \tilde{F} \left((\tilde{F}^{-1} \circ F)(x, y, t) \right)$$

and $\tilde{F}^{-1} \circ F(x, y, t) = (t, x, y)$, hence

$$P \circ F(x, y, t) = P \circ \tilde{F} \left((t, x, y) \right) = y ,$$

now it is clear that the homogeneous degree of P is one.

Definition 2.3.53 (Homogeneous degree) A polynomial $P : \mathbb{G} \longrightarrow \mathbb{R}$ is *homogeneous of degree* $\alpha > 0$ if $P(\delta_r s) = r^\alpha P(s)$ for any $s \in \mathbb{G}$ and $r > 0$.

Note that all polynomials of homogeneous degree 0 are constants. In fact, if $P : \mathbb{G} \longrightarrow \mathbb{R}$ is of homogeneous degree 0, we have

$$P \circ \delta_r \circ F = P \circ F \circ \Lambda_r = \tilde{P} \circ \Lambda_r : \mathbb{R}^q \longrightarrow \mathbb{R}^q$$

and $\tilde{P}(\Lambda_r x) = \tilde{P}(x)$ implies

$$d\tilde{P}(x) = d\tilde{P}(\Lambda_r x) \circ \Lambda_r \longrightarrow 0 \quad \text{as} \quad r \to 0^+$$

for any $x \in \mathbb{R}^q$, hence $\tilde{P}$ is a constant function and P is also.

Left invariant vector fields

Here we obtain a standard representation in $\mathbb{R}^q$ of left invariant vector fields in $\mathcal{G}$ via graded coordinates. To get this representation, we will basically follow the approach adopted in [174], Chapter XIII, Section 5.

Let us fix a system of graded coordinates (F, W). We aim to obtain an explicit canonical representation of the vector fields $\tilde{W}_k = F_*^{-1} W_k \in \Gamma(T\mathbb{R}^q)$ for any $k = 1, \ldots, q$. We will need to consider translations read on $\mathbb{R}^q$ and a representation of the BCH formula (2.18) in graded coordinates.

Definition 2.3.54 (Coordinate translations) Let (F, W) be a system of graded coordinates and choose $x \in \mathbb{R}^q$. We say that the map

$$\tilde{l}_x = F^{-1} \circ l_{F(x)} \circ F : \mathbb{R}^q \longrightarrow \mathbb{R}^q$$

is the *coordinate translation* of x with respect to (F, W).

Now we write the coordinate translation $\tilde{l}_x$ in graded coordinates:

$$\tilde{l}_x y = F^{-1}\left(F(x)\, F(y)\right) = \sum_{j=1}^{q} P_j(x, y)\, e_j\,. \tag{2.40}$$

where by formula (2.18) we know that P_j are polynomials. Let us check that P_j are homogeneous polynomials of degree d_j. We have

$$\begin{aligned} \textstyle\sum_{j=1}^q P_j(x,y)\, r^{d_j}\, e_j &= \Lambda_r \left(\textstyle\sum_{j=1}^q P_j(x,y)\, e_j\right) = \Lambda_r \left(F^{-1}\left(F(x)\, F(y)\right)\right) \\ &= F^{-1}\left(\delta_r \left(F(x)\, F(y)\right)\right) = F^{-1}\left(\delta_r F(x) \delta_r F(y)\right) = F^{-1}\left(F(\Lambda_r x) F(\Lambda_r y)\right) \\ &= \textstyle\sum_{j=1}^q P_j(\Lambda_r x, \Lambda_r y)\, e_j\,, \end{aligned}$$

hence from both the first and the last term of the chain of equalities we deduce

$$P_j(\Lambda_r x, \Lambda_r y) = r^{d_j}\, P_j(x,y)\,.$$

The first observation is that $\tilde{W}_k$ is left invariant with respect to coordinate translations, due to the fact that W_k is also left invariant with respect to translations. Utilizing the representation of vector field $\tilde{W}_k$ as a derivation on a smooth map $\varphi : \mathbb{R}^q \longrightarrow \mathbb{R}$ and considering the translated map $y \longrightarrow \varphi \circ \tilde{l}_x(y)$, we obtain

$$\tilde{W}_k \varphi(x) = \partial_{y_k}(\varphi \circ \tilde{l}_x)(0) = \sum_{j=1}^{q} \partial_{x_j}\varphi(x)\, \partial_{y_k}\tilde{l}^j_x(0)\,.$$

We deduce from (2.40) that $\partial_{y_k}\tilde{l}^j_x(0) = \partial_{y_k}P_j(x,\cdot)(0) = a_{kj}(x)$ and the homogeneity yields

$$r^{d_j}\,\partial_{y_k}P_j(x,y)\Big|_{y=0} = \partial_{y_k}P_j(\Lambda_r x, \Lambda_r y)\Big|_{y=0} = r^{d_k}\,\partial_{y_k}P_j(\Lambda_r x,\cdot)(0)\,,$$

hence the polynomials a_{kj} are of homogeneous degree $d_j - d_k$. As a result, noting that $d_k > d_j$ implies $a_{kj}(x) = 0$ and that $d_k = d_j$ yields $da_{kj}(x) = 0$ for any $x \in \mathbb{R}^q$, we conclude that

$$\tilde{W}_k\varphi(x) = \sum_{d_k = d_j} \partial_{x_j}\varphi(x)\, c_{kj} + \sum_{d_k < d_j} \partial_{x_j}\varphi(x)\, a_{kj}(x)\,, \tag{2.41}$$

where c_{kj} are constants. Now we use the condition

$$\tilde{W}_k\varphi(0) = \frac{d}{dt}\left[\varphi \circ F^{-1}\left(\exp tW_k\right)\right]\Big|_{t=0} = \frac{d}{dt}\varphi(t\, e_k)\Big|_{t=0} = \partial_{x_k}\varphi(0)\,.$$

The last formula, together with (2.41), the condition $a_{kj}(0) = 0$ whenever $d_k < d_j$ and the arbitrary choice of φ, yield that

$$\tilde{W}_k\varphi(x) = \partial_{x_k}\varphi(x) + \sum_{d_k<d_j} \partial_{x_j}\varphi(x)\, a_{kj}(x) = \partial_{x_k}\varphi(x) + \sum_{j=m_{d_k}+1}^{q} \partial_{x_j}\varphi(x)\, a_{kj}(x)\,.$$

From condition $a_{kj}(\Lambda_r x) = r^{d_j - d_k}\, a_{kj}(x)$, with $d_j - d_k > 0$, we deduce that variables x_l with $d_l > d_j - d_k$ cannot appear in the polynomial expression of a_{kj}, then a_{kj} does not depend on x_l whenever $d_l \geq d_j$, i.e.

$$a_{kj}(x) = a_{kj}(x_1, \ldots, x_{j-1})\,.$$

Finally, we have proved that

$$\tilde{W}_k = \partial_{x_k} + \sum_{j=m_{d_k}+1}^{q} a_{kj}(x_1,\ldots,x_{j-1})\,\partial_{x_j}\,. \tag{2.42}$$

2.4 H-BV functions

Throughout the section we will denote by Ω an open subset of $\mathbb{G}$.

Definition 2.4.1 (Horizontal divergence) Let $(X_1, \ldots, X_m)$ be a basis of left invariant vector fields of V_1 and let $\varphi \in \Gamma(H\Omega)$. Writing $\varphi = \sum_{i=1}^m \varphi^j X_j$, the *horizontal divergence* (in short H-*divergence*) of φ is defined as follows

$$\mathrm{div}_H\, \varphi = \sum_{j=1}^m X_j \varphi^j \,.$$

Remark 2.4.2 In previous definition it is not used any Riemannian metric. Furthermore, it does not depend on the basis $(X_1, \ldots, X_m)$. Let $(Y_1, \ldots, Y_m)$ be another basis of left invariant vector fields of V_1. Then we have the relations $X_j = c_j^i Y_i$, where c_j^i are constants. Supposing that $\varphi = \sum_{j=1}^m \varphi^j X_j = \sum_{j=1}^m \left(\sum_{i=1}^m c_j^i\, \varphi^j \right) Y_j = \sum_{j=1}^m \tilde{\varphi}^j Y_j$, we have

$$\mathrm{div}_H\, \varphi = \sum_{j=1}^m Y_j \tilde{\varphi}^j = \sum_{i,j=1}^m Y_i(c_j^i\, \varphi^j) = \sum_{i,j=1}^m c_j^i\, Y_i \varphi^j = \sum_{j=1}^m X_j \varphi^j \,.$$

Definition 2.4.3 (H-BV functions) We say that a function $u \in L^1(\Omega)$ is a function of H-*bounded variation* (in short, a *H-BV function*) if

$$|D_H u|(\Omega) := \sup \left\{ \int_\Omega u\, \mathrm{div}_H \phi\, dv_g \,\middle|\, \phi \in \Gamma_c(H\Omega),\ |\phi| \leq 1 \right\} < \infty \,,$$

We denote respectively by $BV_H(\Omega)$ and $BV_{\mathrm{loc},H}(\Omega)$ the space of all functions of H-bounded variation and of locally H-bounded variation.

Remark 2.4.4 Notice that in the definition of H-BV function we have employed the Riemannian volume. However, in view of Proposition 2.3.47 our notion of H-BV function coincides with the usual one adopted in the literature once it is interpreted as referred to a system of graded coordinates.

Precisely, let us read the vector fields $X_j \in \Gamma(H\Omega)$ in $\mathbb{R}^q$, defining $\tilde{X}_i = F_*^{-1} X_i \in \Gamma(T\tilde{\Omega})$, where (F, W) is a system of graded coordinates and $\tilde{\Omega} = F^{-1}(\Omega) \subset \mathbb{R}^q$. We consider $\varphi = \sum_{j=1}^m \varphi^j X_j \in \Gamma_c(H\Omega)$, where $\tilde{\varphi}^i = \varphi^i \circ F$ for any $i = 1, \ldots, m$. Formula (2.9) yields

$$\tilde{X}_i \tilde{\varphi}^i = F_*^{-1} X_i(\varphi^i \circ F) = (X_i \varphi^i) \circ F$$

for any $i = 1, \ldots, m$. As a consequence of Definition 2.4.1, we have established

$$(\mathrm{div}_H \varphi) \circ F = \sum_{j=1}^m \tilde{X}_j \tilde{\varphi}^j \,, \tag{2.43}$$

consequently, by (2.1) and Proposition 2.3.47 it follows that

$$\int_\Omega u \,\mathrm{div}_H \phi \, dv_g = \int_\Omega u \,\mathrm{div}_H \phi \, dF_\sharp \mathcal{L}^q = \int_{\tilde\Omega} \tilde u \sum_{j=1}^m \tilde X_j \tilde\varphi_j \, d\mathcal{L}^q$$

where $\tilde u = u \circ F$ is the H-BV function read in graded coordinates. The last expression in the chain of equalities corresponds to the standard definition given in the literature, see [31], [69], [71], [72], [79].

Remark 2.4.5 The use of a left invariant metric reveals some advantages when one looks for some symmetry properties on the group. We will see in Chapter 5 that the existence of a large class of horizontal isometries on the group depends on the choice of the graded metric.

In Subsection 2.3.1 we have seen that different bases (2.27) and (2.28) induce isomorphic representations on the Heisenberg group. But this correspondence is not longer true from a metric point of view when we regard these bases are orthonormal. In fact, if we consider the graded metrics g_1 and g_2 on $\mathfrak{h}_{2n+1}$ such that (2.27) and (2.28) are orthonormal bases, respectively, it is clear that the metrics g_1 and g_2 are different. Now, if we think of $A \subset \mathbb{R}^q$ as a measurable subset in $\mathbb{G}$ with respect to the coordinates (2.27) we will not see the different value of the measure taking coordinates associated to (2.28). This apparently ambiguous situation can be clarified considering indeed different sets $F_1(A)$ and $F_2(A)$ in $\mathbb{G}$, where (F_1, W) and (F_2, S) are systems of graded coordinates associated to the bases (2.27) and (2.27), respectively.

We also observe that in view of Remark 2.4.2 the Definition 2.4.3 is independent of any frame of vector fields. As a result, the variational measure $|D_H u|$ depends only on the restriction of the left invariant metric g to $H\Omega$.

By Riesz Representation Theorem we get the existence of a nonnegative Radon measure $|D_H u|$ and a Borel section ν of $H\Omega$ such that $|D_H u|$-a.e. we have $|\nu| = 1$ and for any horizontal vector field $\phi \in \Gamma(H\Omega)$ we have

$$\int_\Omega u \,\mathrm{div}_H \phi \, dv_g = -\int_\Omega g(\phi, \nu) \, d\,|D_H u| \,. \tag{2.44}$$

Some remarks here are in order, since the canonical Riesz theorem deals with linear operators on spaces of continuous functions. In this case the space is $\Gamma(H\Omega)$ and we have used the scalar product in each fiber of the tangent spaces (indeed, strictly speaking ν should be thought of as a section of the cotangent bundle). Using local coordinates it is not hard to prove the extension of Riesz theorem we have used. The "vector" measure $\nu\,|D_H u|$, acting on bounded Borel sections ϕ of $H\Omega$ as in (2.44) is denoted by $D_H u$. Splitting $|D_H u|$ in absolutely continuous part $|D_H u|^a$ and singular part $|D_H u|^s$ with respect to the volume measure, we have the Radon-Nikodým decomposition $D_H u = D_H^a u + D_H^s u$, with $D_H^a u = \nu |D_H u|^a$, $D_H^s u = \nu |D_H u|^s$. We denote by $\nabla_H u$ the density of $D_H^a u$ with respect to the volume measure $\mathcal{H}^Q$. Note that

$$\int_E \nabla_H u \, dv_g = \nu \, |D_H u|^a(E)$$

for any $E \in \mathcal{B}(\Omega)$. Therefore the Borel map $\nabla_H u$ is a section of $H\Omega$.

Remark 2.4.6 For a.e. $x \in \Omega$ we have

$$\lim_{r \to 0^+} \frac{|D_H^s u|(U_{x,r})}{r^Q} = 0 \, .$$

Indeed, notice that from Radon-Nikodým Theorem we get a Borel subset $N \subset \Omega$ such that $|N| = 0$ and $|D_H^s u|(N^c) = 0$. Therefore, if we had a measurable subset $A \subset \Omega$, with $|A| > 0$ and

$$\limsup_{r \to 0^+} \frac{|D_H^s u|(U_{x,r})}{|U_1| \, r^Q} > 0 \, ,$$

for any $x \in A$ we would get $A' \subset A$ and $\lambda > 0$ such that $|D_H^s u|(A') \geq \lambda \, |A'| > 0$, see for instance Theorem 2.10.17 and Theorem 2.10.18 of [55]. Hence

$$|D_H^s u|(A' \setminus N) \geq \lambda |A' \setminus N| > 0 \, ,$$

which contradicts $|D_H^s u|(N^c) = 0$.

Proposition 2.4.7 *For every orthonormal basis $(X_1, \ldots, X_m)$ of $H\Omega$ we have*

$$\operatorname{div} \phi = \operatorname{div}_H \phi \, ,$$

where $\phi \in \Gamma(H\Omega)$ and div is the Riemannian divergence with respect to a graded metric.

Proof. We complete the horizontal orthonormal frame $(X_1 \ldots, X_m)$ to an orthonormal adapted basis $(X_1 \ldots, X_m, Y_{m+1}, \ldots, Y_q)$, so we are considering a graded metric. By definition of Riemannian divergence we have

$$\operatorname{div} \phi = \operatorname{Tr} D\phi = \sum_{i=1}^{m} g(D_{X_i}\phi, X_i) + \sum_{i=m+1}^{q} g(D_{Y_i}\phi, Y_i)$$

where D is the Riemannian connection. We choose $\phi \in \Gamma(H\Omega)$, with the representation $\phi = \sum_{i=1}^m \phi^i \, X_i$ for some smooth functions ϕ^i. By properties of Riemannian connection (using the summation convention) we have

$$g(D_{X_i}\phi, X_i) = g(X_i \phi^l \, X_l + \phi^l D_{X_i} X_l, X_i) = X_i \phi^i + \phi^l g(D_{X_i} X_l, X_i) \, ,$$

$$g(D_{X_i} X_l, X_i) = g([X_i, X_l], X_i) + g(D_{X_l} X_i, X_i) = 0 \, .$$

The last equation holds because $[X_i, X_j] \in V_2$ is orthogonal to $X_i \in V_1$ and

$$2g(D_{X_l}X_i, X_i) = X_l\,(g(X_i, X_i)) = 0\,.$$

Reasoning as above we get

$$g(D_{Y_i}\phi, Y_i) = g(Y_i\phi^l\, X_l + \phi^l D_{Y_i}X_l, Y_i) = \phi^l\, g(D_{Y_i}X_l, Y_i) = 0\,,$$

and this completes the proof. □

In view of Proposition 2.4.7 the H-divergence in Definition 2.4.3 can be replaced by the Riemannian divergence with respect to a graded metric (see Definition 2.3.30). This independence of the particular frame of vector fields cannot occur in general CC-spaces. In fact, the lack of a homogeneous structure forces the use of a particular frame of vector fields. However, with this fixed frame it is possible to construct a nonnegative matrix $A(x)$ (which should be interpreted as a degenerate Riemannian controvariant metric) and introduce the space $BV_A(\Omega)$, similarly to ours when we replace the div_H with the Riemannian divergence, see Definition 2.1.5 and Proposition 2.1.7 of [69].

Definition 2.4.8 We say that a measurable set $E \subset \Omega$ has *H-finite perimeter* in Ω when

$$P_H(E,\Omega) = |\partial E|_H(\Omega) = \sup\left\{\int_E \mathrm{div}_H\phi\; dv_g \,\middle|\, \phi \in \Gamma(H\Omega),\ |\phi| \le 1\right\} < \infty\,.$$

If $\Omega = \mathbb{G}$ we simply say that E has H-finite perimeter.

We will use both the notations $P_H(E,\Omega)$ and $|\partial E|_H$ to denote the perimeter measure. By previous discussion, $P_H(E, A)$ is the restriction to open sets A of a finite Borel measure in Ω. It is clear that if E has H-finite perimeter in Ω and $\mathbf{1}_E \in L^1(\Omega)$, then $\mathbf{1}_E \in BV_H(\Omega)$ and $|D_H\mathbf{1}_E|(F) = P_H(E, F)$, for any Borel set $F \subset \Omega$.

For a set of H-finite perimeter it is possible to introduce the notion of generalized inward normal.

Definition 2.4.9 (Generalized inward normal) Let E be a set of H-finite perimeter in Ω. The *generalized inward normal* to E is the measurable section ν_E of $H\Omega$ such that $D_H\mathbf{1}_E = \nu_E\,|D_H\mathbf{1}_E|$.

By the standard polar decomposition (Corollary 1.29 of [6]) we have that $|\nu_E(p)| = 1$ for $|D_H\mathbf{1}_E|$-a.e. $p \in \Omega$. and the formula of integration by parts (2.44) gives

$$\int_E \mathrm{div}_H\phi\; dv_g = -\int_\Omega \langle\phi, \nu_E\rangle\; d\,|\partial E|_H\,. \tag{2.45}$$

Now we point out some compatibility properties of the perimeter measure with respect to dilations and translations. Let E be a set of H-finite perimeter. Thus, directly from definition of perimeter measure we obtain

$$|\partial E|_H(\delta_r A) = r^{Q-1}\,|\partial\left(\delta_{1/r}E\right)|_H(A) \quad \text{and} \quad |\partial E|_H(l_p A) = |\partial\left(l_{p^{-1}}E\right)|_H(A) \tag{2.46}$$

for any open set $A \subset \mathbb{G}$. Clearly, these properties can be extended with no difficulties to any $|\partial E|_H$-measurable set of $\mathbb{G}$.

Definition 2.4.10 (H-reduced boundary) Let E be a set of H-finite perimeter in Ω. We say that a point $p \in \Omega$ belongs to the H-*reduced boundary* of E if

$$\lim_{r\to 0^+} \fint_{B_{p,r}} \nu_E \, d|\partial E|_H = \nu_E(p) \quad \text{and} \quad |\nu_E(p)| = 1\,. \tag{2.47}$$

The H-reduced boundary of E is denoted by $\partial_{*H}E$.

By a recent result of L. Ambrosio, [5], the H-perimeter measure is an asymptotically doubling measure, according to Definition 2.1.21. This result holds in a metric measure space that admits a $(1,1)$-Poincaré inequality and it is Ahlfors regular with respect to the distance. In a sub-Riemannian group the previous conditions hold for the CC-distance ρ of the group. Now we point out that if the asymptotically doubling property holds for (X,μ,ρ), then for any bilipschitz equivalent d the space (X,μ,d) is asymptotically doubling, according to Definition 2.1.21. Thus, for any homogeneous distance of the group the H-perimeter measure is asymptotically doubling and by Theorem 2.1.22 the family of closed balls in $\mathbb{G}$ form a $|\partial E|_H$-Vitali relation with respect to any homogeneous distance. In view of Theorem 2.9.8 of [55] and the previous discussion it is clear that for any homogeneous distance and $|\partial E|_H$-a.e. $p \in \mathbb{G}$ the conditions (2.47) hold. Thus, the H-reduced boundary $\partial_{*H}E$ is defined independently of the homogeneous distance up to $|\partial E|_H$-negligible sets and we have

$$|\partial E|_H(\mathbb{G} \setminus \partial_{*H}E) = 0\,. \tag{2.48}$$

2.5 Some general results

In this section we recall some important general theorems that will be used in the thesis. The open ball of center x and radius r with respect to the CC-distance of the group will be denoted by $U_{x,r}$. For the sake of simplicity we will simply write $|A| = v_g(A)$, for the Riemannian volume of measurable subsets.

We start recalling the coarea formula for H-BV functions, see [69], [79], [134], [141].

Theorem 2.5.1 (Coarea formula) *For any $u \in BV_H(\Omega)$ the following formula holds*

$$|D_H u|(\Omega) = \int_{\mathbb{R}} |\partial E_t|_H(\Omega)\, dt\,, \tag{2.49}$$

where $E_t = \{x \in \Omega \mid u(x) > t\}$.

A crucial tool in the Analysis on sub-Riemannain groups is the Poincaré inequality. This theorem holds for general vector fields that satisfy the *Chow condition*, see [100].

Theorem 2.5.2 (Poincaré inequality) *There exists a constant $C > 0$ such that for any C^∞ smooth map $w : \Omega \longrightarrow \mathbb{R}$ and any ball $U_{x,r}$ compactly contained in Ω, we have*

$$\int_{U_{x,r}} |w(z) - w_{U_{x,r}}|\, dz \le C\, r\, |D_H w|(U_{x,r})\,. \tag{2.50}$$

Now, we state an important theorem about the smooth approximation of H-BV functions, see either Theorem 2.2.2 of [69] or Theorem 1.14 of [79].

Theorem 2.5.3 (Smooth approximation) *Let $u : \Omega \longrightarrow \mathbb{R}$ be an H-BV function. Then there exists a sequence (u_k) of smooth functions such that*

1. *$u_k \longrightarrow u$ in $L^1(\Omega)$;*
2. *$|D_H u_k|(\Omega) \longrightarrow |D_H u|(\Omega)$.*

In view of (2.50) and Theorem 2.5.3 we obtain the following theorem.

Theorem 2.5.4 *Let $w : \Omega \longrightarrow \mathbb{R}$ be a locally H-BV function. Then for any ball $U_{x,r}$ compactly contained in Ω we have*

$$\int_{U_{x,r}} |w(z) - w_{U_{x,r}}|\, dz \le C\, r\, |D_H w|(U_{x,r})\,. \tag{2.51}$$

An important consequence of (2.51) is the local isoperimetric inequality for sets of H-finite perimeter.

Theorem 2.5.5 (Isoperimetric estimate) *Let E be a set of H-finite perimeter. Then for any $U_{x,r} \subset \mathbb{G}$ we have*

$$\min\{|U_{x,r} \cap E|, |U_{x,r} \setminus E|\} \le C\, r\, P_H(E, U_{x,r})\,. \tag{2.52}$$

It is a general fact that the Poincaré inequality (2.50) implies a Sobolev-Poincaré inequality, see for instance Theorem 2 of [66] or Theorem 1.15 (II) of [79]. This inequality can be extended to H-BV functions via Theorem 2.5.3.

Theorem 2.5.6 *Let $w : \Omega \longrightarrow \mathbb{R}$ be a locally H-BV function. Then*

$$\left(\fint_{U_{x,r}} |w(z) - w_{U_{x,r}}|^{1^*} \right)^{1/1^*} \leq C\, r\, \frac{|D_H w|(U_{x,r})}{|U_{x,r}|}\,, \tag{2.53}$$

whenever $U_{x,r}$ is compactly contained in Ω and $1^* = Q/(Q-1)$.

The following theorem is a consequence of Theorem 1.28 and Theorem 1.15 of [79].

Theorem 2.5.7 (Compact embedding) *Let U be a Carnot-Caratheodory ball of $\mathbb{G}$. Then for any $q \in [1, 1^*[$ the inclusion $BV_H(U) \hookrightarrow L^q(U)$ is compact.*

Proposition 2.5.8 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz map which vanishes at the origin and let $u \in [BV_H(\Omega)]^n$. Then $f \circ u : \Omega \longrightarrow \mathbb{R}$ is a H-BV function and*

$$|D_H(f\circ u)| \leq \operatorname{Lip}(f) \sum_{l=1}^{n} |D_H u^l|\,. \tag{2.54}$$

Proof. Let ϕ be a standard mollifier in $\mathbb{R}^n$ and consider $f_k(x) = f * \phi_{\varepsilon_k} - f * \phi_{\varepsilon_k}(0)$ for $x \in \mathbb{R}^n$, with $\varepsilon_k \to 0^+$ as $k \to \infty$. Then, $f_k(0) = 0$ for any $k \in \mathbb{N}$, (f_k) converges to f uniformly on bounded sets of $\mathbb{R}^n$, the Lipschitz constants of f_k are uniformly bounded by the Lipschitz constant of f. Now, we take smooth maps $(u_k^l)_k$ for any $l = 1, \ldots, n$, which approximate u^l as in Theorem 2.5.3, and consider the composition $h_k = f_k \circ u_k \in C^\infty(\Omega)$, where $u_k = (u_k^l)$. One can easily verify that under these conditions $h_k \longrightarrow f\circ u$ in $L^1(\Omega)$. In order to get the estimate (2.54) we consider

$$\int h_k \operatorname{div}\varphi = -\int \sum_{i=1}^{m} \varphi^i X_i(h_k) = -\sum_{l=1}^{n} \int (\partial_{x_l} f_k)\circ u \sum_{i=1}^{m} \varphi^i\, X_i u_k^l\,,$$

where $\varphi = \sum_{i=1}^m \varphi^i X_i$, $|\varphi| \leq 1$ and $(X_1, \ldots, X_m)$ is an orthonormal basis of $H\Omega$. In view of the last equality we have

$$\left| \int h_k \operatorname{div}\varphi \right| \leq L \sum_{l=1}^{n} \int \Big[\sum_{i=1}^{m} (X_i u_k^l)^2 \Big]^{1/2} = L \sum_{l=1}^{n} |D_H u_k^l|\,.$$

Letting $k \to \infty$, the thesis follows by condition 2 of Theorem 2.5.3 and the convergence in $L^1(\Omega)$ of (h_k). □

Definition 2.5.9 (Maximal operator) We consider a nonnegative Radon measure ν in Ω. For each $r > 0$ the *restricted maximal function* of ν is defined as follows

$$M_r\nu(x) := \sup \left\{ \frac{\nu(U_{x,t})}{|U_{x,t}|} :\ 0 < t < r,\, U_{x,t} \subset \Omega \right\} \qquad x \in \Omega\,.$$

The *maximal function* of ν is defined as $M\nu(x) = \sup_{r>0} M_r\nu(x)$. If the measure ν is induced by a locally integrable function $f : \Omega \longrightarrow \mathbb{R}$, we define analogously

$$M_r f(x) := \sup \left\{ \fint_{U_{x,t}} |f(y)| \, dy : \; 0 < t < r, \, U_{x,t} \subset \Omega \right\}$$

and $Mf(x) = \sup_{r>0} M_r f(x)$.

It is well known that the maximal operator is (1,1)-weakly continuous, i.e. there exists a constant $C > 0$ such that

$$|\{x \in E \mid M\nu(x) > t\}| \leq \frac{C}{t} \, \nu(E) \,, \tag{2.55}$$

for any Borel set $E \subset \Omega$ and any $t > 0$, see for instance [10]. Inequality (2.55) implies that if ν is a finite measure, then $M\nu$ is finite a.e. in Ω.

Chapter 3

Calculus on sub-Riemannian groups

This chapter is mainly devoted to the concept of "H-differentiability" and to some related applications. In this setting the notion of differentiability can be formulated in a purely intrinsic way, using the operation of the group and the homogeneous structure given by dilations. With this notion we develop to some extent a "Calculus on sub-Riemannian groups", showing some basic theorems of classical analysis, as the chain rule formula and the inverse mapping theorem. Clearly these results generalize the classical ones of Euclidean spaces. However, in the proof of the inverse mapping theorem we will follow a novel approach.

The privileged role played by differentiability in classical Geometric Measure Theory still reveals a potentially rich variety of applications in the geometry of sub-Riemannian groups. With this tool we are also able to define in any codimension different "intrinsic" notions of rectifiable set. For instance, the notion of $\mathbb{G}$-rectifiability introduced in [71], [73], has been proved to be the "right" concept to study sets of H-finite perimeter. An important structure theorem holds in 2-steps sub-Riemannian groups: all sets of H-finite perimeter are $\mathbb{G}$-rectifiable (Definition 3.5.2), see [73].

It is well understood that the classical Rademacher Theorem on differentiability of Lipschitz maps is a powerful tool in classical Geometric Measure Theory, [55], [131]. An important part of the chapter is also devoted to the proof of a.e. differentiability of Lipschitz maps in the sub-Riemannian case. In a remarkable paper [154] P. Pansu proved that any Lipschitz map $f : A \longrightarrow \mathbb{M}$ is a.e. H-differentiable provided that A is an open subset of $\mathbb{G}$. We extend the Pansu result to a slightly more general situation, requiring that A is only measurable. This generalization requires some effort, since no Lipschitz extension theorem is presently known in this general setting. Although we follow essentially the Pansu approach, our proof involves some nontrivial technical adjustments due to the fact that the interior of A could be empty, [124]. This extension was first proved in [177], where some technical details were overlooked and subsequently corrected in [184]. The motivation for this extension comes from the need of considering a manageable version of the area formula, [124], and to get basic properties for $(N, \mathbb{G})$-rectifiable sets, when N is a stratified group (Definition 3.5.4). We also prove by a counterexample that the hypothesis for differentiability of Lipschitz maps are basically sharp: if the target group has a left invariant distance which is not homogeneous with respect to dilations, then it is possible to construct a nowhere differentiable Lipschitz map, [111]. Now, let us look more closely to the content of this chapter.

In Section 3.1 we introduce H-linear maps and we study their properties. We prove that H-linear maps are indeed linear, if they are read between the corresponding Lie algebras, so they form a subclass of all linear maps. This should give a naive explanation of why the geometry of these groups is "rigid", see Remark 4.3.8 and Theorem 4.4.6. Moreover, in Theorem 3.1.12 we provide a simple metric characterization of H-linear maps and we prove their "contact property".

In Section 3.2 the notion of H-differential for maps $f : A \subset \mathbb{G} \longrightarrow \mathbb{M}$ is given. We show that the differential of Lipschitz maps does not depend on any Lipschitz extension that coincides in a set with the same density point, Proposition 3.2.4 and we prove the chain rule formula for composition of differentiable maps. We introduce H-continuously differentiable maps of any order, observing that real valued C^1 maps are C^1_H (Proposition 3.2.8). However, this implication is no longer true for group valued maps as we show in Examples 3.2.9 and 3.2.10.

In Section 3.3 we obtain the inverse mapping theorem for H-continuously differentiable maps of sub-Riemannian groups (Theorem 3.3.3). Its proof follows an entirely different argument with respect to the standard one. In fact, the classical argument to obtain the bilipschitz property in a neighbourhood of a point where the map has invertible differential strongly relies on the commutativity of Euclidean spaces. Here we adapt the general linearization procedure of Lemma 3.2.2 in [55] to C^1_H smooth maps, where the additional information on regularity of differential $x \longrightarrow d_H f(x)$ gives the Lipschitz estimate (3.15) in an open ball, instead of a measurable set.

The core of Section 3.4 is Theorem 3.4.11, i.e. Lipschitz maps of sub-Riemannian groups are a.e. H-differentiable. The main difficulty in proving this theorem arises from the fact that a Lipschitz extension theorem for maps of sub-Riemannian groups is still not known. So, when we fix a point $x \in A \cap \mathcal{I}(A)$ and a direction $w \in \mathcal{G}$, it might happen that $x \exp(tw) \notin A$ for many $t > 0$ and so we are not able to consider the difference quotient of f in that direction. The leading idea is to consider the "generating property" of bases (v_i) of V_1 (see Proposition 2.3.22) and to select all density points x whose curves $J_i(t) = x \exp(tv_i)$ intersect A in one dimensional sets which have density 1 at $t = 0$, getting a set of full measure in A. At these points we are able to approximate any curve $c(t) = \exp(\delta_t z)$, $z \in \mathcal{G}$, with a path built with projections on A of horizontal lines with controlled distance. All of this procedure is performed by induction. Finally, the difference quotient of f is approximated by the difference quotient along these paths, where f is defined and uniformly differentiable along the horizontal directions.

Section 3.5 is devoted to the presentation and discussion of different notions of "intrinsic" rectifiability. In the cycle of papers [71], [72], [73] B. Franchi, R. Serapioni and F. Serra Cassano have introduced and studied the notion of $\mathbb{G}$-rectifiability, where regular hypersurfaces are seen as level sets of real valued C^1_H maps with nonvanishing H-differential. With this notion they have proved the celebrated De Giorgi Rectifiability Theorem for sets of H-finite perimeter in sub-Riemannian groups of step 2, [73]. It turns out that this notion fits the geometry of the group. Another independent notion of rectifiability is given in [156], where rectifiable sets are regarded as Lipschitz images of subsets contained is some subgroup. This is a reasonable extension of the Federer notion of rectifiable set, see 3.2.14 of [55]. However, the problem of establishing some equivalence between this notion and the $\mathbb{G}$-rectifiability seems to be an hard question. We extend the notion of $\mathbb{G}$-rectifiability in higher codimension, introducing the $(\mathbb{G},\mathbb{M})$-rectifiability, namely, regular sets are regarded as level sets of maps in $C^1_H(\mathbb{G},\mathbb{M})$ with surjective H-differential. Note that $\mathbb{G}$-rectifiability corresponds to the case $\mathbb{M}=\mathbb{R}$. Clearly the class of $(\mathbb{G},\mathbb{M})$-rectifiable sets depends on $\mathbb{M}$. So by means of $\mathbb{M}$ we can consider several geometries to be investigated in $\mathbb{G}$. But it may happen that some $(\mathbb{G},\mathbb{M})$-rectifiable classes are empty. In this perspective the "right" choice of $\mathbb{M}$ should yield an as large as possible class of $(\mathbb{G},\mathbb{M})$-rectifiable sets. For instance, if $\mathbb{G}=\mathbb{H}^{2n+1}$ it is convenient to choose $\mathbb{M}=\mathbb{R}^k$. Hence we obtain nontrivial classes of rectifiable sets of Hausdorff dimension $2n+2-k$ and topological dimension $2n+1-k$, for any $k=1,\dots,2n$. Furthermore, there exists also a rich class of $(\mathbb{R}^k,\mathbb{H}^{2n+1})$-rectifiable sets of Hausdorff dimension k and topological dimension k for any $k=1,\dots,n$. The last assertion is due to the existence of "horizontal surfaces" in $\mathbb{H}^{2n+1}$ whenever their dimension is less than $n+1$. Notice that horizontal curves are included in $(\mathbb{R},\mathbb{H}^{2n+1})$- rectifiable objects. It turns out that both definitions of rectifiability we have adopted complete the picture of rectifiable sets in $\mathbb{H}^{2n+1}$.

In Section 3.6 we present a counterexample to a.e. H-differentiability of Lipschitz maps (Theorem 3.4.11) as soon as we replace the homogeneous distance in the target with another left invariant distance that is not homogeneous with respect to dilations. This is accomplished by taking the identity map of the three dimensional Heisenberg group $I:\mathbb{H}^3\longrightarrow\mathbb{H}^3$ and building such a particular non homogeneous left invariant distance on the codomain. More precisely, we show a slightly stronger fact, i.e. that the map I is also not metrically differentiable, according to Definition 3.6.2, so in particular it is not differentiable in the sense of Definition 3.2.1. We mention that if $f:A\longrightarrow Y$, with $A\subset\mathbb{R}^n$, is a metric space valued Lipschitz map, then in [7], [110] and [115] it was proved that f is a.e. metrically differentiable. In view of our counterexample it follows that there is no hope to extend these Lipschitz differentiability results when A is a subset of some stratified group.

3.1 H-linear maps

The notion of differentiability can be modeled with respect to a fixed family of maps with suitable geometric properties. Then, one requires that a differentiable function at a fixed point has an approximation of the first order with a map of such a family. Clearly these class of maps constitutes just the family of intrinsic differentials. This general idea was pursued in general metric spaces in the remarkable paper [35].

In case of stratified groups this class of differentials is formed by homogeneous group homomorphisms, i.e. H-linear maps. These maps play the same role of linear maps in Euclidean spaces, indeed the class of H-linear maps coincides with that of linear maps when the group is an Euclidean space (i.e. an abelian sub-Riemannian group). We will see in Proposition 3.1.3 that in general H-linear maps can be seen as a subclass of all linear maps.

Definition 3.1.1 Let $L : \mathbb{G} \longrightarrow \mathbb{M}$ be a map of stratified groups. We say that L is *homogeneous* if $\delta_r(Lx) = L(\delta_r x)$ for every $r > 0$.

Definition 3.1.2 (H-linear maps) We say that $L : \mathbb{G} \longrightarrow \mathbb{M}$ is a *horizontal linear map* (shortly, H-linear map) if it is a homogeneous Lie group homomorphism.

Proposition 3.1.3 *Let $\mathbb{G}$ and $\mathbb{M}$ be nilpotent simply connected Lie groups and let $L : \mathbb{G} \longrightarrow \mathbb{M}$ be a continuous group homomorphism. Then L can be read between the Lie algebras as $\tilde{L} = \ln \circ L \circ \exp : \mathcal{G} \longrightarrow \mathcal{M}$ and $\tilde{L}$ is an algebra homomorphism.*

Proof. Since L is continuous, then Theorem 3.39 of [187] implies that it is C^∞. Thus, by Theorem 3.32 of [187] we have that L can be written as $\exp \circ dL(e) \circ \ln$, where $dL(e)$ is an algebra homomorphism, so the proof is complete. □

Definition 3.1.4 We denote by $\mathrm{HL}(\mathbb{G}, \mathbb{M})$ the class of H-linear maps between $\mathbb{G}$ and $\mathbb{M}$. Given $T, L \in \mathrm{HL}(\mathbb{G}, \mathbb{M})$ and $t \in \mathbb{R}$ we define the new functions $\delta_t T$, $T \cdot L$, $-T : \mathbb{G} \longrightarrow \mathbb{M}$ as $\delta_t T(u) = \delta_t(T(u))$, $T \cdot L(u) = T(u)L(u)$, $-T(u) = (T(u))^{-1}$ for any $u \in \mathbb{G}$. We define $\mathrm{HL}(\mathcal{G}, \mathcal{M})$ as the set of all maps $L : \mathcal{G} \longrightarrow \mathcal{M}$ such that $\exp \circ L \circ \ln \in \mathrm{HL}(\mathbb{G}, \mathbb{M})$.

Remark 3.1.5 (Group of H-linear maps) It turns out that $\mathrm{HL}(\mathbb{G}, \mathbb{M})$ has a natural structure of Lie group with respect to the operation introduced in the previous definition. We also notice that any map of $\mathrm{HL}(\mathbb{G}, \mathbb{M})$ induces uniquely a map of $\mathrm{HL}(\mathcal{G}, \mathcal{M})$ and viceversa. We will prove that any $T \in \mathrm{HL}(\mathcal{G}, \mathcal{M})$ is linear, preserves the bracket operation and $L(V_1) \subset W_1$. Finally, we point out that in both $\mathrm{HL}(\mathbb{G}, \mathbb{M})$ and $\mathrm{HL}(\mathcal{G}, \mathcal{M})$ there is a natural group of dilations, $T \longrightarrow \delta_\lambda T$, $\lambda > 0$.

Definition 3.1.6 Given $T, L \in \mathrm{HL}(\mathbb{G}, \mathbb{M})$ we define

$$\rho(T, L) = \sup_{d(u) \leq 1} \rho\left(T(u), L(u)\right)$$

as the distance between T and L. If L is identically equal to the unit element of $\mathbb{M}$, the distance between T and L corresponds to the *norm* of T, denoted with $\rho(T)$. If we do not need to emphasize the distance that defines the norm, we can simply denote it by $\|T\|$. Analogous definitions hold for maps in $\mathrm{HL}(\mathcal{G},\mathcal{M})$.

Remark 3.1.7 The norm defined above on the group $\mathrm{HL}(\mathbb{G},\mathbb{M})$ induces a homogeneous distance that makes the group a complete metric space.

Proposition 3.1.8 *Any function $T \in \mathrm{HL}(\mathbb{G},\mathbb{M})$ is continuous and the distance of Definition 3.1.6 is a finite number, making $\mathrm{HL}(\mathbb{G},\mathbb{M})$ a complete metric space. Moreover, for any $u \in \mathbb{G}$ we have the estimate $\rho(T(u)) \leq \rho(T)\, d(u)$.*

PROOF. Fix a basis $\{v_i \mid i = 1,\dots,m\}$ of V_1. By Proposition 2.3.22, after a rescaling we obtain that

$$E = \left\{ \prod_{s=1}^{\gamma} \exp(a_s v_{i_s}) \mid (a_s) \subset U \right\} \supset \{u \in \mathbb{G} \mid d(u) \leq 1\},$$

where $U \subset \mathbb{R}^\gamma$ is a bounded neighbourhood of the origin. By triangle inequality we get the estimate

$$\rho(T) \leq \left(\sup_{a\in U} |a|\right) \sum_{i=1}^{\gamma} \rho\left(T(v_{i_s})\right) < \infty .$$

The homogeneity of ρ implies the inequality $\rho\left(T(u)\right) \leq \rho(T)\, d(u)$ for every $u \in \mathbb{G}$. Considering the map $T^{-1} \cdot L \in \mathrm{HL}(\mathbb{G},\mathbb{M})$ we have proved that the distance between T and L is finite. Of course $\rho(T) = 0$ implies that T is the null map, the triangle inequality and symmetry property of the distance follow directly from that of the metric ρ in $\mathbb{M}$. The homogeneity of ρ on $\mathbb{G}$ gives the homogeneity of the distance in $\mathrm{HL}(\mathbb{G},\mathbb{M})$. Even the continuity is straightforward from the same inequality. The completeness of $\mathrm{HL}(\mathbb{G},\mathbb{M})$ easily follows by the completeness of $\mathbb{M}$. $\Box$

Corollary 3.1.9 *Let $L : \mathbb{G} \longrightarrow \mathbb{M}$ be an injective H-linear map and $L(\mathbb{G}) = \mathbb{S}$. Then $\mathbb{S}$ is a stratified subgroup of $\mathbb{M}$ and $L^{-1} : \mathbb{S} \longrightarrow \mathbb{G}$ is H-linear with*

$$d\left(L^{-1}(y)\right) \leq \|L^{-1}\|\, \rho(y), \qquad \|L^{-1}\| < \infty . \tag{3.1}$$

PROOF. Clearly $\mathbb{S}$ is a subgroup of $\mathbb{M}$ and the contact property $L(\mathbb{V}_1) \subset \mathbb{W}_1$ implies the stratification. In fact, denoting $\tilde{L} = dL(e)$ we have

$$[\tilde{L}(V_i), \tilde{L}(V_1)] = \tilde{L}([V_i, V_1]) = \tilde{L}(V_{i+1}),$$

so $\mathbb{S}$ is a stratified subgroup and $L^{-1} : \mathbb{S} \longrightarrow \mathbb{G}$ is H-linear. Finally, Proposition 3.1.8 yields the estimate (3.1). $\Box$

Corollary 3.1.10 *Consider* $L, T \in \mathrm{HL}(\mathbb{G},\mathbb{M})$ *and* $S \in \mathrm{HL}(\mathbb{M},\mathbb{T})$*, where* $(\mathbb{G},d)$*,* $(\mathbb{M},\rho)$*,* $(\mathbb{T},\nu)$ *are stratified Lie groups. Then* $S{\circ}L \in \mathrm{HL}(\mathbb{G},\mathbb{T})$*,* $L\cdot T \in \mathrm{HL}(\mathbb{G},\mathbb{M})$ *and*

$$\|S{\circ}L\| \leq \|S\|\,\|L\|\,, \qquad \|L\cdot T\| \leq \|L\| + \|T\|\,. \tag{3.2}$$

Proof. It is an easy computation, using Proposition 3.1.8 and the triangle inequality. □

Corollary 3.1.11 *Any map* $L \in \mathrm{HL}(\mathcal{G},\mathcal{M})$ *is an algebra homomorphism.*

Proof. Proposition 3.1.8 implies the continuity and Proposition 3.1.3 yields the thesis. □

Theorem 3.1.12 (Characterization) *Any homomorphism* $L:\mathbb{G}\longrightarrow\mathbb{M}$ *is an H-linear map if and only it is a Lipschitz map and in this case it has the contact property* $L(\mathbb{V}_j)\subset\mathbb{W}_j$ *for every* $j=1,\dots,\iota$*.*

Proof. Proposition 3.1.8 implies the Lipschitz property of L if it is H-linear. Viceversa, consider a Lipschitz homomorphism $L:\mathbb{G}\longrightarrow\mathbb{M}$. We introduce the auxiliary homogeneous norm

$$\|x\| = \sum_{j=1}^{\iota} |x_j|^{1/j}\,,$$

where $x=\exp\left(\sum_{j=1}^{\iota} x_j\right)\in\mathbb{G}$, $x_j\in V_j$ and $|\cdot|$ is a norm on $\mathcal{G}$. Reasoning as in the proof of Proposition 2.3.37 we easily obtain that $c_1\|\cdot\|\leq d(\cdot,e)\leq c_2\|\cdot\|$ with $c_1,c_2>0$. We choose $v\in\mathbb{V}_1$ and write $L=\exp\left(\sum_{j=1}^{\iota} L_i\right)$, where $L_i:\mathbb{G}\longrightarrow V_i$. By the Lipschitz property we have

$$\|\delta_{1/t}L(\delta_t v)\| = \sum_{j=1}^{m} |t^{1-j}L_i(v)| \leq \mathrm{Lip}(L)$$

for any $t>0$. Then we have $L_i(v)=0$ for every $i=2,\dots,\iota$ and $L(v)\in\mathbb{W}_1$, where $\mathcal{M}=W_1\oplus\cdots\oplus W_v$ and $\mathbb{W}_j=\exp W_j$. Therefore the homomorphism property yields $L(\mathbb{V}_j)\subset\mathbb{W}_j$, for any $j=1,\dots,\iota$. As a result, defining $\tilde{L}=\ln\circ L\exp$, $x=\exp\left(\sum_{j=1}^{\iota} x_j\right)$ and the linear maps $\tilde{L}_j=L_j\circ\exp$, where $j=1,\dots,\iota$, we have

$$\begin{aligned} L(\delta_t x) &= \exp\Big(\sum_{j=1}^{m}\tilde{L}_j\left(\ln(\delta_t x)\right)\Big) = \exp\Big(\sum_{j=1}^{m}\tilde{L}_j\Big(\sum_{l=1} t^l x_l\Big)\Big) \\ &= \exp\Big(\sum_{j=1}^{m} t^j\tilde{L}_j(x_j)\Big) = \exp\Big(\delta_t\sum_{j=1}^{m}\tilde{L}_j(x_j)\Big) \qquad (3.3) \\ &= \exp\Big(\delta_t\tilde{L}\Big(\sum_{j=1}^{m} x_j\Big)\Big) = \delta_t\left(L(x)\right)\,, \qquad (3.4) \end{aligned}$$

where the equalities of (3.3) and the first one of (3.4) follow by the fact that $\tilde{L}(x_j) \in W_j$ for every $j = 1, \ldots, \iota$. The above chain of equalities proves the homogeneity of L, so the proof is complete. $\Box$

Remark 3.1.13 Notice that from the previous proof we deduce also that Lipschitz homomorphisms of graded groups are H-linear. In fact, the hypothesis that $\mathbb{G}$ and $\mathbb{M}$ are stratified was used only in the opposite implication.

The following example is taken from [151].

Example 3.1.14 We consider a basis of (X, Y, T) of the Heisenberg algebra $\mathfrak{h}_3$, with $[X, Y] = T$. Then the group operation in coordinates is as follows

$$(x, y, t) \cdot (x', y', t') = \big(x + x', y + y', t + t' + (xy' - yx')/2\big) \ .$$

It is easy to check that all H-linear maps $L : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ can be represented with respect to the basis (X, Y, T) with matrices of the following form

$$[L] = \begin{pmatrix} a^1_1 & a^1_2 & 0 \\ a^2_1 & a^2_2 & 0 \\ 0 & 0 & \det(A) \end{pmatrix} . \tag{3.5}$$

where $A = (a^i_j)_{i,j=1,2}$.

Remark 3.1.15 Notice that any H-linear map can be represented by a matrix with diagonal blocks. This basically follows from the contact property stated in Theorem 3.1.12. In fact, if we have the gradings $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$, $\mathcal{M} = W_1 \oplus \cdots \oplus W_\upsilon$ the contact property of H-linear maps implies

$$L_{|V_i} : V_i \longrightarrow W_i \tag{3.6}$$

for any $i \geq 1$ (taking into account that spaces V_i and W_i are null spaces when i is greater than the degree of nilpotency of the group). We point out that the general explicit computation of the coefficients of L with respect to a fixed basis can be very involved. This is due to the fact that the group operation given by the BCH formula (2.18) becomes a large polynomial expression as the step of nilpotence of the group increases. So this general expression for the matrix seems to be an open computational problem.

Let us consider another simple example.

Example 3.1.16 Let $L : \mathbb{E}^2 \longrightarrow \mathbb{H}^3$ be an H-linear map and consider the canonical basis (e_1, e_2) of the Euclidean space $\mathbb{E}^2$ and the basis (X, Y, Z) of $\mathbb{H}^3$ used in the previous example. Then, in view of Remark 3.1.15 the representation of L with respect to the above fixed bases is as follows

$$[L] = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \\ 0 & 0 \end{pmatrix} . \tag{3.7}$$

3.2 The instrinsic differential

In this section we introduce the concept of H-differential for maps of graded groups endowed with an homogeneous distance. The metric spaces $(\mathbb{G}, d)$, $(\mathbb{M}, \rho)$ and $(\mathbb{P}, \nu)$ will indicate graded groups with their corresponding homogeneous distances. We will denote by A and Ω a measurable subset and an open subset of $\mathbb{G}$, respectively.

Definition 3.2.1 (H-Differentiability) We say that $f : A \longrightarrow \mathbb{M}$ is H-*differentiable* (or simply *differentiable*) at $x \in \mathcal{I}(A) \cap A$ if there exists an H-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ such that

$$\lim_{y \in A,\ y \to x} \frac{\rho(f(x)^{-1}f(y), L(x^{-1}y))}{d(x,y)} = 0\,. \tag{3.8}$$

Notice that when A is an open set, Definition 3.2.1 coincides with Pansu definition of differentiability [154]. Indeed this notion is also called *Pansu differentiability.* We simply speak of differentiable functions, due to the fact that in stratified groups it is understood that the use of dilations, of the group operations and of the homogeneous distance, are exactly what we need to define an “intrinsic” concept of differentiability. Furthermore, when the group $\mathbb{G}$ is an Euclidean space, Definition 3.2.1 coincides with the classical definition of differentiability. However, we will often use the terminology H-*differentiability*, when we want to emphasize the “intrinsic” notion in the sense of Definition 3.2.1. The prefix “H” stands for “horizontal”, indeed in the proof of Theorem 3.4.11 we will see that the intrinsic differential is entirely reconstructed by derivatives along “horizontal” directions. From the other side, the same principle holds for H-linear maps, due to the fact that any element of the group can be written as a finite product of a fixed basis of horizontal elements (Proposition 2.3.22).

Next, we show that the H-linear map of Definition 3.2.1 is unique.

Proposition 3.2.2 (Uniqueness) *Let $f : A \longrightarrow \mathbb{M}$ and let $x \in \mathcal{I}(A) \cap A$. If f satisfies limit (3.8) with respect to H-linear maps L and L', then $L = L'$.*

PROOF. Let $\omega \in \mathbb{G}$ be an arbitrary element with $d(\omega) = 1$. By Lemma 2.1.15 we know that $d(x\delta_t\omega, A) = o(t)$. Let us choose $y_t \in A$ such that $d(x\delta_t\omega, A) + t^2 > d(x\delta_t\omega, y_t)$ and write the estimate

$$\rho\left((-L)L'(\omega)\right) \le \frac{\rho\left((-L)L'(x^{-1}y_t)\right)}{t} + \frac{\rho\left((-L)L'(y_t^{-1}x\delta_t\omega)\right)}{t}\,. \tag{3.9}$$

The second of the above addenda goes to zero as $t \to 0^+$, due to both the behaviour of y_t and the Lipschitz property of $(-L)L'$. The first one can be estimated as follows

$$\frac{\rho\left((-L)L'(x^{-1}y_t)\right)}{t} \le \frac{\rho\left(L'(x^{-1}y_t), f(x)^{-1}f(y_t)\right)}{t} + \frac{\rho\left(L(x^{-1}y_t), f(x)^{-1}f(y_t)\right)}{t}\,.$$

By previous estimate and taking into account that $t^{-1}d(x, y_t)$ is bounded and f is differentiable with respect to both L and L', the thesis follows. □

Definition 3.2.3 (H-differential) Let $f : A \longrightarrow \mathbb{M}$ be differentiable at $x \in \mathcal{I}(A) \cap A$. We denote by $d_H f(x)$ the unique H-linear map L, which satisfies (3.8). We will call $d_H f(x)$ the H-*differential* of f at x, or simply *differential* when it will be clear from the context that it is referred to sub-Riemannian groups.

In Section 3.4 a uniqueness result for maps with different domains will be needed. The following proposition shows that in the class of Lipschitz maps the differential is unique and essentially independent of the domain.

Proposition 3.2.4 *Let $f : A \subset \mathbb{G} \longrightarrow \mathbb{M}$, $g : B \subset \mathbb{G} \longrightarrow \mathbb{M}$ be Lipschitz maps, with $x \in \mathcal{I}(A \cap B) \cap (A \cap B)$, $f = g$ on $A \cap B$ and suppose that f satisfies (3.8). Then the map g is differentiable at x and*

$$\lim_{y \in B,\, y \to x} \frac{\rho(g(x)^{-1} g(y), L(x^{-1}y))}{d(x,y)} = 0\,.$$

The proof of the above proposition can be obtained similarly to that of Proposition 3.2.2, again exploiting Lemma 2.1.15.

Proposition 3.2.5 (Chain rule) *Let $f : A \longrightarrow \mathbb{P}$ be differentiable at $x \in \mathcal{I}(A) \cap A$ and $g : f(A) \longrightarrow \mathbb{M}$ differentiable at $f(x) \in \mathcal{I}\left(f(A)\right) \cap f(A)$. Then $g \circ f : A \longrightarrow \mathbb{M}$ is differentiable at x, with differential $d_H(g \circ f)(x) = d_H g(y) \circ d_H f(x)$.*

Proof. Let us define $h = g \circ f$, $L = d_H g(y) \circ d_H f(x)$, $y = f(x)$ and let us fix $\varepsilon > 0$. By hypothesis there exists $\delta > 0$ such that

$$\rho\left(h(x)^{-1}h(u), L(x^{-1}u)\right) \leq \rho\left(h(x)^{-1}h(u), d_H g(y)(y^{-1}f(u))\right) \quad (3.10)$$
$$+\|d_H g(y)\|\ \nu\big(d_H f(x)(x^{-1}u), y^{-1}f(u)\big) \leq \varepsilon\, \nu\left(y, f(u)\right) + \|d_H g(y)\|\ \varepsilon\, d(x,u)\,, \quad (3.11)$$

whenever $d(x,u), \nu\left(y, f(u)\right) \leq \delta$. The differentiability of f at x implies in that

$$\nu\left(y, f(u)\right) \leq \left(\|d_H f(x)\| + 1\right) d(x,u) \leq \delta$$

whenever $d(u,x) \leq \delta'$, for some $\delta' \in]0, \delta[$. Replacing the latter inequality in (3.11) the thesis follows. □

Definition 3.2.6 (C^1_H-maps) We say that $f : \Omega \longrightarrow \mathbb{M}$ is H-*continuously differentiable* in Ω if it is differentiable at any $x \in \Omega$ and $d_H f : \Omega \longrightarrow \mathrm{HL}(\mathbb{G}, \mathbb{M})$ is continuous. We denote by $C^1_H(\Omega, \mathbb{M})$ the space of all continuously differentiable maps. When $\mathbb{M} = \mathbb{R}$ we simply write $C^1_H(\Omega)$.

We mention that when $\mathbb{M} = \mathbb{R}$, the class $C^1_H(\Omega)$ corresponds to the one introduced in [71], [72].

Remark 3.2.7 It would be very natural to give a notion of C^1_H regular map claiming that only derivatives along horizontal directions exist and are continuous. But presently it is not clear if the previous condition is sufficient to guarantee the existence of the H-differential. In other words the problem of finding reasonable sufficient conditions for the H-differentiability at a given point is an open question.

In the case of real valued maps we have a precise relation between C^1 smoothness and C^1_H smoothness.

Proposition 3.2.8 *The following inclusion holds $C^1(\Omega) \subset C^1_H(\Omega)$ and for any $f \in C^1(\Omega)$ we have $d_H f(x)(v) = df(x)(v_1)$ whenever $x \in \Omega$ and $v = \sum_{j=1}^{\iota} v_j$ with $v_j \in H^j_x\mathbb{G}$ for any $j = 1, \dots \iota$.*

PROOF. By definition of H-differentiability we have to prove the existence of the following limit

$$\lim_{r \to 0^+} \frac{f\left(x \exp(\delta_r v)\right) - f(p)}{r} = d_H f(x)(v)\,, \tag{3.12}$$

uniformly on $v \in \exp^{-1}(B_1) \subset \mathcal{G}$. Let us define the map

$$r \longrightarrow f\left(x \exp(\delta_r v)\right) = f\Big(x \exp\Big(\sum_{j=1}^{\iota} r^j v_j\Big)\Big) = \psi(r, v)\,,$$

where $v = \sum_{j=1}^{\iota} v_j$ and $v_j \in V_j$. Clearly the map ψ is C^1 and in particular it is partially differentiable with respect to r at the point 0. Hence the uniform convergence of (3.12) follows, obtaining

$$\frac{\partial \psi}{\partial r}(0, v) = d_H f(x)(v) = df(x)(v_1),$$

where $v = \sum_{j=1}^{\iota} v_j$ and $v_j \in H^j_x\mathbb{G}$. $\square$

Example 3.2.9 However the inclusion in the previous proposition cannot be extended to the case of group valued maps. Consider the Heisenberg group $\mathbb{H}^3$ with exponential coordinates $(F, (X, Y, T))$ and the nontrivial Lie relation $[X, Y] = Z$. Let us consider the curve $\gamma \in C^\infty(\mathbb{R}, \mathbb{H}^3)$ defined as

$$F^{-1}\gamma(t) = t\, e_3 \in \mathbb{R}^3 \quad \text{ for any } t \in \mathbb{R},$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. We utilize the homogeneous distance d_∞ constructed in Example 2.3.38. By the BCH formula (2.18) we have

$$\frac{d_\infty\left(\gamma(t), \gamma(\tau)\right)}{|t - \tau|} = N\left(\delta_{|t-\tau|}\left(\gamma(t)^{-1}\gamma(\tau)\right)\right) = \frac{1}{|t - \tau|}\,,$$

where the map $N : \mathbb{H}^3 \longrightarrow \mathbb{R}$ is defined in the Example 2.3.38. Then $\gamma : \mathbb{R} \longrightarrow \mathbb{H}^3$ is nowhere H-differentiable, according to Definition 3.2.1.

Example 3.2.10 Beyond the previous example there is a general fact that we want to illustrate as a class of examples. Take $\tilde{z} \in V_k$ with $k \geq 1$ and notice that by the group operation in the algebra (2.18) we have

$$\delta_t(-\tilde{z}) \circledcirc \delta_{t'}\tilde{z} = \left(t'|t'|^{k-1} - t|t|^{k-1}\right)\tilde{z} = \delta_{\sqrt[k]{\left|t'|t'|^{k-1}-t|t|^{k-1}\right|}}\ \sigma_{\left(t'|t'|^{k-1}-t|t|^{k-1}\right)}\tilde{z}$$

where $t \neq t'$. Then, posing $z = \exp \tilde{z}$ we obtain

$$\delta_{-t}z\ \delta_{t'}z = \delta_{\sqrt[k]{\left|t'|t'|^{k-1}-t|t|^{k-1}\right|}}\ \sigma_{\left(t'|t'|^{k-1}-t|t|^{k-1}\right)}z \tag{3.13}$$

that implies

$$d\left(\delta_{-t}z\,\delta_{t'}z\right) = \sqrt[k]{\left|t'|t'|^{k-1} - t|t|^{k-1}\right|}\,d(z)\,. \tag{3.14}$$

Notice that if $z \in V_1$ formula (3.13) yields

$$\delta_{-t}z\ \delta_{t'}z = \delta_{t'-t}z.$$

It follows that the smooth curve $t \longrightarrow \delta_t z \in \mathbb{G}$ is rectifiable if and only if $z \in \mathbb{V}_1$.

Remark 3.2.11 By preceding examples it is clear that H-differentiability requires a geometric constraint on the map and not only the simple smoothness. One can also observe that the curve considered in Example 3.2.9 is not rectifiable in the sense of Definition 2.1.10. This can suggest a natural condition on a C^1 map $f : \mathbb{G} \longrightarrow \mathbb{M}$ in order to be H-differentiable: for any rectifiable curve γ of $\mathbb{G}$ the image curve $f \circ \gamma$ is rectifiable in $\mathbb{M}$. Notice that the Lipschitz property implies the condition above and we will see in Theorem 3.4.11 that Lipschitz maps of stratified groups are a.e. H-differentiable.

In Remark 3.1.5 we have seen that $\mathrm{HL}(\mathbb{G},\mathbb{M})$ is endowed with a natural structure of Lie group with dilations and also of a homogeneous distance. This allows us to intrinsically define higher order differentiability.

Definition 3.2.12 By induction on $k \geq 2$ we say that $f : \Omega \longrightarrow \mathbb{M}$ is H-*continuously k-differentiable* if the $(k-1)$ H-differential $d_H^{k-1}f(x) : \Omega \longrightarrow \mathrm{HL}\left(\mathbb{G}, \mathrm{HL}^{k-2}(\mathbb{G},\mathbb{M})\right)$ is H-continuously differentiable, where also $\mathrm{HL}^k(\mathbb{G},\mathbb{M}) = \mathrm{HL}\left(\mathbb{G}, \mathrm{HL}^{k-1}(\mathbb{G},\mathbb{M})\right)$ is defined by induction.

The previous definition of differentiability could be used in order to find further properties for C_H^k smooth functions. This certainly runs away from the studies accomplished in this thesis, but it remains however an interesting object to be investigated. We will deal with higher order differentiability in Chapter 8, concerning real valued functions of higher order variation.

3.3 Inverse mapping theorem

In this section we prove the inverse mapping theorem on a graded group $\mathbb{G}$. We denote by Ω an open subset of $\mathbb{G}$. We start with the following definition.

Definition 3.3.1 Let $L \in \mathrm{HL}(\mathbb{G},\mathbb{G})$. Then we define the number

$$\|L\|^- = \min_{u\in\mathbb{G},\, d(u)=1} d(Lu)\,.$$

The next lemma is the core of the proof of the inverse mapping theorem. In particular, it implies that H-continuously differentiable maps with non-singular differential are locally bilipschitz maps.

Lemma 3.3.2 *Let $f \in C^1_H(\Omega,\mathbb{G})$. Then for every $x \in A$ and $\varepsilon > 0$, there exists $\delta = \delta(x,\varepsilon) > 0$ such that*

$$d(z,w)\left(\|d_Hf(x)\|^- - \varepsilon\right) \le d(f(z),f(w)) \le \left(\|d_Hf(x)\| + \varepsilon\right) d(z,w) \tag{3.15}$$

for any $z,w \in B_{x,\delta}$.

PROOF. Let $\varepsilon = 2\varepsilon_1 > 0$ and let $K \subset A$ be a compact neighbourhood of $x \in \Omega$, with $\mathrm{dist}(K,\Omega^c) = 2\tau > 0$. We choose a sequence $(s_k) \subset]0,\tau[$, with $s_k \to 0^+$ as $k \to \infty$ and we define the open sets

$$O_k = \left\{y \in \mathrm{Int}(K) \mid d\left(f(z), f(y)\, d_Hf(y)(y^{-1}z)\right) < \varepsilon_1\, d(z,y) \ \text{ for each } z \in \overline{B}_{y,s_k}\right\}.$$

We consider the compact set $K_\tau = \{y \in \Omega \mid \mathrm{dist}(y,K) \le \tau\}$, so $\mathrm{dist}(K_\tau,\Omega^c) \ge \tau$. The function

$$K_\tau \times K \ni (z,y) \longrightarrow d\left(f(z), f(y)d_Hf(y)(y^{-1}z)\right) - \varepsilon_1 d(z,y)$$

is uniformly continuous on the compact $K_\tau \times K$, hence for any $k \in \mathbb{N}$ the map

$$\mathrm{Int}(K) \ni y \longrightarrow \max_{z\in\overline{B}_{y,s_k}} \left\{d\left(f(z), f(y)\, d_Hf(y)(y^{-1}z)\right) - \varepsilon_1 d(z,y)\right\}$$

is continuous, and therefore O_k is an open set for any $k \in \mathbb{N}$. The differentiability of f in Ω implies that $\{O_k \mid k \in \mathbb{N}\}$ is a covering of $\mathrm{Int}(K)$, in particular there exists $j \in \mathbb{N}$ such that $x \in O_j$. Now we can choose $\delta \in]0, s_j/2[$ such that $B_{x,\delta} \subset O_j$ and

$$\|d_Hf(w)\|^- \ge -\varepsilon_1 + \|d_Hf(x)\|^-, \quad \|d_Hf(w)\| \le \varepsilon_1 + \|d_Hf(x)\| \tag{3.16}$$

for any $w \in B_{x,\delta}$. For each couple $z,w \in B_{x,\delta}$ inequalities

$$d\left(f(z), f(w)\right) \le d\left(f(z), f(w)\, d_Hf(w)(w^{-1}z)\right) + d\left(d_Hf(w)(w^{-1}z)\right),$$

$$d\left(f(z),f(w)\right)\geq d\left(d_Hf(w)(w^{-1}z)\right)-d\left(f(z),f(w)\,d_Hf(w)(w^{-1}z)\right)\,,$$

and the fact that $w\in O_j$, so $d(w,z)<2\delta<s_j$, imply

$$d\left(f(z),f(w)\right)\leq\varepsilon_1\,d(z,w)+d\left(d_Hf(w)(w^{-1}z)\right)\leq\left(\varepsilon_1+\|d_Hf(w)\|\right)\,d(z,w)\,,$$

$$d\left(f(z),f(w)\right)\geq d\left(d_Hf(w)(w^{-1}z)\right)-\varepsilon_1\,d(z,w)\geq\left(\|d_Hf(w)\|^{-}-\varepsilon_1\right)\;d(z,w)\,.$$

The latter inequalities together with (3.16) give the assertion (3.15). □

Theorem 3.3.3 (Inverse mapping theorem) *Let $f\in C^1_H(\Omega,\mathbb{G})$, with $x_0\in\Omega$ and suppose that $d_Hf(x_0)$ is invertible. Then there exist neighbourhoods U and V, respectively of x_0 and $f(x_0)$, such that $f:U\longrightarrow V$ has an inverse function $\varphi:V\longrightarrow U$ which is H-continuously differentiable and $d_H\varphi(f(y))=d_Hf(y)^{-1}$ for every $y\in U$.*

Proof. We know that $L=d_Hf(x_0):\mathbb{G}\longrightarrow\mathbb{G}$ is an isomorphism, in particular

$$0<2\alpha=\|L\|^{-}\leq\|L\|=\beta/2\,.$$

By Lemma 3.3.2, there exists a positive $\delta<\min\{\alpha,\beta/2\}$ such that

$$\alpha\,d(z,w)\leq d(f(z),f(w))\leq\beta\,d(z,w)$$

for any $z,w\in B_{x_0,\delta}$. By continuity of differential we can also suppose δ so small that $d_Hf(x)$ is invertible for any $x\in B_{x_0,\delta}$. Defining $V=f(B_{x_0,\delta})$ and $U=B_{x_0,\delta}$, we obtain that $f:U\longrightarrow V$ is a differentiable homeomorphism, with inverse mapping $\varphi:V\longrightarrow U$. Now we choose $v,y\in V$, where $v=f(u)$ and $y=f(x)$ with $u,x\in U$ and we fix an arbitrary $\varepsilon>0$. Then there exists $\sigma>0$, with $B_{y,\sigma}\subset V$, such that

$$\begin{aligned}&d\left(\varphi(y)^{-1}\varphi(v),d_Hf(x)^{-1}(y^{-1}v)\right)\leq\|d_Hf(x)^{-1}\|\,d\left(d_Hf(x)\left(\varphi(y)^{-1}\varphi(v)\right),y^{-1}v\right)\\&=\|d_Hf(x)^{-1}\|\,d\left(d_Hf(x)(x^{-1}u),f(x)^{-1}f(u)\right)\leq\|d_Hf(x)^{-1}\|\,\varepsilon\,d(x,u)\\&\leq\frac{\varepsilon}{\alpha}\,\|d_Hf(x)^{-1}\|\,d(y,v)\,,\end{aligned}$$

whenever $v\in B_{y,\sigma}$. This implies the differentiability of φ at y, with differential $d_H\varphi(y)=d_Hf(x)^{-1}$. The previous formula gives immediately the continuity of $d_H\varphi$. So the proof is complete. □

Remark 3.3.4 Here a remarkable difference with the Euclidean case occurs. Indeed, from Theorem 3.3.3 we cannot recover the Implicit Function Theorem in an easy way. This is clear already in the simple case of a map $u\in C^1_H(\mathbb{H}^3,\mathbb{R})$ with nonsingular H-differential. Indeed, denoting by $\mathfrak{p}_1(x)=x_1$ the canonical projection on the first component, we have that any map $\tilde{u}\in C^1(\mathbb{H}^3,\mathbb{R}^3)$ such that $\mathfrak{p}_1\circ\tilde{u}=u$ cannot have an invertible H-differential simply because $\mathbb{H}^3$ is not commutative. It turns out that our intrinsic version of the Inverse Mapping Theorem cannot be applied. However this does not exclude another more “intrinsic” way to accomplish the extension $\tilde{u}$.

3.4 Differentiability of Lipschitz maps

In this section we will be concerned with differentiability of Lipschitz maps in sub-Riemannian groups. Here it is crucial assuming that $\mathbb{G}$ is a stratified group. In fact, on stratified groups the Chow condition holds (see Remark 2.3.21), so the "generating property" of V_1 holds (see Proposition 2.3.22), that is one of the key points in the proof of Theorem 3.4.11. In our assumptions the map $f : A \longrightarrow \mathbb{M}$ is Lipschitz on a closed subset $A \subset \mathbb{G}$, where $\mathbb{M}$ is another stratified group. Since the target metric space $\mathbb{M}$ is complete and f is a Lipschitz function this assumption does not affect the generality of the domain. We also point out that in view of Proposition 3.2.4 the last assumption does not modify the differential of f. Throughout the section we will denote by d and ρ the homogeneous distances of $\mathbb{G}$ and $\mathbb{M}$, respectively.

As we have explained in the beginning of the chapter, the lack of a Lipschitz extension theorem makes important the shape of the domain around the point where we consider the differentiability. A first information about the existence of points of the domain along arbitrary directions is given in the subsequent statements.

Proposition 3.4.1 *Consider a summable function $g : \mathbb{G} \longrightarrow \mathbb{R}$ and $z \in \mathbb{G}$. Then*

$$\int_{\mathbb{G}} |g(y\delta_t z) - g(y)|\, d\mathcal{H}_d^Q(y) \longrightarrow 0 \qquad as \qquad t \to 0\,.$$

Proof. By an isometric change of variable, the map g can be read on $\mathcal{G}$ where it is $\mathcal{L}^q$-measurable. Then we can use the standard density arguments to achieve the theorem. The density argument works because the Lebesgue measure is preserved under translations of the group. The isometric change of variable does not change the value of the integral. □

Corollary 3.4.2 *Let $A \subset \mathbb{G}$ be a compact set and let (τ_j) be an infinitesimal sequence. Then there exists a subsequence (t_l) such that, $\lim_{t_l \to 0} \mathbf{1}_A(y\delta_{t_l} z) = 1$, for $\mathcal{H}_d^Q$-a.e. $y \in A$.*

Proof. It is enough to apply Proposition 3.4.1 to $g = \mathbf{1}_A$. □

The following Lemma is a particular case of Theorem 2.10.1 in [178].

Lemma 3.4.3 *Let Z_1, Z_2 be two subspaces whose direct sum gives $\mathcal{G}$ and Z_1 with dimension 1. Then there are open neighbourhoods of the origin $\Omega_1 \subset Z_1$, $\Omega_2 \subset Z_2$ and an open $U \subset \mathbb{G}$, $U \ni e$, such that the map $\phi : \Omega_2 \times \Omega_1 \longrightarrow U$, defined as $\phi(\omega, z) = \exp \omega \exp z$, is a diffeomorphism.*

Proposition 3.4.4 (Linear density) *Let $v \in \mathcal{G}$ and $T_{x,v} = \{s \in \mathbb{R} \mid x\exp(sv) \in A\}$, then $0 \in \mathcal{I}(T_{x,v})$ for $\mathcal{H}_d^Q$-a.e. $x \in A$.*

PROOF. Consider the map $\phi : \Omega_2 \times \Omega_1 \longrightarrow U$ of Lemma 3.4.3, where Z_1 is the space spanned by v and Z_2 is the complement factor. Covering A with a countable family of translated neighbourhoods $\{y_i U\}$ it is not restrictive to assume that $A \subset U$. Thus, identifying $\Omega_2 \times \Omega_1$ with $\mathbb{R}^q$, by 3.1.3(5) of [55] applied to the measurable set $\phi^{-1}(A) \subset \Omega_2 \times \Omega_1$ we obtain that for $\mathcal{L}^q$-a.e. $(\omega, z) \in \Omega_2 \times \Omega_1$ the set $\{\tau \mid (\omega, \tau v) \notin \phi^{-1}(A)\}$ has density zero at t. Then the set

$$T_{\phi(\omega,tv),v} = \{s \mid \phi(\omega, tv)\exp(sv) \notin A\}$$

$$= \{s \mid \phi(\omega, (t+s)v) \notin A\} = \{\tau \mid \phi(\omega, \tau v) \notin A\} - t$$

has density zero at $s = 0$. □

Remark 3.4.5 It is important to observe that only when $v \in V_1$ (v is a horizontal vector) we have $T_{x,v} = \{s \in \mathbb{R} \mid x\exp(sv) \in A\} = \{s \in \mathbb{R} \mid x\,\delta_s(\exp v) \in A\}$. This fact will be useful in the proof of Theorem 3.4.11, for the construction of the approximating path (see discussion before the Theorem).

Lemma 3.4.6 (Horizontal extension) *Consider $v \in V_1$ and a Lipschitz function $f : A \subset U \longrightarrow \mathbb{M}$, with U as in the Lemma 3.4.3. Then there exists a function $f^v : U \longrightarrow \mathbb{M}$ extending f, which is Lip(f)-Lipschitz on any segment $\{y\exp(tv) \mid tv \in \Omega_1\} \subset U$ for any $y \in \exp(\Omega_2) \subset U$.*

PROOF. Let $\phi : \Omega_2 \times \Omega_1 \longrightarrow U$ be as in the Lemma 3.4.3. For any $\omega \in \Omega_2$ we will extend the map $\phi(\omega, \cdot)$ to all of Ω_1. The set $Z_\omega = \{tv \in \Omega_1 \mid \phi(\omega, tv) \in A\}$ is closed in Ω_1, so $Z_\omega^c \cap \Omega_1$ is a countable disjoint union of open intervals. Thus, we can define $f^v(\omega, \cdot)$ on any bounded interval of $Z_\omega^c \cap \Omega_1$ joining with a geodesic the values of $f(\omega, \cdot)$ on the boundary of the interval (Carnot groups are geodesically complete metric spaces, [86]) and putting constant values on the unbounded intervals, if they exist. This extension of $f^v(\omega, \cdot)$ is Lip(f)-Lipschitz on the segment $\phi(\omega, \Omega_1)$, because we are using the Carnot-Carathéodory metric (length metric) and $\phi(\omega, tv) = \exp\omega\exp(tv)$ is a radial geodesic in $(\mathbb{G}, d)$, being $v \in V_1$. □

Remark 3.4.7 Under the hypotheses of Lemma 3.4.6 we make the following two observations: the extension f^v is not necessarily continuous on U and if $u = \delta_a v$, for some $a \in \mathbb{R}$, we have $f^u = f^v$. The map $\ln \circ \phi : \Omega_2 \times \Omega_1 \longrightarrow \mathcal{G}$, being differentiable, is locally Lipschitz with respect to the Euclidean metric on $\Omega_2 \times \Omega_1$ and the scalar product on $\mathcal{G}$. This implies a *Lusin property* for the map $\exp \circ \phi$, that is, $\mathcal{L}^q$-negligible sets of $\Omega_2 \times \Omega_1$ are mapped into $\mathcal{L}^q$-negligible sets of $\mathcal{G}$. But $\mathcal{L}^q$ is proportional to $\mathcal{H}_d^Q$ on $\mathcal{G}$, so the Lusin property holds for ϕ.

Using the extension lemma and the H-differentiability of rectifiable curves proved in [154] we get the existence of partial derivatives along horizontal directions.

Proposition 3.4.8 (Horizontal derivatives) *Assume that hypotheses of Lemma 3.4.6 hold. then, for $\mathcal{H}_d^Q$-a.e. $x \in U$ there exists*

$$\lim_{t\to 0} \delta_{1/t}\left(f^v(x)^{-1} f^v(x\exp(tv))\right) = \partial_H f^v(x)\,(\exp(v)) \in \exp(W_1).$$

In particular f has partial derivative along v for $\mathcal{H}_d^Q$-a.e. $x \in A$.

Proof. Consider $f^v : U \longrightarrow \mathcal{M}$ and define the Lipschitz curve

$$J_\omega(t) = f^v\left(\phi(\omega, tv)\right) \quad \text{for any } tv \in \Omega_1\,.$$

The Proposition 4.2 of [154] gives the differentiability of J_ω for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ (in the sense of definition 3.2.1) and moreover the derivative is in W_1. So the derivative is a horizontal direction of $\mathcal{M}$. Now by a Fubini argument we get the partial differentiability of f for $\mathcal{L}^q$-a.e. $(\omega, t) \in \Omega_2 \times \Omega_1$ and by Remark 3.4.7 the $\mathcal{H}_d^Q$-a.e. partial differentiability follows. □

Proposition 3.4.9 *Define $T_{x,v} = \{t \in \mathbb{R} \mid x\exp(tv) \in A\}$, with $v \in \mathcal{G}$. Then for any $\tau \in \mathbb{R}$ the map $\beta : \mathbb{G} \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined as $\beta(x) = \inf_{s\in T_{x,v}} |s-\tau|$ is lower semicontinuous (where is assumed $\inf \emptyset = +\infty$).*

Proof. Choose $\sigma > 0$ and $x \in A$ such that $\beta(x) > \sigma$. Fix σ_1 such that $\beta(x) > \sigma_1 > \sigma$, so $x\exp(tv) \notin A$ for any $t \in [\tau - \sigma_1, \tau + \sigma_1]$. By the closedness of A together with the continuity of the map $x\exp(tv)$ with respect to the variables (x, t), there exists $\varepsilon > 0$ such that $y\exp(tv) \notin A$ for any $y \in B_{x,\varepsilon}$ and any $t \in [\tau - \sigma_1, \tau + \sigma_1]$. Then for any $y \in B_{x,\varepsilon}$ it follows $\beta(y) \ge \sigma_1 > \sigma$. □

Corollary 3.4.10 *The map β is finite for $\mathcal{H}_d^Q$-a.e. $y \in A$ and $y\exp(\beta(y)v) \in A$.*

Proof. This is a straightforward consequence of Proposition 3.4.4. □

The next theorem constitutes an extension of the classical Rademacher's Theorem to sub-Riemannian groups.

Theorem 3.4.11 (H-differentiability) *Let $f : A \longrightarrow \mathbb{M}$ be a Lipschitz map, where A is a measurable subset of $\mathbb{G}$. Then f is H-differentiable $\mathcal{H}_d^Q$-a.e.*

Proof. *Step 1, (Existence and uniform convergence of partial derivatives)*
By Proposition 2.3.22 and a suitable rescaling we can find an open bounded set $M \subset \mathbb{R}^\gamma$, with $0 \in M$, such that $E = \{\prod_{s=1}^{\gamma} \exp(a_s v_{i_s}) \mid (a_s) \subset M\} \supset \overline{B_1}$, where the products of the elements are understood in ordered sense and $\{v_i \mid i = 1, \dots m\}$ is a basis of V_1. By the σ-compactness of $\mathbb{G}$, $\mathcal{H}_d^Q$ being a Radon measure on $\mathbb{G}$, we can assume that A is compact. Thus, considering U as in Lemma 3.4.3 we cover A with a finite open covering $\{y_i U\}$ and translating f on $(y_i U) \cap A$, the invariance of

differentiability under translations allows to assume $A \subset U$. Applying Proposition 3.4.8 we have the partial derivatives $\partial_H f^{v_i}(y)(\exp(v_i)) \in \exp(W_1)$ of the extension f^{v_i} for $\mathcal{H}_d^Q$-a.e. $y \in A \subset U$, for $i = 1, \ldots, m$. Thus, for any $\varepsilon > 0$, Egorov theorem and the partial differentiability of f give a closed subset $A_{1\varepsilon} \subset A$ such that $\mathcal{H}_d^Q(A \setminus A_{1\varepsilon}) \leq \varepsilon/3$ and the limits

$$\lim_{t \to 0} \delta_{1/t} \left(f(y)^{-1} f^{v_{i_s}} \left(y \exp(t v_{i_s}) \right) \right) = \partial_H f^{v_{i_s}}(y) \left(\exp(v_{i_s}) \right) ,$$

with $s \in \{1, \ldots, \gamma\}$, $y \in A_{1\varepsilon}$, are uniform. Defining $u_{i_s} = \exp(a_s v_{i_s})$ we have that

$$\lim_{t \to 0} \delta_{1/t}(f(y)^{-1} f^{v_{i_s}}(y \delta_t u_{i_s})) = \partial_H f^{v_{i_s}}(y)(u_{i_s}) = \delta_{a_s} \partial_H f^{v_{i_s}}(y) \left(\exp(v_{i_s}) \right) ,$$

for any $s \in \{1, \ldots, \gamma\}$ and $y \in A_{1\varepsilon}$. The uniformity of the limit holds even when $a \in M$. In fact, the following equality holds

$$\begin{aligned} &\rho\Big(\delta_{1/t} \left(f(y)^{-1} f(y \delta_t u_{i_s}) \right), \partial_H f^{v_{i_s}}(y)(u_{i_s}) \Big) \\ &= a_s \rho\Big(\delta_{1/(a_s t)} \left(f(y)^{-1} f(y \delta_{a_s t} v_{i_s}) \right), \partial_H f^{v_{i_s}}(y)(\exp(v_{i_s})) \Big) . \end{aligned}$$

For any $\zeta \neq 0$ and any $s = 1, \ldots, \gamma$ we define the map

$$\beta(y, \zeta, v_{i_s}) = \inf_{t \in T_{y, v_{i_s}}} |t - \zeta| ,$$

by Proposition 3.4.9 this map is a measurable function. Proposition 3.4.4 and Lemma 2.1.15 imply that the quotient $|\zeta - \beta(y, \zeta, v_{i_s})|/\zeta$ tends to zero as $\zeta \to 0$ for $\mathcal{H}_d^Q$-a.e. $y \in A$. Then, by Egorov theorem we get a uniform convergence, for $s = 1, \ldots, \gamma$, in another closed subset $A_{2\varepsilon} \subset A$ such that $\mathcal{H}_d^Q(A \setminus A_{2\varepsilon}) \leq \varepsilon/3$. Define the measurable map

$$\theta_t(y) = \sup_{u \in B_{y,t} \setminus \{y\}} \left(d(u, A)/d(u, y) \right)$$

for $t > 0$ and use again Lemma 2.1.15 to obtain that $\theta_t(y) \to 0$ as $t \to 0^+$ for $\mathcal{H}_d^Q$-a.e. $y \in A$. Using Egorov theorem we are able to find a closed set $A_{3\varepsilon} \subset A$ such that $\mathcal{H}_d^Q(A \setminus A_{3\varepsilon}) \leq \varepsilon/3$ and $\theta_t(y)$ goes to zero uniformly on $A_{3\varepsilon}$ as $t \to 0$. Now consider $A_\varepsilon = A_{1\varepsilon} \cap A_{2\varepsilon} \cap A_{3\varepsilon}$ and $x \in \mathcal{I}(A_\varepsilon)$. Notice that A_ε does not depend on the vector $a = (a_s) \in M$, moreover $\mathcal{H}_d^Q(A \setminus A_\varepsilon) \leq \varepsilon$. We want to prove the convergence of the following limit

$$\begin{aligned} &\lim_{x \delta_t z \in A,\, t \to 0} \delta_{1/t}(f(x)^{-1} f(x \delta_t z)) = \prod_{s=1}^{\gamma} \partial_H f^{v_{i_s}}(x)(u_{i_s}) \\ &= \prod_{s=1}^{\gamma} \delta_{a_{i_s}} \partial_H f^{v_{i_s}}(x)(\exp(v_{i_s})) \end{aligned} \tag{3.17}$$

uniformly with respect to $a \in M$ and $z = \prod_{s=1}^{\gamma} \exp(a_s v_{i_s}) = \prod_{s=1}^{\gamma} u_{i_s}$. By Lemma 2.1.15 with $A = A_\varepsilon$ and $y = x\delta_t u_{i_1}$, we can choose $u^t \in A_\varepsilon$ such that

$$d(x\delta_t u_{i_1}, u^t) \le d(x\delta_t u_{i_1}, x)\, \theta_{ct}(x)\,, \tag{3.18}$$

where $c = \sup_{a\in M,\, l=1,\dots,\gamma} d\left(\exp(a_1 v_{i_1}) \cdots \exp(a_l v_{i_l})\right)$. Representing $u^t = x\delta_t u^t_{i_1}$, the left invariance and the homogeneity of the distance give

$$d\left(u_{i_1}, u^t_{i_1}\right) = \frac{d\left(x\delta_t u_{i_1}, x\delta_t u^t_{i_1}\right)}{t} \le c\,\theta_{ct}(x) \to 0 \qquad \text{as} \qquad t \to 0$$

Then the convergence of $u^t_{i_1}$ to u_{i_1} is uniform with respect to $a \in M$. Now by induction suppose that vectors $(w^t_{i_j})$ are defined for any $j \le s < \gamma$ such that $x\delta_t u^t_{i_1} \cdots u^t_{i_j} \in A_\varepsilon$ and $d(u^t_{i_j}, u_{i_j}) \to 0$, uniformly with respect to $a \in M$ (for simplicity of notation we have omitted the parenthesis after the symbol of dilation δ_t, being understood that all subsequent terms are considered dilated). Again from Lemma 2.1.15 with $A = A_\varepsilon$ and $y = x\delta_t u^t_{i_1} \cdots u^t_{i_s} u_{i_{s+1}}$, we find another family of points in A_ε, which can be represented as $x\delta_t u^t_{i_1} \cdots u^t_{i_s} u^t_{i_{s+1}}$ for a suitable $u^t_{i_{s+1}}$ and with the property

$$d(x\delta_t u^t_{i_1} \cdots u^t_{i_s} u_{i_{s+1}}, x\delta_t u^t_{i_1} \cdots u^t_{i_s} u^t_{i_{s+1}}) \le 3c\, t\, \theta_{3ct}(x)\,, \tag{3.19}$$

for t small enough, depending on s. The estimate (3.19) is independent of $a \in M$. From inequality (3.19), by the left invariance and the homogeneity of the distance, we deduce

$$d(u_{i_{s+1}}, u^t_{i_{s+1}}) = \frac{d(x\delta_t u^t_{i_1} \cdots u^t_{i_s} u_{i_{s+1}}, x\delta_t u^t_{i_1} \cdots u^t_{i_s} u^t_{i_{s+1}})}{t} \le 3c\,\theta_{3ct}(x) \longrightarrow 0$$

as $t \to 0^+$ and uniformly on $a \in M$. Finally we consider

$$\delta_{1/t}\left(f(x)^{-1} f(x\delta_t u_{i_1} \cdots u_{i_\gamma})\right) = \Big(\prod_{s=1}^{\gamma} D^t_s B^t_s\Big) G^t$$

where $z = u_{i_1} \cdots u_{i_\gamma} = \prod_{s=1}^{\gamma_1} \exp(a_s v_{i_s})$ and we have defined :

$$D^t_s = \delta_{1/t}\left(f(x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}})^{-1} f^{v_{i_s}}(x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}} u_{i_s})\right)\,,$$

$$B^t_s = \delta_{1/t}\left(f^{v_{i_s}}(x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}} u_{i_s})^{-1} f(x\delta_t u^t_{i_1} \cdots u^t_{i_s})\right)\,,$$

$$G^t = \delta_{1/t}\left(f(x\delta_t u^t_{i_1} \cdots u^t_{i_\gamma})^{-1} f(x\delta_t u_{i_1} \cdots u_{i_\gamma})\right)\,.$$

We observe that $x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}} \in A_\varepsilon$ for $s = 1, \dots, \gamma$, so $D^t_s \to \partial_H f^{v_i}(u_{i_s})$ as $t \to 0$ and uniformly when $a \in M$. It remains to be seen that B^t_s, $s = 1, \dots, \gamma$, and G^t go to the unit element as $t \to 0$, uniformly as $a \in U$. Denote $y_s^t = x\delta_t u_{i_1}^t \cdots u_{i_{s-1}}^t \in A_\varepsilon$ and $\omega_{i_s} = \lg u_{i_s}$; in view of Corollary 3.4.10 we see that $y_s^t \exp(\beta(y_s^t, t, w_{i_s})w_{i_s}) \in A$, then we can further decompose $B_s^t = F_s^t N_s^t$, where

$$F_s^t = \delta_{1/t}\left(f^{v_{i_s}}(x\delta_t u_{i_1}^t \cdots u_{i_{s-1}}^t u_{i_s})^{-1} f\left(x(\delta_t u_{i_1}^t \cdots u_{i_{s-1}}^t)\exp(\beta(y_s^t, t, w_{i_s})w_{i_s})\right)\right),$$

$$N_s^t = \delta_{1/t}\left(f\left(x(\delta_t u_{i_1}^t \cdots u_{i_{s-1}}^t)\exp(\beta(y_s^t, t, w_{i_s})w_{i_s})\right)^{-1} f(x\delta_t u_{i_1}^t \cdots u_{i_s}^t)\right).$$

We have seen that $\beta(y, \zeta, v_{i_s})/\zeta \to 1$ as $\zeta \to 0$, uniformly in $y \in A_\varepsilon$, then

$$\beta(y_s^t, a_s t, v_{i_s})/a_s t \to 1\,,$$

when a varies in M. Moreover

$$a_s\, \beta(y, t, w_{i_s}) = \beta(y, a_s t, v_{i_s})\,, \qquad s \in \{1, \ldots, \gamma\}$$

so the following estimates hold

$$\begin{aligned}
\rho(F_s^t) &\le \mathrm{Lip}(f)\, \frac{d\left(\delta_t u_{i_s}, \exp(\beta(y_s^t, t, w_{i_s})w_{i_s})\right)}{t} \\
&= \mathrm{Lip}(f)\, d\left(\exp(w_{i_s}), \exp\left((\beta(y_s^t, t, w_{i_s})/t)w_{i_s}\right)\right) \\
&= \mathrm{Lip}(f)\, a_s\, d\Big(\exp(v_{i_s}), \exp\left((\beta(y_{sl}, a_s t_l, v_{i_s})/(a_s t_l))v_{i_s}\right)\Big) \\
&\le \mathrm{Lip}(f)\, \left(\sup_{a \in U} |a|\right)\, d\Big(\exp(v_{i_s}), \exp\left((\beta(y_{sl}, a_s t_l, v_{i_s})/(a_s t_l))v_{i_s}\right)\Big)\,,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\rho(N_s^t) &\le \mathrm{Lip}(f)\, \frac{d\left(\delta_t u_{i_s}^t, \exp(\beta(y_s^t, t, w_{i_s})w_{i_s})\right)}{t} \\
&= \mathrm{Lip}(f)\, d\left(u_{i_s}^t, \exp\left((\beta(y_s^t, t, w_{i_s})/t)w_{i_s}\right)\right) \\
&= \mathrm{Lip}(f)\, d\left(u_{i_s}^t, \exp\left((\beta(y_s^t, a_s t, v_{i_s})/(a_s t))w_{i_s}\right)\right)\,.
\end{aligned} \tag{3.21}$$

The first of these two estimates follows by Lemma 3.4.6, whereas the second is due to the fact that the points $x(\delta_t u_{i_1}^t \cdots u_{i_{s-1}}^t)\exp(\beta(y_s^t, t, w_{i_s})w_{i_s})$ and $x\delta_t u_{i_1}^t \cdots u_{i_s}^t$ are in A, where f is Lipschitz. Both last right terms of equations (3.20), (3.21) go to zero uniformly as $a \in M$. The same reasoning yields

$$\rho(G^t) \le \mathrm{Lip}(f)\, d(u_{i_1}^t \cdots u_{i_\gamma}^t, u_{i_1} \cdots u_{i_\gamma}) \longrightarrow 0\,, \tag{3.22}$$

where we have used the uniform convergence of any $u_{i_s}^t$ for $s = 1, \ldots, \gamma$. Now we remember that $x \in \mathcal{I}(A_\varepsilon)$ and ε is arbitrary, so there exists a null set $N \subset A$ such that for any $x \in A \setminus N$ the equation (3.17) holds uniformly with respect to $a \in U$.
Step 2, (H-linearity and construction of differential)

One finds easily that partial derivatives are 1-homogeneous under dilations. We want to prove the homomorphism property of partial derivatives, that is $\partial_H f(y)(u\omega) = \partial_H f(y)(u)\partial_H f(y)(\omega)$. To get this equality we use step 1, but we need at least of an infinitesimal sequence $(t_l) \subset \mathbb{R} \setminus \{0\}$, which connects the three directions in the following sense : for $\mathcal{H}^Q_d$-a.e. $y \in A$ we have $y\delta_{t_l}(u\omega)$, $y\delta_{t_l}u$, $y\delta_{t_l}\omega \in A$. In fact, equation (3.17) is not trivial when we have directions $z \in E$ such that $x\delta_{t_j}z \in A$ and $t_j \to 0$. To obtain the sequence (t_l) it is enough to consider the three arbitrary directions $u\omega$, u, $\omega \in \mathbb{G}$ and iterate Corollary 3.4.2 for any direction, extracting further subsequences. In this situation, with $u = \prod_{s=1}^{\gamma_1} \exp(b_s v_{i_s})$ and $\omega = \prod_{s=1}^{\gamma_2} \exp(c_s v_{i_s})$, applying twice step 1 it follows

$$\begin{aligned}&\lim_{x\delta_t(u\omega)\in A,\, t\to 0} \delta_{1/t}\Big(f(x)^{-1}f(x\delta_t(u\omega))\Big)\\ &= \prod_{s=1}^{\gamma_1} \delta_{b_s}\Big(\partial_H f(x)\,(\exp(v_{i_s}))\Big) \prod_{s=1}^{\gamma_2} \delta_{c_s}\Big(\partial_H f(x)\,(\exp(v_{i_s}))\Big)\,,\end{aligned}$$

it follows

$$\partial_H f(x)(u\omega) = \lim_{x\delta_t z\in A,\, t\to 0} \delta_{1/t}\Big(f(x)^{-1}f(x\delta_t(u\omega))\Big) = \partial_H f(x)(u)\partial_H f(x)(\omega) \tag{3.23}$$

and directly from equation (3.17) we infer

$$\lim_{x\delta_t z\in A,\, t\to 0} \delta_{1/t}\Big(f(x)^{-1}f(x\delta_t(u^{-1}))\Big) = \left(\partial_H f(x)(u)\right)^{-1}. \tag{3.24}$$

Now we want to define the differential map $d_H f(y)$ globally on $\mathbb{G}$ for $\mathcal{H}^Q_d$-a.e. $y \in A$. Consider the countable dense subset $D_0 = \{\prod_{s=1}^{\gamma} \exp(b_s v_{i_s}) \mid (b_s) \in \mathbb{Q}^\gamma\} \subset \mathbb{G}$. Define the countable set given by $D = \{\omega_1 \cdots \omega_j \mid j \in \mathbb{N},\ \omega_i \in D_0,\ i = 1, \ldots, j\}$. For any $\omega \in D$, in view of Corollary 3.4.2 we get a sequence (depending on ω) which allows us to apply step 1, defining the partial derivative of f on direction ω for any $y \in A \setminus N_\omega$, where $\mathcal{H}^Q_d(N_\omega) = 0$. In fact, for all $\omega \in D$ we have at least an infinitesimal sequence of points (t_j) such that $y\delta_{t_j}\omega \in A$ for all $y \in A \setminus \bigcup_{\omega' \in D} N_{\omega'}$ and the limit (3.17) with $x = y$ and $z = \omega$ is taken on the nonempty set $\{y\delta_{t_j}\omega\}$ with y as accumulation point. Thus, for $\mathcal{H}^Q_d$-a.e. $y \in A$ and $\omega \in D$, the partial derivative

$$L_y(\omega) = \lim_{t\to 0,\, A\ni x\delta_t\omega} \delta_{1/t}\left(f(y)^{-1}f(y\delta_t\omega)\right)$$

is well defined. By density we extend L_y to all of $\mathbb{G}$, setting $L_y(z) = \lim_{l\to\infty} L_y(\omega_l)$ whenever $(\omega_l) \subset D$ and $\omega_l \to z$. In view of equations (3.23) and (3.24) the sequence $L_y(\omega_l)$ is convergent and the extension is well defined, so choosing another sequence $(z_l) \subset D$ which converges to z we obtain

$$\rho\left(L_y(\omega_l)^{-1}L_y(z_l)\right) = \rho\left(L_y(\omega_l^{-1})L_y(z_l)\right) = \rho\left(L_y(\omega_l^{-1}z_l)\right) \le \mathrm{Lip}(f)\, d(\omega_l^{-1}z_l) \to 0$$

as $l \to \infty$, because $\omega_l z_l \in D$. The latter inequality also proves that $L_y(\omega_l)$ is a Cauchy sequence whenever (ω_l) is convergent. We have defined $L_y : \mathbb{G} \longrightarrow \mathbb{M}$ for $\mathcal{H}_d^Q$-a.e. $y \in A$. By definition of L_y and equations (3.23), (3.24) the H-linearity of L_y follows.

Step 3, (Differentiability)

In step 1 we have proved that for $\mathcal{H}_d^Q$-a.e. $y \in A$ it follows

$$\rho\Big(\delta_{1/t}\left(f(y)^{-1}f(y\delta_t z)\right), \prod_{s=1}^{\gamma} \delta_{a_s} \partial_H f^{v_{i_s}}(y)\left(\exp(v_{i_s})\right)\Big) \longrightarrow 0 \qquad \text{as} \qquad t \to 0\,, \tag{3.25}$$

uniformly when $z = \prod_{s=1}^{\gamma_1} \delta_{a_s} v_{i_s}$, $a \in M$, and $y\delta_t z \in A$.

We want to prove that the uniform limit (3.25) implies the differentiability. Assume by contradiction the existence of $\sigma > 0$ and $(z_l) \subset \mathbb{G}$ such that $z_l \to 0$ and

$$\rho\left(f(y)^{-1}f(yz_l), L_y(z_l)\right) \geq \sigma d(z_l)\,,$$

define $z_l = \delta_{t_l} w_l$, with $t_l = d(z_l)$, obtaining

$$\rho\left(\delta_{1/t_l}\left(f(y)^{-1}f(y\delta_{t_l} w_l)\right), L_y(w_l)\right) \geq \sigma\,. \tag{3.26}$$

Represent $w_l = \prod_{s=1}^{\gamma} \exp(b_s^l v_{i_s})$, $(d(w_l) = 1)$, and consider rational vectors $(b_s^{lj}) \in \mathbb{Q}^\gamma \cap M$ such that $\omega_{lj} = \prod_{s=1}^{\gamma} \exp(b_s^{lj} v_{i_s}) \in D_0$ and $\omega_{lj} \to \omega_l$ as $j \to \infty$. The explicit definition of L_y implies $L_y(\omega_{lj}) = \prod_{s=1}^{\gamma} \delta_{b_s^{lj}}\left(\partial_H f^{v_{i_s}}(y)(\exp(v_{i_s}))\right)$. As we have seen in Subsection 3.1, any H-linear map is continuous, then

$$\begin{aligned} L_y(\omega_l) &= \lim_{j\to\infty} L_y(\omega_{lj}) \\ &= \lim_{j\to\infty} \prod_{s=1}^{\gamma} \delta_{b_s^{lj}}\Big(\partial_H f^{v_{i_s}}(x)\Big(\exp(v_{i_s})\Big)\Big) = \prod_{s=1}^{\gamma} \delta_{b_s^l}\Big(\partial_H f^{v_{i_s}}(x)(v_{i_s})\Big)\,. \end{aligned}$$

Replacing $L_y(\omega_l)$ in equation (3.26) we have

$$\rho\left(\delta_{1/t_l}\left(f(y)^{-1}f(y\delta_{t_l} w_l)\right), \prod_{s=1}^{\gamma} \delta_{b_s^l}\left(\partial_H f^{v_{i_s}}(x)\Big(\exp(v_{i_s})\Big)\right)\right) \geq \sigma\,,$$

so from uniform convergence of equation (3.25) it follows

$$\rho\left(\delta_{1/t_l}\left(f(y)^{-1}f(y\delta_{t_l} w_l)\right), \prod_{s=1}^{\gamma} \delta_{b_s^l}\Big(\partial_H f^{v_{i_s}}(x)\Big(\exp(v_{i_s})\Big)\Big)\right) \longrightarrow 0\,,$$

which is a contradiction. This concludes the proof of differentiability. □

Remark 3.4.12 By Proposition 3.2.4 the differential does not depend on the explicit construction we have done in Theorem 3.4.11, where the basis (v_i) and the extensions f^{v_i} were involved. Our choice of (v_i) can be interpreted as the choice of a fixed coordinate system where the differential is represented.

3.5 Rectifiability

By means of real valued C^1_H maps we can define an intrinsic definition of regular hypersurface and of rectifiability. These notions are due to B. Franchi, R. Serapioni and F. Serra Cassano, [71], [72], [73].

In this section $\mathbb{G}$ and $\mathbb{M}$ will denote two sub-Riemannian groups and Ω will be assumed to be an open subset of $\mathbb{G}$.

Definition 3.5.1 ($\mathbb{G}$-regular hypersurface) We say that $\Sigma \subset \Omega$ is a $\mathbb{G}$-*regular hypersurface* if there exists a map $f \in C^1_H(\Omega)$ such that $\Sigma = f^{-1}(0)$ and

$$d_H f(p) : \mathbb{G} \longrightarrow \mathbb{R}$$

is a nonvanishing H-linear map for any $p \in \Sigma$.

Definition 3.5.2 ($\mathbb{G}$-rectifiability) We say that a subset $S \subset \Omega$ is $\mathbb{G}$-*rectifiable*, if there exists a sequence of $\mathbb{G}$-regular hypersurfaces $\{\Sigma_j\}$ such that

$$\mathcal{H}^{Q-1}_\rho\Big(S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) = 0\,,$$

where ρ is the CC-distance of the group.

Notice that the previous definition in the terminology introduced by Federer in 3.2.14 of [55] would have been translated as "countably (Q-1) $\mathbb{G}$-rectifiability". But for the aims of the thesis, we do not need to make any distinction between $\mathbb{G}$-rectifiability and the countably (Q-1) $\mathbb{G}$-rectifiability.

Remark 3.5.3 It is important to emphasize the lack of an equivalent notion of $\mathbb{G}$-rectifiability by means of Lipschitz parametrizations defined on subsets of Euclidean spaces, as it is done classically, see Definitions in 3.2.14 of [55]. For instance, the Heisenberg group $\mathbb{H}^3$ is purely k-unrectifiable whenever $k \geq 2$ (see [7] and the characterization of pure unrectifiability given in Section 4.4 of the present thesis).

However, in [156] a notion of rectifiability "modeled" on the group is proposed, as it is stated in the next definition.

Definition 3.5.4 ($(N, \mathbb{G})$-rectifiability) Let $\mathbb{P}$ be a sub-Riemannian group and let $N \subset \mathbb{P}$ be a subgroup. A subset $S \subset \Omega$ is $(N, \mathbb{G})$-rectifiable if there exist a Lipschitz map $f : A \longrightarrow \mathbb{G}$, with $A \subset N$ and such that $S = f(A)$.

The previous definition clearly generalizes the classical one, where N is an Euclidean space, but several questions arise. In fact, the subgroup N is graded, but it may not be stratified. It is not presently clear whether a differentiability theorem of Lipschitz maps can be proved when the domain is only a graded group. This fact is of crucial importance, because if N is stratified the a.e. H-differentiability of Lipschitz maps, that we have proved in Theorem 3.4.11, can be used to get information on the $(N, \mathbb{G})$-rectifiable set, obtaining for instance the existence $\mathcal{H}^Q_d$-a.e. of tangent spaces, where Q is the Hausdorff dimension of N and one could go on as in [7], [110]. Moreover the area formula (4.20) gives a way to compute the intrinsic measure of the set (see Example 4.3.7). The easiest example where subgroups are not stratified occurs for "vertical" subgroups $\mathbb{H}^3$, i.e. all subgroups of topological dimension 2. It is not difficult to see that these groups are not stratified and not connected by rectifiable curves with respect to the CC-distance of $\mathbb{H}^3$. However, this situation has its own interest due to the fact that $\mathbb{H}^3$ has a rich family of $\mathbb{H}^3$-regular hypersurfaces and it is a hard question to establish if they are $(N, \mathbb{H}^3)$-rectifiable for some suitable N. It is natural to expect that N is a vertical subgroup due to the fact that it has the same topological and Hausdorff dimensions of $\mathbb{H}^3$-regular hypersurfaces. Notice also that vertical subgroups form a particular class of $\mathbb{H}^3$-regular hypersurfaces without characteristic points.

Next, we present novel definitions of regular surfaces and rectifiable surfaces, that somehow extend Definition 3.5.1 and Definition 3.5.2 to higher codimension.

Definition 3.5.5 ($(\mathbb{G}, \mathbb{M})$-regular surface) A subset $\Sigma \subset \Omega$ is a $(\mathbb{G}, \mathbb{M})$-*regular surface* if there exist $f \in C^1_H(\Omega, \mathbb{M})$ such that $f^{-1}(e) = \Sigma$ and

$$d_H f(p) : \mathbb{G} \longrightarrow \mathbb{M}$$

is a surjective H-linear map for any $p \in \Sigma$.

It is apparent that the notion of $(\mathbb{G}, \mathbb{M})$-regularity in higher codimension allows us a certain amount of freedom in the choice of $\mathbb{M}$, but not all codomains are "good" to be considered. For instance, the family of $(\mathbb{H}^{2n+1}, \mathbb{H}^{2m+1})$-regular surfaces is empty whenever $n > m$. This follows by the fact that there are no surjective H-linear maps between $\mathbb{H}^{2n+1}$ onto $\mathbb{H}^{2m+1}$, see Proposition 6.3.3.

As soon as we have a surjective H-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ a canonical example of $(\mathbb{G}, \mathbb{M})$-regular surface can be given by choosing the subgroup $N = L^{-1}(0) \subset \mathbb{G}$ which is clearly a $(\mathbb{G}, \mathbb{M})$-regular surface. Furthermore, in view of Proposition 6.1.5 the Hausdorff dimension of N is $Q-P$, where Q and P are the Hausdorff dimensions of $\mathbb{G}$ and $\mathbb{M}$, respectively.

These observations suggest that $(\mathbb{G}, \mathbb{M})$-regular surfaces must possess topological dimension $q - p$ and Hausdorff dimension $Q - P$, where q and p are the topological dimensions of $\mathbb{G}$ and $\mathbb{M}$, respectively. As a result, in the Heisenberg group $\mathbb{H}^3$ there is no hope to recover smooth horizontal curves (which are rectifiable) as $(\mathbb{H}^3, \mathbb{M})$-regular curves, for some $\mathbb{M}$. In fact, the topological dimension of $\mathbb{M}$ has to be 2, then $\mathbb{M} = \mathbb{R}^2$ and the Hausdorff dimension of the curve is forced to be $Q - P = 2$, but all horizontal curves have Hausdorff dimension 1 with respect to the CC-distance. From the other side, rectifiable curves can be seen as $(\mathbb{R},\mathbb{H}^3)$-rectifiable objects, according to Definition 3.5.4, where the Lipschitz parametrization is given by the curve itself. So, somehow both Definition 3.5.4 and the following Definition 3.5.6 are able to supply a suitable notion of rectifiable surface in higher codimension.

Definition 3.5.6 ($(\mathbb{G},\mathbb{M})$-rectifiability) We say that $S\subset\Omega$ is $(\mathbb{G},\mathbb{M})$-*rectifiable*, if there exists a sequence of $(\mathbb{G},\mathbb{M})$-regular surfaces $\{\Sigma_j\}$ such that

$$\mathcal{H}_\rho^{Q-P}\Big(S\setminus\bigcup_{j\in\mathbb{N}}\Sigma_j\Big)=0\,,$$

where ρ is the CC-distance of the group.

It would be very interesting to investigate to what extent the previously mentioned definitions are able to cover all "rectifiable objects" of the group.

3.6 A counterexample

In this section we present a counterexample to Lipschitz differentiability (Theorem 3.4.11), when slightly general distances on the target are considered. We notice that differentiability between sub-Riemannian groups implies metric differentiability (Definition 3.6.2), when one consider the target group as a metric space. So we will build the counterexample proving that metric differentiability fails for a suitable non homogeneous left invariant distance on the target.
We begin with the following definitions.

Definition 3.6.1 Let $\mathbb{G}$ be a graded group. We say that a map $\nu:\mathbb{G}\longrightarrow[0,+\infty[$ is a *homogeneous seminorm* if for each $x,y\in\mathbb{G}$ and $r>0$ we have

1. $\nu(\delta_r x)=r\,\nu(x)\,,$
2. $\nu(xy)\leq\nu(x)+\nu(y)\,.$

Definition 3.6.2 Let (Y,ρ) and $(\mathbb{G},d)$ be a metric space and a graded group, respectively. We say that a map $f:A\longrightarrow Y$, where A is an open subset of $\mathbb{G}$, is *metrically differentiable* at $x\in A$, if there exists a homogeneous seminorm ν_x such that

$$\frac{\rho\left(f(x\delta_t v),f(x)\right)}{t}\longrightarrow\nu_x(v)\quad\text{as}\quad t\to0^+\,,$$

uniformly in v which varies in a compact neighbourhood of the unit element.

Remark 3.6.3 We point out that in [155] it is shown that bilipschitz maps are a.e. metric differentiable on stratified groups if one allows the direction v to vary only on the elements of V_1, namely the *horizontal directions*. The latter result directly applies also to Lipschitz maps. In fact, if $f : \mathbb{G} \longrightarrow Y$ is a metric valued Lipschitz map, we consider the bilipschitz map $F : \mathbb{G} \longrightarrow \mathbb{G} \times Y$, with $F(x) = (x, f(x))$. Hence the metric differentiability of F along horizontal directions, with the product distance on $\mathbb{G} \times Y$, implies the same type of differentiability for f. From this fact, it is clear that we will consider a nonhorizontal direction in order to show that the metric differentiability does not hold in general.

We consider the 3-dimensional Heisenberg group $\mathbb{H}^3$, which can be linearly identified with $\mathbb{R}^3$. Elements $\eta, \xi \in \mathbb{H}^3$ are represented as $\xi = (z, t)$, $\eta = (w, \tau)$, where $z = (z_1, z_2)$, $w = (w_1, w_2)$ belong to $\mathbb{R}^2$. The nonabelian operation on $\mathbb{H}^3$ reads as follows

$$(z, t)(w, \tau) = (z + w, t + \tau + 2(z_1 w_2 - z_2 w_1)) \,.$$

In this case the nonhorizontal directions are of the type $(0, 0, s)$, with $s \neq 0$. We consider $G : \mathbb{H}^3 \longrightarrow \mathbb{R}$, defined as $G(z, t) = |z| \vee \sqrt{|t|}$, where the symbol $\vee$ denotes the "maximum" operation. It is known that $d(\xi, \eta) = G(\xi^{-1}\eta)$, for $\xi, \eta \in \mathbb{H}^3$, yields a left invariant distance on the Heisenberg group, see for instance [71]. The dilations $\delta_r : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ are defined as $\delta_r((z, t)) = (rz, r^2 t)$. It is clear that these dilations scale homogeneously with the distance d, so $(\mathbb{H}^3, d)$ is a stratified group with a homogeneous distance d.

Our aim is to build a left invariant distance ρ on $\mathbb{H}^3$ such that the identity map $I : (\mathbb{H}^3, d) \longrightarrow (\mathbb{H}^3, \rho)$ is a 1-Lipschitz function and the metric differentiability fails. We have seen that a homogeneous distance in the Heisenberg group can be defined as $d(\xi, \eta) = G(\xi^{-1}\eta)$, where $G(z, t) = |z| \vee \sqrt{|t|}$. We obtain our counterexample replacing the square root function in the definition of G with a concave map $g : [0, +\infty[\longrightarrow [0, +\infty[$ such that the function $S : \mathbb{H}^3 \longrightarrow \mathbb{R}$, $S(z, t) = |z| \vee g(|t|)$ satisfies the following three claims:

1. the function $S : \mathbb{H}^3 \longrightarrow \mathbb{R}$ yields a left invariant metric on $\mathbb{H}^3$ which is defined as $\rho(\xi, \eta) = S(\xi^{-1}\eta)$, $\xi, \eta \in \mathbb{H}^3$.

2. the map $I : (\mathbb{H}^3, d) \longrightarrow (\mathbb{H}^3, \rho)$ is 1-Lipschitz,

3. if we consider the nonhorizontal direction $v = (0, 0, 1) \in \mathbb{H}^3$, then for any $\zeta \in \mathbb{H}^3$ there does not exist the limit of
$$\frac{\rho(I(\zeta\delta_t v), I(\zeta))}{t} = \frac{\rho(\delta_t v, 0)}{t} \qquad \text{as} \qquad t \to 0_+ \,,$$
in fact, we reach the maximal possible oscillation of the quotient
$$\limsup_{t \to 0_+} \frac{\rho(I(\zeta\delta_t v), I(\zeta))}{t} = 1 \,, \qquad \liminf_{t \to 0_+} \frac{\rho(I(\zeta\delta_t v), I(\zeta))}{t} = 0 \,.$$

Claim 3 says in particular that the 1-Lipschitz map $I : (\mathbb{H}^3, d) \longrightarrow (\mathbb{H}^3, \rho)$ is not metrically differentiable at any point of $\mathbb{H}^3$. The following two theorems will prove the existence of a map $g : [0, +\infty[\longrightarrow [0, +\infty[$ such that our claims are satisfied and in this way establish the counterexample.

Theorem 3.6.4 *Let $\kappa : [0, +\infty[\longrightarrow [0, +\infty[$ be a convex, strictly increasing function, which is continuous at the origin and satisfies $\kappa(0) = 0$. Then, defining $h(t) = \kappa(t) + t^2$, the concave map $g = h^{-1}$ yields a function $S(z, t) = |z| \vee g(|t|)$ which satisfies claims 1 and 2.*

PROOF. The convexity and the continuity at the origin of κ imply $\kappa(t)+\kappa(s) \leq \kappa(t+s)$ for any $t, s \geq 0$, hence

$$h(t + s) \geq h(t) + h(s) + 2ts \quad \text{for} \quad t, s \geq 0\,. \tag{3.27}$$

The function $h(t) = \kappa(t) + t^2$ is strictly monotone, thus $g = h^{-1}$ is well defined and $S(z, t) = |z| \vee g(|t|)$ also. The triangle inequality for the function $\rho(\xi, \eta) = S(\xi^{-1}\eta)$ is equivalent to $S(\xi\eta) \leq S(\xi) + S(\eta)$, for every $\xi, \eta \in \mathbb{H}^3$. We denote $\xi = (z, t)$, $\eta = (w, \tau)$, where $z = (z_1, z_2)$ and $w = (w_1, w_2)$, then

$$S(\xi\eta) = |z + w| \vee g(|t + \tau + 2(z_1w_2 - z_2w_1)|)\,.$$

If $|z + w| \geq g(|t + \tau + 2(z_1w_2 - z_2w_1)|)$, then we clearly have

$$S(\xi\eta) = |z + w| \leq |z| + |w| \leq S(\xi) + S(\eta)\,.$$

So, our inequality holds if we prove that

$$g(|t + \tau + 2(z_1w_2 - z_2w_1)|) \leq S(\xi) + S(\eta)\,. \tag{3.28}$$

We have

$$|t + \tau + 2(z_1w_2 - z_2w_1)| \leq |t| + |\tau| + 2|(z_1, z_2) \cdot (w_2, -w_1)| \leq |t| + |\tau| + 2|z||w|$$

and $|t| = h(g(|t|)) \leq h(S(\xi))$, $|\tau| = h(g(|\tau|)) \leq h(S(\eta))$, hence

$$|t + \tau + 2(z_1w_2 - z_2w_1)| \leq h(S(\xi)) + h(S(\eta)) + 2S(\xi)S(\eta)\,.$$

The latter inequality and property (3.27) give $|t+\tau+2(z_1w_2-z_2w_1)| \leq h(S(\xi)+S(\eta))$, which corresponds to $g(|t + \tau + 2(z_1w_2 - z_2w_1)|) \leq S(\xi) + S(\eta)$. It remains to prove $I : (\mathbb{H}^3, d) \longrightarrow (\mathbb{H}^3, \rho)$ is 1-Lipschitz. This fact is equivalent to show that $S \leq G$ which is true if $g(|t|) \leq \sqrt{t}$, that is $|t| \leq h(\sqrt{|t|}) = \kappa(\sqrt{t}) + |t|$. So the proof is complete. □

Now, among all the maps κ which enjoy the properties assumed in the preceding lemma, we want to find a particular one which produces the oscillation required in

Claim 3. We notice that if $v = (0,0,1) \in \mathbb{H}^3$, then $\rho(I(\zeta\delta_t v), I(\zeta)) = \rho(\delta_t v, 0) = g(t^2)$, so Claim 3 is equivalent to require the following

$$\limsup_{t\to 0_+} \frac{g(t^2)}{t} = 1\,, \qquad \liminf_{t\to 0_+} \frac{g(t^2)}{t} = 0\,, \tag{3.29}$$

where $g = h^{-1}$ and $h(t) = \kappa(t) + t^2$.

Theorem 3.6.5 *There exists $\kappa : [0,+\infty[\longrightarrow [0,+\infty[$, which is continuous, strictly increasing and convex, with $\kappa(0) = 0$, such that, defining $g = h^{-1}$, with $h(t) = \kappa(t) + t^2$, $t \geq 0$, the upper and lower limits as given in (3.29) hold.*

Proof. It is easy to see that the requirement (3.29) for g is equivalent to the condition

$$\limsup_{t\to 0_+} \frac{\kappa(t)}{t^2} = +\infty \ \text{ and } \ \liminf_{t\to 0_+} \frac{\kappa(t)}{t^2} = 0\,, \tag{3.30}$$

on the corresponding function κ. To find such a κ, we use the following simple observation. If we are given an affine, increasing function κ that vanishes at some positive number t' very close to zero, then the quotient $\kappa(t)/t^2$ oscillates a lot. Indeed, if t declines from 1 towards t' then the quotient first gets very large and then approaches zero. Stopping shortly before t', we can connect κ to another affine function with smaller but still positive slope that vanishes much closer to zero. Thus, the quotient considered oscillates along the new function even more and the combined function is convex.

To make this argument precise, we fix two positive sequences $(\varepsilon_l) \subset]0,1[$, $(m_l) \subset]0,+\infty[$, with $\varepsilon_l \to 0$ and $m_l \to +\infty$ as $l \to \infty$. We consider an arbitrary number $b_0 > 0$ and choose $t_0, a_0 > 0$ such that $t_0\varepsilon_0 < b_0$, $a_0 < \varepsilon_0 t_0^2$. Then, we define $\kappa_0(t) = a_0 + b_0(t - t_0)$, observing that $\kappa_0(t_0)/t_0^2 < \varepsilon_0$. We consider $\beta_1 = a_0/t_0 < t_0\varepsilon_0 < b_0$ and fix $\tau_1 \in]0, t_0[$ such that $\beta_1/\tau_1 > m_1$. We observe that

$$\lim_{b\to\beta_1^+} \frac{b}{\tau_1} + \frac{(\beta_1 - b)t_0}{\tau_1^2} = \frac{\beta_1}{\tau_1} > m_1\,, \quad \lim_{b\to\beta_1^+} \frac{t_0(b-\beta_1)}{b^2} = 0$$

hence we can choose $b_1 \in]\beta_1, b_0[$ such that

$$\frac{b_1}{\tau_1} + \frac{(\beta_1 - b_1)t_0}{\tau_1^2} > m_1 \quad \text{and} \quad \frac{t_0(b_1 - \beta_1)}{b_1^2} < \frac{1}{2}\,. \tag{3.31}$$

Now, we define $\kappa_1(t) = t_0(\beta_1 - b_1) + b_1 t$, so by the first inequality (3.31) we have $\kappa_1(\tau_1)/\tau_1^2 > m_1$ and $\kappa_1(t_0) = \beta_1 t_0 = a_0 = \kappa_0(t_0)$. We note that $\kappa_1(\bar{t}) = 0$ if and only if $\bar{t} = t_0(b_1 - \beta_1)/b_1 > 0$. By the second inequality of (4) we get $\bar{t} < b_1/2$ and since $\kappa_1(\tau_1) > 0$ we infer that $\bar{t} < \tau_1$. Thus, we can choose $t_1 \in]\bar{t}, \min(\tau_1, b_1/2)[$ such that $\kappa_1(t_1) < \varepsilon_1 t_1^2$ and $t_1\varepsilon_1 < b_1$. Defining $a_1 = \kappa_1(t_1)$, we see that $\kappa_1(t) = a_1 + b_1(t - t_1)$

and we have shown that for every $b_0, a_0, t_0, m_1 > 0$, with $a_0/t_0 < b_0$, for each $\varepsilon_1 > 0$ and $m_1 \in \mathbb{R}$ there exist $t_1 < \tau_1$ in $]0, t_0[$ and $a_1 > 0$, $b_1 \in]0, b_0[$ such that

$$\begin{cases} \kappa_1(t_0) = \kappa_0(t_0)\,, & \kappa_1(\tau_1)/\tau_1^2 > m_1\,, \\ \kappa_1(t_1)/t_1^2 < \varepsilon_1\,, & \kappa_1(t_1)/t_1 < b_1 < b_0\,, \quad t_1 < b_1/2\,. \end{cases}$$

This procedure can be iterated by induction, obtaining for each $j \geq 1$ that there exists $\tau_j, t_j > 0$, $\tau_j \in]t_j, t_{j-1}[$, and $a_j, b_j > 0$ such that the map $\kappa_j(t) = a_j + b_j(t - t_j)$ satisfies

$$\begin{cases} \kappa_j(t_{j-1}) = \kappa_{j-1}(t_{j-1}) & b_j < b_{j-1} \\ \kappa_j(\tau_j)/\tau_j^2 > m_j & \kappa_j(t_j)/t_j^2 < \varepsilon_j\,, \quad t_j < 2^{-j} b_j\,. \end{cases} \tag{3.32}$$

We define

$$\kappa(t) = \kappa_0(t)\, \mathbf{1}_{[t_0, +\infty[}(t) + \sum_{j=1}^{\infty} \kappa_j(t)\, \mathbf{1}_{[t_j, tj-1[}(t)\,,$$

observing that $t_j < b_j/2^j < b_0/2^j \to 0$ as $j \to \infty$, so by conditions (3.32) κ is a strictly increasing convex map defined on $]0, +\infty[$. The convexity follows from the continuity and from the fact that the sequence of slopes (b_j) decreases as the intervals get close to the origin. By the construction of κ we have that

$$\liminf_{t \to 0_+} \frac{\kappa(t)}{t^2} \leq \limsup_{j \to \infty} \frac{\kappa(t_j)}{t_j^2} \leq \lim_{j \to \infty} \varepsilon_j = 0\,, \tag{3.33}$$

$$\limsup_{t \to 0_+} \frac{\kappa(t)}{t^2} \geq \liminf_{j \to \infty} \frac{\kappa(\tau_j)}{\tau_j^2} \geq \lim_{j \to \infty} m_j = +\infty\,. \tag{3.34}$$

The sequence $(\kappa(t_j))$ converges to zero as $j \to \infty$ and κ is monotone, so $\kappa(t) \to 0$ as $t \to 0_+$ and κ is continuous at the origin. Thus, we have proved the existence of a strictly increasing convex map $\kappa : [0, +\infty[\longrightarrow [0, +\infty[$ which is continuous at the origin with $\kappa(0) = 0$ and which satisfies (3.33) and (3.34). These two conditions are of course just (3.30), so our proof is finished. □

Chapter 4

Area formulae

In this chapter we present the area formula for Lipschitz maps both in a general metric context and in the sub-Riemannian one. As an application, we characterize a wide class of sub-Riemannian groups that are purely unrectifiable and we prove that nonisomorphic sub-Riemannian groups cannot have bilipschitz equivalent pieces of positive measure.

We mention some related results in the literature. If $f : X \longrightarrow Y$ is a Lipschitz map, where X is a subset of $\mathbb{R}^n$, it was proved in [7], [110], [115] that the map is a.e. metrically differentiable (according to Definition 3.6.2). In papers [7], [110], it was also proved that the area formula holds, with a suitable notion of jacobian. When X and Y are sub-Riemannian groups the area formula has been proved in [124], [156], [184]. A first example of purely unrectifiable sub-Riemannian group was given in [7]. About the nonexistence of bilipschitz parametrizations for different stratified groups we mention results of [154] and [168].

In Section 4.1 we present the general metric setting to organize the area formula for Lipschitz maps. One of the main reasons of this abstract presentation is to emphasize that the core of area formula is the notion of jacobian. We adopt a notion of "metric jacobian" (Definition 4.1.4) that was already considered in [154], when X and Y are sub-Riemannian groups. With this notion we obtain a general metric formulation of the area formula (Theorem 4.1.7). This result could appear a bit tautological, because it consists in the integration of the density of $f^\sharp \mathcal{H}^k_\rho$ (Definition 4.1.2) with respect to $\mathcal{H}^k_d$, where the density is by definition the jacobian of f. On the other hand, it is curious to notice that the minimal conditions on f, X and $\mathcal{H}^k_d$ in Section 4.1 are sufficient to formulate the area formula in a purely metric context. Furthermore, this approach also provides a novel and unified method to prove the area formula in several contexts, simply by proving that the jacobian coincides with the "metric" one. Notice that we do not assume any differentiability-type theorem for f, but we suppose that there exists a countable covering $\{E_i\}$ of the set of points where the jacobian is positive, such that the restriction $f_{|E_i}$ is injective. This last condition in general can be deduced from some differentiability-type theorem, see Remark 4.3.5.

In Section 4.2 we study another definition of jacobian for H-linear maps of sub-Riemannian groups. This notion is well adapted to the geometry of the groups, taking into account the algebraic structure and the metric structure (Definition 4.2.1). We can view this notion as a natural extension of the one introduced in [7]. Basically this definition requires that the area formula holds in principle for any H-linear map. We also show that the H-jacobian is proportional to the classical jacobian (4.13).

In Section 4.3 we prove the area formula for Lipschitz maps between sub-Riemannian groups. We will present two proofs of this formula. The first one, based on the general area formula in metric spaces and the second one, based on a more classical approach. In both proofs we will need of Proposition 4.3.1 and Proposition 4.3.3 that represent the core of the sub-Riemannian area formula. Concerning the more classical approach, we utilize the notion of jacobian given in Section 4.2, that provides also a way to either compute or represent the measure of the image of an injective Lipschitz map (see Example 4.3.7). Notice that if the Lipschitz map takes values in the same sub-Riemannian group, the area formula reduces to a change of variable, that it was first proved in [177]. Concerning this classical approach, we mention that our definition of jacobian (Definition 4.2.1) allows us to avoid the decomposition of the differential as a product of a symmetric linear map and an isometry, and to follow a bit more intrinsic computation. A delicate part in the proof of the area formula is to show that the image of points with noninjective differential is negligible. To do this, we generalize the method used in [6] for the Euclidean case. We get an estimate on the number of balls we need to cover $f(B_{x,r})$, exploiting the fact that the image of $d_H f(x)$ at a singular point x is a subgroup of Hausdorff dimension smaller than Q, where Q is the Hausdorff dimension of $\mathbb{G}$.

In Section 4.4 we provide a general criterion to characterize nonabelian sub-Riemannian groups which are purely unrectifiable (Theorem 4.4.4), according to the definition given in 3.2.14 of [55]. We mention that first examples of purely unrectifiable sub-Riemannian groups were given in [7], considering the three dimensional Heisenberg group. Our approach relies on the area estimate (4.29) applied to Lipschitz maps $f : A \longrightarrow \mathbb{M}$, where $A \subset \mathbb{R}^k$. Observing that H-linear maps are in particular group homomorphisms, they may be injective maps only if there exist abelian subgroups of $\mathbb{M}$ with topological dimension k. If this is not allowed by the nonabelian structure, then any H-linear map has nontrivial kernel and we always have $J_Q(d_H f(x)) = 0$ in formula (4.29), whence the purely k-unrectifiability follows. With an analogous procedure, in Theorem 4.4.6 we show that two nonisomorphic sub-Riemannian groups cannot be bilipschitz equivalent, even if we consider two arbitrary measurable subsets with positive measure. This is basically a "rigidity theorem", namely, bilipschitz classes of sub-Riemannian groups contain only one group up to isomorphisms.

4.1 Area formula in metric spaces

In this section we present the general metric setting to organize the metric area formula. Let (X,d) and (Y,ρ) be complete metric spaces and let $f : X \longrightarrow Y$ be a Lipschitz map. Throughout the section we will also assume that (X,d) is separable. Our basic assumptions are the following:

$(A1)$ the measure $\mathcal{H}_d^k$ is finite on bounded sets,

$(A2)$ for $\mathcal{H}_d^k$-a.e. $x \in X$ we have lower density estimate

$$\liminf_{r \to 0^+} \frac{\mathcal{H}_d^k(D_{x,r})}{r^k} > 0\,.$$

Theorem 2.10.18(3) of [55] yields

$$\limsup_{r \to 0^+} \frac{\mathcal{H}_d^k(D_{x,r})}{r^k} \le \omega_k\,, \tag{4.1}$$

for $\mathcal{H}_d^k$-a.e. $x \in X$. Then estimate (4.1) and assumption (A2) imply that the measure $\mathcal{H}_d^k$ is a.e. asymptotically doubling, i.e.

$$\lim_{r \to 0^+} \frac{\mathcal{H}_d^k(D_{x,2r})}{\mathcal{H}_d^k(D_{x,r})} < +\infty \tag{4.2}$$

for $\mathcal{H}_d^k$-a.e. $x \in X$. This last property of $\mathcal{H}_d^k$ is crucial in order to differentiate the pull-back measure (Definition 4.1.2) with respect to $\mathcal{H}_d^k$.

Remark 4.1.1 Notice that if (X,d) is a k-rectifiable metric space, then for $\mathcal{H}_d^k$-a.e. $x \in X$ we have

$$\lim_{r \to 0^+} \frac{\mathcal{H}_d^k(D_{x,r})}{r^k} = \omega_k\,,$$

(see [110]), then condition (A2) holds. When (X,d) is a sub-Riemannian group of homogeneous dimension $k = Q$, we simply have $\mathcal{H}_d^Q(D_{x,r}) = r^Q \mathcal{H}_d^Q(D_1)$ and (A2) trivially holds.

Our first observation is that a Lipschitz map $f : X \longrightarrow Y$ induces a new measure on X which is absolutely continuous with respect to $\mathcal{H}_d^k$.

Definition 4.1.2 Let $f : X \longrightarrow Y$ be a Lipschitz map. We define the *pull-back measure* on X as follows

$$f^\sharp \mathcal{H}_\rho^k(A) = \mathcal{H}_\rho^k(f(A))$$

for any $A \subset X$.

It is a standard fact that

$$f^\sharp \mathcal{H}^k_\rho(A) \leq \mathrm{Lip}(f)^k\, \mathcal{H}^k_d(A)\,, \tag{4.3}$$

then $f^\sharp \mathcal{H}^k_\rho$ is absolutely continuous with respect to $\mathcal{H}^k_d$. The metric notion of jacobian in Definition 4.1.4 is motivated by Theorem 2.9.5 and Theorem 2.9.7 of [55], which concern differentation of measures and integration of densities, respectively. Besides the asymptotically doubling property (4.2), these theorems also require that the measure $f^\sharp \mathcal{H}^k_\rho$ is both finite on bounded sets and Borel regular. The first condition follows from (A1) and (4.3). The second condition follows by the so-called "Carathéodory's criterion" (see 2.3.2(9) of [55]): *a measure μ on a metric space X such that it is additive on open sets with positive distance is a Borel measure.*

By virtue of the result 2.2.13 in [55] we know that every Borel set A is mapped into an $\mathcal{H}^k_\rho$-measurable set $f(A)$. In the case when f is injective the additivity property holds, hence $f^\sharp \mathcal{H}^k_\rho$ is a Borel measure. Finally, the Borel regularity of $\mathcal{H}^k_d$ and the estimate (4.3) imply the Borel regularity of $f^\sharp \mathcal{H}^k_\rho$. The previous arguments can be summarized in the following proposition.

Proposition 4.1.3 *Let $f : X \longrightarrow Y$ be an injective Lipschitz map. Under the assumption (A1) the measure $f^\sharp \mathcal{H}^k_\rho$ is a Borel regular measure on X and it is both absolutely continuous with respect to $\mathcal{H}^k_d$ and finite on bounded sets.*

Definition 4.1.4 (Metric jacobian) Let $f : X \longrightarrow Y$ be a Lipschitz map and let $x \in X$. The *metric jacobian* of f at x is defined as follows

$$J_f(x) = \liminf_{r \to 0^+} \frac{f^\sharp \mathcal{H}^k_\rho(D_{x,r})}{\mathcal{H}^k_d(D_{x,r})}. \tag{4.4}$$

Remark 4.1.5 Notice that in view of (4.3) we have $J_f(x) < +\infty$ for any $x \in X$.

Theorem 4.1.6 *Let $f : X \longrightarrow Y$ be an injective Lipschitz map and assume (A1) and (A2). Then for any $\mathcal{H}^k_d$-measurable subset $A \subset X$ the following formula holds*

$$\int_A J_f(x)\, d\mathcal{H}^k_d(x) = \mathcal{H}^k_\rho\left(f(A)\right)\,. \tag{4.5}$$

PROOF. We have seen in the above discussion that assumption (A2) yields the estimate (4.2) for $\mathcal{H}^k_d$-a.e. $x \in X$. Due to Theorem 2.1.22, the family of closed balls in X is an $\mathcal{H}^k_d$-Vitali relation. Moreover, by Proposition 4.1.3 the measure $f^\sharp \mathcal{H}^k_\rho$ is Borel regular and absolutely continuous with respect to $\mathcal{H}^k_d$. Hence we are in the position to apply Theorem 2.9.5 and Theorem 2.9.7 of [55], obtaining

$$\limsup_{r \to 0^+} \frac{f^\sharp \mathcal{H}^k_\rho(D_{x,r})}{\mathcal{H}^k_d(D_{x,r})} = \liminf_{r \to 0^+} \frac{f^\sharp \mathcal{H}^k_\rho(D_{x,r})}{\mathcal{H}^k_d(D_{x,r})} = J_f(x)$$

for $\mathcal{H}^k_d$-a.e. $x \in X$ and the integration formula (4.5). □

Theorem 4.1.7 (Area formula) *Let $f : A \longrightarrow Y$ be a Lipschitz map, where A is a closed subset of X and assume (A1) and (A2). If there exists a disjoint family of $\mathcal{H}^k_d$-measurable subsets $\{E_i\}_{i\in\mathbb{N}}$ which covers A, such that*

$$\mathcal{H}^k_d\left(A \setminus \bigcup_{i\in\mathbb{N}} E_i\right) = 0\,,$$

$f_{|E_i}$ is injective for every $i \geq 1$ and for $\mathcal{H}^k_d$-a.e. $x \in E_0$ we have $J_f(x) = 0$, then the following formula holds

$$\int_A J_f(x)\, d\mathcal{H}^k_d(x) = \int_Y N(f, A, y)\, d\mathcal{H}^k_\rho(y)\,, \tag{4.6}$$

where the jacobian J_f is referred to the complete metric space A.

PROOF. For every closed subset $F \subset X$ we will use the notation $D^F_{x,r} = F \cap D_{x,r}$ to indicate the closed ball relative to the complete metric space F. Let us fix $\varepsilon > 0$ and consider a sequence of closed sets $C_i \subset E_i$ such that $\mathcal{H}^k_d(E_i \setminus C_i) \leq \varepsilon 2^{-i}$ for any $i \in \mathbb{N}$. In order to apply Theorem 4.1.6 to the maps $f_i = f_{|C_i} : C_i \longrightarrow Y$, we have to make sure that (A2) holds replacing X with the complete metric space C_i. In view of our assumptions the estimate (4.2) holds $\mathcal{H}^k_d$-a.e. in X. Thus, by Theorem 2.1.22 closed balls of X form an $\mathcal{H}^k_d$-Vitali relation, hence Theorem 2.9.8 of [55] applied to $\mathbf{1}_{C_i}$ yields

$$\frac{\mathcal{H}^k_d(D^{C_i}_{x,r})}{\mathcal{H}^k_d(D_{x,r})} \longrightarrow 1 \quad \text{as} \quad r \to 0^+\,, \tag{4.7}$$

for $\mathcal{H}^k_d$-a.e. $x \in C$, that implies

$$\liminf_{r\to0^+} \frac{\mathcal{H}^k_d(D^{C_i}_{x,r})}{r^k} = \liminf_{r\to0^+} \frac{\mathcal{H}^k_d(D^{C_i}_{x,r})}{\mathcal{H}^k_d(D_{x,r})}\frac{\mathcal{H}^k_d(D_{x,r})}{r^k} = \liminf_{r\to0^+} \frac{\mathcal{H}^k_d(D_{x,r})}{r^k} > 0\,,$$

for $\mathcal{H}^k_d$-a.e. $x \in C_i$. Next, we check that $J_f(x) = J_{f_i}(x)$ for $\mathcal{H}^k_d$-a.e. $x \in C_i$. By (4.7) and (4.3) we have

$$\begin{aligned}
J_{f_i}(x) &= \liminf_{r\to0^+} \frac{f_i^\sharp\mathcal{H}^k_\rho(D^{C_i}_{x,r})}{\mathcal{H}^k_d(D^{C_i}_{x,r})} \leq \liminf_{r\to0^+} \frac{f^\sharp\mathcal{H}^k_\rho(D^{A}_{x,r})}{\mathcal{H}^k_d(D^{C_i}_{x,r})} = \liminf_{r\to0^+} \frac{f^\sharp\mathcal{H}^k_\rho(D^{A}_{x,r})}{\mathcal{H}^k_d(D^{A}_{x,r})} \\
&= J_f(x) \leq \liminf_{r\to0^+}\left(\frac{f^\sharp\mathcal{H}^k_\rho(D^A_{x,r} \setminus D^{C_i}_{x,r})}{\mathcal{H}^k_d(D^{C_i}_{x,r})} + \frac{f^\sharp\mathcal{H}^k_\rho(D^{C_i}_{x,r})}{\mathcal{H}^k_d(D^{C_i}_{x,r})}\right) \\
&\leq \liminf_{r\to0^+}\left(\mathrm{Lip}(f)^k\frac{\mathcal{H}^k_d(D^A_{x,r} \setminus D^{C_i}_{x,r})}{\mathcal{H}^k_d(D^{C_i}_{x,r})} + \frac{f_i^\sharp\mathcal{H}^k_\rho(D^{C_i}_{x,r})}{\mathcal{H}^k_d(D^{C_i}_{x,r})}\right) = J_{f_i}(x),
\end{aligned}$$

for $\mathcal{H}^k_d$-a.e. $x \in C_i$. Then Theorem 4.1.6 applied to f_i yields

$$\int_{C_i} J_f(x)\, d\mathcal{H}^k_d(x) = \int_{C_i} J_{f_i}(x)\, d\mathcal{H}^k_d(x) = \int_Y \mathbf{1}_{f(C_i)}(y)\, d\mathcal{H}^k_\rho(y)\,, \tag{4.8}$$

for any $i \in \mathbb{N}$. Adding formula (4.8) over all $i \geq 1$ we obtain

$$\int_F J_f(x)\, d\mathcal{H}^k_d(x) = \int_Y N(f, F, y)\, d\mathcal{H}^k_\rho(y) \tag{4.9}$$

where $F = \bigcup_{i\geq 1} C_i$. From the hypothesis $A = \bigcup_{i\in\mathbb{N}} E_i$ we conclude that

$$\mathcal{H}^k_d\Big((A \setminus E_0) \setminus F\Big) \leq 2\varepsilon\,.$$

From estimate 2.10.25 of [55] we conclude that

$$\int_Y N(f, Z, y)\, d\mathcal{H}^k_\rho(y) = 0$$

whenever $\mathcal{H}^k_d(Z) = 0$, hence by virtue of Beppo Levi Convergence Theorem for nonnegative increasing sequences of maps, taking an increasing sequence of Borel sets (F_j) constructed as above such that (4.9) holds and which are associated to an infinitesimal sequence (ε_j), we obtain

$$\int_A J_f(x)\, d\mathcal{H}^k_d(x) = \int_{A\setminus E_0} J_f(x)\, d\mathcal{H}^k_d(x) = \int_Y N(f, A \setminus E_0, y)\, d\mathcal{H}^k_\rho(y)$$

If we prove that $\mathcal{H}^k_\rho(f(E_0)) = 0$, then

$$\int_Y N(f, A \setminus E_0, y)\, d\mathcal{H}^k_\rho(y) = \int_Y N(f, A, y)\, d\mathcal{H}^k_\rho(y)\,,$$

and the set additive property of the multiplicity function $N(f, \cdot, y)$ leads us to (4.6). We use again the fact that the family of closed balls in X is an $\mathcal{H}^k_d$-Vitali relation. Thus, by Lemma 2.1.24 applied to $\mu = \mathcal{H}^k_d$ and $\nu = f^\sharp\mathcal{H}^k_\rho$ we obtain that

$$f^\sharp\mathcal{H}^k_\rho(F) \leq \alpha\, \mathcal{H}^k_d(F)$$

whenever F is an $\mathcal{H}^k_d$-measurable subset of $\{x \in A \mid J_f(x) \leq \alpha\}$. By definition of metric jacobian and the fact that $J_f(x) = 0$ for any $x \in E_0$ we can choose α arbitrarily small and $F = E_0 \cap D_{p,n}$ for some fixed $p \in X$ and $n \in \mathbb{N}$. Letting $\alpha \to 0^+$ we obtain $\mathcal{H}^k_\rho\left(f(E_0 \cap D_{p,n})\right) = 0$, where $n \in \mathbb{N}$ is arbitrary. Then, considering $n \to \infty$ we conclude that $\mathcal{H}^k_\rho\left(f(E_0)\right) = 0$ and the thesis follows. $\Box$

Remark 4.1.8 Note that under the assumptions (A1) and (A2) the metric jacobian is uniquely defined up to sets of $\mathcal{H}^k_d$-negligible measure. In fact, if formula (4.6) holds with another notion of jacobian $\tilde{J}_f$, then we have

$$\int_{D^A_{p,r}} J_f(x)\, d\mathcal{H}^k_d(x) = \int_{D^A_{p,r}} \tilde{J}_f(x)\, d\mathcal{H}^k_d(x)$$

for any $p \in A$ and $r > 0$, where $D^A_{p,r} = D_{p,r} \cap A$. By Theorem 2.1.22 the family of closed balls forms a $\mathcal{H}^k_d$-Vitali relation, hence Theorem 2.9.8 of [55] referred to implies that for $\mathcal{H}^k_d$-a.e. $p \in A$ we have

$$\begin{aligned} J_f(p) &= \lim_{r\to 0^+} \frac{1}{\mathcal{H}^k_d(D_{p,r})} \int_{D^A_{p,r}} J_f(x)\, d\mathcal{H}^k_d(x) \\ &= \lim_{r\to 0^+} \frac{1}{\mathcal{H}^k_d(D_{p,r})} \int_{D^A_{p,r}} \tilde{J}_f(x) d\mathcal{H}^k_d(x) = \tilde{J}_f(p)\,. \end{aligned}$$

We also point out that the assumption on closedness of the domain A in Theorem 4.1.7 is not restrictive. In fact, a Lipschitz map defined on an arbitrary subset and with values in a complete metric space can always be extended to the closure of its domain.

Next, we present an example where the abstract conditions of Theorem 4.1.7 are satisfied. To do this, we will need of both results in [7] and [110]. We will utilize the following notion of jacobian, taken from [7].

Definition 4.1.9 (Normed jacobian) Let ν be a seminorm on $\mathbb{R}^k$. The *normed jacobian* of ν is defined as follows

$$\mathbf{J}_k(\nu) = \frac{\omega_k}{\mathcal{H}^k_{|\cdot|}\left(\{v \in \mathbb{R}^k \mid \nu(v) \le 1\}\right)}\,,$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^k$.

Proposition 4.1.10 *Let $f : A \longrightarrow Y$ be a Lipschitz map, where A is a closed subset of $\mathbb{R}^k$ and Y is a metric space. Then hypotheses of Theorem 4.1.7 are satisfied.*

Proof. It is known that f is a.e. metrically differentiable on A, see [7], [110], [115]. As a consequence, by Lemma 4 of [110] the set where f is metrically differentiable and the metric differential is a norm admits a partition $\{E_j\}_{j\ge 1}$, where $f_{|E_j}$ is injective for any $j \ge 1$. We define $E_0 = A \setminus \bigcup_{j\ge1} E_j$. It remains to prove that $J_f(x) = 0$ for $\mathcal{H}^k_d$-a.e. $x \in E_0$. The validity of area formula with respect to the notion of normed jacobian of Definition 4.1.9 (see Theorem 5.1 of [7]) and Remark 4.1.8 imply that the metric jacobian coincides with the normed jacobian. By the fact that $mdf(x)$ is not a norm for $\mathcal{H}^k_d$-a.e. $x \in E_0$ we conclude that $\mathcal{H}^k_{|\cdot|}\big(\{v \in \mathbb{R}^k \mid mdf(x,v) \le 1\}\big) = \infty$ for $\mathcal{H}^k_d$-a.e. $x \in E_0$. Thus, Definition 4.1.9 leads us to the conclusion. □

Remark 4.1.11 It is curious that in Proposition 4.1.10 we have proved assumptions of Theorem 4.1.7 using the validity of area formula of [7]. It would be interesting to prove Proposition 4.1.10 using directly the notion of metric jacobian.

4.2 Jacobians

This section is devoted to the notion of H-jacobian for H-linear maps. This notion is essentially inspired by the work [7]. We will show an explicit formula that connects the H-jacobian and the classical one (4.13). In the sequel, we suppose that $\mathbb{G}$ and $\mathbb{M}$ are sub-Riemannian groups and a system of graded coordinates (F, W) will be assumed on $\mathbb{M}$.

Definition 4.2.1 (H-jacobian) Let $L \in \mathrm{HL}(\mathbb{G}, \mathbb{M})$. The *horizontal jacobian* $J_Q(L)$ of L is defined as follows

$$J_Q(L) = \frac{\mathcal{H}^Q_\rho(L(B_1))}{\mathcal{H}^Q_d(B_1)}\,.$$

We will also say in short H-*jacobian*.

A covering argument together with the homogeneity and the homomorphism property of L shows that the above definition is independent of the set we consider, hence we can replace the set B_1 with any measurable set with positive finite measure.

In the next proposition we show that the H-jacobian is zero for noninjective H-linear maps and we provide a formula for the Hausdorff dimension of its image.

Proposition 4.2.2 *Let ρ be a homogeneous distance of $\mathbb{M}$ and let $L \in \mathrm{HL}(\mathbb{G}, \mathbb{M})$. We denote by q_0 the topological dimension of $\mathbb{S} = L(\mathbb{G})$. Then, the Hausdorff dimension of $\mathbb{S}$ in the metric ρ is $Q_0 = \sum_{j=1}^{\iota} j\,\dim\left(L(\mathbb{V}_j)\right)$ and*

$$\mathcal{H}^{Q_0}_\rho \llcorner \mathbb{S} = \alpha_{\mathbb{S}}\, \mathcal{H}^{q_0}_{|\cdot|} \llcorner F^{-1}(\mathbb{S}) \tag{4.10}$$

where $\alpha_{\mathbb{S}} = \mathcal{H}^{Q_0}_\rho(\mathbb{S} \cap B^\rho_1)/\mathcal{H}^{q_0}_{|\cdot|}\left(F^{-1}(\mathbb{S} \cap B^\rho_1)\right)$ and $\dim\left(L(\mathbb{V}_j)\right)$ is the topological dimension of $L(\mathbb{V}_j)$.

Proof. We define $\tilde{L} = F^{-1} \circ L \circ F : \mathbb{R}^q \longrightarrow \mathbb{R}^p$, observing that

$$\tilde{\mathbb{S}} = \tilde{L}(\mathbb{R}^q) = \tilde{L}(\mathbb{R}^{n_1}) \oplus \cdots \oplus \tilde{L}(\mathbb{R}^{n_\iota}) \subset \mathbb{R}^p$$

where $\mathbb{R}^{n_j} = F^{-1}(\mathbb{V}_j)$ and the variables in $\mathbb{R}^{n_j}$ have degree j for every $j = 1, \ldots, \iota$. It follows that the restriction of the coordinate dilation Λ_r to $\tilde{\mathbb{S}}$ has jacobian

$$\mathcal{J}_{q_0}\left(\Lambda_{r|\tilde{\mathbb{S}}}\right) = r^{\sum_{j=1}^{\iota} j\,\dim(L(\mathbb{V}_j))} = r^{Q_0}\,, \tag{4.11}$$

where we have defined $Q_0 = \sum_{j=1}^{\iota} j \dim(L(\mathbb{V}_j))$. By definition of coordinate dilations we have

$$B_r^\rho \cap \mathbb{S} = \delta_r (B_1^\rho \cap \mathbb{S}) = F \circ \Lambda_r \left(F^{-1}(B_1^\rho \cap \mathbb{S})\right) .$$

We denote by $\tilde{v}_g$ the Riemannian volume restricted to $\mathbb{S}$. Thus, Proposition 2.3.47 and the previous formula yields

$$\tilde{v}_g(\mathbb{S} \cap B_r^\rho) = F_\sharp^{-1} \tilde{v}_g \left(\Lambda_r \left(F^{-1}(B_1^\rho \cap \mathbb{S})\right)\right) = \mathcal{H}_{|\cdot|}^{q_0} \left(\Lambda_r \left(F^{-1}(B_1^\rho \cap \mathbb{S})\right)\right).$$

Due to (4.11) the previous relation becomes

$$\tilde{v}_g(\mathbb{S} \cap B_r^\rho) = r^{Q_0}\, \mathcal{H}_{|\cdot|}^{q_0} \left(F^{-1}(B_1^\rho \cap \mathbb{S})\right) \tag{4.12}$$

and observing that for every $x \in \mathbb{S}$ the translation $l_x : \mathbb{S} \longrightarrow \mathbb{S}$ is an isometry, it follows that

$$\tilde{v}_g(\mathbb{S} \cap B_{x,r}^\rho) = \tilde{v}_g(\mathbb{S} \cap B_r^\rho) = r^{Q_0}\, \mathcal{H}_{|\cdot|}^{q_0} \left(F^{-1}(B_1^\rho \cap \mathbb{S})\right) .$$

From the last formula we deduce that $\mathcal{H}_\rho^{Q_0} \llcorner \mathbb{S}$ is a locally finite measure that is also left invariant. Thus, the measures $\mathcal{H}_\rho^{Q_0} \llcorner \mathbb{S}$ and $\mathcal{H}_{|\cdot|}^{q_0} \llcorner F^{-1}(\mathbb{S})$ are proportional and the thesis follows. □

Proposition 4.2.3 *Let $L : \mathbb{G} \longrightarrow \mathbb{M}$ be an injective H-linear map, with $\mathbb{S} = L(\mathbb{G})$. Then the H-jacobian of the map is given by the formula*

$$J_Q(L) = \alpha_{\mathbb{S}}\, \beta_Q\, \mathcal{J}_q \left(F^{-1} \circ L \circ F\right) , \tag{4.13}$$

where $\alpha_{\mathbb{S}} = \mathcal{H}_\rho^Q (\mathbb{S} \cap B_1^\rho) / \mathcal{H}_{|\cdot|}^q \left(F^{-1}(\mathbb{S} \cap B_1^\rho)\right)$, $\beta_Q = \mathcal{L}^q(\tilde{B}_1)/\mathcal{H}_d^Q(B_1)$, $\tilde{B}_1 = F^{-1}(B_1)$ and $\mathcal{J}_q$ denotes the classical jacobian according to Definition 2.3.40.

Proof. From Proposition 4.2.2 and the injectivity of L the space $\mathbb{S}$ has topological dimension q and Hausdorff dimension Q, moreover we have

$$\mathcal{H}_\rho^Q (L(B_1)) = \alpha_{\mathbb{S}}\, \mathcal{H}_{|\cdot|}^q (F^{-1} \circ L(B_1)) .$$

The Euclidean area formula for linear maps yields

$$\mathcal{H}_{|\cdot|}^q \left(F^{-1} \circ L \circ F(\tilde{B}_1)\right) = \mathcal{J}_q \left(F^{-1} \circ L \circ F\right)\, \mathcal{L}^q(\tilde{B}_1) ,$$

so the proof is complete. □

Remark 4.2.4 The coefficient

$$\alpha_{\mathbb{S}}\, \beta_Q = \frac{\mathcal{H}_\rho^Q (\mathbb{S} \cap B_1^\rho)\, \mathcal{L}^q(\tilde{B}_1)}{\mathcal{H}_{|\cdot|}^q (F^{-1}(\mathbb{S} \cap B_1^\rho))\, \mathcal{H}_d^Q(B_1)}$$

represents a "distortion factor", which depends on both the measures $\mathcal{H}_d^Q$, $\mathcal{H}_\rho^Q$ and on the subspace $\mathbb{S}$ we consider. Notice that if $\mathbb{G} = \mathbb{M}$ and then $d{=}\rho$, $\mathbb{S} = \mathcal{G}$, we get $\mathcal{H}_\rho^Q (\mathbb{S} \cap B_1^\rho) = \mathcal{H}_d^Q(B_1)$ and $\mathcal{H}_{|\cdot|}^q \left(F^{-1}(\mathbb{S} \cap B_1^\rho)\right) = \mathcal{L}^q(\tilde{B}_1)$ and the distortion factor reduces to one.

4.3 Sub-Riemannian area formula

In this section we prove the sub-Riemannian area formula for Lipschitz maps. We denote by $\mathbb{G}$ and $\mathbb{M}$ two sub-Riemannian groups.

Proposition 4.3.1 *Let $f : A \subset \mathbb{G} \longrightarrow \mathbb{M}$ be a measurable function, $\lambda > 1$, and*

$$E = \{x \in \mathcal{I}(A) \mid \textit{there exists } d_H f(x) : \mathbb{G} \longrightarrow \mathbb{M} \textit{ and is injective}\}.$$

Then E has a measurable countable partition $\mathcal{F}$, such that for any $T \in \mathcal{F}$ there is an injective H-linear map $\varphi : \mathbb{G} \longrightarrow \mathbb{M}$ with the following properties

$$\lambda^{-1}\rho(\varphi(z)) \leq \rho(d_H f(x)(z)) \leq \lambda\,\rho(\varphi(z)) \quad \textit{for any } z \in \mathbb{G} \textit{ and any } x \in T \tag{4.14}$$

$$Lip\left(f_{|T} \circ (\varphi_{|T})^{-1}\right) \leq \lambda \;\; \textit{and} \;\; Lip\left(\varphi_{|T} \circ (f_{|T})^{-1}\right)\right) \leq \lambda. \tag{4.15}$$

Proof. By linearity of H-linear maps when represented between Lie algebras (Corollary 3.1.11) we get a countable dense subset $\tilde{\mathcal{K}}$ of $\mathrm{HL}(\mathcal{G}, \mathcal{M})$. The set $\tilde{\mathcal{K}}$ has the isometric correspondent $\mathcal{K} = \{\varphi \in \mathrm{HL}(\mathbb{G}, \mathbb{M}) \mid \varphi = \exp \circ \tilde{\varphi} \circ \ln : \mathbb{G} \longrightarrow \mathbb{M},\ \tilde{\varphi} \in \tilde{\mathcal{K}}\}$. Choose $\varepsilon > 0$ such that $\lambda^{-1} + \varepsilon < 1 < \lambda - \varepsilon$ and define the measurable set $S(\varphi, k) = \{y \in E \mid (\star) \text{ holds}\}$ with $\varphi \in \mathcal{K}$ and $k \in \mathbb{N}$, where

$$(\star) \qquad \begin{cases} (\lambda^{-1} + \varepsilon)\,\rho\,(\varphi(z)) \leq \rho\,(d_H f(y)(z)) \leq (\lambda - \varepsilon)\,\rho\,(\varphi(z)) & \forall z \in \mathbb{G} \\ \rho\left(f(z), f(y)\, d_H f(y)(y^{-1}z)\right) \leq \varepsilon\,\rho(\varphi(y^{-1}z)) & \forall z \in B_{y,1/k}. \end{cases}$$

We will prove that every $y \in E$ is contained in $S(\varphi, k)$ for some $k \in \mathbb{N}$ and $\varphi \in \mathcal{K}$. Define $\tilde{L} = \ln \circ\, d_H f(y) \circ \exp$ and choose a positive $\varepsilon_1 < \min_{|w|=1} |\tilde{L}|$, where $|\cdot|$ is the norm of the fixed scalar product on the Lie algebras. We can find $\tilde{\varphi} \in \tilde{\mathcal{K}}$ such that $\|\tilde{L} - \tilde{\varphi}\| \leq \varepsilon_1$ as linear maps, so $\tilde{\varphi}$ has to be injective on $\mathfrak{g}$. The maps $\tilde{\varphi} : \mathcal{G} \longrightarrow \mathcal{M}$ and $\tilde{L} : \mathcal{G} \longrightarrow \mathcal{M}$ are injective, so by Corollary 3.1.9 the maps $\tilde{\varphi}^{-1}$ and $\tilde{L}^{-1}$ are H-linear. We accomplish our calculations for $\tilde{\varphi}$ due to the equality $\rho(\tilde{\varphi}(\ln z)) = \rho(\varphi(z))$, for any $z \in \mathbb{G}$, where $\varphi = \exp \circ \tilde{\varphi} \circ \ln \in \mathcal{K}$. By estimate (2.11) we obtain

$$\rho(\tilde{L}, \tilde{\varphi}) \leq C\, \|\tilde{L} - \tilde{\varphi}\|^{1/\iota} \leq C\, \varepsilon_1^{1/\iota},$$

where ι is the degree of nilpotency of $\mathbb{M}$. The estimates (3.2) imply

$$\rho(\tilde{L} \circ \tilde{\varphi}^{-1}) = \rho\left((\tilde{\varphi} \cdot (-\tilde{\varphi}) \cdot \tilde{L}) \circ \tilde{\varphi}^{-1})\right) \leq 1 + \rho(\tilde{\varphi}, \tilde{L})\, d(\tilde{\varphi}^{-1}),$$

$$d(\tilde{\varphi}^{-1}) = \rho(\tilde{L}^{-1} \circ \tilde{L} \circ \tilde{\varphi}^{-1}) \leq d(\tilde{L}^{-1})\, \rho(\tilde{L} \circ \tilde{\varphi}^{-1}),$$

hence, choosing ε_1 small enough, depending on $\tilde{L}, C, \varepsilon$ and λ, we have

$$\rho(\tilde{L} \circ \tilde{\varphi}^{-1}) \leq \frac{1}{1 - \rho(\tilde{\varphi}, \tilde{L}) d(\tilde{L}^{-1})} \leq \frac{1}{1 - C\varepsilon_1^{1/m} d(\tilde{L}^{-1})} < \lambda - \varepsilon,$$

$$\rho(\tilde{\varphi}\circ\tilde{L}^{-1}) = \rho\left((\tilde{L}\cdot(-\tilde{L})\cdot\tilde{\varphi})\circ\tilde{L}^{-1})\right) \le 1 + \rho(\tilde{L},\tilde{\varphi})\, d(\tilde{L}^{-1})$$
$$\le 1 + C\varepsilon_1^{1/m} d(\tilde{L}^{-1}) < (\lambda^{-1}+\varepsilon)^{-1}$$

and the last two inequalities prove the first estimate of $(\star)$. The definition of differentiability and the Lipschitz property of φ^{-1} leads to the second estimate of $(\star)$ for k large depending on φ and ε. From the σ-compactness of $\mathbb{G}$ the set $S(\varphi,k)$ has a countable partition of measurable sets $T\subset S(\varphi,k)$ with $\mathrm{diam}(T)\le 1/k$, so if we prove properties (4.14) and (4.15) for any T, we have finished the proof. Consider two points $u,y\in T\subset S(\varphi,k)$, by the definition of $S(\varphi,k)$, the first equation of $(\star)$ leads to (4.14). The second equation of $(\star)$ relatively to y gives

$$\rho(f(u),f(y)) \le \rho(d_H f(y)(y^{-1}u)) + \varepsilon\,\rho(\varphi(y^{-1}u))\,, \tag{4.16}$$

$$\rho(f(u),f(y)) \ge \rho(d_H f(y)(y^{-1}u)) - \varepsilon\,\rho(\varphi(y^{-1}u))\,, \tag{4.17}$$

adding the first one of $(\star)$, with $z=y^{-1}u$, to both equations (4.16) and (4.17) we get (4.15). $\square$

An important tool for Proposition 4.3.3 is the following, see for instance [45].

Lemma 4.3.2 *Let (X,d,μ) be an Ahlfors regular space of dimension Q. Then, any ball B of radius R can be covered by at most $C\,(R/r)^Q$ balls of radius r, with C depending only on the regularity constants for X.*

The next proposition is an extension of the Sard Theorem in stratified groups when the dimension of the target is larger than that of the domain.

Proposition 4.3.3 *Let $f: A\longrightarrow \mathbb{M}$ be a Lipschitz map and $A\subset\mathbb{G}$ a measurable set. If the differential of f is non-injective at $\mathcal{H}^Q_d$-a.e. point of A, then $\mathcal{H}^Q_\rho(f(A))=0$.*

Proof. Clearly it is not restrictive to assume that A contains only the points where f is differentiable and the differential is singular. So, let consider a point $x\in A$ where $d_Hf(x)$ is not injective and let $M_x = d_Hf(x)(\mathbb{G})$ be the corresponding subgroup of $\mathcal{M}$. From Proposition 4.2.2 it follows in particular that M_x is an Ahlfors regular space of dimension Q_x. The singularity of $d_Hf(x)$ implies $Q_x\le Q-1$. Denote with C_x the constant of Lemma 4.3.2 applied to $X=M_x$ and define the family of sets

$$E_j = \{x\in A \mid C_x\le j\,\}\cap B_j \qquad \text{with } j\in\mathbb{N}$$

Consider $x\in E_j$ and $\varepsilon>0$; denote with $I^\rho_r(E)$ the open set of points with distance from E less than r in the metric ρ. By differentiability we obtain

$$f(B_{x,r}) \subset f(x) I^\rho_{\varepsilon r}(d_Hf(x)(B_r)) \tag{4.18}$$

for any $r \le r_{x,\varepsilon}$. Observe that $d_H f(x)(B_r) \subset B^\rho_{cr} \cap M_x$, where $c = 2\mathrm{Lip}(f)$, then using Lemma 4.3.2 we find $N \le C_x \varepsilon^{-Q_x}$ balls $B^l_\varepsilon \subset M_x$ of radius $c\varepsilon r$ which cover $B^\rho_{cr} \cap M_x$. Defining $\overline{c}_Q = \mathcal{H}^Q_\rho(B^\rho_1)$ we see that the inclusion

$$I^\rho_{\varepsilon r}(B^\rho_{cr} \cap M_x) \subset \bigcup_{l=1}^{N} I^\rho_{\varepsilon r}(B^l_\varepsilon)$$

implies

$$\mathcal{H}^Q_{\rho,\infty}\left(I^\rho_{\varepsilon r}(B^\rho_{cr} \cap M_x)\right) \le j\varepsilon^{-Q_x}\overline{c}_Q(c+1)^Q(\varepsilon r)^Q \le j\varepsilon\overline{c}_Q(c+1)^Q r^Q = j\varepsilon C_Q \mathcal{H}^Q_d(B_r)$$

then for any $r \le r_{x,\varepsilon}$ and $x \in E_j$ it follows

$$\mathcal{H}^Q_\rho(f(B_{x,r})) \le j\varepsilon C_Q \mathcal{H}^Q_d(B_r). \tag{4.19}$$

Now we fix $j \in \mathbb{N}$ and consider the covering $\{B_{x,r} \mid x \in E_j$ and (4.18) holds for some $r \le r_{x,\varepsilon}/5 \le 1\}$. By a Vitali procedure we can extract a disjoint family of balls B_{x_l,r_l} contained in $I^d_1(E_j)$ and such that $E_j \subset \bigcup_{l=1}^\infty B_{x_l,5r_l}$ (see [45]). The estimate (4.19) proves

$$\mathcal{H}^Q_\rho(f(E_j)) \le j\varepsilon C_Q \mathcal{H}^Q_d(I^d_1(E_j))$$

The free choice and the independence of ε and j lead us to the conclusion. □

Now we prove the area formula as a corollary of the general formulation we have given in metric spaces (Theorem 4.1.7).

Theorem 4.3.4 (Area formula) *Let $A \subset \mathbb{G}$ be a measurable set and $f : A \longrightarrow \mathbb{M}$ be a Lipschitz map. Then the following formula holds*

$$\int_A J_Q(d_H f(x))\, d\mathcal{H}^Q_d(x) = \int_{\mathbb{M}} N(f,A,y)\, d\mathcal{H}^Q_\rho(y)\,. \tag{4.20}$$

Proof. According to Remark 4.1.1, condition (A2) is trivially satisfied and the measure $\mathcal{H}^Q_d$ is finite on bounded sets. By Proposition 4.3.1 referred to some $\overline{\lambda} > 1$ we obtain a decomposition of the domain $\mathcal{F} \cup \{E_0\}$, where E_0 is the set of points where the differential is not injective and for any $T \in \mathcal{F}$ the restriction $f_{|T}$ is injective. Moreover, in view of estimate (4.19) in Proposition 4.3.3 we have $J_f(x) = 0$ for $\mathcal{H}^Q_d$-a.e. in $x \in E_0$. Then hypothesis of Theorem 4.1.7 are satisfied and the metric area formula follows. In order to achieve (4.20), it remains to prove that metric jacobian and H-jacobian coincide $\mathcal{H}^Q_d$-a.e.

To do this, first of all we can assume that up to a negligible set our map is H-differentiable everywhere, due to the fact that negligible sets are mapped into negligible sets. So we can decompose the domain A by the covering $\mathcal{F}_\lambda \cup \{E_0\}$, where $\mathcal{F}_\lambda = \{T_\lambda\}$ is the covering $\mathcal{F}$ in Proposition 4.3.1 referred to $\lambda > 1$. Define $A_1 = A \setminus E_0$ and take a sequence $\lambda_n \to 1^+$. Notice that

$$\mathcal{H}_d^Q\Big(\bigcup_{n\in\mathbb{N}}\Big(A_1 \setminus \bigcup_{T_{\lambda_n}\in\mathcal{F}_{\lambda_n}} \mathcal{I}(T_{\lambda_n})\Big)\Big) = 0\,,$$

it follows that for a.e. $x \in A_1$ there exists a sequence of sets $\{T_{\lambda_n}(x)\}_{n\in\mathbb{N}}$ such that $x \in \mathcal{I}(T_{\lambda_n}(x))$. By Proposition 4.3.1 there exist H-linear maps $\varphi_n : \mathbb{G} \longrightarrow \mathbb{M}$ such that conditions (4.14) and (4.15) with φ and T replaced by φ_n and $T_{\lambda_n}(x)$ hold, respectively. For ease of notation we write $T_{\lambda_n} = T_{\lambda_n}(x)$. Then we get

$$\lambda_n^{-Q} J_Q(\varphi_n) = \lim_{r\to 0^+} \lambda_n^{-Q}\frac{\mathcal{H}_\rho^Q(\varphi_n(D_{x,r}))}{\mathcal{H}_d^Q(D_{x,r})} \tag{4.21}$$

$$= \lim_{r\to 0^+} \lambda_n^{-Q}\frac{\mathcal{H}_\rho^Q(\varphi_n(D_{x,r}\cap T_{\lambda_n}))}{\mathcal{H}_d^Q(D_{x,r})} \le \limsup_{r\to 0^+}\frac{\mathcal{H}_\rho^Q(f(D_{x,r}\cap T_{\lambda_n}))}{\mathcal{H}_d^Q(D_{x,r})} \tag{4.22}$$

$$\le \limsup_{r\to 0^+}\frac{\mathcal{H}_\rho^Q(f(D_{x,r}\cap A))}{\mathcal{H}_d^Q(D_{x,r}\cap A)} = J_f(x) \tag{4.23}$$

$$\le \limsup_{r\to 0^+}\frac{\mathcal{H}_\rho^Q(f(D_{x,r}\cap T_{\lambda_n}))}{\mathcal{H}_d^Q(D_{x,r})} \le \lim_{r\to 0^+} \lambda_n^{Q}\frac{\mathcal{H}_\rho^Q(\varphi_n(D_{x,r}\cap T_{\lambda_n}))}{\mathcal{H}_d^Q(D_{x,r})} \tag{4.24}$$

$$= \lambda_n^Q J_Q(\varphi_n)\,. \tag{4.25}$$

The first equality of (4.22) follows observing that $x \in \mathcal{I}(T_{\lambda_n}(x))$ and

$$\mathcal{H}_\rho^Q(\varphi_n(D_{x,r}\cap T_{\lambda_n})) = J_Q(\varphi_n)\mathcal{H}_d^Q(D_{x,r}\cap T_{\lambda_n})\,.$$

The inequality of (4.22) follows by (4.15) replacing φ and λ by φ_n and λ_n, respectively. From the fact that $x \in \mathcal{I}(A)$ and $T_{\lambda_n} \subset A$ we deduce the first inequality of (4.23). Observing that

$$\begin{aligned}\mathcal{H}_\rho^Q(f(D_{x,r}\cap A)) &\le \mathcal{H}_\rho^Q(f(D_{x,r}\setminus T_{\lambda_n})) + \mathcal{H}_\rho^Q(f(D_{x,r}\cap T_{\lambda_n}))\\ &\le \mathrm{Lip}(f)^Q\,\mathcal{H}_d^Q(D_{x,r}\setminus T_{\lambda_n}) + \mathcal{H}_\rho^Q(f(D_{x,r}\cap T_{\lambda_n}))\end{aligned}$$

and using the fact that $x \in \mathcal{I}(T_{\lambda_n}) \subset \mathcal{I}(A)$ the first inequality of (4.24) follows. We can deduce the second inequality of (4.24) from the analogous argument used for the inequality (4.22). By (4.14) applied to the sequence (φ_n) we get a subsequence $(\varphi_{\alpha(n)})$ uniformly converging to an H-linear map $\varphi : \mathbb{G} \longrightarrow \mathbb{M}$ such that $\rho(\varphi(z)) = \rho(d_H f(x)(z))$ whenever $z \in \mathbb{G}$ and $J_Q(d_H f(x)) = J_Q(\varphi)$, therefore the convergence of $J_Q(\varphi_n)$ to $J_Q(\varphi)$ yields $J_f(x) = J_Q(d_H f(x))$. Thus, we have proved that $J_f(x) = J_Q(d_H f(x))$ for a.e. $x \in A$ and our claim follows. □

Remark 4.3.5 Theorem 4.1.10 and Theorem 4.3.4 confirm the general fact according to which whenever a differentiability type theorem holds, it is possible to get a partition $\{E_i\}$ with the properties required in Theorem 4.1.7. We can say that Theorem 4.1.7 moves the difficulty in the proof of Area formula into the difficulty of obtaining the existence of the covering $\{E_j\}$ required in the same theorem. When the metric space has a differentiable structure, as for sub-Riemannian groups, there are natural notions of jacobian, as we have seen. In this case one has also to check that the notion of metric jacobian coincides with the one given by the differentiable structure. This is done in the proof of Theorem 4.3.4.

Now, for the sake of completeness we present the proof of Theorem 4.3.4, following path close to the classical one adopted in Euclidean spaces, see [124].

Proof of Theorem 4.3.4 We start observing that (4.20) holds when A is negligible, because Lipschitz map have the Lusin property, i.e. it maps negligible sets into negligible sets. Thus, in view of Theorem 3.4.11, we can exclude from the beginning the null subset of A where the function is not differentiable, assuming the differentiability at any point of A. We define the set $A' = \{x \in A \mid d_H f(x) \text{ is injective}\}$ and $Z = A \setminus A'$. The set additivity of $N(f, \cdot, y)$ gives

$$\int_{\mathbb{P}} N(f, A', y)\, d\mathcal{H}^Q_\rho(y) + \int_{\mathbb{P}} N(f, Z, y)\, d\mathcal{H}^Q_\rho(y) = \int_{\mathbb{P}} N(f, A, y)\, d\mathcal{H}^Q_\rho(y)\,,$$

so the proof is achieved if we show the following equalities

$$\int_{\mathbb{P}} N(f, A', y)\, d\mathcal{H}^Q_\rho(y) = \int_A J_Q(d_H f(x))\, d\mathcal{H}^Q_d(x)\,, \tag{4.26}$$

$$\int_{\mathbb{P}} N(f, Z, y)\, d\mathcal{H}^Q_\rho(y) = 0\,. \tag{4.27}$$

We start from (4.26), applying Proposition 4.3.1 we get a measurable countable partition $\mathcal{F}$ of A' where we have an approximation of f controlled by a parameter $\lambda > 1$. Consider an element $T \in \mathcal{F}$ contained in some $S(\varphi, k)$; the equation (4.14) implies

$$\lambda^{-Q}\mathcal{H}^Q_\rho(\varphi(T)) \le \mathcal{H}^Q_\rho((d_H f(x)\circ\varphi^{-1}\circ\varphi)(T)) \le \lambda^Q \mathcal{H}^Q_\rho(\varphi(T)) \quad \text{for any } x \in T$$

By definition of H-jacobian, taking the average on T of the above inequality we find

$$\lambda^{-Q}\mathcal{H}^Q_\rho(\varphi(T)) \le \int_T J_Q(d_H f(x))\, d\mathcal{H}^Q_d(x) \le \lambda^Q \mathcal{H}^Q_\rho(\varphi(T))$$

using (4.15)

$$\lambda^{-2Q}\mathcal{H}^Q_\rho(f(T)) \le \int_T J_Q(d_H f(x)) d\mathcal{H}^Q_d(x) \le \lambda^{2Q}\mathcal{H}^Q_\rho(f(T)). \tag{4.28}$$

The map f is injective on T, so adding (4.28) on all these sets it follows

$$\lambda^{-2Q}\int_{\mathbb{P}} N(f,A',y)d\mathcal{H}^Q_\rho(y) \le \int_{A'} J_Q(d_Hf(x))\,d\mathcal{H}^Q_d(x) \le \lambda^{2Q}\int_{\mathbb{P}} N(f,A',y)d\mathcal{H}^Q_\rho(y).$$

Letting $\lambda \to 1^+$ we have (4.26). The equation (4.27) follows directly from Proposition 4.3.3. □

Corollary 4.3.6 *Given a Lipschitz map $f : A \subset \mathbb{G} \longrightarrow \mathbb{P}$ and a summable function $u : A \subset \mathbb{G} \longrightarrow \mathbb{R}$ we have*

$$\int_A u(x)\,J_Q(d_Hf(x))\;d\mathcal{H}^Q_d(x) = \int_{\mathbb{P}} \sum_{x\in f^{-1}(y)} u(x)\;\mathcal{H}^Q_\rho(y)\,.$$

PROOF. We use the standard argument of approximating u with finite linear combinations of characteristic functions, see for example [53]. □

Example 4.3.7 We consider the Heisenberg group $\mathbb{H}^5$, with horizontal vector fields $X_i = \partial_{x^i} - \frac{y^i}{2}\partial_z$ and $Y_i = \partial_{y^i} + \frac{x^i}{2}\partial_z$, for $i = 1, 2$. We have $[X_i, Y_i] = Z = \partial_z$ for $i = 1, 2$, getting a basis of $\mathbb{R}^5$, which can be identified with the Lie algebra of $\mathbb{H}^5$. Thus, an element of $\mathbb{H}^5$ can be written as $\exp\left(\sum_{i=1}^2 (x^i\,X_i + y^i\,Y_i) + z\,Z\right)$, where $\exp : \mathbb{R}^5 \longrightarrow \mathbb{H}^5$. Then, we represent an element of $\mathbb{H}^5$ as $(x, y, z) \in \mathbb{R}^5$, with $x = (x^1, x^2)$ and $y = (y^1, y^2)$. The BCH formula (2.18) gives the explicit group operation (denoted with $\circledcirc$) in our coordinates

$$(x,y,z)\circledcirc(\xi,\eta,\zeta) = \left(x+\xi,\, y+\eta,\, z+\zeta+\frac{(x^1\eta^1 + x^2\eta^2 - y^1\xi^1 - y^2\xi^2)}{2}\right).$$

The restriction of the operation to the subset $\mathbb{G} = \{(x, y, z) \in \mathbb{H}^5 \mid x^2 = 0\}$ gives

$$(x^1,y,z)\circledcirc(\xi^1,\eta,\zeta) = \left(x^1+\xi^1,\, y+\eta,\, z+\zeta+\frac{(x^1\eta^1 - y^1\xi^1)}{2}\right),$$

so $\mathbb{G}$ is a subgroup of $\mathbb{H}^5$. Moreover $\mathbb{G}$ is a stratified group. In fact, the horizontal space $V_1 = \mathrm{span}(X_1, Y_1, \partial_{y^2})$ is left invariant under the translations of the subgroup and $[X_1, Y_1] = Z$, so the generating condition is achieved with $V_2 = \mathrm{span}(Z)$.

Consider an injective Lipschitz map $f : A \subset \mathbb{G} \longrightarrow \mathbb{H}^5$ and fix $S = f(A)$. The set $S \subset \mathbb{H}^5$ can be seen as a hypersurface of $\mathbb{H}^5$ with Hausdorff dimension 5 ($\mathbb{H}^5$ has Hausdorff dimension 6). In view of the differentiability (Theorem 3.4.11), there exists a tangent hyperplane to S in $\mathcal{H}^5_d$-a.e. $y \in S$, $T_y(S) = d_Hf(x)(\mathbb{G})$, with $y = f(x)$ and the Area formula gives

$$\mathcal{H}^5_d(S) = \int_A J_5(d_Hf(x))\,d\mathcal{H}^5_d(x)\,.$$

Remark 4.3.8 It is worth to observe that even if either the Hausdorff dimension or the topological dimension of $\mathbb{G}$ is less than the Hausdorff dimension of the target $\mathbb{M}$, it may happen that there does not exist a Lipschitz map $f : \mathbb{G} \longrightarrow \mathbb{M}$ with injective differential at some differentiability point. In fact, recalling that $\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ and $\mathcal{M} = W_1 \oplus W_2 \oplus \cdots \oplus W_m$, it suffices that the geometric constraint $\dim(V_{j_0}) > \dim(W_{j_0})$ holds for some $j_0 \leq \min\{m, n\}$, so the contact property of any H-linear maps $L : \mathbb{G} \longrightarrow \mathbb{M}$ implies the inclusion $L(V_{j_0}) \subset W_{j_0}$, therefore L cannot be injective. In this case the area formula is a straightforward consequence only of Proposition 4.3.3. This remark points out the typical rigidity of stratified geometry. In other words the conditions we assumed on the stratification prevent any Lipschitz embedding of $\mathbb{G}$ into $\mathbb{M}$.

4.4 Unrectifiable metric spaces and rigidity

In this section we apply the area formula to characterize purely k-unrectifiable sub-Riemannian groups. We will also prove that bilipschitz equivalent sub-Riemannian groups are isomorphic. This shows how the algebraic structure of the group affects its metric structure, and viceversa.

Definition 4.4.1 We say that a metric space (X, d) is *purely k-unrectifiable* if for any Lipschitz map $f : A \longrightarrow X$ with $A \subset \mathbb{R}^k$, we have $\mathcal{H}^k_d(f(A)) = 0$.

Our target metric space is a fixed sub-Riemannian group $(\mathbb{M}, d)$. Let us consider a Lipschitz map $f : A \longrightarrow \mathbb{M}$, where A is a subset of $\mathbb{R}^k$. The Lipschitz condition on f and the completeness of $\mathbb{M}$ allow us to assume that A is a closed set.

The area formula (4.20) easily gives

$$\mathcal{H}^Q_\rho\left(f(A)\right) \leq \int_A J_Q\left(d_H f(x)\right)\, d\mathcal{H}^Q_d(x)\,. \tag{4.29}$$

Therefore, if we prove that under suitable algebraic conditions on $\mathbb{M}$ any H-linear map $L : \mathbb{R}^k \longrightarrow \mathbb{M}$ has nontrivial kernel, then $J_Q(L) = 0$ and the estimate (4.29) implies that $\mathbb{M}$ is purely k-unrectifiable.

We fix the grading $\mathcal{M} = W_1 \oplus W_2 \oplus \cdots \oplus W_\iota$ for the group $\mathbb{M}$. Notice that the Euclidean space $\mathbb{R}^k$ can be seen as an abelian sub-Riemannian group with the easiest grading $\mathbb{R}^k = V_1$. Now, let us consider an H-linear map $L : \mathbb{R}^k \longrightarrow \mathcal{M}$ read in the Lie algebras. In view of Theorem 3.1.12 it follows that $L(\mathbb{R}^k) \subset W_1$, so if W_1 does not contain k-dimensional subalgebras of $\mathcal{M}$ then L cannot be injective. We have proved the following theorem.

Proposition 4.4.2 *Let $\mathbb{M}$ be a sub-Riemannian group with $\mathcal{M} = W_1 \oplus W_2 \oplus \cdots \oplus W_\iota$ and suppose that there do not exist k-dimensional Lie subalgebras contained in W_1. Then $\mathbb{M}$ is purely k-unrectifiable.*

For instance whenever $\dim(W_1) < k$ hypothesis of Proposition 4.4.2 is fulfilled. Let us read the unrectifiability result of [7] with this criterion. We consider the Heisenberg group $\mathbb{H}^3$ where the grading is $\mathfrak{h}_3 = V_1 \oplus V_2$, with $\dim(V_1) = 2$ and $\dim(V_2) = 1$. From Proposition 4.4.2 it follows that $\mathbb{H}^3$ is purely unrectifiable for any $k > 2$. Now, observing that V_1 is not a subalgebra of $\mathfrak{h}_3$, because $[X, Y] \notin V_1$ whenever X, Y are linearly independent vectors of V_1, we obtain that $\mathbb{H}^3$ is also purely 2-unrectifiable.

Let us give the following converse of Proposition 4.4.2.

Proposition 4.4.3 *In the assumptions of Proposition 4.4.2, if there exists a k-dimensional subalgebra $\mathcal{S}$ of W_1, then $\mathbb{M}$ is not purely k-unrectifiable.*

Proof. We recall that any subalgebra $\mathcal{S}$ induces a subgroup $\mathbb{S}$ of $\mathbb{M}$, whose Lie algebra is exactly $\mathcal{S}$. This is easily seen defining $\exp \mathcal{S} = \mathbb{S}$ and using the BCH formula (2.18), see also Theorem 2.5.2 of [178]. Moreover, the condition $\mathcal{S} \subset W_1$ implies $[\mathcal{S}, \mathcal{S}] = 0$, so $\mathbb{S}$ is an abelian subgroup $\mathbb{M}$, then it can be identified with $\mathbb{R}^k$ and the identification $i : \mathbb{R}^k \hookrightarrow \mathbb{S} \subset \mathbb{M}$ is an injective H-linear map. Thus, the area formula (4.20) yields

$$\mathcal{H}^k_\rho\,(i(A)) = J_k(L)\mathcal{H}^k_{|\cdot|}(A) > 0\,,$$

whenever $\mathcal{H}^k_{|\cdot|}(A) > 0$, where $\mathcal{H}^k_{|\cdot|}$ indicates the k-dimensional Hausdorff measure in $\mathbb{R}^k$ with respect to the Euclidean norm. □

Joining Propositions 4.4.2 and 4.4.3 we get the following characterization.

Theorem 4.4.4 *Let $\mathbb{M}$ be a sub-Riemannian group with $\mathcal{M} = W_1 \oplus W_2 \oplus \cdots \oplus W_\iota$. Then $\mathbb{M}$ is purely k-unrectifiable if and only if there do not exist k-dimensional Lie subalgebras contained in W_1.*

Notions of rectifiability and pure unrectifiability according to 3.2.14 of [55] can naturally be extended to the sub-Riemannian setting replacing the Euclidean space $\mathbb{R}^k$ with some sub-Riemannian group. This approach is followed in [156]. With these notions Theorem 4.4.4 could be analogously extended replacing $\mathbb{R}^k$ with another sub-Riemannian group as a model space.

Definition 4.4.5 We say that two sub-Riemannian groups are *isomorphic* if there exists an invertible H-linear map between them.

The next application is a "rigidity result" for sub-Riemannian groups.

Theorem 4.4.6 *Let $\mathbb{G}$ and $\mathbb{M}$ be two nonisomorphic sub-Riemannian groups and let $A \subset \mathbb{G}$ and $B \subset \mathbb{M}$ be two subsets with positive measures with respect to the Haar measure of the groups. Then there does not exist a bilipschitz map $f : A \longrightarrow B$.*

Proof. By contradiction, we suppose that there exists a bilipschitz map $f : A \longrightarrow B$, where $A \subset \mathbb{G}$ and $B \subset \mathbb{M}$ are both subset with positive measure. We divide A into three disjoint subsets A_0, A_1 and A_2, where A_0 is the subset of points where f is not H-differentiable, A_1 is the subset of points where the H-differential of f is surjective and A_2 is the subset of points where the H-differential of f is not surjective. As a consequence of Theorem 3.4.11 we know that $\mathcal{H}^Q(A_0) = 0$. We fix $x \in A_1$ and $L = d_H f(x) : \mathbb{G} \longrightarrow \mathbb{M}$. In view of our assumption we know that L cannot be an isomorphism, hence it cannot be injective. By the expression of the gradings $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$ and $\mathcal{M} = W_1 \oplus \cdots \oplus W_\upsilon$ together with the surjectivity of L we can establish the condition $\iota \geq \upsilon$. Moreover, the contact property of Theorem 3.1.12 yields $L : \mathbb{V}_i \longrightarrow \mathbb{W}_i$ for any $i = 1, \ldots, \upsilon$. The Hausdorff dimension of $\mathbb{G}$ is given by $Q = \sum_{j=1}^{\iota} j \dim(V_j)$, as it has been shown in Subsection 2.3.2, hence the equalities $L(V_i) = W_i$ for any $i = 1, \ldots, \upsilon$ imply

$$Q = \sum_{j=1}^{\iota} j \dim(V_j) > \sum_{j=1}^{\upsilon} j \dim(L(V_j)) = P ,$$

where P is the Hausdorff dimension of $\mathbb{M}$. As a consequence of Definition 4.2.1 we obtain $J_Q(L) = 0$, hence $J_Q(d_H f(x)) = 0$ for any $x \in A_1$ and the area formula (4.20) yields

$$\mathcal{H}^Q_\rho(B_1) = \int_A J_Q\left(d_H f(x)\right)\, d\mathcal{H}^Q_d(x) = 0 ,$$

where $B_1 = f(A_1)$. The bilipschitz property of f gives $\mathcal{H}^Q_d(A_1) = 0$. Now we define $g = f^{-1} : B_2 \longrightarrow A_2$, where $B_2 = f(A_2)$ and consider the subset $B_2' \subset B_2$ where g is H-differentiable. Theorem 3.4.11 implies that $\mathcal{H}^Q_d(B_2 \setminus B_2') = 0$, hence we have $\mathcal{H}^Q_d(A_2 \setminus A_2') = 0$, where we have defined $A_2' = g(B_2')$. By differentiating the map $\mathrm{id}_A = g \circ f : A_2' \longrightarrow A_2'$ and using Proposition 3.2.5 we obtain

$$\mathrm{id}_{\mathbb{G}} = d_H g(f(x)) \circ d_H f(x) , \tag{4.30}$$

for any $x \in A_2'$. The non surjectivity of $d_H f(x)$ and relation (4.30) imply that $d_H g(y)$ is non injective for any $y \in B_2'$. Then, reasoning as before we obtain $\mathcal{H}^Q(A_2') = 0$. As a consequence, we have proved that $\mathcal{H}^Q(A) = 0$, that contradicts our hypothesis, then the map f cannot exist. □

Remark 4.4.7 Note that $\mathbb{G}$ and $\mathbb{M}$ may have the same Hausdorff dimension even if they are not isomorphic. The statement of Theorem 4.4.6 can also be read as follows: let $A \subset \mathbb{G}$ and $B \subset \mathbb{M}$ be subsets with positive measure such that there exists a bilipschitz map $f : A \longrightarrow B$. Then $\mathbb{G}$ and $\mathbb{M}$ are isomorphic.

Chapter 5

Rotations in sub-Riemannian groups

In this chapter we introduce some novel concepts on sub-Riemannian groups first introduced in [126] and which are strictly related to the graded metric of the group. Through these concepts it will be apparent that not all graded metrics are really "suitable" for the geometry of the group. A key notion of the chapter is that of "horizontal isometry", e.g. an H-linear map that is also an isometry with respect to the graded metric (Definition 5.1.1). So, a good graded metric should yield a large group of horizontal isometries that, roughly speaking, amounts to a space with many symmetries. With the notion of "$\mathcal{R}$-rotational group" (Definition 5.1.4) we single out all sub-Riemannian groups that have enough symmetries. In fact, we will see in Chapter 6 and Chapter 7 that the generalized coarea formulae (6.42) and (7.19) take a particular simplified form in rotational groups with $\mathcal{R}$-invariant distances, see (6.45), (7.23) and Definition 5.1.10. We also point out that by Proposition 5.1.12 any class $\mathcal{R}$ of horizontal isometries admits a corresponding $\mathcal{R}$-invariant distance, that is the CC-distance with respect to the graded metric.

The previous notions were motivated by the question of finding a class of sub-Riemannian groups where the "metric factor" (Definition 5.2.2) is a geometrical constant independent from the direction to which is referred. The metric factor appears in the generalized coarea formulae (6.42) and (7.19), in the expression of the perimeter measure (6.31) and in the formula for the spherical Hausdorff measure of C^1 hypersurfaces (7.17). It amounts to the measure of the unit ball of codimension one in a sub-Riemannian group. For instance, in the n-dimensional Euclidean space it coincides with the measure ω_{n-1} of the $(n-1)$-dimensional Euclidean unit ball. Due to the anisotropy of a general homogeneous distance the metric factor may depend on the direction in which is calculated. In Proposition 5.2.5 we prove that $\mathcal{R}$-rotational groups admit an $\mathcal{R}$-invariant distance where this dependence does not occur.

Let us give a brief summary of the chapter. In Section 5.1 we introduce the definition of horizontal isometry, $\mathcal{R}$-rotational group and $\mathcal{R}$-invariant distance. In Proposition 5.1.8 we prove that there exists a graded metric on $\mathbb{H}^{2n+1}$ such that it is an $\mathcal{R}$-rotational group. In Remark 5.1.9 we point out that, using sophisticated results in the literature it is possible to show that all H-type groups are $\mathcal{R}$-rotational. In Proposition 5.1.12 we prove that for any given class $\mathcal{R}$ of horizontal isometries the associated CC-distance is $\mathcal{R}$-invariant.

In Section 5.2 we introduce the notion of metric factor, showing that in some examples it can be explicitly calculated. In Proposition 5.2.5 we prove that the metric factor of $\mathcal{R}$-rotational groups with respect to an $\mathcal{R}$-invariant distance is a dimensional constant only related to the graded metric of the group and to the homogeneous distance to which is referred.

5.1 Horizontal isometries and rotational groups

In this chapter we will assume that $\mathbb{G}$ is a graded group endowed with graded metric.

Definition 5.1.1 (Horizontal isometry) Let $T \in \mathrm{HL}(\mathbb{G}, \mathbb{G})$ be an H-linear map. We say that T is a *horizontal isometry* if the differential $dT(e) : \mathcal{G} \longrightarrow \mathcal{G}$ is an isometry.

Notice that any horizontal isometry is in particular an isometry of $\mathbb{G}$ in the classical sense of Riemannian Geometry.

Definition 5.1.2 Let $\mathbb{G}$ be a simply connected nilpotent Lie group. We mean by a *subspace* of $\mathbb{G}$ the image of a subspace of $\mathcal{G}$ under the exponential map.

By Theorem 2.3.10 there is a bijective correspondence between subspaces of $\mathbb{G}$ and the ones of $\mathcal{G}$. Note that in general subspaces of $\mathbb{G}$ are not subgroups.

Definition 5.1.3 We say that Π is a *vertical hyperplane of* $\mathcal{G}$ if it is the orthogonal space of some horizontal vector. A *vertical hyperplane* $\mathcal{L}$ of $\mathbb{G}$ is the image of a vertical hyperplane of $\mathcal{G}$ under the exponential map.

Definition 5.1.4 ($\mathcal{R}$-rotational group) We say that a sub-Riemannian group $\mathbb{G}$ is $\mathcal{R}$-*rotational*, if there exists a graded metric g and a class $\mathcal{R}$ of horizontal isometries with respect to g such that for any couple of vertical hyperplanes $\mathcal{L}$ and $\mathcal{L}'$ of $\mathbb{G}$ we have some $T \in \mathcal{R}$ such that $T(\mathcal{L}) = \mathcal{L}'$. We will simply say *rotational group*, when the class $\mathcal{R}$ is understood.

Remark 5.1.5 Notice that in the above definition we could have required equivalently that for any couple of vertical hyperplanes Π and Π' of $\mathcal{G}$ there exists $T \in \mathcal{R}$ such that $dT(e)\Pi = \Pi'$.

Example 5.1.6 The Euclidean space $\mathbb{E}^n$ endowed with the canonical Riemannian metric is a rotational group. In fact, any hyperplane is vertical, then we can choose $\mathcal{R}$ as the class of all Euclidean isometries of $\mathbb{E}^n$. It follows that Euclidean spaces are $\mathcal{R}$-rotational.

We point out that the existence of horizontal isometries in a graded group is a rather delicate question. In Example 5.1.7 we show that horizontal isometries cannot always be obtained by isometries of $\mathcal{G}$. In other words, if we consider an isometry $I : \mathcal{G} \longrightarrow \mathcal{G}$ it may happen that there not exist an H-linear map $T : \mathbb{G} \longrightarrow \mathbb{G}$ such that $dT(e) = I$. This fact strongly depends on the compatibility of the left invariant Riemannian metric with the algebraic structure of the group.

Example 5.1.7 We consider the Heisenberg algebra $\mathfrak{h}_3$ and $\tilde{T} : \mathfrak{h}_3 \longrightarrow \mathfrak{h}_3$. We take the following matrix representation of $\tilde{T}$

$$[\tilde{T}] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.1}$$

with respect to a basis (X, Y, Z) of $\mathfrak{h}_3$ with $[X, Y] = Z$. A left invariant metric that makes (X, Y, Z) orthonormal is a graded metric (see Definition 2.3.30). Now we define $\exp \circ T \circ \exp^{-1} : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$, observing that $dT(e) = \tilde{T}$. Then $dT(e)$ is an isometry, but from Example 3.1.14 the matrix representation of T contradicts the H-linearity.

In the same notation of the previous example we can show easily an example of H-linear map that cannot be a horizontal isometry. It suffices to consider the following matrix representation

$$[\tilde{L}] = \begin{pmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \tag{5.2}$$

with $|\alpha| \notin \{0, 1\}$. Clearly we have $|Z| \neq \alpha^2 |Z| = |T(Z)|$ for any graded metric g, where $|W| = \sqrt{g(W, W)}$, therefore T is not an isometry.

However, in the following proposition we will show that Heisenberg groups are important examples of rotational sub-Riemannian groups.

Proposition 5.1.8 (Rotational Heisenberg group) *There exist a graded metric and a class $\mathcal{R}$ of horizontal isometries that make $\mathbb{H}^{2n+1}$ an $\mathcal{R}$-rotational group.*

PROOF. We will refer to the basis $(X_1, \dots, X_n, Y_1, \dots Y_n, Z)$ of Remark 2.3.27, where $\mathbb{H}^{2n+1}$ can be thought of as $\mathbb{C}^n \times \mathbb{R}$, where the group operation in exponential coordinates (Definition 2.3.13) is given as follows

$$(z, s) \cdot (w, t) = \Big(z + w, s + t + 2\mathrm{Im}\langle z, w \rangle \Big) . \tag{5.3}$$

We will find a class of horizontal isometries represented in this system of exponential coordinates. Notice that our basis is adapted to the grading of $\mathcal{G}$, so if we choose a metric such that the basis is orthonormal, then the metric is graded and our coordinates are indeed graded coordinates (Definition 2.3.43). In the sequel we will refer to this graded metric. We consider a unitary operator $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and define the map

$$\tilde{T} : \mathbb{C}^n \times \mathbb{R} \longrightarrow \mathbb{C}^n \times \mathbb{R}\,, \quad (z,s) \longrightarrow (U(z),s)\ .$$

Directly from the definition of $\tilde{T}$ it is clear that it is 1-homogeneous with respect to the group of dilations. Now we check that $\tilde{T}$ is a group homomorphism. By the fact that U preserves the Hermitian product we have

$$\begin{aligned}
&\tilde{T}\Big((z,s)\cdot(w,t)\Big) = \tilde{T}\Big(z+w, s+t+2\mathrm{Im}\langle z,w\rangle\Big) \\
&= \Big(U(z)+U(w), s+t+2\mathrm{Im}\langle z,w\rangle\Big) = \Big(U(z)+U(w), s+t+2\mathrm{Im}\langle U(z),U(w)\rangle\Big) \\
&= (U(z),s)\cdot(U(w),t) = \tilde{T}(z,s)\cdot\tilde{T}(w,t)\,.
\end{aligned}$$

Denoting by $F : \mathbb{R}^{2n+1} \longrightarrow \mathbb{H}^{2n+1}$ the system of graded coordinates defined by

$$F(\xi) = \exp\Big[\Big(\sum_{j=1}^{n} \xi_j\, X_j + \xi_{n+j}\, Y_j\Big) + \xi_{2n+1} Z\Big]$$

we define $T = F\circ\tilde{T}\circ F^{-1} : \mathbb{H}^{2n+1} \longrightarrow \mathbb{H}^{2n+1}$. We can check immediately that $dT(e)$ is represented by $\tilde{T}$ with respect to our orthonormal basis, then it is an isometry, due to the fact that $\tilde{T}$ is an Euclidean isometry on $\mathbb{R}^{2n+1}$ with respect to the standard real scalar product. We have proved that T is a horizontal isometry.

It remains to prove that this class of horizontal isometries is sufficiently large to give the rotational property of Definition 5.1.4. Vertical hyperplanes in $\mathbb{H}^{2n+1}$ can be characterized in our coordinates as products $\Pi \times \mathbb{R}$, where Π is a real $2n-1$ dimensional space of $\mathbb{C}^n$. We consider hyperplanes Π and Π' of $\mathbb{C}^n$ and observe that they can be characterized by two unit vectors of $\mathbb{C}^n$. Then there exists a unitary transformation $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $U(\Pi) = \Pi'$, so

$$\tilde{T}(\Pi\times\mathbb{R}) = \Pi'\times\mathbb{R}$$

where and $\tilde{T}$ is defined as above. Since the hyperplane Π and Π' are arbitrary, defining $\mathcal{L} = F(\Pi\times\mathbb{R})$, $\mathcal{L}' = F(\Pi'\times\mathbb{R})$ we get $T(\mathcal{L}) = \mathcal{L}'$, where $T = F\circ\tilde{T}\circ F^{-1}$ is a horizontal isometry. □

Remark 5.1.9 (Rotational H-type group) The result of Proposition 5.1.8 can be achieved also in general groups of Heisenberg type. These are 2-step groups endowed with a scalar product $\langle\,,\,\rangle$ and a linear map $J : V_2 \longrightarrow \mathrm{End}(V_1)$ with the following properties

1. $\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle$ for any $X, Y \in V_1$ and $Z \in V_2$
2. $J_Z^2 = -|Z|^2 I$,

see [104], [105], [106] for more information. Let us consider the group

$$G = \left\{ (\phi,\psi) \in O(V_2) \times O(V_1) \mid J_{\phi(v)}(\psi(x)) = \psi(J_v(x)) \right\},$$

where $O(V_1)$ and $O(V_2)$ denote the group of isometries in V_1 and V_2, respectively. In Proposition 5 of [162], C. Riehm proves that the maps of (ϕ,ψ) of G are homomorphisms, hence G corresponds to a group of horizontal isometries according to our definition. Furthermore, denoting by G_{V_1} the projection of G in $O(V_1)$, in [161] there is a precise characterization of H-type groups where G_{V_1} is transitive on the sphere $V_1^* = \{v \in V_1 \mid |v| = 1\}$. In view of Definition 5.1.4, groups with this transitive property on V_1^* are $\mathcal{R}$-rotational with $\mathcal{R} = G$.

Definition 5.1.10 ($\mathcal{R}$-invariant distance) Let $\mathcal{R}$ be a set of horizontal isometries and let B_1 be the open unit ball with respect to a fixed homogeneous distance d. We say that d is *$\mathcal{R}$-invariant* if for any $T \in \mathcal{R}$ we have $T(B_1) = B_1$.

Example 5.1.11 Let us consider the homogeneous distance d_∞ of $\mathbb{H}^{2n+1}$ introduced in Example 2.3.38. We recall that this distance was defined by means of graded coordinates associated to the basis (2.23) with $\alpha = -4$. In Proposition 5.1.8 we have seen that horizontal isometries with respect to these coordinates can be represented as $T(z,t) = (U(z), t)$, where U is a unitary operator. Here the graded metric is the one which makes the basis (2.23) orthonormal. Thus, by definition of d_∞ we have

$$d_\infty\left(F\left(T(z,t)\right)\right) = d_\infty\left(F\left(z,t\right)\right), \tag{5.4}$$

where F is the transformation relative to the graded coordinates. The formula (5.4) yields the $\mathcal{R}$-invariance of d_∞.

In the following proposition we show that whenever we have a class $\mathcal{R}$ of horizontal isometries we can always define an $\mathcal{R}$-invariant distance.

Proposition 5.1.12 *Let g be the graded metric of a sub-Riemannian group $\mathbb{G}$ and let ρ be the CC-distance of $\mathbb{G}$ with respect to g. We consider the class $\mathcal{R}$ of all horizontal isometries with respect to g. Then ρ is $\mathcal{R}$-invariant.*

Proof. It suffices to notice that horizontal isometries bring horizontal curves into horizontal curves and preserve their length. Then any $T \in \mathcal{R}$ is an isometry of $\mathbb{G}$ with respect to the CC-distance. In particular, the $\mathcal{R}$-invariance of ρ follows. □

5.2 Metric factor

Lemma 5.2.1 *Let $\mathcal{L}$ be a hyperplane of $\mathbb{G}$ and let B_1 be the open unit ball with respect to a homogeneous distance d. Then, for any couple of graded coordinates (F_1, W) and (F_2, V) we have*

$$\mathcal{H}^{q-1}_{|\cdot|}\left(F_1^{-1}(\mathcal{L}\cap B_1)\right) = \mathcal{H}^{q-1}_{|\cdot|}\left(F_2^{-1}(\mathcal{L}\cap B_1)\right) .$$

Proof. In view of Definition 2.3.43 we have

$$F_1(x) = \exp\Big(\sum_{j=1}^{q} x^j W_j\Big), \quad F_2(y) = \exp\Big(\sum_{j=1}^{q} y^j V_j\Big),$$

where (W_j) and (V_j) are adapted orthonormal bases of $\mathcal{G}$. Then we can write $F_1 = F_2 \circ I^{-1}$, where I is an isometry of $\mathbb{R}^q$. It follows

$$F_1^{-1}(\mathcal{L}\cap B_1) = I \circ F_2^{-1}(\mathcal{L}\cap B_1)$$

that yields our claim. $\Box$

Definition 5.2.2 (Metric factor) Consider a vector $\nu \in \mathcal{G}\setminus\{0\}$ and its orthogonal hyperplane $\mathcal{L}$ in $\mathbb{G}$. We fix a system of graded coordinates (F, W) and define

$$\theta^g_{Q-1}(\nu) = \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}\cap B_1)\right) . \tag{5.5}$$

We call $\theta^g_{Q-1}(\nu)$ the *metric factor* of the homogeneous distance d with respect to the direction ν.

Remark 5.2.3 In view of the Lemma 5.2.1, the above definition does not depend on the choice of graded coordinates. So the number $\theta^g_{Q-1}(\nu)$ depends only on the homogeneous distance d, the direction of ν and the graded metric. We can also easily observe that the function $\nu \to \theta^g_{Q-1}(\nu)$ is uniformly bounded from above and below by positive constants.

In order to emphasize the dependence of the metric factor on the direction ν, we present a simple example where ν affects the metric factor.

Example 5.2.4 Let us consider the Euclidean space $\mathbb{E}^2$, with homogeneous distance $\eta(x) = \max\{|x_1|, |x_2|\}$, where (x_1, x_2) are Euclidean coordinates. We observe that $\mathbb{E}^2$ is an abelian 2-dimensional stratified group, where the canonical Riemannian metric is trivially graded. We denote by $L(\alpha)$ the straight line which contains the origin and whose direction is $\alpha \in \mathbb{T}^1$, where $\mathbb{T}^1$ is the 1-dimensional torus. In this case, by definition of $\theta_1(\alpha)$, we have

$$\theta_1(\alpha) = \mathcal{H}^1_{|\cdot|}\Big(L(\alpha + \frac{\pi}{2}) \cap \{x \in E^2 \mid \max\{|x_1|, |x_2|\} < 1\}\Big) ,$$

By a direct computation we have

$$\theta_1(\alpha) = \begin{cases} 2(\cos\alpha)^{-1} & -\frac{\pi}{4} \le \alpha \le \frac{\pi}{4} \\ 2(\sin\alpha)^{-1} & \frac{\pi}{4} \le \alpha \le \frac{3}{4}\pi \\ 2|\cos\alpha|^{-1} & -\frac{3}{4}\pi \le \alpha \le \frac{5}{4}\pi \\ 2|\sin\alpha|^{-1} & \frac{5}{4}\pi \le \alpha \le \frac{7}{4}\pi \end{cases} .$$

In the following proposition we show that an $\mathcal{R}$-invariant distance of an $\mathcal{R}$-rotational group has a constant metric factor.

Proposition 5.2.5 *Let $\mathbb{G}$ be an $\mathcal{R}$-rotational group and let d be an $\mathcal{R}$-invariant distance of $\mathbb{G}$. Then there exists $\alpha_{Q-1} > 0$ such that*

$$\theta^g_{Q-1}(\nu) = \alpha_{Q-1}$$

for any $\nu \in V_1 \setminus \{0\}$.

Proof. Let g be the graded metric that gives the rotational property of $\mathbb{G}$ and let $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ be the map associated to a system of graded coordinates (F, W) relatively to the metric g. Let ν and ν' be two horizontal directions of $\mathcal{G}$ with the corresponding vertical hyperplanes $\mathcal{L}$ and $\mathcal{L}'$ in $\mathbb{G}$. By definition of metric factor we have only to prove that

$$\mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}\cap B_1)\right) = \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}'\cap B_1)\right) . \tag{5.6}$$

In view of the rotational assumption on $\mathbb{G}$ there exists a horizontal isometry $T \in \mathcal{R}$ such that $T(\mathcal{L}) = \mathcal{L}'$. By virtue of the $\mathcal{R}$-invariance of d we have

$$F^{-1}(\mathcal{L}'\cap B_1) = F^{-1}\circ T\,(\mathcal{L}\cap B_1)\,,$$

then defining $I = F^{-1}\circ T\circ F$ and observing that I is an isometry of $\mathbb{R}^q$ equation (5.6) follows. $\Box$

Remark 5.2.6 The number α_{Q-1} in Proposition 5.2.5 amounts to the measure of the intersection between the unit ball and a vertical hyperplane, that is independent of the vertical section we consider. We can consider α_{Q-1} as a geometrical constant associated to the $\mathcal{R}$-invariant distance.

Example 5.2.7 Let us consider $\mathbb{E}^n$ with standard coordinates $x = (x_i)$ and the classical Euclidean norm $\eta(x) = |x| = \sqrt{\sum_{i=1}^n x_i^2}$. In this case we have

$$\theta_{n-1}(\nu(x)) = \mathcal{H}^{n-1}_{|\cdot|}\left(\Pi_x \cap \{y\in\mathbb{E}^n \mid |y|<1\}\right) = \mathcal{H}^{n-1}_{|\cdot|}\left(\{y\in\mathbb{E}^{n-1} \mid |y|<1\}\right) = \omega_{n-1}$$

Example 5.2.8 Let us consider the distance d of Example 5.1.11. By calculations of Lemma 4.5 (iii) in [71] we have that the corresponding metric factor is $\alpha_{Q-1} = 2\omega_{2n-1}$.

Chapter 6

Coarea type formulae

This chapter is devoted to the problem of coarea formula for Lipschitz maps between sub-Riemannian groups and to some related consequences. It is well known that this formula holds for Lipschitz maps of Euclidean spaces, see Theorem 3.2.11 of [55]. In the proof of the classical result the relevant aspect consists in the fact that any Euclidean space can be regarded as an isometric product of two orthogonal subspaces. Such a decomposition enters into the proof when one considers the tangent space of the level set and its orthogonal space. This fact in turn allows us to parametrize the level set by a Lipschitz map between the two subspaces and to apply the Euclidean area formula. Nonabelian sub-Riemannian groups in general do not possess such an isometric decomposition (see Proposition 2.3.28 and Remark 3.3.4) and our approach follows a genuinely different method. Here we emphasize a basic distinction between the coarea formula for real valued maps and for group valued maps.

In the first case, we have very general "variational" coarea formulae for functions of bounded variation of both CC-spaces and metric spaces, where the perimeter measure of upper level sets represents the surface measure of level sets, see [69], [79], [134], [141]. So it is natural to wonder whether one is allowed to replace the perimeter measure with a "suitable" Hausdorff measure in the case of Lipschitz maps, as it was raised in Remark 4.9 of [141]. We answer this question through the theory of sets of H-finite perimeter, obtaining the coarea formula (6.42) in all groups where a rectifiability theorem for the perimeter measure holds, namely generating groups (Definition 6.4.8). Due to results of [73], the class of generating groups encompasses all sub-Riemannian groups of step 2.

In the second case, we are able to prove a general inequality for group valued Lipschitz maps $f : A \longrightarrow \mathbb{M}$, namely

$$\int_{\mathbb{M}} \mathcal{H}^{Q-P}\left(A \cap f^{-1}(\xi)\right) \, d\mathcal{H}^P(\xi) \le \int_A C_P(d_H f(x)) \, d\mathcal{H}^Q(x)\,, \tag{6.1}$$

where $A \subset \mathbb{G}$ is measurable and $\mathbb{G}$, $\mathbb{M}$ are sub-Riemannian groups, [125]. Actually, the validity of the equality in (6.1) is a completely open question and it seems that none of the classical methods can be used to solve this problem. The philosophical reason for this difficulty is that we are considering Lipschitz maps between different types of geometries. So the validity of the coarea formula for group valued maps would imply a huge family of coarea formulae, where the Euclidean one would correspond to the simplest case of abelian geometries. Due to this general formulation we emphasize the existence of cases where the group valued coarea formula holds, but it is trivial. This is shown in Theorem 6.3.4, considering two different Heisenberg groups. This strange phenomenon indeed agrees perfectly with the fact that there are no $(\mathbb{H}^{2n+1},\mathbb{H}^{2m+1})$-rectifiable surfaces in $\mathbb{H}^{2n+1}$ when $n>m$, (Section 3.5). In a way, this confirms the compatibility between the formulation of the general coarea formula and our notion of rectifiability in higher codimension. Now we give a brief summary of the chapter.

In Section 6.1 we utilize the general Carathéodory construction to introduce the measure Φ^a that includes a family of possible measures, as the Hausdorff measure and the spherical Hausdorff measure, according to Definition 6.1.1. In this way we are able to obtain a general version of the coarea inequality (6.1), namely (6.14). Another element of this inequality is the H-coarea factor $C_P(L)$ for H-linear maps. Basically we extend the notion of H-coarea factor given in [7] to the sub-Riemannian context. In Definition 6.1.3 we introduce this notion, that replaces the classical one of coarea factor $\mathcal{C}_p(L)$ for linear maps of Hilbert spaces (Definition 2.3.41). In Proposition 6.1.5 we show that $C_P(L)$ and $\mathcal{C}_p(L)$ are indeed proportional by a dimensional constant that takes into account the homogeneous distances of the groups (6.4).

In Section 6.2 we prove the coarea inequality (6.14). This is an important result of the chapter and its consequences will be used in Sections 6.3, 6.4 and 6.6. Our technique is based on differentation theorems for measures. Precisely, we extend the blow-up method used in Lemma 2.96 of [6], reaching explicit estimates. The main result that leads to (6.14) is Theorem 6.2.4, where we obtain the upper estimate of the density for the family of "coarea measures" ν_t (Definition 6.2.1), with a constant independent of $t>0$. We show that this constant is exactly the H-coarea factor of the differential of the map at the point of blow-up. Integrating the upper density estimate (6.8) and letting $t\to 0^+$ the coarea inequality (6.14) follows.

Section 6.3 is devoted to some direct applications of (6.14). In Theorem 6.3.1 we obtain a weak version of the classical Sard Theorem, proving that for $\mathcal{H}^P$-a.e. level set of a Lipschitz map between stratified groups the set of singular points is $\mathcal{H}^{Q-P}$-negligible, [125]. We point out that also in the Euclidean case it is not possible to get more information on level sets of Lipschitz maps. Theorem 6.3.1 will be an important tool in Chapter 7 in order to prove the coarea formula (7.19). Another consequence of (6.14) is Theorem 6.3.4, where we obtain the trivial coarea formula (6.17) for Lipschitz maps between different Heisenberg groups. Notice that to obtain Theorem 6.3.4 we follow the same principle adopted in Section 4.4, i.e. from algebraic conditions given by the groups we obtain information on their "metric compatibility". As the algebraic conditions on the group affect the H-jacobian of the differential, here the same phenomenon happens to the H-coarea factor.

In Section 6.4 we extend the representation of the perimeter measure with the spherical Hausdorff measure to any homogeneous distance. This is done in Theorem 6.4.11 which starts from the result of Theorem 6.4.7, proved in [73], which is referred to the CC-distance. The proof of Theorem 6.4.11 has the interesting feature of not relying on an explicit form of the homogeneous distance, but only on its abstract properties. Our formula is as follows

$$|\partial E|_H = \frac{\theta^g_{Q-1}(\nu_E)}{\omega_{Q-1}}\, \mathcal{S}^{Q-1} \llcorner \partial_{*H} E\,, \tag{6.2}$$

where $\theta^g_{Q-1}(\nu_E)$ is the metric factor introduced in Chapter 5. Formula (6.2) is obtained for all generating groups.

The main result of Section 6.5 is the generalized coarea formula (6.42) for locally Lipschitz maps $u : \mathbb{G} \longrightarrow \mathbb{R}$. Its validity rests on different results. We first consider the coarea formula for H-BV functions (2.49) where the perimeter measure of the upper level sets $E_t = \{x \in \mathbb{G} \mid u(x) > t\}$ is considered. Clearly for a.e. t the set E_t has locally H-finite perimeter, then it is possible to replace its perimeter measure with the spherical Hausdorff measure according to (6.2). Here a crucial point of the proof occurs: we have to prove that the H-reduced boundary $\partial_{*H} E_t$ covers $\mathcal{H}^{Q-1}$ almost all of the level set $u^{-1}(t)$. This is done in Theorem 6.5.1 where it is proved that for a.e. $t \in \mathbb{R}$ we have $\mathcal{H}^{Q-1}(u^{-1}(t) \setminus \partial_{*H} E_t) = 0$. In the same theorem a natural relation between the H-differential of u and the generalized inward normal to E_t is also provided, namely

$$\nu_{E_t}(p) = \frac{\nabla_H u(p)}{|\nabla_H u(p)|}\,,$$

for $\mathcal{H}^{Q-1}$-a.e. $p \in u^{-1}(t)$ and a.e. $t \in \mathbb{R}$. We mention that the proof of this theorem stems from a careful application of several results, as Theorem 6.3.1 of Section 6.3, formula (2.48), Theorem 4.2 of [5] and Lemma 2.31 of [73]. By this theorem the coarea formula is easily proved. The subsequent coarea formulae (6.45), (6.46) and (6.47) follow applying results of Section 5.2, where it is proved that the metric factor of rotational groups is constant. We mention that in this particular case another proof of the coarea formula can be given using directly the coarea inequality (6.14), without exploiting Theorem 6.5.1, [125].

In Section 6.6 we are concerned with the estimate of the characteristic set of C^1 hypersurfaces. We mention that the size of the characteristic set is of great importance in the study of trace theorems in the sub-Riemannian setting. For instance, M.Derridj proved in [51] that the characteristic set of a C^∞ hypersurface is negligible with respect to the Euclidean surface measure and by this result he proved the existence of a measurable trace on $\partial\Omega$ for Sobolev maps with respect to horizontal vector fields. In this picture, characteristic points play the role of cusps where it is not possible to consider the trace map. In the theory of sets of H-finite perimeter a precise estimate of the size of the characteristic set allows us to answer the following natural question, raised in [71] and [73]: are all C^1 hypersurfaces $\mathbb{G}$-rectifiable? The answer to this question follows essentially by proving that the set of characteristic points is $\mathcal{H}^{Q-1}$-negligible. This fact was first proved in [12] for Heisenberg groups and subsequently generalized in [73] to sub-Riemannian groups of step 2. In both cases the proofs are based on covering arguments. In Theorem 6.6.2 we extend these results to any sub-Riemannian group using a different argument. Our proof relies on the weak Sard-type Theorem proved in Section 6.3 and on the observation that characteristic points of regular level sets can be regarded as those points where the H-differential of the map vanishes, Lemma 6.6.1. As a result, in every sub-Riemannian group the hypersurfaces of class C^1 are $\mathbb{G}$-rectifiable, according to Definition 3.5.2. Another important consequence of Theorem 6.6.2 is the estimate (7.52) that answers a conjecture raised by D. Danielli, N. Garofalo and D.M. Nhieu in [42]. More details on this major consequence are given in Chapter 7.

6.1 Carathéodory measures and coarea factor

In this section we introduce some additional notions that will be used throughout the chapter. We will assume that $\mathbb{G}$ and $\mathbb{M}$ are stratified groups with homogeneous distances d and ρ and Hausdorff dimension Q and P, respectively.

Definition 6.1.1 We fix a compact neighbourhood $D \subset \mathbb{G}$ of the unit element and define the family $\mathcal{F}_0 = \{x\delta_r D \mid x \in \mathbb{G},\ r > 0\}$. Given $a \geq 0$ we apply the construction of Definition 2.1.17 with $\mathcal{F}$ equal to either $\mathcal{F}_0$ or $\mathcal{P}(\mathbb{G})$, denoting with Φ^a the corresponding measure on $\mathbb{G}$.

Proposition 6.1.2 *The measure Φ^a defined above satisfies the estimate (2.4) and the following ones*

1. $\Phi^a(\delta_r E) = r^a \Phi^a(E)$ *for* $E \subset \mathbb{G}$, $r > 0$
2. $\Phi^a_t(\delta_r E) \leq r^a \Phi^a_t(E)$, *for* $E \subset \mathbb{G}$, $r, t > 0$ *and* $r < 1$
3. $\Phi^a(xE) = \Phi^a(E)$, *for any* $x \in \mathbb{G}$ *(left invariance)*

Proof. In case $\mathcal{F} = \mathcal{P}(\mathbb{G})$ clearly $\Phi^a = \mathcal{H}^a$, so (2.4) is trivial. If $\mathcal{F} = \mathcal{F}_0$ it is enough to observe that there exist two positive constants c_1 and c_2 such that $B_{c_1} \subset D \subset B_{c_2}$ and compare Φ^a with $\mathcal{S}^a$. Properties 1 and 2 follow from the fact that for any $s, r > 0$ and $x \in \mathbb{G}$ one has $\operatorname{diam}(\delta_r E) = r \operatorname{diam}(E)$ and $\delta_s(x\delta_r D) = \delta_s x \delta_{sr} D \in \mathcal{F}$. Finally, by the left invariance of the homogeneous metric Property 3 follows. □

The following definition is essentially taken from [7].

Definition 6.1.3 (H-coarea factor) Consider an H-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$, with $Q \geq P$. The *horizontal coarea factor* $C_P(L)$ of L is the unique constant such that

$$\Phi^Q(B_1)\, C_P(L) = \int_{\mathbb{M}} \Phi^{Q-P}\left(B_1 \cap L^{-1}(\xi)\right)\, d\Phi^P(\xi)\,. \tag{6.3}$$

We will also say in short H-coarea factor.

Remark 6.1.4 Notice that measures Φ^P, Φ^Q and Φ^{Q-P} can be built independently, choosing different families $\mathcal{F}_0$, according to Definition 6.1.1.

In view of the following proposition the definition of H-coarea factor is well posed.

Proposition 6.1.5 *Let $L \in \mathrm{HL}(\mathbb{G}, \mathbb{M})$ and let (F, W) and $(\tilde{F}, \tilde{W})$ be two systems of graded coordinates on $\mathbb{G}$ and $\mathbb{M}$, respectively. Then there exists a unique nonnegative constant $C_P(L)$ such that (6.3) holds and the number $C_P(L)$ is positive if and only if L is surjective. In this case we have*

$$C_P(L) = \frac{\alpha_{Q-P}\, \beta_P}{\beta_Q}\, \mathcal{C}_p(\tilde{F}^{-1} \circ L \circ F)\,, \tag{6.4}$$

where posing $N = L^{-1}(0)$ we have defined $\alpha_{Q-P} = \Phi^{Q-P} \llcorner N(B_1^d)/F_\sharp \mathcal{H}^{q-p}_{|\cdot|} \llcorner N(B_1^d)$, $\beta_P = \Phi^P(B_1^\rho)/\tilde{F}_\sharp \mathcal{L}^p(B_1^\rho)$ and $\beta_Q = \Phi^Q(B_1^d)/F_\sharp \mathcal{L}^q(B_1^d)$.

Proof. Let us fix a system of graded coordinates (F, W), according to Definition 2.3.43. We proceed similarly to the proof of Proposition 4.2.2. If we read the dilation δ_r restricted to the subspace $L(\mathbb{G})$ as coordinate dilation with respect to graded coordinates it is easy to see that its jacobian is $r^{P'}$, where $P' = \sum_{i=1}^m i \dim\left(L(V_i)\right)$. It follows that

$$F_\sharp \mathcal{H}^{p'}_{|\cdot|}\left(B_r^\rho \cap L(\mathbb{G})\right) = r^{P'}\, F_\sharp \mathcal{H}^{p'}_{|\cdot|}\left(B_1^\rho \cap L(\mathbb{G})\right)\,,$$

where p' is the topological dimension of $L(\mathbb{G})$. In the case L is not surjective it follows that

$$P' = \sum_{i=1}^m i \dim\left(L(V_i)\right) < \sum_{i=1}^m i \dim(W_i) = P\,,$$

hence the Hausdorff dimension of $L(\mathbb{G})$ is less than P and by (2.4) and (6.3) it follows that $C_P(L) = 0$. Now assume that L is surjective. We start proving that Φ^{Q-P} is proportional to $F_\sharp \mathcal{H}^{q-p}_{|\cdot|}$ on the subgroup $N = L^{-1}(0)$. Note that N has topological dimension $q-p$ and a graded structure $\mathcal{N} = U_1 \oplus U_2 \oplus \cdots \oplus U_\iota$, where U_i is a subspace of V_i for any $i = 1, \ldots, \iota$. Reasoning as above we have that

$$F_\sharp \mathcal{H}^{q-p}_{|\cdot|}\left(B_r^d \cap N\right) = r^{Q'}\, F_\sharp \mathcal{H}^{q-p}_{|\cdot|}\left(B_1^d \cap N\right)\,, \tag{6.5}$$

where $Q' = \sum_{i=1}^n i\dim(U_i)$. The fact that L is surjective implies that $n \geq m$, $\dim(V_i) \geq \dim(W_i)$ and $\dim(U_i) = \dim(V_i) - \dim(W_i)$, $i = 1, \dots, m$, so

$$Q' = \sum_{i=1}^{n} i\dim(U_i) = \sum_{i=1}^{m} i\,(\dim(V_i) - \dim(W_i)) + \sum_{i=m+1}^{n} i\dim(V_i) = Q - P\,.$$

It is clear that $\Phi^{Q-P} \llcorner N$ is a left invariant measure on N, because the metric d restricted to N is still left invariant. We have to check that it is also locally finite. It is clear that $F_\sharp \mathcal{H}^{q-p}_{|\cdot|} \llcorner N$ is locally finite. Moreover it is left invariant due to the fact that coordinate translations (Definition 2.3.54) restricted to the subspace $F^{-1}(N)$ preserve the Lebesgue measure. This in turn follows by Proposition 2.3.47 observing that translations of $\mathbb{G}$ restricted to N preserve the Riemannian volume restricted to N. By (6.5) it follows that $\Phi^{Q-P} \llcorner N$ is locally finite and hence it is proportional to $\mathcal{H}^{q-p}_{|\cdot|} \llcorner N$, namely

$$\Phi^{Q-P} \llcorner N = \alpha_{Q,P}\ F_\sharp \mathcal{H}^{q-p}_{|\cdot|} \llcorner N\,, \tag{6.6}$$

where $\alpha_{Q-P} = \Phi^{Q-P} \llcorner N(B_1^d) / F_\sharp \mathcal{H}^{q-p}_{|\cdot|} \llcorner N(B_1^d)$. Notice that for any $\xi \in \mathbb{M}$ we can write $L^{-1}(\xi) = xN$, where $L(x) = \xi$, so taking into account that left translations are isometries, one concludes that the constant $\alpha_{Q,P}$ remains unchanged if one replaces N with $L^{-1}(\xi)$ in formula (6.6). As a result we find that the measure

$$\nu(A) = \int_{\mathbb{M}} \Phi^{Q-P}\left(A \cap L^{-1}(\xi)\right)\, d\Phi^P(\xi)$$

is positive on open bounded sets, while inequality (2.7) guarantees that ν is finite on the sets $A \subset \mathbb{G}$ with Φ^Q-finite measure. By a change of variable involving left translations it is not difficult to see that ν is a left invariant measure on $\mathbb{G}$, so there exists a positive constant $C_P(L)$ such that $\nu = C_P(L)\Phi^Q$. Now we want to compute explicitly the H-coarea factor $C_P(L)$. We know that Φ^P is proportional to the Lebesgue measure $\mathcal{L}^p$ on $\mathbb{M}$. Thus, we obtain

$$\int_{\mathbb{M}} \Phi^{Q-P}\left(B_1^d \cap L^{-1}(\xi)\right)\, d\Phi^P(\xi) = \alpha_{Q-P}\,\beta_P \int_{\mathbb{M}} F_\sharp \mathcal{H}^{q-p}_{|\cdot|}\left(B_1^d \cap L^{-1}(\xi)\right)\, d\mathcal{L}^p(\xi)\,,$$

where $\beta_P = \Phi^P(B_1^\rho)/\tilde{F}_\sharp \mathcal{L}^p(B_1^\rho)$. From the classical coarea formula we get

$$\int_{\mathbb{M}} \Phi^{Q-P}\left(B_1^d \cap L^{-1}(\xi)\right)\, d\Phi^P(\xi) = \frac{\alpha_{Q-P}\,\beta_P}{\beta_Q}\, \mathcal{C}_p(L)\, \Phi^Q(B_1^d)\,,$$

where $\beta_Q = \Phi^Q(B_1^d)/\mathcal{L}^q(B_1^d)$. Finally, formula (6.3) leads us to the claim. $\Box$

Remark 6.1.6 If $\mathbb{G}$ and $\mathbb{M}$ are Euclidean spaces it follows

$$C_P(L) = \det(LL^*)^{1/2} = \mathcal{C}_p(L)\,,$$

where L is a linear map. Therefore, the H-coarea factor coincides with the classical coarea factor of Definition 2.3.41. For H-linear maps, by (2.7), we always have

$$C_P(L) \le \frac{\omega_{Q-P}\,\omega_P\,\Theta_{Q-P}\,\Theta_P\,\Theta_Q}{\omega_Q\,\Phi^Q(B_1)}\ \mathrm{Lip}(L)\,. \tag{6.7}$$

6.2 Coarea inequality

This section is devoted to the proof of coarea inequality. In the sequel the set $A \subset \mathbb{G}$ will be assumed to be closed and $f : A \longrightarrow \mathbb{M}$ will be a Lipschitz map. Notice that the map $\xi \longrightarrow \Phi_t^{Q-P}\left(A \cap f^{-1}(\xi)\right)$ is a Borel map for any $t > 0$, hence we can state the following definition.

Definition 6.2.1 Let t be a positive number. We define the measure ν_t on $\mathbb{G}$ as follows: for any $D \subset \mathbb{G}$

$$\nu_t(D) = \int_{\mathbb{M}} \Phi_t^{Q-P}\left(D \cap A \cap f^{-1}(\xi)\right)\, d\Phi^P(\xi)\,.$$

By the general coarea estimate (2.7) the measure ν_t is locally finite uniformly in $t > 0$.

Definition 6.2.2 For each map $f : A \longrightarrow \mathbb{M}$ and $x_0 \in A$, we define the *r-rescaled of f at x_0* as the map $f_{x_0,r} : \delta_{1/r}(x_0^{-1}A) \longrightarrow \mathbb{M}$ defined as

$$f_{x_0,r}(y) = \delta_{1/r}\left(f(x_0)^{-1} f(x_0\delta_r y)\right)\,.$$

Proposition 6.2.3 *Consider a map $f : A \longrightarrow \mathbb{M}$, a differentiability point $x_0 \in \mathcal{I}(A)$ and a sequence of positive numbers (r_j) which tends to zero. For every $\zeta \in \mathbb{M}$, $j \in \mathbb{N}$ define the compact set*

$$K_j(\zeta) = \overline{\bigcup_{m \ge j}\left(D_1 \cap f_{x_0,r_m}^{-1}(\zeta) \cap \delta_{1/r_m}(x_0^{-1}A)\right)}\,.$$

Then it follows $\bigcap_{j\ge1} K_j \subset D_1 \cap d_H f(x_0)^{-1}(\zeta)$.

PROOF. Pick an element $y \in \bigcap_{j\ge1} K_j$, getting a subsequence (ρ_l) of (r_j) and a sequence (y_l) such that $y_l \in D_1 \cap f_{x_0,\rho_l}^{-1}(\zeta) \cap \delta_{1/\rho_l}(x_0^{-1}A)$, $y_l \to y$. Thus, by definition of differentiability it follows

$$f_{x_0,\rho_l}(y_l) \to d_H f(x_0)(y)\,,$$

so $f_{x_0,\rho_l}(y_l) = \zeta$ for every $l \in \mathbb{N}$ yields $\zeta = d_H f(x_0)(y)$. □

Theorem 6.2.4 (Density estimate) *In the above assumptions, for any $t > 0$ we have*

$$\limsup_{r\to0} \frac{\nu_t(D_{x_0,r})}{\Phi^Q(D_{x_0,r})} \le C_P(d_H f(x_0))\,. \tag{6.8}$$

PROOF. We start considering the quotient

$$\nu_t(D_{x_0,r})r^{-Q} = \int_{\mathbb{M}} \Phi_t^{Q-P}\left(A \cap D_{x_0,r} \cap f^{-1}(\xi)\right) r^{-Q}\, d\Phi^P(\xi).$$

The map $T_{x_0,r} : \mathbb{G} \longrightarrow \mathbb{G}$, $y \longrightarrow x_0\delta_r y$ is the composition of an isometry and a dilation δ_r. Thus, choosing $r < 1$, by property 2 of Proposition 6.1.2 it follows

$$\Phi_t^{Q-P}\left(A \cap D_{x_0,r} \cap f^{-1}(\xi)\right) = \Phi_t^{Q-P}\Big(T_{x_0,r}\left(A_{x_0,r}(\xi)\right)\Big) \le r^{Q-P}\,\Phi_t^{Q-P}(A_{x_0,r}(\xi))\,,$$

where $A_{x_0,r}(\xi) = \{y \in D_1 \mid f(x_0\delta_r y) = \xi\} \cap \delta_{1/r}(x_0^{-1}A)$. This implies

$$\nu_t(D_{x_0,r})r^{-Q} \le \int_{\mathbb{M}} \Phi_t^{Q-P}(A_{x_0,r}(\xi))\, r^{-P}\, d\Phi^P(\xi).$$

Defining $R_{x_0,r} : \mathbb{M} \longrightarrow \mathbb{M}$, $\xi \longrightarrow \delta_{1/r}(f(x_0)^{-1}\xi) = \zeta$ and using property 1 of Proposition 6.1.2 we obtain $(R_{x_0,r})_\sharp(\Phi^P) = r^P\,\Phi^P$, hence

$$\nu_t(D_{x_0,r})r^{-Q} \le \int_{\mathbb{M}} \Phi_t^{Q-P}\left(A_{x_0,r}(R^{-1}_{x_0,r}(\zeta)\right)\, d\Phi^P(\zeta)\,.$$

By the definition of r-rescaled function we have

$$\begin{aligned} A_{x_0,r}\left(R^{-1}_{x_0,r}(\zeta)\right) &= \Big\{y \in D_1 \mid f(x_0\delta_r y) = f(x_0)\delta_r\zeta\Big\} \cap \delta_{1/r}(x_0^{-1}A) \\ &= D_1 \cap f^{-1}_{x_0,r}(\zeta) \cap \delta_{1/r}(x_0^{-1}A)\,. \end{aligned}$$

Now we notice that the family of functions $\{f_{x_0,r}\}_{r>0}$ is uniformly Lipschitz with bound $\mathrm{Lip}(f) = h$ on the Lipschitz constants, hence we have $f_{x_0,r}(D_1) \subset D_h$ for any $r > 0$ and

$$\nu_t(D_{x_0,r})r^{-Q} \le \int_{D_h} \Phi_t^{Q-P}\left(D_1 \cap f^{-1}_{x_0,r}(\zeta) \cap \delta_{1/r}(x_0^{-1}A)\right)\, d\Phi^P(\zeta)\,. \tag{6.9}$$

We choose a sequence (r_j) such that $r_j \to 0$ and for each $j \in \mathbb{N}$ define the functions

$$g_j^t(\zeta) = \Phi_t^{Q-P}\Big(D_1 \cap f^{-1}_{x_0,r_j}(\zeta) \cap \delta_{1/r_j}(x_0^{-1}A)\Big) \tag{6.10}$$

and the following decreasing sequence of compact sets

$$K_j(\zeta) = \overline{\bigcup_{m\ge j}\left(D_1 \cap f^{-1}_{x_0,r_m}(\zeta) \cap \delta_{1/r_m}(x_0^{-1}A)\right)}\,.$$

In view of Proposition 6.2.3 we obtain

$$\bigcap_{j\ge 1} K_j(\zeta) \subset D_1 \cap L^{-1}(\zeta)\,,$$

where $L = d_H f(x_0)$ is the differential of f at x_0. By results of paragraph 2.10.20 in [55] it follows

$$\limsup_{j\to\infty} g_j^t(\zeta) \leq \lim_{j\to\infty} \Phi_t^{Q-P}\left(K_j(\zeta)\right) \leq \Phi_\tau^{Q-P}\Big(\bigcap_{j\geq 1} K_j(\zeta)\Big) \leq \Phi_\tau^{Q-P}\big(D_1 \cap L^{-1}(\zeta)\big) \tag{6.11}$$

with $\tau < t$. Each measure Φ_τ^a, with $\tau, a > 0$, is finite on bounded sets, then the sequence of nonnegative functions $(g_j^t)_{j\in\mathbb{N}}$ is uniformly bounded by $\Phi_\tau^{Q-P}(D_1)$ on D_h. This fact together with the Fatou Theorem and inequality (6.11) implies

$$\limsup_{j\to\infty} \int_{D_h} g_j^t(\zeta)\, d\Phi^P(\zeta) \leq \int_{D_h} \Phi_\tau^{Q-P}\left(D_1 \cap L^{-1}(\xi)\right)\, d\Phi^P(\zeta)\,. \tag{6.12}$$

Joining inequalities (6.9), (6.10), (6.12) and taking into account the inequality $\Phi_\tau^a \leq \Phi^a$ it follows

$$\limsup_{j\to\infty} \nu_t(D_{x_0,r_j}) r_j^{-Q} \leq \int_{\mathbb{M}} \Phi^{Q-P}\left(D_1 \cap L^{-1}(\zeta)\right)\, d\Phi^P(\zeta)\,.$$

The arbitrary choice of the sequence (r_k) and Definition 6.1.3 yield

$$\limsup_{r\to 0} \nu_t(D_{x_0,r}) r^{-Q} \leq C_P\left(d_H f(x_0)\right)\, \Phi^Q(D_1)\,, \tag{6.13}$$

finally, by inequality (6.13) and the property 1 of Proposition 6.1.2 the proof is complete. □

Theorem 6.2.5 (Coarea inequality) *Let $A \subset \mathbb{G}$ be a measurable set and consider a Lipschitz map $f : A \longrightarrow \mathbb{M}$. Then we have*

$$\int_{\mathbb{M}} \Phi^{Q-P}\left(A \cap f^{-1}(\xi)\right)\, d\Phi^P(\xi) \leq \int_A C_P(d_H f(x))\, d\Phi^Q(x)\,. \tag{6.14}$$

PROOF. We start proving the measurability of $g(x) = C_P(d_x f)$. For any $t > 0$ we consider the Borel function defined on H-linear maps

$$L \longrightarrow \Phi_t^{Q-P}\left(L^{-1}(0) \cap D_1\right)\,.$$

The limit as $t \to 0$ is a measurable function, so by the measurability of $x \longrightarrow d_x f$ and the representation (6.4) one concludes this verification. Furthermore, in view of (6.7) the map g is bounded. Now we define $A' \subset \mathcal{I}(A) \cap A$ as the set of differentiability points, hence by Theorem 3.4.11 we have $\Phi^Q(A \setminus A') = 0$ and by (2.7) it follows

$$\int_M \Phi^{Q-P}\left(A \cap f^{-1}(\xi)\right)\, d\Phi^P(\xi) \leq \int_M \Phi^{Q-P}\left(A' \cap f^{-1}(\xi)\right)\, d\Phi^P(\xi)\,. \tag{6.15}$$

Consider a measurable step function $\varphi = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i} \geq g$, $\alpha_i \geq 0$, $\bigsqcup_{i=1}^{k} A_i = A'$ (disjoint union). By estimate (6.8), for any $i = 1, \ldots, k$ we have

$$\liminf_{r \to 0} \frac{\nu_t(D_{x,r})}{\Phi^Q(D_{x,r})} \leq \alpha_i$$

for each $x \in A_i$. Inequality (2.7) implies the absolute continuity of the measure ν_t with respect to Φ^Q, so for every $i = 1, \ldots, k$ we can apply Lemma 2.1.24, getting

$$\nu_t(A_i) \leq \alpha_i \, \Phi^Q(A_i) \, .$$

Since our estimates are independent of $t > 0$, we can allow $t \to 0$. Therefore, summing over $i = 1, \ldots, k$ we find

$$\int_M \Phi^{Q-P} \left(A' \cap f^{-1}(\xi) \right) \, d\Phi^P(\xi) \leq \int_{A'} \varphi(x) \, d\Phi^Q(x) \, .$$

By (6.15) and the measurability of g the proof is complete. □

6.3 Some applications

The classical Sard Theorem states that for sufficiently smooth maps, almost every level set has an empty set of singular points. An analogous statement for Lipschitz maps is to require that for a.e. level set the subset of singular points is negligible with respect to the surface measure. In the following theorem we prove this statement for Lipschitz maps of sub-Riemannian groups.

Theorem 6.3.1 (Sard-type Theorem) *Let $f : A \longrightarrow \mathbb{M}$ be a Lipschitz map, where A is a closed subset of $\mathbb{G}$. We denote the set of singular points as follows*

$$S = \{ x \in A \mid d_H f(x) \text{ is not surjective} \} \, .$$

Then, for $\mathcal{H}^P$-a.e. $\xi \in \mathbb{M}$ we have $\mathcal{H}^{Q-P} \left(S \cap f^{-1}(\xi) \right) = 0$.

The proof follows immediately from coarea inequality (6.1), by taking $A = S$. Indeed, we get

$$\int_{\mathbb{M}} \mathcal{H}^{Q-P} \left(S \cap f^{-1}(\xi) \right) \, d\mathcal{H}^P(\xi) = 0 \, .$$

As a result, in almost every fiber the set of non-singular points has full measure.

To better understand the meaning of "singular point" we consider the C^1 case. Let $u \in C^1(\Omega)$ and $t \in \mathbb{R}$ be a regular value of u, where Ω is an open subset of $\mathbb{G}$. In Lemma 6.6.1 we will prove that singular points coincide with characteristic points: we will precisely show that $S \cap u^{-1}(t) = C\left(u^{-1}(t)\right)$, where $C\left(u^{-1}(t)\right)$ is the characteristic set of the hypersurface $u^{-1}(t)$. It turns out that singular points, e.g.

those points where the H-differential vanishes, represent the class of characteristic points even when the surface is less regular, since it can be the level set of a Lipschitz map with respect to the CC-distance.

We also notice that C^1 real valued maps are in particular Lipschitz with respect to the CC-distance, hence we can apply our weak version of Sard's Theorem, obtaining that in a.e. fiber the set of characteristic points is negligible for the $Q-1$ Hausdorff measure. We will use this simple observation in Theorem 6.6.2 to prove that the set of characteristic points of a C^1 hypersurface is $\mathcal{H}^{Q-1}$-negligible. Our Sard type theorem will be also crucial in the proof of Theorem 6.5.1 that provides the main tool to prove the coarea formula (6.42).

The coarea inequality (6.1) can be also used to know if there exists only a trivial coarea formula for two given stratified groups. In the next proposition we show that if all H-linear maps between the groups are not surjective, then only a trivial coarea formula holds between the groups, namely a vanishing identity.

Proposition 6.3.2 *Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups such that any H-linear map $L \in \mathrm{HL}(\mathbb{G}, \mathbb{M})$ is not surjective. Then for any Lipschitz map $f : A \longrightarrow \mathbb{M}$, where A is a measurable subset of $\mathbb{G}$, the coarea formula holds and it is trivial*

$$\int_A C_P(d_H f(x))\, d\mathcal{H}^Q(x) = 0 = \int_{\mathbb{M}} \mathcal{H}^{Q-P}\left(A \cap f^{-1}(\xi)\right)\, d\mathcal{H}^P(\xi)\,.$$

Proof. By Theorem 3.4.11 the map f is differentiable a.e. in A and the differential $d_H f(x) : \mathbb{G} \longrightarrow \mathbb{M}$ is an H-linear map. Our assumption yield that any H-linear map of $\mathrm{HL}(\mathbb{G}, \mathbb{M})$ is not surjective, hence by Proposition 6.1.5 we have $C_P(d_H f(x)) = 0$ for $\mathcal{H}^Q$-a.e. $x \in A$. Thus, the coarea inequality (6.14) implies

$$\int_{\mathbb{M}} \mathcal{H}^{Q-P}\left(A \cap f^{-1}(\xi)\right)\, d\mathcal{H}^P(\xi) \leq \int_A C_P(d_H f(x))\, d\mathcal{H}^Q(x) = 0\,. \quad \square$$

The hypotheses of the previous proposition are satisfied when $\mathbb{G} = \mathbb{H}^{2n+1}$ and $\mathbb{M} = \mathbb{H}^{2m+1}$ with $n > m$.

Proposition 6.3.3 *Any H-linear map $T \in \mathrm{HL}(\mathbb{H}^{2n+1}, \mathbb{H}^{2m+1})$, with $n > m$, is not surjective.*

Proof. We use the exponential coordinates of Remark 2.3.27, then the product operation is as follows

$$(z,s) \cdot (w,t) = (z + w, s + t + 2\mathrm{Im}\langle z, w\rangle)\ ,$$

where (z,s), $(w,s) \in \mathbb{C}^n \times \mathbb{R}$. By Remark 3.1.15 we have $T(z,s) = (Az, \lambda s)$, where $A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is a linear map with respect to the field of real numbers and $\lambda \in \mathbb{R}$.

The homomorphism property implies that

$$\begin{aligned} & T(z+w, s+t+2\mathrm{Im}\langle z,w\rangle) = (Az+Aw, \lambda(s+t)+2\lambda\mathrm{Im}\langle z,w\rangle) \\ & = T(z,s)\cdot L(w,t) = (Az+Aw, \lambda s+\lambda t+2\mathrm{Im}\langle Az, Aw\rangle)\,, \end{aligned}$$

then

$$\lambda\,\mathrm{Im}\langle z,w\rangle = \mathrm{Im}\langle Az, Aw\rangle \tag{6.16}$$

for any $z,w\in\mathbb{C}^n$. By the fact that $n>m$ we can take a non vanishing u in the kernel of A. Replacing $z=u$ and $w=iu$ in (6.16) we obtain that $\lambda=0$, this in turn implies that T is not surjective. □

Proposition 6.3.2 and Proposition 6.3.3 yield the following theorem.

Theorem 6.3.4 *Let $f: A \longrightarrow \mathbb{H}^{2m+1}$ be a Lipschitz map, where A is a measurable subset of $\mathbb{H}^{2n+1}$ and $n>m$. Then the coarea formula holds and it is trivial*

$$\int_A C_P(d_H f(x))\, d\mathcal{H}^{2n+2}(x) = 0 = \int_{\mathbb{H}^{2m+1}} \mathcal{H}^{2(n-m)}\left(A\cap f^{-1}(\xi)\right)\, d\mathcal{H}^{2m+2}(\xi)\,. \tag{6.17}$$

6.4 Representation of the perimeter measure

In this section we find the representation of the perimeter measure with respect to the spherical Hausdorff measure built with an arbitrary homogeneous distance. This is done in all groups where a Blow-up Theorem holds, namely generating groups. This general representation will be used in Section 6.5 in order to obtain a general formulation of the coarea formula for real valued Lipschitz maps.

Definition 6.4.1 Let $\mathbb{G}$ be a graded group and let $E\subset\mathbb{G}$ and $p\in\mathbb{G}$. The *r-rescaled* of E at p is the set

$$E_{p,r} = \delta_{1/r}(p^{-1}E)\,.$$

In formula (6.34) we will see the connection between the notion of rescaled set and the one of rescaled map (Definition 6.2.2).

Remark 6.4.2 It is not difficult to check that if E is a set of H-finite perimeter in $\mathbb{G}$, then for any $p\in\mathbb{G}$ and $r>0$ the set $E_{p,r}$ is also and the following formula holds

$$\left(\delta_{1/r}\circ l_p^{-1}\right)_\sharp |\partial E|_H = r^{Q-1}\,|\partial E_{p,r}|_H \tag{6.18}$$

In the next definition we introduce the notion of vertical half spaces.

Definition 6.4.3 Let $p\in\mathbb{G}$ and $\nu\in V_1\setminus\{0\}$. The *vertical half spaces* at $p\in\mathbb{G}$ relative to ν are defined as follows

$$\begin{aligned} S_g^+(p,\nu) &= \exp\left(\left\{v\in T_p\mathbb{G}\;\middle|\; g(p)(\nu(p),v)>0\right\}\right) \\ S_g^-(p,\nu) &= \exp\left(\left\{v\in T_p\mathbb{G}\;\middle|\; g(p)(\nu(p),v)<0\right\}\right)\,. \end{aligned}$$

When $p = e$ we will simply write $S_g^+(\nu)$ and $S_g^+(\nu)$.

Remark 6.4.4 Note that the notion of half space is strictly related to the graded metric we consider. We also point out that by definition of vertical half space we get the following equalities

$$S_g^+(p,\nu) = l_p\left(S_g^+(\nu)\right) \quad S_g^-(p,\nu) = l_p\left(S_g^-(\nu)\right) .$$

The previous relations follow observing that

$$dl_p\left(\left\{v \in T_e\mathbb{G} \,\middle|\, g(e)(\nu(e),v) > 0\right\}\right) = \left\{v \in T_p\mathbb{G} \,\middle|\, g(p)(\nu(p),v) > 0\right\}$$

and $l_p = \exp \circ dl_p \circ \exp^{-1}$.

Now one can wonder whether the notion of half space yields two different notions of intrinsic normal. In fact, if we look at $S_g^+(X)$, with $X \in V_1 \setminus \{0\}$, as a set of H-finite perimeter we have a natural notion of normal to $S_g^+(X)$ by taking the generalized inward normal $\nu_{S_g^+(X)}$. On the other hand, if we consider $\partial S_g^+(X)$ as a regular hypersurface of $\mathbb{G}$ we can also adopt the notion of horizontal normal to $\partial S_g^+(X)$ at the unit element $e \in \partial S_g^+(X)$ given in Definition 2.2.9, that clearly yields the direction X. In the following lemma we check that these two notions do coincide.

Lemma 6.4.5 *Let $X \in V_1 \setminus \{0\}$. Then we have*

$$\nu_{S_g^+(X)}(p') = \frac{X}{|X|}$$

for any $p' \in \partial S_g^+(X)$ and $\nu_{S_g^+(X)}(p') = 0$ otherwise.

Proof. Let (F, W) be a system of graded coordinates, where $(W_1, \ldots, W_m)$ is an orthonormal basis of the first layer V_1 and $W_1 = X/|X|$. For any $i = 1, \ldots, m$ we consider the vector fields $\tilde{W}_i = F_*^{-1} W_i \in \Gamma(T\mathbb{R}^q)$ and the maps $\tilde{\varphi}^i = \varphi^i \circ F$, where $\varphi = \sum_{j=1}^m \varphi^j\, W_j$. For ease of notation we denote $S_g^+(X) = S^+$. Proposition 2.3.47 and formula (2.1) give

$$\int_{S^+} \mathrm{div}_H \varphi \; dv_g = \int_{S^+} \mathrm{div}_H \varphi \; dF_\sharp \mathcal{L}^q = \int_{F^{-1}(S^+)} (\mathrm{div}_H \varphi) \circ F \; d\mathcal{L}^q \,.$$

By our choice of W_1, we obtain $S^+ = F(\Pi_1^+)$, where

$$\Pi_1^+ = \left\{x \in \mathbb{R}^q \,\middle|\, x_1 > 0\right\} ,$$

therefore, exploiting formula (2.43) and Proposition 2.3.47 we get

$$\int_{S^+} \mathrm{div}_H \varphi \; dv_g = \int_{\Pi_1^+} \sum_{j=1}^m \tilde{W}_j \tilde{\varphi}^j \; d\mathcal{L}^q = -\int_{\partial \Pi_1^+} \sum_{j=1}^m \tilde{\varphi}^j \, \langle \tilde{W}_j, e_1 \rangle \, d\mathcal{H}_{|\cdot|}^{q-1} ,$$

where (e_j) is the canonical basis of $\mathbb{R}^q$. Hence, formulae (2.1) and (2.42) imply

$$\int_{\partial\Pi_1^+} \tilde{\varphi}^1 \, d\mathcal{H}^{q-1}_{|\cdot|} = \int_{\mathbb{G}} \langle \varphi, \nu_{S^+} \rangle \, d|\partial S^+|_H = \int_{\mathbb{R}^q} \sum_{j=1}^m \tilde{\varphi}^j \, \tilde{\nu}^j \, dF^{-1}_\sharp |\partial S^+|_H \,, \tag{6.19}$$

where $\tilde{\nu}^j = \nu^j \circ F$ and $\nu^j = \langle \nu_{S^+}, W_j \rangle$ for any $i = 1, \dots, m$. The validity of (6.19) for any test function φ yields the equality of vector measures on $\mathbb{R}^q$

$$e_1 \, \mathcal{H}^{q-1} \llcorner \partial\Pi_1^+ = \Big(\sum_{j=1}^m \tilde{\nu}^j \, e_j \Big) F^{-1}_\sharp |\partial S^+|_H \,. \tag{6.20}$$

In particular, $\tilde{\nu}^j = 0$ for any $i = 2, \dots, m$ and

$$\tilde{\nu}^1 \circ F^{-1} = \langle \nu_{S^+}, W_1 \rangle = \langle \nu_{S^+}, \frac{X}{|X|} \rangle = 1 \,.$$

By the fact that $|\nu_{S^+}| = 1$ the thesis follows. $\square$

Remark 6.4.6 Notice that formula (6.20) also yields

$$|\partial S_g^+(X)|_H(B_1) = \mathcal{H}^{q-1}_{|\cdot|} \left(F^{-1} \left(B_1 \cap \partial S_g^+(X) \right) \right) = \theta^g_{Q-1}(X) \,. \tag{6.21}$$

Now we state the Blow-up Theorem for the perimeter measure. This result corresponds to Theorem 3.1 of [73].

Theorem 6.4.7 (Blow-up of perimeter measure) *We consider a set of locally H-finite perimeter $E \subset \mathbb{G}$ and a point $p \in \partial_{*H} E$. If $\mathbb{G}$ is a 2 step sub-Riemannian group, we have*

$$\lim_{r \to 0^+} |\partial E_{p,r}|_H(U_R) = |\partial S_g^+ \left(\nu_E(p) \right)|_H(U_R) \tag{6.22}$$

for any $R > 0$ and the following weak* convergence of vector valued Radon measures holds

$$\nu_{E_{p,r}} \, |\partial E_{p,r}|_H \rightharpoonup \nu_E(p) \, |\partial S_g^+ \left(\nu_E(p) \right)|_H \quad as \quad r \to 0 \,. \tag{6.23}$$

We have denoted by $U_{p,r}$ the open ball in the CC-distance associated to the graded metric g.

In order to emphasize that our representation of the perimeter measure with respect to a homogeneous distance is valid whenever the previous Blow-up Theorem holds we give the following definition.

Definition 6.4.8 We say that a sub-Riemannian group is *generating* if the statement of Theorem 6.4.7 holds for this group.

Remark 6.4.9 Due to results of [73] all 2-step groups are generating. In Section 6.5 we will prove a general coarea formula for real valued Lipschitz maps defined on generating groups.

Remark 6.4.10 Notice that the notation $S_g^+(\nu_E(p))$ in Theorem 6.4.7 must be properly interpreted. Indeed, according to Definition 6.4.3 one has to consider the left invariant vector field $Z \in V_1$ such that $Z(p) = \nu_E(p)$, obtaining $S_g^+(\nu_E(p)) = S_g^+(Z)$. The point p only indicates that the direction of the horizontal normal depends on the point we consider in ∂E.

The following theorem is the main result of the section. We will see that Theorem 6.4.12 is its straightforward consequence.

Theorem 6.4.11 *Let d be a homogeneous distance on a generating group $\mathbb{G}$ and assume that E is a set of locally H-finite perimeter. Then for $|\partial E|_H$-a.e. $p \in \mathbb{G}$ we have*

$$\lim_{r\to 0^+} |\partial E_{p,r}|_H(B_1) = \lim_{r\to 0^+} \frac{|\partial E|_H(B_{p,r})}{r^{Q-1}} = \theta^g_{Q-1}\left(\nu_E(p)\right) \tag{6.24}$$

where the open balls $B_{p,r}$ are defined with respect to the metric d and $\theta^g_{Q-1}\left(\nu_E(p)\right)$ is the metric factor of d with respect to the horizontal direction $\nu_E(p)$, according to Definition 5.2.2.

Proof. In view of the discussion of Section 2.4, concerning the independence of $\partial_{*H}E$ with respect to the homogeneous distance to which is referred (Definition 2.4.10), relation (2.48) holds when the reduced boundary is referred to d. Thus, it suffices to prove that limit (6.24) holds for each point $p \in \partial^*_H E$. By formula (6.21) we see that $|\partial S_g^+\left(\nu_E(p)\right)|_H$ is finite on compact sets, then by (6.22) we can choose $R > 0$ such that for some $\rho > 0$ we have

$$\sup_{0<r<\rho} |\partial E_{p,r}|_H(\overline{U}_R) < +\infty$$

By the weak*-compactness of Radon measures, see Theorem 1.59 of [6], there exists an infinitesimal sequence $(r_k) \subset]0,\rho[$ and a Radon measure σ such that

$$|\partial E_{p,r_k}|_H \llcorner \overline{U}_R \rightharpoonup \sigma \quad \text{as} \quad r_k \to 0^+ .$$

Since the measure σ is finite, then for a.e. $t \in]0,T[$ such that $B_T \subset U_R$ we have $\sigma(\partial B_t) = 0$. We choose $\tau \in]0,T[$ such that $\sigma(\partial B_\tau) = 0$. Since $p \in \partial_{*H}E$ we can use (6.23) and observing that the test function $\phi(w) = \nu_E(p)\mathbf{1}_{B_\tau}(w)$ has discontinuities in ∂B_τ, that is σ-negligible, then we can utilize Proposition 1.62(b) of [6] that implies

$$\int_{B_\tau} g\left(\nu_{E_{p,r_k}}, \nu_E(p)\right) d|\partial E_{p,r_k}|_H \longrightarrow g\left(\nu_E(p), \nu_E(p)\right) \, |\partial S_g^+\left(\nu_E(p)\right)|_H(B_\tau)$$

as $r_k \to 0$. From the definition of H-reduced boundary we know that $\nu_E(p)$ is a unit vector and from (6.18) and (6.21) we deduce that

$$|\partial S_g^+(\nu_E(p))|_H(B_\tau) = \tau^{Q-1}\,\theta^g_{Q-1}(\nu_E(p))\,,$$

where the factor τ^{Q-1} is the jacobian of dilation δ_τ restricted to the vertical hyperplane Π_p. Thus, we obtain

$$\int_{B_\tau} g\Big(\nu_{E_{p,r_k}}, \nu_E(p)\Big)\, d|\partial E_{p,r_k}|_H \longrightarrow \tau^{Q-1}\,\theta^g_{Q-1}(\nu_E(p))\,. \tag{6.25}$$

Now we fix $0 < \lambda < \lambda' < \tau$ such that $\sigma(\partial B_\lambda) = 0$ and choose a cut-off function ψ such that

$$\mathbf{1}_{\overline{B}_\lambda} \leq \psi \leq \mathbf{1}_{B^c_{\lambda'}}\,.$$

By a direct calculation, using formula (2.45) and the property of homogeneous distances $B_{tr} = \delta_r B_t$, we obtain

$$\int_{B_\tau} g\Big(\nu_{E_{p,r_k}}, \phi\psi\Big)\; d\,|\partial E_{p,r_k}|_H = \frac{1}{r_k^{Q-1}} \int_{B_{p,\tau r_k}} g(\nu_E, \phi_{p,k}\psi_{p,k})\; d\,|\partial E|_H\,, \tag{6.26}$$

where $\phi_{p,k} = \phi \circ \delta_{1/r_k} \circ l_{p^{-1}}$ and $\psi_{p,k} = \psi \circ \delta_{1/r_k} \circ l_{p^{-1}}$. Since $p \in \partial_{*H} E$ we have

$$\lim_{r_k \to 0^+} \fint_{B_{p,\lambda r_k}} g(\nu_E, \nu_E(p))\; d\,|\partial E|_H = 1$$

then, by properties (2.46)

$$\limsup_{r_k \to 0^+} \frac{1}{r_k^{Q-1}} \int_{B_{p,\lambda r_k}} g(\nu_E, \nu_E(p))\; d\,|\partial E|_H = \lambda^{Q-1} \limsup_{r_k \to 0^+} \frac{|\partial E|_H(B_{p,r_k})}{r_k^{Q-1}}\,. \tag{6.27}$$

We define $\overline{\psi} = 1 - \psi$. Then, observing that $\phi_{p,k}\psi_{p,k} = \nu_E(p)$ on $B_{p,\lambda r_k}$ and applying (6.26) we obtain

$$\begin{aligned}&\frac{1}{r_k^{Q-1}} \int_{B_{p,\lambda r_k}} g(\nu_E, \nu_E(p))\; d\,|\partial E|_H = \int_{B_\tau} g\Big(\nu_{E_{p,r_k}}, \nu_E(p)\Big)\, d|\partial E_{p,r_k}|_H \\ &- \int_{B_\tau} g\Big(\nu_{E_{p,r_k}}, \phi\overline{\psi}\Big)\; d\,|\partial E_{p,r_k}|_H - \frac{1}{r_k^{Q-1}} \int_{B_{p,\tau r_k} \setminus B_{p,\lambda r_k}} g(\nu_E, \phi_{p,k}\psi_{p,k})\; d\,|\partial E|_H\,.\end{aligned}$$

Hence, by equality (6.27) and (6.25) it follows

$$\begin{aligned}&\lambda^{Q-1} \limsup_{r_k \to 0^+} \frac{|\partial E|_H(B_{p,r_k})}{r_k^{Q-1}} \leq \tau^{Q-1}\,\theta^g_{Q-1}(\nu_E(p)) + \limsup_{r_k \to 0^+} |\partial E_{p,r_k}|_H(B_\tau \setminus \overline{B}_\lambda) \\ &+ \limsup_{r_k \to 0^+} \frac{|\partial E|_H\,(\delta_{r_k}(B_{p,\tau} \setminus B_{p,\lambda}))}{r_k^{Q-1}}\,.\end{aligned}$$

By virtue of the choice of λ and properties (2.46) we have

$$\limsup_{r_k\to 0^+} \frac{|\partial E|_H(B_{p,r_k})}{r_k^{Q-1}} \le \left(\frac{\tau}{\lambda}\right)^{Q-1} \theta^g_{Q-1}(\nu_E(p)) + \lambda^{1-Q}\,\sigma(B_\tau \setminus \overline{B}_\lambda)$$
$$+\lambda^{1-Q}\,|\partial E|_H\left(B_{p,\tau} \setminus \overline{B}_{p,\lambda}\right)\,. \tag{6.28}$$

It is possible to choose a sequence $(\lambda_k) \subset]0,\tau[$ such that $\sigma(\partial B_{\lambda_k}) = 0$ and $\lambda_k \to \tau$. Hence, from the last inequality it follows

$$\limsup_{r_k\to 0^+} \frac{|\partial E|_H(B_{p,r_k})}{r_k^{Q-1}} \le \theta^g_{Q-1}(\nu_E(p))\,. \tag{6.29}$$

It is a straightforward calculation from definition of the perimeter measure to notice that

$$|\partial E_{p,r}|_H(B_1) = \frac{|\partial E|_H(B_{p,r})}{r^{Q-1}}\,,$$

therefore by (6.23) and the semicontinuity of the total variation with respect to the weak* convergence of measures we have

$$\liminf_{r\to 0^+} \frac{|\partial E|_H(B_{p,r})}{r^{Q-1}} \ge \mathcal{H}^{q-1}_{|\cdot|}(F^{-1}(B_1)\cap \Pi_p) = \theta^g_{Q-1}(\nu_E(p))\,. \tag{6.30}$$

By virtue of (6.29) and (6.30) we can conclude that

$$\lim_{r\to 0^+} \frac{|\partial E|_H(B_{p,r})}{r^{Q-1}} = \theta^g_{Q-1}(\nu_E(p))\,,$$

so the thesis follows. □

Theorem 6.4.12 (Representation) *Let $E \subset \mathbb{G}$ be a set of locally H-finite perimeter and let $\mathbb{G}$ be a generating group. Then we can represent the perimeter measure as follows*

$$|\partial E|_H = \frac{\theta^g_{Q-1}(\nu_E)}{\omega_{Q-1}}\,\mathcal{S}^{Q-1} \llcorner \partial_{*H} E\,, \tag{6.31}$$

where $\mathcal{S}^{Q-1}$ and $\theta^g_{Q-1}(\nu_E)$ refer to the same homogeneous distance.

Proof. The perimeter measure is finite on bounded sets, then for a.e. $r > 0$ we have $|\partial E|_H(\partial B_{p,r}) = 0$ and $|\partial E|_H(B_{p,r}) = |\partial E|_H(D_{p,r})$. Then the family $\mathcal{C} = \{B_{p,r} \mid |\partial E|_H(B_{p,r}) = |\partial E|_H(D_{p,r})\}$ is fine at each $p \in \mathbb{G}$, i.e. defining $I_p = \{r \mid B_{p,r} \in \mathcal{C}\}$ we have $\inf I_p = 0$. In view of Theorem 6.4.11 it follows

$$\lim_{r\in I_p, r\to 0^+} \frac{|\partial E|_H(D_{p,r})}{r^{Q-1}} = \theta^g_{Q-1}(\nu_E(p))\,,$$

for $|\partial E|_H$-a.e. $p \in \mathbb{G}$. Now we note that any homogeneous distance has the property $\operatorname{diam}(B_r) = 2r$, for any $r > 0$. In fact, for any point $p \in \exp(V_1)$ we have $\delta_2 p = \exp(2 \ln p)$, hence the homogeneity of dilations allows us to get the required property. Taking into account (2.48), we can apply Theorems 2.10.17(2) and 2.10.18(1) of [55] to the measure $|\partial E|_H$ restricted to $\partial^*_H E$, so the proof follows by a standard argument, observing that $\nu \longrightarrow \theta^g_{Q-1}(\nu)$ is measurable and then it can be approximated by measurable steps functions. $\square$

6.5 Coarea formula

This section is devoted to the proof of the coarea formula for real valued Lipschitz maps on generating groups.

Theorem 6.5.1 *Let $\mathbb{G}$ be a sub-Riemannian group and let $u : \mathbb{G} \longrightarrow \mathbb{R}$ be a locally Lipschitz map. Then for a.e. $t \in \mathbb{R}$ we have*

$$\mathcal{H}^{Q-1}(u^{-1}(t) \setminus \partial_{*H} E_t) = 0 \qquad \textit{and} \tag{6.32}$$

$$\nu_{E_t}(p) = \frac{\nabla_H u(p)}{|\nabla_H u(p)|}\,, \tag{6.33}$$

for $\mathcal{H}^{Q-1}$-a.e. $p \in u^{-1}(t)$, where $E_t = \{x \in \mathbb{G} \mid u(x) > t\}$.

Proof. For every $t \in \mathbb{R}$ we denote by $\mathcal{D}_t$ the set of points $p \in u^{-1}(t)$ such that u is H-differentiable at p and $d_H u(p)$ is nonvanishing. In view of Theorem 6.3.1 and of (2.49) we derive that for a.e. $t \in \mathbb{R}$ we have $\mathcal{H}^{Q-1}(u^{-1}(t) \setminus \mathcal{D}_t) = 0$ and the set E_t has locally H-finite perimeter. Now we pick one of these t. The first thing we want to prove is that $\mathcal{D}_t \subset \partial^* E_t$, where $\partial^* E_t$ is the essential boundary (Definition 2.1.16). To see this we fix $p \in \mathcal{D}_t$ and observe that Definition 6.2.2 and Definition 6.4.1 yield

$$(E_t)_{p,r} = \Big\{ x \in \mathbb{G} \mid u_{p,r}(x) > 0 \Big\}\,. \tag{6.34}$$

The H-differentiability at p implies the uniform convergence on compact sets of $u_{p,r}$ to $d_H u(p)$ and this in turn yields the following L^1_{loc}-convergence

$$(E_t)_{p,r} \longrightarrow S^+_g\left(\nabla_H u(p)\right). \tag{6.35}$$

From previous limit we deduce the following

$$\frac{v_g(B_{p,r} \cap E_t)}{r^Q} = v_g\left(B_1 \cap (E_t)_{p,r}\right) \longrightarrow v_g\left(B_1 \cap S^+_g\left(\nabla_H u(p)\right)\right) > 0$$

and analogously

$$\frac{v_g(B_{p,r} \setminus E_t)}{r^Q} = v_g\left(B_1 \setminus (E_t)_{p,r}\right) \longrightarrow v_g\left(B_1 \cap S^-_g\left(\nabla_H u(p)\right)\right) > 0\,,$$

so our first claim is achieved. Using the fact that $\mathcal{H}^{Q-1}(u^{-1}(t) \setminus \mathcal{D}_t) = 0$ we also obtain

$$\mathcal{H}^{Q-1}(u^{-1}(t) \setminus \partial^* E_t) = 0\,. \tag{6.36}$$

Now we recall the general representation of the perimeter measure for a set E of H-finite perimeter

$$|\partial E|_H(A) = \int_{A\cap\partial^* E} \theta \; d\mathcal{H}^{Q-1} \tag{6.37}$$

where $\theta \geq c > 0$ is a Borel map and A is an arbitrary Borel set. This formula is proved in Theorem 4.2 of [5], where the more general context of metric measure spaces is considered. By formulae (2.48) and (6.37) we get

$$c\,\mathcal{H}^{Q-1}(\partial^* E_t \setminus \partial_{*H} E_t) \leq |\partial E_t|_H\,(\partial^* E_t \setminus \partial_{*H} E_t) = 0\,,$$

then $\mathcal{H}^{Q-1}(\partial^* E_t \setminus \partial_{*H} E_t) = 0$ and by (6.36) we obtain (6.32). In order to establish (6.33) for $\mathcal{H}^{Q-1}$-a.e. $p \in u^{-1}(t)$ we can limit ourselves to prove the formula for a point $p \in \mathcal{D}_t \cap \partial_{*H} E_t$. From now on, we denote by E the set E_t. Now we use the fact that $p \in \partial_{*H} E$. By Lemma 2.31 of [73] we obtain a constant c_0, only depending on the group, such that

$$|\partial E|_H(U_{p,r}) \leq c_0\, r^{Q-1} \tag{6.38}$$

for any $r \in (0, r_0)$, where $r_0 > 0$ depends on p. We recall that $U_{p,r}$ represents the open ball with respect to the CC-distance. Due to the fact that the CC-distance is a homogeneous distance there exists a constant $c_1 > 1$ such that $D_1 \subset U_{c_1}$ (Proposition 2.3.37), then from formulae (6.18) and (6.38) we deduce that

$$|\partial E_{p,r}|_H(D_1) = \frac{|\partial E|_H(B_{p,r})}{r^{Q-1}} \leq \frac{|\partial E|_H(U_{p,c_1 r})}{r^{Q-1}} \leq\; c_0\, c_1^{Q-1} \tag{6.39}$$

for any $0 < r < r_0/c_1$, where D_1 is the closed unit ball with respect to the homogeneous distance d. By the weak*-compactness of Radon measures, see Theorem 1.59 of [6], there exists an infinitesimal sequence $(r_k) \subset (0, r_0/c_1)$ and a Radon measure σ such that

$$|\partial E_{p,r_k}|_H \llcorner D_1 \;\rightharpoonup\; \sigma \quad \text{as} \quad r_k \to 0^+\,.$$

By the finiteness of σ there exists $\tau \in (0,1)$ such that $\sigma(\mathrm{Fr}(B_\tau)) = 0$. By the uniform estimate (6.39) and the L^1_{loc} convergence (6.35) we derive the following weak*-convergence

$$\nu_{E_{p,r_k}}\, |\partial E_{p,r_k}|_H \llcorner D_1 \;\rightharpoonup\; \frac{\nabla_H u(p)}{|\nabla_H u(p)|}\, |\partial S_g^+\left(\nabla_H u(p)\right)|_H \llcorner D_1 \quad \text{as} \quad r_k \to 0^+\,.$$

By Proposition 1.62(b) of [6] it follows that

$$\int_{B_\tau} \langle \nu_{E_{p,r_k}}, \nu_E(p)\rangle\, d|\partial E_{p,r_k}|_H \longrightarrow \left\langle \frac{\nabla_H u(p)}{|\nabla_H u(p)|}, \nu_E(p)\right\rangle |\partial S_g^+\left(\nabla_H u(p)\right)|_H(B_\tau). \tag{6.40}$$

Using formula (2.45) and the homogeneity of d, it is a direct calculation to obtain

$$\int_{B_\tau} \langle \nu_{E_{p,r_k}}, \nu_E(p)\rangle \, d|\partial E_{p,r_k}|_H = \frac{1}{r_k^{Q-1}} \int_{B_{p,\tau r_k}} \langle \nu_E, \nu_E(p)\rangle \, d|\partial E|_H \,. \tag{6.41}$$

By definition of H-reduced boundary we know that

$$\lim_{r_k \to 0^+} \fint_{B_{p,\tau r_k}} \langle \nu_E, \nu_E(p)\rangle \, d|\partial E|_H = 1,$$

hence the limit (6.40) and equality (6.41) imply

$$\begin{aligned}
\int_{B_\tau} \langle \nu_{E_{p,r_k}}, \nu_E(p)\rangle \, d|\partial E_{p,r_k}|_H &= \left(\fint_{B_{p,\tau r_k}} \langle \nu_E, \nu_E(p)\rangle \, d\,|\partial E|_H \right) \frac{|\partial E|_H(B_{p,\tau r_k})}{r_k^{Q-1}} \\
&= \left(\tau^{Q-1} + \mathrm{o}(1)\right) |\partial E_{p,r_k}|_H(B_1) \longrightarrow \left\langle \frac{\nabla_H u(p)}{|\nabla_H u(p)|}, \nu_E(p) \right\rangle |\partial S_g^+ \left(\nabla_H u(p)\right)|_H(B_\tau).
\end{aligned}$$

By the invariance of $S_g^+\left(\nabla_H u(p)\right)$ under dilations we get

$$\lim_{r_k \to 0^+} |\partial E_{p,r_k}|_H(B_1) = \left\langle \frac{\nabla_H u(p)}{|\nabla_H u(p)|}, \nu_E(p) \right\rangle |\partial S_g^+ \left(\nabla_H u(p)\right)|_H(B_1)\,,$$

and the lower semicontinuity of the perimeter measure yields

$$\liminf_{r_k \to 0^+} |\partial E_{p,r_k}|_H(B_1) \geq |\partial S_g^+ \left(\nabla_H u(p)\right)|_H(B_1)\,.$$

By the last two limits we obtain that

$$1 \leq \left\langle \frac{\nabla_H u(p)}{|\nabla_H u(p)|}, \nu_E(p) \right\rangle$$

then the thesis follows. □

Theorem 6.5.2 (Generalized coarea formula) *Let $u : \mathbb{G} \longrightarrow \mathbb{R}$ be a locally Lipschitz map, where $\mathbb{G}$ is a generating group. Then for any nonnegative measurable map $h : \mathbb{G} \longrightarrow \overline{\mathbb{R}}$ we have*

$$\int_{\mathbb{G}} h(w)\, |\nabla_H u|(w)\, dv_g(w) = \int_{\mathbb{R}} \int_{u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_{E_t}(w))}{\omega_{Q-1}}\, h(w)\, d\mathcal{S}^{Q-1}(w)\, dt\,, \tag{6.42}$$

where $\mathcal{S}^{Q-1}$, θ^g_{Q-1} refer to the same homogeneous distance and ν_{E_t} is the generalized inward normal to the set $E_t = \{x \in \mathbb{G} \mid u(x) > t\}$.

PROOF. By the a.e. differentiability of Lipschitz maps it is not difficult to see that $|D_H u| = |\nabla_H u|\, v_g$, where $D_H u$ is the distributional derivative of u regarded as a measure. Thus, formulae (2.49) and (6.31) immediately yield

$$\int_U |\nabla_H u|\, dv_g = \int_{\mathbb{R}} \int_{\partial_{*H} E_t \cap U} \frac{\theta^g_{Q-1}(\nu_{E_t})}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}\, dt \tag{6.43}$$

Now, by (6.32) the previous formula becomes

$$\int_U |\nabla_H u|\, dv_g = \int_{\mathbb{R}} \int_{u^{-1}(t) \cap U} \frac{\theta^g_{Q-1}(\nu_{E_t})}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}\, dt \tag{6.44}$$

The Borel regularity of the spherical Hausdorff measure yields

$$\int_A |\nabla_H u|\, dv_g = \int_{\mathbb{R}} \int_{u^{-1}(t) \cap A} \frac{\theta^g_{Q-1}(\nu_{E_t})}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}\, dt$$

for any measurable set $A \subset \mathbb{G}$. Finally, taking an increasing sequence of nonnegative step functions which converges pointwise to a nonnegative measurable map h and applying the Beppo Levi Convergence Theorem, the thesis follows. □

In the next theorem we show that the general coarea formula (6.42) has a simpler form in rotational groups.

Theorem 6.5.3 *Let $\mathbb{G}$ be an $\mathcal{R}$-rotational group endowed with an $\mathcal{R}$-invariant homogeneous distance and let $u : \mathbb{G} \longrightarrow \mathbb{R}$ be a locally Lipschitz map. Then for any nonnegative measurable map $h : \mathbb{G} \longrightarrow \overline{\mathbb{R}}$ we have*

$$\int_{\mathbb{G}} h(w)\, |\nabla_H u|(w)\, dv_g(w) = \frac{\alpha_{Q-1}}{\omega_{Q-1}} \int_{\mathbb{R}} \int_{u^{-1}(t)} h(w)\, d\mathcal{S}^{Q-1}(w)\, dt\,, \tag{6.45}$$

where α_{Q-1} is given by Proposition 5.2.5 and it is referred to the $\mathcal{R}$-invariant distance together with $\mathcal{S}^{Q-1}$.

PROOF. Since the generalized inward normal of a set of H-finite perimeter takes values in $H\mathbb{G}$, then formula (6.45) follows from Proposition 5.2.5 and (6.42). □

The coarea formula can be particularized in Heisenberg groups, which are rotational groups and possess the homogeneous distance d_∞ where the factor α_{Q-1} is computed explicitly.

Corollary 6.5.4 *Let ρ be the CC-distance in $\mathbb{H}^{2n+1}$ and let $u : \mathbb{H}^{2n+1} \longrightarrow \mathbb{R}$ be a locally Lipschitz map. We consider the graded metric g associated to the basis of Proposition 5.1.8. Then for any nonnegative measurable map $h : \mathbb{H}^{2n+1} \longrightarrow \overline{\mathbb{R}}$ we have*

$$\int_{\mathbb{H}^{2n+1}} h(w)\, |\nabla_H u|(w)\, dv_g(w) = \frac{\alpha_{Q-1}}{\omega_{Q-1}} \int_{\mathbb{R}} \int_{u^{-1}(t)} h(w)\, d\mathcal{S}^{Q-1}_\rho(w)\, dt\,, \tag{6.46}$$

where α_{Q-1} is given by Proposition 5.2.5 and it is referred to ρ.

PROOF. By Proposition 5.1.8 the group $\mathbb{H}^{2n+1}$ is $\mathcal{R}$-rotational and the class $\mathcal{R}$ of horizontal isometries is described in the same proposition. By Proposition 5.1.12 the CC-distance with respect to g is $\mathcal{R}$-invariant, then we can apply Theorem 6.5.3 and obtain formula (6.46). □

Corollary 6.5.5 *Let d_∞ be the distance introduced in Example 5.1.11 and consider a locally Lipschitz map $u : \mathbb{H}^{2n+1} \longrightarrow \mathbb{R}$. We refer to the graded metric g on $\mathbb{H}^{2n+1}$ associated to the basis of Remark 2.3.27. Then for any nonnegative measurable map $h : \mathbb{H}^{2n+1} \longrightarrow \overline{\mathbb{R}}$ we have*

$$\int_{\mathbb{H}^{2n+1}} h(w)\, |\nabla_H u|(w)\, dv_g(w) = \frac{2\,\omega_{2n-1}}{\omega_{Q-1}} \int_{\mathbb{R}} \int_{u^{-1}(t)} h(w)\, d\mathcal{S}_d^{Q-1}(w)\, dt\,. \tag{6.47}$$

PROOF. As we have observed in the proof of Corollary 6.5.4 the Heisenberg group with the metric g is $\mathcal{R}$-invariant. From Example 5.1.11 we know that our distance d is $\mathcal{R}$-invariant, so by Proposition 5.2.5 we get a constant metric factor α_{Q-1}. In view of Example 5.2.8 we know that $\alpha_{Q-1} = 2\omega_{2n-1}$, so applying (6.45) we get (6.47). □

6.6 Characteristic set of C^1 hypersurfaces

In this section we utilize the weak Sard-type Theorem of Section 6.3 in order to study the characteristic set of C^1 hypersurfaces on sub-Riemannian groups.

Let us fix some notation that will be used throughout the section. We consider an adapted orthonormal basis $(W_1, \ldots, W_q)$ of the stratified algebra $\mathcal{G}$ and we define the associated graded coordinates by $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ (see Definition 2.3.43). We denote by ρ the CC-distance of $\mathbb{G}$ (see Definition 2.3.33). We fix a map $u : O \longrightarrow \mathbb{R}$ of class C^1 on the open bounded set $O \subset \mathbb{G}$, with $e \in O$ and $u(e) = 0$. We assume that there exists j_0 such that $W_{j_0}u(p) \neq 0$ for any $p \in O$. Hence $\Sigma = u^{-1}(0)$ is a C^1 hypersurface in O and $e \in \Sigma$. We recall that for any $j = 1, \ldots, \iota$ the subspace $H_p^j\mathbb{G} \subset T_p\mathbb{G}$ is a translation of $H_e\mathbb{G}$ at $p \in \mathbb{G}$ (Definition 2.3.16).

Lemma 6.6.1 *In the notation above, we have*

$$C(\Sigma) = \{p \in \Sigma \mid d_H u(p) : T_p\mathbb{G} \longrightarrow \mathbb{R} \text{ is the null map}\}\,.$$

PROOF. By definition of Σ it follows that $du(p)(\nu) = 0$ if and only if $\nu \in T_p\Sigma$. Now assume that $p \in C(\Sigma)$. Then $H_p\mathbb{G} \subset T_p\Sigma$, so $du(p)(v_1) = 0$ for any $v_1 \in V_1(p)$. By Proposition 3.2.8 it follows that $d_H u(p)(v) = du(p)(v_1)$ whenever $v = \sum_{j=1}^{\iota} v_j$ and $v_j \in H_p^j\mathbb{G}$ for any $j = 1, \ldots \iota$. Therefore $d_H u(p)$ is the null map. Viceversa, if $d_H u(p)$ is the null map, then $du(p)(v_1) = 0$ whenever $v_1 \in H_p\mathbb{G}$, namely $H_p\mathbb{G} \subset T_p\Sigma$. □

Theorem 6.6.2 *Let $\mathbb{G}$ be a sub-Riemannian group. Then, for any C^1 hypersurface Σ of an open subset $\Omega \subset \mathbb{G}$ we have $\mathcal{H}_\rho^{Q-1}(C(\Sigma)) = 0$, where ρ is the CC-distance.*

Proof. Notice that any C^1 hypersurface of Ω can be written as a countable union of C^1 bounded pieces S_i, with $i \in \mathbb{N}$. In view of Proposition 2.3.39 the translations are isometries with respect to ρ and it is also easy to observe that $l_p C(S_i) = C(l_p(S_i))$ for any $p \in \mathbb{G}$. It follows that

$$\mathcal{H}_\rho^{Q-1}\left(C(S_i)\right) = \mathcal{H}_\rho^{Q-1}\left(l_p C(S_i)\right) = \mathcal{H}_\rho^{Q-1}\left(C(l_p S_i)\right).$$

So the thesis follows if we prove that for a suitable small hypersurface $\Sigma \ni e$ we have $\mathcal{H}_\rho^{Q-1}\left(C(\Sigma)\right) = 0$. To do this, it is not restrictive to assume the hypotheses and the notation fixed in the beginning of the section. Let $\tilde{O} = F^{-1}(O) \subset \mathbb{R}^q$ and $\tilde{\Sigma} = F^{-1}(\Sigma) \subset \mathbb{R}^q$ and observe that $\tilde{u}^{-1}(0) = \tilde{\Sigma}$ where $\tilde{u} = u \circ F : \tilde{O} \longrightarrow \mathbb{R}$ is a C^1 map. We define the hyperplane

$$\Pi_0 = \Big\{ x \in \mathbb{R}^q \mid x_{j_0} = 0 \Big\}.$$

By the implicit function theorem there exists an open subset $A \subset \Pi_0$ containing the origin and a C^1 map $\varphi : A \longrightarrow \mathbb{R}$ such that $\tilde{u}(\xi, \varphi(\xi)) = 0$ for any $\xi \in A$, where we have posed $\xi = \sum_{j \neq j_0} x_j \, e_j$, $\;(\xi, \varphi(\xi)) = \sum_{j \neq j_0} \xi_j \, e_j + \varphi(\xi) \, e_{j_0}$ and (e_j) is the canonical basis of $\mathbb{R}^q$. The map $\phi : A \longrightarrow \tilde{O}$, defined by $\xi \longrightarrow (\xi, \varphi(\xi))$ has the image contained in $\tilde{\Sigma}$.

Now we define $G : \mathbb{R} \times A \longrightarrow \mathbb{G}$ by $(t, \xi) \longrightarrow l_{\exp tW_{j_0}}\left(F(\phi(\xi))\right)$ and we note that

$$\partial_t G(0) = W_{j_0}(e) \quad \text{and} \quad \partial_{\xi_j} G(0) = W_j(e) + \varphi_{\xi_j}(0)\, W_{j_0}(e),$$

for any $j \neq j_0$. It follows that there exist $\varepsilon > 0$, $\tilde{A} \subset A$ and $U \subset \mathbb{G}$, with $0 \in \tilde{A}$ and $e \in U$, such that

$$G : (-\varepsilon, \varepsilon) \times \tilde{A} \longrightarrow U$$

is invertible. Let us consider the projection $\mathfrak{p}_1(x) = x_1$ for any $x \in \mathbb{R}^q$ and define the C^1 map $\tau : U \longrightarrow (-\varepsilon, \varepsilon)$ by $\tau(p) = \mathfrak{p}_1\left(G^{-1}(p)\right)$. The map $\tau \circ F^{-1}$ is clearly Lipschitz with respect to the Euclidean distance. It follows that τ is Lipschitz with respect to the Riemannian distance of $\mathbb{G}$. Observing that in general $\rho \geq d_g$, where d_g is the Riemannian distance, we obtain that τ is Lipschitz with respect to ρ. Up to a choice of a smaller neighbourhood of the origin $\tilde{O}$ we can suppose that $\phi(\tilde{A}) = \tilde{\Sigma}$. Now, in view of Theorem 6.3.1 for a.e. $t \in (-\varepsilon, \varepsilon)$ we have $\mathcal{H}_\rho^{Q-1}(\tau^{-1}(t) \cap S) = 0$, where

$$S = \{ p \in U \mid d_H \tau(p) \text{ is vanishing} \}.$$

By the fact that G is invertible it follows that $d\tau(p)$ is nonvanishing at any $p \in U$ and by Lemma 6.6.1 we have $C\left(\tau^{-1}(t)\right) = \tau^{-1}(t) \cap S$, therefore it follows

$$\mathcal{H}_\rho^{Q-1}\left(C(\tau^{-1}(t))\right) = 0,$$

for a.e. $t \in (-\varepsilon, \varepsilon)$. By definition of $\tau^{-1}(t)$ we have

$$\tau^{-1}(t) = \left\{ p \in U \mid G^{-1}(p) \in \{t\} \times \tilde{A} \right\} = l_{\exp t W_{j_0}} \left(F(\phi(\tilde{A})) \right) = l_{\exp t W_{j_0}} (\Sigma) \ .$$

Thus, for a.e. $t \in (-\varepsilon, \varepsilon)$ we have

$$\begin{aligned} 0 &= \mathcal{H}_\rho^{Q-1} \left(C(\tau^{-1}(t)) \right) = \mathcal{H}_\rho^{Q-1} \left(C \left(l_{\exp t W_{j_0}} \Sigma \right) \right) \\ &= \mathcal{H}_\rho^{Q-1} \left(l_{\exp t W_{j_0}} (C(\Sigma)) \right) = \mathcal{H}_\rho^{Q-1} (C(\Sigma)) \end{aligned}$$

and the thesis follows. $\square$

Chapter 7

Blow-up Therems on regular hypersurfaces

In the previous chapter we have seen how the validity of a Blow-up Theorem implies the representation of the perimeter measure by the spherical Hausdorff measure built with respect to any homogeneous distance. By this fact and Theorem 6.5.1 we have also established a generalized coarea formula for scalar Lipschitz maps with respect to the CC-distance. All these results hold in generating groups (Definition 6.4.8), so if we want to extend their validity to every sub-Riemannian group we have to prove that any sub-Riemannian group is generating. Unfortunately, this seems to be a difficult open issue.

In this chapter we tackle this problem considering more regular domains and hypersurfaces. Under these strengthened conditions we will prove a Blow-up Theorem for the Riemannian measure of C^1 hypersurfaces of arbitrary sub-Riemannian groups (Theorem 7.1.2). This leads us to a formula to represent the spherical Hausdorff measure of a C^1 hypersurface $\Sigma \subset \mathbb{G}$, where $\mathbb{G}$ is an arbitrary sub-Riemannian group. Our formula is as follows

$$\mathcal{S}^{Q-1}(\Sigma) = \int_\Sigma \frac{\omega_{Q-1}}{\theta^g_{Q-1}(\nu_H(x))}\, |\nu_H(x)|\, d\sigma_g(x)\,, \tag{7.1}$$

where both the spherical Hausdorff measure $\mathcal{S}^{Q-1}$ and the metric factor $\theta^g_{Q-1}(\nu_H(x))$ are considered with respect to the same homogeneous distance, $\sigma_g = \mathcal{H}^{q-1}_{d_g}$ and d_g is the Riemannian distance associated to the graded metric g. Formula (7.1) has been obtained in [126] through the limit

$$\lim_{r\to 0} \frac{\sigma_g(\Sigma \cap B_{p,r})}{r^{Q-1}} = \frac{\theta^g_{Q-1}(\nu_H(p))}{|\nu_H(p)|} \tag{7.2}$$

at noncharacteristic points $p \in C(\Sigma)$, hence its validity holds in principle for hypersurfaces where the characteristic set is $\mathcal{H}^{Q-1}$-negligible. By Proposition 2.2.10

we notice that (7.1) formally suggests that the set of characteristic points must be $\mathcal{H}^{Q-1}$-negligible, as we have proved in Theorem 6.6.2. Just by this last result in Theorem 7.1.3 we are able to extend the validity of (7.1) to any C^1 hypersurface of an arbitrary sub-Riemannian group. Formula (7.1) also fits a general estimate for the Hausdorff dimension of hypersurfaces in CC-spaces proved by M. Gromov in [86]. Precisely, if Q is the Hausdorff dimension of an equiregular CC-space with topological dimension q, then the Hausdorff dimension of a compact subset with topological dimension $q-1$ is always greater than or equal to $Q-1$, see formula $(*)$ at p.152 of [86]. By virtue of (7.1) the previously mentioned estimate becomes an equality when the equiregular CC-space is a sub-Riemannian group and the compact subset of topological dimension $q-1$ is of class C^1. In fact, formula (7.1) implies that these compact subsets have $\mathcal{S}^{Q-1}$- finite measure. In particular we have proved that the intrinsic Hausdorff dimension of C^1 hypersurfaces of sub-Riemannian groups is exactly $Q-1$.

A further consequence of Theorem 7.1.2 is a version of the Riemannian coarea formula in sub-Riemannian groups

$$\int_{\mathbb{G}} h(w)\,|\nabla_H u|(w)\,dv_g(w) = \int_{\mathbb{R}}\int_{u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_H(w))}{\omega_{Q-1}}\,h(w)\,d\mathcal{S}^{Q-1}(w)\,dt\,, \tag{7.3}$$

where $u:\mathbb{G}\longrightarrow\mathbb{R}$ is a locally Lipschitz map with respect to the Riemannian distance d_g and $h:\mathbb{G}\longrightarrow\overline{\mathbb{R}}$ is a nonnegative measurable map. Another important tool to get (7.3) is Theorem 6.3.1, by which the set of characteristic points of a.e. level set is $\mathcal{H}^{Q-1}$-negligible. The coarea formula (7.3) was first obtained by P. Pansu in the Heisenberg group, using the Carnot-Carathéodory distance, [152], and it was extended to general stratified groups for smooth functions by J. Heinonen, [92]. In the case $\mathbb{G}$ is an Euclidean space $\mathbb{E}^n$, with the classical Riemannian metric, formula (7.3) yields an extension of the classical Euclidean coarea formula

$$\int_{\mathbb{E}^n} h(x)\,|\nabla u|(x)\,dx = \int_{\mathbb{R}}\int_{u^{-1}(t)} \frac{\theta_{n-1}(\nu(x))}{\omega_{n-1}}\,h(x)\,d\mathcal{H}^{n-1}_{\|\cdot\|}(x)\,dt\,,$$

where $\theta_{n-1}(\nu(x))$ and $\mathcal{H}^{n-1}_{\|\cdot\|}$ are considered with respect to the same Banach norm, ∇u is the Euclidean gradient and ν is the unit normal to the level set. We stress the fact that our Blow-up Theorem for C^1 hypersurfaces yields (7.3) in any sub-Riemannian group, provided that the map u is Lipschitz with respect to the Riemannian distance. However, in the sub-Riemannian context it would be natural to assume that u is Lipschitz with respect to the CC-distance (or equivalently any homogeneous distance). As we have seen in Chapter 6, the coarea formula under this weaker conditions holds for generating groups, where a Blow-up Theorem for the perimeter measure holds. We also mention that another type of coarea formula for metric space valued Lipschitz maps on Euclidean normed spaces (or rectifiable subsets) is proved in [7], where the role of the metric factor is replaced by the notion of coarea factor, corresponding to Definition 6.1.3. In Theorem 7.3.1, using the same technique of the Blow-up

Theorem for C^1 hypersurfaces, we obtain different blow-up estimates adapted to the case of $C^{1,1}$ hypersurfaces. As a result, we derive a sharp upper estimate of the Hausdorff dimension of the characteristic set of $C^{1,1}$ hypersurfaces in 2-step graded groups endowed with a homogeneous distance. This result fits the ones obtained in [12] for the class of Heisenberg groups.

A further application of the blow-up technique together with the method used to prove Theorem 6.4.11 allows us to achieve Theorem 7.4.2, where we obtain the Blow-up Theorem for the perimeter measure of C^1 domains. Here the C^1 regularity of the set permits us to avoid the use of any isoperimetric inequality, which is an essential tool when the set is of H-finite perimeter, [47], [71] and [73]. Precisely, Theorem 7.4.2 holds in graded groups endowed with a homogeneous distance. This class of groups clearly encompasses all sub-Riemannian groups, where it is always possible to consider the CC-distance as a homogeneous distance. A first important consequence of Theorem 7.4.2 arises in connection with a conjecture stated in [42]. In this paper it is shown that every C^2 compact domain E of the Heisenberg group satisfies the following estimates

$$c\,\mathcal{H}^{Q-1}(\partial E) \le P_H(E,\mathbb{H}^{2n+1}) \le C\,\mathcal{H}^{Q-1}(\partial E)\,, \tag{7.4}$$

where Q is the Hausdorff dimension of $\mathbb{H}^{2n+1}$ and $c, C > 0$ are dimensional constants. Here the authors of [42] conjecture the validity of estimates (7.4) for any sub-Riemannian group, under suitable regularity assumptions on the domain E. By the Blow-up for the perimeter measure of C^1 domains (Theorem 7.4.2) and the fact that characteristic points are $\mathcal{H}^{Q-1}$-negligible for C^1 hypersurfaces (Theorem 6.6.2) we positively answer the conjecture extending (7.4) to any sub-Riemannian group and any C^1 closed set E as follows

$$\frac{\underline{\theta}_{Q-1}}{\omega_{Q-1}}\ \mathcal{H}^{Q-1}(\partial E\cap\Omega)\ \le P_H(E,\Omega) \le \frac{2^Q\,\overline{\theta}_{Q-1}}{\omega_{Q-1}}\ \mathcal{H}^{Q-1}(\partial E\cap\Omega)\,, \tag{7.5}$$

where $\Omega \subset \mathbb{G}$ is an arbitrary bounded open set. In addition, we can provide explicit formulae for the dimensional constants $\underline{\theta}_{Q-1}$, $\overline{\theta}_{Q-1}$, which are related to the graded metric used for the perimeter measure and to the homogeneous distance used to build the Hausdorff measure:

$$\underline{\theta}_{Q-1} = \inf_{\nu\in V_1} \theta^g_{Q-1}(\nu) \quad \text{and} \quad \overline{\theta}_{Q-1} = \sup_{\nu\in V_1} \theta^g_{Q-1}(\nu)\,.$$

Formula (7.5) is a straightforward consequence of a more precise result, corresponding to Theorem 7.4.4, where we obtain the following representation of the perimeter measure

$$|\partial E|_H = \frac{\theta^g_{Q-1}(\nu_H)}{\omega_{Q-1}}\,\mathcal{S}^{Q-1}\llcorner\partial E \tag{7.6}$$

for any C^1 closed subset E of a sub-Riemannian group. We mention that in the particular case $\mathbb{G} = \mathbb{H}^{2n+1}$ and $d = d_\infty$ (see Example 2.3.38), the representation (7.6) first appeared in [71]. The general validity of (7.6) for any C^1 closed subset of an arbitrary sub-Riemannian group and with respect to an arbitrary homogeneous distance answers a question that has been raised in [72] and [73]. This is the question of finding a relation between the perimeter measure of a set and the spherical Hausdorff measure of its boundary in general sub-Riemannian groups, under suitable regularity assumptions. By Theorem 6.6.2, Proposition 7.4.3 and formulae (2.45), (7.6) we obtain an intrinsic version of the divergence theorem for C^1 subsets (7.54), that becomes

$$\int_E \mathrm{div}_H \phi \, dv_g = -\frac{\alpha_{Q-1}}{\omega_{Q-1}} \int_{\partial E} \left\langle \phi, \frac{\nu_H}{|\nu_H|} \right\rangle d\mathcal{S}_\rho^{Q-1} .$$

on $\mathcal{R}$-rotational groups, where ρ is the CC-distance with respect to the graded metric that makes the group $\mathcal{R}$-rotational, α_{Q-1} is the constant metric factor (see Proposition 5.2.5) and ν_H is the horizontal normal. Another immediate consequence of (7.6) joined with (7.16) is the relation

$$|\partial E|_H = |\nu_H| \, \sigma_g \llcorner \partial E \,, \tag{7.7}$$

that connects the perimeter measure of a C^1 set with the Riemannian surface measure of its boundary. It is interesting to notice that the previous formula depends only on the graded metric of $\mathbb{G}$ and the horizontal subbundle $H\mathbb{G}$. We point out that if we consider the set E as a subset of $\mathbb{R}^q$ with respect to a system of graded coordinates we can exploit the classical divergence theorem for C^1 sets obtaining a version of (7.7), as it is shown in [31] concerning the general context of CC-spaces. In this case the integration by parts in $\mathbb{R}^q$ yields the following term in place of the right hand side of (7.7)

$$\int_{\partial E} \Big(\sum_{j=1}^m \langle \nu, X_j \rangle^2 \Big)^{1/2} \mathcal{H}_{|\cdot|}^{q-1} \tag{7.8}$$

where $\mathcal{H}_{|\cdot|}^{q-1}$ is the $q-1$ dimensional Euclidean Hausdorff measure, ν is the unit normal to E and X_i are the vector fields in $\mathbb{R}^q$ which span the horizontal subbundle $H\mathbb{G}$. But the term (7.8) is not immediately recognizable as an intrinsic object of the group, due to the presence of the Euclidean scalar product and the measure $\mathcal{H}_{|\cdot|}^{q-1}$. We also point out that (2.45), (7.7), Proposition 7.4.3 and Theorem 6.6.2 imply the following version of the intrinsic divergence theorem for C^1 sets of sub-Riemannian groups

$$\int_E \mathrm{div}_H \phi \, dv_g = - \int_{\partial E} \langle \phi, \nu_H \rangle \, d\sigma_g \,. \tag{7.9}$$

Let us give a synthetic overview of the chapter.

The main result of Section 7.1 is Theorem 7.1.2, where the blow-up with respect to the Riemannian measure of C^1 hypersurfaces is proved in any graded group endowed with a homogeneous distance. As a consequence, in Theorem 7.1.3 we obtain the relationship between the $Q-1$ dimensional spherical Hausdorff measure and the Riemannian measure of C^1 hypersurfaces, (7.16), (7.17). We recall that the validity of the previously mentioned formulae also relies on Theorem 6.6.2, where we have proved that characteristic points are $\mathcal{H}^{Q-1}$-negligible.

In Section 7.2 we prove the coarea formula (7.3). The main results used for its proof are the representation formula (7.16) and Theorem 6.3.1. In Corollary 7.2.3 we extend the classical Euclidean coarea formula to the case when the $n-1$ dimensional Hausdorff measure is built with a Banach norm on $\mathbb{E}^n$. As we have seen in the previous chapter the coarea formula takes a simpler form in rotational groups. Here the same simplification occurs in Theorem 7.2.4 and analogous theorems could be stated for Heisenberg groups. Finally, in (7.24) we present a formulation of coarea formula where only the restriction of the graded metric onto the horizontal subbundle is involved. This presentation agrees with the philosophical principle of sub-Riemannian Geometry according to which all information is contained in the horizontal subbundle and in all its related structures.

The relevant result of Section 7.3 is the application of the blow-up technique developed in Theorem 7.1.2 to the characteristic points of the hypersurface. This is done in Theorem 7.3.1, where $C^{1,1}$ hypersurfaces in groups of step 2 are considered. By this theorem it is easily proved that the $Q-2$ dimensional Hausdorff measure of the characteristic set is comparable with its Riemannian surface measure (7.34) and the upper estimate (7.35) of its Hausdorff dimension follows. Finally, by results of [12] for any $\alpha > 0$ it is possible to find a $C^{1,1}$ hypersurface Σ_α in the Heisenberg group with Hausdorff dimension greater than or equal to $Q-2-\alpha$, this in turn implies that our upper estimate (7.35) is sharp.

In Section 7.4 we prove that noncharacteristic points of the boundary of a C^1 set are in the H-reduced boundary. This is obtained by Proposition 7.4.1 and Theorem 7.4.2, where we also show that at these points the Blow-up Theorem holds, namely, limits (7.41) and (7.42) hold. The proof of these limits is the main result of the section. The C^1 regularity of the set E allows us to use also the notion of horizontal normal ν_H to ∂E. In Proposition 7.4.3 we check that ν_H has the same direction of the generalized inward normal ν_E, as one can expect. By the previously mentioned Blow-up Theorem and the key result of Theorem 6.6.2 we easily achieve Theorem 7.4.4, where formula (7.6) is proved. This formula joined with Theorem 6.6.2 yields (7.5), and joined with (7.16), yields (7.7). As an immediate consequence, we obtain the divergence theorems (7.54), (7.55) and (7.56).

7.1 Blow-up of the Riemannian surface measure

Throughout the chapter we will denote by $\mathbb{M}$ a graded group endowed with both a homogeneous distance and a graded metric. The symbol $\mathbb{G}$ will denote a sub-Riemannian group. The Riemannian measure of hypersurfaces is $\sigma_g = \mathcal{H}^{q-1}_{d_g}$, where d_g is the Riemannian distance corresponding to the graded metric g.

Lemma 7.1.1 *Let $O \subset \mathbb{M}$ be an open bounded neighbourhood of $e \in \mathbb{G}$. We consider a C^1 map $u : O \longrightarrow \mathbb{R}$ such that $u(e) = 0$ and we assume that $du(p) : T_p\mathbb{M} \longrightarrow \mathbb{R}$ is surjective for every $p \in O$. Then, for every $p \in \Sigma = u^{-1}(0)$ and $Z \in T_p\mathbb{M}$ we have*

$$d_H u(p)(Z) = |\nabla u(p)| \, \langle \nu_H(p), Z \rangle_p \quad \textit{and} \quad \nu_H(p) = \frac{\nabla_H u(p)}{|\nabla u(p)|}\,, \tag{7.10}$$

where $\nu_H(p)$ is the orthogonal projection of $\nu(p)$ onto $H_p\mathbb{M}$ and $\nu(p)$ is the unit normal to Σ.

Proof. By Proposition 3.2.8 we know that $d_H u(p)(v) = du(p)(v_1)$ for every $v = \sum_{j=1}^{\iota} v_j$, where $v_j \in H_p^j\mathbb{G}$. It is standard to recognize that $\nu(p) = \nabla u(p)/|\nabla u(p)|$, where $\nu(p)$ is the unit normal to Σ. Now, by definition of horizontal normal $\nu_H(p)$ (Definition 2.2.9) and horizontal gradient $\nabla_H u(p)$ (Definition 2.2.7), equations of (7.10) easily follow. □

Theorem 7.1.2 (Blow-up) *Let Σ be a hypersurface of class C^1 in $\Omega \subset \mathbb{M}$ and let $p \in \Sigma$ such that $\nu_H(p) \neq 0$. Then we have*

$$\lim_{r \to 0} \frac{\sigma_g(\Sigma \cap B_{p,r})}{r^{Q-1}} = \frac{\theta^g_{Q-1}(\nu_H(p))}{|\nu_H(p)|}\,, \tag{7.11}$$

Proof. Up to a translation we can suppose that p is the unit element $e \in \mathbb{G}$. By Lemma 7.1.1 we can represent Σ by a C^1 map $u : O \longrightarrow \mathbb{R}$, where $O \subset \Omega$ is an open bounded neighbourhood of e and $d_H u(p)$ is surjective for any $p \in O$. We fix a system of graded coordinates (F, W) where $W_1(e) = \nu_H(e)/|\nu_H(e)|$. Then, taking into account that $(W_1, \dots, W_m)$ spans $H_e\mathbb{G}$ and formulae (7.10), we have

$$W_1 u(e) = d_H u(e)(W_1) = |\nabla u(e)|\, |\nu_H(e)| \langle W_1, W_1 \rangle = |\nabla_H u(e)| \tag{7.12}$$

$$W_j u(e) = d_H u(e)(W_j) = |\nabla u(e)|\, |\nu_H(e)| \langle W_1, W_j \rangle_e = 0 \tag{7.13}$$

for any $j = 2, \dots, m$. Let us define $\tilde{u} = u \circ F : \tilde{O} \longrightarrow \mathbb{R}$, where $\tilde{O} = F^{-1}(O) \subset \mathbb{R}^q$. We note that $\tilde{u}^{-1}(0) = \tilde{\Sigma}$ with $\tilde{\Sigma} = F^{-1}(\Sigma) \subset \mathbb{R}^q$. We define the hyperplane

$$\Pi_1 = \Big\{ x \in \mathbb{R}^q \mid x_1 = 0 \Big\}\,.$$

By the implicit function theorem there exists an open subset $A \subset \Pi_1$ containing the origin and a C^1 map $\varphi : A \longrightarrow \mathbb{R}$ such that $\tilde{u}\,(\varphi(\xi), \xi) = 0$ for any $\xi \in A$, where we have posed $\xi = \sum_{j=2}^{q} x_j\, e_j$ and (e_j) is the canonical basis of $\mathbb{R}^q$. Now we consider the parametrization $\tilde\phi : A \longrightarrow \tilde\Sigma$, $\xi \longrightarrow (\varphi(\xi), \xi)$ and the map $\phi = F \circ \tilde\phi$. Denoting $\tilde B_r = F^{-1}(B_r)$, for a suitable small $r_0 > 0$ we have

$$\sigma_g(\Sigma \cap B_r) = \sigma_g\left(\phi\left(A \cap \tilde\phi^{-1}(\tilde B_r)\right)\right) = \int_{\tilde\phi^{-1}(\tilde B_r)} \sqrt{\det\left(h_{ij}(\tilde\phi(\xi))\right)}\, d\xi\,,$$

for any $r < r_0$, where h_{ij} denotes the graded metric g restricted to Σ with respect to the coordinates ξ. Now we consider the restriction of the coordinate dilation Λ_r to the hyperplane Π_1 and we denote it by $\tilde\Lambda_r$. We make the change of variable $\xi = \tilde\Lambda_r \xi'$, observing that the jacobian of $\tilde\Lambda_r$ is r^{Q-1}. We obtain

$$\sigma_g(\Sigma \cap B_r) = r^{Q-1} \int_{\tilde\Lambda_{1/r}\tilde\phi^{-1}(\tilde B_r)} \sqrt{\det\left(h_{ij}(\phi(\tilde\Lambda_r \xi'))\right)}\, d\xi'\,. \tag{7.14}$$

Now, we analyze the domain of integration $\tilde\Lambda_{1/r}\phi^{-1}(\tilde B_r) \subset \Pi_1$ as $r \to 0$. We have the representation

$$\tilde\Lambda_{1/r}\tilde\phi^{-1}(\tilde B_r) = \{\xi \in \Pi_1 \mid \left(\varphi(\tilde\Lambda_r \xi)\, r^{-1}, \xi\right) \in \tilde B_1\}\,.$$

By (7.13) it follows that

$$\partial_{x_j}\varphi(0) = -\frac{\partial_{x_j}\tilde u(0)}{\partial_{x_1}\tilde u(0)} = 0 \quad \text{for } j = 2, \ldots, m\,,$$

hence, by Taylor formula we get

$$\varphi(\tilde\Lambda_r \xi) r^{-1} = \sum_{i=m+1}^{q} \partial_{x_i}\tilde u(0) r^{d_i - 1}\xi_i + R(\delta_r \xi) r^{-1}\,,$$

where $R(v)|v|^{-1} \to 0$ as $|v| \to 0$ and $|\cdot|$ is the Euclidean norm on Π_1. For any $i > m$ we have $d_i \geq 2$, then $\varphi(\delta_r \xi) r^{-1} \to 0$ as $r \to 0$, uniformly in ξ, which varies in a bounded set. Hence, for any $\xi \in \tilde B_1 \cap \Pi_1$ we have

$$\mathbf{1}_{\tilde\Lambda_{1/r}\phi^{-1}(B_r)}(\xi) \longrightarrow 1 \quad \text{as} \quad r \to 0\,,$$

whereas for any $\xi \in \Pi_1 \setminus \overline{\tilde B_1}$ we get

$$\mathbf{1}_{\tilde\Lambda_{1/r}\phi^{-1}(B_r)}(\xi) \longrightarrow 0 \quad \text{as} \quad r \to 0\,,$$

so by (7.14) and The Lebesgue Convergence Theorem it follows

$$\lim_{r \to 0} \frac{\sigma_g(\Sigma \cap B_r)}{r^{Q-1}} = \int_{\tilde B_1 \cap \Pi_1} \sqrt{\det\left(h_{ij}(e)\right)}\, d\xi\,. \tag{7.15}$$

Now we explicitly compute the number $\det(h_{ij}(e))$. We have

$$h_{ij}(\phi(\xi)) = \left\langle \frac{\partial\phi}{\partial\xi_i}, \frac{\partial\phi}{\partial\xi_j} \right\rangle_{\phi(\xi)}.$$

Denoting by g_{ij} the graded metric with respect to the coordinates $x \in A$, we note that $g_{ij}(e) = \delta_{ij}$. Then, by (7.12) it follows that

$$\sqrt{\det(h_{ij}(e))} = \sqrt{\det\left(\left\langle \frac{\partial\phi}{\partial\xi_i}, \frac{\partial\phi}{\partial\xi_j} \right\rangle_e\right)} = \frac{|\nabla\tilde{u}(0)|}{|\partial_{x_1}\tilde{u}(0)|} = \frac{|\nabla u(e)|}{|\nabla_H u(e)|}.$$

Finally, by (7.10) and observing that $\mathcal{H}^{q-1}_{|\cdot|}(\tilde{B}_1 \cap \Pi_1) = \theta^g_{Q-1}(\nu_H(e))$ (see Definition 5.2.2), the limit (7.15) leads us to the conclusion. $\square$

Theorem 7.1.3 *Let Σ be a hypersurface of class C^1 in $\Omega \subset \mathbb{M}$. Then we have*

$$\frac{\theta^g_{Q-1}(\nu_H)}{\omega_{Q-1}} \mathcal{S}^{Q-1} \llcorner \Sigma = |\nu_H| \, \sigma_g \llcorner \Sigma, \tag{7.16}$$

$$\mathcal{S}^{Q-1} \llcorner \Sigma = \frac{\omega_{Q-1}}{\theta^g_{Q-1}(\nu_H)} |\nu_H| \, \sigma_g \llcorner \Sigma. \tag{7.17}$$

Proof. Theorem 7.1.2 implies that for any $p \in \Sigma \setminus C(\Sigma)$ we have

$$\lim_{r\to 0} \frac{\sigma_g(\Sigma \cap B_{p,r})}{\omega_{Q-1} r^{Q-1}} = \frac{\theta^g_{Q-1}(\nu(p))}{\omega_{Q-1} |\nu_H(p)|}.$$

Due to Theorem 6.6.2 we have $\mathcal{S}^{Q-1}(C(\Sigma)) = 0$. Observing that $\sigma_g(\Sigma \cap B_{p,r}) = \sigma_g(\Sigma \cap D_{p,r})$ for a.e. $r > 0$ and using theorems on measure derivatives, see for instance Theorems 2.10.17 (2) and 2.10.18 (1) of [55], the proof follows by a standard argument. $\square$

7.2 Coarea formula on sub-Riemannian groups

In this section we apply the relation (7.16) between the Riemannian surface measure and the $Q-1$ dimensional spherical Hausdorff measure of C^1 hypersurfaces to the study of an intrinsic version of the coarea formula in sub-Riemannian groups.

We begin the section recalling a classical result, see 13.4 of [25].

Theorem 7.2.1 (Riemannian coarea formula) *Let (M, g) be a Riemannian manifold and let $u : M \longrightarrow \mathbb{R}$ be a Lipschitz map with respect to the Riemannian distance. Then for any summable map $h : \mathbb{M} \longrightarrow \mathbb{R}$ we have*

$$\int_{\mathbb{M}} h \, |\nabla u| \, dv_g = \int_{\mathbb{R}} \int_{u^{-1}(t)} h \, d\sigma_g \, dt. \tag{7.18}$$

In the following theorem we extend the Riemannian coarea formula to sub-Riemannian groups, where the measure of level sets is computed by the spherical Hausdorff measure with respect to the homogeneous distance.

Theorem 7.2.2 (Generalized coarea formula) *Let $u : \mathbb{G} \longrightarrow \mathbb{R}$ be a locally Lipschitz map with respect to the Riemannian distance of $\mathbb{G}$. Then for any nonnegative measurable map $h : \mathbb{G} \longrightarrow \overline{\mathbb{R}}$ we have*

$$\int_{\mathbb{G}} h(w)\, |\nabla_H u|(w)\, dv_g(w) = \int_{\mathbb{R}} \int_{u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_H(w))}{\omega_{Q-1}}\, h(w)\, d\mathcal{S}^{Q-1}(w)\, dt\,, \tag{7.19}$$

where the spherical Hausdorff measure and the metric factor are understood with respect to the same homogeneous distance.

Proof. Without loss of generality, we can assume that u is a Lipschitz map on a bounded Borel set E. Moreover, we can extend u to a Lipschitz map on $\mathbb{G}$. The Whitney Extension Theorem (see 3.1.15 of [55]) ensures that for any $\varepsilon > 0$ there exists a map $\tilde{u} : \mathbb{G} \longrightarrow \mathbb{R}$ of class C^1 such that, defining the Borel set

$$E' = \Big\{x \in \mathbb{G} \mid u(x) = \tilde{u}(x)\Big\}\,,$$

we have $v_g(E \setminus E') \le \varepsilon$. Thus, the gradients of u and $\tilde{u}$ coincide a.e. on E'. In view of formulae (7.10) and (7.18) we obtain

$$\int_E |\nabla_H u|\, dv_g = \int_{\mathbb{R}} \Big(\int_{E \cap u^{-1}(t)} |\nu_H|\, d\sigma_g \Big)\, dt\,,$$

for any measurable subset $E \subset \mathbb{G}$. Hence, the general coarea estimate (2.6) implies

$$0 \le \int_E |\nabla u|\, dv_g - \int_{\mathbb{R}} \Big(\int_{E' \cap \tilde{u}^{-1}(t)} |\tilde{\nu}_H|\, d\sigma_g \Big)\, dt \le C\, \mathrm{Lip}(u)\, \varepsilon\,,$$

where C is a dimensional constant and $\tilde{\nu}_H$ is the horizontal normal of the level sets of $\tilde{u}$. By the fact that $\mathbb{G}$ is a stratified group we can apply Theorem 6.3.1, getting that the set of characteristic points is $\mathcal{S}^{Q-1}$-negligible for a.e. level set of $\tilde{u}$. Furthermore, by the classical Sard Theorem the set of critical values of $\tilde{u}$ is negligible. Thus, formula (7.16) yields

$$0 \le \int_E |\nabla u|\, dv_g - \int_{\mathbb{R}} \Big(\int_{E' \cap \tilde{u}^{-1}(t)} \frac{\theta^g_{Q-1}(\tilde{\nu}_H(x))}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}(x) \Big)\, dt \le C\, \mathrm{Lip}(u)\, \varepsilon\,. \tag{7.20}$$

Let us observe that $E' \cap u^{-1}(t) = E' \cap \tilde{u}^{-1}(t)$ and for a.e. level set we have $\nabla u = \nabla \tilde{u}$ outside of a $\mathcal{S}^{Q-1}$-negligible set. Thus, for a.e. $t \in \mathbb{R}$ the following equality holds for $\mathcal{S}^{Q-1}$-a.e. $x \in u^{-1}(t)$

$$\theta^g_{Q-1}(\tilde{\nu}_H(x)) = \theta^g_{Q-1}(\nu_H(x))\,.$$

Hence, inequality (7.20) becomes

$$0 \leq \int_E |\nabla u|\, dv_g - \int_{\mathbb{R}} \Big(\int_{E' \cap u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_H(x))}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}(x) \Big)\, dt \leq \, C\, \mathrm{Lip}(u)\, \varepsilon\,.$$

Again, using (2.6) and observing that in view of (5.5) the function $\theta^g_{Q-1}(\cdot)$ is bounded, we get

$$\int_{(E \setminus E') \cap u^{-1}(t)} \theta^g_{Q-1}(\nu_H(x))\, d\mathcal{S}^{Q-1}(x) \Big)\, dt \leq \, C'\, \mathrm{Lip}(u)\, \varepsilon\,.$$

Finally, in joining the last two inequalities we arrive at

$$-C'\mathrm{Lip}(u)\varepsilon \leq \int_E |\nabla u|\, dv_g - \int_{\mathbb{R}} \Big(\int_{E \cap u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_H(x))}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}(x) \Big)\, dt \leq C\mathrm{Lip}(u)\varepsilon\,.$$

Letting $\varepsilon \to 0$ it follows

$$\int_E |\nabla_H u|\, dv_g = \int_{\mathbb{R}} \Big(\int_{E \cap u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_H(x))}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}(x) \Big)\, dt\,. \tag{7.21}$$

Now, by a standard argument, taking an increasing sequence of nonnegative step functions that converge to h and applying the Beppo Levi-Monotone convergence theorem the thesis follows. □

Corollary 7.2.3 *Let $(\mathbb{E}^n, \|\cdot\|)$ be the Euclidean space endowed with a norm and let $u : \mathbb{E}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz map. Then for any nonnegative measurable map $h : \mathbb{E}^n \longrightarrow \overline{\mathbb{R}}$ we have*

$$\int_{\mathbb{E}^n} h(x)\, |\nabla u|(x)\, dx = \int_{\mathbb{R}} \int_{u^{-1}(t)} \frac{\theta_{n-1}(\nu(x))}{\omega_{n-1}}\, h(x)\, d\mathcal{H}^{n-1}_{\|\cdot\|}(x)\, dt\,, \tag{7.22}$$

where ∇u is the Euclidean gradient, ν is the normal direction to the level set and $\theta_{n-1}(\nu(x))$ is defined with respect to $\|\cdot\|$ as the Hausdorff measure $\mathcal{H}^{n-1}_{\|\cdot\|}$.

Proof. Formula (7.22) follows directly from (7.19), observing that $Q = n$ and that any direction in $\mathbb{E}^n$ is horizontal. Thus, the horizontal gradient coincides with the Euclidean gradient and the horizontal normal ν_H coincides with the normal ν to the level set. Now, we recall that the spherical Hausdorff measure coincides with the Hasudorff measure on rectifiable subsets of a normed space. This fact follows from the isodiametric inequality of finite dimensional normed spaces, see [25]. Thus, for a.e. level set of f we can replace the $\mathcal{S}^{n-1}_{\|\cdot\|}$ in formula (7.19) with $\mathcal{H}^{n-1}_{\|\cdot\|}$. This completes the proof. □

In the next theorem we apply the generalized coarea formula in rotational groups with invariant homogeneous distances.

Theorem 7.2.4 *Let $\mathbb{G}$ be an $\mathcal{R}$-rotational group endowed with an $\mathcal{R}$-invariant homogeneous distance d and let $u : \mathbb{G} \longrightarrow \mathbb{R}$ be a locally Lipschitz map with respect to the Riemannian distance. Then for any nonnegative measurable map $h : \mathbb{G} \longrightarrow \overline{\mathbb{R}}$ we have*

$$\int_A h(x)\, |\nabla_H u|(x)\, dv_g(x) = \frac{\alpha_{Q-1}}{\omega_{Q-1}} \int_{\mathbb{R}} \int_{u^{-1}(t)} h(x)\, d\mathcal{S}^{Q-1} dt\,, \qquad (7.23)$$

where α_{Q-1} is given by Proposition 5.2.5 and it is referred to d together with $\mathcal{S}^{Q-1}$.

PROOF. By virtue of Theorem 7.2.2 we have

$$\int_{\mathbb{G}} h(w)\, |\nabla_H u|(w)\, dv_g(w) = \int_{\mathbb{R}} \int_{u^{-1}(t)} \frac{\theta^g_{Q-1}(\nu_H(w))}{\omega_{Q-1}}\, h(w)\, d\mathcal{S}^{Q-1}(w)\, dt\,.$$

Proposition 5.2.5 yields $\theta^g_{Q-1}(\nu_H(x)) = \alpha_{Q-1}$ and this concludes the proof. □

Remark 7.2.5 Theorem 7.2.4 yields simplified versions of the coarea formula in the Heisenberg group with suitable homogeneous distances, with formulae analogous to (6.46) and (6.47).

In the sub-Riemannian context it is natural to require formulae where only the restriction of the metric g to the horizontal subbundle $H\mathbb{G}$ is involved. Next, we will apply this principle to formula (7.19). Let us define the factor

$$\beta_{Q-1}(\nu) = \frac{\mathcal{S}^Q(B_1)}{v_g(B_1)} \frac{\theta^g_{Q-1}(\nu)}{\omega_{Q-1}}$$

and observe that the left invariance of both v_g and $\mathcal{S}^Q$ implies

$$\int_{\mathbb{G}} h(w)\, |\nabla_H u|(w)\, d\mathcal{S}^Q(w) = \int_{\mathbb{R}} \int_{u^{-1}(t)} \beta_{Q-1}(\nu_H(w))\, h(w)\, d\mathcal{S}^{Q-1}(w)\, dt\,. \qquad (7.24)$$

It is not difficult to recognize that the left hand side of (7.24) involves only the restriction of g to $H\mathbb{G}$. In the following proposition we check that even the constant $\beta_{Q-1}(\nu)$ depends only on the restriction of the graded metric g to $H\mathbb{G}$.

Proposition 7.2.6 *Let g and $\tilde{g}$ be two graded metrics on the graded group $\mathbb{M}$ and suppose that $g(e)(v, w) = \tilde{g}(e)(v, w)$ for any $v, w \in H_e\mathbb{M}$. Then for any $\nu \in H_e\mathbb{M} \setminus \{0\}$ we have*

$$\frac{\theta^g_{Q-1}(\nu)}{v_g(B_1)} = \frac{\theta^{\tilde{g}}_{Q-1}(\nu)}{v_{\tilde{g}}(B_1)}\,.$$

PROOF. Let $\mathcal{L}$ the vertical hyperplane in $\mathbb{M}$ orthogonal to ν. We define W_1 to be the unit vector parallel to ν and we complete it to an adapted orthonormal basis $(W_1, \dots, W_q)$ of $\mathcal{M}$ with respect to g. Let $F : \mathbb{R}^q \longrightarrow \mathbb{M}$ be the associated system of graded coordinates (Definition 2.3.43). By the hypothesis on g and $\tilde{g}$ we can consider an adapted orthonormal basis $(Y_1, \dots, Y_q)$ of $\mathcal{M}$ with respect to $\tilde{g}$ such that $Y_i = W_i$ for any $i = 1, \dots, m$. To this latter basis we associate the graded coordinates defined by $T : \mathbb{R}^q \longrightarrow \mathbb{M}$. Let C be the $q \times q$ matrix defined by relations $W_i = c_i^j Y_j$ and notice that it has the following form

$$C = \begin{pmatrix} I_m & C_1 \\ O & C_2 \end{pmatrix}, \tag{7.25}$$

where I_m is the $m \times m$ identity matrix, $C_1 \in M_{m,q-m}(\mathbb{R})$, $C_2 \in M_{q-m,q-m}(\mathbb{R})$ and $O \in M_{q-m,m}(\mathbb{R})$ is the null matrix. By definition of the maps F and T and identifying the matrix C with its corresponding map, we have $F = T \circ C$. Now we read the hyperplane $\mathcal{L}$ in the two systems of coordinates, getting the following relations

$$\Pi = F^{-1}(\mathcal{L}) = \Big\{\xi \in \mathbb{R}^q \mid \xi_1 = 0\Big\} = C^{-1}T^{-1}(\mathcal{L}) = C^{-1}S\,.$$

From (7.25) we notice that the restriction of C to Π has the determinant equal to $\det C_2$ and that it maps Π into S. It follows

$$\begin{aligned} \theta^{\tilde{g}}_{Q-1}(\nu) &= \mathcal{H}^{q-1}_{|\cdot|}\left(T^{-1}(B_1 \cap \mathcal{L})\right) \\ = \mathcal{H}^{q-1}_{|\cdot|}\left(C \circ F^{-1}(B_1 \cap \mathcal{L})\right) &= \mathcal{H}^{q-1}_{|\cdot|}\left(C\left(F^{-1}(B_1) \cap \Pi\right)\right) \\ = |\det C_2|\, \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(B_1 \cap \mathcal{L})\right) &= |\det C_2|\, \theta^{g}_{Q-1}(\nu). \end{aligned} \tag{7.26}$$

Proposition 2.3.47 implies that $v_{\tilde{g}}(B_1) = \mathcal{L}^q\left(T^{-1}(B_1)\right)$, then we have

$$\begin{aligned} v_{\tilde{g}}(B_1) &= \mathcal{L}^q\left(T^{-1}(B_1)\right) = \mathcal{L}^q(C \circ F^{-1}(B_1)) \\ &= |\det C|\, \mathcal{L}^q\left(F^{-1}(B_1)\right) = |\det C_2|\, v_g(B_1)\,, \end{aligned} \tag{7.27}$$

where the last equality follows by (7.25) and Proposition 2.3.47. Finally, equations (7.26) and (7.27) yield the thesis. □

7.3 Characteristic set of $C^{1,1}$ hypersurfaces

In this section we study the size of the characteristic set of $C^{1,1}$ hypersurfaces in 2-step graded groups endowed with a homogeneous distance. Throughout the section Ω will denote an open subset of $\mathbb{M}$.

Theorem 7.3.1 (Blow-up estimates) *Let $\mathcal{M} = V_1 \oplus V_2$ be the graded algebra of $\mathbb{M}$ and let $\Sigma \subset \Omega$ be a $C^{1,1}_{loc}$ hypersurface with $p \in C(\Sigma)$. Then there exist a neighbourhood U of p in Σ such that for any $p' \in C(\Sigma) \cap U$ we have*

$$0 < c \le \liminf_{r \to 0^+} \frac{\sigma_g(\Sigma \cap B_{p',r})}{r^{Q-2}} \le \limsup_{r \to 0^+} \frac{\sigma_g(\Sigma \cap B_{p',r})}{r^{Q-2}} \le C\,. \tag{7.28}$$

where c depends on U and C is a geometrical constant independent of Σ.

PROOF. We represent Σ in a neighbourhood $O_p = l_p(O)$ of p as $O_p \cap u^{-1}(t) \subset \Sigma$, where O is an open neighbourhood of $e \in \mathbb{G}$ and $u \in C^{1,1}(O_p)$, with $u(p) = 0$. Let (F, W) be system of graded coordinates. Since $p \in C(\Sigma)$, then $\nabla_H u(p) = 0$ and $\sum_{l=m+1}^{q} W_l u(p)\, W_l(p) \neq 0$, where $(W_{m+1}, \ldots, W_q)$ is an orthogonal basis of V_2. We can also assume that $W_{m+1}u(p) = |\nabla u(p)|$ and $W_j u(p) = 0$ for any $j = m+1, \ldots, q$. We define the map $\tilde{u} = u \circ l_p \circ F : \tilde{O} \longrightarrow \mathbb{R}^q$, where $\tilde{O} = F^{-1}(O)$, obtaining

$$\partial_{x_j} \tilde{u}(0) = W_j u(p) = \delta_{jm+1} \left| \nabla u(p) \right| . \tag{7.29}$$

We consider the hyperplane

$$\Pi_{m+1} = \left\{ x \in \mathbb{R}^q \mid x_{m+1} = 0 \right\} .$$

By the implicit function theorem there exists an open subset $A \subset \Pi_{m+1}$ containing the origin and a C^1 map $\varphi : A \longrightarrow \mathbb{R}$ such that

$$\tilde{u}\left(\xi, \varphi(\xi, \eta), \eta\right) = 0 \qquad \text{for any} \quad (\xi, \eta) \in A ,$$

where we have posed $(\xi, \eta) = \sum_{j=1}^{m} x_j\, e_j + \sum_{j=m+2}^{q} x_j\, e_j$ and (e_j) is the canonical basis of $\mathbb{R}^q$. Let us define $\tilde{\phi}(\xi, \eta) = (\xi, \varphi(\xi, \eta), \eta)$ for any $(\xi, \eta) \in A$ and $\phi = F \circ \tilde{\phi}$.

Now, by the fact that translations are isometries with respect to the Riemannian metric, for a suitable small $r_0 > 0$ we have

$$\begin{aligned} \sigma_g(\Sigma \cap B_{p,r}) &= \sigma_g\left(l_{p^{-1}}(\Sigma) \cap B_r\right) = \sigma_g\left(\phi(A) \cap B_r\right) = \sigma_g\left(\phi\left(\tilde{\phi}^{-1}(A \cap \tilde{B}_r)\right)\right) \\ &= \int_{\phi^{-1}(\tilde{B}_r)} \sqrt{\det\left(h_{ij}(\phi(\xi, \eta))\right)}\, d\xi d\eta , \end{aligned}$$

for any $r < r_0$, where $\tilde{B}_r = F^{-1}(B_r)$ and h_{ij} denotes the graded metric g restricted to $l_{p^{-1}}(\Sigma)$ with respect to the coordinates (ξ, η). Now we consider the restriction of the coordinate dilation Λ_r to the hyperplane Π_{m+1} and denote it by $\tilde{\Lambda}_r$. It is easy to notice that the jacobian of $\tilde{\Lambda}_r$ is r^{Q-2}, hence by a change of variable $(\xi, \eta) = \tilde{\Lambda}_r(\xi', \eta')$ we get

$$\sigma_g(\Sigma \cap B_{p,r}) = r^{Q-2} \int_{\tilde{\Lambda}_{1/r}\phi^{-1}(\tilde{B}_r)} \sqrt{\det\left(h_{ij}(\phi(\tilde{\Lambda}_r(\xi', \eta')))\right)}\, d\xi' d\eta' . \tag{7.30}$$

Next, we study the shape of the domain $\tilde{\Lambda}_{1/r}\phi^{-1}(\tilde{B}_r)$ as $r \to 0$. We have the representation

$$\tilde{\Lambda}_{1/r}\phi^{-1}\delta_r(\tilde{B_1}) = \left\{ (\xi, \eta) \in \Pi_{m+1} \middle| \ \left(\xi, \varphi\left(\delta_r(\xi, \eta)\right) r^{-2}, \eta\right) \in \tilde{B_1} \right\} . \tag{7.31}$$

By (7.29) it follows

$$\partial_{x_k}\varphi(0) = -\frac{\partial_{x_k}\tilde{u}(0)}{\partial_{x_{m+1}}\tilde{u}(0)} = 0 \qquad \partial_{x_l}\varphi(0) = -\frac{\partial_{x_l}\tilde{u}(0)}{\partial_{x_{m+1}}\tilde{u}(0)} = 0$$

for any $k = 1, \dots, m$ and $l = m+2, \dots, q$. Hence, we have proved that $\nabla\varphi(0) = 0$ and by Taylor formula for $C^{1,1}$ functions we obtain

$$\varphi\left(\tilde{\Lambda}_r(\eta,\xi)\right) = \theta\left(r\xi, r^2\eta)\right)\ |(r\xi, r^2\eta)|^2 \tag{7.32}$$

where $|\cdot|$ is the Euclidean norm on Π_{m+1} and θ is a map which is bounded by the Lipschitz constant of $\nabla\varphi$. Let $C \subset \mathbb{R}^q$ be an open Euclidean ball contained in $\tilde{B}_1$ and let us define the set

$$E = \{(\xi,\eta) \in \Pi_{m+1} \mid\ (\xi, L|\xi|^2, \eta) \in C\},$$

where $L = 2\|\theta\|_\infty$. Now, we aim to prove that for any $(\xi,\eta) \in E$

$$\mathbf{1}_{\delta_{1/r}\phi^{-1}(B_r)}\left((\xi,\eta)\right) \longrightarrow 1 \quad \text{as} \quad r \to 0\,. \tag{7.33}$$

Consider $(\xi,\eta) \in E$ and choose $r_1 \in (0, r_0)$ such that for any $r < r_1$ and $(\xi,\eta) \in E$ we have $|(\xi, r\eta)| \leq \sqrt{2}|\xi|$. Then, by equation (7.32) for any $r \in (0, r_1)$ we get

$$r^{-2}|\varphi\left(\delta_r(y,z')\right)| = |\theta\left((ry, r^2 z')\right)|\ |(y, rz')|^2 \leq 2\|\theta\|_\infty\,|y|^2 = L\,|y|^2\,.$$

Since C is convex and $\tilde{\Lambda}_{1/r}\phi^{-1}\delta_r(\tilde{B}_1)$ has representation (7.31) it follows that

$$E \subset \tilde{\Lambda}_{1/r}\phi^{-1}\delta_r(\tilde{B}_1) \quad \text{for any} \quad r \in (0, r_1)\,,$$

and the limit (7.33) is proved. In view of Fatou Theorem and (7.33) we obtain

$$\liminf_{r\to 0} \int_{\delta_{1/r}\phi^{-1}(\tilde{B}_r)} \sqrt{\det\left(h_{ij}(\phi(\delta_r(\xi,\eta)))\right)}\, d\xi d\eta \geq \int_E \sqrt{\det\left(h_{ij}(e)\right)}\, d\xi d\eta\,,$$

where

$$\sqrt{\det\left(h_{ij}(e)\right)} = \sqrt{\det\left(\left\langle \frac{\partial\phi}{\partial\xi_i}, \frac{\partial\phi}{\partial\xi_j}\right\rangle_e\right)} = \frac{|\nabla\tilde{u}(0)|}{|\partial_{x_{m+1}}\tilde{u}(0)|} = \frac{|\nabla u(p)|}{|W_{m+1}u(p)|} = 1\,.$$

We observe that the size of the open set E depends on $\|\theta\|_\infty$, so the constant $c = \mathcal{H}^{q-1}_{|\cdot|}(E)$ can be chosen independent of all points $p' \in C(\Sigma) \cap U$, where U is a bounded open neighbourhood of p in Σ and by (7.30) c_1 satisfies our claim. To get the upper estimate we observe directly from the representation (7.31) that there exists a bounded sets $F \subset \Pi_{m+1}$ which contains $\tilde{\Lambda}_{1/r}\phi^{-1}(\tilde{B}_r)$ for any $r > 0$. Thus, we can choose $C = \mathcal{H}^{q-1}_{|\cdot|}(F)$ independent of $p \in C(\Sigma)$. □

Theorem 7.3.2 *Let $\Sigma \subset \Omega$ be a $C^{1,1}_{loc}$ hypersurface in a step 2 group $\mathbb{M}$. Then there exists a countable open covering $\{U_j\}$ of $C(\Sigma)$ and positive constants c_j and C such that*

$$c_j\,\mathcal{S}^{Q-2}\left(C(U_j\cap\Sigma)\right) \leq \sigma_g(C(U_j\cap\Sigma) \leq C\,\mathcal{S}^{Q-2}\left(C(U_j\cap\Sigma)\right) \tag{7.34}$$

for any $j \in \mathbb{N}$ and we have

$$\mathcal{H}_d\text{–}\dim\left(C(\Sigma)\right) \leq Q - 2\,. \tag{7.35}$$

PROOF. We adopt the notation of Theorem 2.10.18 in [55], where $V = \Sigma$, $\mu = \sigma_g \llcorner \Sigma$ and F is the family of balls with respect to the homogeneous metric d and $\zeta(B_{x,r}) = r^\alpha$ for any $x \in \mathbb{G}$ and $r > 0$. By Theorem 7.3.1 Theorems 2.10.17(2), 2.10.18(1) of [55] we have a countable open covering $\{U_j\}$ of $C(\Sigma)$ and positive constants c_j, C such that

$$c_j\, \mathcal{S}^{Q-2}\left(C(U_j \cap \Sigma)\right) \leq \sigma_g\left(C(U_j \cap \Sigma)\right) \leq C\, \mathcal{S}^{Q-2}\left(C(U_j \cap \Sigma)\right).$$

These estimates imply in particular that $\mathcal{S}^{Q-2}\left(C(U_j \cap \Sigma)\right)$ is finite for every $j \in \mathbb{N}$, then estimate (7.35) follows. □

Remark 7.3.3 As a consequence of (7.34), the characteristic set of a $C^{1,1}_{loc}$ hypersurface is a countable union of subsets with $\mathcal{H}^{Q-2}$-finite measure.

We observe that the CC distance ρ is always greater than or equal to the Riemannian distance d_g, in the case both of them are built with the same graded metric. Hence, for any set $E \subset \mathbb{M}$ and $\alpha > 0$ we have $\mathcal{H}^\alpha_{d_g}(E) \leq \mathcal{H}^\alpha_\rho(E)$. So the following inequality holds

$$\mathcal{H}_{d_g}\text{–}\dim(E) \leq \mathcal{H}_\rho\text{–}\dim(E)\,. \tag{7.36}$$

Now, by Theorem 1.4(1) of [12], for any $\alpha > 0$ there exists a $C^{1,1}$ hypersurface Σ_α in the Heisenberg group $\mathbb{H}^n$ such that $\mathcal{H}_{|\cdot|}\text{–}\dim(C(\Sigma_\alpha)) \geq 2n - \alpha$, where $|\cdot|$ is the Euclidean norm in $\mathbb{H}^n$, viewed as a vector space. It is clear that $\mathcal{H}_{d_g}\text{–}\dim(C(\Sigma)) = \mathcal{H}_{|\cdot|}\text{–}\dim(C(\Sigma))$, so by (7.36) we get

$$\mathcal{H}_\rho\text{–}\dim(C(\Sigma_\alpha)) \geq 2n - \alpha = Q - 2 - \alpha\,, \tag{7.37}$$

where $Q = 2n + 2$ is the Hausdorff dimension of $\mathbb{H}^n$ with respect to a homogeneous distance. Thus, by virtue of Theorem 7.3.2 we get

$$Q - 2 - \alpha \leq \mathcal{H}_\rho\text{–}\dim(C(\Sigma_\alpha)) \leq Q - 2\,,$$

hence the estimate (7.35) is optimal.

7.4 Perimeter measure

In this section we study the perimeter measure of C^1 domains in sub-Riemannian groups, obtaining its representation in terms of the $Q-1$ spherical Hausdorff measure of the topological boundary (7.6). Several consequences of this formula are given.

In the sequel, subsets with C^1 boundary and nonempty interior will be simply called C^1 subsets.

Proposition 7.4.1 *Let E be a C^1 closed subset of $\mathbb{M}$. Then we have*

$$\partial^* E \setminus C(\partial E) = \partial E \setminus C(\partial E)\,. \tag{7.38}$$

PROOF. The inclusion $\partial^* E \subset \partial E$ is immediate. So consider $p \in \partial E \setminus C(\partial E)$. For a suitable $r_0 > 0$ and any $r \in (0, r_0)$ we have

$$B_{p,r} \cap E = l_p \left(\left\{ \exp(w) \in B_r \,\middle|\, u(p \exp w)) \geq 0 \right\} \right) ,$$

where $u \in C^1(O, \mathbb{R})$ has nonvanishing gradient on the open bounded neighbourhood O of p and $u(p) = 0$. Proposition 3.2.8 and formula (7.10) yield

$$u(p \exp w) = d_H u(p)(\exp(w)) + o\left(d\left(\exp(w)\right)\right) = |\nabla u(p)| \left\langle \nu_H(p), w \right\rangle_p + o\left(d\left(\exp(w)\right)\right) ,$$

where $\nu_H(p)$ is the horizontal normal of ∂E at p. Then it follows

$$B_{p,r} \cap E = l_p \left(\left\{ p' \in B_r \,\middle|\, \langle \nu_H(p), \ln p' \rangle_p + o\left(d(p', e)\right) \geq 0 \right\} \right) .$$

Utilizing Definition 6.4.1 and posing $\tilde{B}_r = \ln B_r$ we notice that

$$\begin{aligned} v_g(B_{p,r} \cap E) &= r^Q\, v_g(B_1 \cap E_{p,r}) \\ &= r^Q\, v_g \left(\exp \left(\left\{ w \in \tilde{B}_1 \,\middle|\, \langle \nu_H(p), w \rangle + o(1) \geq 0 \right\} \right) \right) , \end{aligned} \tag{7.39}$$

where $E_{p,r} = \delta_{1/r}(p^{-1} E)$ is the r-rescaled of E at p (Definition 6.4.1). Due to the fact that $p \notin C(\partial E)$ we have $\nu_H(p) \neq 0$ (see Proposition 2.2.10). Thus, by the Lebesgue Convergence Theorem and Definition 6.4.3 it follows that

$$\lim_{r \to 0^+} \frac{v_g(B_{p,r} \cap E)}{r^Q} = v_g\left(B_1 \cap S_g^+\left(\nu_H(p)\right)\right) > 0 . \tag{7.40}$$

From the expression

$$B_{p,r} \setminus E = l_p \left(\left\{ \exp(w) \in B_r \,\middle|\, u(p \exp w)) < 0 \right\} \right)$$

and reasoning in the same way as before we deduce that

$$\lim_{r \to 0^+} \frac{v_g(B_{p,r} \setminus E)}{r^Q} = v_g\left(B_1 \cap S_g^-\left(\nu_H(p)\right)\right) > 0 .$$

Observing that $v_g(B_{p,r}) = v_g(B_1)\, r^Q$ and keeping in mind the definition of essential boundary, our claim follows. □

The following theorem is the main result of the section. We prove the blow-up in any graded group endowed with a homogeneous distance, provided that the domain E is of class C^1.

Theorem 7.4.2 (Blow-up) *Let E be a C^1 closed subset of $\mathbb{M}$. Then for any $p \in \partial E \setminus C(\partial E)$ we have*

$$\lim_{r\to 0^+} |\partial E_{p,r}|_H(B_1) = \lim_{r\to 0^+} \frac{|\partial E|_H(B_{p,r})}{r^{Q-1}} = \theta^g_{Q-1}\left(\nu_H(p)\right) \tag{7.41}$$

with the following weak* convergence of vector valued Radon measures

$$\nu_{E_{p,r}}\, |\partial E_{p,r}|_H \rightharpoonup \frac{\nu_H(p)}{|\nu_H(p)|}\, |\partial S^+_g(\nu_H(p))|_H \quad as \quad r\to 0\,, \tag{7.42}$$

where $\nu_H(p)$ is the horizontal normal to ∂E at p (Definition 2.2.9) and the couple $B_{p,r}$, $\theta^g_{Q-1}\left(\nu_H(p)\right)$ is considered with respect to the same homogeneous distance.

PROOF. Let us fix a system of graded coordinates (F,W) and a point $p \in E\setminus C(\partial E)$. We start proving the limit (7.41). We fix the notation $E_p = p^{-1}E$. By the C^1 regularity of E, we can represent $\Sigma = O\cap \partial E_p$ by a C^1 map $u: O\longrightarrow \mathbb{R}$, where $O\subset \mathbb{M}$ is an open bounded neighbourhood of e and $d_H u(s)$ is surjective for any $s\in O$. By virtue of formula (7.10) we have

$$d_H u(e)(Z) = |\nabla u(e)|\, \langle \nu_H(e), Z\rangle_p\,,$$

where $\nu_H(e)$ is the horizontal normal to ∂E at $p\in\partial E\setminus\{0\}$ translated to $e\in\mathbb{M}$. Choosing our graded coordinates such that $W_1(e) = \nu_H(e)/|\nu_H(e)|$ and taking into account (7.10) it follows that

$$\begin{aligned} W_1 u(e) &= d_H u(e)(W_1) = |\nabla u(e)|\, |\nu_H(e)| \langle W_1, W_1\rangle = |\nabla_H u(e)| \\ W_j u(e) &= d_H u(e)(W_j) = |\nabla u(e)|\, |\nu_H(e)| \langle W_1, W_j\rangle_e = 0 \end{aligned} \tag{7.43}$$

for any $j=2,\dots,m$. Let us define $\tilde u = u\circ F: \tilde O\longrightarrow\mathbb{R}$, where $\tilde O = F^{-1}(O)\subset\mathbb{R}^q$. We note that $\tilde u^{-1}(0) = \tilde\Sigma$ with $\tilde\Sigma = F^{-1}(\Sigma)\subset\mathbb{R}^q$. Consider the hyperplane

$$\Pi_1 = \Big\{x\in\mathbb{R}^q \mid x_1 = 0\Big\}\,.$$

By the implicit function theorem there exists an open subset $A\subset \Pi_1$ containing the origin and a C^1 map $\varphi: A\longrightarrow\mathbb{R}$ such that $\tilde u\left(\varphi(\xi),\xi\right) = 0$ for any $\xi\in A$, where we have posed $\xi = \sum_{j=2}^q x_j\, e_j$ and (e_j) is the canonical basis of $\mathbb{R}^q$. Now we consider the parametrization $\tilde\phi(\xi) = (\varphi(\xi),\xi)$ and the map $\phi = F\circ\tilde\phi$. Formula (7.49) applied to E_p yields

$$|\partial E_p|_H(B_r) = F^{-1}_\sharp |\partial E_p|_H(\tilde B_r) = \int_{\tilde B_r\cap\partial\tilde E_p} |w_{\tilde E_p}|\, d\mathcal{H}^{q-1}$$

for any $r\in(0,r_0)$, where $r_0>0$ is suitable small and $\tilde B_r = F^{-1}(B_r)$. Hence, from the parametrization of $O\cap\partial\tilde E_p$ by the map ϕ we deduce that

$$|\partial E_p|_H(B_r) = \int_{\tilde\phi^{-1}(\tilde B_r)} |w_{\tilde E_p}|(\tilde\phi(\xi))\, \mathcal{J}_{q-1}(d\phi(\xi)) d\xi$$

and the change of variable by the coordinate dilation Λ_r restricted to the hyperplane Π_1 yields

$$|\partial E_p|_H(B_r) = r^{Q-1} \int_{\Lambda_{1/r}\tilde{\phi}^{-1}(\tilde{B}_r)} |w_{\tilde{E}_p}|(\tilde{\phi}(\Lambda_r \xi))\, \mathcal{J}_{q-1}\left(d\phi(\Lambda_r(\xi))\right) d\xi\,.$$

Now, proceeding as in the proof of Theorem 7.1.2 we see that

$$\mathcal{L}^{q-1}\left(\Lambda_{1/r}\tilde{\phi}^{-1}(\tilde{B}_r)\right) \longrightarrow \theta^g_{Q-1}\left(\nu_H(e)\right) \quad \text{as} \quad r \to 0^+$$

therefore

$$|\partial E_{p,r}|_H(B_1) = \frac{|\partial E_p|_H(B_r)}{r^{Q-1}} \longrightarrow \theta^g_{Q-1}\left(\nu_H(e)\right)\, |w_{\tilde{E}_p}|(0)\, \mathcal{J}_{q-1}\left(d\phi(0)\right)\,, \tag{7.44}$$

as $r \to 0^+$. We notice that the differential $dF : \mathbb{R}^q \longrightarrow \mathcal{G}$ is an isometry and that the unit inward normal ν_{E_p} is orthogonal to $T_e\partial E_p$, therefore the unit normal $\nu_{\tilde{E}}(0)$ to $\tilde{E}_p$ at the origin satisfies the relation $F_*\nu_{\tilde{E}_p}(e) = \nu_{E_p}(e)$ and

$$\langle \nu_{E_p}, W_i\rangle_e = \langle F_*\nu_{\tilde{E}_p}, F_*\tilde{W}_i\rangle_e = \langle \nu_{\tilde{E}_p}, \tilde{W}_i\rangle_0\,. \tag{7.45}$$

By virtue of (7.45) we see that

$$|w_{\tilde{E}_p}|(0) = \sqrt{\sum_{j=1}^m \langle \nu_{\tilde{E}_p}, \tilde{W}_i\rangle_0^2} = |\nu_H(e)| \tag{7.46}$$

and by (7.43) we get

$$\partial_{x_j}\varphi(0) = -\frac{\partial_{x_j}\tilde{u}(0)}{\partial_{x_1}\tilde{u}(0)} = 0 \quad \text{for any } j = 2, \ldots, m\,,$$

that yields

$$\mathcal{J}_{q-1}\left(d\phi(0)\right) = \frac{|\nabla \tilde{u}(0)|}{|\partial_{x_1}\tilde{u}(0)|} = \frac{|\nabla u(e)|}{|\nabla_H u(e)|} = \frac{1}{|\nu_H(e)|}\,, \tag{7.47}$$

where the latter equality follows from formula (7.10). In view of formulae (7.44), (7.46) and (7.47) we have established (7.41).

Now, we adopt the notation used in the proof of Proposition 7.4.1. First, we want to prove that the rescaled set $E_{p,r}$ converges to $S^+_g(\nu_H(e))$ in $L^1_{loc}(\mathbb{M})$. If we replace B_1 with B_R in (7.39) and we apply (7.40), then for any $R > 0$ we have

$$\lim_{r\to 0^+} v_g\left(\exp\left(\left\{w \in \tilde{B}_R \,\Big|\, \langle \nu_H(e), w\rangle + o(1) > 0\right\}\right)\right) = v_g\left(B_R \cap S^+_g\left(\nu_H(e)\right)\right)\,.$$

Let us pick a test function $\varphi \in \Gamma_c(H\mathbb{G})$ and observe that the integral

$$\int_{\mathbb{G}} \langle \varphi, \nu_{E_{p,r}} \rangle \, d|\partial E_{p,r}|_H = \int_{E_{p,r}} \mathrm{div}_H \varphi \, dv_g$$

converges to

$$\int_{S_g^+(\nu_H(e))} \mathrm{div}_H \varphi \; dv_g = \int_{\mathbb{G}} \langle \varphi, \nu_{S_g^+(\nu_H(e))} \rangle \, d|\partial S_g^+(\nu_H(e))|_H \, .$$

as $r \to 0^+$. From limit (7.41) and weak* compactness we deduce the existence of a weak* converging subsequence

$$\nu_{E_{p,r_k}} |\partial E_{p,r_k}|_H \rightharpoonup \nu_{S_g^+(\nu_H(e))} |\partial S_g^+(\nu_H(e))|_H \, ,$$

then the above convergence holds as $r \to 0^+$. By Lemma 6.4.5 the previous limit becomes

$$\nu_{E_{p,r}} |\partial E_{p,r_k}|_H \rightharpoonup \frac{\nu_H(e)}{|\nu_H(e)|} |\partial S_g^+(\nu_H(e))|_H \quad \text{as} \quad r \to 0^+ \, .$$

Let us write $\nu_{H,\partial E_p}(e) = \nu_H(e)$ to stress that the horizontal normal is relative to $p^{-1}\partial E$. Then it is clear the relation $dl_p \nu_{H,\partial E_p}(e) = \nu_{H,\partial E}(p) \in T_p\mathbb{M}$ that corresponds to the horizontal normal $\nu_H(p)$ to ∂E at p. We use the same notation $\nu_H(p)$ to denote the left invariant vector field of $\mathcal{G}$ that coincides with $\nu_H(p)$ at p (according to Remark 6.4.10), so from the last limit we infer (7.42). □

Given a C^1 closed subset $E \subset \mathbb{M}$ and looking at its boundary as a C^1 hypersurface ∂E, there are defined the horizontal normal $\nu_H(p)$ at $p \in \partial E \setminus C(\partial E)$ and the generalized inward normal $\nu_E(p)$, due to (7.48). In the following proposition we prove that these two normal vectors have the same direction, completing the previous theorem.

Proposition 7.4.3 *In the assumptions of Theorem 7.4.2 we have*

$$\nu_E(p) = \frac{\nu_H(p)}{|\nu_H(p)|} \quad \text{and} \quad \partial E \setminus C(\partial E) = \partial_{*H} E \setminus C(\partial E) \tag{7.48}$$

where $\nu_E(p)$ is the generalized inward normal (Definition 2.4.9) and $\nu_H(p)$ is the horizontal normal to ∂E at p (Definition 2.2.9).

Proof. We adopt the notation used in the proof of Theorem 7.4.2. By definition of generalized inward normal it is not difficult to check that for any $\varphi \in \Gamma_c(HB_{p,r})$

$$\int_{B_{p,r}} \langle \nu_E, \varphi \rangle \, d|\partial E|_H = r^{Q-1} \int_{B_1} \langle \nu_{E_{p,r}}, \varphi \circ l_p \circ \delta_r \rangle \, d|\partial E_{p,r}|_H$$

and by formulae (2.1) and (6.18) we have

$$\int_{B_{p,r}} \langle \nu_E, \varphi \rangle \, d|\partial E|_H = \int_{B_1} \langle \nu_E, \varphi \rangle \circ l_p \circ \delta_r \; d\big(\delta_{1/r} \circ l_p^{-1}\big)_\sharp |\partial E|_H$$
$$= r^{Q-1} \int_{B_1} \langle \nu_E \circ l_p \circ \delta_r, \varphi \circ l_p \circ \delta_r \rangle \; d|\partial E_{p,r}|_H$$

therefore $\nu_{E_{p,r}} = \nu_E \circ l_p \circ \delta_r$ as vector measurable maps, that indeed are continuous. By previous equality we infer that

$$\fint_{B_{p,r}} \langle \nu_E, \varphi \rangle \, d|\partial E|_H = \frac{r^{Q-1}}{|\partial E|_H(B_{p,r})} \int_{B_1} \langle \nu_{E_{p,r}}, \varphi \circ l_p \circ \delta_r \rangle \; d|\partial E_{p,r}|_H$$
$$\fint_{B_1} \langle \nu_{E_{p,r}}, \varphi \circ l_p \circ \delta_r - \psi \rangle \; d|\partial E_{p,r}|_H + \fint_{B_1} \langle \nu_{E_{p,r}}, \psi \rangle \; d|\partial E_{p,r}|_H$$

whenever $\psi \in \Gamma_c(HB_{p,r})$. By virtue of (6.21) and (7.41) we obtain

$$|\partial E_{p,r}|_H(B_1) \longrightarrow |\partial S_g^+ \left(\nu_H(p)\right)|_H \, .$$

Hence, the weak convergence (7.42) and (7.50) yield

$$\limsup_{r \to 0^+} \left| \fint_{B_{p,r}} \langle \nu_E, \varphi \rangle \, d|\partial E|_H - \fint_{B_1} \langle \nu_{E_{p,r}}, \psi \rangle \; d|\partial E_{p,r}|_H \right|$$
$$= \left| \langle \nu_E(p), \varphi(p) \rangle - \fint_{B_1} \left\langle \frac{\nu_H(p)}{|\nu_H(p)|}, \psi \right\rangle \; d|\partial S_g^+ \left(\nu_H(p)\right)|_H \right|$$
$$\le \limsup_{r \to 0^+} \|\varphi \circ l_p \circ \delta_r - \psi\|_{L^\infty(B_1)} = \|\varphi(p) - \psi\|_{L^\infty(B_1)} \, .$$

Replacing ψ with ψ_k in the last estimate, where $(\psi_k) \subset \Gamma_c(HB_1)$ and $\psi_k \to \varphi(p)\mathbf{1}_{B_1}$ we obtain

$$\langle \nu_E(p), \varphi(p) \rangle = \left\langle \frac{\nu_H(p)}{|\nu_H(p)|}, \varphi(p) \right\rangle$$

and the arbitrary choice of φ yields the equality $\nu_E(p) = \nu_H(p)/|\nu_H(p)|$. Now we have to prove that $p \in \partial_{*H}E$. We check the continuity of ν_E, that is defined in principle as a measurable map by the Riesz Representation Theorem. Let $\varphi = \sum_{j=1}^m \varphi^j \, W_j \in \Gamma_c(H\mathbb{M})$ and apply Proposition 2.3.47, formula (2.1) and (2.43), obtaining

$$\int_{\mathbb{R}^q} \langle \nu_E, \varphi \rangle \circ F \, dF_\sharp^{-1} |\partial E|_H = \int_{\mathbb{M}} \langle \nu_E, \varphi \rangle \, d|\partial E|_H = -\int_E \operatorname{div}_H \varphi \, dv_g$$
$$= -\int_{\tilde{E}} \sum_{j=1}^m \tilde{W}_j \tilde{\varphi}_j \, d\mathcal{L}^q = \int_{\partial \tilde{E}} \sum_{j=1}^m \tilde{\varphi}^j \langle \nu_{\tilde{E}}, \tilde{W}_j \rangle \, d\mathcal{H}^{q-1} \, ,$$

where $\tilde{E} = F^{-1}(E)$, $\tilde{W}_j = F_*^{-1} W_j \in \Gamma(T\mathbb{R}^q)$ and $\nu_{\tilde{E}}$ is the unit inward normal to $\tilde{E}$. In view of the arbitrary choice of φ we get the equality of vector measures

$$\nu_E \circ F \, F_\sharp^{-1} |\partial E|_H = w_{\tilde{E}} \mathcal{H}^{q-1} \llcorner \partial \tilde{E} \,, \tag{7.49}$$

where $w_{\tilde{E}} = \sum_{j=1}^m \langle \nu_{\tilde{E}} \, \tilde{W}_j \rangle \, e_j$. From continuity of $\nu_{\tilde{E}}$ we deduce that ν_E is also continuous. Next we prove that $p \in \partial_{*H} E$. Let us apply the change of variable formula (2.1)

$$\int_{B_{p,r}} \nu_E \, d|\partial E|_H = \int_{B_1} \nu_E \circ l_p \circ \delta_r \; d \left(\delta_{1/r} \circ l_p^{-1} \right)_\sharp |\partial E|_H$$

and formula (6.18), obtaining

$$⨍_{B_{p,r}} \nu_E \, d|\partial E|_H = \frac{r^{Q-1}}{|\partial E|_H (B_{p,r})} \int_{B_1} \nu_E \circ l_p \circ \delta_r \; d|\partial E_{p,r}|_H = ⨍_{B_1} \nu_E \circ l_p \circ \delta_r \; d|\partial E_{p,r}|_H$$

The continuity of ν_E implies

$$\lim_{r \to 0^+} ⨍_{B_{p,r}} \nu_E \, d\,|\partial E|_H = \nu_E(p) \tag{7.50}$$

hence the first equality of (7.48) yields our claim. $\square$

Theorem 7.4.2 will be an essential tool in the proof of the next result, where an explicit representation for the perimeter measure of C^1 domains is given, (7.51). Another crucial tool to obtain this representation formula is Theorem 6.6.2, where it is proved that the characteristic set of C^1 hypersurfaces is $\mathcal{H}^{Q-1}$-negligible. Note that Theorem 6.6.2 is proved in stratified groups, instead of general graded groups with a homogeneous distance. In fact, the main result used in its proof is Theorem 6.3.1 that follows from the coarea inequality (6.1), which in turn was proved using the a.e. H-differentiability of Lipschitz maps. This last result can be proved only for stratified groups as we have discussed in Chapter 3. Just for this reason the following theorem is proved in sub-Riemannian groups instead of general graded groups endowed with a homogeneous distance.

Theorem 7.4.4 *Let E be a C^1 closed subset of the sub-Riemannian group $\mathbb{G}$. Then we can represent the perimeter measure as follows*

$$|\partial E|_H = \frac{\theta^g_{Q-1}(\nu_H)}{\omega_{Q-1}} \, \mathcal{S}^{Q-1} \llcorner \partial E \,, \tag{7.51}$$

where $\mathcal{S}^{Q-1}$ and $\theta^g_{Q-1}(\nu_H)$ refer to the same homogeneous distance.

PROOF In view of Theorem 6.6.2 we have $\mathcal{H}^{Q-1}(C(\partial E)) = 0$. Thus, by (7.48) the rest of the proof follows exactly as it is done in Theorem 6.4.12 using the limit (7.41) at points of $\partial_{*H}E \setminus C(\partial E)$. $\square$

By Remark 5.2.3 the following numbers

$$\underline{\theta}_{Q-1} = \inf_{\nu\in V_1} \theta^g_{Q-1}(\nu) \quad \text{and} \quad \overline{\theta}_{Q-1} = \sup_{\nu\in V_1} \theta^g_{Q-1}(\nu)$$

are finite positive constants. Thus, by virtue of Theorem 7.4.4 we immediately obtain the following result.

Theorem 7.4.5 *For any C^1 closed subset $E \subset \mathbb{G}$ we have the following estimates*

$$\frac{\underline{\theta}_{Q-1}}{\omega_{Q-1}}\ \mathcal{H}^{Q-1}(\partial E \cap \Omega)\ \le P_H(E,\Omega) \le\ \frac{2^Q\,\overline{\theta}_{Q-1}}{\omega_{Q-1}}\ \mathcal{H}^{Q-1}(\partial E \cap \Omega) \tag{7.52}$$

for any bounded open set $\Omega \subset \mathbb{G}$.

The following theorem is a straightforward consequence of (7.16) and (7.51).

Theorem 7.4.6 *For any C^1 closed subset $E \subset \mathbb{G}$ we have*

$$|\partial E|_H = |\nu_H|\, \sigma_g \llcorner \partial E\,. \tag{7.53}$$

The previous theorems complete the picture of relations among perimeter measure of C^1 domains of sub-Riemannian groups, the Riemannian surface measure of their boundary and the $Q-1$ dimensional spherical Hausdorff measure of their boundary with respect to a homogeneous distance.

We also mention that other notions of surface measure can be considered in the sub-Riemannian context, for instance in [141] the notion of Minkowski content of a hypersurface is extended to CC-spaces and the equality between X-perimeter of a smooth domain and the Minkowski content of its boundary is proved. Here the X-perimeter is referred to a general system of vector fields, see [31].

Another simple consequence of (7.51) is the following divergence theorem for C^1 sets of sub-Riemannian groups, that immediately follows using formula (2.45) and Proposition 7.4.3

Theorem 7.4.7 (Divergence Theorem) *Let E be a C^1 closed subset of a sub-Riemannian group $\mathbb{G}$. Then for any $\phi \in \Gamma_c(H\mathbb{G})$ we have*

$$\int_E \mathrm{div}_H \phi\, dv_g = -\int_{\partial E} \left\langle \phi, \frac{\nu_H}{|\nu_H|} \right\rangle \frac{\theta^g_{Q-1}(\nu_H)}{\omega_{Q-1}}\, d\mathcal{S}^{Q-1}\,. \tag{7.54}$$

By Proposition 5.2.5 and Proposition 5.1.12 we easily see that the previous formula becomes

$$\int_E \mathrm{div}_H \phi \, dv_g = -\frac{\alpha_{Q-1}}{\omega_{Q-1}} \int_{\partial E} \left\langle \phi, \frac{\nu_H}{|\nu_H|} \right\rangle d\mathcal{S}_\rho^{Q-1} \tag{7.55}$$

in $\mathcal{R}$-rotational groups, where ρ is CC-distance relative to the graded metric that makes the group $\mathcal{R}$-rotational and α_{Q-1} is the constant metric factor defined in Proposition 5.2.5. In a similar way, taking into account (7.53) we obtain another version of the divergence theorem

$$\int_E \mathrm{div}_H \phi \, dv_g = -\int_{\partial E} \langle \phi, \nu_H \rangle \, d\sigma_g \, . \tag{7.56}$$

Chapter 8

Weak differentiability of H-BV functions

In this last chapter we focus the attention on the real-valued maps defined on sub-Riemannian groups such that their distributional derivatives along horizontal directions are finite measures, namely H-BV functions. When all these weak derivatives up to an order greater than one are finite measures, we have functions of H-bounded higher order variation (Definition 8.5.1). In the corresponding Euclidean theory it is well known that they are a.e. approximately differentiable up to the order of their finite variation and that their approximate discontinuity set is rectifiable. To have an account of the classical theory we refer the reader to the works [27], [55], [185] and to the historical note in [6]. So the first natural question arising from the Euclidean context is to study the approximate differentiability and the properties of the approximate discontinuity set in the framework of sub-Riemannian groups.

We will accomplish this study using some tools of classical Analysis that have been recently extended to the context of sub-Riemannian geometries. In fact, in the last few years there has been a strong development of the theory of Sobolev spaces in the sub-Riemannian context and also in the metric one: important results such as Poincaré inequalities, embedding theorems, representation formulae, trace theorems, compactness results and much more, hold in sub-Riemannian groups when formulated in intrinsic terms, e.g. using left translations, dilations and the CC-distance, see [42], [66], [67], [79], [91], [100], [140] (in Section 2.5 we have collected some of these results).

In Section 8.1 we introduce the notion of approximate continuity and of approximate differentiability for locally summable maps. Following Federer's definition, we also define a weaker notion of approximate differentiability that will be useful in the proof of Theorem 8.2.2.

Section 8.2 deals with the approximate differentiability of H-BV functions. Here the problem consists in the fact that an integral inequality of type (8.3) is not still known for nonabelian sub-Riemannian groups, therefore the classical approach described in [6] cannot be applied. We will utilize two notions of approximate differentiability: the stronger one of Definition 8.1.2 and the weaker one of (8.2). However we will always refer to approximate differentiability meaning the stronger notion. The idea of our approach is to prove first that H-BV functions are a.e. approximately differentiable with respect to the weaker notion and subsequently, by a bootstrap argument based on the Poincaré inequality, to achieve the approximate differentiability. We also obtain that the approximate differential coincides a.e. with the density of the absolutely continuous part of the H-BV vector measure.

In Section 8.3 we prove that the approximate discontinuity set of an H-BV function is contained in a countable union of the essential boundaries of sets with H-finite perimeter, up to an $\mathcal{H}^{Q-1}$-negligible set, (8.16). Note that by results of [5] we can conclude that S_u is contained in a countable union of sets with $\mathcal{H}^{Q-1}$-finite measure. Furthermore, whenever a rectifiability theorem for sets of H-finite perimeter in sub-Riemannian groups holds, we immediately achieve the $\mathbb{G}$-rectifiability of S_u when u is H-BV. Here rectifiability theorem means that the H-reduced boundary of H-finite perimeter sets is $\mathbb{G}$-rectifiable. This result is known only for sub-Riemannian groups of step two, [73], and it is an open question for groups of higher step.

In Section 8.4 we present an important integral inequality, named "representation formula" by the authors of [67]. This will be a crucial tool in the Section 8.5, concerning the proof of higher order differentiability. In order to keep the chapter more self-contained, we will give a proof of this formula adapting to our case the proof of Theorem 1 in [67], that holds for the much more general spaces of homogeneous type which satisfy the Poincaré inequality.

In Section 8.5 we prove the approximate differentiability of higher order. Notice that the case of H-BV^2 maps correspond to a weak Alexandrov type differentiability (Theorem 8.5.6). Our method is based on two crucial estimates: first, in a suitable point x we estimate the difference $|u(y) - u(x)|$ utilizing the maximal function, see (8.8). Second, we use the representation formula (8.19) in the form (8.23) in order to obtain information on the behavior of $|D_H v|$, where $v = |D_H u|$. This procedure is applied to the function u once we have subtracted a suitable polynomial in such a way that the densities of the absolutely continuous parts of the horizontal measure derivatives are vanishing at the point. By the isomorphism between polynomials and left invariant differential operators the above mentioned polynomial is uniquely defined (Proposition 8.5.3) and it corresponds to the "intrinsic" Taylor expansion of the map at the fixed point. We point out that some difficulties come from the non-commutativity of left invariant differential operators. Here we exploit the important Poincaré-Birkhoff-Witt Theorem, which provides a manageable basis for the algebra of left invariant differential operators.

In Section 8.6 we construct a nontrivial class of H-BV^2 functions in the Heisenberg group arising as inf-convolutions of the cost function $d(x, y)^2/2$, where the distance d is constructed with a suitable gauge norm.

8.1 Weak notions of regularity

Here we introduce some weak notions of limit and differential for measurable functions on sub-Riemannian groups. We will utilize the same notation of Section 2.5 for both the Riemannian volume and the metric ball with respect to the CC-distance.

Definition 8.1.1 (Approximate limit) We say that a function $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ has an *approximate limit* $\lambda \in \mathbb{R}^m$ at $x \in \Omega$ if

$$\lim_{r \to 0^+} \fint_{U_{x,r}} |u(y) - \lambda| \, dy = 0 \, .$$

If u does not have an approximate limit at x we say that x is an *approximate discontinuity point* and we denote by S_u the measurable set of all these points, namely the *approximate discontinuity set.*

It is clear that the approximate limit is uniquely defined and that it does not depend on the representative element of u; it will be denoted by $\tilde{u}(x)$. We call the points in $\Omega \setminus S_u$ *approximate continuity points* of u. Since sub-Riemannian groups are doubling spaces we have that S_u is negligible and $u(x) = \tilde{u}(x)$ for a.e. $x \in \Omega$. This follows from Theorem 2.1.22 of the thesis and Theorem 2.9.8 of [55].

There is a weaker, and more canonical, definition of approximate limit (we will refer to [55]). Let us consider a measurable function $u : \Omega \longrightarrow \mathbb{R}$, $x \in \Omega$ and $\lambda \in \mathbb{R}$. We say that λ is the approximate limit of u at x if for any $\varepsilon > 0$ we have that

$$x \in \mathcal{I}(\{z \in \Omega \mid |u(z) - \lambda| < \varepsilon\}) \, .$$

The approximate limit λ is uniquely defined and it is denoted by $\operatorname{ap}\lim_{z \to x} u(z)$. Note that $x \in \Omega \setminus S_u$ implies $\operatorname{ap}\lim_{z \to x} u(z) = \tilde{u}(x)$, but the converse is not true in general, see for instance Remark 3.66 of [6]. However there will be no confusion utilizing the same word (but a different notation) for the two concepts. Moreover, for locally bounded functions u there is a complete equivalence: $\operatorname{ap}\lim_{z \to x} u(z) = \lambda$ implies $x \in \Omega \setminus S_u$ and $\lambda = \tilde{u}(x)$.

In the sequel it will be useful to represent a scalar valued H-linear map $L : \mathbb{G} \longrightarrow \mathbb{R}$ by a horizontal vector of $\mathcal{G}$. This is easily done exploiting the graded metric on the group as follows

$$L(x) = \langle v, \ln x \rangle \quad \text{for any } x \in \mathbb{G}.$$

We will use the notation v^* to indicate the map L.

Definition 8.1.2 (Approximate differential) Consider $u \in L^1_{\text{loc}}(\Omega)$ and a point $x \in \Omega \setminus S_u$. We say that u is *approximately differentiable* at x if there exists an H-linear map $L : \mathbb{G} \longrightarrow \mathbb{R}$ such that

$$\lim_{r \to 0^+} \fint_{U_{x,r}} \frac{|u(z) - \tilde{u}(x) - L(x^{-1}z)|}{r} \, dz = 0 \, . \tag{8.1}$$

The map L is uniquely defined, it is denoted by $d_H u(x)$ and it is called the *approximate differential* of u at x.

We warn the reader that we have used the same symbol for both differentials of Definitions 3.4.8 and 8.1.2. This slight abuse of notation is justified by the fact that differentiability implies approximate differentiability.

In the spirit of [55], a weaker notion of approximate differentiability could be given, saying that the approximate differential of a map $u : A \subset \mathbb{G} \longrightarrow \mathbb{R}$ at $x \in \mathcal{I}(A)$ is the unique H-linear map $L : \mathbb{G} \longrightarrow \mathbb{R}$ such that

$$\operatorname{ap}\lim_{y \to x} \frac{u(y) - u(x) - L(x^{-1}y)}{d(x,y)} = 0\,. \tag{8.2}$$

We point out that the approximate differentiability implies the existence of the approximate limit (8.2), as it will be proved in Proposition 8.2.1, but already in the Euclidean case the converse is not true, see for instance Remark 3.66 of [6].

8.2 First order differentiability

In this section we prove the approximate differentiability of H-BV functions. We point out that the validity of the following inequality

$$\int_{U_{x,r}} \frac{|u(x) - u(z)|}{d(x,z)}\, dz \le C \int_0^1 \frac{|D_H u|(U_{x,\lambda r t})}{t^Q}\, dt\,, \tag{8.3}$$

where $\lambda, C > 0$ are dimensional constants, would imply a slightly stronger approximate differentiability via classical methods described in [6]. Unfortunately, this seems to be an open question.

Proposition 8.2.1 *Let $u : \Omega \longrightarrow \mathbb{R}$ be a Borel map. Then the following statements are equivalent:*

1. *for a.e. $x \in \Omega$ there exists an H-linear map $L_x : \mathbb{G} \longrightarrow \mathbb{R}$ such that*

$$\operatorname{ap}\lim_{y \to x} \frac{u(y) - u(x) - L_x(x^{-1}y)}{d(x,y)} = 0\,, \tag{8.4}$$

2. *u is countably Lipschitz up to a negligible set, i.e. there exists a countable family of Borel subsets $\{A_i \mid A_i \subset \Omega,\ i \in \mathbb{N}\}$ such that and for each $i \in \mathbb{N}$ the restriction $u_{|A_i}$ is a Lipschitz map and we have $\left|\Omega \setminus \bigcup_{i \in \mathbb{N}} A_i\right| = 0$.*

Furthermore 1. and 2. hold if u is approximately differentiable a.e. in Ω.

PROOF. We start proving that property 1 is implied by the approximate differentiability. Assume that u is approximately differentiable at $x \in \Omega$ with approximate differential $d_H u(x)$. Let us fix $\varepsilon > 0$ and consider the set

$$E_{x,\rho} = \left\{ z \in U_{x,\rho} \Big| |u(z) - u(x) - d_H u(x)(x^{-1}z)| > \varepsilon\, d(x,z) \right\} .$$

In order to get (8.4) we have to prove that

$$\lim_{\rho \to 0^+} |E_{x,\rho}|\, \rho^{-Q} = 0 . \tag{8.5}$$

Let us define the maps $T_{x,\rho}(z) = \delta_{1/\rho}(x^{-1}z)$ and

$$R_{x,\rho}(z) = \frac{|u(x\delta_\rho z) - u(x) - d_H u(x)(\delta_\rho z)|}{\rho} ,$$

observing that

$$T_{x,\rho}(E_{x,\rho}) = \left\{ y \in U_1 \Big| \ R_{x,\rho}(y) > \varepsilon\, d(y) \right\} := A_\rho .$$

Hence we have $|E_{x,\rho}|\rho^{-Q} = |A_\rho|$, so (8.5) follows if we prove that

$$\lim_{\rho \to 0^+} |A_\rho| = 0 . \tag{8.6}$$

By hypothesis, making a change of variable we get

$$\lim_{\rho \to 0^+} \fint_{U_{x,\rho}} \frac{|u(z) - u(x) - d_H u(x)(x^{-1}z)|}{\rho}\, dz = \lim_{\rho \to 0^+} \fint_{U_1} R_{x,\rho}(z)\, dz = 0 . \tag{8.7}$$

For each $t \in]0,1[$ we have

$$\int_{A_\rho \setminus U_t} R_{x,\rho} \geq \ |A_\rho \setminus U_t|\, \varepsilon\, t ,$$

so in view of (8.7) we obtain $|A_\rho \setminus U_t| \longrightarrow 0$ as $\rho \to 0^+$. It follows that

$$\limsup_{\rho \to 0^+} |A_\rho| \leq \limsup_{\rho \to 0^+} |A_\rho \setminus U_t| + |U_t| = |U_t| .$$

Finally, letting $t \to 0$ equation (8.6) follows, so statement 1 is proved. The fact that statement 1 implies statement 2 can be proved as in Theorem 3.1.8 of [55], see also Theorem 6 in [177]. Now, let us prove that statement 2 implies 1. By Theorem 3.4.11 we know that $u_{|A_i}$ is a.e. differentiable. Let us indicate by $\mathcal{D}_u(A_i)$ the subset of $\mathcal{I}(A_i)$ where $u_{|A_i}$ is differentiable in A_i. Clearly, we have

$$\left| \Omega \setminus \bigcup_{i \in \mathbb{N}} \mathcal{D}_u(A_i) \right| = 0 .$$

Consider $x \in \mathcal{D}_u(A_i)$ and choose $\varepsilon > 0$. Then there exists $\delta > 0$ such that for any $z \in A_i \cap U_{x,\delta}$ we get

$$R(z) = \frac{|u(z) - u(x) - L(x^{-1}z)|}{d(z,x)} < \varepsilon\,,$$

with $L = du_{|A_i}(x)$. From the last inequality it follows that

$$U_{x,r} \cap \{z \in \Omega \mid R(z) \geq \varepsilon\} \subset U_{x,r} \setminus A_i$$

for any $r \leq \delta$. Hence we get

$$\limsup_{r\to 0^+} \frac{|U_{x,r} \cap \{z \in \Omega \mid R(z) \geq \varepsilon\}|}{|U_{x,r}|} \leq \limsup_{r\to 0^+} \frac{|U_{x,r} \setminus A_i|}{|U_{x,r}|} = 0\,,$$

in view of the fact that x is a density point of A_i. □

Theorem 8.2.2 *Let $u : \Omega \longrightarrow \mathbb{R}$ be a locally H-BV function. Then, u is approximately differentiable a.e. in Ω and the differential corresponds to the density of the absolutely continuous part of $D_H u$, i.e. $d_H u(x) = \nabla_H u(x)^*$ for a.e. $x \in \Omega$.*

Proof. We first prove that u is countably Lipschitz up to a negligible set. To see this, we use a standard technique which is well known in the study of metric Sobolev spaces, see for instance Theorem 3.2 of [91]. Let us fix $t > 0$ and define the open subset

$$\Omega_t = \{z \in \Omega \mid \operatorname{dist}(z, \Omega^c) > t\}\,.$$

We want to prove that u is countably Lipschitz on Ω_t. We cover Ω_t with a countable union of open balls $\{P_j \mid j \in \mathbb{N}\}$ with center in Ω_t and radius $t/4$. Let us consider $j \in \mathbb{N}$ and two approximate continuity points $x, y \in P_j$. For each $i \in \mathbb{Z}$ we define the balls $B_i(x) = U_{x,2^{-i}d(x,y)}$ and $B_i(y) = U_{y,2^{-i}d(x,y)}$. Notice that $B_i(x)$ and $B_i(y)$ are compactly contained in Ω for any $i \geq -1$. We have

$$|\tilde{u}(x) - u_{B_0(x)}| \leq \sum_{i=0}^{\infty} |u_{B_{i+1}(x)} - u_{B_i(x)}| \leq \sum_{i=0}^{\infty} \fint_{B_{i+1}(x)} |u(z) - u_{B_i(x)}|\, dz$$

and by Poincaré inequality (2.51) it follows

$$\leq C\, d(x,y) \sum_{i=0}^{\infty} 2^{-i-1} \frac{|D_H u|(B_{i+1}(x))}{|B_{i+1}(x)|} \leq C\, d(x,y)\, M_{d(x,y)} |D_H u|(x)\,.$$

In the same way we get

$$|\tilde{u}(y) - u_{B_0(y)}| \leq C\, d(x,y)\, M_{d(x,y)} |D_H u|(y)\,.$$

We proceed analogously, getting

$$
\begin{aligned}
|u_{B_0(x)} - u_{B_0(y)}| &\le |u_{B_0(x)} - u_{B_{-1}(x)}| + |u_{B_{-1}(x)} - u_{B_0(y)}| \\
&\le \fint_{B_0(x)} |u(z) - u_{B_{-1}(x)}|\, dz + \fint_{B_0(y)} |u(z) - u_{B_{-1}(x)}|\, dz \\
\le 2^{Q+1} \fint_{B_{-1}(x)} |u(z) - u_{B_{-1}(x)}|\, dz &\le C\, 2^{Q+2}\, d(x,y)\, M_{2d(x,y)}|D_H u|(x)\,.
\end{aligned}
$$

Finally, we obtain

$$
|\tilde{u}(x) - \tilde{u}(y)| \le c\, d(x,y)\, \big(M_{2d(x,y)}|D_H u|(x) + M_{2d(x,y)}|D_H u|(y)\big) \tag{8.8}
$$

with $c = (2^{Q+2}+2)\, C$. Now, let us consider the decomposition

$$
P_j = N_j \cup \left(\bigcup_{l \in \mathbb{N}} E_{jl} \right)
$$

where E_{jl} is the Borel set of all approximate continuity points $z \in P_j$ such that $M|D_H u|(z) \le l$ and $N_j = S_u \cup \{ z \in P_j \mid M|D_H u|(z) = +\infty \}$. Then N_j is a negligible set and by (8.8) it follows that

$$
|\tilde{u}(x) - \tilde{u}(y)| \le 2\, c\, l\, d(x,y) \qquad \forall x,\, y \in E_{jl}
$$

and any $j, l \in \mathbb{N}$. This gives the countably Lipschitz property of u in Ω_t. Observing that Ω is a countable union of $\Omega_{1/k}$, with $k \in \mathbb{N}\backslash\{0\}$ we obtain the countably Lipschitz property of u in Ω. In view of Proposition 8.2.1 the countably Lipschitz property yields the existence of an H-linear map $L_x : \mathbb{G} \longrightarrow \mathbb{R}$ such that for a.e. $x \in \Omega$ we have

$$
\operatorname{ap}\lim_{y \to x} \frac{u(y) - \tilde{u}(x) - L_x(x^{-1}y)}{d(x,y)} = 0\,. \tag{8.9}
$$

In order to prove the a.e. approximate differentiability, we select a point $x \in \Omega \setminus (S_u \cup S_{\nabla_H u})$ such that (8.9) holds and

$$
\lim_{r \to 0^+} \frac{|D_H^s u|(U_{x,r})}{r^Q} = 0\,. \tag{8.10}
$$

In view of Remark 2.4.6 the set of points which do not satisfy (8.10) is negligible, so the set of selected points with all the above properties has full measure in Ω. We fix $\varepsilon > 0$ and consider the set

$$
F_{x,r} = \left\{ y \in U_{x,r} \mid |u(y) - \tilde{u}(x) - L_x(x^{-1}y)| > \varepsilon d(x,y) \right\}\,,
$$

observing that

$$Z_{x,r} := \delta_{1/r}(x^{-1}F_{x,r}) = \left\{ z \in U_1 \mid \frac{|u(x\delta_r z) - \tilde{u}(x) - L_x(\delta_r z)|}{r} > \varepsilon \right\} . \tag{8.11}$$

In view of (8.10) we have that $|F_{x,r}|\, r^{-Q} \longrightarrow 0$ as $r \to 0^+$, therefore

$$|Z_{x,r}| = |\delta_{1/r}(x^{-1}F_{x,r})| = r^{-Q}|x^{-1}F_{x,r}| = r^{-Q}|F_{x,r}| \longrightarrow 0 \quad \text{as} \quad r \to 0^+ . \tag{8.12}$$

Now, we consider the difference $S_x = \nabla_H u(x)^* - L_x$ and define the maps

$$v(y) = u(y) - \tilde{u}(x) - \nabla_H u(x)^*(x^{-1}y) ,$$

$$w_{x,r}(z) = \frac{v(x\delta_r z) + S_x(\delta_r z)}{r} = v_{x,r}(z) + S_x(z) ,$$

observing that $\tilde{v}(x) = 0$, $|D_H v_{x,r}|(U_1) \longrightarrow 0$ as $r \to 0^+$ and

$$Z_{x,r} = \{ z \in U_1 \mid |w_{x,r}(z)| > \varepsilon \} .$$

Thus, by (8.12) it follows that $w_{x,r} \to 0$ in measure as $r \to 0^+$. Since $v_{x,r}$ is an H-BV function, we can apply Poincaré inequality (2.51), getting

$$\int_{U_1} |v_{x,r}(z) - m_{x,r}|\, dz \le C\, |D_H v_{x,r}|(U_1) \longrightarrow 0 \quad \text{as} \quad r \to 0^+ , \tag{8.13}$$

where $m_{x,r} = ⨍_{U_1} v_{x,r}$. Then, we obtain

$$\int_{U_1} |w_{x,r}(z) - m_{x,r} - S_x(z)|\, dz \longrightarrow 0 \quad \text{as} \quad r \to 0^+ .$$

It follows that $m_{x,r} + S_x$ converges to zero in measure on U_1 as $r \to 0^+$. This easily implies that $m_{x,r} \to 0$ and $S_x = 0$. So, $\nabla_H u(x)^* = L_x$ and in view of (8.13) we get

$$\frac{1}{r} ⨍_{U_{x,r}} |v(z)|\, dz = ⨍_{U_1} |v_{x,r}(z)|\, dz \longrightarrow 0 \quad \text{as} \quad r \to 0^+ ,$$

which proves the approximate differentiability of u at x with $d_H u(x) = \nabla_H u(x)^*$. □

8.3 Size of S_u

In this section we study the regularity of the approximate discontinuity set S_u when u is an H-BV function. The following two lemmas are crucial to prove Theorem 8.3.3. They are the sub-Riemannian version of Lemma 3.74 and Lemma 3.75 of [6]. We give the proof of them in order to emphasize the main steps, where the relevant sub-Riemannian general theorems are needed. Furthermore, since Lemma 3.74 in [6] is proved using the Besicovitch Covering Theorem, which may fail in general sub-Riemannian groups, we show another simpler way to prove it, adopting the Vitali Covering Theorem for doubling spaces.

In the sequel we will use the *upper Q dimensional density* of a measure μ at a point $x \in \mathbb{G}$, defined as follows

$$\theta^*_Q(\mu, x) = \limsup_{r\to 0^+} \frac{\mu(B_{x,r})}{|B_{x,r}|}.$$

In the case $\mu = v_g \llcorner E$, where $E \subset \mathbb{G}$ is a measurable set, we will write $\theta^*_Q(E, x)$.

Lemma 8.3.1 *Let (E_h) be a sequence of measurable subsets of Ω, such that $|E_h| \longrightarrow 0$ and $P_H(E_h, \Omega) \longrightarrow 0$ as $h \to \infty$. Then, for any $\alpha > 0$ we have*

$$\mathcal{H}^{Q-1}\left(\bigcap_{h=1}^{\infty} \{x \in \Omega \mid \theta^*_Q(E_h, x) \geq \alpha\}\right) = 0\,.$$

Proof. Let us fix $\delta > 0$ and $\alpha \in]0, 1[$. We consider a Borel set $E \subset \Omega$ such that $|E| < |U_1|\, \alpha\, \delta^Q/2$ and define

$$E^\alpha = \{x \in \Omega \mid \theta^*_Q(E, x) \geq \alpha\}\,.$$

For any $x \in E^\alpha$ the estimate

$$\frac{|U_{x,\delta} \cap E|}{|U_{x,\delta}|} \leq \frac{|E|}{|U_1|\, \delta^Q} < \frac{\alpha}{2}$$

implies the existence of a radius $r_x \in]0, \delta[$ such that $|U_{x,r_x} \cap E| = \alpha |U_{x,r_x}|/2$. Thus, in view of (2.52) we get

$$\frac{\alpha}{2}\, |U_1|\, r_x^Q = |U_{x,r_x} \cap E| \leq C\, r_x\, P_H(E, U_{x,r_x}). \tag{8.14}$$

Now, let us consider an open subset $\Omega' \Subset \Omega$, with $0 < \delta < \operatorname{dist}(\Omega', \partial\Omega)$ and $|E| \leq |U_1|\, \alpha\, \delta^Q/2$. Using a well known covering theorem for the family $\{U_{x,r_x} \mid x \in \Omega' \cap E^\alpha\}$ (Corollary 2.8.5 in [55]), we get a countable disjoint subfamily

$$\{B_j \mid B_j = U_{x_j, r_{x_j}}, j \in \mathbb{N}\}$$

such that $\Omega' \cap E^\alpha \subset \bigcup_{h=1}^\infty 5B_j$, where $5B_j$ is the ball of center x_j and radius $5r_{x_j}$. Therefore, the estimate (8.14) implies

$$\mathcal{H}^{Q-1}_{10\delta}(\Omega' \cap E^\alpha) \leq 5^{Q-1} \sum_{i=1}^{\infty} r_{x_i}^{Q-1} \leq \frac{2C5^{Q-1}}{\alpha\, |U_1|} \sum_{i=1}^{\infty} P_H(E, B_i) \leq \frac{C'}{\alpha} P_H(E, \Omega)\,.$$

We fix the sequence $\delta_i = (2|E_i|/|U_1|\alpha)^{1/Q}$, observing that $\delta_i \leq \delta$ for i large, hence

$$\mathcal{H}^{Q-1}_{10\delta_i}\left(\Omega' \cap \bigcap_{h=1}^{\infty} E_h^{\alpha}\right) \leq \frac{C'}{\alpha} P_H(E_i, \Omega)\,.$$

Thus, letting first $\delta_i \to 0^+$ and then $\Omega' \uparrow \Omega$ the conclusion follows. □

Lemma 8.3.2 *Let $u : \Omega \longrightarrow \mathbb{R}$ be an H-BV function. Then, the set*

$$L = \left\{ x \in \Omega \mid \limsup_{r\to 0^+} \fint_{U_{x,r}} |u(y)|^{1^*}\, dy = \infty \right\}$$

is $\mathcal{H}^{Q-1}$-negligible, where $1^* = Q/(Q-1)$.

Proof. In view of Proposition 2.5.8 we can assume that $u \geq 0$ (replacing u by $|u|$). We define the set

$$D = \left\{ y \in \Omega \mid \limsup_{r\to 0^+} \frac{|D_H u|(U_{x,r})}{r^{Q-1}} = \infty \right\},$$

observing that by Theorem 2.10.17 and Theorem 2.10.18 of [55] and the fact that $|D_H u|(\Omega) < \infty$, we have $\mathcal{H}^{Q-1}(D) = 0$. For any integer $h \in \mathbb{N}$ we can choose $t_h \in]h, h+1[$ such that

$$P_H(E_{t_h}, \Omega) \leq \int_h^{h+1} P_H(E_t, \Omega)\, dt\,,$$

where $E_t = \{x \in \Omega \mid u(x) > t\}$, for each $t \geq 0$. By (2.49) we have

$$\sum_{h=0}^{\infty} P_H(E_{t_h}, \Omega) \leq \int_0^{\infty} P_H(E_t, \Omega) dt = |D_H u|(\Omega) < \infty\,.$$

Then, we apply Lemma 8.3.1 to the sequence (E_{t_h}) with $\alpha = 1$, getting

$$\mathcal{H}^{Q-1}\Big(\bigcap_{h=0}^{\infty} F_h\Big) = 0\,,$$

where we have defined $F_h = \{x \in \Omega \mid \theta^*_Q(E_{t_h}, x) = 1\}$. We want to prove that $L \subset D \cup \bigcap_{h=0}^{\infty} F_h$. In order to do that, we consider $x \notin D \cup \bigcap_{h=0}^{\infty} F_h$ and we prove that $x \notin L$. We define the constants $c_{x,r}$ to be the mean value of u on $U_{x,r}$ and apply the Sobolev-Poincaré inequality (2.53) obtaining

$$\fint_{U_{x,r}} |u(z) - c_{x,r}|^{1^*}\, dz \leq C \left(\frac{|D_H u|(U_{x,r})}{r^{Q-1}}\right)^{1^*}\,. \tag{8.15}$$

Notice that if $\limsup_{r\to 0^+} c_{x,r} < \infty$ then (8.15) implies $x \notin L$. Then, reasoning by contradiction, suppose that there exists a sequence c_{x,r_j} such that $r_j \to 0^+$ and $c_{x,r_j} \to \infty$ as $j \to \infty$. We define the function $v_j(y) = u(x\delta_{r_j} y) - c_{x,r_j}$, observing that $|D_H v_j|(U_1) = |D_H u|(U_{x,r_j})\, r_j^{1-Q}$. Since the sequence $|D_H v_j|(U_1)$, $j \in \mathbb{N}$, is bounded, Theorem 2.5.7 implies the convergence a.e. of (v_j) to a function $w \in L^1(U_1)$, possibly extracting a subsequence. As a consequence, $u(x\delta_{r_j} y) \to +\infty$ as $j \to \infty$ for a.e. $y \in U_1$, and therefore

$$|U_1| = \lim_{j\to\infty} |\{z \in U_1 \mid u(x\delta_{r_j} z) > t_h\}| = \lim_{j\to\infty} \frac{|\{y \in U_{x,r_j} \mid u(y) > t_h\}|}{r_j^Q}\,.$$

This implies $x \in \bigcap_{h=1}^\infty F_h$, contradicting the initial assumption. □

Theorem 8.3.3 *Let $u : \Omega \longrightarrow$ be an H-BV function. Then there exists an $\mathcal{H}^{Q-1}$-negligible set $L \subset \mathbb{G}$, such that*

$$S_u \setminus L \subset \bigcup_{j\in\mathbb{N}} \partial^* E_j\,, \tag{8.16}$$

where E_j has H-finite perimeter in Ω for every $j \in \mathbb{N}$.

Proof. We define $E_t = \{x \in \Omega \mid u(x) > t\}$ for $t \in \mathbb{R}$. By coarea formula (2.49) the set of numbers $t \in \mathbb{R}$ such that $P_H(E_t, \Omega) < \infty$ has full measure in $\mathbb{R}$, then it is possible to consider a countable dense subset $D \subset \mathbb{R}$ such that $P_H(E_t, \Omega) < \infty$ for any $t \in D$. Notice that from general results about sets of finite perimeter in Ahlfors metric spaces, see Theorem 4.2 in [5], we have that $\mathcal{H}^{Q-1}(E_t) < \infty$ for any $t \in D$. So, in view of Lemma 8.3.2 it suffices to prove the following inclusion

$$S_u \setminus L \subset \bigcup_{t\in D} \partial^* E_t\,, \tag{8.17}$$

where $L = \{x \in \Omega \mid \limsup_{r\to 0^+} \fint_{U_{x,r}} |u(y)|^{1^*}\, dy = \infty\}$. Let us consider a point $x \notin \bigcup_{t\in D} \partial^* E_t \cup L$. Then, for any positive t we have

$$\limsup_{r\to 0^+} \frac{|E_t|}{|U_{x,r}|} \le \frac{1}{t}\ \limsup_{r\to 0^+} \fint_{U_{x,r}} |u|\,, \tag{8.18}$$

hence, for any $t \in D$ sufficiently large such that the right hand side of (8.18) is less than one, it must be $\theta^*_Q(E_t, x) = 0$. Analogously, for $t \in D \cap (-\infty, 0)$ with $|t|$ large enough we have $\theta^*_Q(E_t^c, x) = 0$, so $\theta^*_Q(E_t, x) = 1$. This means that

$$\tau = \sup\{t \in D \mid \theta^*_Q(E_t, x) = 1\}$$

is a real number. Since D is dense in $\mathbb{R}$ and $t \longrightarrow |E_t|$ is a decreasing map it follows that $\theta^*_Q(E_t, x) = 0$ for any $t > \tau$ and $\theta^*_Q(E_t^c, x) = 0$ for any $t < \tau$. By virtue of this fact it follows that for any $\varepsilon > 0$ we have $|F_\varepsilon \cap U_{x,r}| = o(r^Q)$, where $F_\varepsilon = \{y \in \Omega \mid |u(y) - \tau| > \varepsilon\}$. Finally

$$\limsup_{r\to 0^+} \fint_{U_{x,r}} |u(y) - \tau|\, dy \le \varepsilon + \limsup_{r\to 0^+} \frac{1}{|U_{x,r}|} \int_{F_\varepsilon} |u(y) - \tau|\, dy$$

$$\le \varepsilon + \limsup_{r\to 0^+} \left(\frac{|F_\varepsilon|}{|U_{x,r}|}\right)^{1/Q} \left(\fint_{U_{x,r}} |u(y) - \tau|^{1^*}\, dy\right)^{1/1^*} = \varepsilon\,.$$

Letting $\varepsilon \to 0^+$, we obtain that $x \notin S_u$, so the inclusion (8.17) is proved. □

8.4 Representation formula

In this section we prove the "representation formula" for H-BV functions. We recall that the metric ball of center $x \in \mathbb{G}$ and radius $r > 0$ with respect to the CC-distance is denoted by $U_{x,r}$.

Theorem 8.4.1 (Representation formula) *There exists a dimensional constant $C > 0$ such that*

$$|\tilde{w}(x) - w_{U_{x,r}}| \le C \int_{U_{x,r}} \frac{1}{\rho(x,y)^{Q-1}}\, d|D_H w|(y). \tag{8.19}$$

for any $w \in BV_H(U_{x,r})$ and $x \notin S_w$, where ρ is the CC-distance of the group.

PROOF. Let us define the radii $r_j = r2^{-j}$ and $w_{U_{x,r_j}} = \fint_{U_{x,r_j}} w dv_g$ for every $j \in \mathbb{N}$. By the fact that $x \notin S_w$ and the Poincaré inequality (2.50), we obtain the following chain of estimates

$$\begin{aligned}
&|w(x) - w_{U_{x,r}}| \le \sum_{j\in\mathbb{N}} |w_{U_{x,r_{j+1}}} - w_{U_{x,r_j}}| \le \sum_{j\in\mathbb{N}} \fint_{U_{x,r_{j+1}}} |w - w_{U_{x,r_j}}| \\
&\le 2^Q \sum_{j\in\mathbb{N}} \fint_{U_{x,r_j}} |w - w_{U_{x,r_j}}| \le \frac{2^Q\, C}{v_g(U_1)} \sum_{j\in\mathbb{N}} r_j^{1-Q} \int_{U_{x,r_j}} d|D_H w| \\
&= \frac{2^Q\, C}{v_g(U_1)} \int_{U_{x,r}} \sum_{j\in\mathbb{N}} r_j^{1-Q} \mathbf{1}_{U_{x,r_j}}\, d|D_H w| = \frac{2^Q\, C}{v_g(U_1)} \int_{U_{x,r}} \sum_{2^j < r/\rho(x,y)} r_j^{1-Q} d|D_H w|\,.
\end{aligned}$$

Now we fix $\varepsilon = (Q-1)/2 > 0$ and observe that for any $y \in U_{x,r_j}$ we have

$$r_j^{1-Q} \le \left(\frac{\rho(x,y)}{r_j}\right)^{\varepsilon} \rho(x,y)^{1-Q}\,,$$

hence the previous inequalities yield

$$|w(x) - w_{U_{x,r}}| \le \frac{2^Q\,C}{v_g(U_1)} \int_{U_{x,r}} \rho(x,y)^{1-Q} \sum_{2^j < r/\rho(x,y)} \left(\frac{\rho(x,y)}{r_j}\right)^{\varepsilon} d|D_H w|\,. \tag{8.20}$$

Now fix N_y as the maximum integer $j \in \mathbb{N}$ such that $2^j < r/\rho(x,y)$, obtaining

$$\begin{aligned}
&\sum_{2^j < r/\rho(x,y)} \left(\frac{\rho(x,y)}{r_j}\right)^{\varepsilon} = \left(\frac{\rho(x,y)}{r}\right)^{\varepsilon} \sum_{j=0}^{N_y} 2^{\varepsilon\, j} = \left(\frac{\rho(x,y)}{r}\right)^{\varepsilon} \frac{2^{\varepsilon\, N_y+\varepsilon} - 1}{2^{\varepsilon} - 1} \\
&\le \left(\frac{\rho(x,y)}{r}\right)^{\varepsilon} 2^{\varepsilon\, N_y} \, \frac{2^{\varepsilon}}{2^{\varepsilon}-1} \le \frac{2^{\varepsilon}}{2^{\varepsilon}-1} = c_Q\,.
\end{aligned}$$

Then the estimate (8.20) becomes

$$|w(x) - w_{U_{x,r}}| \le \frac{2^Q\,C\,c_Q}{v_g(U_1)} \int_{B_{x,r}} \rho(x,y)^{1-Q} d|D_H w|$$

and the thesis follows. □

In the sequel we will need of other versions of formula (8.19). By Fubini's Theorem for products of Radon measures we have

$$\begin{aligned}
&\int_{U_{x,r}} \rho(x,y)^{1-Q}\, d|D_H w|(y) = \int_{U_{x,r}} (Q-1) \left(\int_{\rho(x,y)}^{\infty} t^{-Q}\, dt\right) d|D_H w|(y) \\
&= (Q-1) \int_{U_{x,r}} \left(\int_0^{\infty} t^{-Q} \mathbf{1}_{\{\rho(x,y)<t\}}\, dt\right) d|D_H w|(y) \\
&= (Q-1) \int_0^{+\infty} \frac{|D_H w|(U_{x,r} \cap U_{x,t})}{t^Q}\, dt \\
&= (Q-1) \int_0^r \frac{|D_H w|(U_{x,t})}{t^Q} dt + \frac{|D_H w|(U_{x,r})}{r^{(Q-1)}}\,.
\end{aligned} \tag{8.21}$$

so that (8.19) becomes

$$|\tilde{w}(x) - w_{U_{x,r}}| \le C \left[(Q-1) \int_0^r \frac{|D_H w|(U_{x,t})}{t^Q} dt + \frac{|D_H w|(U_{x,r})}{r^{(Q-1)}}\right]\,. \tag{8.22}$$

Furthermore, we notice that in the case $\tilde{w}(x) = 0$ the inequality (8.19) yields

$$|w_{U_{x,s}}| \le C \int_{U_{x,r}} \frac{1}{\rho(x,y)^{Q-1}}\, d|D_H w|(y).$$

for every $0 < s < r$. By Definition 2.5.9 for the restricted maximal function and by equality (8.21) we arrive at the following integral estimate

$$|M_r w(x)| \le C \left[(Q-1) \int_0^r \frac{|D_H w|(U_{x,t})}{t^Q} dt + \frac{|D_H w|(U_{x,r})}{r^{(Q-1)}}\right]\,. \tag{8.23}$$

8.5 Higher order differentiability of H-BV^k functions

In this section we study the differentiability properties of maps with higher order H-bounded variation. The method to accomplish this study is substantially different from that one employed for H-BV functions. Particularly interesting is the case of maps with second H-bounded variation, in view of potential applications to the theory of convex functions on stratified groups (see [44] and [123]).

We begin with the definition of high order H-BV function.

Definition 8.5.1 Let us fix an orthonormal frame $(X_1, \dots, X_m)$ of $H\Omega$. By induction on $k \geq 2$ and taking into account the definition of H-BV with $k = 1$, we say that a Borel map $u : \Omega \longrightarrow \mathbb{R}$ has *H-bounded k-variation* (in short, H-BV^k) if for any $i = 1, \dots, m$ the distributional derivatives $X_i u$ are representable by functions with H-bounded $(k-1)$-variation. We denote by $BV_H^k(\Omega)$ the space of all H-BV^k functions.

Remark 8.5.2 The notion of H-BV^k function does not depend on the choice of the orthonormal frame $(X_1, \dots, X_m)$.

The Poincaré-Birkhoff-Witt Theorem (shortly PBW Theorem) states that for any basis $(W_1, W_2, \dots W_q)$ of $\mathcal{G}$ regarded as frame of first order differential operators, the algebra of left invariant differential operators on $\mathbb{G}$ has a basis formed by the following ordered terms

$$W^\alpha = W_1^{i_1} \cdot \cdots \cdot W_q^{i_q} ,$$

where $\alpha = (i_1, \dots, i_q)$ varies in $\mathbb{N}^q$, see Chapter 1.C of [59]. Analogously as we have done for polynomials, we define the degree of a left invariant differential operator $Z = \sum_\alpha c_\alpha W^\alpha$ as

$$\deg_H(Z) = \max \Big\{ \sum_{k=1}^q d_k \alpha_k \mid c_\alpha \neq 0 \Big\} ,$$

where d_k is the degree of the coordinate y_k and (F, W) is the corresponding system of graded coordinates (Definition 2.3.43). The space $\mathcal{A}_k(\mathbb{G})$ represents the space of left invariant differential operators of homogeneous degree less than or equal to k. This analogy between polynomials and differential operators is not only formal, as the following proposition shows.

Proposition 8.5.3 *There exists an isomorphism $L : \mathcal{P}_{H,k}(\mathbb{G}) \longrightarrow \mathcal{A}_k(\mathbb{G})$, given by*

$$L(P) = \sum_{\deg_H(W^\alpha) \leq k} W^\alpha P(0) \, W^\alpha .$$

For the proof of this fact we refer the reader to Proposition 1.30 of [59].

In order to deal with higher order differentiability theorems we make some preliminary considerations. Let us consider a basis $\{W^\alpha \mid \deg_H(W^\alpha) \le k\}$ of $\mathcal{A}_k(\mathbb{G})$ and $u \in BV_H^k(\Omega)$, where $(W_1, \ldots, W_q)$ is an adapted basis of $\mathcal{G}$. We denote $W_i = X_i$, with $i = 1, \ldots, m$, where $(X_1, \ldots, X_m)$ is a fixed horizontal orthonormal frame. Our aim is to find out a polynomial $P : \mathbb{G} \longrightarrow \mathbb{R}$ which approximates u at a fixed point $x \in \Omega$ with order k. In view of the last proposition it is natural to look for a substitute of homogeneous derivatives W^α of u at x, with $\deg_H(W^\alpha) \le k$. Our first observation is that due to the stratification of $\mathcal{G}$ the operators W^α with $\deg_H(W^\alpha) \le l$ are linear combinations of operators $X_{\gamma_1} \cdots X_{\gamma_l}$ with $1 \le \gamma_i \le m$ and $l \le k$. Therefore the distributional derivatives $D_W^\alpha u$ are measures whenever $\deg_H(W^\alpha) \le k$. So, taking into account the preceding observation and the fact that vector fields W_i have vanishing divergence, we state the following definition.

Definition 8.5.4 Let $u \in BV_H^k(\Omega)$. For any $\alpha \in \mathbb{N}^q$, we consider the following multi-index Radon measures $D_W^\alpha u$ with $\deg_H(W^\alpha) \le k$

$$\int_\Omega \phi \, dD_W^\alpha u = (-1)^{|\alpha|} \int_\Omega u \; W_l^{\alpha_l} \cdots W_1^{\alpha_1} \phi \qquad \forall \phi \in C_c^\infty(\Omega).$$

By Radon-Nikodým Theorem we have $D_W^\alpha u = (D_W^\alpha u)^a + (D_W^\alpha u)^s$, where the addenda are respectively the absolutely continuous part and the singular part of the measure $D_W^\alpha u$ with respect to the volume measure. We define the *weak mixed derivatives* as the summable maps $\nabla_W^\alpha u$ such that

$$(D_W^\alpha u)^a = \nabla_W^\alpha u \; \mathcal{H}^q \, .$$

Our substitute for the α-derivative of u is $\tilde{u}_{W^\alpha}(x)$, which is the approximate limit of $\nabla_W^\alpha u$ at points $x \in \Omega \setminus S_{\nabla_W^\alpha u}$. Now, let us consider the differential operator $X_{\gamma_1} \cdots X_{\gamma_l} u$ where $1 \le \gamma_i \le m$ and $1 \le l \le k$. By virtue of PBW Theorem, there exist coefficients $\{c_\gamma^\alpha\}$ such that

$$X_{\gamma_1} \cdots X_{\gamma_l} u = \sum_{j=1}^{N_k} c_{\gamma,\alpha} \; W^\alpha u \, , \tag{8.24}$$

where $N_k = \dim(\mathcal{A}_k(\mathbb{G}))$.

Definition 8.5.5 Let $u \in BV_H^k(\Omega)$. Utilizing the above notation, we denote by u_γ the density of the absolutely continuous part of the measure $X_{\gamma_1} \cdots X_{\gamma_l} u$, where $\gamma \in \{1, \ldots, m\}^l$ and $l \le k$.

Decomposing the singular and absolutely continuous part of both the measures in (8.24), we obtain the following equality of summable maps

$$u_\gamma = \sum_{j=1}^{N_k} c_{\gamma,\alpha} \;\; \nabla_W^\alpha u \, . \tag{8.25}$$

The next theorem is the main result of this section and can be regarded as a weak extension of Alexandrov differentiability theorem to the setting of non-Riemannian geometries.

Theorem 8.5.6 (Alexandrov) *Let $u \in BV_H^2(\Omega)$. Then for a.e. $x \in \Omega$ there exists a polynomial $P_{[x]}$ with $\deg_H(P_{[x]}) \le 2$, such that*

$$\lim_{r\to 0^+} \frac{1}{r^2} \fint_{U_{x,r}} |u - P_{[x]}| = 0 \tag{8.26}$$

Proof. First of all, we fix a point $x \notin \bigcup_{\deg_H(W^\alpha)\le 2} S_{\nabla_W^\alpha u}$ such that (8.25) and the limit

$$\lim_{r\to 0^+} \frac{|\,(X_{\gamma_1}X_{\gamma_2}u)^s\,|(U_{x,r})}{r^Q} = 0 \tag{8.27}$$

holds for every $\gamma \in \{1,\dots,m\}^l$, with $l = 1,2$. By the previous discussion and Remark 2.4.6, the set of points where these conditions do not occur is negligible. Due to Proposition 8.5.3, there exists a unique polynomial $P_{[x]} = P$ which satisfies the condition $W^\alpha P(x) = \tilde u_{W^\alpha}(x)$, whenever $\deg_H(W^\alpha) \le 2$. Now, let us define $w = u - P$. By relation (8.25) we observe that $\tilde w_\gamma(x) = 0$ for any $\gamma \in \{1,\dots,m\}^l$, $l = 0,1,2$. This means that

$$\tilde w(x) = 0\,, \quad \tilde w_i(x) = 0\,, \quad \tilde w_{ij}(x) = 0 \tag{8.28}$$

for any $i,j = 1,\dots,m$. We consider the summable map

$$v = |D_H w| = \Big(\sum_{i=1}^m w_i^2\Big)^{1/2}.$$

By Proposition 2.5.8 it follows that

$$|D_H v| \le \sum_{i=1}^m |D_H w_i|\,,$$

hence conditions (8.27) and (8.28) yield

$$|D_H v|(U_{x,r}) = o(r^Q)\,. \tag{8.29}$$

We can fix $r_0 > 0$ small enough such that $U_{x,4r_0} \subset \Omega$, so we will consider all $r \in]0, r_0[$. By the standard telescopic estimate (8.8), for a.e. $y \in U_{x,r}$ we have

$$|\tilde w(y)| \le C\,[M_{2r}v(x) + M_{2r}v(y)]\,d(x,y)\,,$$

therefore, taking the average over $U_{x,r}$ and dividing by r^2 we obtain

$$\frac{1}{r^2}\fint_{U_{x,r}} |w(y)|\,dy \le C\left(\frac{M_{2r}v(x)}{r} + \frac{1}{r}\fint_{U_{x,r}} M_{2r}v(y)\,dy\right).$$

Thus, in order to prove (8.26) we show that the maps

$$a(r) = r^{-1} M_{2r} v(x)\,, \qquad b(r) = r^{-1} \fint_{U_{x,r}} M_{2r} v(y)\, dy$$

go to zero as $r \to 0^+$. Since also $\tilde{v}(x) = 0$, inequality (8.23) gives

$$|M_r v(x)| \le C \left[(Q-1) \int_0^r \frac{|D_H v|(U_{x,t})}{t^Q} dt + \frac{|D_H v|(U_{x,r})}{r^{(Q-1)}} \right].$$

By (8.29) and the last estimate we get that $a(r) \to 0$ as $r \to 0^+$. Let us consider the estimate

$$b(r) \le \frac{1}{r} \fint_{U_{x,r}} |M_{2r} v(y) - \tilde{v}(y)|\, dy + \frac{1}{r} \fint_{U_{x,r}} |v(y)|\, dy\,,$$

observing that

$$\frac{1}{r} \fint_{U_{x,r}} |v(y)|\, dy \le r^{-1} M_r v(x) \le a(r) \longrightarrow 0 \quad \text{as} \quad r \to 0^+\,.$$

In view of inequality

$$|M_{2r} v(y) - \tilde{v}(y)| \le M_{2r}[v - \tilde{v}(y)](y)\,, \tag{8.30}$$

and applying inequality (8.23) to the map $z \longrightarrow v(z) - \tilde{v}(y)$ we get

$$M_{2r}[v - \tilde{v}(y)](y) \le C \left[(Q-1) \int_0^{2r} \frac{|D_H v|(U_{y,t})}{t^Q} dt + \frac{|D_H v|(U_{y,2r})}{(2r)^{(Q-1)}} \right]. \tag{8.31}$$

Thus, estimates (8.30) and (8.31) yield

$$\frac{1}{r} \fint_{U_{x,r}} |M_{2r} v(y) - \tilde{v}(y)| dy \le \frac{C}{r} \fint_{U_{x,r}} \left[(Q-1) \int_0^{2r} \frac{|D_H v|(U_{y,t})}{t^Q} dt + \frac{|D_H v|(U_{y,2r})}{(2r)^{(Q-1)}} \right] dy\,.$$

Now, in order to get the thesis, we have to prove that both terms

$$\alpha(r) = \frac{1}{r} \fint_{U_{x,r}} \int_0^{2r} \left(\frac{|D_H v|(U_{y,t})}{t^Q} dt \right) dy\,, \quad \beta(r) = \frac{1}{r} \fint_{U_{x,r}} \frac{|D_H v|(U_{y,2r})}{(2r)^{(Q-1)}} dy$$

are infinitesimal as $r \to 0^+$. By Fubini's Theorem we have

$$\begin{aligned} \alpha(r) &= \frac{r^{-1}}{|U_{x,r}|} \int_0^{2r} \frac{dt}{t^Q} \int_{U_{x,r}} \left(\int_{U_{x,3r}} \mathbf{1}_{U_{y,t}}(z)\, d|D_H v|(z) \right) dy \\ &= \frac{r^{-1}}{|U_{x,r}|} \int_0^{2r} \frac{dt}{t^Q} \int_{U_{x,3r}} |U_{x,r} \cap U_{z,t}|\, d|D_H v|(z) \end{aligned}$$

$$= \frac{|U_1| r^{-1}}{|U_{x,r}|} \int_0^{2r} \int_{U_{x,3r}} \frac{|U_{x,r} \cap U_{z,t}|}{|U_{z,t}|} \, d|D_H v|(z) \leq 3^Q \, 2 \, \frac{|D_H v|(U_{x,3r})}{(3r)^Q} \, .$$

By (8.29) the last term goes to zero as $r \to 0$, so $\lim_{r \to 0} \alpha(r) = 0$. Similarly, we have

$$\beta(r) = \frac{1}{2^{Q-1} r^Q |U_{x,r}|} \int_{U_{x,r}} \left(\int_{U_{x,3r}} \mathbf{1}_{U_{y,2r}}(z) \, d|D_H v|(z) \right) dy$$

$$= \frac{1}{2^{Q-1} r^Q} \int_{U_{x,3r}} \frac{|U_{z,2r} \cap U_{x,r}|}{|U_{x,r}|} \, d|D_H v|(z) \leq \frac{3^Q}{2^{Q-1}} \frac{|D_H v|(U_{x,3r})}{(3r)^Q} \, .$$

Again, utilizing (8.29) on the last term we get $\lim_{r \to 0^+} \beta(r) = 0$, so the thesis follows. □

The arguments used for second order differentiability of H-BV^2 functions can be extended with some additional efforts to higher order differentiability.

Theorem 8.5.7 *Let $u \in BV_H^k(\Omega)$ and $1 \leq l \leq k$. Then for a.e. $x \in \Omega$ there exists a polynomial $P_{[x]}$, with $\deg_H(P_{[x]}) \leq l$, such that*

$$\lim_{r \to 0^+} \frac{1}{r^l} \fint_{U_{x,r}} |u - P_{[x]}| = 0. \tag{8.32}$$

Proof. We prove the theorem by induction on $k \geq 2$. Theorem 8.2.2 and Theorem 8.5.6 give us the validity of induction hypothesis for $k = 2$. Now, let us consider $u \in BV_H^k(\Omega)$ with $k \geq 3$. Clearly we have $X_i X_j u \in BV_H^{k-2}$ for any $i, j = 1, \ldots, m$. By induction hypothesis for a.e. $x \in \Omega$ there exist polynomials $R_{[x,ij]}$, with h-deg$(R_{[x,ij]}) \leq k - 2$, such that

$$\fint_{U_{x,r}} |u_{ij} - R_{[x,ij]}| = o(r^{k-2}) \tag{8.33}$$

and

$$W^\beta R_{[x,ij]}(x) = \tilde{u}_{ij W^\beta}(x) \, , \quad \text{whenever } \ d(\beta) \leq k - 2 \, . \tag{8.34}$$

Moreover, for a.e. $x \in \Omega$ there exists a polynomial $P_{[x]}$, with $\deg_H(P_{[x]}) \leq k$, such that

$$W^\alpha P_{[x]}(x) = \tilde{u}_{W^\alpha}(x) \, , \quad \text{whenever } \ \deg_H(W^\alpha) \leq k \, . \tag{8.35}$$

The PBW Theorem yields the distributional relations

$$W^\beta X_i X_j = \sum_{\deg_H(W^\alpha) \leq k} c^\beta_{ij,\alpha} W^\alpha \tag{8.36}$$

for any $i, j = 1, \ldots, m$ and $\beta \in \mathbb{N}^q$ with $\deg_H(W^\beta) \leq k - 2$. Thus, relations (8.34), (8.35), (8.36) and the following equality

$$(W^\beta X_i X_j u)^a = (W^\beta u_{ij})^a = \tilde{u}_{ij W^\beta}$$

imply

$$W^\beta R_{[x,ij]}(x) = \sum_{\deg_H(W^\alpha)\le k} c^\beta_{ij,\alpha}\tilde u_{W^\alpha}(x) = \sum_{\deg_H(W^\alpha)\le k} c^\beta_{ij,\alpha} W^\alpha P_{[x]}(x) = W^\beta X_i X_j P(x)\,,$$

whenever $\deg_H(W^\beta)\le k-2$. Thus, Proposition 8.5.3 yields $R_{[x,ij]} = X_iX_jP$.

Now, let us define $w = u - P$ and $v = |D_H w|$, obtaining the following inequalities of measures

$$|D_H v| \le \sum_{i=1}^m |D_H X_i w| \le \sum_{i,j=1}^m |X_j X_i w|\,. \tag{8.37}$$

By the fact that $u\in BV^k_H(\Omega)$, with $k\ge 3$, the distributional derivatives X_jX_iw are represented by integrable functions w_{ij}. So, equality $R_{[x,ij]} = X_iX_jP$ and the inductive formula (8.33) yield

$$\frac{|X_jX_iw|(U_{x,r})}{|U_{x,r}|} = \fint_{U_{x,r}} |w_{ij}| = \fint_{U_{x,r}} |u_{ij} - R_{[x,ij]}| = o(r^{k-2}),$$

hence (8.37) implies

$$|D_H v|(U_{x,r}) = o(r^{Q+k-2})\,. \tag{8.38}$$

Now, the rest of the proof proceeds analogously to Theorem 8.5.6, replacing property (8.29) with (8.38). This last observation leads us to the conclusion. □

8.6 A class of H-BV^2 functions

In this section we present a way to construct explicit examples of H-BV^2 functions arising from the inf-convolution of the so-called "gauge distance" in the Heisenberg group $\mathbb{H}^{2n+1}$.

We begin with some elementary remarks about distributional derivatives along vector fields. In the following preliminary considerations the set Ω will be an open subset of $\mathbb{R}^q$ with the Euclidean metric.

Let $X:\Omega\longrightarrow\mathbb{R}^q$ be a locally Lipschitz vector field; then, the following chain rule

$$D_X(h\circ u) = h'(u)D_X u \tag{8.39}$$

holds whenever $h:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, $u:\Omega\longrightarrow\mathbb{R}$ is continuous and D_Xu is representable by a Radon measure in Ω as follows:

$$\int_\Omega u\, X^*\varphi\, d\mathcal{L}^q = \int_\Omega \varphi\, dD_X u \qquad \forall\varphi\in C^\infty_c(\Omega),$$

where $X^* = -X - \mathrm{div}X$ is the formal adjoint of X. Analogously, the product rule

$$D_X(uv) = vD_Xu + uD_Xv \tag{8.40}$$

holds whenever $u : \Omega \longrightarrow \mathbb{R}$ is locally integrable (or locally bounded) and $D_X u$ is representable in Ω by a Radon measure, $v : \Omega \longrightarrow \mathbb{R}$ is continuous and $D_X v$ is representable in Ω by a locally bounded (or locally summable) function. The proofs of (8.39) and (8.40) can be achieved by approximations of the following type.

Proposition 8.6.1 *Let $u \in \Xi$, where Ξ is either $C(\Omega)$, $L^1_{loc}(\Omega)$ or $L^\infty_{loc}(\Omega)$, respectively. Then there exists a sequence of smooth functions (u_l) such that*

$$|D_X u_l|(\Omega) \leq |D_X u|(\Omega) + \frac{1}{l} \tag{8.41}$$

and either (u_l) uniformly converges to u on compact sets, or it converges to u in $L^1_{loc}(\Omega)$, or it is locally uniformly locally bounded, respectively.

The estimate (8.41) is proved in [69], [79]. One considers a locally finite open cover $\{A_i\}$, where $A_i = \Omega_{i+1} \setminus \overline{\Omega}_{i-1}$ and

$$\Omega_i = \left\{ x \in \Omega \middle|, |x| < i,\ \mathrm{dist}(x, \Omega^c) > \frac{1}{i+1} \right\}$$

for any $i \in \mathbb{N}$, with $\Omega_{-1} = \emptyset$. A smooth partition of unity $\{\psi_i\}$ is defined with respect to $\{A_i\}$, hence the candidate to be the approximating functions is as follows

$$u_l := \sum_{i=0}^{\infty} (u\psi_i) * \phi_{\varepsilon_i}$$

with $\varepsilon_i = \varepsilon_i(l)$ small enough. Since $\sup_i \varepsilon_i(l)$ tends to 0 as $l \to \infty$ all L^p convergence properties of the sequence follow directly from this representation. Notice also that when $D_X u << \mathcal{L}^q$, we get the $L^1_{\mathrm{loc}}(\Omega)$ convergence of the densities of $D_X u_l$ to the density of $D_X u$, see either Proposition 1.2.2 of [69] or Theorem A.2 of [79].

Now, the proof of (8.39) can be achieved by approximation of u with the sequence (u_l) of Proposition 8.6.1, so that $D_X u_l$ weakly converges to $D_X u$ in the topology of Radon measures and u_l converges to u uniformly on compact sets of Ω. The proof of (8.40) is similar and requires either the L^1_{loc} convergence of u_l to u when $u \in L^1_{\mathrm{loc}}$, or the additional uniform local bound, when $u \in L^\infty_{\mathrm{loc}}$, and the L^1_{loc} convergence of densities $D_X v_l$ to $D_X v$, when $D_X v \in L^1_{\mathrm{loc}}$, or the additional uniform local bound, when $D_X v \in L^\infty(\Omega)$, together with the uniform convergence of v_l to v on compact sets of Ω.

Lemma 8.6.2 *Let u, $v : \Omega \longrightarrow \mathbb{R}$ be continuous functions, $\gamma \in \mathbb{R}$ and let $X : \Omega \longrightarrow \mathbb{R}^q$ be a locally Lipschitz vector field. Then*

$$D_X u \leq \gamma\, \mathcal{L}^q, \quad D_X v \leq \gamma\, \mathcal{L}^q \qquad \Longrightarrow \qquad D_X(u \wedge v) \leq \gamma\, \mathcal{L}^q. \tag{8.42}$$

If $D_X u$ and $D_X v$ are representable by $L^\infty_{loc}(\Omega)$ functions, then

$$D_{XX} u \leq \gamma\, \mathcal{L}^q, \quad D_{XX} v \leq \gamma\, \mathcal{L}^q \qquad \Longrightarrow \qquad D_{XX}(u \wedge v) \leq \gamma\, \mathcal{L}^q. \tag{8.43}$$

PROOF. In order to show (8.42), it suffices to approximate $u \wedge v$ by $u + h_\epsilon(u - v)$, where $h_\epsilon \in C^\infty(\mathbb{R})$, $-1 \leq h'_\epsilon \leq 0$, $h_\epsilon(t) \to -t^+$ uniformly as $\epsilon \to 0^+$. Indeed, the chain rule (8.39) gives

$$D_X(u + h_\epsilon(u - v)) = \left(1 + h'_\epsilon(u - v)\right) D_X u - h'_\epsilon(u - v) D_X v \leq \gamma\, \mathcal{L}^q.$$

The implication (8.43) follows by the same argument, noticing that the functions h_ϵ can be chosen to be concave. We have

$$\begin{aligned}
&D_{XX}(u + h_\epsilon(u - v)) \\
&= \left(1 + h'_\epsilon(u - v)\right) D_{XX} u - h'_\epsilon(u - v) D_{XX} v + h''_\epsilon(u - v)(D_X u - D_X v)^2 \\
&\leq \left(1 + h'_\epsilon(u - v)\right) D_{XX} u - h'_\epsilon(u - v) D_{XX} v \leq \gamma\, \mathcal{L}^q.
\end{aligned}$$

□

Now we particularize our study to $\mathbb{H}^{2n+1}$ (we recall that $\mathbb{H}^n$ is isomorphic to $\mathbb{R}^{2n+1}$). To denote elements of $\mathbb{H}_{2n+1}$ we consider the coordinates $(\xi, t) = (\xi_1, \ldots, \xi_{2n}, t)$. The following family of vector fields

$$X_i = \partial_{\xi_i} + 2\xi_{n+i}\partial_t, \quad Y_i = \partial_{\xi_{n+i}} - 2\xi_i\partial_t, \qquad i = 1, \ldots, n \tag{8.44}$$

can be considered as a horizontal orthonormal frame of $H\mathbb{H}^{2n+1}$, so

$$\nabla_H u = \sum_{i=1}^{n} X_i u\, X_i + Y_i u\, Y_i$$

whenever u is smooth. The only nontrivial bracket relations are

$$[X_i, Y_i] = -4\, Z = -4\, \partial_t, \qquad i = 1, \ldots, n\,.$$

Via the BCH formula our vector fields induce the following group operation

$$x\, x' = \left(\xi + \xi', t + t' + 2\sum_{i=1}^{n} \xi_{n+i}\xi'_i - \xi_i\xi'_{n+i}\right).$$

Now for any element $x = (\xi, t) \in \mathbb{H}^{2n+1}$ we define the following gauge norm

$$\|(\xi, t)\| = \sqrt[4]{|\xi|^4 + t^2}\,.$$

A non-trivial fact is that $d(x, y) = \|x^{-1}y\|$ yields a left invariant distance on $\mathbb{H}^{2n+1}$, see [113]. In the following we define $c(x, y) = d(x, y)^2$ and we consider a function u arising from the inf-convolution of c. Precisely, we assume that there exist a bounded family $\{y_i\}_{i\in I} \subset \mathbb{H}^{2n+1}$ and $t_i \in \mathbb{R}$ such that

$$u(x) = \inf_{i\in I} c(x, y_i) + t_i \qquad \forall x \in \mathbb{H}^{2n+1}. \tag{8.45}$$

Inf-convolution formulas of this type appear in several fields, for instance in the representation theory of viscosity solutions, in the related field of dynamic programming (see for instance [33], [127]) and in the theory of optimal transportation problems. In the latter theory, functions representable as in (8.45) are called *c-concave* (see [164], [159]). In these theories it is well known that in many situations the function u inherits from c a one-sided estimate on the second distributional derivative; for instance, this is the case when $c(x,y) = h(x-y)$ and $h : \mathbb{R}^q \longrightarrow \mathbb{R}$ is a $C^{1,1}_{\mathrm{loc}}$ function (see for instance [76]). In the following theorem we extend this result to the Heisenberg setting, thus getting a non-trivial class of examples of H-BV^2-functions.

Theorem 8.6.3 *Let $\Omega \subset \mathbb{H}^{2n+1}$ be a bounded open set. The function u defined in (8.45) is Lipschitz and belongs to $BV^2_H(\Omega)$.*

PROOF. Since the family $\{y_i\}_{i\in I}$ is bounded it is easy to check that $c(\cdot, y_i)$ are uniformly Lipschitz in Ω, therefore u is a Lipschitz function in Ω. Notice also that, since $\mathbb{H}^{2n+1}$ is separable, we can assume I to be finite or countable with no loss of generality.

The essential fact leading to the H-BV^2 property consists in the following pointwise estimates on $\mathbb{H}^{2n+1} \setminus \{0\}$

$$|X_iX_jc(\cdot,e)|,\ |X_iY_jc(\cdot,e)|,\ |Y_iY_jc(\cdot,e)|,\ |Y_jX_ic(\cdot,e)| \le 10 \tag{8.46}$$

for any $i,j = 1,\ldots,n$. This can be checked by direct calculation. Notice that the identity $c(z,y) = c(y^{-1}z,e)$ yields

$$T[c(\cdot,y)](z) = T[c(y^{-1}\cdot,e)](z) = Tc(\cdot,e)(y^{-1}z)$$

for any left invariant vector field T. It follows that the previous estimates hold replacing the unit element e with any $y \in \mathbb{H}^{2n+1}$, getting

$$|PPc(\cdot,y)| \le \gamma \quad \text{on } \mathbb{H}^{2n+1} \setminus \{y\}\,,$$

whenever $P = \sum_{i=1}^n a_iX_i + b_iY_i$ and $\sum_{i=1}^n a_i^2 + b_i^2 \le 1$, with $\gamma = 20n^2$. By applying Lemma 8.6.2 we obtain that

$$D_{PP}u \le \gamma\, \mathcal{L}^{2n+1}$$

first for finite families and then, by a limiting argument, for countable families. In particular $D_{PP}u$ is representable in Ω by a Radon measure for any P of the above form. By polarization identity, taking $P = (X_i \pm X_j)/2$, $P = (X_i \pm Y_j)/2$ and $P = (Y_i \pm Y_j)/2$, respectively, we obtain that $D_{X_iX_j}u$, $D_{Y_iY_j}u$ $D_{X_iY_j+Y_jX_i}u$ are Radon measures for any $i,j = 1,\ldots,n$. In particular $D_{X_iY_j}u$ is a measure whenever $i \ne j$. Again, exploiting (8.46) and the non-trivial bracket relations we obtain $|D_Zc(\cdot,y)| \le 5\mathcal{L}^{2n+1}$, so that Lemma 8.6.2 yields that D_Zu is representable in Ω by a Radon measure (actually absolutely continuous with respect to $\mathcal{L}^{2n+1}$). Finally, the relation $D_{X_iY_i+Y_iX_i} + D_Z = 2D_{X_iY_i}$ yields that $D_{X_iY_i}u$ is a Radon measure. □

Chapter 9

Basic notation and terminology

$\subset$	set inclusion
A^c	complement set
$\overline{A}$	topological closure
$\mathbf{1}_A$	characteristic function of a set A
$\overline{\mathbb{R}}$	extended real numbers, Section 2.1
$\mathbb{R}^n$	n-dimensional space of Euclidean coordinates
$\mathbb{E}^n$	n-dimensional Euclidean space
$\mathbb{H}^{2n+1}$	Heisenberg group, Definition 2.3.23
$\mathbb{G}$	nilpotent Lie group, Definition 2.3.8
$\mathcal{P}(X)$	class of all subsets of X, Section 2.1
$f_\sharp \mu$	image measure, Definition 2.1.5
$f\ \mu$	measure induced by the map f and the measure μ, Definition 2.1.7
$\fint_E u\, d\mu$	averaged integral, Section 2.1
$\mathcal{I}(A)$	set of density points, Definition 2.1.14
$\mathrm{Lip}(f)$	Lipschitz constant of the map f, Definition 2.1.9
$N(f, A, y)$	multiplicity function, Definition 2.1.11
$\dim(V)$	dimension of linear space V
$\mathrm{span}\{X_i\}$	linear space generated by vectors X_i
$M_{n,m}(\mathbb{K})$	$n \times m$ matrices over the field $\mathbb{K}$
$B_{x,r}$	open ball of center x and radius r, Definition 2.1.8
$B^d_{x,r}$	open ball with respect to the distance d, Definition 2.1.8
$D_{x,r}$	closed ball of center x and radius r, Definition 2.1.8
$D^d_{x,r}$	closed ball with respect to the distance d, Definition 2.1.8
$\mathrm{diam}(E)$	diameter of a set, Section 2.1
Φ^a	Hausdorff-type measure, Definition 2.1.17
$\mathcal{H}^a$	Hausdorff measure, Definition 2.1.17
$\mathcal{H}^a_{\lvert\cdot\rvert}$	Hausdorff measure with respect to the Euclidean norm
$\mathcal{S}^a$	spherical Hausdorff measure, Section 2.1

$\lvert E\rvert$	Haar measure of a subset E
$C^k(\Omega, N)$	continuously k-differentiable functions, Definition 2.2.1
$\Gamma(TM)$	space of vector fields, Definition 2.2.2
f_*X	image of the vector field X under f, Definition 2.2.3
$\mathfrak{p}_H$	horizontal Riemannian projection, Definition 2.2.7
$\Gamma(HM)$	space of horizontal vector fields, Definition 2.2.6
$\mathcal{G}$	Lie algebra of left invariant vector fields, Definition 2.3.2
$\mathfrak{h}_{2n+1}$	Heisenberg algebra, Definition 2.3.23
V_j	subspace of $\mathcal{G}$ with degree j, Definition 2.3.16
$\mathbb{V}_j$	subset of $\mathbb{G}$ with elements of degree j, Definition 2.3.16
$T_p\mathbb{G}$	tangent space of $\mathbb{G}$ at the point p
$H_p^j\mathbb{G}$	subspace of vectors of degree j at the point p, Definition 2.3.16
e	unit element of a Lie group, Section 2.3
l_p	left translation, Definition 2.3.1
exp	exponential map of Lie groups, Definition 2.3.6
$\circledcirc$	operation between vectors of the Lie algebra, (2.18)
δ_r	dilation, Definition 2.3.18
σ_t	sign map, Definition 2.3.18
Λ_r	coordinate dilation, Definition 2.3.45
v_g	Riemannian volume, Subsection 2.3.2
$\langle X, Y\rangle_p$	Riemannian metric, (2.31)
σ_g	Riemannian measure on hypersurfaces, Chapter 7
$l_g(\cdot)$	length of a curve, Definition 2.2.18
$\deg_H(P)$	homogeneous degree of a polynomial P, Definition 2.3.50
$\mathcal{P}_{H,k}(\mathbb{G})$	space of polynomials P with $\deg_H(P) \leq k$, Definition 2.3.50
$\deg_H(Z)$	homogeneous degree of a differential operator Z, Section 8.5
$\mathcal{A}_k(\mathbb{G})$	space of differential operators Z with $\deg_H(Z) \leq k$, Section 8.5
$C(\Sigma)$	characteristic set of a C^1 hypersurface, Definition 2.2.8
div_H	horizontal divergence, Definition 2.4.1
D_H	horizontal distributional gradient
$\lvert D_H u\rvert$	variational measure associated to an H-BV function, Definition 2.4.3
$\lvert\partial E\rvert_H$	perimeter measure, Definition 2.4.8
$P_H(E,\cdot)$	perimeter measure, Definition 2.4.8
$E_{p,r}$	r-rescaled of E at p, Definition 6.4.1
$f_{x,r}$	r-rescaled of f at x, Definition 6.2.2
∇_H	horizontal gradient
$\partial_{*H}E$	H-reduced boundary, Definition 2.4.10
ν_E	generalized inward normal, Definition 2.4.9
ν_H	horizontal normal, Definition 2.2.9
$D_H u$	vector measure of an H-BV function, Section 2.4
$D_H^a u$	absolutely continuous part of the H-BV measure, Section 2.4

$D^s_H u$	singular part of the H-BV measure, Section 2.4
$\mathrm{HL}(\mathbb{G},\mathbb{M})$	group of H-linear maps, Definition 3.1.4
$C^1_H(\Omega,\mathbb{M})$	C^1_H maps, Definition 3.2.6
$\theta^g_{Q-1}(\nu)$	metric factor, Definition 5.2.2
$\mathcal{R}$	subset of horizontal isometries, Definition 5.1.1
$\mathcal{J}_q(L)$	jacobian, Definition 2.3.40
$J_Q(L)$	H-jacobian, Definition 4.2.1
$\mathbf{J}_k(\nu)$	normed jacobian, Definition 4.1.9
$J_f(x)$	metric jacobian, Definition 4.1.4
$\mathcal{C}_p(L)$	coarea factor, Definition 2.3.40
$C_P(L)$	H-coarea factor, Definition 6.1.3
$L^p_\mu(X,N)$	p-summable maps with respect to the measure μ, Section 2.1
δ_{ij}	Kronecker delta

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