A COUNTEREXAMPLE TO WELL-POSEDNESS OF ENTROPY SOLUTIONS TO THE COMPRESSIBLE EULER SYSTEM

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Abstract. We deal with entropy solutions to the Cauchy problem for the isentropic compressible Euler equations in the space-periodic case. In more than one space dimension, the methods developed by De Lellis-Székelyhidi enable us to show failure of uniqueness on a finite time-interval for entropy solutions starting from any continuously differentiable initial density and suitably constructed bounded initial linear momenta.

1. Introduction

In this note, we deal with the Cauchy Problem for the isentropic compressible Euler equations in the space-periodic setting. Given any continuously differentiable initial density, we can construct bounded initial linear momenta for which admissible solutions are not unique in more than one space dimension.

We first introduce the isentropic compressible Euler equations of gas dynamics in n space dimensions, $n \geq 2$ (cf. Section 3.3 of [3]). They are obtained as a simplification of the full compressible Euler equations, by assuming the entropy to be constant. The state of the gas will be described through the state vector

$$V = (\rho, m)$$

whose components are the density ρ and the linear momentum m. The balance laws in force are for mass and linear momentum. The resulting system, which consists of $n+1$ equations, takes the form:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x m \ = \ 0 \\ \partial_t m + \mathrm{div}_x \left(\frac{m \otimes m}{\rho} \right) + \nabla_x [p(\rho)] \ = \ 0 \\ \rho(\cdot, 0) \ = \ \rho^0 \\ m(\cdot, 0) \ = \ m^0 \end{cases} . \tag{1.1}$$

The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas in question. A common choice is the polytropic pressure law

$$p(\rho) = k\rho^\gamma$$

with constants $k > 0$ and $\gamma > 1$. The set of admissible values is $P = \{\rho > 0\}$ (cf. [3] and [19]). The system is hyperbolic if

$$p'(\rho) > 0.$$

In addition, thermodynamically admissible processes must also satisfy an additional constraint coming from the energy inequality

$$\partial_t \left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho} \right) + \operatorname{div}_x \left[\left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \right] \leq 0 \quad (1.2)$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2\varepsilon'(r)$. The physical region for (1.1) is $\{(\rho, m) | \, |m| \leq R\rho\}$, for some constant $R > 0$. For $\rho > 0$, $v = m/\rho$ represents the velocity of the fluid.

We will consider, from now on, the case of general pressure laws given by a function p on $[0, \infty[$, that we always assume to be continuously differentiable on $[0, \infty[$. The crucial requirement we impose upon p is that it has to be strictly increasing on $[0, \infty[$. Such a condition is meaningful from a physical viewpoint since it is a consequence of the principles of thermodynamics.

Now, we will disclose the content of this note. Using some techniques introduced by De Lellis-Székelyhidi (cf. [4] and [5]) we can consider any continuously differentiable periodic initial density ρ^0 and exhibit suitable periodic initial momenta m^0 for which space-periodic weak admissible solutions of (1.1) are not unique on some finite time-interval.

Theorem 1.1. *Let $n \geq 2$. Then, for any given function p and any given continuously differentiable periodic initial density ρ^0, there exist a bounded periodic initial momentum m^0 and a positive time $\overline{T}$ for which there are infinitely many space-periodic admissible solutions (ρ, m) of (1.1) on $\mathbb{R}^n \times [0, \overline{T}[$ with $\rho \in C^1(\mathbb{R}^n \times [0, \overline{T}[)$.*

Remark 1.2. *Indeed, in order to prove Theorem 1.1, it would be enough to assume that the initial density is a Hölder continuous periodic function: $\rho^0 \in C^{0,\alpha}(\mathbb{R}^n)$ (cf. Proof of Proposition 7.1).*

Some connected results are obtained in [5] (cf. Theorem 2 therein) as a further consequence of their analysis on the incompressible Euler equations. Inspired by their approach, we adapt and apply directly to (1.1) the method of convex integration combined with Tartar's programme on oscillation phenomena in conservation laws (see [21] and [13]). In this way, we can show failure of uniqueness of admissible solutions to the compressible Euler equations starting from any given continuously differentiable initial density. For a survey on these h-principle-type results in fluid dynamics we refer the reader to [6].

The paper is organised as follows. Section 2 is an overview on the definitions of weak and admissible solutions to (1.1) and gives a first glimpse on how Theorem 1.1 is achieved. Section 3 is devoted to the reformulation of a simplified version of the isentropic compressible Euler equations as a differential inclusion and to the corresponding geometrical analysis. In Section 4 we state and prove a criterion (Proposition 4.1) to select initial momenta allowing for infinitely many solutions. The proof builds upon a refined version of the Baire category method for differential inclusions developed in [5] and aimed at yielding weakly continuous in time solutions. Section 5 and 6 contain the proofs of the main tools used to prove Proposition 4.1. In Section 7, we show initial momenta satisfying the requirements of Proposition 4.1. Finally, in Section 8 we prove Theorem 1.1 by applying Proposition 4.1.

2. Weak and admissible solutions to the isentropic Euler system

The deceivingly simple-looking system of first-order partial differential equations (1.1) has a long history of important contributions over more than two centuries. We recall a few classical facts on this system (see for instance [3] for more details).

- If ρ^0 and m^0 are "smooth" enough (see Theorem 5.3.1 in [3]), there exists a maximal time interval $[0, T[$ on which there exists a unique "smooth" solution (ρ, m) of (1.1) (for $0 \leq t < T$). In addition, if $T < \infty$, and this is the case in general, (ρ, m) becomes discontinuous as t goes to T.
- If we allow for discontinuous solutions, i.e., for instance, solutions $(\rho, m) \in L^\infty$ satisfying (1.1) in the sense of distributions, then solutions are neither unique nor stable. More precisely, one can exhibit sequences of such solutions which converge weakly in $L^\infty - *$ to functions which do not satisfy (1.1).
- In order to restore the stability of solutions and (possibly) the uniqueness, one may and should impose further restrictions on bounded solutions of (1.1), restrictions which are known as (Lax) entropy inequalities.

This note stems from the problem of better understanding the efficiency of entropy inequalities as selection criteria among weak solutions.

Here, we have chosen to emphasize the case of the flow with *space periodic* boundary conditions. For *space periodic* flows we assume that the fluid fills the entire space $\mathbb{R}^n$ but with the condition that m, ρ are periodic functions of the space variable. The space periodic case is not a physically achievable one, but it is relevant on the physical side as a model for some flows. On the mathematical side, it retains the complexities due to the nonlinear terms (introduced by the kinematics) and therefore it includes many of the difficulties encountered in the general case. However the former is simpler to treat because of the absence of boundaries. Furthermore, using Fourier transform as a tool simplifies the analysis.

Let $Q = [0,1]^n$, $n \geq 2$ be the unit cube in $\mathbb{R}^n$. We denote by $H_p^m(Q)$, $m \in \mathbb{N}$, the space of functions which are in $H_{loc}^m(\mathbb{R}^n)$ and which are periodic with period Q:

$$m(x+l) = m(x) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and every } l \in \mathbb{Z}^n.$$

For $m = 0$, $H_p^0(Q)$ coincides simply with $L^2(Q)$. Analogously, for every functional space X we define $X_p(Q)$ to be the space of functions which are locally (over $\mathbb{R}^n$) in X and are periodic of period Q. The functions in $H_p^m(Q)$ are easily characterized by their Fourier series expansion

$$H_p^m(Q) = \left\{ m \in L_p^2(Q) : \sum_{k \in \mathbb{Z}^n} |k|^{2m} \left|\widehat{m}(k)\right|^2 < \infty \text{ and } \widehat{m}(0) = 0 \right\}, \tag{2.1}$$

where $\widehat{m} : \mathbb{Z}^n \to \mathbb{C}^n$ denotes the Fourier transform of m. We will use the notation $H(Q)$ for $H_p^0(Q)$ and $H_w(Q)$ for the space $H(Q)$ endowed with the weak L^2 topology.

Let T be a fixed positive time. By a *weak solution* of (1.1) on $\mathbb{R}^n \times [0,T[$ we mean a pair $(\rho, m) \in L^\infty([0,T[; L_p^\infty(Q))$ satisfying

$$|m(x,t)| \leq R\rho(x,t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,T[\text{ and some } R > 0, \tag{2.2}$$

and such that the following identities hold for every test functions $\psi \in C_c^\infty([0,T[; C_p^\infty(Q))$, $\phi \in C_c^\infty([0,T[; C_p^\infty(Q))$:

$$\int_0^T \int_Q \left[\rho \partial_t \psi + m \cdot \nabla_x \psi\right] dxdt + \int_Q \rho^0(x)\psi(x,0)dx \;=\; 0 \tag{2.3}$$

$$\begin{aligned} &\int_0^T \int_Q \left[m \cdot \partial_t \phi + \left\langle \frac{m \otimes m}{\rho}, \nabla_x \phi \right\rangle + p(\rho) \operatorname{div}_x \phi \right] dxdt \\ &+ \int_Q m^0(x) \cdot \phi(x,0)dx \;=\; 0. \end{aligned} \tag{2.4}$$

For $n \geq 2$ the only non-trivial entropy is the total energy $\eta = \rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho}$ which corresponds to the flux $\Psi = \left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho}\right) m$.

Then a bounded weak solution (ρ, m) of (1.1) satisfying (1.2) in the sense of distributions, i.e. satisfying the following inequality

$$\int_0^T \int_Q \left[\left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho} \right) \partial_t \varphi + \left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \cdot \nabla_x \varphi \right]$$
$$+ \int_Q \left(\rho^0 \varepsilon(\rho^0) + \frac{1}{2}\frac{|m^0|^2}{\rho} \right) \varphi(\cdot, 0) \ \geq \ 0, \tag{2.5}$$

for every nonnegative $\varphi \in C_c^\infty([0,T[; C_p^\infty(Q))$, is said to be an *entropy* (or *admissible*) solution of (1.1).

The lack of entropies is one of the essential reasons for a very limited understanding of compressible Euler equations in dimensions greater than or equal to 2.

A recent paper by De Lellis-Székelyhidi gives an example in favour of the conjecture that entropy solutions to the multi-dimensional compressible Euler equations are in general not unique. Showing that this conjecture is true has far-reaching consequences. The entropy condition is not sufficient as a selection principle for physical/unique solutions. The non-uniqueness result by De Lellis-Székelyhidi is a byproduct of their new analysis of the incompressible Euler equations based on its formulation as a differential inclusion. They first show that, for some bounded compactly supported initial data, none of the classical admissibility criteria singles out a unique solution to the Cauchy problem for the incompressible Euler equations. As a consequence, by constructing a piecewise constant in space and independent of time density ρ, they look at the compressible isentropic system as a "piecewise incompressible" system (i.e. still incompressible in the support of the velocity field) and thereby exploit the result for the incompressible Euler equations to exhibit bounded initial density and bounded compactly supported initial momenta for which admissible solutions of (1.1) are not unique (in more than one space dimension).

Inspired by their techniques, we give a further counterexample to the well-posedeness of entropy solutions to (1.1). Our result differs in two main aspects: here the initial density can be any given "regular" function and remains "regular" forward in time while in [5] the density allowing for infinitely many admissible solutions must be chosen as piecewise constant in space; on the other hand we are not able to deal with compactly supported momenta (indeed we work in the periodic setting), hence our non-unique entropy solutions are only locally L^2 in contrast with the global-L^2-in-space property of solutions obtained in [5]. Moreover, we have chosen to study the case of the flow in a cube of $\mathbb{R}^n$ with *space periodic* boundary conditions. This case leads to many technical simplifications while retaining the main structure of the problem.

More precisely, we are able to analyze the compressible Euler equations in the framework of convex integration. This method works well with systems of nonlinear PDEs such that the convex envelope (in an appropriate sense) of each small domain of the submanifold representing the PDE in the jet-space (see [8] for more details) is big enough. In our case, we consider a simplification of system (1.1), namely the semi-stationary associated problem, whose submanifold allows a convex integration approach leading us to recover the result of Theorem 1.1.

We are interested in the semi-stationary Cauchy problem associated with the isentropic Euler equations (simply set to 0 the time derivative of the density in (1.1) and drop the initial condition for ρ):

$$\begin{cases} \operatorname{div}_x m \ = \ 0 \\ \partial_t m + \operatorname{div}_x \left(\frac{m\otimes m}{\rho}\right) + \nabla_x [p(\rho)] \ = \ 0 \\ m(\cdot,0) \ = \ m^0. \end{cases} \tag{2.6}$$

A pair $(\rho, m) \in L^\infty_p(Q) \times L^\infty([0,T[; L^\infty_p(Q))$ is a *weak solution* on $\mathbb{R}^n \times [0,T[$ of (2.6) if $m(\cdot,t)$ is weakly-divergence free for almost every $0<t<T$ and satisfies the following bound

$$|m(x,t)| \le R\rho(x) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,T[\text{ and some } R>0, \tag{2.7}$$

and if the following identity holds for every $\phi \in C^\infty_c([0,T[; C^\infty_p(Q))$:

$$\begin{aligned} &\int_0^T \int_Q \left[m\cdot\partial_t\phi + \left\langle \frac{m\otimes m}{\rho}, \nabla_x\phi \right\rangle + p(\rho)\operatorname{div}_x \phi \right] dxdt \\ &+ \int_Q m^0(x)\cdot\phi(x,0)dx \ = \ 0. \end{aligned} \tag{2.8}$$

A general observation suggests us that a non-uniqueness result for weak solutions of (2.6) whose momentum's magnitude satisfies some suitable constraint could lead us to a non-uniqueness result for entropy solutions of the isentropic Euler equations (1.1). Indeed, the entropy solutions we construct in Theorem 1.1 come from *some* weak solutions of (2.6).

Theorem 2.1. *Let $n \ge 2$. Then, for any given function p, any given density $\rho_0 \in C^1_p(Q)$ and any given finite positive time T, there exists a bounded initial momentum m^0 for which there are infinitely many weak solutions $(\rho, m) \in C^1_p(Q) \times C([0,T]; H_w(Q))$ of (2.6) on $\mathbb{R}^n \times [0,T[$ with density $\rho(x) = \rho_0(x)$.*
In particular, the obtained weak solutions m satisfy

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad a.e. \text{ in } \mathbb{R}^n \times [0,T[, \tag{2.9}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad a.e. \text{ in } \mathbb{R}^n, \tag{2.10}$$

for some smooth function χ.

An easy computation shows how, by properly choosing the function χ in (2.9)-(2.10), the solutions (ρ_0, m) of (2.6) obtained in Theorem 2.1 satisfy the admissibility condition (2.5).

Theorem 2.2. *Under the same assumptions of Theorem 2.1, there exists a maximal time* $\overline{T} > 0$ *such that the weak solutions* (ρ, m) *of* (2.6) *(coming from Theorem 2.1) satisfy the admissibility condition* (2.5) *on* $[0, \overline{T}[$.

Our construction yields initial data m^0 for which the nonuniqueness result of Theorem 1.1 holds on any time interval $[0, T[$, with $T \leq \overline{T}$. However, as pointed out before, for sufficiently regular initial data, classical results give the local uniqueness of smooth solutions. Thus, *a fortiori*, the initial momenta considered in our examples have necessarily a certain degree of irregularity.

3. Geometrical analysis

This section is devoted to a qualitative analysis of the isentropic compressible Euler equations in a semi-stationary regime (i.e. (2.6)).

As in [4] we will interpret the system (2.6) in terms of a differential inclusion, so that it can be studied in the framework combining the plane wave analysis of Tartar, the convex integration of Gromov and the Baire's arguments.

3.1. **Differential inclusion.**

The system (2.6) can indeed be naturally expressed as a linear system of partial differential equations coupled with a pointwise nonlinear constraint, usually called *differential inclusion*.

The following Lemma, based on Lemma 2 in [5], gives such a reformulation. We will denote by $\mathcal{S}^n$ the space of symmetric $n \times n$ matrices, by $\mathcal{S}^n_0$ the subspace of $\mathcal{S}^n$ of matrices with null trace, and by I_n the $n \times n$ identity matrix.

Lemma 3.1. *Let* $m \in L^\infty([0,T]; L^\infty_p(Q;\mathbb{R}^n))$, $U \in L^\infty([0,T]; L^\infty_p(Q;\mathcal{S}^n_0))$ *and* $q \in L^\infty([0,T]; L^\infty(Q;\mathbb{R}^+))$ *such that*

$$\begin{aligned} \operatorname{div}_x m &= 0 \\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0. \end{aligned} \tag{3.1}$$

If (m, U, q) *solve* (3.1) *and in addition there exists* $\rho \in L^\infty_p(\mathbb{R}^n;\mathbb{R}^+)$ *such that* (2.7) *holds and*

$$\begin{aligned} U &= \frac{m \otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n \quad \textit{a.e. in } \mathbb{R}^n \times [0,T], \\ q &= p(\rho) + \frac{|m|^2}{n\rho} \quad \textit{a.e. in } \mathbb{R}^n \times [0,T], \end{aligned} \tag{3.2}$$

then m and ρ solve (2.6) *distributionally. Conversely, if m and ρ are weak solutions of* (2.6)*, then m, $U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n$ and $q = p(\rho) + \frac{|m|^2}{n\rho}$ solve* (3.1)-(3.2)*.*

In Lemma 3.1 we made clear the distinction between the augmented system (3.1), whose linearity allows a plane wave analysis, and the non-linear pointwise constraint (3.2), which leads us to study the graph below.

For any given $\rho \in]0,\infty[$, we define the following graph

$$K_\rho := \Bigg\{ (m,U,q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n, \\ q = p(\rho) + \frac{|m|^2}{n\rho} \Bigg\}. \tag{3.3}$$

The key of the forthcoming analysis is the behaviour of the graph K_ρ with respect to the wave vectors associated with the linear system (3.1): are differential and algebraic constraints in some sense compatible?

For our purposes, it is convenient to consider "slices" of the graph K_ρ, by considering vectors m whose modulus is subject to some ρ-depending condition. Thus, for any given $\chi \in \mathbb{R}^+$, we define:

$$K_{\rho,\chi} := \Bigg\{ (m,U,q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n, \\ q = p(\rho) + \frac{|m|^2}{n\rho}, |m|^2 = \rho\chi \Bigg\}. \tag{3.4}$$

3.2. **Wave cone.** Following Tartar's framework [21], we consider a system of first order linear PDEs

$$\sum_i A_i \partial_i z = 0 \tag{3.5}$$

where z is a vector valued function and the A_i are matrices. Then, *planewave* solutions to (3.5) are solutions of the form

$$z(x) = a h(x\cdot\xi), \tag{3.6}$$

with $h : \mathbb{R} \to \mathbb{R}$. In order to find such solutions, we have to solve the relation $\sum_i \xi_i A_i a = 0$, where ξ_i is the oscillation frequency in the direction i. The set of directions a for which a solution $\xi \neq 0$ exists is called *wave cone* Λ of the system (3.5): equivalently Λ characterizes the directions of one dimensional high frequency oscillations compatible with (3.5).

The system (3.1) can be analyzed in this framework. Consider the $(n+1) \times (n+1)$ symmetric matrix in block form

$$M = \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix}. \tag{3.7}$$

Note that, with the new coordinates $y = (x,t) \in \mathbb{R}^{n+1}$, the system (3.1) can be easily rewritten as $\operatorname{div}_y M = 0$ (the divergence of M in space-time is zero). Thus, the wave cone associated with the system (3.1) is equal to

$$\Lambda = \left\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : \det \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix} = 0 \right\}. \tag{3.8}$$

Indeed, the relation $\sum_i \xi_i A_i a = 0$ for the system (3.1) reads simply as $M \cdot (\xi, c) = 0$, where $(\xi, c) \in \mathbb{R}^n \times \mathbb{R}$ (ξ is the space-frequency and c the time-frequency): this equation admits a non-trivial solution if M has null determinant, hence (3.8).

3.3. **Convex hull and geometric setup.** Given a cone Λ, we say that K is convex with respect to Λ if, for any two points $A, B \in K$ with $B - A \in \Lambda$, the whole segment $[A, B]$ belongs to K. The Λ-convex hull of $K_{\rho,\chi}$ is the smallest Λ-convex set $K^\Lambda_{\rho,\chi}$ containing $K_{\rho,\chi}$, i.e. the set of states obtained by mixture of states of $K_{\rho,\chi}$ through oscillations in Λ-directions (Gromov [11], who works in the more general setting of jet bundles, calls this the P- convex hull). The key point in Gromov's method of convex integration (which is a far reaching generalization of the work of Nash [17] and Kuiper [14] on isometric immersions) is that (3.5) coupled with a pointwise nonlinear constraint of the form $z \in K$ a.e. admits many interesting solutions provided that the Λ-convex hull of K, K^Λ, is sufficiently large. In applications to elliptic and parabolic systems we always have $K^\Lambda = K$ so that Gromov's approach does not directly apply. For other applications to partial differential equations it turns out that one can work with the Λ-convex hull defined by duality. More precisely, a point does not belong to the Λ- convex hull defined by duality if and only if there exists a Λ-convex function which separates it from K. A crucial fact is that the second notion is much weaker. This surprising fact is illustrated in [13].

In our case, the wave cone is quite large, therefore it is sufficient to consider the stronger notion of Λ-convex hull, indeed it coincides with the whole convex hull of $K_{\rho,\chi}$.

Lemma 3.2. *For any $S \in \mathcal{S}^n$ let $\lambda_{max}(S)$ denote the largest eigenvalue of S. For $(\rho, m, U) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}_0^n$ let*

$$e(\rho, m, U) := \lambda_{max}\left(\frac{m \otimes m}{\rho} - U \right). \tag{3.9}$$

Then, for any given $\rho, \chi \in \mathbb{R}^+$, the following holds

(i) $e(\rho, \cdot, \cdot) : \mathbb{R}^n \times \mathcal{S}_0^n \to \mathbb{R}$ *is convex;*
(ii) $\frac{|m|^2}{n\rho} \leq e(\rho, m, U)$*, with equality if and only if* $U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n$*;*
(iii) $|U|_\infty \leq (n-1) e(\rho, m, U)$*, with* $|U|_\infty$ *being the operator norm of the matrix;*
(iv) *the* $\frac{\chi}{n}$*-sublevel set of* e *defines the convex hull of* $K_{\rho,\chi}$*, i.e.*

$$K_{\rho,\chi}^{co} = \left\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : e(\rho, m, U) \leq \frac{\chi}{n}, \right.$$
$$\left. q = p(\rho) + \frac{\chi}{n} \right\} \tag{3.10}$$

and $K_{\rho,\chi} = K_{\rho,\chi}^{co} \cap \{|m|^2 = \rho\chi\}$.

For the proof of (i)-(iv) we refer the reader to the proof of Lemma 3.2 in [5]: the arguments there can be easily adapted to our case.

We observe that, for any $\rho, \chi \in \mathbb{R}^+$, the convex hull $K_{\rho,\chi}^{co}$ lives in the hyperplane H of $\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+$ defined by $H := \left\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : q = p(\rho) + \frac{\chi}{n} \right\}$. Therefore, the interior of $K_{\rho,\chi}^{co}$ as a subset of $\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+$ is empty. This seems to prevent us from working in the classical framework of convex integration, but we can overcome this apparent obstacle.

For any $\rho, \chi \in \mathbb{R}^+$, we define the *hyperinterior* of $K_{\rho,\chi}^{co}$, and we denote it with "hint $K_{\rho,\chi}^{co}$", as the following set

$$\text{hint } K_{\rho,\chi}^{co} := \left\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : e(\rho, m, U) < \frac{\chi}{n}, \right.$$
$$\left. q = p(\rho) + \frac{\chi}{n} \right\}. \tag{3.11}$$

In the framework of convex integration, the larger the Λ-convex hull of $K_{\rho,\chi}$ is, the bigger the breathing space will be. How to "quantify" the meaning of a "large" Λ-convex hull in our context? The previous definition provides an answer: the Λ-convex hull of $K_{\rho,\chi}$ will be "large" if its hyperinterior is nonempty. The wave cone of the semi-stationary Euler isentropic system is wide enough to ensure that the Λ-convex hull of $K_{\rho,\chi}$ coincides with the convex hull of $K_{\rho,\chi}$ and has a nonempty hyperinterior. As a consequence, we can construct irregular solutions oscillating along any fixed direction. For our purposes, it will be convenient to restrict to some *special directions* in Λ, consisting of matrices of rank 2, which are not stationary in time, but are associated with a constant pressure.

Lemma 3.3. *Let $c, d \in \mathbb{R}^n$ with $|c| = |d|$ and $c \neq d$, and let $\rho \in \mathbb{R}^+$. Then $\left(c - d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right) \in \Lambda$.*

Proof. Since the vector $\left(c + d, -\left(\frac{|c|^2 + c\cdot d}{\rho}\right)\right)$ is in the kernel of the matrix

$$C = \begin{pmatrix} \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho} & c - d \\ c - d & 0 \end{pmatrix},$$

C has indeed determinant zero, hence $\left(c - d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right) \in \Lambda$. □

Now, we introduce some important tools: they allow us to prove that $K^{\Lambda}_{\rho,\chi} = K^{co}_{\rho,\chi}$ is sufficiently large, thus providing us room to find many solutions for (3.1)-(3.2).

As first, we define the *admissible segments* as segments in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ whose directions belong to the wave cone Λ for the linear system of PDEs (3.1) and are indeed *special directions* in the sense specified by Lemma 3.3.

Definition 3.4. *Given $\rho, \chi \in \mathbb{R}^+$ we call σ an admissible segment for (ρ, χ) if σ is a line segment in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ satisfying the following conditions:*

- *σ is contained in the hyperinterior of $K^{co}_{\rho,\chi}$;*
- *σ is parallel to $\left(c - d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right)$ for some $c, d \in \mathbb{R}^n$ with $|c|^2 = |d|^2 = \rho\chi$ and $c \neq \pm d$.*

The admissible segments defined above correspond to suitable plane-wave solutions of (3.1). The following Lemma ensures that, for any $\rho, \chi \in \mathbb{R}^+$, the hyperinterior of $K^{co}_{\rho,\chi}$ is " sufficiently round " with respect to the special directions: given any point in the hyperinterior of $K^{co}_{\rho,\chi}$, it can be seen as the midpoint of a sufficiently large admissible segment for (ρ, χ).

Lemma 3.5. *There exists a constant $F = F(n) > 0$ such that for any $\rho, \chi \in \mathbb{R}^+$ and for any $z = (m, U, q) \in \text{hint}\, K^{co}_{\rho,\chi}$ there exists an admissible line segment for (ρ, χ)*

$$\sigma = \left[(m, U, q) - (\overline{m}, \overline{U}, 0), (m, U, q) + (\overline{m}, \overline{U}, 0)\right] \tag{3.12}$$

such that

$$|\overline{m}| \geq \frac{F}{\sqrt{\rho\chi}} \left(\rho\chi - |m|^2\right).$$

The proof rests on a clever application of Carathéodory's theorem for convex sets and can be carried out, with minor modifications, as in [5] (cf. Lemma 6 therein).

As an easy consequence of the previous Lemma, we can finally establish that the Λ-convex hull of $K_{\rho,\chi}$ coincides with $K^{co}_{\rho,\chi}$.

Proposition 3.6. *For all given $\rho,\chi\in\mathbb{R}^+$, the Λ-convex hull of $K_{\rho,\chi}$ coincides with the convex hull of $K_{\rho,\chi}$.*

***Proof*.** Recall that, given $\rho,\chi\in\mathbb{R}^+$, we denote the Λ-convex hull of $K_{\rho,\chi}$ with $K^{\Lambda}_{\rho,\chi}$. Of course $K^{\Lambda}_{\rho,\chi}\subset K^{co}_{\rho,\chi}$, hence we have to prove the opposite inclusion, i.e. $K^{co}_{\rho,\chi}\subset K^{\Lambda}_{\rho,\chi}$. For every $z\in K^{co}_{\rho,\chi}$ we can follow the procedure in the proof of Lemma 3.5 (cf. [5]) and write it as $z=\sum_j\lambda_j z_j$, with $(z_j)_{1\leq j\leq N+1}$ in $K_{\rho,\chi}$, $(\lambda_j)_{1\leq j\leq N+1}$ in $[0,1]$ and $\sum_j\lambda_j=1$. Again, we can assume that $\lambda_1=\max_j\lambda_j$. In case $\lambda_1=1$ then $z=z_1\in K_{\rho,\chi}\subset K^{\Lambda}_{\rho,\chi}$ and we can already conclude. Otherwise (i.e. when $\lambda_1\in(0,1)$) we can argue as in Lemma 3.5 so to find an admissible segment σ for (ρ,χ) of the form (3.12). Since we aim at writing z as a Λ-barycenter of elements of $K_{\rho,\chi}$, we "play" with these admissible segments by prolongations and iterative constructions until we get segments with extremes lying in $K_{\rho,\chi}$. More precisely: we extend the segment σ until we meet $\partial\,\mathrm{hint}K^{co}_{\rho,\chi}$ thus obtaining z as the barycenter of two points (w_0,w_1) with $(w_0-w_1)\in\Lambda$ and such that every $w_i=(m_i,U_i,q_i)$, $i=0,1$, satisfies either $|m_i|^2=\rho\chi$ or $|m_i|^2<\rho\chi$ and $e(\rho,m_i,U_i)=\chi/n$.

In the first case, $U_i-\left(\frac{m_i\otimes m_i}{\rho}-\frac{|m_i|^2}{n\rho}I_n\right)\geq 0$, and since it is a null-trace-matrix it is identically zero, whence $w_i\in K_{\rho,\chi}$ $\Big($note that in the construction of σ the q-direction remains unchanged, hence $q_i=p(\rho)+\frac{\chi}{n}\Big)$.

In the second case, i.e. when $|m_i|^2<\rho\chi$ and $e(\rho,m_i,U_i)=\chi/n$, we apply again Lemma 3.5 and a limit procedure to express w_i as barycentre of $(w_{i,0},w_{i,1})$ with $(w_{i,0}-w_{i,1})\in\Lambda$ and such that every $w_{i,k}=(m_{i,k},U_{i,k},q_{i,k})$, $k=0,1$, will satisfy either $|m_{i,k}|^2=\rho\chi$ or $\lambda_2(\rho,m_{i,k},U_{i,k})=e(\rho,m_{i,k},U_{i,k})=\chi/n$, where $\lambda_1(\rho,m,U)\geq\lambda_2(\rho,m,U)\geq \ldots\ldots\geq\lambda_n(\rho,m,U)$ denote the ordered eigenvalues of the matrix $\frac{m\otimes m}{\rho}-U$ (note that $\lambda_1(\rho,m,U)=e(\rho,m,U)$). Now, we iterate this procedure of constructing suitable admissible segments for (ρ,χ) until we have written z as Λ-barycenter of points (m,U,q) satisfying either $|m|^2=\rho\chi$ or $\lambda_n(\rho,m,U)=\chi/n$ and therefore all belonging to $K_{\rho,\chi}$ as desired. $\square$

4. A criterion for the existence of infinitely many solutions

The following Proposition provides a criterion to recognize initial data m^0 which allow for many weak admissible solutions to (1.1). Its proof relies deeply on the geometrical analysis carried out in Section 3. The underlying idea comes from convex integration. The general principle of this method, developed for partial differential equations by Gromov [11] and for ordinary differential equations by Filippov [10], consists in the following steps: given a nonlinear equation $\mathcal{E}(z)$,

- (i) we rewrite it as $(\mathcal{L}(z) \wedge z \in K)$ where $\mathcal{L}$ is a linear equation;
- (ii) we introduce a strict subsolution z_0 of the system, i.e. satisfying a relaxed system $(\mathcal{L}(z_0) \wedge z \in \mathcal{U})$;
- (iii) we construct a sequence $(z_k)_{k\in\mathbb{N}}$ approaching K but staying in $\mathcal{U}$;
- (iv) we pass to the limit, possibly modifying the sequence (z_k) in order to ensure a suitable convergence.

Step (i) has already been done in Section 3.1. The choice of z_0 will be specified in Sections 7-8. Here, we define the notion of subsolution for an appropriate set $\mathcal{U}$, we construct an improving sequence and we pass to the limit. The way how we construct the approximating sequence will be described in Section 6 using some tools from Section 5.

One crucial step in convex integration is the passage from open sets K to general sets. This can be done in different ways, e.g. by the Baire category theorem (cf. [18]), a refinement of it using Baire-1 functions or the Banach-Mazur game [12] or by direct construction [20]. Whatever approach one uses the basic theme is the same: at each step of the construction one adds a highly oscillatory correction whose frequency is much larger and whose amplitude is much smaller than those of the previous corrections.

In this section, we achieve our goals following some Baire category arguments as in [4]: they are morally close to the methods developed by Bressan and Flores in [1] and by Kirchheim in [12].

In our framework the initial data will be constructed starting from solutions to the convexified (or relaxed) problem associated to (2.6), i.e. solutions to the linearized system (3.1) satisfying a “relaxed” nonlinear constraint (3.2) (i.e. belonging to the hyperinterior of the convex hull of the “constraint set”), which we will call *subsolutions*.

As in [4], our application shows that the Baire theory is comparable in terms of results to the method of convex integration and they have many similarities: they are both based on an approximation approach to tackle problems while the difference lies only in the limit arguments, i.e. on the way the exact solution is obtained from better and better approximate ones. These similarities are clarified by Kirchheim in [12], where the continuity points of a first category Baire function are considered; a comparison between the two methods is drawn by Sychev in [20].

Here, the topological reasoning of Baire theory is preferred to the iteration technique of convex integration, since the first has the advantage to provide us directly with infinitely many different solutions.

Proposition 4.1. *Let* $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ *be a given density function and let* T *be any finite positive time. Assume there exist* (m_0, U_0, q_0) *continuous space-periodic solutions of* (3.1) *on* $\mathbb{R}^n\times]0,T[$ *with*

$$m_0 \in C([0,T];H_w(Q)), \tag{4.1}$$

and a function $\chi \in C^\infty([0,T];\mathbb{R}^+)$ *such that*

$$e(\rho_0(x), m_0(x,t), U_0(x,t)) < \frac{\chi(t)}{n} \qquad \text{for all } (x,t)\in\mathbb{R}^n\times]0,T[, \tag{4.2}$$

$$q_0(x,t) = p(\rho_0(x)) + \frac{\chi(t)}{n} \qquad \text{for all } (x,t)\in\mathbb{R}^n\times]0,T[. \tag{4.3}$$

Then there exist infinitely many weak solutions (ρ, m) *of the system* (2.6) *in* $\mathbb{R}^n\times[0,T[$ *with density* $\rho(x) = \rho_0(x)$ *and such that*

$$m \in C([0,T];H_w(Q)), \tag{4.4}$$

$$m(\cdot,t) = m_0(\cdot,t) \qquad \text{for } t = 0, T \text{ and for a.e. } x\in\mathbb{R}^n, \tag{4.5}$$

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \qquad \text{for a.e. } (x,t)\in\mathbb{R}^n\times]0,T[. \tag{4.6}$$

4.1. **The space of subsolutions.** We define the space of *subsolutions* as follows. Let ρ_0 and χ be given as in the assumptions of Proposition 4.1. Let m_0 be a vector field as in Proposition 4.1 with associated modified pressure q_0 and consider space-periodic momentum fields $m : \mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ which satisfy

$$\operatorname{div} m = 0, \tag{4.7}$$

the initial and boundary conditions

$$m(x,0) = m_0(x,0), \tag{4.8}$$

$$m(x,T) = m_0(x,T), \tag{4.9}$$

$$\tag{4.10}$$

and such that there exists a continuous space-periodic matrix field $U : \mathbb{R}^n\times]0,T[\to \mathcal{S}^n_0$ with

$$\begin{aligned} &e(\rho_0(x), m(x,t), U(x,t)) < \frac{\chi}{n} \quad \text{for all } (x,t)\in\mathbb{R}^n\times]0,T[,\\ &\partial_t m + \operatorname{div}_x U + \nabla_x q_0 = 0 \quad \text{in } \mathbb{R}^n\times[0,T]. \end{aligned} \tag{4.11}$$

Definition 4.2. *Let* X_0 *be the set of such linear momentum fields, i.e.*

$$X_0 = \Big\{ m \in C^0(]0,T[;C^0_p(Q)) \cap C([0,T];H_w(Q)) : \\ (4.7) - (4.11) \text{ are satisfied} \Big\} \tag{4.12}$$

and let X *be the closure of* X_0 *in* $C([0,T];H_w(Q)$*. Then* X_0 *will be the space of strict subsolutions.*

As ρ_0 is continuous and periodic on $\mathbb{R}^n$ and χ is smooth on $[0,T]$, there exists a constant G such that $\chi(t)\int_Q \rho_0(x)dx \le G$ for all $t\in[0,T]$. Since for any $m\in X_0$ with associated matrix field U we have that (see Lemma 3.2- (ii))

$$\int_Q |m(x,t)|^2\,dx \le \int_Q n\rho_0(x)e(\rho_0(x),m(x,t),U(x,t))dx$$
$$< \chi(t)\int_Q \rho_0(x)dx \text{ for all } t\in[0,T],$$

we can observe that X_0 consists of functions $m:[0,T]\to H(Q)$ taking values in a bounded subset B of $H(Q)$. Without loss of generality, we can assume that B is weakly closed. Then, B in its weak topology is metrizable and, if we let d_B be a metric on B inducing the weak topology, we have that (B,d_B) is a compact metric space. Moreover, we can define on $Y:=C([0,T],(B,d_B))$ a metric d naturally induced by d_B via

$$d(f_1,f_2):=\max_{t\in[0,T]} d_B(f_1(\cdot,t),f_2(\cdot,t)). \tag{4.13}$$

Note that the topology induced on Y by d is equivalent to the topology of Y as a subset of $C([0,T];H_w)$. In addition, the space (Y,d) is complete. Finally, X is the closure in (Y,d) of X_0 and hence (X,d) is as well a complete metric space.

Lemma 4.3. *If $m\in X$ is such that $|m(x,t)|^2=\rho_0(x)\chi(t)$ for almost every $(x,t)\in\mathbb{R}^n\times]0,T[$, then the pair (ρ_0,m) is a weak solution of* (2.6) *in $\mathbb{R}^n\times[0,T[$ satisfying* (4.4)-(4.5)-(4.6).

Proof. Let $m\in X$ be such that $|m(x,t)|^2=\rho_0(x)\chi(t)$ for almost every $(x,t)\in\mathbb{R}^n\times]0,T[$. By density of X_0, there exists a sequence $\{m_k\}\subset X_0$ such that $m_k\overset{d}{\to} m$ in X. For any $m_k\in X_0$ let U_k be the associated smooth matrix field enjoying (4.11). Thanks to Lemma 3.2 (iii) and (4.11), the following pointwise estimate holds for the sequence $\{U_k\}$

$$|U_k|_\infty \le (n-1)e(\rho_0,m_k,U_k) < \frac{(n-1)\chi}{n}.$$

As a consequence, $\{U_k\}$ is uniformly bounded in $L^\infty([0,T];L^\infty_p(Q))$; by possibly extracting a subsequence, we have that

$$U_k \overset{*}{\rightharpoonup} U \text{ in } L^\infty([0,T];L^\infty_p(Q)).$$

Note that $\overline{\mathrm{hint}K^{co}_{\rho_0,\chi}}=K^{co}_{\rho_0,\chi}$ is a convex and compact set by Lemma 3.2-(i)-(ii)-(iii). Hence, $m\in X$ with associated matrix field U solves (3.1) on $\mathbb{R}^n\times[0,T]$ for $q=q_0$ and (m,U,q_0) takes values in $K^{co}_{\rho_0,\chi}$ almost everywhere. If, in addition, $|m(x,t)|^2=\rho_0(x)\chi(t)$, then $(m,U,q_0)(x,t)\in K_{\rho,\chi}$ a.e. in $\mathbb{R}^n\times[0,T]$ (cf. Lemma 3.2-(iv)). Lemma 3.1 allows us to conclude that (ρ_0,m) is a weak solution of (2.6) in $\mathbb{R}^n\times[0,T[$. Finally, since $m_k\to m$ in $C([0,T];H_w(Q))$ and $|m(x,t)|^2=\rho_0(x)\chi(t)$ for almost every $(x,t)\in\mathbb{R}^n\times]0,T[$, we see that m satisfies also (4.4)-(4.5)-(4.6). $\square$

Now, we will argue as in [4] exploiting Baire category techniques to combine weak and strong convergence (see also [12]).

Lemma 4.4. *The identity map $I : (X, d) \to L^2([0, T]; H(Q))$ defined by $m \to m$ is a Baire-1 map, and therefore the set of points of continuity is residual in (X, d).*

***Proof*.** Let $\phi_r(x, t) = r^{-(n+1)}\phi(rx, rt)$ be any regular spacetime convolution kernel. For each fixed $m \in X$, we have

$$\phi_r * m \to m \quad \text{strongly in } L^2(H) \text{ as } r \to 0.$$

On the other hand, for each $r > 0$ and $m_k \in X$,

$$m_k \overset{d}{\to} m \text{ implies } \phi_r * m_k \to \phi_r * m \text{ in } L^2(H).$$

Therefore, each map $I_r : (X, d) \to L^2(H)$, $m \to \phi_r * m$ is continuous, and $I(m) = \lim_{r\to 0} I_r(m)$ for all $m \in X$. This shows that $I : (X, d) \to L^2(H)$ is a pointwise limit of continuous maps; hence it is a Baire-1 map. As a consequence, the set of points of continuity of I is residual in (X, d) (cf. [18]). □

4.2. **Proof of Proposition 4.1.** We aim to show that all points of continuity of the identity map correspond to solutions of (2.6) enjoying the requirements of Proposition 4.1: Lemma 4.4 will then allow us to prove Proposition 4.1 once we know that the cardinality of X is infinite. In light of Lemma 4.3, for our purposes it suffices to prove the following claim:

CLAIM. If $m \in X$ is a point of continuity of I, then

$$|m(x, t)|^2 = \rho_0(x)\chi(t) \text{ for almost every } (x, t) \in \mathbb{R}^n \times]0, T[. \tag{4.14}$$

□

Note that proving (4.14) is equivalent to prove that $\|m\|_{L^2(Q\times[0,T])} = \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx\right)^{1/2}$, since for any $m \in X$ we have $|m(x, t)|^2 \leq \rho_0(x)\chi(t)$ for almost all $(x, t) \in \mathbb{R}^n \times [0, T]$. Thanks to this remark, the claim is reduced to the following lemma (cf. Lemma 4.6 in [4]), which provides a strategy to move towards the boundary of X_0: given $m \in X_0$, we will be able to approach it with a sequence inside X_0 but closer than m to the boundary of X_0.

Lemma 4.5. *Let ρ_0, χ be given functions as in Proposition 4.1. Then, there exists a constant $\beta = \beta(n)$ such that, given $m \in X_0$, there exists a* *sequence* $\{m_k\} \subset X_0$ *with the following properties*

$$\|m_k\|^2_{L^2(Q\times[0,T])} \geq \|m\|^2_{L^2(Q\times[0,T])} + \beta\left(\int_Q\int_0^T \rho_0(x)\chi(t)dtdx - \|m\|^2_{L^2(Q\times[0,T])}\right)^2 \tag{4.15}$$

and

$$m_k \to m \textit{ in } C([0,T], H_w(Q)). \tag{4.16}$$

The proof is postponed to Section 6. Let us show how Lemma 4.5 implies the claim. As in the claim, assume that $m \in X$ is a point of continuity of the identity map I. Let $\{m_k\} \subset X_0$ be a fixed sequence that converges to m in $C([0,T], H_w(Q))$. Using Lemma 4.5 and a standard diagonal argument, we can find a second sequence $\{\widetilde{m}_k\}$ yet converging to m in X and satisfying

$$\liminf_{k\to\infty} \|\widetilde{m}_k\|^2_{L^2(Q\times[0,T])} \geq \liminf_{k\to\infty}\Bigg(\|m_k\|^2_{L^2(Q\times[0,T])} + \beta\left(\int_Q\int_0^T \rho_0(x)\chi(t)dtdx - \|m_k\|^2_{L^2(Q\times[0,T])}\right)^2\Bigg).$$

According to the hypothesis, I is continuous at m, therefore both m_k and $\widetilde{m}_k$ converge strongly to m and

$$\|m\|^2_{L^2(Q\times[0,T])} \geq \|m\|^2_{L^2(Q\times[0,T])} + \beta\left(\int_Q\int_0^T \rho_0(x)\chi(t)dtdx - \|m\|^2_{L^2(Q\times[0,T])}\right)^2.$$

Hence $\|m\|_{L^2(Q\times[0,T])} = \left(\int_Q\int_0^T \rho_0(x)\chi(t)dtdx\right)^{1/2}$ and the claim holds true. Finally, since the assumptions of Proposition 4.1 ensure that X_0 is nonempty, by Lemma 4.5 we can see that the cardinality of X is infinite whence the cardinality of any residual set in X is infinite. In particular, the set of continuity points of I is infinite: this and the claim conclude the proof of Proposition 4.1.

5. Localized oscillating solutions

The wild solutions are made by adding one dimensional oscillating functions in different directions $\lambda \in \Lambda$. For that it is needed to localize the waves. More precisely, the proof of Lemma 4.5 relies on the construction of solutions to the linear system (3.1), localized in space-time and oscillating between two states in $K^{co}_{\rho_0,\chi}$ along a given special direction $\lambda \in \Lambda$. Aiming at compactly supported solutions, one faces the problem of localizing vector valued functions: this is bypassed thanks to the construction of a "localizing" potential for the conservation laws

(3.1). This approach is inherited from [5]. As in [4] it could be realized for every $\lambda \in \Lambda$, but in our framework it is convenient to restrict only to *special* Λ-directions (cf. [5]): this restriction will allow us to localize the oscillations at constant pressure.

Why oscillations at constant pressure are meaningful for us and needed in the proof of Lemma 4.5?

Owing to Section 3, in the variables $y = (x,t) \in \mathbb{R}^{n+1}$, the system (3.1) is equivalent to $\operatorname{div}_y M = 0$, where $M \in \mathcal{S}^{n+1}$ is defined via the linear map

$$\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \ni (m, U, q) \longmapsto M = \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix}. \tag{5.1}$$

More precisely, this map builds an identification between the set of solutions (m, U, q) to (3.1) and the set of symmetric $(n+1)\times(n+1)$ matrices M with $M_{(n+1)(n+1)} = 0$ and $\operatorname{tr}(M) = q$.

Therefore, solutions of (3.1) with $q \equiv 0$ correspond to matrix fields $M : \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$\operatorname{div}_y M = 0, \quad M^{\mathrm{T}} = M, \quad M_{(n+1)(n+1)} = 0, \quad \operatorname{tr}(M) = 0. \tag{5.2}$$

Moreover, given a density ρ and two states $(c, U_c, q_c), (d, U_d, q_d) \in K_\rho$ with non collinear momentum vector fields c and d having same magnitude ($|c| = |d|$), and hence same pressure ($q_c = q_d$), then the corresponding matrices M_c and M_d have the following form

$$M_c = \begin{pmatrix} \frac{c\otimes c}{\rho} + p(\rho)I_n & c \\ c & 0 \end{pmatrix} \text{ and } M_d = \begin{pmatrix} \frac{d\otimes d}{\rho} + p(\rho)I_n & d \\ d & 0 \end{pmatrix}$$

and satisfy

$$M_c - M_d = \begin{pmatrix} \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho} & c-d \\ c-d & 0 \end{pmatrix}.$$

Finally note that $\operatorname{tr}(M_c - M_d) = 0$ and $M_c - M_d \in \Lambda$ corresponds to a *special direction.*

The following Proposition provides a potential for solutions of (3.1) oscillating between two states M_c and M_d at constant pressure. It is an easy adaptation to our framework of Proposition 4 in [5].

Proposition 5.1. *Let $c, d \in \mathbb{R}^n$ such that $|c| = |d|$ and $c \neq d$. Let also $\rho \in \mathbb{R}$. Then there exists a matrix-valued, constant coefficient, homogeneous linear differential operator of order* 3

$$A(\partial) : C_c^\infty(\mathbb{R}^{n+1}) \to C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{(n+1)\times(n+1)})$$

such that $M = A(\partial)\phi$ satisfies (5.2) *for all $\phi \in C_c^\infty(\mathbb{R}^{n+1})$. Moreover there exists $\eta \in \mathbb{R}^{n+1}$ such that*

- *η is not parallel to e_{n+1};*

- *if* $\phi(y) = \psi(y \cdot \eta)$, *then*
$$A(\partial)\phi(y) = (M_c - M_d)\phi'''(y \cdot \eta).$$

We also report Lemma 7 from [5]: it ensures that the oscillations of the planewaves generated in proposition 5.1 have a certain size in terms of an appropriate norm-type-functional.

Lemma 5.2. *Let* $\eta \in \mathbb{R}^{n+1}$ *be a vector which is not parallel to* e_{n+1}*. Then for any bounded open set* $B \subset \mathbb{R}^n$
$$\lim_{N\to\infty} \int_B \sin^2(N\eta \cdot (x,t))dx = \frac{1}{2}|B|$$
uniformly in $t \in \mathbb{R}$.

For the proof we refer the reader to [5].

6. The improvement step

We are now about to prove one of the cornerstones of the costruction. Before moving forward, let us resume the plan. We have already identified a relaxed problem by introducing *subsolutions*. Then, we have proved a sort of "h-principle" (even if there is no homotopy here) according to which, the space of *subsolutions* can be "reduced" to the space of solutions or, equivalently, the typical (in Baire's sense) *subsolution* is a solution. Once assumed that a *subsolution* exists, the proof of our "h-principle" builds upon Lemma 4.5 combined with Baire category arguments. Indeed, we could also prove Proposition 4.1 by applying iteratively Lemma 4.5 and thus constructing a converging sequence of subsolutions approaching $K_{\rho,\chi}$: this would correspond to the convex integration approach. So two steps are left in order to conclude our argument: showing the existence of a "starting" *subsolution* and prove Lemma 4.5.

This section is devoted to the second task, the proof of Lemma 4.5, while in next section we will exhibit a "concrete" *subsolution*.

What follows will be quite technical, therefore we first would like to recall the plan: we will add fast oscillations in allowed directions so to let $|m|^2$ increase in average. The proof is inspired by [4]-[5].

***Proof*.** [Proof of Lemma 4.5] Let us fix the domain $\Omega := Q \times [0,T]$. We look for a sequence $\{m_k\} \subset X_0$, with associated matrix fields $\{U_k\}$, which improves m in the sense of (4.15) and has the form
$$(m_k, U_k) = (m, U) + \sum_j (\widetilde{m}_{k,j}, \widetilde{U}_{k,j}) \tag{6.1}$$
where every $z_{k,j} = (\widetilde{m}_{k,j}, \widetilde{U}_{k,j})$ is compactly supported in some suitable ball $B_{k,j}(x_{k,j}, t_{k,j}) \subset \Omega$. We proceed as follows.

Step 1. Let $m \in X_0$ with associated matrix field U. By Lemma 3.5, for any $(x,t) \in \Omega$ we can find a line segment $\sigma_{(x,t)} := [(m(x,t), U(x,t), q_0(x)) - \lambda_{(x,t)}, (m(x,t), U(x,t), q_0(x)) + \lambda_{(x,t)}]$ admissible for $(\rho_0(x), \chi(t))$ and with direction

$$\lambda_{(x,t)} = (\overline{m}(x,t), \overline{U}(x,t), 0)$$

such that

$$|\overline{m}(x,t)| \geq \frac{F}{\sqrt{\rho_0(x)\chi(t)}} \left(\rho_0(x)\chi(t) - |m(x,t)|^2 \right). \tag{6.2}$$

Since $z := (m, U)$ and $K^{co}_{\rho_0,\chi}$ are uniformly continuous in (x,t), there exists an $\varepsilon > 0$ such that for any $(x,t), (x_0,t_0) \in \Omega$ with $|x - x_0| + |t - t_0| < \varepsilon$, we have

$$(z(x,t), q_0(x)) \pm (\overline{m}(x_0,t_0), \overline{U}(x_0,t_0), 0) \subset \mathrm{hint} K^{co}_{\rho_0,\chi}. \tag{6.3}$$

Step 2. Fix $(x_0, t_0) \in \Omega$ for the moment. Now, let $0 \leq \phi_{r_0} \leq 1$ be a smooth cutoff function on Ω with support contained in a ball $B_{r_0}(x_0,t_0) \subset \Omega$ for some $r_0 > 0$, identically 1 on $B_{r_0/2}(x_0,t_0)$ and strictly less than 1 outside. Thanks to Proposition 5.1 and the identification $(m, U, q) \to M$, for the admissible line segment $\sigma_{(x_0,t_0)}$, there exist an operator A_0 and a direction $\eta_0 \in \mathbb{R}^{n+1}$ not parallel to e_{n+1}, such that for any $k \in \mathbb{N}$

$$A_0 \left(\frac{\cos(k\eta_0 \cdot (x,t))}{k^3} \right) = \lambda_{(x_0,t_0)} \sin(k\eta_0 \cdot (x,t)),$$

and such that the pair $(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})$ defined by

$$(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})(x,t) := A_0 \left[\phi_{r_0}(x,t) k^{-3} \cos(k\eta_0 \cdot (x,t)) \right]$$

satisfies (3.1) with $q \equiv 0$. Note that $(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})$ is supported in the ball $B_{r_0}(x_0,t_0)$ and that

$$\begin{aligned} &\left\| (\widetilde{m}_{k,0}, \widetilde{U}_{k,0}) - \phi_{r_0} \left(\overline{m}(x_0,t_0), \overline{U}(x_0,t_0) \right) \sin(k\eta_0 \cdot (x,t)) \right\|_\infty \\ &\leq const \left(A_0, \eta_0, \|\phi_0\|_{C^3} \right) \frac{1}{k} \end{aligned} \tag{6.4}$$

since A_0 is a linear differential operator of homogeneous degree 3. Furthermore, for all $(x,t) \in B_{r_0/2}(x_0,t_0)$, we have

$$|\widetilde{m}_{k,0}(x,t)|^2 = |\overline{m}(x_0,t_0)|^2 \sin^2(k\eta_0 \cdot (x,t)).$$

Since $\eta_0 \in \mathbb{R}^{n+1}$ is not parallel to e_{n+1}, from Lemma 5.2 we can see that

$$\lim_{k\to\infty} \int_{B_{r_0/2}(x_0,t_0)} |\widetilde{m}_{k,0}(x,t)|^2 \, dx = \frac{1}{2} \int_{B_{r_0/2}(x_0,t_0)} |\overline{m}(x_0,t_0)|^2 \, dx$$

uniformly in t. In particular, using (6.2), we obtain

$$\lim_{k\to\infty}\int_{B_{r_0/2}(x_0,t_0)} |\widetilde{m}_{k,0}(x,t)|^2\,dxdt \geq \frac{F^2}{2\rho_0(x_0)\chi(t_0)}\left(\rho_0(x_0)\chi(t_0)) - |m(x_0,t_0)|^2\right)^2 \left|B_{r_0/2}(x_0,t_0)\right|. \quad (6.5)$$

Step 3. Next, observe that since m is uniformly continuous, there exists an $\bar{r} > 0$ such that for any $r < \bar{r}$ there exists a finite family of pairwise disjoint balls $B_{r_j}(x_j,t_j) \subset \Omega$ with $r_j < \bar{r}$ such that

$$\int_\Omega \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right)^2 dxdt \leq 2\sum_j \left(\rho_0(x_j)\chi(t_j) - |m(x_j,t_j)|^2\right)^2 \left|B_{r_j}(x_j,t_j)\right|. \quad (6.6)$$

Fix $s > 0$ with $s < \min\{\bar{r},\varepsilon\}$ and choose a finite family of pairwise disjoint balls $B_{r_j}(x_j,t_j) \subset \Omega$ with radii $r_j < s$ such that (6.6) holds. In each ball $B_{2r_j}(x_j,t_j)$ we apply the construction of *Step 2* to obtain, for every $k \in \mathbb{N}$, a pair $(\widetilde{m}_{k,j}, \widetilde{U}_{k,j})$.

Final step. Letting (m_k, U_k) be as in (6.1), we observe that the sum therein consists of finitely many terms. Therefore from (6.3) and (6.4) we deduce that there exists $k_0 \in \mathbb{N}$ such that

$$m_k \in X_0 \text{ for all } k \geq k_0. \quad (6.7)$$

Moreover, owing to (6.5) and (6.6) we can write

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |m_k(x,t)-m(x,t)|^2\,dxdt &= \sum_j \lim_{k\to\infty}\int_\Omega |\widetilde{m}_{k,j}(x,t)|^2\,dxdt\\ &\geq \sum_j \frac{F^2}{2\rho_0(x_j)\chi(t_j)}\left(\rho_0(x_j)\chi(t_j)) - |m(x_j,t_j)|^2\right)^2 \left|B_{r_j}(x_j,t_j)\right|\\ &\geq C\int_\Omega \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right)^2 dxdt. \quad (6.8)\end{aligned}$$

Since $m_k \overset{d}{\to} m$, due to (6.8) we have

$$\begin{aligned}\liminf_{k\to\infty}\|m_k\|^2_{L^2(\Omega)} &= \|m\|_2^2 + \liminf_{k\to\infty}\|m_k - m\|_2^2\\ &\geq \|m\|_2^2 + C\int_\Omega \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right)^2 dxdt, \quad (6.9)\end{aligned}$$

which gives (4.15) with $\beta = \beta(n) = \beta(F(n))$. □

7. Construction of suitable initial data

In this section we show the existence of a *subsolution* in the sense of Definition 4.2. Since the subsolution we aim to construct has to be space-periodic, it will be enough to work on the building brick Q and then extend the costruction periodically to $\mathbb{R}^n$.

The idea to work in the space-periodic setting has been recently adopted by Wiedemann [22] in order to construct global solutions to the incompressible Euler equations.

Proposition 7.1. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function as in Proposition 4.1 and let T be any given positive time. Then, there exist a smooth function $\widetilde{\chi}:\mathbb{R}\to\mathbb{R}^+$, a continuous periodic matrix field $\widetilde{U}:\mathbb{R}^n\to\mathcal{S}^n_0$ and a function $\widetilde{q}\in C^1(\mathbb{R};C^1_p(\mathbb{R}^n))$ such that*

$$\operatorname{div}_x \widetilde{U} + \nabla_x \widetilde{q} = 0 \qquad \text{on} \quad \mathbb{R}^n\times\mathbb{R} \tag{7.1}$$

and

$$e(\rho_0(x),0,\widetilde{U}(x)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t)\in\mathbb{R}^n\times[0,T[\tag{7.2}$$

$$\widetilde{q}(x,t) = p(\rho_0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } x\in\mathbb{R}^n\times\mathbb{R}. \tag{7.3}$$

Proof. [Proposition 7.1] Let us define $\widetilde{U}$ componentwise by its Fourier transform as follows:

$$\begin{aligned}
\widehat{\widetilde{U}}_{ij}(k) &:= \left(\frac{nk_ik_j}{(n-1)\,|k|^2}\right)\widehat{p(\rho_0(k))} \text{ if } i\neq j,\\
\widehat{\widetilde{U}}_{ii}(k) &:= \left(\frac{nk_i^2-|k|^2}{(n-1)\,|k|^2}\right)\widehat{p(\rho_0(k))}.
\end{aligned} \tag{7.4}$$

for every $k\neq 0$, and $\widehat{\widetilde{U}}(0)=0$. Clearly $\widehat{\widetilde{U}}_{ij}$ thus defined is symmetric and trace-free. Moreover, since $p(\rho_0)\in C^1_p(\mathbb{R}^n)$, standard elliptic regularity arguments allow us to conclude that $\widetilde{U}$ is a continuous periodic matrix field. Next, notice that

$$\left\|e(\rho_0(x),0,\widetilde{U}(x))\right\|_\infty = \left\|\lambda_{max}(-\widetilde{U})\right\|_\infty = \widetilde{\lambda} \tag{7.5}$$

for some positive constant $\widetilde{\lambda}$. Therefore, we can choose any smooth function $\widetilde{\chi}$ on $\mathbb{R}$ such that $\widetilde{\chi} > n\widetilde{\lambda}$ on $[0,T]$ in order to ensure (7.2). Now, let $\widetilde{q}$ be defined exactly as in (7.3) for the choice of $\widetilde{\chi}$ just done. It remains to show that (7.1) holds. In light of (7.3), we can write equation (7.1) in Fourier space as

$$\sum_{j=1}^n k_j\widehat{\widetilde{U}}_{ij} = k_i\widehat{p(\rho_0)} \tag{7.6}$$

for $k\in\mathbb{Z}^n$. It is easy to check that $\widehat{\widetilde{U}}$ as defined by (7.4) solves (7.6) and hence $\widetilde{U}$ and $\widetilde{q}$ satisfy (7.1)

□

Remark 7.2. *We note that the Hölder continuity of ρ_0 would be enough to argue as in the previous proof in order to infer the continuity of $\widetilde{U}$.*

Proposition 7.3. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function as in Proposition 4.1 and let T be any given positive time. There exist triples $(\overline{m}, \overline{U}, \overline{q})$ solving (3.1) distributionally on $\mathbb{R}^n \times \mathbb{R}$ enjoying the following properties:*

$$(\overline{m}, \overline{U}, \overline{q}) \text{ is continuous in } \mathbb{R}^n \times (\mathbb{R}\setminus\{0\}) \text{ and } \overline{m} \in C(\mathbb{R}; H_w(\mathbb{R}^n)), \tag{7.7}$$

$$\overline{U}(\cdot, t) = \widetilde{U}(\cdot) \text{ for } t = -T, T \tag{7.8}$$

and

$$\overline{q}(x) = p(\rho_0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{7.9}$$

$$e(\rho_0(x), \overline{m}(x,t), \overline{U}(x,t)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times ([-T, 0[\cap]0, T]). \tag{7.10}$$

Moreover

$$|\overline{m}(x,0)|^2 = \rho_0(x)\chi(0) \text{ a.e. in } \mathbb{R}^n. \tag{7.11}$$

Proof. [Proposition 7.3] We first choose $\overline{q} := \tilde{q}$ given by Proposition 7.1. This choice already yields (7.9).

Now, in analogy with Definition 4.2 we consider the space X_0 defined as the set of continuous vector fields $m : \mathbb{R}^n \times]-T, T[\to \mathbb{R}^n$ in $C^0(]-T, T[; C^0_p(Q))$ to which there exists a continuous space-periodic matrix field $U : \mathbb{R}^n \times]-T, T[\to \mathcal{S}^n_0$ such that

$$\begin{aligned} \operatorname{div}_x m &= 0, \\ \partial_t m + \operatorname{div}_x U + \nabla_x \overline{q} &= 0, \end{aligned} \tag{7.12}$$

$$\operatorname{supp}(m) \subset Q \times [-T/2, T/2[\tag{7.13}$$

$$U(\cdot, t) = \widetilde{U}(\cdot) \text{ for all } t \in [-T, T[\setminus[-T/2, T/2] \tag{7.14}$$

and

$$e(\rho_0(x), m(x,t), U(x,t)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times]-T, T[. \tag{7.15}$$

As in Section 4.1, X_0 consists of functions $m :]-T, T[\to H$ taking values in a bounded set $B \subset H$. On B the weak topology of L^2 is metrizable, and correspondingly we find a metric d on $C(]-T, T[; B)$ inducing the topology of $C(]-T, T[; H_w(Q))$.

Next we note that with minor modifications the proof of Lemma 4.5 leads to the following claim:

Claim: Let $Q_0 \subset Q$ be given. Let $m \in X_0$ with associated matrix field U and let $\alpha > 0$ such that

$$\int_{Q_0} \Big[|m(x,0)|^2 - (\rho_0(x)\widetilde{\chi}(0)) \Big] \, dx < -\alpha$$

Then, for any $\delta > 0$ there exists a sequence $m_k \in X_0$ with associated smooth matrix field U_k such that

$$\operatorname{supp}(m_k - m, U_k - U) \subset Q_0 \times [-\delta, \delta],$$

$$m_k \overset{d}{\to} m,$$

and

$$\liminf_{k\to\infty} \int_{Q_0} |m_k(x,0)|^2 \geq \int_{Q_0} |m(x,0)|^2 \, dx + \beta\alpha^2.$$

Fix an exhausting sequence of bounded open subsets $Q_k \subset Q_{k+1} \subset Q$, each compactly contained in Ω, and such that $|Q_{k+1} \backslash Q_k| \leq 2^{-k}$. Let also γ_ε be a standard mollifying kernel in $\mathbb{R}^n$ (the unusual notation γ_ε for the standard mollifying kernel is aimed at avoiding confusion between it and the density function). Using the claim above we construct inductively a sequence of momentum vector fields $m_k \in X_0$, associated matrix fields U_k and a sequence of numbers $\eta_k < 2^{-k}$ as follows.

First of all let $m_1 \equiv 0$, $U_1(x,t) = \widetilde{U}(x)$ for all $(x,t) \in \mathbb{R}^{n+1}$ and having obtained (m_1, U_1), ..., (m_k, U_k), $\eta_1, ..., \eta_{k-1}$ we choose $\eta_k < 2^{-k}$ in such a way that

$$\|m_k - m_k * \gamma_{\eta_k}\|_{L^2} < 2^{-k}. \tag{7.16}$$

Then, we set

$$\alpha_k = -\int_{Q_k} [|m_k(x,0)|^2 - \rho_0(x)\widetilde{\chi}(0))] dx.$$

Note that (7.15) ensures $\alpha_k > 0$. Then, we apply the claim with Q_k, $\alpha = \alpha_k$ and $\delta = 2^{-k}T$ to obtain $m_{k+1} \in X_0$ and associated smooth matrix field U_{k+1} such that

$$\operatorname{supp}(m_{k+1} - m_k, U_{k+1} - U_k) \subset Q_k \times [-2^{-k}T, 2^{-k}T], \tag{7.17}$$

$$d(m_{k+1}, m_k) < 2^{-k}, \tag{7.18}$$

$$\int_{Q_k} |m_{k+1}(x,0)|^2 \, dx \geq \int_{Q_k} |m_k(x,0)|^2 \, dx + \beta\alpha_k^2. \tag{7.19}$$

Since d induces the topology of $C(]-T,T[; H_w(\Omega))$ we can also require that

$$\left\|(m_k - m_{k+1}) * \gamma_{\eta_j}\right\|_{L^2(\Omega)} < 2^{-k} \text{ for all } j \leq k \text{ for } t = 0. \tag{7.20}$$

From (7.18) we infer the existence of a function $\overline{m} \in C(]-T,T[,H_w(\Omega))$ such that

$$m_k \xrightarrow{d} \overline{m}.$$

Besides, (7.17) implies that for any compact subset S of $Q\times]-T,0[\cup]0,T[$ there exists k_0 such that $(m_k,U_k)|_S = (m_{k_0},U_{k_0})|_S$ for all $k > k_0$. Hence (m_k,U_k) converges in $C^0_{\text{loc}}(Q\times]-T,0[\cup]0,T[)$ to a continuous pair $(\overline{m},\overline{U})$ solving equations (7.12) in $\mathbb{R}^n\times]-T,0[\cup]0,T[$ and such that (7.7)-(7.10) hold. In order to conclude, we show that also (7.11) holds for $\overline{m}$.

As first, we observe that (7.19) yields

$$\alpha_{k+1} \le \alpha_k - \beta\alpha_k^2 + |Q_{k+1}\backslash Q_k| \le \alpha_k - \beta\alpha_k^2 + 2^{-k},$$

from which we deduce that

$$\alpha_k \to 0 \text{ as } k \to \infty.$$

This, together with the following inequality

$$0 \ge \int_Q \Big[|m_k(x,0)|^2 - \rho_0(x)\chi(0)\Big]\,dx \ge -(\alpha_k + C\,|Q\backslash Q_k|) \ge -(\alpha_k + C2^{-k}),$$

implies that

$$\lim_{k\uparrow\infty}\int_\Omega \Big[|m_k(x,0)|^2 - \rho_0(x)\chi(0)\Big]\,dx = 0. \tag{7.21}$$

On the other hand, owing to (7.16) and (7.20), we can write for $t = 0$ and for every k

$$\begin{aligned}
&\|m_k - \overline{m}\|_{L^2}\\
&\le \|m_k - m_k * \gamma_{\eta_k}\|_{L^2} + \|m_k * \gamma_{\eta_k} - \overline{m} * \gamma_{\eta_k}\|_{L^2} + \|\overline{m} * \gamma_{\eta_k} - \overline{m}\|_{L^2}\\
&\le 2^{-k} + \sum_{j=0}^{\infty}\|m_{k+j} * \gamma_{\eta_k} - m_{k+j+1} * \gamma_{\eta_k}\|_{L^2} + 2^{-k}\\
&\le 2^{-(k-2)}.
\end{aligned} \tag{7.22}$$

Finally, (7.22) implies that $m_k(\cdot,0) \to \overline{m}(\cdot,0)$ strongly in $H(Q)$ as $k \to \infty$, which together with (7.21) gives

$$|\overline{m}(x,0)|^2 = \rho_0(x)\chi(0) \text{ for almost every } x \in \mathbb{R}^n.$$

□

8. Proof of the main Theorems

Proof. [Proof of Theorem 2.1] Let T be any finite positive time and $\rho_0 \in C^1_p(Q)$ be a given density function. Let also $(\overline{m},\overline{U},\overline{q})$ be as in Proposition 7.3. Then, define $\chi(t) := \widetilde{\chi}(t)$, $q_0(x) := \overline{q}(x)$,

$$m_0(x,t) = \begin{cases} \overline{m}(x,t) \text{ for } t \in [0,T] \\ \overline{m}(x,t-2T) \text{ for } t \in [T,2T], \end{cases} \tag{8.23}$$

$$U_0(x,t) = \begin{cases} \overline{U}(x,t) \text{ for } t \in [0,T] \\ \overline{U}(x,t-2T) \text{ for } t \in [T,2T]. \end{cases} \tag{8.24}$$

For this choices, the quadruple (m_0, U_0, q_0, χ) satisfies the assumptions of Proposition 4.1. Therefore, there exist infinitely many solutions $m \in C([0,2T], H_w(Q))$ of (2.6) in $\mathbb{R}^n \times [0,2T[$ with density ρ_0, such that

$$m(x,0) = \overline{m}(x,0) = m(x,2T) \text{ for a.e. } x \in \Omega$$

and

$$|m(\cdot,t)|^2 = \rho_0(\cdot)\chi(0) \text{ for almost every } (x,t) \in \mathbb{R}^n \times]0,2T[. \tag{8.25}$$

Since $|m_0(\cdot,0)|^2 = \rho_0(\cdot)\chi(0)$ a.e. in $\mathbb{R}^n$ as well, it is enough to define $m^0(x) = m_0(x,0)$ to satisfy also (2.10) and hence conclude the proof. □

Proof. [Proof of Theorem 2.2] Under the assumptions of Theorem 2.1, we have proven the existence of a bounded initial momentum m^0 allowing for infinitely many solutions $m \in C([0,T]; H_w(Q))$ of (2.6) on $\mathbb{R}^n \times [0,T[$ with density ρ_0. Moreover, the proof (see Proof of Proposition 7.1) showed that for any smooth function $\chi : \mathbb{R} \to \mathbb{R}^+$ with $\chi > n\widetilde{\lambda} > 0$ the following holds

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad \text{a.e. in } \mathbb{R}^n \times [0,T[, \tag{8.26}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \mathbb{R}^n. \tag{8.27}$$

Now, we claim that there exist constants $C_1, C_2 > 0$ such that choosing the function $\chi(t) > n\widetilde{\lambda}$ on $[0,T[$ among solutions of the following differential inequality

$$\chi'(t) \leq -C_1\chi^{1/2}(t) - C_2\chi^{3/2}(t), \tag{8.28}$$

then the weak solutions (ρ_0, m) of (2.6) obtained in Theorem 2.1 will also satisfy the admissibility condition (2.5) on $\mathbb{R}^n \times [0,T[$. Of course, there is an issue of compatibility between the differential inequality (8.28) and the condition $\chi > n\widetilde{\lambda}$: this motivates the existence of a time $\overline{T} > 0$ defining the maximal time-interval in which the admissibility condition indeed holds.

Let T be any finite positive time. As first, we aim to prove the claim. Since $m \in C([0,T]; H_w(Q))$ is divergence-free and fulfills (8.33)-(8.34) and ρ_0 is time-independent, (2.5) reduces to the following inequality

$$\frac{1}{2}\chi'(t) + m\cdot\nabla\left(\varepsilon(\rho_0(x)) + \frac{p(\rho_0(x))}{\rho_0(x)}\right) + \frac{\chi(t)}{2} m\cdot\nabla\left(\frac{1}{\rho_0(x)}\right) \leq 0, \tag{8.29}$$

intended in the sense of (space-periodic) distributions on $\mathbb{R}^n \times [0,T]$. As $\rho_0 \in C^1_p(Q)$, there exists a constant c_0^2 with $\rho_0 \leq c_0^2$ on $\mathbb{R}^n$, whence (see (8.33)-(8.34))

$$|m(x,t)| \leq c_0\sqrt{\chi(t)} \text{ a.e. on } \mathbb{R}^n \times [0,T[. \tag{8.30}$$

Similarly we can find constants $c_1, c_2 > 0$ with

$$\left|\nabla\left(\varepsilon(\rho_0(x)) + \frac{p(\rho_0(x))}{\rho_0(x)}\right)\right| \le c_1 \text{ a.e. in } \mathbb{R}^n \tag{8.31}$$

$$\left|\nabla\left(\frac{1}{\rho_0(x)}\right)\right| \le c_2 \text{ a.e. in } \mathbb{R}^n. \tag{8.32}$$

As a conseguence of (8.30)-(8.32), (8.29) holds as soon as χ satisfies

$$\chi'(t) \le -2c_1c_0\chi^{1/2}(t) - c_2c_0\chi^{3/2}(t) \text{ on } [0, T[.$$

Therefore, by choosing $C_1 := 2c_1c_0$ and $C_2 := c_2c_0$ we can conclude the proof of the claim.

Now, it remains to show the existence of a function χ as in the claim, i.e. that both the differential inequality (8.28) and the condition $\chi > n\widetilde{\lambda}$ can hold true on some suitable time-interval. To this aim, we can consider the equality in (8.28), couple it with the initial condition $\chi(0) = \chi_0$ for some constant $\chi_0 > n\widetilde{\lambda}$ and then solve the resulting Cauchy problem. For the obtained solution χ, there exists a positive time $\overline{T}$ such that $\chi(t) > n\widetilde{\lambda}$ on $[0, \overline{T}[$.

Finally, applying the claim on the time-interval $[0, \overline{T}[$ we conclude that the admissibility condition holds on $\mathbb{R}^n \times [0, \overline{T}[$ as desired. □

Proof. [Proof of Theorem 1.1] The proof of Theorem 1.1 strongly relies on Theorems 2.1-2.2. Given a continuously differentiable initial density ρ^0 we apply Theorems 2.1-2.2 for $\rho_0(x) := \rho^0(x)$ thus obtaining a positive time $\overline{T}$ and a bounded initial momentum m^0 allowing for infinitely many solutions $m \in C([0, T]; H_w(Q))$ of (2.6) on $\mathbb{R}^n \times [0, \overline{T}[$ with density ρ^0 and such that the following holds

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad \text{a.e. in } \mathbb{R}^n \times [0, \overline{T}[, \tag{8.33}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \mathbb{R}^n, \tag{8.34}$$

for a suitable smooth function $\chi : [0, \overline{T}] \to \mathbb{R}^+$. Now, define $\rho(x, t) = \rho_0(x)\mathbf{1}_{[0,\overline{T}[}(t)$. This shows that (2.4) holds. To prove (2.3) observe that ρ is independent of t and m is weakly divergence-free for almost every $0 < t < \overline{T}$. Therefore, the pair (ρ, m) is a weak solution of (1.1) with initial data (ρ^0, m^0). Finally, we can also prove (2.5): each solution obtained is also admissible. Indeed, for $\rho(x, t) = \rho_0(x)\mathbf{1}_{[0,\overline{T}[}(t)$, (2.5) is ensured by Theorem 2.2. □

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