Some results about the existence of critical points for the Willmore functional

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ABSTRACT. Using a perturbative approach, it is shown existence and multiplicity of critical points for the Willmore functional in $(\mathbb{R}^3, g_\epsilon)$ -where g_ϵ is a metric close to the euclidean one. With the same technique it is proved a non existence result in general Riemannian manifold (M, g) of dimension three.

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1 Introduction

The goal of this paper is to study the Willmore functional.

One of the most important classes of submanifolds is the set of immersions whose mean curvature is null: the minimal submanifolds. We will deal with a generalization of the minimal submanifolds, known in literature as "Willmore surfaces". The topic is classical and goes back to Blaschke (1929) and Thomsen (1923): recall that given $\mathring{M} \hookrightarrow \mathbb{R}^3$ an immersed compact oriented surface, the Willmore functional is defined as

$$I(\mathring{M}) := \int_{\mathring{M}} H^2 d\Sigma,$$

where H and $d\Sigma$ are the mean curvature and the area form of $\mathring{M}$.

From now on we adopt the convention that H is the sum of the principal curvatures.
As Blaschke and Thomsen proved, this functional is invariant under conformal transformations of $\mathbb{R}^3$.
The second fundamental property (due to Willmore) asserts that the standard spheres S^ρ_p are the points of strict global minimum (hence they form a critical manifold - i.e. a manifold made of critical points) for I:

$$I(\mathring{M}) := \int_{\mathring{M}} H^2 d\Sigma \geq 16\pi; \quad I(\mathring{M}) = 16\pi \Leftrightarrow \mathring{M} = S^\rho_p. \tag{1}$$

The proofs of the last facts can be found in [Will] (pag. 271 and pag. 276-279).

A surface which makes the Willmore functional stationary with respect to normal variations is called a *Willmore surface*.

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The functional relative to immersions in $\mathbb{R}^3$ and S^3 has been deeply studied with remarkable results (look at [Will] Chap. 7) and recently a lot of generalizations for immersions of larger dimension and codimension have been considered (see the papers [SiL], [BK], [ZG] and [Riv]). Finally, in the last years, the flow generated by the L^2-differential of the functional has been analyzed ([Sim], [KS1], [KS2]).

The Willmore functional has lots of applications in biology, general relativity, string theory and elasticity theory: in the study of lipid bilayer membranes it is called "Hellfrich energy", in general relativity it is linked with the "Hawking mass", in string theory it appears in the "Polyakov extrinsic action" and in nonlinear elasticity theory it arises as Γ-limit of some energy functionals (see [FJM]). We also mention the classical fact that in the mean curvature flow analysis one has

$$\frac{d}{dt}Vol(\mathring{M}) = -\int_{\mathring{M}} H^2 d\Sigma$$

where $\mathring{M}$ is the evolving submanifold with respect to the parameter t and $Vol(\mathring{M}) := \int_{\mathring{M}} d\Sigma$ is its the area.

While, as we remarked, there is an extensive study for immersions into $\mathbb{R}^n$ or S^n, very little is known for general ambient manifolds (apart from the case of minimal surfaces). The aim of this paper is to give some existence (resp. non existence) results for curved metrics in $\mathbb{R}^3$, close and asymptotic to the flat one (resp. in general Riemannian manifolds).

We consider the following direct generalization of the Willmore functional: let (M, g) be a Riemannian manifold of dimension three and $(\mathring{M}, \mathring{g}) \hookrightarrow (M, g)$ a compact oriented isometrically immersed surface. We will study the functional

$$I(\mathring{M}) := \int_{\mathring{M}} H^2 d\Sigma$$

and we will call it *Willmore functional*.

Denote with $\mathring{N}$ the inward normal unit vector to $\mathring{M}$ and fix a function $f \in C^\infty(\mathring{M})$. Consider the normal perturbation

$$\mathring{M}_f[t] := \{Exp_p(tf\mathring{N}) | p \in \mathring{M}\},$$

where Exp_p is the exponential map in (M, g) of center p.
We say that $\mathring{M}$ is a *critical point* of I (or a *Willmore surface*) if

$$\frac{d}{dt} I(\mathring{M}_f[t])|_{t=0} = 0 \quad \forall f \in C^\infty(\mathring{M}).$$

The Willmore functional is L^2-differentiable and it turns out that (from formula (29) pag. 9 in [PV], with an easy computation-notice the difference in the convetions: opposite sign convention about the normal vector $\mathring{N}$ and different convention on the mean curvature H)

$$I'(\mathring{M}) = 2\triangle_{\mathring{M}} H + H\big(H^2 - 4D + 2R_{\mu\nu}\mathring{N}^\mu \mathring{N}^\nu\big),$$

where $\triangle_{\mathring{M}}$ is the Laplacian on $\mathring{M}$, $R_{\mu\nu}$ is the Ricci curvature of M and D is the product of the principal curvatures of the surface (for more details look at the section "Notations and conventions").
As a result, $\mathring{M}$ is a critical point of I if and only if the following fourth order nonlinear PDE is satisfied

$$2\triangle_{\mathring{M}} H + H(H^2 - 4D + 2R_{\mu\nu}\mathring{N}^\mu \mathring{N}^\nu) = 0.$$

We will prove some existence and multiplicity results for the Willmore functional in the ambient manifold

$$(\mathbb{R}^3, g_\epsilon) \quad \text{with } g_\epsilon = \delta + \epsilon h, \quad \lim_{|p|\to\infty} |h_{\mu\nu}(p)| = 0 \tag{2}$$

where δ is the euclidean scalar product and h is a symmetric smooth bilinear form. Moreover we will show a non-existence result in a general ambient manifold (M, g) of dimension three.

Let us discuss now some more details.
The Willmore functional with ambient manifold $(\mathbb{R}^3, g_\epsilon)$ will be called I_ϵ, and we will look for critical points of I_ϵ which are perturbed standard spheres of $\mathbb{R}^3$. Denote with S^2 the standard unit sphere and let S^ρ_p be the standard sphere of $\mathbb{R}^3$ with center p and radius ρ

$$S^\rho_p := \{x \in \mathbb{R}^3 : \|x - p\|_{\text{euclidean}} = \rho\}$$

parametrized by $\Theta \in S^2 \mapsto p + \rho\Theta$.
Let us define the perturbation (for more details see Section "Notations and Conventions", here we are quite sketchy). Given a small $w \in C^{4,\alpha}(S^2)$, the perturbed standard sphere $S^\rho_p(w)$ is defined as the image of

$$\Theta \mapsto p + \rho(1 - w(\Theta))\Theta.$$

The main results of this paper are Theorem 1.1 and Theorem 1.2 below, which will be proved in Subsection 5.1.
Before stating them observe (see Remark 3.7) that the scalar curvature of $(\mathbb{R}^3, g_\epsilon)$ can be written as

$$R_{g_\epsilon} = \epsilon R_1 + o(\epsilon), \quad \text{where } R_1 := \sum_{\mu\nu} D_{\mu\nu} h_{\mu\nu} - \triangle \mathrm{Tr} h. \tag{3}$$

Theorem 1.1. *Let g_ϵ be as in (2) and let R_1 be defined in (3). Suppose that*
(i) There exists $\bar{p} \in \mathbb{R}^3$ such that $R_1(\bar{p}) \neq 0$;
(ii) There exist $C > 0$ and $\alpha > 2$ such that

$$|D_\lambda h_{\mu\nu}(p)| < \frac{C}{|p|^\alpha} \quad \forall \lambda, \mu, \nu = 1 \dots 3.$$

Then, for ϵ small enough, there exist $(p_\epsilon, \rho_\epsilon) \in \mathbb{R}^3 \oplus \mathbb{R}^+$ and $w_\epsilon \in C^{4,\alpha}(S^2)$ with $\|w_\epsilon\|_{C^{4,\alpha}} \to 0$ as $\epsilon \to 0$, such that the perturbed sphere $S^{\rho_\epsilon}_{p_\epsilon}(w_\epsilon)$ is a critical point of the Willmore functional I_ϵ.

With more (but mild) assumptions, we are able to show the existence of two critical points:

Theorem 1.2. *Under the assumptions of Theorem 1.1, suppose that there exist two points $p_1, p_2 \in \mathbb{R}^3$ such that $R_1(p_1) > 0$ and $R_1(p_2) < 0$.*

Then, for ϵ small enough, there exist $(p^1_\epsilon, \rho^1_\epsilon)$, $(p^2_\epsilon, \rho^2_\epsilon) \in \mathbb{R}^3 \oplus \mathbb{R}^+$ and w^1_ϵ, $w^2_\epsilon \in C^{4,\alpha}(S^2)$ with $\|w^i_\epsilon\|_{C^{4,\alpha}} \to 0$ as $\epsilon \to 0$, such that the perturbed spheres $S^{\rho^1_\epsilon}_{p^1_\epsilon}(w^1_\epsilon)$, $S^{\rho^2_\epsilon}_{p^2_\epsilon}(w^2_\epsilon)$ are distinct critical points of the Willmore functional I_ϵ.

Remark 1.1. *The condition (i) of Theorem 1.1 is not redundant. In fact it is easy to give examples of non null bilinear forms $h_{\mu\nu}$ vanishing a infinity and which satisfy (ii), such that $R_1 \equiv 0$.*
Chosen h such that $h_{\mu\nu} = 0$ for $\mu \neq \nu$ and $h_{33} = 0$, the linear PDE $R_1 \equiv 0$ takes the form:

$$D_{22}h_{11} + D_{33}h_{11} = -D_{11}h_{22} - D_{33}h_{22}.$$

Suppose that h_{22} is a non identically zero fast decreasing function. It is immediate to see that chosen $h_{11}(x, y, z) = -h_{22}(y, x, z)$, the non null metric h vanishes at infinity and satisfies both (ii) and the PDE.

In Subsection 5.2, we will observe that the Willmore functional is invariant under isometries of the ambient manifold (Theorem 5.1). Hence, assuming that the perturbation h of the euclidean metric possesses some symmetries and that the critical points found with the previous Theorems are not invariant under those isometries, we will show the existence of infinitely many stationary points (see Theorem 5.2 and the subsequent examples of Section 5).

The non-existence result concerns perturbed geodesic spheres of small radius defined as follows: fixed $p \in M$ and denoted with Exp_p the exponential map with center p, for small ρ the geodesic sphere $S_{p,\rho}$ is well defined and can be parametrized by

$$\Theta \in S^2 \subset T_pM \mapsto Exp_p[\rho\Theta].$$

Analogously to the previous case, fixed $p \in M$, $\rho > 0$ and a small $C^{4,\alpha}(S^2)$ function w, the perturbed geodesic sphere $S_{p,\rho}(w)$ is the surface parametrized by

$$\Theta \in S^2 \mapsto Exp_p[\rho\big(1 - w(\Theta)\big)\Theta].$$

Now we can state the non existence result:

Theorem 1.3. *Let (M, g) be the ambient Riemannian manifold. Assume that the scalar curvature of M at the point $\bar{p}$ is not null:*

$$R(\bar{p}) \neq 0.$$

Then there exist $\rho_0 > 0$ and $r > 0$ such that for radius $\rho < \rho_0$ and perturbation $w \in C^{4,\alpha}(S^2)$ with $\|w\|_{C^{4,\alpha}(S^2)} < r$, the surfaces $S_{\bar{p},\rho}(w)$ are not *critical points of the Willmore functional I.*

The interest of the previous Theorem resides in the difference with the flat case $M = \mathbb{R}^3$, where all the spheres are point of global minimum (see (1)).

The methods used to prove the above results rely on the same technique: the Lyapunov-Schmidt reduction (for more details see Subsection 2.1). Since the ideas are similar, here we only discuss the existence part. We are quite informal since we just want to motivate (for the details see Section 4).
As we remarked, (1) implies that the Willmore functional in the euclidean space $\mathbb{R}^3$ possesses a critical manifold Z made of standard spheres S^ρ_p. The tangent space to Z at S^ρ_p is composed of constant and affine functions on S^ρ_p so, with a pull back via the parametrization, on S^2.
As we will point out in Remark 3.3, the second derivative of I_0 at S^ρ_p is

$$I_0''(S^\rho_p)[w] = \frac{2}{\rho^3}\triangle_{S^2}(\triangle_{S^2} + 2)[w],$$

which is a Fredholm operator of index zero and whose Kernel is made of the constant and affine functions; exactly the tangent space to Z.
So, considered $C^{4,\alpha}(S^2)$ as a subspace of $L^2(S^2)$ and called

$$C^{4,\alpha}(S^2)^\perp := C^{4,\alpha}(S^2) \cap Ker[\triangle_{S^2}(\triangle_{S^2} + 2)]^\perp,$$

it follows that $I_0''|_{C^{4,\alpha}(S^2)^\perp}$ is invertible on its image and one can apply the Lyapunov-Schmidt reduction. Thanks to this reduction, the critical points of I_ϵ in a neighbourhood of Z are exactly the stationary points of a function $\Phi_\epsilon : Z \to \mathbb{R}$ of *finitely many* variables (we remark that in a neighbourhood of Z the condition is necessary and sufficient for the existence of critical points of I_ϵ).

In order to study the function Φ_ϵ, in Section 3 we will compute explicit formulas of the Willmore functional.
More precisely, in the perturbative setting $(\mathbb{R}^3, g_\epsilon)$, we will calculate an expansion of I_ϵ on the standard spheres S^ρ_p (Lemma 3.5):

$$I_\epsilon(S^\rho_p) = 16\pi + 2\epsilon \int_{S^2} \Big[\mathrm{Tr}h - 3h_{\mu\nu}\Theta^\mu\Theta^\nu + \rho A_{\mu\mu\lambda}\Theta^\lambda - \rho A_{\mu\nu\lambda}\Theta^\mu\Theta^\nu\Theta^\lambda\Big] d\Sigma_0 + o(\epsilon).$$

Here

$$A_{\mu\nu\lambda} := [D_\mu h_{\lambda\nu} + D_\lambda h_{\nu\mu} - D_\nu h_{\mu\lambda}],$$

Θ^μ are the Cartesian coordinates of $\Theta \in S^2$ and the remainder $o(\epsilon)$ is uniform on compact subsets of $\mathbb{R}^3 \oplus \mathbb{R}^+ \ni (p, \rho)$. Of course, in the integral, h is evaluated at the points of S^ρ_p.

With a Taylor expansion, we will get a formula for spheres of small radius (Lemma 3.6):

$$I_\epsilon(S^\rho_p) = 16\pi - \frac{8\pi}{3} R_1(p)\rho^2\epsilon + o(\rho^2)\epsilon + o(\epsilon),$$

where the remainder $o(\epsilon)$ is uniform on compact subsets of $\mathbb{R}^3 \oplus \mathbb{R}^+ \ni (p, \rho)$.

In Lemma 4.3, we will prove that $\Phi_\epsilon(S^\rho_p) = I_\epsilon(S^\rho_p) + o(\epsilon)$; then, using the above expansions and the assumptions on h, we will show that $\Phi_\epsilon \to 16\pi + o(\epsilon)$ if $\rho \to 0$ or if $(p, \rho) \to \infty$. From the conditions on

R_1, it follows that Φ_ϵ has a point of global minimum or/and a point of global maximum (resp. Theorem 1.1/Theorem 1.2); hence we obtain the existence of critical points for I_ϵ.

Since the abstract perturbation method gives also a necessary condition for the (local) existence of critical points, for the non existence result we will proceed in an analogous way.
First of all, even in a general manifold, when ρ is small it is possible to show that we are in the above perturbative regime. Then we will show that, near the points where the scalar curvature is non zero, Φ_ϵ is strictly monotone in ρ; by the necessary condition we obtain the non existence as well.

Notations and conventions

1) $\mathbb{R}^+$ denotes the set of strictly positive real numbers.

2) Let (M, g) be a 3-dimensional Riemannian manifold.
· First we make the following convention: the Greek index letters, such as $\mu, \nu, \iota, \ldots$, range from 1 to 3 while the Latin index letters, such as $i, j, k, \ldots$, will run from 1 to 2.
· About the Riemann curvature tensor we adopt the convention of [Will]: denoting $\mathfrak{X}(M)$ the set of the vector fields on M, $\forall X, Y, Z \in \mathfrak{X}(M)$

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(X,Y,Z,W) := g(R(Z,W)Y, X);$$

chosen in p an orthonormal frame E_μ, the Ricci curvature tensor is

$$\begin{aligned} Ric_p(v_1, v_2) &:= \sum_1^3 R(E_\mu, v_1, E_\mu, v_2) = \sum_1^3 g(R_p(E_\mu, v_2)v_1, E_\mu) \\ &= -\sum_1^3 g(R_p(v_2, E_\mu)v_1, E_\mu) \quad \forall v_1, v_2 \in T_pM. \end{aligned} \tag{4}$$

Recall the definitions of the Hessian and the Laplace-Beltrami operator on a function w:

$$Hess(w)_{\mu\nu} := \nabla_\mu \nabla_\nu w$$

$$\triangle := g^{\mu\nu} \nabla_\mu \nabla_\nu w.$$

3) Let $(\mathring{M}, \mathring{g}) \hookrightarrow (M, g)$ be an isometrically immersed surface. Recall the notion of second fundamental form $\mathring{h}$: fix a point p and an orthonormal base Z_1, Z_2 of $T_p\mathring{M}$; the inward normal unit vector is denoted as $\mathring{N}$. By the Weingarten equation $\mathring{h}_{ij} = -g(\nabla_{Z_i} \mathring{N}, Z_j)$.
Call k_1 and k_2 the principal curvatures (the eigenvalues of the second fundamental form with respect to the first fundamental form of $\mathring{M}$, i.e. the roots of $\det(\mathring{h}_{ij} - k\mathring{g}_{ij}) = 0$). We adopt the convention that the mean curvature is defined as $H := k_1 + k_2$. Sometimes we will call $D := k_1 k_2$.

4) As mentioned in the introduction, throughout this paper we will focus our attention on perturbed spheres.
· First, let us define the perturbed standard sphere $S_p^\rho(w) \subset \mathbb{R}^3$ we will use to prove the existence result. We denote with S^2 the standard unit sphere in the euclidean 3-dimensional space , $\Theta \in S^2$ is the radial unit vector with components Θ^μ parametrized by the polar coordinates $0 < \theta^1 < \pi$ and $0 < \theta^2 < 2\pi$ chosen in order to satisfy

$$\begin{cases} \Theta^1 = \sin\theta^1 \cos\theta^2 \\ \Theta^2 = \sin\theta^1 \sin\theta^2 \\ \Theta^3 = \cos\theta^1. \end{cases}$$

We call Θ_i the coordinate vector fields on S^2

$$\Theta_1 := \frac{\partial\Theta}{\partial\theta^1}, \quad \Theta_2 := \frac{\partial\Theta}{\partial\theta^2}$$

and $\bar{\theta}_i$ the corresponding normalized ones

$$\bar{\theta_1} := \frac{\Theta_1}{\|\Theta_1\|}, \quad \bar{\theta_2} := \frac{\Theta_2}{\|\Theta_2\|}.$$

The standard sphere in $\mathbb{R}^3$ with center p and radius $\rho > 0$ is denoted by S_p^ρ; we parametrize it as $(\theta^1, \theta^2) \mapsto p + \rho\Theta(\theta^1, \theta^2)$ and call θ_i the coordinate vector fields

$$\theta_1 := \rho\frac{\partial\Theta}{\partial\theta^1}, \quad \theta_2 := \rho\frac{\partial\Theta}{\partial\theta^2}.$$

The perturbed spheres will be normal graphs on standard spheres by a function w which belongs to a suitable function space. Let us introduce the function space which has been chosen by technical reasons (to apply Schauder estimates in Lemma 4.1).
Denote $C^{4,\alpha}(S^2)$ (or simply $C^{4,\alpha}$) the set of the C^4 functions on S^2 whose fourth derivatives, with respect to the tangent vector fields, are α-Hölder ($0 < \alpha < 1$). The Laplace-Beltrami operator on S^2 is denoted by $\triangle_{S^2}$ or, if there is no confusion, as $\triangle$. The fourth order elliptic operator $\triangle(\triangle + 2)$ induces a splitting of $L^2(S^2)$:

$$L^2(S^2) = Ker[\triangle(\triangle + 2)] \oplus Ker[\triangle(\triangle + 2)]^\perp$$

(the splitting makes sense because the kernel is finite dimensional, so it is closed).
If we consider $C^{4,\alpha}(S^2)$ as a subspace of $L^2(S^2)$, we can define

$$C^{4,\alpha}(S^2)^\perp := C^{4,\alpha}(S^2) \cap Ker[\triangle(\triangle + 2)]^\perp.$$

We explicitly observe that $C^{4,\alpha}(S^2)^\perp$ is a Banach space with respect to the $C^{4,\alpha}$ norm; it is the space from which we will draw the perturbations w. If there is no confusion $C^{4,\alpha}(S^2)^\perp$ will be called simply $C^{4,\alpha\perp}$.
Now we can define the perturbed spheres we will use to prove existence of critical points: fix $\rho > 0$ and a small $C^{4,\alpha\perp}$ function w; the perturbed sphere $S_p^\rho(w)$ is the surface parametrized by

$$\Theta \in S^2 \mapsto p + \rho\big(1 - w(\Theta)\big)\Theta.$$

$\cdot$ Now let us define the perturbed geodesic spheres $S_{p,\rho}(w)$ in the Riemannian manifold (M, g); we will use them to prove the non-existence result.
Once a point $p \in M$ is fixed we can consider the exponential map Exp_p with center p. For $\rho > 0$ small enough, the sphere $\rho S^2 \subset T_pM$ is contained in the radius of injectivity of the exponential. We call $S_{p,\rho}$ the geodesic sphere of center p and radius ρ. This hypersurface can be parametrized by

$$\Theta \in S^2 \subset T_pM \mapsto Exp_p[\rho\Theta].$$

Analogously to the previous case, fix $p \in M$, $\rho > 0$ and a small $C^{4,\alpha}(S^2)$ function w; the perturbed geodesic sphere $S_{p,\rho}(w)$ is the surface parametrized by

$$\Theta \in S^2 \mapsto Exp_p[\rho\big(1 - w(\Theta)\big)\Theta].$$

The tangent vector fields on $S_{p,\rho}(w)$ induced by the canonical polar coordinates on S^2 are denoted by Z_i.

5) $\cdot$ Following the notation of [PX], given $a \in \mathbb{N}$, any expression of the form $L_p^{(a)}(w)$ denotes a linear combination of the function w together with its derivatives with respect to the tangent vector fields Θ_i up to order a. The coefficients of $L_p^{(a)}$ might depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $C > 0$ independent on $\rho \in (0, 1)$ and $p \in M$ such that

$$\|L_p^{(a)}(w)\|_{C^{k,\alpha}(S^2)} \leq C\|w\|_{C^{k+a,\alpha}(S^2)}.$$

· Similarly, given $b \in \mathbb{N}$, any expression of the form $Q_p^{(b)(a)}(w)$ denotes a nonlinear operator in the function w together with its derivatives with respect to the tangent vector fields Θ_i up to order a such that, for all $p \in M$, $Q_p^{(b)(a)}(0) = 0$. The coefficients of the Taylor expansion of $Q_p^{(b)(a)}(w)$ in powers of w and its partial derivatives might depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $C > 0$ independent on $\rho \in (0,1)$ and $p \in M$ such that

$$(5)\quad \|Q_p^{(b)(a)}(w_2) - Q_p^{(b)(a)}(w_1)\|_{C^{k,\alpha}(S^2)} \leq c\big(\|w_2\|_{C^{k+a,\alpha}(S^2)} + \|w_1\|_{C^{k+a,\alpha}(S^2)}\big)^{b-1} \times \|w_2 - w_1\|_{C^{k+a,\alpha}(S^2)},$$

provided $\|w_l\|_{C^a(S^2)} \leq 1$, $l = 1, 2$.

· We also agree that any term denoted by $O_p(\rho^d)$ is a smooth function on S^2 that might depend on p but which is bounded by a constant (independent on p) times ρ^d in C^k topology, for all $k \in N$.

6) Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc.. Also constants with subscripts are allowed to vary.

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2 Preliminary results

2.1 A perturbative method: the Lyapunov-Schmidt reduction

In this subsection we briefly summarize the perturbative method which is extensively treated with proofs and examples in [AM].

Given an Hilbert space H, let $I_\epsilon : H \to \mathbb{R}$ be a C^2 functional of the form

$$I_\epsilon(u) = I_0(u) + \epsilon G(u),$$

where $I_0 \in C^2(H, \mathbb{R})$ plays the role of the unperturbed functional and $G \in C^2(H, \mathbb{R})$ is the perturbation.

We first suppose that there exists a finite dimensional smooth manifold Z made of critical points of I_0: $I_0'(z) = 0$ for all $z \in Z$. The set Z will be called *critical manifold* (of I_0).

Under suitable non degeneracy assumptions on I_0 it is known that near Z there exists a perturbed manifold Z^ϵ such that the critical points of I_ϵ constrained on Z^ϵ give rise to stationary points of I_ϵ. More precisely, the non degeneracy conditions are

(ND) for all $z \in Z$, $T_z Z = Ker[I_0''(z)]$,

(Fr) for all $z \in Z$, $I_0''(z)$ is a Fredholm operator of index zero;

and the fundamental tool is the following Theorem:

Theorem 2.1. *Suppose I_0 possesses a non degenerate (satisfying (ND) and (Fr)) critical manifold Z of dimension d.*

Given a compact subset Z_c of Z, there exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$ there is a smooth function

$$w_\epsilon(z) : Z_c \to H$$

such that

(i) for $\epsilon = 0$ it results $w_\epsilon(z) = 0$, $\forall z \in Z_c$;

(ii) $w_\epsilon(z)$ is orthogonal to $T_z Z$, $\forall z \in Z_c$;

(iii) the manifold

$$Z^\epsilon = \{z + w_\epsilon(z) : z \in Z_c\}$$

is a natural constraint for I'_ϵ. Namely, denoting

$$\Phi_\epsilon(z) = I_\epsilon(z + w_\epsilon(z)) : Z_c \to \mathbb{R}$$

the constriction of I_ϵ to Z^ϵ, if z_ϵ is a critical point of Φ_ϵ then $u_\epsilon = z_\epsilon + w_\epsilon(z_\epsilon)$ is a critical point of I_ϵ.

Thanks to this fundamental Theorem, in order to find critical points of I_ϵ, we can reduce ourselves to study Φ_ϵ which is a function in *finitely* many variables.

If we are slightly more accurate, it can be shown that the function $w_\epsilon(z)$ is of order $O(\epsilon)$ as $\epsilon \to 0$ uniformly in z varying in the compact Z_c. Thanks to this fact, by a Taylor expansion it is easy to see that

$$\Phi_\epsilon(z) = I_\epsilon(z) + o(\epsilon).$$

The last formula suggests that, in order to prove the existence of critical points, it will be enough to know the perturbed functional I_ϵ on the critical manifold Z.

2.2 Geometric expansions

In this subsection we recall the well-known expansions of the first and second fundamental form and the mean curvature for the geodesic perturbed spheres $S_{p,\rho}(w)$ introduced in the previous "notations and conventions". For the proofs we refer to [PX]. Recall that Θ_i are the coordinate vector fields on S^2 (induced by polar coordinates) and Z_i are the corresponding coordinate vector fields on $S_{p,\rho}(w)$. The derivatives of w with respect to Θ_i are denoted by w_i.

Let $\mathring{g}$ denote the first fundamental form on $S_{p,\rho}(w)$ induced by the immersion in (M, g). In the next Lemma we find an expansion of the components $\mathring{g}_{ij} := g_p(Z_i, Z_j)$:

Lemma 2.2. *The first fundamental form on $S_{p,\rho}(w)$ has the following expansion:*

$$\begin{aligned}(1-w)^{-2}\rho^{-2}\mathring{g}_{ij} &= g(\Theta_i, \Theta_j) + (1-w)^{-2} w_i w_j + \frac{1}{3} g(R_p(\Theta, \Theta_i)\Theta, \Theta_j)\rho^2(1-w)^2 \\ &\quad + O_p(\rho^3) + \rho^3 L_p^{(2)}(w) + \rho^3 Q_p^{(2)(2)}(w),\end{aligned}$$

where all curvature terms and scalar products are evaluated at p (since we are in normal coordinates, at p the metric is euclidean).

Let $\mathring{h}$ denote the second fundamental form on $S_{p,\rho}(w)$ induced by the immersion in (M, g) and $\mathring{N}$ the inward normal unit vector to $S_{p,\rho}(w)$; by the Weingarten equation $\mathring{h}_{ij} = -g(\nabla_{Z_i}\mathring{N}, Z_j)$.

Lemma 2.3. *The second fundamental form on $S_{p,\rho}(w)$ has the following expansion:*

$$\begin{aligned}\mathring{h}_{ij} &= \rho(1-w) g(\Theta_i, \Theta_j) + \rho(Hess_{g_{S^2}} w)_{ij} + \frac{2}{3} g(R_p(\Theta, \Theta_i)\Theta, \Theta_j)\rho^3(1-w)^3 \\ &\quad + O_p(\rho^4) + \rho^3 L_p^{(2)}(w) + \rho Q_p^{(2)(2)}(w)\end{aligned}$$

where, as usual, all curvature terms and scalar products are evaluated at p.

Recall that the mean curvature H is the trace of $\mathring{h}$ with respect to the metric $\mathring{g}$: $H = \mathring{h}_{ij}\mathring{g}^{ij}$. Collecting the two previous Lemmas we obtain the following

Lemma 2.4. *The mean curvature of the hyper-surface $S_{p,\rho}(w)$ can be expanded as*

$$\begin{aligned}H(S_{p,\rho}(w)) &= \frac{2}{\rho} + \frac{1}{\rho}(2 + \triangle_{S^2})w - \frac{1}{3} Ric_p(\Theta, \Theta)\rho(1-w) \\ &\quad + O_p(\rho^2) + \rho^2 L_p^{(2)}(w) + \frac{1}{\rho} Q_p^{(2)(2)}(w), \qquad (6)\end{aligned}$$

where Ric_p is the Ricci tensor computed at p.

3 The expansions of the Willmore functional

In this section we present the geometric computations needed to show the non-existence and the existence results. In the first subsection we derive the expansions of the Willmore functional and its differential on the perturbed geodesic spheres $S_{p,\rho}(w)$ immersed in the Riemannian manifold (M, g). In the second subsection we will compute the Willmore functional on the standard spheres S^ρ_p immersed in $\mathbb{R}^3$ endowed with the metric $g_\epsilon = \delta + \epsilon h$.

3.1 The Willmore functional on perturbed geodesic spheres $S_{p,\rho}(w)$

Let us start with the expansion of the Willmore functional I.

Proposition 3.1. *The Willmore functional on the surfaces $S_{p,\rho}(w)$ can be expanded in ρ and w as follows:*

$$I(S_{p,\rho}(w)) = 16\pi - \frac{8\pi}{3}R(p)\rho^2 + \int_{S^2}(Q^{(2)(2)}_p(w) + \rho^2 L^{(2)}_p(w))d\Theta + O_p(\rho^3), \tag{7}$$

where $R(p)$ is the scalar curvature of (M, g) evaluated at p.

Proof. Recall the definition of the Willmore functional:

$$I(S_{p,\rho}(w)) := \int_{S_{p,\rho}(w)} H^2 d\Sigma$$

where $d\Sigma$ is the area form of $S_{p,\rho}(w)$.
Using the expansion of the metric $\mathring{g}$ of Lemma 2.2, it is easy to see that

$$\det[\mathring{g}] = \|\Theta_2\|^2\rho^4\Big[(1-4w) - \frac{1}{3}Ric_p(\Theta,\Theta)\rho^2 + O_p(\rho^3) + \rho^2 L^{(2)}_p(w) + Q^{(2)(2)}_p(w)\Big].$$

Taking the square root we obtain the area form in the polar coordinates θ^1, θ^2:

$$\begin{aligned} d\Sigma &= \sqrt{\det[\mathring{g}]}d\theta^1\wedge d\theta^2 \\ &= \|\Theta_2\|\rho^2\Big\{1 - 2w - \frac{1}{6}Ric_p(\Theta,\Theta)\rho^2 + O_p(\rho^3) + \rho^2 L^{(2)}_p(w) + Q^{(2)(2)}_p(w)\Big\}d\theta^1\wedge d\theta^2 \\ &= \rho^2\Big\{1 - 2w - \frac{1}{6}Ric_p(\Theta,\Theta)\rho^2 + O_p(\rho^3) + \rho^2 L^{(2)}_p(w) + Q^{(2)(2)}_p(w)\Big)d\Theta, \end{aligned} \tag{8}$$

where $d\Theta := \|\Theta_2\|d\theta^1\wedge d\theta^2$ is the area form on S^2.

From (6) we get the expansion of H^2:

$$H^2 = \frac{4}{\rho^2} + \frac{4}{\rho^2}(2 + \triangle_{S^2})w - \frac{4}{3}Ric_p(\Theta,\Theta) + O_p(\rho) + L^{(2)}_p(w) + \frac{1}{\rho^2}Q^{(2)(2)}_p(w). \tag{9}$$

Collecting the above identities (8) and (9) we get

(10)
$$I(S_{p,\rho}(w)) = 16\pi + 4\int_{S^2}(\triangle_{S^2}w)d\Theta - 2\rho^2\int_{S^2}Ric_p(\Theta,\Theta)d\Theta + \int_{S^2}\big(\rho^2 L^{(2)}_p(w) + Q^{(2)(2)}_p(w)\big)d\Theta + O_p(\rho^3).$$

Recall that the Laplacian of a function integrated on a compact surface without boundary is null, so the second term $\int_{S^2}(\triangle_{S^2}w)d\Theta$ is zero.

Let us compute the term $\int_{S^2}Ric_p(\Theta,\Theta)d\Theta$. Choose Cartesian coordinates $x^\mu, \mu = 1\ldots 3$ on T_pM, and recall that (see [Bren] pag. 28)

$$\int_{S^2} x^\mu x^\nu d\Theta = \frac{4\pi}{3}\delta^{\mu\nu}.$$

By a straightforward computation

$$\int_{S^2} Ric_p(\Theta,\Theta)d\Theta = \int_{S^2} R_{\mu\nu}x^\mu x^\nu d\Theta = R_{\mu\mu}\int_{S^2} x^\mu x^\mu d\Theta = \frac{4\pi}{3}R(p).$$

At last, we can expand the Willmore functional I as

$$I(S_{p,\rho}(w)) = 16\pi - \frac{8\pi}{3}R(p)\rho^2 + \int_{S^2}\big(Q_p^{(2)(2)}(w) + \rho^2 L_p^{(2)}(w)\big)d\Theta + O_p(\rho^3).$$

■

Now we obtain an expansion for the L^2-differential of the Willmore functional, with respect to normal variations, on the perturbed geodesic spheres:

Proposition 3.2. *The L^2-differential of the Willmore functional on the surfaces $S_{p,\rho}(w)$ can be expanded as*

$$\rho^2 I'(S_{p,\rho}(w)) = \frac{2}{\rho}\triangle_{S^2}(\triangle_{S^2}+2)w + O_p(\rho) + \rho L_p^{(4)}(w) + \frac{1}{\rho}Q_p^{(2)(4)}(w). \tag{11}$$

Proof. In [PV], Parthasarathy and Viswanathan computed the L^2-differential of the Willmore functional for a general immersed submanifold; in the case of a surface $\mathring{M}$ immersed in the 3-dimensional Riemannian manifold M, formula (29) pag. 9 and a simple computation give the expression of the differential (notice the difference in the convention about H and about the sign of $\mathring{N}$):

$$I'(\mathring{M}) = 2\triangle_{\mathring{M}}H + H\big(H^2 - 4D + 2R_{\mu\nu}\mathring{N}^\mu\mathring{N}^\nu\big) \tag{12}$$

where $\triangle_{\mathring{M}}$ is the Laplacian on $\mathring{M}$, $R_{\mu\nu}$ is the Ricci curvature of M and $\mathring{N}$ is the normal unit vector to the surface $\mathring{M}$. Recall that k_1 and k_2 are the principal curvatures, $D := k_1k_2$ and in our convention $H = k_1 + k_2$. Observe that

$$D := k_1k_2 = \frac{\det\mathring{h}_{ij}}{\det\mathring{g}_{ij}}.$$

Let us compute I' on the surfaces $S_p^\rho(w)$. From the expansion of $\mathring{h}_{ij}$ and $\mathring{g}_{ij}$ of Lemmas 2.2 and 2.3, with simple computations we get

$$D = \frac{1}{\rho^2}\Big((1 + 2w + \triangle_{S^2}w) + O_p(\rho^2) + \rho^2 L_p(w) + Q_p^{(2)}(w)\Big). \tag{13}$$

Collecting this formula and the expansion of H of Lemma 2.4, we obtain

$$H^2 - 4D = O_p(\rho^0) + L_p^{(2)}(w) + \frac{1}{\rho^2}Q_p^{(2)(2)}(w).$$

From [PX] pag.10, we have

$$\mathring{N} = O_p(\rho^0) + L_p^{(2)}(w) + Q_p^{(2)(2)}(w),$$

so it follows

$$H^2 - 4D + 2R_{\mu\nu}\mathring{N}^\mu\mathring{N}^\nu = O_p(\rho^0) + L_p^{(2)}(w) + \frac{1}{\rho^2}Q_p^{(2)(2)}(w),$$

and

$$H\big(H^2 - 4D + 2R_{\mu\nu}\mathring{N}^\mu\mathring{N}^\nu\big) = O_p\left(\frac{1}{\rho}\right) + \frac{1}{\rho}L_p^{(2)}(w) + \frac{1}{\rho^3}Q_p^{(2)(2)}(w).$$

Using this formula and the expansion of H of Lemma 2.4, the expression of the differential given in (12) for the surface $S_{p,\rho}(w)$ becomes

(14)

$$I'(S_{p,\rho}(w)) = \frac{2}{\rho}\triangle_{\mathring{g}}(2+\triangle_{S^2})w + \triangle_{\mathring{g}}\Big(O_p(\rho) + \rho L_p^{(2)}(w) + \frac{1}{\rho}Q_p^{(2)(2)}(w)\Big) + O_p\left(\frac{1}{\rho}\right) + \frac{1}{\rho}L_p^{(2)}(w) + \frac{1}{\rho^3}Q_p^{(2)(2)}(w).$$

Now we want to write the Laplacian $\triangle_{\mathring{g}}$ on $S_{p,\rho}(w)$ in a convenient way, in terms of $\triangle_{S^2}$.
Recall that given $u \in C^\infty(S_{p,\rho}(w))$

$$\triangle_{\mathring{g}} u = \mathring{g}^{ij} \nabla_i \nabla_j u = \mathring{g}^{ij}(u_{ij} - \mathring{\Gamma}^k_{ij} u_k)$$

where u_{ij}, u_k are the derivatives of u with respect to the coordinates vector fields and $\mathring{\Gamma}^k_{ij}$ are the Christoffel symbols of $S_{p,\rho}(w)$. From the expression of the first fundamental form of $S_{p,\rho}(w)$ we see that

$$\mathring{\Gamma}^k_{ij} = \Gamma^k_{ij} + O(\rho^2) + L^{(2)}_p(w) + Q^{(2)(2)}_p(w),$$

where Γ^k_{ij} are the Christoffel symbols of S^2 in the usual polar coordinate system. From

$$\mathring{g}^{ij} = \frac{1+2w}{\rho^2}[g^{ij}_{S^2} + O_p(\rho^2) + \rho^2 L^{(2)}_p(w) + Q^{(2)(2)}(w)],$$

we get

$$\begin{aligned}\triangle_{\mathring{g}} u &= \mathring{g}^{ij}(u_{ij} - \mathring{\Gamma}^k_{ij} u_k) \\ &= \frac{1}{\rho^2}\triangle_{S^2} u + O_p(\rho^0) + \rho^{-2} L^{(2)}_p(w) + \rho^{-2} Q^{(2)(2)}(w).\end{aligned}$$

We can at last conclude that

$$I'(S_{p,\rho}(w)) = \frac{2}{\rho^3}\triangle_{S^2}(\triangle_{S^2}+2)w + O_p\left(\frac{1}{\rho}\right) + \frac{1}{\rho} L^{(4)}_p(w) + \frac{1}{\rho^3} Q^{(2)(4)}_p(w).$$

■

Remark 3.3. *In formula (11), $O_p(\rho)$ and $L^{(4)}_p(w)$ are polynomial expressions (in ρ and w respectively) whose coefficients are curvature terms of the ambient metric g evaluated at p. Therefore if the ambient manifold is $\mathbb{R}^3$ endowed with euclidean metric, those terms are null. Obviously, in euclidean metric the perturbed geodesic spheres $S_{p,\rho}(w)$ coincide with the perturbed standard spheres $S^\rho_p(w)$; denoting with I_0 the Willmore functional in euclidean metric we get*

$$I_0'(S^\rho_p(w)) = \frac{2}{\rho^3}\triangle_{S^2}(\triangle_{S^2}+2)w + \frac{1}{\rho^3} Q^{(2)(4)}_p(w).$$

Just taking the linearization, the second differential is

$$I_0''(S^\rho_p)[w] = \frac{2}{\rho^3}\triangle_{S^2}(\triangle_{S^2}+2)[w].$$

Notice that the expansion is consistent with the formula (7.40) page 289 in [Will] (the notation about the mean curvature is different).

3.2 The Willmore functional on standard spheres S^ρ_p

In this subsection we consider $\mathbb{R}^3$ endowed with the perturbed metric $g_\epsilon = \delta + \epsilon h$ and compute the corresponding Willmore functional I_ϵ on the standard spheres S^ρ_p.
Recall that

$$I_\epsilon(S^\rho_p) := \int_{S^\rho_p} H^2_\epsilon d\Sigma_\epsilon,$$

where H_ϵ and $d\Sigma_\epsilon$ are respectively the mean curvature and the area form of S^ρ_p with respect to the metric g_ϵ. In the following, $\nu_0 = -\Theta$ will denote the inward normal unit vector to S^ρ_p in euclidean metric and

$$\nu_\epsilon = \nu_0 + \epsilon N + o(\epsilon)$$

the normal unit vector in metric g_ϵ.

Lemma 3.4. *The Willmore functional I_ϵ computed on the standard spheres S_p^ρ has the following expansion*

$$I_\epsilon(S_p^\rho) = 16\pi + 2\epsilon \int_{S^2} \Big[h\big(\bar{\theta_1},\bar{\theta_1}\big) + h(\bar{\theta_2},\bar{\theta_2}) - 2h(\nu_0,\nu_0) - \rho(\bar{\theta}_1^\mu\bar{\theta}_1^\nu + \bar{\theta}_2^\mu\bar{\theta}_2^\nu)\nu_0^\lambda A_{\mu\nu\lambda}\Big] d\Sigma_0 + o(\epsilon) \tag{15}$$

where

$$A_{\mu\nu\lambda} := [D_\mu h_{\nu\lambda} + D_\lambda h_{\mu\nu} - D_\nu h_{\lambda\mu}]$$

and the remainder $o(\epsilon)$ is uniform on compact sets of $\mathbb{R}^3 \oplus \mathbb{R}^+ \ni (p,\rho)$. In the integral, h is evaluated at the points of S_p^ρ.

Proof. Let us start with expanding in terms of ϵ the geometric quantities of interest. We will use the classical notation to denote the coefficients of the first fundamental form $\mathring{g}_{\epsilon ij}$ of S_p^ρ. Denoting with $(.,.)$ the euclidean scalar product, we have

$$\begin{aligned} E_\epsilon &= g_\epsilon(\theta_1,\theta_1) = (\theta_1,\theta_1) + \epsilon h(\theta_1,\theta_1) = E_0 + \epsilon h(\theta_1,\theta_1) \\ F_\epsilon &= g_\epsilon(\theta_1,\theta_2) = F_0 + \epsilon h(\theta_1,\theta_2) = \epsilon h(\theta_1,\theta_2) \\ G_\epsilon &= g_\epsilon(\theta_2,\theta_2) = G_0 + \epsilon h(\theta_2,\theta_2), \end{aligned}$$

where E_0, F_0, G_0 are the corresponding quantities in euclidean metric.
Recall that the area form is $d\Sigma_\epsilon = \sqrt{E_\epsilon G_\epsilon - F_\epsilon^2}\, d\theta^1 \wedge d\theta^2$; by a Taylor expansion in ϵ we get

$$d\Sigma_\epsilon = d\Sigma_0 + \frac{\epsilon}{2}\frac{E_0 h(\theta_2,\theta_2) + G_0 h(\theta_1,\theta_1)}{\sqrt{E_0 G_0}} d\theta^1 \wedge d\theta^2 + o(\epsilon), \tag{16}$$

where $d\Sigma_0$ is the area form in euclidean metric and the remainder $o(\epsilon)$ is uniform on compact sets of $\mathbb{R}^3 \oplus \mathbb{R}^+ \ni (p,\rho)$.

Now expand the second fundamental form.
Recall that $\nu_\epsilon = \nu_0 + \epsilon N + o(\epsilon)$ is the normal unit vector to S_p^ρ in metric g_ϵ; from the orthogonality conditions $g_\epsilon(\theta_1,\nu_\epsilon) = 0$ and $g_\epsilon(\theta_2,\nu_e)$ we get

$$\begin{aligned} (N,\theta_1) &= -h(\nu_0,\theta_1) \\ (N,\theta_2) &= -h(\nu_0,\theta_2). \end{aligned}$$

Imposing the normalization condition $g_\epsilon(\nu_\epsilon,\nu_\epsilon) = 1$ we obtain

$$(N,\nu_0) = -\frac{1}{2}h(\nu_0,\nu_0).$$

Collecting the formulas above, being $(\bar{\theta_1},\bar{\theta_2},\nu_0)$ an orthonormal frame in euclidean metric, we can represent N as

$$N = -h(\nu_0,\bar{\theta_1})\bar{\theta_1} - h(\nu_0,\bar{\theta_2})\bar{\theta_2} - \frac{1}{2}h(\nu_0,\nu_0)\nu_0. \tag{17}$$

Knowing the normal unit vector ν_ϵ we can evaluate the coefficients of the second fundamental form

$$\mathring{h}_{\epsilon ij} := -g_\epsilon(\nabla_{\theta_i}\nu_\epsilon,\theta_j), \tag{18}$$

where ∇ is the connection on $\mathbb{R}^3$ endowed with the metric g_ϵ.
By linearity, denoting with $\frac{\partial}{\partial x^\lambda}$ the standard euclidean frame in $\mathbb{R}^3$,

$$\begin{aligned} \nabla_{\theta_i}\nu_\epsilon &= \theta_i^\mu \nabla_\mu(\nu_\epsilon^\lambda \frac{\partial}{\partial x^\lambda}) = \theta_i^\mu \frac{\partial \nu_\epsilon^\lambda}{\partial x^\mu}\frac{\partial}{\partial x^\lambda} + \theta_i^\mu \nu_\epsilon^\lambda \Gamma_{\mu\lambda}^\nu \frac{\partial}{\partial x^\nu} \\ &= \frac{\partial \nu_\epsilon}{\partial \theta^i} + \theta_i^\mu \nu_\epsilon^\lambda \Gamma_{\mu\lambda}^\nu \frac{\partial}{\partial x^\nu}, \end{aligned} \tag{19}$$

where $\Gamma^{\nu}_{\mu\lambda}$ are the Christoffel symbols of $\mathbb{R}^3$ with metric g_ϵ.
Let us find an expansion in ϵ of $\Gamma^{\nu}_{\mu\lambda}$. By definition

$$\Gamma^{\nu}_{\mu\lambda} = \frac{1}{2} g^{\nu\sigma}[D_\mu g_{\lambda\sigma} + D_\lambda g_{\sigma\mu} - D_\sigma g_{\mu\lambda}].$$

Observe that $g^{\mu\sigma} = \delta^{\mu\sigma} - \epsilon h_{\mu\sigma} + o(\epsilon)$ and $D_\mu g_{\lambda\sigma} = \epsilon D_\mu h_{\lambda\sigma}$, so

$$\Gamma^{\nu}_{\mu\lambda} = \frac{1}{2}\epsilon\delta^{\nu\sigma}[D_\mu h_{\lambda\sigma} + D_\lambda h_{\sigma\mu} - D_\sigma h_{\mu\lambda}] + o(\epsilon).$$

Putting this expansion in (19) and using (18), the second fundamental form becomes

$$\mathring{h}_{\epsilon ij} = \left(\frac{\partial \nu_0}{\partial\theta^i}, \theta_j\right) - \epsilon\left[h\left(\frac{\partial \nu_0}{\partial\theta^i}, \theta_j\right) + \left(\frac{\partial N}{\partial\theta^i}, \theta_j\right)\right] - \frac{1}{2}\epsilon\theta^{\mu}_i\theta^{\nu}_j\nu^{\lambda}_0 A_{\mu\nu\lambda}. \tag{20}$$

From the expressions of the first and second fundamental forms, with simple computations, we obtain the mean curvature $H_\epsilon = \mathring{h}_{\epsilon ij}\mathring{g}^{ij}_\epsilon$:

$$\begin{aligned} H_\epsilon &= H_0 - \epsilon\left(\frac{E_0 h(\theta_2,\theta_2) + G_0 h(\theta_1,\theta_1)}{(E_0G_0)^2}(E_0 n_0 + G_0 l_0)\right) - \epsilon\left(\frac{E_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\theta_2\right) + \left(\frac{\partial N}{\partial\theta^2},\theta_2\right) + \frac{1}{2}\theta^{\mu}_2\theta^{\nu}_2\nu^{\lambda}_0 A_{\mu\nu\lambda}\right]}{E_0G_0}\right.\\ &\quad\left. + \frac{G_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\theta_1\right) + \left(\frac{\partial N}{\partial\theta^1},\theta_1\right) + \frac{1}{2}\theta^{\mu}_1\theta^{\nu}_1\nu^{\lambda}_0 A_{\mu\nu\lambda}\right] - n_0 h(\theta_1,\theta_1) - l_0 h(\theta_2,\theta_2)}{E_0G_0}\right) + o(\epsilon) \end{aligned}$$

and its square

$$\begin{aligned} H^2_\epsilon &= H^2_0 - 2\epsilon H_0\left(\frac{E_0 h(\theta_2,\theta_2) + G_0 h(\theta_1,\theta_1)}{(E_0G_0)^2}(E_0 n_0 + G_0 l_0)\right)\\ &\quad -2\epsilon H_0\left(\frac{E_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\theta_2\right) + \left(\frac{\partial N}{\partial\theta^2},\theta_2\right)\right] + G_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\theta_1\right) + \left(\frac{\partial N}{\partial\theta^1},\theta_1\right)\right] - n_0 h(\theta_1,\theta_1) - l_0 h(\theta_2,\theta_2)}{E_0G_0}\right)\\ &\quad -\epsilon H_0\left(\frac{(E_0\theta^{\mu}_2\theta^{\nu}_2 + G_0\theta^{\mu}_1\theta^{\nu}_1)\nu^{\lambda}_0 A_{\mu\nu\lambda}}{E_0G_0}\right) + o(\epsilon), \end{aligned} \tag{21}$$

where $l_0, m_0 (= 0), n_0$ and H_0 are the coefficients of the second fundamental form and the mean curvature of S^{ρ}_p in euclidean metric ($l_0 = \mathring{h}_{011}, m_0 = \mathring{h}_{012}, n_0 = \mathring{h}_{022}$).

Collecting the formulas of H^2_ϵ and $d\Sigma_\epsilon$, observing that $I_0(S^{\rho}_p) = 16\pi$ and recalling that $H_0 = \frac{l_0G_0 - 2m_0F_0 + n_0E_0}{E_0G_0 - F_0^2}$, we obtain the expansion of the Willmore functional

$$\begin{aligned} I_\epsilon(S^{\rho}_p) &= 16\pi - \frac{3}{2}\epsilon\int H^2_0 \frac{E_0 h(\theta_2,\theta_2) + G_0 h(\theta_1,\theta_1)}{\sqrt{E_0G_0}} d\theta^1 d\theta^2\\ &\quad -2\epsilon\int H_0\left(\frac{E_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\theta_2\right) + \left(\frac{\partial N}{\partial\theta^2},\theta_2\right)\right] + G_0\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\theta_1\right) + \left(\frac{\partial N}{\partial\theta^1},\theta_1\right)\right] - n_0 h(\theta_1,\theta_1) - l_0 h(\theta_2,\theta_2)}{\sqrt{E_0G_0}}\right) d\theta^1 d\theta^2\\ &\quad -\epsilon\int H_0\left(\frac{(E_0\theta^{\mu}_2\theta^{\nu}_2 + G_0\theta^{\mu}_1\theta^{\nu}_1)\nu^{\lambda}_0 A_{\mu\nu\lambda}}{\sqrt{E_0G_0}}\right) d\theta^1 d\theta^2 \quad + o(\epsilon), \end{aligned} \tag{22}$$

where all the integrals are computed on $(0,\pi)\times(0,2\pi)\ni(\theta^1,\theta^2)$.
The coefficients of the unperturbed first fundamental form are

$$\begin{aligned} E_0 &= \rho^2 \\ F_0 &= 0 \\ G_0 &= \rho^2\sin^2\theta^1, \end{aligned}$$

those of the unperturbed second fundamental form are

$$\begin{aligned} l_0 &= \rho \\ m_0 &= 0 \\ n_0 &= \rho\sin^2\theta^1 \end{aligned}$$

and the unperturbed mean curvature is

$$H_0 = \frac{2}{\rho}.$$

Now we use these expressions to simplify (22). Let us start from the first integral:

$$-\frac{3}{2}\epsilon\int H_0^2\frac{E_0 h(\theta_2,\theta_2)+G_0 h(\theta_1,\theta_1)}{\sqrt{E_0G_0}}d\theta^1 d\theta^2 = -6\epsilon\int_{S^2}\Big[h(\bar{\theta_1},\bar{\theta_1})+h(\bar{\theta_2},\bar{\theta_2})\Big]d\Sigma_0, \tag{23}$$

where $d\Sigma_0$ is the standard area form on S^2. The second integral becomes

$$-4\epsilon\int_{(0,\pi)\times(0,2\pi)}\left[h\left(\frac{\partial\nu_0}{\partial\theta^2},\bar{\theta_2}\right)+\left(\frac{\partial N}{\partial\theta^2},\bar{\theta_2}\right)\right]d\theta^1 d\theta^2 - 4\epsilon\int_{S^2}\left[h\left(\frac{\partial\nu_0}{\partial\theta^1},\bar{\theta_1}\right)+\left(\frac{\partial N}{\partial\theta^1},\bar{\theta_1}\right)-h(\bar{\theta_1},\bar{\theta_1})-h(\bar{\theta_2},\bar{\theta_2})\right]d\Sigma_0.$$

Observe that $\frac{\partial\nu_0}{\partial\theta^1}=-\bar{\theta_1}$ and $\frac{\partial\nu_0}{\partial\theta^2}=-\sin\theta^1\bar{\theta_2}$, so we can go on

$$= -4\epsilon\int_{S^2}\left[\frac{1}{\sin\theta^1}\left(\frac{\partial N}{\partial\theta^2},\bar{\theta_2}\right)+\left(\frac{\partial N}{\partial\theta^1},\bar{\theta_1}\right)\right]d\Sigma_0 + 8\epsilon\int_{S^2}\Big[h(\bar{\theta_1},\bar{\theta_1})+h(\bar{\theta_2},\bar{\theta_2})\Big]d\Sigma_0. \tag{24}$$

Let us try to make explicit $\left(\frac{\partial N}{\partial\theta^2},\bar{\theta_2}\right)$ and $\left(\frac{\partial N}{\partial\theta^1},\bar{\theta_1}\right)$ in terms of known quantities. Of course we have

$$\left(\frac{\partial N}{\partial\theta^2},\bar{\theta_2}\right) = \frac{\partial}{\partial\theta^2}(N,\bar{\theta_2}) - \left(N,\frac{\partial\bar{\theta_2}}{\partial\theta^2}\right).$$

Observe that $\frac{\partial\bar{\theta_2}}{\partial\theta^2}=-\cos\theta^1\bar{\theta_1}+\sin\theta^1\nu_0$. From the representation (17) of N, we get

$$\left(N,\frac{\partial\bar{\theta_2}}{\partial\theta^2}\right) = h(\bar{\theta_1},\nu_0)\cos\theta^1 - \frac{1}{2}h(\nu_0,\nu_0)\sin\theta^1,$$

so

$$\left(\frac{\partial N}{\partial\theta^2},\bar{\theta_2}\right) = -\frac{\partial}{\partial\theta^2}h(\nu_0,\bar{\theta_2}) - h(\bar{\theta_1},\nu_0)\cos\theta^1 + \frac{1}{2}h(\nu_0,\nu_0)\sin\theta^1.$$

In an analogous way, observing that $\frac{\partial\bar{\theta_1}}{\partial\theta^1}=\nu_0$ and $(N,\bar{\theta_1})=-h(\nu_0,\bar{\theta_1})$ we get

$$\left(\frac{\partial N}{\partial\theta^1},\bar{\theta_1}\right) = \frac{\partial}{\partial\theta^1}(N,\bar{\theta_1}) - \left(N,\frac{\partial\bar{\theta_1}}{\partial\theta^1}\right) = -\frac{\partial}{\partial\theta^1}h(\nu_0,\bar{\theta_1}) + \frac{1}{2}h(\nu_0,\nu_0).$$

Summing up the last two quantities we obtain

$$\frac{1}{\sin\theta^1}\left(\frac{\partial N}{\partial\theta^2},\bar{\theta_2}\right)+\left(\frac{\partial N}{\partial\theta^1},\bar{\theta_1}\right) = -\frac{1}{\sin\theta^1}\frac{\partial}{\partial\theta^2}h(\nu_0,\bar{\theta_2}) - \frac{\cos\theta^1}{\sin\theta^1}h(\bar{\theta_1},\nu_0) - \frac{\partial}{\partial\theta^1}h(\nu_0,\bar{\theta_1}) + h(\nu_0,\nu_0). \tag{25}$$

We want to integrate this expression on S^2. Observe that

$$\int_{S^2}\left[\frac{1}{\sin\theta^1}\frac{\partial}{\partial\theta^2}h(\nu_0,\bar{\theta_2})\right]d\Sigma_0=\int_{(0,\pi)\times(0,2\pi)}\frac{\partial}{\partial\theta^2}h(\nu_0,\bar{\theta_2})d\theta^1d\theta^2=0$$

because $h(\nu_0,\bar{\theta_2})$ is a 2π-periodic function in θ^2. The integral of the third term is

$$\int_{S^2}\left[-\frac{\partial}{\partial\theta^1}h(\nu_0,\bar{\theta_1})\right]d\Sigma_0=\int_0^{2\pi}\Big(\int_0^{\pi}-\frac{\partial}{\partial\theta^1}h(\nu_0,\bar{\theta_1})\sin\theta^1d\theta^1\Big)d\theta^2;$$

integrating by parts and using that $0=\sin\pi=\sin 0$ we get

$$=\int_{(0,\pi)\times(0,2\pi)}h(\nu_0,\bar{\theta_1})\cos\theta^1d\theta^1d\theta^2.$$

So the integrals of the second and third terms of (25) delete each other and we obtain

$$\int_{S^2}\left[\frac{1}{\sin\theta^1}\Big(\frac{\partial N}{\partial\theta^2},\bar{\theta_2}\Big)+\Big(\frac{\partial N}{\partial\theta^1},\bar{\theta_1}\Big)\right]d\Sigma_0=\int_{S^2}h(\nu_0,\nu_0)d\Sigma_0. \tag{26}$$

In this way we simplified the first two integrals of (22). The last integral can be rewritten as:

$$-\epsilon\int_{(0,\pi)\times(0,2\pi)}H_0\Bigg(\frac{(E_0\theta_2^\mu\theta_2^\nu+G_0\theta_1^\mu\theta_1^\nu)\nu_0^\lambda A_{\mu\nu\lambda}}{\sqrt{E_0G_0}}\Bigg)d\theta^1d\theta^2=-2\epsilon\rho\int_{S^2}\Big[(\bar{\theta}_2^\mu\bar{\theta}_2^\nu+\bar{\theta}_1^\mu\bar{\theta}_1^\nu)\nu_0^\lambda A_{\mu\nu\lambda}\Big]d\Sigma_0. \tag{27}$$

So, collecting the above formulas (23), (24) together with (26), and (27) we obtain that the Willmore functional presented as in (22) becomes

$$I_\epsilon(S_p^\rho)=16\pi+2\epsilon\int_{S^2}\Big[h\big(\bar{\theta_1},\bar{\theta_1}\big)+h(\bar{\theta_2},\bar{\theta_2})-2h(\nu_0,\nu_0)-\rho(\bar{\theta}_1^\mu\bar{\theta}_1^\nu+\bar{\theta}_2^\mu\bar{\theta}_2^\nu)\nu_0^\lambda A_{\mu\nu\lambda}\Big]d\Sigma_0+o(\epsilon).$$

Since h and its derivatives are bounded on compact sets, fixed the compact $Z_c\subseteq\mathbb{R}^3\oplus\mathbb{R}^+$, the remainder $o(\epsilon)$ is uniform for $(p,\rho)\in Z_c$. ■

Writing the vectors $\bar{\theta_1},\bar{\theta_2},\nu_0$ in terms of the affine functions on S^2 we obtain a more synthetic expansion for I_ϵ:

Lemma 3.5. *The Willmore functional I_ϵ relative to the metric $g_\epsilon=\delta+\epsilon h$ and evaluated on the standard spheres S_p^ρ has the following expansion*

$$I_\epsilon(S_p^\rho)=16\pi+2\epsilon\int_{S^2}\Big[Trh-3h_{\mu\nu}\Theta^\mu\Theta^\nu+\rho A_{\mu\mu\lambda}\Theta^\lambda-\rho A_{\mu\nu\lambda}\Theta^\mu\Theta^\nu\Theta^\lambda\Big]d\Sigma_0+o(\epsilon), \tag{28}$$

where Θ^μ are the Cartesian coordinates of $\Theta\in S^2$ and the remainder $o(\epsilon)$ is uniform on compact subsets of $\mathbb{R}^3\oplus\mathbb{R}^+\ni(p,\rho)$. Of course, in the integral, h is evaluated at the points of S_p^ρ.

Proof. Let us rewrite the integrands in terms of the Cartesian coordinates Θ^λ. Observe that

$$\begin{aligned}\bar{\theta_1}&=\Bigg(\frac{\Theta^1\Theta^3}{\sqrt{(\Theta^1)^2+(\Theta^2)^2}},\frac{\Theta^2\Theta^3}{\sqrt{(\Theta^1)^2+(\Theta^2)^2}},-\sqrt{(\Theta^1)^2+(\Theta^2)^2}\Bigg)\\ \bar{\theta_2}&=\Bigg(-\frac{\Theta^2}{\sqrt{(\Theta^1)^2+(\Theta^2)^2}},\frac{\Theta^1}{\sqrt{(\Theta^1)^2+(\Theta^2)^2}},0\Bigg)\\ \nu_0&=-\Theta\end{aligned}$$

where $\Theta^\lambda,\lambda=1\dots3$ are the Cartesian coordinates of $\Theta\in S^2$.
Using the above formulas and observing that $(\Theta^3)^2=1-(\Theta^1)^2-(\Theta^2)^2$, the first two integrands of (15) can be rewritten as

$$h\big(\bar{\theta_1},\bar{\theta_1}\big)+h(\bar{\theta_2},\bar{\theta_2})=\mathrm{Tr}h-h_{\mu\nu}\Theta^\mu\Theta^\nu,$$

where $\mathrm{Tr}h = \sum_\mu h_{\mu\mu}$, as always the indexes run in $1, \dots, 3$ and repeated indexes are added. Noticing that $h(\nu_0, \nu_0) = h_{\mu\nu}\Theta^\mu\Theta^\nu$, the first part of the integrand becomes

$$h\big(\bar\theta_1, \bar\theta_1\big) + h(\bar\theta_2, \bar\theta_2) - 2h(\nu_0, \nu_0) = \mathrm{Tr}h - 3h_{\mu\nu}\Theta^\mu\Theta^\nu. \tag{29}$$

The second part, analogously becomes

$$-\rho(\bar\theta_1^\mu\bar\theta_1^\nu + \bar\theta_2^\mu\bar\theta_2^\nu)\nu_0^\lambda A_{\mu\nu\lambda} = \big(\rho\Theta^\lambda\sum_\mu A_{\mu\mu\lambda}\big) - \big(\rho A_{\mu\nu\lambda}\Theta^\mu\Theta^\nu\Theta^\lambda\big). \tag{30}$$

Collecting the above formulas (29) and (30), we get the desired expression of I_ϵ:

$$I_\epsilon(S_p^\rho) = 16\pi + 2\epsilon\int_{S^2}\Big[\mathrm{Tr}h - 3h_{\mu\nu}\Theta^\mu\Theta^\nu + \rho A_{\mu\mu\lambda}\Theta^\lambda - \rho A_{\mu\nu\lambda}\Theta^\mu\Theta^\nu\Theta^\lambda\Big]d\Sigma_0 + o(\epsilon). \tag{31}$$

■

Now we want to obtain an expansion of $I_\epsilon(S_p^\rho)$ for spheres of small radius. Observe that $\mathrm{Tr}h$, $h_{\mu\nu}$ and $A_{\mu\nu\lambda}$ are real functions on $\mathbb{R}^3$ and, in the integral (28), they are evaluated at the points $p + \rho\Theta$ of S_p^ρ. By a Taylor expansion we get

$$\begin{aligned}
h_{\mu\nu}(p+\rho\Theta) &= h_{\mu\nu}[p] + \rho(D_\lambda h_{\mu\nu})[p]\Theta^\lambda + \frac{1}{2}\rho^2(D^2_{\lambda\eta}h_{\mu\nu})[p]\Theta^\lambda\Theta^\eta + o(\rho^2)\\
\mathrm{Tr}h(p+\rho\Theta) &= \mathrm{Tr}h[p] + \rho(D_\lambda\mathrm{Tr}h)[p]\Theta^\lambda + \frac{1}{2}\rho^2(D^2_{\lambda\eta}\mathrm{Tr}h)[p]\Theta^\lambda\Theta^\eta + o(\rho^2)\\
\rho A_{\mu\nu\lambda} &= \rho A_{\mu\nu\lambda}[p] + \rho^2(D_\eta A_{\mu\nu\lambda})[p]\Theta^\eta + o(\rho^2),
\end{aligned}$$

where, as always, D_λ denotes the partial derivative in $\mathbb{R}^3$ with respect to x^λ. Thanks to this expansions we show the following

Lemma 3.6. *For spheres of small radius, the Willmore functional I_ϵ has the following expansion in ϵ and ρ:*

$$I_\epsilon(S_p^\rho) = 16\pi - \frac{8\pi}{3}R_1(p)\rho^2\epsilon + o(\rho^2)\epsilon + o(\epsilon),$$

where

$$R_1 := \sum_{\mu\nu} D_{\mu\nu}h_{\mu\nu} - \triangle\, Trh \tag{32}$$

and where the remainder $o(\epsilon)$ is uniform on compact subsets of $\mathbb{R}^3 \oplus \mathbb{R}^+ \ni (p, \rho)$.

Proof. Using the previous Taylor expansions we get

$$\begin{aligned}
I_\epsilon(S_p^\rho) = {} & 16\pi + 2\epsilon\int_{S^2}\Big[\mathrm{Tr}h[p] - 3h_{\mu\nu}[p]\Theta^\mu\Theta^\nu\Big]d\Sigma_0\\
&+2\rho\epsilon\int_{S^2}\Big[(D_\lambda\mathrm{Tr}h)[p]\Theta^\lambda - 3(D_\lambda h_{\mu\nu})[p]\Theta^\lambda\Theta^\mu\Theta^\nu + A_{\mu\mu\lambda}[p]\Theta^\lambda - A_{\mu\nu\lambda}[p]\Theta^\mu\Theta^\nu\Theta^\lambda\Big]d\Sigma_0\\
&+\rho^2\epsilon\int_{S^2}\Big[(D_{\lambda\eta}\mathrm{Tr}h)[p]\Theta^\lambda\Theta^\eta - 3(D_{\lambda\eta}h_{\mu\nu})[p]\Theta^\lambda\Theta^\eta\Theta^\mu\Theta^\nu\Big]d\Sigma_0\\
&+2\rho^2\epsilon\int_{S^2}\Big[(D_\eta A_{\mu\mu\lambda})[p]\Theta^\eta\Theta^\lambda - (D_\eta A_{\mu\nu\lambda})[p]\Theta^\eta\Theta^\mu\Theta^\nu\Theta^\lambda\Big]d\Sigma_0\\
&+o(\rho^2)\epsilon + o(\epsilon).
\end{aligned}$$

From the formulas of Brendle (see [Bren] pag. 28):

$$\begin{aligned}\int_{S^2} \Theta^\mu d\Sigma_0 &= 0 \\ \int_{S^2} \Theta^\mu \Theta^\nu d\Sigma_0 &= \frac{4\pi}{3}\delta^{\mu\nu} \\ \int_{S^2} \Theta^\mu \Theta^\nu \Theta^\lambda d\Sigma_0 &= 0 \\ \int_{S^2} \Theta^\mu \Theta^\nu \Theta^\lambda \Theta^\eta d\Sigma_0 &= \frac{4\pi}{15}(\delta^{\mu\nu}\delta^{\lambda\eta} + \delta^{\mu\lambda}\delta^{\nu\eta} + \delta^{\mu\eta}\delta^{\nu\lambda}),\end{aligned}$$

we obtain

$$\begin{aligned}&\int_{S^2} \Big[(\mathrm{Tr}h[p] - 3h_{\mu\nu}[p]\Theta^\mu\Theta^\nu\Big] d\Sigma_0 = 4\pi(\mathrm{Tr}h)[p] - 3h_{\mu\nu}[p]\int_{S^2}\Theta^\mu\Theta^\nu d\Sigma_0 = 0 \\ &\int_{S^2}\Big[(D_\lambda \mathrm{Tr}h)[p]\Theta^\lambda - 3D_\lambda(h_{\mu\nu})[p]\Theta^\lambda\Theta^\mu\Theta^\nu\Big] d\Sigma_0 = 0 \\ &\int_{S^2}\Big[A_{\mu\mu\lambda}[p]\Theta^\lambda - A_{\mu\nu\lambda}[p]\Theta^\mu\Theta^\nu\Theta^\lambda\Big] d\Sigma_0 = 0\end{aligned}$$

and

$$\int_{S^2}\Big[(D_{\lambda\eta}\mathrm{Tr}h)[p]\Theta^\lambda\Theta^\eta - 3(D_{\lambda\eta}h_{\mu\nu})[p]\Theta^\lambda\Theta^\eta\Theta^\mu\Theta^\nu\Big] d\Sigma_0 = \frac{4\pi}{3}(\triangle \mathrm{Tr}h)[p] - \frac{4\pi}{5}(D_{\lambda\eta}h_{\mu\nu})[p](\delta^{\mu\nu}\delta^{\lambda\eta} + \delta^{\mu\lambda}\delta^{\nu\eta} + \delta^{\mu\eta}\delta^{\nu\lambda}).$$

Observing that

$$\begin{aligned}(D_{\lambda\eta}h_{\mu\nu})[p](\delta^{\mu\nu}\delta^{\lambda\eta}) &= (\triangle \mathrm{Tr}h)[p] \\ (D_{\lambda\eta}h_{\mu\nu})[p](\delta^{\mu\lambda}\delta^{\nu\eta} + \delta^{\mu\eta}\delta^{\nu\lambda}) &= 2\sum_{\mu\nu}(D_{\mu\nu}h_{\mu\nu})[p],\end{aligned}$$

we can write

$$\int_{S^2} D_{\lambda\eta}(\mathrm{Tr}h)[p]\Theta^\lambda\Theta^\eta - 3(D_{\lambda\eta}h_{\mu\nu})[p]\Theta^\lambda\Theta^\eta\Theta^\mu\Theta^\nu d\Sigma_0 = \frac{8\pi}{15}(\triangle \mathrm{Tr}h)[p] - \frac{8\pi}{5}\sum_{\mu\nu}(D_{\mu\nu}h_{\mu\nu})[p].$$

We still have to study the last integral:

$$2\int_{S^2}\Big[(D_\eta A_{\mu\mu\lambda})[p]\Theta^\eta\Theta^\lambda - (D_\eta A_{\mu\nu\lambda})[p]\Theta^\eta\Theta^\mu\Theta^\nu\Theta^\lambda\Big] d\Sigma_0 = \frac{8\pi}{3}(D_\lambda A_{\mu\mu\lambda})[p] - \frac{8\pi}{15}(D_\eta A_{\mu\nu\lambda})[p](\delta^{\eta\mu}\delta^{\nu\lambda} + \delta^{\eta\nu}\delta^{\mu\lambda} + \delta^{\eta\lambda}\delta^{\mu\nu}).$$

Recalling that $A_{\mu\nu\lambda} := [D_\mu h_{\nu\lambda} + D_\lambda h_{\mu\nu} - D_\nu h_{\lambda\mu}]$, we have

$$\begin{aligned}\frac{8\pi}{3}D_\lambda(A_{\mu\mu\lambda})[p] &= \frac{8\pi}{3}[D^2_{\lambda\mu}h_{\nu\lambda} + D^2_{\lambda\lambda}h_{\mu\mu} - D^2_{\lambda\mu}h_{\lambda\mu}] = \frac{8\pi}{3}D^2_{\lambda\lambda}h_{\mu\mu} \\ &= \frac{8\pi}{3}\triangle \mathrm{Tr}h(p)\end{aligned}$$

and, with the same trick,

$$-\frac{8\pi}{15}D_\eta(A_{\mu\nu\lambda})[p](\delta^{\eta\mu}\delta^{\nu\lambda} + \delta^{\eta\nu}\delta^{\mu\lambda} + \delta^{\eta\lambda}\delta^{\mu\nu}) = -\frac{8\pi}{15}\triangle \mathrm{Tr}h - \frac{16\pi}{15}\sum_{\mu\nu}D^2_{\mu\nu}h_{\mu\nu}.$$

Collecting the previous formulas, we finally obtain

$$I_\epsilon(S^\rho_p) = 16\pi - \frac{8\pi}{3}\rho^2\epsilon\Big[\sum_{\mu\nu}(D^2_{\mu\nu}h_{\mu\nu})[p] - \triangle \mathrm{Tr}h)[p]\Big] + o(\rho^2)\epsilon + o(\epsilon).$$

■

Remark 3.7. *If $R_{g_\epsilon}(p)$ is the scalar curvature of $g_\epsilon = \delta + \epsilon h$ at p, then one has*

$$R_{g_\epsilon}(p) = \epsilon R_1(p) + o(\epsilon).$$

For the proof see [AM] pag. 80.
It follows that the first non constant term in the expansion of the Willmore functional on standard spheres of small radius is the scalar curvature of g_ϵ. This fact is consistent with the expansion (7) *of the functional on geodesic spheres of small radius.*

4 The finite dimensional reduction

In this section we want to prove that the perturbative method described in Section 2.1 can be applied to our problems of existence and non-existence of critical points: we will take Theorem 2.1 as model and we will prove it for our special cases. In the first subsection it is studied the case of perturbed standard spheres $S^\rho_p(w)$ immersed in $(\mathbb{R}^3, g_\epsilon)$ and in the second subsection the case of perturbed geodesic spheres $S_{p,\rho}(w)$ of small radius immersed in the Riemannian manifold (M, g).

4.1 The finite dimensional reduction for $I_\epsilon(S^\rho_p(w))$

Recall that I_ϵ is the Willmore functional with ambient manifold $\mathbb{R}^3$ endowed with the metric $g_\epsilon = \delta + \epsilon h$, so we explicitly observe that I_0 is the Willmore functional in the euclidean space $\mathbb{R}^3$. Thanks to (1), the standard spheres S^ρ_p are critical points of I_0 (more, they are the points of global minimum); hence I_0 possesses a finite dimensional critical manifold $Z \cong \mathbb{R}^3 \oplus \mathbb{R}^+$, in the sense that we identify S^ρ_p with $(p, \rho) \in \mathbb{R}^3 \oplus \mathbb{R}^+$.

NOTATION: In this Subsection we denote
 · P the orthogonal projection

$$P : L^2(S^2) \to Ker[\triangle_{S^2}(\triangle_{S^2} + 2)].$$

Of course it is defined because $Ker[\triangle_{S^2}(\triangle_{S^2} + 2)]$ is finite dimensional, hence closed.
 · $B(0, r)$ the ball of center 0 and radius r in $C^{4,\alpha}(S^2)^\perp$.

Our goal is to find p, ρ and w such that $I'_\epsilon(S^\rho_p(w)) = 0$. Of course such a sphere satisfies the equation (called *auxiliary equation*)

$$PI'_\epsilon(S^\rho_p(w)) = 0.$$

The next Lemma ensures the existence of solutions for the auxiliary equation.

Lemma 4.1. *For each compact subset $Z_c \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$, there exist $\epsilon_0 > 0$ and $r > 0$ with the following property: for all $|\epsilon| \leq \epsilon_0$ and $(p, \rho) \in Z_c$, the auxiliary equation $PI'_\epsilon(S^\rho_p(w)) = 0$ has unique solution $w = w_\epsilon(p, \rho) \in B(0, r) \subseteq C^{4,\alpha}(S^2)^\perp$ such that:*
1) the map $w_\epsilon(.,.) : Z_c \to C^{4,\alpha}(S^2)^\perp$ is of class C^1
2) $\|w_\epsilon(p, \rho)\|_{C^{4,\alpha}(S^2)} \to 0$ for $\epsilon \to 0$ uniformly with respect to $(p, \rho) \in Z_c$;
3) more precisely $\|w_\epsilon(p, \rho)\|_{C^{4,\alpha}(S^2)} = O(\epsilon)$ for $\epsilon \to 0$ uniformly in $(p, \rho) \in Z_c$.

Proof. Using the results in [PV], we wrote the differential of the Willmore functional as in equation (12). If the ambient manifold is $(\mathbb{R}^3, g_\epsilon)$, with an expansion on ϵ one has

$$I'_\epsilon(S^\rho_p(w)) = I'_0(S^\rho_p(w)) + \epsilon G(\epsilon, S^\rho_p(w))$$

where G is a function on $S^\rho_p(w)$ (so via the immersion it is a function on S^2) bounded as $\epsilon \to 0$; observe that G depends on $(\epsilon, p, \rho, w, Dw, D^2w, D^3w, D^4w, \Theta)$ where D^iw denotes the collection of the derivatives

of order i of w with respect to the coordinate vector fields on S^2.
Letting $R_{p,\rho}(w) := I_0'(S_p^\rho(w)) - I_0''(S_p^\rho)[w]$, the auxiliary equation $PI_\epsilon'(S_p^\rho(w)) = 0$ becomes

$$PI_0''(S_p^\rho)[w] + PR_{p,\rho}(w) + \epsilon PG(\epsilon, S_p^\rho(w)) = 0. \tag{33}$$

From Remark 3.3 we have

$$I_0''(S_p^\rho)[w] = \frac{2}{\rho^3} \triangle_{S^2} (\triangle_{S^2} + 2)[w].$$

Observe that $P(I_0''|_{C^{4,\alpha}(S^2)^\perp}) = I_0''|_{C^{4,\alpha}(S^2)^\perp}$ and

$$I_0''|_{C^{4,\alpha}(S^2)^\perp} : C^{4,\alpha}(S^2)^\perp \to C^{0,\alpha}(S^2)^\perp$$

is invertible with continuous inverse.
In fact by Schauder estimates (see [Jost] pag. 264 e pag. 274), if L is an elliptic operator of second order on S^2, $f \in C^{k,\alpha}(S^2)$ and $u \in C^{k+2,\alpha}$ are such that

$$Lu = f,$$

hence there exists a constant C such that

$$\|u\|_{C^{k+2,\alpha}(S^2)} \leq C(\|f\|_{C^{k,\alpha}(S^2)} + \|u\|_{L^2(S^2)}).$$

In our case $u, f \in Ker[L]^\perp$, so the solution u is unique with $\|u\|_{L^2(S^2)} \leq C\|f\|_{L^2(S^2)}$; hence we estimate $\|u\|_{C^{k+2,\alpha}(S^2)}$ with $\|f\|_{C^{k,\alpha}(S^2)}$. Applying the reasoning twice to $\triangle_{S^2}(\triangle_{S^2} + 2)u = f$ with $u, f \in Ker[\triangle_{S^2}(\triangle_{S^2} + 2)]^\perp$ we arrive to the desired estimate:

$$\|u\|_{C^{4,\alpha}(S^2)} \leq C\|f\|_{C^{0,\alpha}(S^2)},$$

which means that $I_0''(S_p^\rho)^{-1}$ exists and is bounded.

Thanks to this observation, *the auxiliary equation* (33) *is equivalent to the fixed point problem*

$$w = F_{\epsilon,p,\rho}(w) := -I_0''(S_p^\rho)^{-1}[\epsilon PG(\epsilon, S_p^\rho(w)) + PR_{p,\rho}(w)]. \tag{34}$$

Once the compact $Z_c \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$ is fixed, we want to show the existence of $\epsilon_0 > 0$, $r > 0$ such that, for all $|\epsilon| \leq \epsilon_0$ and $(p, \rho) \in Z_c$, $F_{\epsilon,p,\rho}$ is a contraction of $B(0, r) \subset C^{4,\alpha}(S^2)^\perp$ in itself.

• Let us show the existence of ϵ and r small enough such that

$$F_{\epsilon,p,\rho} : B(0, r) \to B(0, r).$$

∘ First let us study the second summand of (34): $R_{p,\rho}(w) := I_0'(S_p^\rho(w)) - I_0''(S_p^\rho)[w]$. From Remark 3.3 we have

$$R_{p,\rho}(w) = I_0'(S_p^\rho(w)) - I_0''(S_p^\rho)[w] = \frac{1}{\rho^3} Q_p^{(2)(4)}(w). \tag{35}$$

The estimate (5), tell us that for all $w_1, w_2 \in B(0, 1) \subset C^{4,\alpha}(S^2)^\perp$

$$\|R_{p,\rho}(w_1) - R_{p,\rho}(w_2)\|_{C^{0,\alpha}(S^2)} \leq C(\|w_1\|_{C^{4,\alpha}(S^2)} + \|w_2\|_{C^{4,\alpha}(S^2)})\|w_1 - w_2\|_{C^{4,\alpha}(S^2)}. \tag{36}$$

Where the constant C depends on the compact $Z_c \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$.

∘ The first summand is a little more delicate.
We want to show the existence of a constant C such that

$$\|w\|_{C^{4,\alpha}(S^2)} \leq 1 \quad \Rightarrow \quad \|G(\epsilon, S_p^\rho(w))\|_{C^{0,\alpha}(S^2)} \leq C. \tag{37}$$

As noticed above, $G(\epsilon, S_p^\rho(w))$ is a function on S^2 which depends on $(\epsilon, p, \rho, w, Dw, D^2w, D^3w, D^4w, \Theta)$. For $\|w\|_{C^{4,\alpha}(S^2)}$ small enough, the dependence on the first eight terms is smooth while the dependence on Θ is not only direct (w is function of Θ) and needs a little care; however the direct (or partial if we think in terms of partial derivative) dependence is smooth and the total dependence (in the sense of total derivative) is a least continuous. Once the compact $Z_c \ni (p,\rho)$ is fixed we can bound the derivatives of w if $\|w\|_{C^{4,\alpha}(S^2)} \leq 1$. In this way $G(\epsilon, S_p^\rho(w))$ is a continuous function on a compact set, so it is bounded:

$$\|G(\epsilon, S_p^\rho(w))\|_{C^0(S^2)} \leq C.$$

About the Hölder regularity, by the Lagrange mean value Theorem, we have for all $\Theta, \bar{\Theta} \in S^2$

$$\begin{aligned}
\frac{|G(\Theta)-G(\bar{\Theta})|}{|\Theta-\bar{\Theta}|^\alpha} &= \frac{|G\big(\epsilon,p,\rho,w(\Theta),Dw(\Theta),D^2w(\Theta),D^3w(\Theta),D^4w(\Theta),\Theta\big)}{|\Theta-\bar{\Theta}|^\alpha} \\
&\quad \frac{-G\big(\epsilon,p,\rho,w(\bar{\Theta}),Dw(\bar{\Theta}),D^2w(\bar{\Theta}),D^3w(\bar{\Theta}),D^4w(\bar{\Theta}),\bar{\Theta}\big)|}{|\Theta-\bar{\Theta}|^\alpha} \\
&= \frac{|(\nabla G)\cdot\big(0,0,0,w(\Theta)-w(\bar{\Theta}),Dw(\Theta)-Dw(\bar{\Theta}),D^2w(\Theta)}{|\Theta-\bar{\Theta}|^\alpha} \\
&\quad \frac{-D^2w(\bar{\Theta}),D^3w(\Theta)-D^3w(\bar{\Theta}),D^4w(\Theta)-D^4w(\bar{\Theta}),\Theta-\bar{\Theta}\big)|}{|\Theta-\bar{\Theta}|^\alpha} \\
&\leq C\|w\|_{C^{4,\alpha}(S^2)} + C\|\Theta-\bar{\Theta}\|^{1-\alpha} \\
&\leq C.
\end{aligned}$$

This completes the proof of the estimate (37).

Collecting the estimates (36) and (37), we have

$$\begin{aligned}
\|F_{\epsilon,p,\rho}(w)\|_{C^{4,\alpha}(S^2)} &= \|PI_0''(S_p^\rho)^{-1}[\epsilon PG(\epsilon,S_p^\rho(w)) + PR_{p,\rho}(w)]\|_{C^{4,\alpha}(S^2)} \\
&\leq \|PI_0''(S_p^\rho)^{-1}\| \quad \big(\|\epsilon PG(\epsilon,S_p^\rho(w))\|_{C^{0,\alpha}(S^2)} + \|PR_{p,\rho}(w)\|_{C^{0,\alpha}(S^2)}\big) \\
&\leq \|PI_0''(S_p^\rho)^{-1}\| \quad \big(C\epsilon + C\|w\|^2_{C^{4,\alpha}(S^2)}\big).
\end{aligned}$$

Chosen r and ϵ small enough it is clear that $F_{\epsilon,p,\rho} : B(0,r) \to B(0,r)$.

- Let us show that fixed Z_c, for $\epsilon > 0$ and $r > 0$ small enough,

$$F_{\epsilon,p,\rho} : B(0,r) \to B(0,r) \text{ is a contraction.}$$

$R_{p,\rho}(w)$ has already been estimated in the right way, let us estimate $G(\epsilon, S_p^\rho(w))$.
Again by the mean value Theorem

$$\begin{aligned}
\|G(\epsilon,S_p^\rho(w_1)) - G(\epsilon,S_p^\rho(w_2))\|_{C^0(S^2)} &\leq \|G\big(\epsilon,p,\rho,w_1(\Theta),Dw_1(\Theta),D^2w_1(\Theta),D^3w_1(\Theta),D^4w_1(\Theta),\Theta\big) \\
&\quad -G\big(\epsilon,p,\rho,w_2(\Theta),Dw_2(\Theta),D^2w_2(\Theta),D^3w_2(\Theta),D^4w_2(\Theta),\Theta\big)\|_{C^0(S^2)} \\
&\leq \|(\nabla G)\cdot\big(0,0,0,w_1(\Theta)-w_2(\Theta),Dw_1(\Theta)-Dw_2(\Theta),D^2w_1(\Theta) \\
&\quad -D^2w_2(\Theta),D^3w_1(\Theta)-D^3w_2(\Theta),D^4w_1(\Theta)-D^4w_2(\Theta),0\big)\|_{C^0(S^2)} \\
&\leq C\|w_1-w_2\|_{C^4(S^2)}.
\end{aligned}$$

In order to estimate the Hölder regularity we proceed in the same way; to simplify the notation let us call $G(\epsilon, w_1) := G(\epsilon, S_p^\rho(w_1))$ and D^α the collection of the derivatives up to order 4.

$$\begin{aligned}
& \frac{\|G(\epsilon,w_1)(\Theta) - G(\epsilon,w_2)(\Theta) - (G(\epsilon,w_1)(\bar\Theta) - G(\epsilon,w_2)(\bar\Theta))\|}{\|\Theta-\bar\Theta\|^\alpha} \\
= & \frac{\|\nabla G(\xi(\Theta))\cdot(D^\alpha(w_1-w_2)(\Theta)) - \nabla G(\xi(\bar\Theta))\cdot(D^\alpha(w_1-w_2)(\bar\Theta))\|}{\|\Theta-\bar\Theta\|^\alpha} \\
& \text{add } \pm\nabla G(\xi(\Theta))\cdot(D^\alpha(w_1-w_2)(\bar\Theta)) \\
\leq & \frac{\|\nabla G(\xi(\Theta))\cdot(D^\alpha(w_1-w_2)(\Theta) - D^\alpha(w_1-w_2)(\bar\Theta))\|}{\|\Theta-\bar\Theta\|^\alpha} \\
& + \frac{\|D^\alpha(w_1-w_2)(\bar\Theta)\cdot(\nabla G(\xi(\bar\Theta)) - \nabla G(\xi(\Theta)))\|}{\|\Theta-\bar\Theta\|^\alpha} \\
\leq & C\|w_1-w_2\|_{C^{4,\alpha}(S^2)} + \|\nabla G(\xi(\Theta))\|_{C^{0,\alpha}(S^2)}\|w_1-w_2\|_{C^4(S^2)} \\
\leq & C\|w_1-w_2\|_{C^{4,\alpha}(S^2)}.
\end{aligned}$$

Hence

$$\|G(\epsilon, S_p^\rho(w_1)) - G(\epsilon, S_p^\rho(w_2))\|_{C^{0,\alpha}(S^2)} \leq C\|w_1-w_2\|_{C^{4,\alpha}(S^2)}, \tag{38}$$

where C is a constant depending on $Z_c \subset \mathbb{R}^3 \oplus \mathbb{R}^+$ but not on ϵ.
Collecting (36) and (38) we obtain

$$\begin{aligned}
\|F_{\epsilon,p,\rho}[w_1] - F_{\epsilon,p,\rho}[w_2]\|_{C^{4,\alpha}} \leq\ & \|PI_0''(S_p^\rho)^{-1}\|\big(\epsilon\|PG(\epsilon,S_p^\rho(w_1)) - PG(\epsilon,S_p^\rho(w_2))\|_{C^{0,\alpha}(S^2)} \\
& \qquad + \|PR_{p,\rho}(w_1) - PR_{p,\rho}(w_2)\|_{C^{0,\alpha}(S^2)}\big) \\
\leq\ & \|PI_0''(S_p^\rho)^{-1}\|\big(C\epsilon\|w_1-w_2\|_{C^{4,\alpha}(S^2)} + \\
& \qquad C(\|w_1\|_{C^{4,\alpha}(S^2)} + \|w_2\|_{C^{4,\alpha}(S^2)})\|w_1-w_2\|_{C^{4,\alpha}(S^2)}\big) \\
\leq\ & \|PI_0''(S_p^\rho)^{-1}\|\big(C\epsilon\|w_1-w_2\|_{C^{4,\alpha}(S^2)} + 2rC\|w_1-w_2\|_{C^{4,\alpha}(S^2)}\big).
\end{aligned}$$

We can at least conclude that, fixed $Z_c \subset \mathbb{R}^3 \oplus \mathbb{R}^+ \ni (p,\rho)$, there exist ϵ_0 and r small enough such that

$$F_{\epsilon,p,\rho} : B(0,r) \to B(0,r)$$

is a contraction for all $|\epsilon| \leq \epsilon_0$ e $(p,\rho) \in Z_c$.

So for $|\epsilon| < \epsilon_0$, $\forall (p,\rho) \in Z_c$ there exists $w_\epsilon(p,\rho) \in C^{4,\alpha}(S^2)^\perp$ such that $w_\epsilon(p,\rho) = F_{\epsilon,p,\rho}(w_\epsilon(p,\rho))$, or equivalently which solves the auxiliary equation

$$PI_\epsilon'(S_p^\rho(w_\epsilon(p,\rho))) = 0.$$

• Regularity for $w_\epsilon(p,\rho)$: fixed $w \in B(0,r)$, the map

$$(\epsilon, p, \rho) \mapsto F_{\epsilon,p,\rho}(w)$$

is continuous in (ϵ,p,ρ), so the fixed point $w_\epsilon(p,\rho)$ is a continuous function in these parameters (for the Contraction Mapping Theorem depending on parameters look [Br] pag. 22, 23). For the C^1 regularity of w with respect to (p,ρ) see [AMN] pag. $447-449$.

• Behaviour of $w_\epsilon(p,\rho)$ when $\epsilon \to 0$:
For $\epsilon = 0$ the fixed point is $w_\epsilon(p,\rho) = 0$. By uniform continuity of $\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}(S^2)}$ on compact sets in the variables (ϵ,p,ρ) we have

$$\lim_{\epsilon\to 0}\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}(S^2)} = 0$$

uniformly in $(p,\rho) \in Z_c$.
Let us show that $w_\epsilon(p,\rho)$ not only tends to 0 but $w_\epsilon(p,\rho) = O(\epsilon)$ uniformly in $(p,\rho) \in Z_c$. From Remark 3.3, $R_{p,\rho}(w) = I_0'(S_p^\rho(w)) - I_0''(S_p^\rho)[w] = \frac{1}{\rho^3}Q^{(2)(4)}(w)$, so dividing by ϵ the auxiliary equation (33) we get

$$PI_0''(S_p^\rho)\left[\frac{w}{\epsilon}\right] + \frac{1}{\epsilon\rho^3}Q^{(2)(4)}(w) = -PG(\epsilon, S_p^\rho(w)). \tag{39}$$

We have already observed that the right hand side is bounded in $C^{0,\alpha}$ norm for $|\epsilon| \leq \epsilon_0$.
Now the left hand side: from $\lim_{\epsilon\to 0} \|w_\epsilon(p,\rho)\|_{C^{4,\alpha}} = 0$ uniformly in $(p,\rho) \in Z_c$ and from the estimate (see (5))

$$\|Q^{(2)(4)}(w)\|_{C^{0,\alpha}(S^2)} \leq C\|w\|^2_{C^{4,\alpha}},$$

it follows that the second summand is of higher order than the first one; so for $\epsilon \to 0$, we have that $PI_0''(S_p^\rho)[\frac{w}{\epsilon}]$ is bounded in $C^{0,\alpha}$ norm and, by Schauder estimates, $\frac{w}{\epsilon}$ is bounded in $C^{4,\alpha}$.
We can conclude that

$$\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}} = O(\epsilon)$$

uniformly in Z_c. ■

Thanks to this Lemma, fixed the compact subset $Z_c \subset \mathbb{R}^3 \times \mathbb{R}^+$, we can define the C^1 function $\Phi_\epsilon : Z_c \to \mathbb{R}$

$$\Phi_\epsilon(p,\rho) := I_\epsilon(S_p^\rho(w_\epsilon(p,\rho))). \tag{40}$$

Using Theorem 2.1 as model, let us prove the following

Lemma 4.2. *Fixed a compact set $Z_c \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$, for $|\epsilon| \leq \epsilon_0$ consider the functional $\Phi_\epsilon : Z_c \to \mathbb{R}$. Assume that, for ϵ small enough, Φ_ϵ has a critical point $(p_\epsilon, \rho_\epsilon) \in Z_c$. Then $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))$ is a critical point of I_ϵ.*

PROOF. Recall that the L^2-differential of the Willmore functional on the hypersurface $S_p^\rho(w)$ can be represented by the function $I_\epsilon'(S_p^\rho(w)) \in L^2(S^2)$ (just using the formula of [PV] and the parametrization of $S_p^\rho(w)$). Our goal is to show that $I_\epsilon'(S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))) = 0$. From Lemma 4.1 we already know that $PI_\epsilon'(S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))) = 0$. Let $q_i^\epsilon, i = 1,\ldots,4$ denote an orthonormal frame for $Ker[\triangle_{S^2}(\triangle_{S^2}+2)] \subset L^2(S^2)$ (which is the subspace of constant and affine functions on S^2). We can write

$$I_\epsilon'(S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))) = \sum_{i=1}^4 A_{i,\epsilon} q_i^\epsilon, \tag{41}$$

where

$$A_{i,\epsilon} = \big(I_\epsilon'(S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))), q_i^\epsilon\big)_{L^2(S^2)}. \tag{42}$$

We have to show that $A_{i,\epsilon} = 0$ for $1 \leq i \leq 4$.
By assumption, $(p_\epsilon,\rho_\epsilon)$ is a critical point of the map $(p,\rho) \mapsto \Phi_\epsilon(p,\rho) := I_\epsilon(S_p^\rho(w_\epsilon(p,\rho)))$. Varying the i^{th} coordinate, $S_p^\rho(w_\epsilon(p,\rho))$ describes a curve of immersed spheres in $\mathbb{R}^3$. The derivative $\partial_i S_p^\rho(w_\epsilon(p,\rho))|_{(p_\epsilon,\rho_\epsilon,w_\epsilon(p_\epsilon,\rho_\epsilon))}$ is a vector field in $\mathbb{R}^3$ along the surface $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))$. To compute the derivative of the Willmore functional I_ϵ we are only interested in the orthogonal (to $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))$) component of $\partial_i S_p^\rho(w_\epsilon(p,\rho))|_{(p_\epsilon,\rho_\epsilon,w_\epsilon(p_\epsilon,\rho_\epsilon))}$ (the tangential part gives only a reparametrization of the surface) which can be identified with a function $D_i[S_p^\rho(w_\epsilon(p,\rho))]|_{(p_\epsilon,\rho_\epsilon,w_\epsilon(p_\epsilon,\rho_\epsilon))}$ times the unit normal vector.
The parametrization of $S_p^\rho(w_\epsilon(p,\rho))$ is given by

$$p + \rho\big(1 - w(p,\rho)[\Theta]\big)\Theta = p + \rho\Theta - \rho w(p,\rho)[\Theta]\Theta = S_p^\rho(\Theta) - \rho w(p,\rho)[\Theta]\Theta.$$

From Lemma 4.1, $\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}(S^2)} = O(\epsilon)$ for $\epsilon \to 0$, so the normal vector to $S_p^\rho(w_\epsilon(p,\rho))$ is $\Theta + O(\epsilon)$.
It follows that

$$D_i[S_p^\rho(w_\epsilon(p,\rho))] = q_i^\epsilon - D_i(\rho w(p,\rho)) + O(\epsilon), \tag{43}$$

where $D_i(\rho w(p,\rho))$ denotes the derivative with respect to the i^{th} coordinate. Now from the fact that $(p_\epsilon,\rho_\epsilon)$ is a critical point of Φ_ϵ, we have

$$0 = \int_{S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))} \Big(I_\epsilon'(S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))) \quad D_i[S_p^\rho(w_\epsilon(p,\rho))]|_{(p_\epsilon,\rho_\epsilon,w_\epsilon(p_\epsilon,\rho_\epsilon))}\Big) d\Sigma_\epsilon.$$

From the fact that $\|w_\epsilon(p,\rho)\|_{C^{4,\alpha}(S^2)} = O(\epsilon)$,

$$= \rho_\epsilon^2 \Big(I'_\epsilon(S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))), D_i[S_p^\rho(w_\epsilon(p,\rho))]|_{(p_\epsilon,\rho_\epsilon,w_\epsilon(p_\epsilon,\rho_\epsilon))} \Big)_{L^2(S^2)} + O(\epsilon).$$

From (43), (42) and using (41)

$$= \rho_\epsilon^2 A_{i,\epsilon} - \rho_\epsilon^2 \sum_j A_{j,\epsilon} \big(q_j^\epsilon, D_i(\rho_\epsilon w(p_\epsilon,\rho_\epsilon))\big)_{L^2(S^2)} + O(\epsilon) \quad 1 \le i \le 4. \tag{44}$$

This is a 4×4 homogeneous linear system in $A_{i,\epsilon}$ with matrix $\rho_\epsilon^2\delta_{ij} - \rho_\epsilon^2 \big(q_j^\epsilon, D_i(\rho_\epsilon w(p_\epsilon,\rho_\epsilon))\big)_{L^2(S^2)} + O(\epsilon)$; in order to conclude $A_{i,\epsilon} = 0$ for $1 \le i \le 4$ it is sufficient to show that the matrix is nonsingular. From the orthogonality condition $\big(w_\epsilon(p_\epsilon,\rho_\epsilon), q_j^\epsilon\big)_{L^2(S^2)} = 0$, $j = 1 \dots 4$, differentiating we get

$$\big(D_i w_\epsilon(p_\epsilon,\rho_\epsilon), q_j^\epsilon\big)_{L^2(S^2)} + \big(w_\epsilon(p_\epsilon,\rho_\epsilon), D_i q_j^\epsilon\big)_{L^2(S^2)} = 0, \quad i,j = 1,\dots,4.$$

From $\lim_{\epsilon\to 0} \|w_\epsilon(p_\epsilon,\rho_\epsilon)\|_{C^{4,\alpha}(S^2)} = 0$; we can conclude

$$\lim_{\epsilon\to 0} \big(q_j^\epsilon, D_i(\rho_\epsilon w(p_\epsilon,\rho_\epsilon))\big)_{L^2(S^2)} = 0.$$

By the continuity of the determinant and observing that ρ_ϵ is bounded away from 0 (by assumption $(p_\epsilon,\rho_\epsilon) \in Z_c$ compact set of $\mathbb{R}^3 \times \mathbb{R}^+$) we get $\mathrm{Det}[\rho_\epsilon^2\delta_{ij} + \rho_\epsilon^2\big(q_j^\epsilon, D_i(\rho_\epsilon w(p_\epsilon,\rho_\epsilon))\big)_{L^2(S^2)} + O(\epsilon)] \neq 0$ for ϵ small enough; hence the thesis. ■

In order to find critical points of I_ϵ we are reduced to study stationary points of the reduced functional $\Phi_\epsilon(p,\rho)$.
Using the decay behaviour of w_ϵ as $\epsilon \to 0$, we can further simplify the problem:

Lemma 4.3. *Fixed the compact $Z_c \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$, the reduced functional Φ_ϵ can be expanded as*

$$\Phi_\epsilon(p,\rho) = I_\epsilon(S_p^\rho) + o(\epsilon) \tag{45}$$

where the remainder is $o(\epsilon)$ uniformly on the compact Z_c.

Proof. Let us write the functional as $I_\epsilon(S_p^\rho(w)) = I_0(S_p^\rho(w)) + G(\epsilon, S_p^\rho(w))$. From the formula (7) we observe that

$$I_0(S_p^\rho(w)) - I_0(S_p^\rho) = Q_p^{(2)(2)}(w),$$

hence for the estimate (5)

$$\|I_0(S_p^\rho(w)) - I_0(S_p^\rho)\|_{C^{0,\alpha}} \le C\|w\|^2_{C^{4,\alpha}}$$

and from 3) of Lemma 4.1

$$\|I_0(S_p^\rho(w)) - I_0(S_p^\rho)\|_{C^{0,\alpha}} = o(\epsilon)$$

uniformly on Z_c.

With similar computations to those of the proof of Lemma 4.1, it is possible to show that

$$\|G(\epsilon, S_p^\rho(w_\epsilon(p,\rho))) - G(\epsilon, S_p^\rho)\|_{C^{0,\alpha}} = o(\epsilon)$$

uniformly in Z_c.
Now we can expand $\Phi_\epsilon(p,\rho)$:

$$\begin{aligned}
\Phi_\epsilon(p,\rho) &= I_\epsilon(S_p^\rho(w_\epsilon(p,\rho))) \\
&= I_0(S_p^\rho(w_\epsilon(p,\rho))) + G(\epsilon, S_p^\rho(w_\epsilon(p,\rho))) \\
&= I_0(S_p^\rho) + G(\epsilon, S_p^\rho) + o(\epsilon) \\
&= I_\epsilon(S_p^\rho) + o(\epsilon).
\end{aligned}$$

■

The last Lemma suggests to study $I_\epsilon(S_p^\rho)$, that is the Willmore functional on standard spheres S_p^ρ relative to the perturbed metric $g_\epsilon = \delta + \epsilon h$. Exactly what we did in the previous Section (Lemma 3.5).

4.2 The finite dimensional reduction for $I(S_{p,\rho}(w))$

NOTATION. In this subsection, the functional space will be $C^{4,\alpha}(S^2)^\perp$: the perturbation w will be an element of $C^{4,\alpha}(S^2)^\perp$ and $B(0,r)$ will denote the ball of center 0 and radius r in $C^{4,\alpha}(S^2)^\perp$.

Lemma 4.4. *Fixed a compact subset $Z_c \subseteq M$, there exist $\rho_0 > 0$, $r > 0$ and a map $w_{(.,.)} : Z_c \times [0,\rho_0] \to C^{4,\alpha}(S^2)^\perp$, $(p,\rho) \mapsto w_{p,\rho}$ such that if $S_{p,\rho}(w)$ is a critical point of the Willmore functional I with $(p,\rho,w) \in Z_c \times [0,\rho_0] \times B(0,r)$ then $w = w_{p,\rho}$.*
Moreover the map $w_{(.,.)}$ satisfies the following properties:
· the map $(p,\rho) \mapsto w_{p,\rho}$ is C^1,
· $\|w_{p,\rho}\|_{C^{4,\alpha}(S^2)} = O(\rho^2)$ as $\rho \to 0$ uniformly for $p \in Z_c$,
· $\|\frac{\partial}{\partial\rho} w_{p,\rho}\|_{L^2(S^2)} = O(\rho)$ as $\rho \to 0$ uniformly for $p \in Z_c$.

PROOF. First fix a compact subset $Z_c \subseteq M$; the point p will be an element of Z_c. If

$$I'(S_{p,\rho}(w)) = 0 \quad \text{(equality in } L^2(S^2)),$$

setting $P : L^2(S^2) \to Ker[\triangle_{S^2}(\triangle_{S^2}+2)]^\perp$ the orthogonal projection, a fortiori we have

$$PI'(S_{p,\rho}(w)) = 0;$$

that is, from (11),

$$P\Big(\triangle_{S^2}(\triangle_{S^2}+2)w + O_p(\rho^2) + \rho^2 L_p^{(4)}(w) + Q_p^{(2)(4)}(w)\Big) = 0. \tag{46}$$

Since $\triangle_{S^2}(\triangle_{S^2}+2)$ is invertible on the space orthogonal to the Kernel and $w \in C^{4,\alpha}(S^2)^\perp := Ker[\triangle_{S^2}(\triangle_{S^2}+2)]^\perp \cap C^{4,\alpha}(S^2)$, setting

$$K := [\triangle_{S^2}(\triangle_{S^2}+2)]^{-1} : Ker[\triangle_{S^2}(\triangle_{S^2}+2)]^\perp \subseteq L^2(S^2) \to Ker[\triangle_{S^2}(\triangle_{S^2}+2)]^\perp,$$

the equation (46) is equivalent to the fixed point problem

$$w = K[O_p(\rho^2) + \rho^2 L_p^{(4)}(w) + Q_p^{(2)(4)}(w)] = F_{p,\rho}(w). \tag{47}$$

The projection in the right hand side is included.

We want to solve this fixed point problem using the Contraction Mapping Theorem.
By Schauder estimates,

$$K = [\triangle_{S^2}(\triangle_{S^2}+2)]^{-1} : C^{0,\alpha}(S^2)^\perp \to C^{4,\alpha}(S^2)^\perp$$

is a bounded linear operator (see the proof of Lemma 4.1).
Let us study the three summands of the right hand side separately.

- $O_p(\rho^2)$: its $C^{0,\alpha}(S^2)$ norm is easily controlled by a constant times ρ^2:

$$\|O_p(\rho^2)\|_{C^{0,\alpha}} \leq C_1\rho^2. \tag{48}$$

- $\rho^2 L_p^{(4)}(w)$ is a linear function of w and its derivatives up to 4° order with $C^\infty(S^2)$ coefficients. Taken $w_1, w_2 \in C^{4,\alpha}(S^2)^\perp$, by definition it satisfies the estimate

$$\|L_p^{(4)}(w_1) - L_p^{(4)}(w_2)\|_{C^{0,\alpha}} \leq C_2\|w_1 - w_2\|_{C^{4,\alpha}}. \tag{49}$$

- $Q_p^{(2)(4)}(w)$ is a function at least quadratic in w and its derivatives up to 4° order with coefficients in $C^\infty(S^2)$. Taken $w_1, w_2 \in C^{4,\alpha}(S^2)^\perp$, by definition

$$\|Q_p^{(2)(4)}(w_1) - Q_p^{(2)(4)}(w_2)\|_{C^{0,\alpha}} \leq C_3(\|w_1\|_{C^{4,\alpha}} + \|w_2\|_{C^{4,\alpha}})\|w_1 - w_2\|_{C^{4,\alpha}} \tag{50}$$

when $||w_i||_{C^4(S^2)} \leq 1$, $i=1,2$.

Now we can show that $F_{p,\rho}$ of the equation (47) is a contraction on a ball $B(0,r) \subset C^{4,\alpha}(S^2)^\perp$ small enough.
Thanks to the continuity of $K : C^{0,\alpha}(S^2)^\perp \to C^{4,\alpha}(S^2)^\perp$ and the estimates (48),(49),(50), taken ρ and r small enough, for $\|w\|_{C^{4,\alpha}} \leq r$ we have $\|F_{p,\rho}(w)\|_{C^{4,\alpha}} \leq r$, that is

$$F_{p,\rho} : B(r) \subset C^{4,\alpha\perp} \to B(r).$$

Moreover

$$\begin{aligned}
\|F_{p,\rho}(w_1) - F_{p,\rho}(w_2)\|_{C^{4,\alpha}} &= \|K\| [\rho^2 (L_p^{(4)}(w_1) - L_p^{(4)}(w_2)) + (Q_p^{(2)(4)}(w_1) - Q_p^{(2)(4)}(w_2))]\|_{C^{0,\alpha}} \\
&\leq \|K\| \big(\rho^2 \|L_p^{(4)}(w_1) - L_p^{(4)}(w_2)\|_{C^{0,\alpha}} + \|Q_p^{(2)(4)}(w_1) - Q_p^{(2)(4)}(w_2)\|_{C^{0,\alpha}}\big) \\
&\leq \|K\| \big(\rho^2 C_2 \|w_1 - w_2\|_{C^{4,\alpha}} + C_3(\|w_1\|_{C^{4,\alpha}} + \|w_2\|_{C^{4,\alpha}})\|w_1 - w_2\|_{C^{4,\alpha}}\big).
\end{aligned}$$

Hence for $p \in Z_c$, ρ and r small enough $F_{p,\rho}$ is a contraction and there exists a unique fixed point $w_{p,\rho} \in B(0,r)$. It is equivalent to say that there exist $\rho_0 > 0$ and $r > 0$ such that

$$PI'(S_{p,\rho}(w)) = 0$$

has $w = w_{p,\rho}$ as unique solution in $B(0,r)$. This proves the first part of the thesis.

· The argument about the regularity is exactly the same as in Lemma 4.1 so we are left to study the behaviour of $w_{p,\rho}$ for $\rho \to 0$.
For $\rho = 0$ the fixed point equation is $w = K[Q_p^{(2)(4)}(w)]$ which has unique (in $B(0,r)$) solution $w_{p,0} = 0$. Since w is continuous in (p,ρ),

$$\lim_{\rho \to 0} w_{p,\rho} = 0 \tag{51}$$

uniformly for $p \in Z_c$.

· Let us show that $w_{p,\rho} = O(\rho^2)$ for $\rho \to 0$.
From the equation (46), omitting the projection P, we have

$$\triangle_{S^2}(\triangle_{S^2} + 2)w + Q_p^{(2)(4)}(w) = O_p(\rho^2) + \rho^2 L_p^{(4)}(w).$$

From (51) and thanks to the estimate (50), the quadratic term is of higher order as $\rho \to 0$ so it can be neglected.
Applying $K = \big(\triangle_{S^2}(\triangle_{S^2} + 2)\big)^{-1}$ to both sides and passing to the lim sup of the norms we get

$$\limsup_{\rho \to 0} \|w\|_{C^{4,\alpha}} \leq \|K\| \limsup_{\rho \to 0} \|O_p(\rho^2) + \rho^2 L_p^{(4)}(w)\|_{C^{0,\alpha}}.$$

So dividing both sides by ρ^2

$$\begin{aligned}
\limsup_{\rho \to 0} \frac{\|w\|_{C^{4,\alpha}}}{\rho^2} &\leq \|K\| \limsup_{\rho \to 0} \|O_p(\rho^0) + L_p^{(4)}(w)\|_{C^{0,\alpha}} \\
&\leq C.
\end{aligned}$$

Since the estimate is uniform in Z_c, we can conclude that there exists a constant C such that for all $p \in Z_c$ and for ρ small enough

$$\|w_{p,\rho}\|_{C^{4,\alpha}} \leq C\rho^2. \tag{52}$$

· Let us show that $\|\frac{\partial}{\partial \rho} w_{p,\rho}\|_{L^2(S^2)} = O(\rho)$ as $\rho \to 0$ uniformly in Z_c.
By construction, $w_{p,\rho}$ solves the auxiliary equation $PI'(S_{p,\rho}(w_{p,\rho})) = 0$ which can be written, using (11), as

$$P\Big[\frac{2}{\rho^3}\triangle_{S^2}(\triangle_{S^2} + 2)w_{p,\rho} + O_p\Big(\frac{1}{\rho}\Big) + \frac{1}{\rho}L_p^{(4)}(w_{p,\rho}) + \frac{1}{\rho^3}Q_p^{(2)(4)}(w_{p,\rho})\Big] = 0.$$

In the following we leave unsaid the projection P. Each summand of the left hand side is differentiable in $L^2(S^2)$ with respect to ρ; differentiating we get

$$\frac{2}{\rho^3}\triangle_{S^2}(\triangle_{S^2}+2)\Big[\frac{\partial}{\partial\rho}w_{p,\rho}\Big]+\frac{1}{\rho}L_p^{(4)}\Big(\frac{\partial}{\partial\rho}w_{p,\rho}\Big)+\frac{1}{\rho^3}L_p^{(4)}(w_{p,\rho})L_p^{(4)}\Big(\frac{\partial}{\partial\rho}w_{p,\rho}\Big)=$$

$$=\frac{6}{\rho^4}\triangle_{S^2}(\triangle_{S^2}+2)w+\frac{1}{\rho^2}L_p^{(4)}(w_{p,\rho})+\frac{1}{\rho}\Big(\frac{\partial}{\partial\rho}L_p^{(4)}\Big)(w_{p,\rho})+\frac{3}{\rho^4}Q_p^{(2)(4)}(w_{p,\rho})+\frac{1}{\rho^3}\Big(\frac{\partial}{\partial\rho}Q_p^{(2)(4)}\Big)(w_{p,\rho})+O_p\Big(\frac{1}{\rho^2}\Big).$$

Observe that $\frac{\partial}{\partial\rho}L_p^{(4)}$ and $\frac{\partial}{\partial\rho}Q_p^{(2)(4)}$ are still functions of the same kind. So, remembering that $\|w_{p,\rho}\|_{C^{4,\alpha}(S^2)}=O_p(\rho^2)$ and multiplying both sides by ρ^3, we get

$$2\triangle_{S^2}(\triangle_{S^2}+2)\Big[\frac{\partial}{\partial\rho}w_{p,\rho}\Big]+\rho^2L_p^{(4)}\Big(\frac{\partial}{\partial\rho}w_{p,\rho}\Big)+L_p^{(4)}(w_{p,\rho})L_p^{(4)}\Big(\frac{\partial}{\partial\rho}w_{p,\rho}\Big)=O_p(\rho).$$

Observe that the second and third summand are of higher order as $\rho\to 0$. From the fact that $O_p(\rho)$ is an $O(\rho)$ uniformly on the compact subset Z_c and using the continuity of $[\triangle_{S^2}(\triangle_{S^2}+2)]^{-1}$, we obtain the claim. ■

This Lemma is the key tool to show the non existence result: it implies that fixed a compact $Z_c\subset M$, for small ρ we can consider the function $\Phi(p,\rho):=I(S_{p,\rho}(w_{p,\rho}))$. Moreover if -for $\bar p\in Z_c$ and $\bar\rho, w$ small enough- $S_{\bar p,\bar\rho}(w)$ is a critical point of I, then $w=w_{\bar p,\bar\rho}$ and a fortiori $(\bar p,\bar\rho)$ is a critical point of the constrained functional Φ. Hence it will be enough to prove that Φ has no critical points.

5 Existence, multiplicity and nonexistence of critical points

5.1 Existence of critical points in $(\mathbb{R}^3, g_\epsilon)$

In this Subsection, using the expansions computed in Subsection 3.2, we want to apply Lemma 4.2 in order to find critical points of the Willmore functional with ambient manifold $(\mathbb{R}^3, g_\epsilon)$.

Proof of Theorem 1.1.
For simplicity let assume $R_1(\bar p)>0$ and $\epsilon>0$, the other cases are analogous. From Lemma 3.6 we have the following expansion for small radius spheres:

$$|I_\epsilon(S_p^\rho)-16\pi+\frac{8\pi}{3}R_1(p)\rho^2\epsilon|=|o(\rho^2)\epsilon+o(\epsilon)|.$$

Choose $\bar\rho>0$ and $\epsilon>0$ small enough such that

$$a)\quad |I_\epsilon(S_{\bar p}^{\bar\rho})-16\pi+\frac{8\pi}{3}R_1(\bar p)\bar\rho^2\epsilon|<\frac{1}{3}\pi R_1(\bar p)\bar\rho^2\epsilon.$$

In the sequel $\bar\rho$ is fixed, while ϵ may be chosen smaller. We want to find a compact $K\subset\mathbb{R}^3\oplus\mathbb{R}^+$ such that the point $(\bar p,\bar\rho)$ is in the interior of K, and on the boundary ∂K we have

$$b)\quad \sup_{(p,\rho)\in\partial K}|I_\epsilon(S_p^\rho)-16\pi|<\frac{2}{3}\pi R_1(\bar p)\bar\rho^2\epsilon$$

for ϵ small enough. From the expansion (15), we get

$$|I_\epsilon(S_p^\rho)-16\pi|\le 2\epsilon\Big|\int_{S^2}\Big[h(\bar\theta_1,\bar\theta_1)+h(\bar\theta_2,\bar\theta_2)-2h(\nu_0,\nu_0)-\rho(\bar\theta_1^\mu\bar\theta_1^\nu+\bar\theta_2^\mu\bar\theta_2^\nu)\nu_0^\lambda A_{\mu\nu\lambda}\Big]d\Sigma_0\Big|+o(\epsilon).$$

Setting

$$\|A(p)\|=\sup_{X,Y,Z\in\mathbb{R}^3,1=|X|=|Y|=|Z|}|A_{\mu\nu\lambda}(p)X^\mu Y^\nu Z^\lambda|,$$

the inequality can be rewritten as

$$|I_\epsilon(S^\rho_p) - 16\pi| \leq 8\epsilon \int_{S^2} \|h(p+\rho\Theta)\| d\Sigma_0 + 4\epsilon \int_{S^2} \rho \|A(p+\rho\Theta)\| \; d\Sigma_0 + o(\epsilon).$$

We want to find ρ large enough such that for all $p \in \mathbb{R}^3$,

$$\begin{array}{ll} b_1) & 8\epsilon \int_{S^2} \|h(p+\rho\Theta)\| d\Sigma_0 < \frac{1}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon \quad \text{and} \\ b_2) & 4\epsilon \int_{S^2} \rho\|A(p+\rho\Theta)\| \; d\Sigma_0 < \frac{1}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon. \end{array}$$

Thanks to the decay assumption on h, for all $\delta, \rho > 0$ there exists $r_\delta(\rho)$ such that

$$\begin{aligned} \mathrm{supp}_\delta \|h\| &:= \{p \in \mathbb{R}^3 : \|h(p)\| \geq \delta\} \subseteq B_{0,r_\delta(\rho)} \quad \text{and} \\ \mathrm{supp}_{\frac{\delta}{\rho}} \|A\| &:= \Big\{p \in \mathbb{R}^3 : \|A(p)\| \geq \frac{\delta}{\rho}\Big\} \subseteq B_{0,r_\delta(\rho)}. \end{aligned}$$

Moreover it is easy to show that there exist $c_1(\delta)$, $c_2(\delta) > 0$ such that

$$r_\delta(\rho) \leq c_1(\delta) + c_2(\delta)\rho^{1/\alpha}, \quad \alpha > 2. \tag{53}$$

Now we show that $b_1)$ and $b_2)$ are satisfied for ρ large enough. Taken a sphere S^ρ_p consider the solid angle subtended by the intersection with $B_{0,r_\delta(\rho)}$:

$$\Omega^\rho_p := \{\Theta \in S^2 : p + \rho\Theta \in S^\rho_p \cap B_{0,r_\delta(\rho)}\}.$$

Setting $|\Omega^\rho_p|$ the measure on S^2 of Ω^ρ_p, it is easy to see that

$$|\Omega^\rho_p| \leq 4\pi \Big(\frac{r_\delta(\rho)}{\rho}\Big)^2.$$

From (53) we get

$$|\Omega^\rho_p| \leq 4\pi \Big(\frac{c_1(\delta)^2}{\rho^2} + 2\frac{c_1(\delta)c_2(\delta)\rho^{1/\alpha}}{\rho^2} + \frac{c_2(\delta)^2\rho^{2/\alpha}}{\rho^2}\Big).$$

Observe that once δ is fixed, $\lim_{\rho\to\infty} \rho|\Omega^\rho_p| = 0$.
We can now get the estimates $b_1)$ and $b_2)$. Let us start from $b_1)$:

$$\begin{aligned} 8\epsilon \int_{S^2} \|h(p+\rho\Theta)\| d\Sigma_0 &= 8\epsilon \int_{\Omega^\rho_p} \|h(p+\rho\Theta)\| d\Sigma_0 + 8\epsilon \int_{S^2 \setminus \Omega^\rho_p} \|h(p+\rho\Theta)\| d\Sigma_0 \\ &< 8\epsilon |\Omega^\rho_p| \sup_{\mathbb{R}^3} \|h\| + 32\pi\epsilon\delta. \end{aligned} \tag{54}$$

Analogously the estimate $b_2)$ is

$$\begin{aligned} 4\epsilon \int_{S^2} \rho\|A(p+\rho\Theta)\| d\Sigma_0 &= 4\epsilon \int_{\Omega^\rho_p} \rho\|A(p+\rho\Theta)\| d\Sigma_0 + 4\epsilon \int_{S^2 \setminus \Omega^\rho_p} \rho\|A(p+\rho\Theta)\| d\Sigma_0 \\ &< 4\epsilon\rho |\Omega^\rho_p| \sup_{\mathbb{R}^3} \|A\| + 16\pi\epsilon\delta. \end{aligned} \tag{55}$$

Choose and fix δ such that

$$32\pi\epsilon\delta < \frac{1}{6}\pi R_1(\bar{p})\bar{\rho}^2\epsilon. \tag{56}$$

Since at δ fixed $\lim_{\rho\to\infty} \rho|\Omega^\rho_p| = 0$, for ρ large enough

$$\begin{cases} 8\epsilon|\Omega^\rho_p| \sup_{\mathbb{R}^3} \|h\| < \frac{1}{6}\pi R_1(\bar{p})\bar{\rho}^2\epsilon \quad \text{and} \\ 4\epsilon\rho|\Omega^\rho_p| \sup_{\mathbb{R}^3} \|A\| < \frac{1}{6}\pi R_1(\bar{p})\bar{\rho}^2\epsilon. \end{cases}$$

Collecting the estimates, we can conclude that for ρ large enough $b_1)$ and $b_2)$ are satisfied.
Now we want to fix the compact K. We search it of the form

$$K_{Rr} := \{(p,\rho) \in \mathbb{R}^3 \oplus \mathbb{R}^+ : \frac{1}{R} \leq \rho \leq R, |p| \leq r\}$$

that is a "cylinder" in $\mathbb{R}^3 \oplus \mathbb{R}^+$. For R large enough, thanks to $b_1)$ and $b_2)$, on the upper face of the cylinder ($\rho = R$) the condition $b)$ is verified:

$$|I_\epsilon(S_p^R) - 16\pi| < \frac{2}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon. \tag{57}$$

On the lower face ($\rho = 1/R$) we can use the expansion of Lemma 3.6 :

$$|I_\epsilon(S_p^{1/R}) - 16\pi| = \Big|\frac{8\pi}{3}R_1(p)\frac{1}{R^2}\epsilon + o(1/R^2)\epsilon + o(\epsilon)\Big|.$$

Taken R large enough and ϵ small enough, for all $p \in \mathbb{R}^3$:

$$|I_\epsilon(S_p^{1/R}) - 16\pi| < \frac{2}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon, \tag{58}$$

and $b)$ is satisfied also on the lower face of the cylinder.
Now the lateral face: fix r so large that the spheres of center p with $|p| = r$ and radius $\rho = R$ are disjoint from $B_{0,r_\delta(R)}$. Of course also the spheres with the same center and radius $\rho < R$ are disjoint from $B_{0,r_\delta(R)}$. Hence on the lateral face we have

$$\begin{aligned}|I_\epsilon(S_p^\rho) - 16\pi| &\leq 8\epsilon\int_{S^2}\|h(p+\rho\Theta)\|d\Sigma_0 + 4\epsilon\int_{S^2}\rho\|A(p+\rho\Theta)\|\ d\Sigma_0 + o(\epsilon)\\ &\leq 32\pi\epsilon\delta + 16\pi\epsilon\delta + o(\epsilon)\end{aligned} \tag{59}$$

which, using (56) and taken ϵ small enough, can be bounded by $\frac{2}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon$.

Collecting the estimates (57), (58), (59) we finally can say that $b)$ is verified taking $K = K_{Rr}$ with R, r large enough and ϵ small enough. Taking R, r even larger we can assume $(\bar{p}, \bar{\rho}) \in K$.

Fixed the compact K, we can apply the Reduction Method described in the subsection 4.1 and find critical points of $\Phi_\epsilon(p,\rho) := I_\epsilon(S_p^\rho) + o(\epsilon)$. Since the remainder is of order $o(\epsilon)$ uniformly in $(p,\rho) \in K$, thanks to $b)$, taken ϵ small enough we have

$$\sup_{(p,\rho)\in\partial K}|\Phi_\epsilon(p,\rho) - 16\pi| < \frac{4}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon, \tag{60}$$

and from $a)$

$$|\Phi_\epsilon(\bar{p},\bar{\rho}) - 16\pi + \frac{8\pi}{3}R_1(\bar{p})\bar{\rho}^2\epsilon| < \frac{2}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon. \tag{61}$$

Now we have all the information to show that $\Phi_\epsilon : K \to \mathbb{R}$ has got a global minimum point in the interior of K. First of all $\Phi_\epsilon(\bar{p},\bar{\rho}) < \inf_{(p,\rho)\in\partial K}\Phi_\epsilon(p,\rho)$:

$$\begin{aligned}\Phi_\epsilon(\bar{p},\bar{\rho}) &= 16\pi + \big(\Phi_\epsilon(\bar{p},\bar{\rho}) - 16\pi + \frac{8}{3}\pi R_1(\bar{p})\bar{\rho}^2\epsilon\big) - \frac{8}{3}\pi R_1(\bar{p})\bar{\rho}^2\\ &\leq 16\pi - \frac{8}{3}\pi R_1(\bar{p})\epsilon\bar{\rho}^2 + \frac{2}{3}\pi R_1(\bar{p})\epsilon\bar{\rho}^2\\ &\leq 16\pi - \frac{6}{3}\pi R_1(\bar{p})\epsilon\bar{\rho}^2.\end{aligned}$$

From (60) it follows

$$\inf_{(p,\rho)\in\partial K}\Phi_\epsilon(p,\rho) > 16\pi - \frac{4}{3}R_1(\bar{p})\epsilon\bar{\rho}^2.$$

Hence, as we wanted,

$$\Phi_\epsilon(\bar{p},\bar{\rho}) < \inf_{(p,\rho)\in\partial K_{R,\delta}} \Phi_\epsilon(p,\rho)$$

and the global minimum of Φ_ϵ is in the interior of K, so it is a critical point of Φ_ϵ. In other words for ϵ small enough the reduced functional Φ_ϵ has a critical point $(p_\epsilon, \rho_\epsilon)$; from Lemma 4.2 it follows that the perturbed sphere $S^{\rho_\epsilon}_{p_\epsilon}(w_\epsilon(p_\epsilon,\rho_\epsilon))$ is a critical point for the Willmore functional I_ϵ.
The case $R_1(\bar{p}) < 0$ is similar, one must take the modulus in the inequalities and observe that Φ_ϵ has an interior global maximum instead of a minimum. ■

Proof of Theorem 1.2.

Let $\eta > 0$ be such that $R_1(p_1) > \eta$ and $R_1(p_2) < -\eta$. For simplicity assume $\epsilon > 0$.
Let us repeat the proof of Theorem 1.1 replacing $R_1(\bar{p})$ with η.
Let $\bar{\rho} > 0$ and $\epsilon > 0$ be small enough such that

$$\begin{aligned} a_1)\quad |I_\epsilon(S^{\bar{\rho}}_{p_1}) - 16\pi + \frac{8}{3}\pi R_1(p_1)\epsilon\bar{\rho}^2| &= |o(\bar{\rho}^2)\epsilon + o(\epsilon)| < \frac{1}{3}\pi\eta\epsilon\bar{\rho}^2 \quad \text{and} \\ a_2)\quad |I_\epsilon(S^{\bar{\rho}}_{p_2}) - 16\pi + \frac{8}{3}\pi R_1(p_2)\epsilon\bar{\rho}^2| &= |o(\bar{\rho}^2)\epsilon + o(\epsilon)| < \frac{1}{3}\pi\eta\epsilon\bar{\rho}^2. \end{aligned}$$

In the sequel $\bar{\rho}$ has to be considered as a fixed constant, while ϵ may be chosen smaller.
Exactly as in Theorem 1.1, one constructs the compact $K \subseteq \mathbb{R}^3 \oplus \mathbb{R}^+$ such that on the boundary

$$b)\quad \sup_{(p,\rho)\in\partial K} |I_\epsilon(S^\rho_p) - 16\pi| \leq \frac{2}{3}\pi\eta\bar{\rho}^2\epsilon.$$

Taken the compact large enough, the points $(p_1,\bar{\rho}), (p_2,\bar{\rho})$ are in the interior of K. Fixed the compact K we can apply the Reduction Method and study the reduced functional $\Phi_\epsilon : K \to \mathbb{R}$. Since $\Phi_\epsilon(p,\rho) = I_\epsilon(S^\rho_p) + o(\epsilon)$, taken ϵ small enough

$$|\Phi_\epsilon(p_1,\bar{\rho}) - 16\pi + \frac{8}{3}\pi R_1(p_1)\epsilon\bar{\rho}^2| < \frac{2}{3}\pi\eta\epsilon\bar{\rho}^2, \tag{62}$$

$$|\Phi_\epsilon(p_2,\bar{\rho}) - 16\pi + \frac{8}{3}\pi R_1(p_2)\epsilon\bar{\rho}^2| < \frac{2}{3}\pi\eta\epsilon\bar{\rho}^2, \tag{63}$$

$$\sup_{(p,\rho)\in\partial K} |\Phi_\epsilon(p,\rho) - 16\pi| < \frac{4}{3}\pi\eta\bar{\rho}^2\epsilon. \tag{64}$$

Now we can show that the points of global maximum and minimum of $\Phi_\epsilon : K \to \mathbb{R}$ are in the interior of K. From (62)

$$\begin{aligned} \Phi_\epsilon(p_1,\bar{\rho}) &\leq 16\pi + \big(\Phi_\epsilon(p_1,\bar{\rho}) - 16\pi + \frac{8}{3}\pi R_1(p_1)\epsilon\bar{\rho}^2\big) - \frac{8}{3}\pi R_1(p_1)\epsilon\bar{\rho}^2 \\ &< 16\pi + \frac{2}{3}\pi\eta\epsilon\bar{\rho}^2 - \frac{8}{3}\pi R_1(p_1)\epsilon\bar{\rho}^2 \\ &\leq 16\pi - \frac{6}{3}\pi\eta\epsilon\bar{\rho}^2. \end{aligned}$$

Similarly, using (63)

$$\Phi_\epsilon(p_2,\bar{\rho}) > 16\pi + \frac{6}{3}\pi\eta\epsilon\bar{\rho}^2.$$

Hence the global minimum (resp. maximum) of Φ_ϵ on K is less than $16\pi - \frac{6}{3}\pi\eta\epsilon\bar{\rho}^2$ (resp. bigger than $16\pi + \frac{6}{3}\pi\eta\epsilon\bar{\rho}^2$); from the estimate (64) it follows that the points of global maximum and minimum are in the interior of K so they are critical points of Φ_ϵ. Call these two distinct points $(p^1_\epsilon, \rho^1_\epsilon)$ and $(p^2_\epsilon, \rho^2_\epsilon) \in \mathbb{R}^3 \oplus \mathbb{R}^+$. From Lemma 4.2, the perturbed spheres $S^{\rho^1_\epsilon}_{p^1_\epsilon}(w_\epsilon(p^1_\epsilon,\rho^1_\epsilon))$ and $S^{\rho^2_\epsilon}_{p^2_\epsilon}(w_\epsilon(p^2_\epsilon,\rho^2_\epsilon))$ are distinct critical points of I_ϵ for ϵ small enough. ■

5.2 Symmetries of the metric and multiplicity results

Since the results of this subsection are rather easy and standard, the proofs will be quite sketchy.
We start this subsection with a general property of the Willmore functional: the invariance under isometries of the ambient manifold.

Theorem 5.1. *Let (M,g) be a Riemannian manifold of dimension 3 and $(\mathring{M},\mathring{g})$ a compact, orientable, isometrically immersed submanifold of dimension 2.*
Then the Willmore functional

$$I(\mathring{M}) := \int_{\mathring{M}} H^2 d\Sigma \tag{65}$$

is invariant under isometries of M.

Proof. Fix an isometry $\phi : M \to M$; we have to prove that $I(\phi(\mathring{M})) = I(\mathring{M})$.
Fix a point $p \in \mathring{M}$ and coordinates $x^i, i = 1,2$ on a neighbourhood (in $\mathring{M}$) of p; obviously $(\phi^{-1})^i$ are coordinates on a neighbourhood (in $\phi(\mathring{M})$) of $\phi(p)$. Of course in these coordinates the first fundamental form is invariant:

$$\mathring{g}_{\mathring{M}_{ij}}(p) = \mathring{g}_{\phi(\mathring{M})_{ij}}(\phi(p))$$

because ϕ is an isometry. With a simple computation it is possible to show that also the second fundamental form is invariant:

$$\mathring{h}_{\mathring{M}_{ij}}(p) = \mathring{h}_{\phi(\mathring{M})_{ij}}(\phi(p)).$$

Hence also $H := \mathring{h}_{ij}\mathring{g}^{ij}$ is invariant and we get the thesis with an integration. ∎

Remark 5.1. *Since the Willmore functional I is intrinsic (i.e. invariant under reparametrizations), the reparametrizations of the same surface have no geometrical relevance, hence in the sequel we identify two immersions with the same image.*

Theorem 5.2. *Let $G < Iso(M)$ be a subgroup of the isometries of M.*
Let $(\mathring{M},\mathring{g}) \hookrightarrow (M,g)$ be an isometrically immersed compact orientable surface which represents a critical point for the Willmore functional I. Then the surfaces of the set

$$G(\mathring{M}) := \{\phi(\mathring{M}) : \phi \in G\}$$

are critical points of I. Moreover, setting

$$Crit(I) = \{\mathring{M} \hookrightarrow M : \mathring{M} \textit{ is a critical point of } I\}$$

and $Stab_G(\mathring{M}) := \{\phi \in G : \phi(\mathring{M}) = \mathring{M}\}$, the action of G on M induces an injection from the cosets

$$G/Stab_G(\mathring{M}) \hookrightarrow Crit(I).$$

Proof. Fix $\phi \in G$. First we want to show that if $\mathring{M}$ is a critical point of I also $\phi(\mathring{M}) \in \mathrm{Crit}(I)$. Denote the normal unit vector to $\mathring{M}$ with ν; the normal unit vector to $\phi(\mathring{M})$ is $\phi_*(\nu)$ and a normal perturbation to $\phi(\mathring{M})$ can be written as

$$\phi(\mathring{M})[t] = \phi(\mathring{M}) + tf\phi_*(\nu)$$

with $f \in C^\infty(\mathring{M})$. For $|t|$ small enough, the sum has to be intended in coordinates (for $|t|$ small enough, by compactness of $\mathring{M}$, the points of $\phi(\mathring{M})$ and of $\phi(\mathring{M}) + tf\phi_*(\nu)$ are in the same chart). With a Taylor expansion in t, we get

$$\phi(\mathring{M})[t] = \phi(\mathring{M} + tf\nu) + o(t).$$

By definition, $\phi(\mathring{M})$ is a critical point of I if for all $f \in C^\infty(\mathring{M})$ one has $\frac{d}{dt}I(\phi(\mathring{M})[t]) = 0$.
With a Taylor expansion of I,

$$I(\phi(\mathring{M})[t]) = I(\phi(\mathring{M} + tf\nu)) + o(t).$$

Hence the derivative in t of the functional is

$$\begin{aligned}\frac{d}{dt}I(\phi(\mathring{M})[t])|_{t=0} &= \lim_{t\to 0}\frac{I(\phi(\mathring{M})[t])-I(\phi(\mathring{M}))}{t}\\ &= \lim_{t\to 0}\frac{I(\phi(\mathring{M}+tf\nu))-I(\phi(\mathring{M}))}{t}+\lim_{t\to 0}\frac{o(t)}{t}.\end{aligned}$$

Of course the second summand is null. Let us consider now the first summand: since ϕ is an isometry, from Theorem 5.1 it follows

$$\lim_{t\to 0}\frac{I(\phi(\mathring{M}+tf\nu))-I(\phi(\mathring{M}))}{t}=\lim_{t\to 0}\frac{I(\mathring{M}+tf\nu)-I(\mathring{M})}{t}.$$

But by assumption $\mathring{M}$ is a critical point, so also the first summand is null for all $f\in C^\infty(\mathring{M})$ and $\phi(\mathring{M})$ is a critical point of I.

For the second part of the corollary, denote $\phi\mathrm{Stab}_G(\mathring{M})$ the left coset of ϕ in G. It is sufficient to observe that the map

$$\psi : G/\mathrm{Stab}_G(\mathring{M})\to \mathrm{Crit}(I)\quad \psi(\phi\mathrm{Stab}_G(\mathring{M})):=\phi(\mathring{M})$$

is well defined and injective. ■

Now let us apply these results to the studied case of perturbed metric $\delta+\epsilon h$ in $\mathbb{R}^3$.
We know from Theorem 1.1 that, if R_1 is not identically null and h is asymptotically null in an appropriate way, then the Willmore functional I_ϵ, for small ϵ, has as critical point a certain perturbed sphere $S^\rho_p(w)$. If h has symmetries which do not fix $S^\rho_p(w)$, thanks to Corollary 5.2, we can find other critical points. Let us examine some simple examples:

Example 5.1. ***h invariant under rotations:*** *Assume that*

$$\forall A\in SO(3)\quad h_{A(p)}(Av,Aw)=h_p(v,w)\quad \forall p,v,w\in\mathbb{R}^3.$$

It follows that $\forall A\in SO(3)$ is an isometry of $g_\epsilon=\delta+\epsilon h$.
From Corollary 5.2, if $S^\rho_p(w)$ is a critical point of I_ϵ there is an injection

$$SO(3)/Stab(S^\rho_p(w))\hookrightarrow Crit(I_\epsilon).$$

In particular, if the center of the sphere $p\neq 0$, it is easy to see that there is the injection $S^2\hookrightarrow SO(3)/Stab(S^\rho_p(w))$; in this case we have a non countable set of critical points.

Example 5.2. ***h invariant under rotations around an axis:***
Assume there exists an axis r such that the rotations $SO(2)$ around r are isometries of h:

$$\forall A\in SO(2)\quad h_{A(p)}(Av,Aw)=h_p(v,w)\quad \forall p,v,w\in\mathbb{R}^3.$$

It follows that $\forall A\in SO(2)$ is an isometry of $g_\epsilon=\delta+\epsilon h$.
From Corollary 5.2, if $S^\rho_p(w)$ is a critical point of I_ϵ, then there is an injection

$$SO(2)/Stab(S^\rho_p(w))\hookrightarrow Crit(I_\epsilon).$$

In particular, if the center of the sphere $p\notin r$, it is easy to see that $Stab(S^\rho_p(w))=\{Id\}$; also in this case the set of critical points is not countable.

Example 5.3. ***h even****: Assume*

$$h_{\mu\nu}(x)=h_{\mu\nu}(-x)\quad \forall x\in\mathbb{R}^3.$$

It is immediate to show that the reflection with respect to the origin is an isometry of g_ϵ. If $S^\rho_p(w)$ is a critical point of I_ϵ not invariant under the reflection, then there are at least two critical points.

Example 5.4. ***h invariant under reflections with respect to a plane or an axis:***
Assume there exists an axis r (or a plane π) such that the reflection A with respect to r (resp. π) is an isometry of h:

$$h_{A(p)}(Av, Aw) = h_p(v, w) \quad \forall p, v, w \in \mathbb{R}^3.$$

If $S^\rho_p(w)$ is a critical point of I_ϵ not invariant under the reflection, then there are at least two critical points.

Remark 5.2. *Using the method studied in [AMYam] it should be possible to find metrics h (with far away concentration bumps) such that the critical points of the functional are not invariant under all the isometries.*

5.3 A non existence result in general manifolds

In this subsection we want to prove Theorem 1.3. We start with a Lemma, which asserts that for small perturbation $u \in C^{4,\alpha}(S^2)$ and small radius ρ, the perturbed geodesic sphere $S_{p,\rho}(u)$ can be obtained as a normal graph on an other geodesic sphere $S_{\tilde{p},\tilde{\rho}}$ with perturbation $\tilde{w} \in C^{4,\alpha\perp}$: $S_{p,\rho}(u) = S_{\tilde{p},\tilde{\rho}}(\tilde{w})$.

Lemma 5.3. *Let (M, g) be a Riemannian manifold of dimension three and fix $\bar{p} \in M$. Then there exist $B(0, r_1) \subset C^{4,\alpha}(S^2)$, $\rho_1 > 0$, a compact neighbourhood U of $\bar{p}$ and three continuous functions*
$\cdot$ $p(.) : B(0, r_1) \to U \subset M$,
$\cdot$ $\rho(.,.) : (0, \rho_1) \times B(0, r_1) \to \mathbb{R}^+$,
$\cdot$ $w(.,.) : U \times B(0, r_1) \to C^{4,\alpha}(S^2)^\perp$,
such that for all $\rho < \rho_1$ and $u \in B(0, r_1)$, all the perturbed geodesic spheres $S_{\bar{p},\bar{\rho}}(u)$ can be realized as

$$S_{\bar{p},\bar{\rho}}(u) = S_{p(u),\rho(\bar{\rho},u)}[w(p(u), u)].$$

Proof. Recall that fixed $\bar{p} \in M$ and a compact neighbourhood U of $\bar{p}$ there exist $\rho' > 0$ such that the exponential map Exp_p is defined on the ball $B_{\rho'} \subset T_pM$ for all $p \in U$.
Observe that there exist an even smaller neighbourhood $U' \subseteq U$ and $B(0, r') \subset C^{4,\alpha}(S^2)$ such that for all $p \in U'$, $u \in B(0, r')$ and $\bar{\rho} < \rho'$ there exists a unique $w = w(p, u) \in C^{4,\alpha}(S^2)$ such that

$$S_{\bar{p},\bar{\rho}}(u) = S_{p,\bar{\rho}}(w(p, u)).$$

Our aim is to show that for $u \in C^{4,\alpha}(S^2)$ small enough, we can choose p and vary $\bar{\rho}$ in the right hand side so that $w \in C^{4,\alpha}(S^2)^\perp$.

Call $P : C^{4,\alpha}(S^2) \to Ker[\triangle_{S^2} + 2]$ the orthogonal projection; we want to choose p so that $P[w(p, u)] = 0$. Observe that the map $w : U' \times B(0, r') \to C^{4,\alpha}(S^2)$, $(p, u) \mapsto w(p, u)$ is C^1. Of course also the map

$$F : U' \times B(0, r') \to Ker[\triangle_{S^2} + 2], \quad (p, u) \mapsto P[w(p, u)]$$

is C^1. We have the following facts:
(i) $F(\bar{p}, 0) = 0$
(ii) $\frac{\partial F}{\partial p}(\bar{p}, 0) : T_{\bar{p}}M \to Ker[\triangle_{S^2} + 2]$ is invertible.
(i) follows from the definitions, let us prove (ii).

Recall that $Ker[\triangle_{S^2} + 2]$ is the three dimensional space of affine functions on S^2, whose base is made of the coordinate functions $x^\mu, \mu = 1, 2, 3$ of $\mathbb{R}^3$. Without loss of generality we can assume that U' is contained in a normal coordinate neighbourhood p^μ of center $\bar{p}$. In order to obtain $\frac{\partial F}{\partial p^\mu}(\bar{p}, 0)$, observe that the infinitesimal generator of the translation of $S_{\bar{p},\bar{\rho}}$ in the direction of $\frac{\partial}{\partial p^\mu}$ is a vector field along $S_{\bar{p},\bar{\rho}}$ whose component along the normal unit vector Θ is the affine function p^μ, so as we claimed

$$\frac{\partial F}{\partial p^\mu}(\bar{p}, 0) = p^\mu$$

and $\frac{\partial F}{\partial p}(\bar{p}, 0)$ is invertible.

By the Implicit Function Theorem there exist a neighbourhood $U'' \subset M$ of $\bar{p}$, $B(0,r_1) \subset C^{4,\alpha}(S^2)$ and a C^1 function $p(.) : B(0,r_1) \to U''$ such that $F(p(u),u) = 0$, that is $P[w(p(u),u)] = 0$.

Now it is sufficient to observe that, if $w(p(u),u)$ has a non null component along $Ker[\triangle_{S^2}] = \{$constant functions on S^2 $\}$, we can make it vanish with a small variation of the radius $\bar{\rho}$: just subtract to w its projection on $Ker[\triangle_{S^2}]$ and choose the appropriate new radius. We denote with $\rho(\bar{\rho},u) = \rho(\bar{\rho}, w(p(u),u)) \in \mathbb{R}^+$ the modified radius and with abuse of notation we still denote with $w(p(u),u)$ the modified perturbation. A direct check shows that the map $\rho(.,.) : (0,\rho') \times B(0,r') \to \mathbb{R}^+$ is continuous. At last we can conclude that

$$S_{\bar{p},\bar{\rho}}(u) = S_{p(u),\rho(\bar{\rho},u)}[w(p(u),u)] \quad \text{with } w(p(u),u) \in C^{4,\alpha}(S^2)^{\perp}.$$

■

Now we are in position to prove the non existence result.

Proof of Theorem 1.3.
Since $R(\bar{p}) \neq 0$, there exists $\eta > 0$ and a compact neighbourhood Z_c of $\bar{p}$ such that $|R(p)| > \eta$ for all $p \in Z_c$.

From Lemma 4.4 there exist $\rho_0 > 0$ and a ball $B(0,r) \subset C^{4,\alpha}(S^2)$ such that- for $w \in C^{4,\alpha\perp} \cap B(0,r)$, $p \in Z_c$ and $\rho < \rho_0$- if the perturbed geodesic sphere $S_{p,\rho}(w)$ is a critical point of I then $w = w_{p,\rho}$ with good decay properties as $\rho \to 0$. Moreover, for $p \in Z_c$ and $\rho < \rho_0$ we can consider the C^1 function

$$\Phi(p,\rho) = I(S_{p,\rho}(w_{p,\rho})).$$

Observe that if $S_{\tilde{p},\tilde{\rho}}(w_{\tilde{p},\tilde{\rho}})$ is a critical point for I then a fortiori $(\tilde{p},\tilde{\rho})$ is a critical point of the constricted functional $\Phi(.,.)$.
We have an explicit formula for $\Phi(p,\rho)$: from the expansion (7) we get

$$\Phi(p,\rho) = 16\pi - \frac{8\pi}{3} R(p)\rho^2 + \int_{S^2} (Q^{(2)(2)}_p(w_{p,\rho}) + \rho^2 L^{(2)}_p(w_{p,\rho})) d\Theta + O_p(\rho^3).$$

Differentiating it with respect to ρ and remembering (from Lemma 4.4) that as $\rho \to 0$ one has $\|w_{p,\rho}\|_{C^{4,\alpha}} = O(\rho^2)$ and $\|\frac{\partial}{\partial\rho} w_{p,\rho}\|_{L^2} = O(\rho)$ uniformly for $p \in Z_c$, we get

$$\frac{\partial}{\partial\rho}\Phi(p,\rho) = -\frac{16\pi}{3} R(p)\rho + O(\rho^2)$$

and

$$\left|\frac{\partial}{\partial\rho}\Phi(p,\rho)\right| > \frac{16\pi}{3}\eta\rho + O(\rho^2) \quad \text{for all } p \in Z_c. \tag{66}$$

Where the remainder $O(\rho^2)$ is uniform on Z_c.
From this equation we can say that there exist $\rho_2 \in]0,\rho_0[<$ such that for all $p \in Z_c$ and $\rho < \rho_2$, (p,ρ) is not a critical point of Φ.
Hence

$$\begin{aligned} &\forall w \in C^{4,\alpha}(S^2)^{\perp} \cap B(0,r),\ \rho < \rho_2 \text{ and } p \in Z_c \\ &\Rightarrow S_{p,\rho}(w) \text{ is NOT a critical point of } I. \end{aligned} \tag{67}$$

Now from Lemma 5.3, if $u \in B(0,r_1) \subset C^{4,\alpha}(S^2)$ and $\bar{\rho} < \rho_1$, any perturbed sphere $S_{\bar{p},\bar{\rho}}(u)$ can be realized as

$$S_{\bar{p},\bar{\rho}}(u) = S_{p(u),\rho(\bar{\rho},u)}[w(p(u),u)], \quad w(p(u),u) \in C^{4,\alpha}(S^2)^{\perp}.$$

From the continuity of the functions $p(.)$, $\rho(.,.)$ and $w(.,.)$, there exist $\rho_3 \in]0, \min(\rho_1,\rho_2)[$ and $r_2 \in]0, \min(r,r_1)[$ such that for all $u \in B(0,r_2) \subset C^{4,\alpha}(S^2)$ and $\bar{\rho} < \rho_3$ we have:

$\cdot$ $p(u) \in Z_c$,

$\cdot\ \rho(\bar\rho, u) < \rho_2$ and

$\cdot\ w(p(u), u) \in C^{4,\alpha}(S^2)^\perp \cap B(0, r)$.

It follows that if $u \in B(0, r_2)$ and $\bar\rho < \rho_3$, the sphere $S_{\bar p,\bar\rho}(u)$ can be realized as $S_{p(u),\rho(\bar\rho,u)}[w(p(u), u)]$ which satisfies the assumptions (67); so it is not a critical point of I. ■

Remark 5.3. *Observe the difference with the flat case: thanks to* (1)*, in* $\mathbb{R}^3$ *the spheres of any radius are critical points of the Willmore functional* I*; on the contrary, in the case of ambient metric with non null scalar curvature we have just shown that the geodesic spheres of small radius are not critical points.*

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