PHASE TRANSITIONS AND MINIMAL HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract. The purpose of this paper is to investigate the Cahn-Hillard approximation for entire minimal hypersurfaces in the hyperbolic space. Combining comparison principles with minimization and blow-up arguments, we prove existence results for entire local minimizers with prescribed behaviour at infinity. Then, we study the limit as the length scale tends to zero through a Γ-convergence analysis, obtaining existence of entire minimal hypersurfaces with prescribed boundary at infinity. In particular, we recover some existence results proved in [3] and [21] using geometric measure theory.

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Contents

1. Introduction

Let $(\mathbb{H}^n, g)$ be the hyperbolic space with its standard metric g, represented either with the Poincaré ball or with the half space model. Given a double well potential $W : \mathbb{R} \to \mathbb{R}$, e.g.,

$W(u) = \frac{1}{4}(1-u^2)^2$ and $W_\varepsilon(u) = \frac{1}{\varepsilon^2}W(u)$, $\varepsilon > 0$, we consider the energy functional

$$\mathcal{E}_\varepsilon(u, A) := \int_A \frac{1}{2}\|\nabla_g u\|^2 + W_\varepsilon(u)\, dVol_g, \tag{1.1}$$

where A is a bounded open subset of $\mathbb{H}^n$, $u \in H^1(A;\mathbb{R})$, and where ∇_g, $\|\cdot\|$ and $dVol_g$ are the gradient, the length of tangent vectors and the volume element with respect to the Riemannian metric g, respectively.

The critical points, and therefore in particular the minimizers of such energy are solutions of the corresponding Euler-Lagrange equation

$$\Delta_g u + f_\varepsilon(u) = 0, \tag{1.2}$$

where $f_\varepsilon(u) := -W_\varepsilon'(u)$ and Δ_g is the usual (negative) Laplace-Beltrami operator.

In this paper we focus on entire solutions u_ε (i.e., defined in the whole $\mathbb{H}^n$) of equation (1.2) that are local minimizers of the energy $\mathcal{E}_\varepsilon$ in (1.1), according to the following definition.

Definition 1. *We say that a function $u \in H^1_{loc}(\mathbb{H}^n)$ is a local minimizer of the energy $\mathcal{E}_\varepsilon$ defined in* (1.1)*, if $\mathcal{E}_\varepsilon(u, A) \le \mathcal{E}_\varepsilon(v, A)$ for every open bounded subset A of $\mathbb{H}^n$ and for every $v \in H^1_{loc}(\mathbb{H}^n)$ such that $u - v$ has compact support contained in A.*

In the Euclidean setting, the energy functional (1.1) is usually referred to as the Cahn-Hillard approximation of the Van der Waals phase transition model. The minimizers u_ε of the energy $\mathcal{E}_\varepsilon$ in (1.1) describe smeared phase transitions, and their asymptotic behavior as $\varepsilon \to 0$ provides a good approximation of sharp area minimizing interfaces (actually with constant mean curvature under the usual additional volume constraint). Roughly speaking, as ε tends to zero, local minimizers u_ε tend to ± 1 far from a minimal hypersurface Σ, and make the transition in an ε-neighborhood of their level sets $\Sigma_\varepsilon := \{u_\varepsilon = 0\}$, which in turn provide a good approximation of Σ. We refer the reader to the important paper [25] for a first rigorous result in this direction and to [10], [18] for some extensions. Conversely, under suitable non-degeneracy assumptions, a given minimal hypersurface Σ (or, more generally, a constant mean curvature hypersurface) can be obtained as limit of the zero level sets Σ_ε of solutions u_ε to equation (1.2) (see [26]). Thus, the study of the energy (1.1) provides a bridge between semilinear elliptic equations and minimal hypersurfaces both in the Euclidean space and on Riemannian manifolds. We refer the reader to [32] for a survey on this topic and to [22] for a first result in case of surfaces of higher codimension related to superconductivity. We refer also to [19] and [8] for the analogous link between the gradient flow of (1.1) and the mean curvature flow in codimension one and two, respectively.

The goal of this paper is to investigate this classical connection in hyperbolic space. The first step in this program is the construction of entire solutions u_ε to equation (1.2) with prescribed behavior near the *sphere at infinity* $S^{n-1}(\infty)$. From now on we will assume that the potential W_ε is of the form $W_\varepsilon = \frac{1}{\varepsilon^2}W$, where $W : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ is a C^2 function satisfying the following usual assumptions

$$\begin{aligned} &i) && W(t) = W(-t) \text{ for all } t \in \mathbb{R},\\ &ii) && \min W = 0 \text{ and } \{W(t) = 0\} = \{-1, 1\},\\ &iii) && W''(1) > 0,\\ &iv) && W(t) \text{ is strictly decreasing in } [0,1] \text{ and strictly increasing for } t > 1. \end{aligned} \tag{1.3}$$

We are interested in solutions which are local energy minimizers, taking the two minima ± 1 of the potential W as boundary values on two different open sets $\Omega^\pm$ on the sphere at infinity, and making the transition in $\mathbb{H}^n$. The results we will achieve show that there are plenty of such solutions (actually uncountably many), in analogy with the simpler case of entire bounded harmonic functions in $\mathbb{H}^n$, with arbitrary continuous data at infinity (see [30], [5]). Our results are in striking contrast with what happens in the Euclidean space, where, at least for $n \leq 7$, the nonconstant entire local minimizer of (1.1) is unique up to isometries (see [28], Theorem 2.3).

Once we have constructed entire solutions u_ε that are locally energy minimizers, the second step consists in letting $\varepsilon \to 0$, to obtain a limit function u^* taking only values ± 1 in $\mathbb{H}^n$, in analogy with the asymptotic analysis done in the Euclidean space in [25]. Eventually, when Ω^+ and Ω^- have common boundary $L \subset S^{n-1}(\infty)$, we obtain an existence result for entire minimal hypersurfaces Σ, the jump set of u^* in $\mathbb{H}^n$, with prescribed behavior $\partial\Sigma = L$ at infinity. This result has been originally proved in [3] by methods of geometric measure theory.

For expository convenience we will state all our results using the Poincaré ball model. First we consider the relevant case where the boundary conditions for u_ε are prescribed on two disjoint spherical caps $\Omega^+ := C^+$ and $\Omega^- = C^-$ in $S^{n-1}(\infty)$ with common boundary L.

Let $\Sigma = \Sigma(L)$ be the spherical cap in $\mathbb{H}^n$ touching L orthogonally (i.e., the totally geodesic $\mathbb{H}^{n-1}$ asymptotic to L at infinity), and denote by $\tilde{d}$ the signed hyperbolic distance from Σ with the sign convention $\tilde{d}(x, \Sigma) \to \pm\infty$ as $x \to C^\pm$. With these special boundary conditions, it is possible to perform a one-dimensional reduction of the problem, i.e., to seek for solutions $U_\varepsilon(x) := h_\varepsilon(\tilde{d}(x, \Sigma(L)))$, for a suitable smooth function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$, with $U_\varepsilon(x) \to \pm 1$ as $x \to C^\pm$ (correspondingly $h_\varepsilon(\pm\infty) = \pm 1$). These kind of one dimensional solutions exhibit a jump at infinity on the interface $L = \partial C^\pm$, that looks smeared far from the sphere at infinity. The first result of the paper deals with the construction of such one dimensional solutions U_ε that will be the building blocks to treat the general case.

Theorem 1.1. *Let $\Omega^+ := C^+$ and $\Omega^- := C^-$ be two disjoint spherical caps in $S^{n-1}(\infty)$ with common boundary L, and let $\Sigma(L)$ be the spherical cap in $\mathbb{H}^n$ touching $S^{n-1}(\infty)$ along L orthogonally.*

Then there exists a unique solution $U_\varepsilon \in C^2(\mathbb{H}^n)$ to equation (1.2) satisfying $U_\varepsilon \equiv 0$ on Σ and the boundary conditions $U_\varepsilon(x) \to \pm 1$ as $x \to C^\pm$. Moreover, U_ε is a local minimizer of the functional $\mathcal{E}_\varepsilon$ in (1.1), and it is one-dimensional, i.e., $U_\varepsilon(x) = h_\varepsilon(\tilde{d}(x, \Sigma))$ for a suitable smooth odd increasing function h_ε. Finally, $U_\varepsilon(x) \to sgn(\tilde{d}(x, \Sigma))$ locally uniformly in $\mathbb{H}^n \setminus \Sigma$ as $\varepsilon \to 0$.

In dimension two, existence results for equation (1.2), with the potential $W(u) = \frac{1}{4}(1-u^2)^2$, have been largely exploited both in the physical and in the mathematical community, because of its relevance in the study of the Yang-Mills equations in four dimension. An explicit solution for $\varepsilon = 1$ with two point singularities at the boundary has been found in [13], while more general solutions with two points singularities have been constructed in [27] and [23], using ODE techniques. A one-dimensional solution in $\mathbb{H}^n$ for any n has been constructed only very recently in [9]. The novelty of our result consists in the existence and uniqueness property for solutions vanishing on Σ. Clearly, as $\varepsilon \to 0$ the hypersurface Σ turns out to be the jump set of the limit function $u^*(x) = \mathrm{sgn}(\tilde{d}(x, \Sigma))$, thus a totally geodesic and area minimizing hypersurface. As will be clarified below, the property $U_\varepsilon = 0$ on Σ will be crucial in order to control the zero level set of solutions u_ε for general boundary data, and therefore to prescribe the boundary L at infinity of the limiting minimal surface obtained as $\varepsilon \to 0$.

Now we pass to the case of general boundary data, namely to the case of arbitrary open sets $\Omega^\pm \subset S^{n-1}(\infty)$. In a two dimensional context, a model case is when Ω^+ and Ω^- consist in a finite number of arcs. The corresponding solutions, usually referred to as *multimeron solutions* of the Yang-Mills equations, are solutions of (1.2) with finitely many boundary singularities. They have been conjectured and formally derived in [33] and [15], and rigorously constructed in [20], [11] and [6]. The first existence result for entire solutions in $\mathbb{H}^n$ with general prescribed behavior on the sphere at infinity $S^{n-1}(\infty)$ is the following.

Theorem 1.2. *Let Ω^+ and Ω^- be disjoint open subsets of $S^{n-1}(\infty)$. Then, there exists an entire solution $u_\varepsilon \in C^2(\mathbb{H}^n)\cap C^0(\mathbb{H}^n\cup\Omega^+\cup\Omega^-)$ to equation (1.2), that is a local minimizer of the energy $\mathcal{E}_\varepsilon$ in (1.1) according with Definition 1, and that satisfies the boundary conditions $u_\varepsilon = 1$ on Ω^+, $u_\varepsilon = -1$ on Ω^-. Moreover, the zero level set $\Sigma_\varepsilon := u_\varepsilon^{-1}(0)$ satisfies $\Sigma_\varepsilon \subseteq \overline{conv(F)}$, where $F = S^{n-1}(\infty)\setminus(\Omega^+\cup\Omega^-)$. In addition, $\partial\Omega^+\cap\partial\Omega^- \subseteq \overline{\Sigma_\varepsilon}\cap S^{n-1}(\infty)\subset F$ (where the closure is understood in $\overline{B_1}$ with respect to to the Euclidean topology). In particular, if $\partial\Omega^+ = \partial\Omega^- = F$ then $\overline{\Sigma_\varepsilon}\cap S^{n-1}(\infty) = F$. Finally, for $n\le 7$ there exists $\varepsilon_0>0$ depending only on n, such that for $\varepsilon\le\varepsilon_0$ the zero level set Σ_ε is a C^2 smooth hypersurface.*

To prove Theorem 1.2, the key point is to prescribe the boundary conditions at infinity. To this purpose the main ingredient is the construction of suitable barriers $\overline{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$ (so that $\underline{\psi}_\varepsilon \le u_\varepsilon \le \overline{\psi}_\varepsilon$) with desired behavior at infinity, obtained combining the one dimensional solutions discussed above. It turns out that the location of the zero level set Σ_ε is controlled by the barriers, so that it is trapped into $\overline{\mathrm{conv}(F)}$, the geodesic convex hull of F in $\mathbb{H}^n\cup S^{n-1}(\infty)$. On the other hand, smoothness of Σ_ε in low dimension is indeed a consequence through blow-up analysis of the recent important paper [28]. This existence result combines ideas from [11] and [20], and gives a positive answer to the question, raised in [9], of constructing entire solutions to equation (1.2), taking values in $\{-1,0,+1\}$ on prescribed sets of the sphere at infinity $S^{n-1}(\infty)$.

Our next result deals with solutions exhibiting a prescribed sharp interface $L\subset S^{n-1}(\infty)$. To this purpose, given $B\subset S^{n-1}(\infty)$, we denote by $K(B)$ the cone over B from the origin in the Poincaré ball model, defined by $K(B) := \cup_{\rho<1}\rho B\cup\{0\}$. Moreover, $\tilde{d}(x,K(L))$ will denote now the hyperbolic signed distance function from $K(L)$, taking positive sign in $K(\Omega^+)$ and negative sign in $K(\Omega^-)$.

Theorem 1.3. *Let Ω^+ and Ω^- be disjoint open subsets of $S^{n-1}(\infty)$ with common boundary L, and assume that $L\subset S^{n-1}(\infty)$ is a smooth hypersurface of class C^1.*

Then, there exists an entire solution $u_\varepsilon \in C^2(\mathbb{H}^n)\cap C^0(\mathbb{H}^n\cup S^{n-1}(\infty)\setminus L)$ to equation (1.2), that satisfies the boundary conditions $u_\varepsilon = 1$ on Ω^+, $u_\varepsilon=-1$ on Ω^-, that is a local minimizer of the energy $\mathcal{E}_\varepsilon$ in (1.1), and having the following asymptotic behavior near the sphere at infinity $S^{n-1}(\infty)$

$$u_\varepsilon(x) = h_\varepsilon(\tilde{d}(x,K(L))) + e(x), \qquad \text{where } e(x)\to 0 \text{ as } x\to S^{n-1}(\infty). \tag{1.4}$$

Moreover, the zero level set $\Sigma_\varepsilon := u_\varepsilon^{-1}(0)$ satisfies $\Sigma_\varepsilon \subset \overline{conv(L)}$. Finally, Σ_ε is a C^1 hypersurface near the sphere at infinity with boundary $\partial\Sigma_\varepsilon = L$, touching $S^{n-1}(\infty)$ orthogonally along L.

The asymptotic expansion in (1.4) generalizes the analogous property established in [11] for solutions near isolated singularities in dimension two (with $\varepsilon=1$ and the explicit potential $W(u)=\frac{1}{4}(1-u^2)^2$). Indeed, as we will see in Proposition 4.5, blowing up the solution u_ε around a point of L the sets $\Omega^\pm$ converge (under rescaling) to a pair of half spheres, while u_ε converges to the corresponding one dimensional solution given by Theorem 1.1, and this will be the key step in proving (1.4).

The orthogonality of the zero level set of the minimizers u_ε at the boundary, stated in Theorem 1.3, means that the normals $\nu_{\Sigma_\varepsilon}(P_k)$ at points $P_k \in \Sigma_\varepsilon$ converging to some $P_\infty \in L \subset S^{n-1}(\infty)$ as $k \to \infty$, tend to $\nu_L(P_\infty)$ (see (4.12)). This orthogonality property can be seen as the natural counterpart, in this phase field framework, of the boundary orthogonality proved in [17] for entire minimal hypersurfaces constructed in [3]. For the reader convenience we quote this last result in Theorem 1.5. In fact, in [17] the authors actually give a complete boundary regularity result, that could be interesting to exploit in our context, in order to obtain higher regularity of Σ_ε when L is more regular than C^1. To this purpose, it seems very natural to investigate the asymptotic behavior of the solutions u_ε near the sphere at infinity, either through a PDE approach or through an asymptotic energy expansion based on Γ-convergence, but we will not pursue further this point in the paper.

Let us pass now to describe the second step of our program, consisting in letting $\varepsilon \to 0$, recovering in the limit sharp area minimizing interfaces in hyperbolic space. The language of Γ-convergence, as shown in [25] in the Euclidean case, provides the natural framework to perform this asymptotic analysis (we refer the reader to the book [12] for an extensive introduction to the subject).

Since we are interested in minimal hypersurfaces in hyperbolic space with infinite area, in order to perform our variational approach based on Γ-convergence it is convenient to restrict the energy functionals to bounded domains of $\mathbb{H}^n$. More precisely, we identify $\mathbb{H}^n$ with B_1 according with the Poincaré ball model, and we restrict the energy functionals $\mathcal{E}_\varepsilon(u)$ defined in (1.1) to balls B_R with $0 < R < 1$. Moreover, we fix a boundary condition $u \equiv w_\varepsilon$ on ∂B_R, where w_ε belongs to $H^1_{loc}(B_1)$ with $|w_\varepsilon| \le 1$, having in mind $w_\varepsilon = u_\varepsilon$ for our purposes, where u_ε is the local minimizer constructed in Theorem 1.2.

Let $\varepsilon_m \to 0$, and assume that (up to subsequences)

$$w_{\varepsilon_m} \to w^* \text{ in } L^1_{loc}(B_1), \quad \mu_{\varepsilon_m} := \varepsilon_m\Big(\frac{1}{2}\|\nabla w_{\varepsilon_m}\|^2 + W_{\varepsilon_m}(w_{\varepsilon_m})\Big)\, dVol_g \overset{*}{\rightharpoonup} \mu^*, \tag{1.5}$$

for some $w^* \in BV_{loc}(B_1; \{+1, -1\})$ and for some locally finite positive measure μ^* on B_1.

We are in a position to define the energy functionals $\mathcal{F}_\varepsilon(\,\cdot\,; w_\varepsilon, B_R) : L^1(B_R) \to \mathbb{R}$ as follows

$$\mathcal{F}_\varepsilon(u; w_\varepsilon, B_R) := \begin{cases} \sqrt{2}\varepsilon\, \mathcal{E}_\varepsilon(u, B_R) & \text{if } u \in H^1_{w_\varepsilon}(B_R), \\ \infty & \text{otherwise in } L^1(B_R), \end{cases} \tag{1.6}$$

where $H^1_{w_\varepsilon}(B_R)$ is the set of H^1 functions with trace on ∂B_R equal to w_ε. Note that $\mathcal{F}_\varepsilon$ is lower semicontinuous even if it could be infinite on some $u \in H^1_{w_\varepsilon}(B_R)$, because we impose no growth condition on W_ε at infinity.

Given $v \in L^1(B_R)$, denote by $\tilde{v} = \tilde{v}_{w^*}$ the extension of v to B_1, coinciding with w^* on $B_1 \setminus B_R$. The candidate Γ-limit of the functionals $\mathcal{F}_\varepsilon$ is the functional $\mathcal{F}(\,\cdot\,; w^*, B_R) : L^1(B_R) \to \mathbb{R}$ defined as

$$\mathcal{F}(v; w^*, B_R) := \begin{cases} C_W |\tilde{v}_{w^*}|_{BV_g(\overline{B}_R)} & \text{if } v \in BV(B_R; \{+1, -1\}); \\ +\infty & \text{otherwise in } L^1(B_R), \end{cases} \tag{1.7}$$

where

$$C_W = \int_{-1}^{1} \sqrt{W(s)}\, ds,$$

and $|\cdot|_{BV_g}$ denotes the intrinsic total variation in the hyperbolic space (see Section 2 for the precise definition).

The following result describes the asymptotic behavior of the energy functionals $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$.

Theorem 1.4. *Let $\varepsilon_m \to 0$, and let w_{ε_m} be a sequence of boundary conditions satisfying* (1.5) *for some suitable w^* and μ^*. The following compactness and Γ-convergence result holds.*

i) (Compactness.) *Let $0 < R < 1$ be fixed, and let v_{ε_m} be a sequence in $L^1(B_R)$ with $|v_{\varepsilon_m}| \le 1$ such that $\mathcal{F}_{\varepsilon_m}(v_{\varepsilon_m}; w_{\varepsilon_m}, B_R) \le C$, for some constant C independent of ε_m. Then (up to a subsequence) $v_{\varepsilon_m} \to v^*$ in $L^1(B_R)$ for some $v^* \in BV(B_R; \{+1, -1\})$.*

ii) (Γ-convergence.) *Let $0 < R < 1$ be such that $\mu^*(\partial B_R) = 0$. Then the following Γ-convergence inequalities hold.*
   i) (Γ-liminf inequality.) *Let $v_{\varepsilon_m} \to v$ in $L^1(B_R)$. Then we have $\mathcal{F}(v; w^*, B_R) \le \liminf_{\varepsilon_m} \mathcal{F}_{\varepsilon_m}(v_{\varepsilon_m}; w_{\varepsilon_m}, B_R)$;*
   ii) (Γ-limsup inequality.) *Let $v \in L^1(B_R)$. Then there exists a sequence $v_{\varepsilon_m} \to v$ in $L^1(B_R)$ such that $\mathcal{F}(v; w^*, B_R) \ge \limsup_{\varepsilon_m} \mathcal{F}_{\varepsilon_m}(v_{\varepsilon_m}; w_{\varepsilon_m}, B_R)$.*

This result represents the counterpart in the hyperbolic space of the classical Γ-convergence result [25] for phase transitions in the Euclidean space. As for the Euclidean setting [25], the bound $|v_{\varepsilon_m}| \le 1$ in the compactness statement is very natural because the energy functionals decrease under truncation, and it can be dropped assuming super-quadratic growth conditions on the potential W at infinity, like in the model case $W_\varepsilon(u) = \frac{1}{4\varepsilon^2}(1-u^2)^2$. Note also that the Γ-limsup inequality would fail if in (1.7) we neglect the contribution due to the possible jump between w^* and v across ∂B_R. In addition, the fact that this boundary contribution depends only on w^* is indeed a consequence of the assumption $\mu^*(\partial B_R) = 0$ (see Remark 5.1). The previous Γ-convergence result, applied to the local minimizers u_ε yields the following theorem.

Theorem 1.5. *Let Ω^+ and Ω^- be disjoint open subsets of $S^{n-1}(\infty)$, and let $F := S^{n-1}(\infty) \setminus (\Omega^+ \cup \Omega^-)$. Let $\varepsilon_m \to 0$ and let u_{ε_m} be the locally minimizing entire solutions of* (1.2) *given by Theorem 1.2 and Theorem 1.3. Then the following holds.*

i) *Up to a subsequence, $u_{\varepsilon_m} \to u^*$ in $L^1_{loc}(B_1)$ for some $u^* \in BV_{loc}(B_1; \{-1, 1\})$. Moreover, the jump set S_{u^*} satisfies $S_{u^*} \subset \overline{conv(F)}$, and $\partial\Omega^+ \cap \partial\Omega^- \subseteq \overline{S_{u^*}} \cap S^{n-1}(\infty) \subset F$ (where the closure is understood in $\overline{B_1}$ with respect to to the Euclidean topology). In particular, if $\partial\Omega^+ = \partial\Omega^- = F$ then $\overline{S_{u^*}} \cap S^{n-1}(\infty) = F$.*

ii) *The limit u^* is a local minimizer of the total variation, i.e., $|u^*|_{BV_g(B_R)} \le |v^*|_{BV_g(B_R)}$ for every $v^* \in BV_{loc}(B_1; \{+1, -1\})$ such that the support of $(u^* - v^*)$ is compactly contained in some ball B_R, $0 < R < 1$.*

iii) *The $(n-1)$-current J_{u^*} corresponding to the jump set S_{u^*} is a local mass minimizer, therefore for $n \le 7$ it is a smooth (analytic) hypersurface, while for $n > 7$ it has a singular set Z of dimension $\dim Z \le n - 8$. Finally, if $L := \partial\Omega^+ = \partial\Omega^- = F$ is a C^1 hypersurface, then J_{u^*} is a smooth hypersurface near the sphere at infinity, touching $S^{n-1}(\infty)$ orthogonally along L.*

Compactness of local minimizers follows from Γ-convergence, while the behaviour of the barriers as $\varepsilon_m \to 0$ allows to use all the information on the zero sets Σ_{ε_m} and to control the position of the jump set S_{u^*} and its behaviour at infinity. The minimality property of u^* is a direct consequence of the fact that the minimality of u_ε passes to the limit under Γ-convergence. The last part of the theorem is essentially well known, so we include it just for reader convenience. Indeed, the minimality for the current J_{u^*} corresponding to S_{u^*} is standard and its interior regularity is a consequence of the celebrated regularity results for codimension-one mass minimizing currents (see [14]). On the other hand, the last statement concerning boundary regularity and orthogonality at infinity has been established in [17].

When $\Omega^+ \cup \Omega^-$ is a dense open subset of $S^{n-1}(\infty)$, the existence of a minimal hypersurface asymptotic to F at infinity was originally proved in [3] for $F = \partial\Omega^+ = \partial\Omega^-$ and $F = L$ an immersed smooth hypersurface, while for very irregular (possibly fractal) interfaces $F = \partial\Omega^+ = \partial\Omega^-$ the result has been proved in [21]. Here we consider a more general case without assuming $\Omega^+ \cup \Omega^-$ dense. As a consequence the hypersurface is hinged at infinity only on the contact region $\partial\Omega^+ \cap \partial\Omega^-$, while we expect that J_{u^*} (and indeed also its boundary at infinity) is a minimizer of a suitable free boundary problem.

Now we would like to discuss few possible directions of investigation. In our opinion it would be interesting to extend the phase transition approach to the case of constant mean curvature hypersurfaces with prescribed asymptotic boundary, as constructed in [31] and [16] working with finite perimeter sets, and with the prescribed mean curvature equation respectively. On the other hand, another direction of investigation could be, in the same spirit of [4], to study minimizing solutions to (1.2) on hyperbolic manifolds, i.e. to investigate entire solutions to (1.2) which are invariant under some discrete cocompact subgroup of isometries in $\mathbb{H}^n$. It would be interesting as well to push further this method in order to deal with the vector valued case. In this way, one could obtain minimal surfaces of higher codimension with prescribed behavior at infinity (already constructed in [3] using geometric measure theory) as a limit of solutions of elliptic systems. In the Euclidean framework the picture is quite well developed; the Γ-convergence result has been done in [1], while for the asymptotic analysis of minimizers in the codimension-two case we refer to [22].

Finally, we mention that for $n = 2$ a discrete analogue of our problem is given by the Ising model on hyperbolic graphs (i.e. on Cayley graphs corresponding to discrete cocompact groups of isometries acting on the hyperbolic plane) considered e.g. in [29]. For this model, we expect existence of uncountably many distinct local minimizers of the Hamiltonian which should be the natural discrete counterpart of the ones given by Theorem 1.2. The presence of several local minimizers would be consistent with the existence of uncountably many mutually singular Gibbs measures on the the set of all spin configurations, rigorously proved in [29, Theorem 1], for sufficiently high inverse temperature.

2. Preliminary overview on the hyperbolic space

In this section we will briefly review the hyperbolic space, described according with the *half space model* and the *Poincaré ball model*. For each of these models, we recall the corresponding metric, the volume element, the geodesics, and the notion of sphere at infinity. We introduce in these models our energy functional and the corresponding Euler-Lagrange equation. We do not review the description of the group of isometries of each model in terms of their conformal homeomorphisms and instead we refer the interested reader e.g. to [7], Chapter 3. Finally we recall the basic definitions of BV functions on the hyperbolic space that we will need in the last section of the paper.

2.1. **The half space model.**

In this model, the hyperbolic space $\mathbb{H}^n$ is given by the half space

$$\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\},$$

endowed with the Riemannian metric

$$g := \frac{\sum_{i=1}^n dx_i^2}{x_n^2}.$$

The induced volume element is given by

$$dVol_g := \frac{dx}{x_n^n},$$

where dx denotes the usual Lebesgue measure in $\mathbb{R}^n$.

The compactification of the hyperbolic space is obtained adding to $\mathbb{H}^n$ the so called *Sphere at infinity* $S^{n-1}(\infty)$, that in the half space model is given by

$$S^{n-1}(\infty) := \partial\mathbb{R}^n_+ \cup \{\infty\}.$$

Given two points p and $q \in \mathbb{H}^n$, the geodesic joining p and q is given by an arc of circle or by a segment (joining p and q), contained in the only semi-circle or half line through p and q and touching the hyper-plane $\partial\mathbb{R}^n_+$ orthogonally.

Finally, since $\nabla_g u(x) = x_n^2 \nabla u(x)$, the energy functional (1.1) can be rewritten more explicitly as

$$\mathcal{E}_\varepsilon(u, A) := \int_A \Big(\frac{1}{2} x_n^2 |\nabla u|^2 + W_\varepsilon(u)\Big) \frac{dx}{x_n^n}, \tag{2.1}$$

while, recalling that $f_\varepsilon(s) = -W'(s)$, the corresponding Euler-Lagrange equation (1.2) reads as

$$x_n^2 \Delta u + (2-n) x_n \partial_{x_n} u + f_\varepsilon(u) = 0. \tag{2.2}$$

2.2. **The Poincaré ball model.** In this model the hyperbolic space $\mathbb{H}^n$ is given by the unit ball

$$B_1 := \{x \in \mathbb{R}^n : |x| < 1\},$$

endowed with the Riemannian metric

$$g := \frac{4\sum_{i=1}^n dx_i^2}{(1-|x|^2)^2}.$$

The corresponding volume element is given by

$$dVol_g := \frac{2^n dx}{(1-|x|^2)^n}.$$

The Sphere at infinity $S^{n-1}(\infty)$ in this case is just given by ∂B_1. Moreover, given two points p and $q \in \mathbb{H}^n$, the geodesic joining p and q is given by an arc of circle or by a segment (with extremes p and q), contained in the only circle or chord passing through p and q and touching ∂B_1 orthogonally.

Finally, since $\nabla_g u(x) = \frac{(1-|x|^2)^2}{4} \nabla u(x)$, the energy functional (1.1) is given by

$$\mathcal{E}_\varepsilon(u, A) := \int_A \Big(\frac{1}{8}(1-|x|^2)^2 |\nabla u|^2 + W_\varepsilon(u)\Big) \frac{2^n dx}{(1-|x|^2)^n}, \tag{2.3}$$

while the corresponding Euler-Lagrange equation (1.2) reads as

$$\frac{(1-|x|^2)^n}{2^n} \mathrm{div}\left(\Big(\frac{1-|x|^2}{2}\Big)^{2-n} \nabla u\right) + f_\varepsilon(u) = 0. \tag{2.4}$$

2.3. **BV functions in $\mathbb{H}^n$.** For the general theory of functions of bounded variation, we refer to the standard reference monograph [2], and we refer to [24] for the theory on Riemaniann manifolds ; here we recall some basic definitions and properties we need in the sequel, confining ourselves to BV functions defined on the hyperbolic space $\mathbb{H}^n$.

Given any open set $A \subset\subset \mathbb{H}^n$ compactly contained in $\mathbb{H}^n$, we recall that $u \in BV_g(A)$ if $u \in L^1(A, dVol_g)$, and it has finite total variation $|u|_{BV_g(A)}$, where

$$|u|_{BV_g(A)} := \sup \left\{ \int_A u \, \mathrm{div}_g \Phi \, dVol_g, \, \Phi \in C_0^\infty(A; TA), \, \|\Phi\| \leq 1 \right\} < \infty.$$

Note that, since the hyperbolic metric is locally equivalent to the Euclidean one, we have (for any model of $\mathbb{H}^n$) $BV_g(A) = BV(A)$ with equivalent, but not identical norms. As for the Euclidean case, we say that $u \in BV_{g,loc}(\mathbb{H}^n)$ if u (restricted on A) belongs to $BV_g(A)$ for every open set A compactly contained in $\mathbb{H}$. In this case it turns out that the jump set $S(u)$, i.e., the set of points $x \in A$ which are not Lebesgue points of u (also referred to as the singular set of u), is $(n-1)$-rectifiable, that is there exists a sequence of C^1 hypersurfaces $(M_i)_{i\in\mathbb{N}}$ such that $S(u) \subseteq \cup_i M_i$ up to a set of $\mathcal{H}^{n-1}$-measure zero.

We are interested in functions $u \in BV_{loc,g}(\mathbb{H}^n; \{-1,+1\})$, i.e., functions $u \in BV_{g,loc}(\mathbb{H}^n)$ valued in $\{-1,+1\}$. For such functions we denote by $|D_g u|(A) = |u|_{BV_g(A)}$ the total variation of u on A. It turns out that $|D_g u|(\cdot)$ is a locally finite Borel measure on $\mathbb{H}^n$, and the following representation formula holds

$$|D_g u|(A) = 2\mathcal{H}_g^{n-1}(S(u) \cap A) \qquad \text{for all open set } A \subset\subset \mathbb{H}^n, \tag{2.5}$$

where $\mathcal{H}_g^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure associated to the hyperbolic distance on $\mathbb{H}^n$. Notice that, in the half space model, we have

$$|D_g u|(A) = 2 \int_{S(u)\cap A} \frac{1}{x_n^{n-1}} d\mathcal{H}^{n-1}, \tag{2.6}$$

where $d\mathcal{H}^{n-1}$ denotes now the standard Euclidean $(n-1)$-dimensional Hausdorff measure, thus it is the usual Euclidean formula up to a conformal factor due to the hyperbolic metric.

3. One-dimensional phase transitions

In this section we will construct elementary solutions to equation (1.2), i.e., solutions corresponding to the case when Ω^+ and Ω^- are disjoint spherical caps with common boundary.

We will work mainly in the half space model, where we construct elementary solutions through a one dimensional reduction, then solving an ODE in $\mathbb{R}$ by a minimization argument, in the spirit of [9]. Our method will produce in particular odd solutions h_ε, and this property will be essential in our approach, since it provides the desired asymptotic behaviour as $\varepsilon \to 0$ of the barriers $\underline{\psi}_\varepsilon$, $\overline{\psi}_\varepsilon$ that we will construct in Section 4. Moreover, we give a uniqueness result for solutions of the ODE vanishing at zero, which in turns yields the uniqueness property for elementary solutions vanishing on Σ.

3.1. **One-dimensional reduction and existence for the ODE.** Here we are looking for particular elementary solutions $u(x_1, \ldots, x_n)$ to equation (2.2), which are odd with respect to x_1, and satisfying the boundary condition $u(x) = \mathrm{sgn}(x_1)$ on the hyperplane $\{x_n = 0\}$. More precisely, we construct one dimensional solutions, which are constant on the level sets

$\{\frac{x_1}{x_n} = c\}$ of the distance function from $\Sigma_0 := \{x_1 = 0\} \subset \mathbb{R}^n_+$. Thus, enforcing that the solution takes the form

$$u_\varepsilon(x) = g_\varepsilon\Big(\frac{x_1}{x_n}\Big),$$

we obtain the following boundary value problem for $g_\varepsilon(\xi)$, $\xi = \frac{x_1}{x_n}$,

$$\begin{cases} (1+\xi^2)g_\varepsilon''(\xi) + n\xi g_\varepsilon'(\xi) = -f_\varepsilon(g_\varepsilon(\xi)); \\ g_\varepsilon(\pm\infty) = \pm 1. \end{cases} \tag{3.1}$$

Since the signed distance $\tilde d$ from Σ_0 satisfies $\tilde d(x, \Sigma_0) = \sinh^{-1}(\xi)$, it is convenient to set $\tau = \sinh^{-1}(\xi)$ and to define $h_\varepsilon(\tau) = g_\varepsilon(\xi)$, so that h_ε has to solve

$$\begin{cases} h_\varepsilon''(\tau) + (n-1)\tanh\tau\, h_\varepsilon'(\tau) = -f_\varepsilon(h_\varepsilon(\tau)); \\ h_\varepsilon(\pm\infty) = \pm 1. \end{cases} \tag{3.2}$$

Such equation is the Euler-Lagrange equation of the energy functional

$$E_\varepsilon(h) = \int \Big(\frac{1}{2}h'^2 + W_\varepsilon(h)\Big)\cosh^{n-1}\tau\, d\tau. \tag{3.3}$$

Proposition 3.1. *Let $\varepsilon > 0$ and let $f_\varepsilon = -W_\varepsilon'$, with W satisfying the assumptions in* (1.3)*. Then problem* (3.2) *admits a solution h_ε which is odd and strictly increasing.*

Moreover, $E_\varepsilon(h_\varepsilon) \le C/\varepsilon$ for some positive constant C independent of ε, and $h_\varepsilon(\tau) \to sgn(\tau)$ locally uniformly in $\mathbb{R}\setminus\{0\}$ as $\varepsilon \to 0$.

Proof. In order to find a solution of equation (3.2), we consider the following minimization problem

$$\min\{E_\varepsilon(h),\, h \in H^1_{loc}(\mathbb{R}_+),\, h(0) = 0\}. \tag{3.4}$$

Let as first prove that the minimum problem (3.4) admits a minimizer h_ε^+ which is increasing and satisfies $h_\varepsilon(+\infty) = 1$.

Let $h^+_{\varepsilon,k}$ be a minimizing sequence for (3.4). Since the potential W is even, we may assume without loss of generality (taking the absolute value if necessary) that $h^+_{\varepsilon,k}$ are positive. Since $h^+_{\varepsilon,k}$ have finite energy and $h^+_{\varepsilon,k}(0) = 0$ we easily deduce that $h^+_{\varepsilon,k}$ is bounded in $H^1_{loc}(\mathbb{R}_+)$, i.e., it is bounded in $H^1(0,M)$ for every positive M. Therefore, in view of the compact embedding $H^1_{loc} \hookrightarrow C^0_{loc}$, a diagonal argument yields that (up to a subsequence) $h^+_{\varepsilon,k}$ converges locally uniformly to some continuous function h^+_ε, with $h^+_\varepsilon(0) = 0$, and $h^+_{\varepsilon,k} \rightharpoonup h^+_\varepsilon$ in $H^1(0,M)$ for every M, so that in particular h^+_ε belongs to $H^1_{loc}(\mathbb{R}_+)$. Since for all positive M the functional E_ε is weakly lower semicontinuous in $H^1(0,M)$, and since $h^+_{\varepsilon,k}$ is a minimizing sequence, we have

$$\begin{aligned} &\int_0^M \Big(\frac{1}{2}(h^+_\varepsilon)'^2 + W_\varepsilon(h^+_\varepsilon)\Big)\cosh^{n-1}\tau\, d\tau \\ &\qquad \le \liminf_k \int_0^M \Big(\frac{1}{2}(h^+_{\varepsilon,k})'^2 + W_\varepsilon(h^+_{\varepsilon,k})\Big)\cosh^{n-1}\tau\, d\tau \le \lim_k E_\varepsilon(h^+_{\varepsilon,k}) = \inf E_\varepsilon. \end{aligned} \tag{3.5}$$

Since h^+_ε is an admissible function in the minimum problem (3.4), passing to the limit for $M \to \infty$ in (3.5) we conclude that h^+_ε is a minimum point. Clearly $h^+_\varepsilon \not\equiv 0$ and since it has finite energy, by a truncation argument we also deduce that $0 \le h_\varepsilon^+ \le 1$ and that $\limsup_\tau h_\varepsilon^+ = 1$ as $\tau \to \infty$.

Setting $w_\varepsilon(\tau) := \min\{\tau/\varepsilon, 1\}$ we have that $E_\varepsilon(w_\varepsilon) \le C\varepsilon^{-1}$ for some $C > 0$ independent of ε, and therefore

$$E_\varepsilon(h_\varepsilon^+) \le E_\varepsilon(w_\varepsilon) \le \frac{C}{\varepsilon}. \tag{3.6}$$

Let us prove by a contradiction argument that h_ε^+ is non decreasing. Since h_ε^+ is continuous, $0 \le h_\varepsilon^+ \le 1$ and $\limsup_\tau h_\varepsilon^+ = 1$ as $\tau \to \infty$, we may assume by contradiction that there exist three points $\tau_1 < \tau_2 < \tau_3$ with $0 \le h_\varepsilon^+(\tau_2) < h_\varepsilon^+(\tau_1) = h_\varepsilon^+(\tau_3) < 1$. Set $w(\tau) = \max\{h_\varepsilon^+(\tau), h_\varepsilon^+(\tau_3)\}$ for every $\tau \in (\tau_1, \tau_3)$. Then, replacing h_ε^+ with w in (τ_1, τ_3) we obtain an admissible function $\tilde{h}$, with $E_\varepsilon(\tilde{h}) < E_\varepsilon(h_\varepsilon^+)$, which is in contradiction with the minimality of h_ε^+. Now we claim that we have $(h_\varepsilon^+(\tau))' > 0$ in $[0, \infty)$. Indeed, if $\tau = 0$ then $(h_\varepsilon^+(\tau))' > 0$ because otherwise we would have by ODE uniqueness $h_\varepsilon^+ \equiv 0$, which contradicts $E_\varepsilon(\tilde{h}_\varepsilon^+) < \frac{C}{\varepsilon}$. While for $\tau > 0$, if $(h_\varepsilon^+(\tau))' = 0$ equation (3.2) would imply $(h_\varepsilon^+(\tau))'' < 0$, which is in contradiction with the monotonicity of h_ε^+ because $(h_\varepsilon^+)'$ would be negative just after τ.

By the fact that h_ε^+ is bounded and increasing we deduce that it admits limit for $\tau \to \infty$; moreover $h_\varepsilon^+(+\infty) = 1$ because $E_\varepsilon(h_\varepsilon^+) < C/\varepsilon$ and $W(t) = 0$ exactly on $\{t = \pm 1\}$. Analogously, since $E_\varepsilon(h_\varepsilon^+) < C/\varepsilon$ it also follows that h_ε^+ converges to 1 locally uniformly in $\mathbb{R}_+$.

Finally, we define $h_\varepsilon(\tau)$ as the odd reflection of h_ε^+, i.e., $h_\varepsilon(\tau) := \operatorname{sgn}(\tau) h_\varepsilon^+(|\tau|)$, hence $h_\varepsilon \in C^1(\mathbb{R})$. By minimality we have that h_ε^+ solves the equation in $(0, +\infty)$, and therefore h_ε is a solution of (3.2), and it has all the desired properties. □

3.2. **A uniqueness property for solutions of the ODE.**

In this paragraph we provide a variational characterization for the solutions of problem (3.2). As a consequence we obtain that there exists a unique solution vanishing at zero. We start with the following lemma.

Lemma 3.2. *Every solution to problem* (3.2) *is strictly increasing and has finite energy.*

Proof. Let k_ε be a solution of (3.2). In order to prove that it is strictly increasing, it is clearly enough to show that $k_\varepsilon'(\tau) \ne 0$ for every $\tau \in \mathbb{R}$. To this purpose, set $V(h, h') := 1/2(h'(\tau))^2 - W(h)$. It is easy to see that

$$\frac{d}{d\tau} V(k_\varepsilon(\tau), k_\varepsilon'(\tau)) = -(n-1) \tanh \tau (k_\varepsilon'(\tau))^2. \tag{3.7}$$

Assume by contradiction that $k_\varepsilon'(\bar{\tau}) = 0$ for some $\bar{\tau} \in \mathbb{R}$. Then we clearly have $|k_\varepsilon(\bar{\tau})| \ne 1$ by ODE uniqueness, and therefore, $V(k_\varepsilon(\bar{\tau}), k_\varepsilon'(\bar{\tau})) = \alpha < 0$. We consider only the case $\bar{\tau} > 0$, the other case being analogous. By (3.7) we deduce $V(k_\varepsilon(\tau), k_\varepsilon'(\tau)) \le \alpha < 0$ for every $\tau \ge \bar{\tau}$, which clearly gives a contradiction since $\liminf_{\tau \to \infty} V(k_\varepsilon(\tau), k_\varepsilon(\tau)') \ge \liminf_{\tau \to \infty} -W(k_\varepsilon(\tau)) = 0$.

Let us prove now that k_ε as finite energy. Multiplying both sides of (3.2) by $k_\varepsilon' \cosh^{n-1} \tau$ we have

$$\frac{d}{d\tau}\left(\frac{1}{2}(k_\varepsilon')^2 \cosh^{n-1}\tau\right) + \frac{n-1}{2}(k_\varepsilon')^2 \cosh^{n-1}\tau \tanh\tau + f_\varepsilon(k_\varepsilon) k_\varepsilon' \cosh^{n-1}\tau = 0. \tag{3.8}$$

Integrating equation (3.8) between $\bar\tau := k_\varepsilon^{-1}(0)$ and $\tau \in \mathbb{R}$, we deduce

$$\left(\frac{1}{2}(k'_\varepsilon(\tau))^2\cosh^{n-1}\tau\right) + \int_{\bar\tau}^{\tau}\left(\frac{n-1}{2}(k'_\varepsilon(s))^2\tanh s + f_\varepsilon(k_\varepsilon(s))k'_\varepsilon(s)\right)\cosh^{n-1}s\,ds = \\ = \left(\frac{1}{2}(k'_\varepsilon(\bar\tau))^2\cosh^{n-1}\bar\tau\right). \tag{3.9}$$

Since $f_\varepsilon(t)t \geq 0$ for $|t| \leq 1$ and $k'_\varepsilon > 0$, we easily obtain that for every $\tau \in \mathbb{R}$

$$\left(\frac{1}{2}(k'_\varepsilon(\tau))^2\cosh^{n-1}\tau\right) + \int_{\bar\tau}^{\tau} f_\varepsilon(k_\varepsilon(s))k'_\varepsilon(s)\cosh^{n-1}s\,ds \geq 0.$$

Since $\tanh(s) \to \pm 1$ as $s \to \pm\infty$, (3.9) yields

$$\int_{\mathbb{R}}\frac{1}{2}(k'_\varepsilon(\tau))^2\cosh^{n-1}(\tau)\,d\tau < \infty, \qquad (k'_\varepsilon(\tau))^2\cosh^{n-1}(\tau) \leq C < \infty \text{ for every } \tau \in \mathbb{R}. \tag{3.10}$$

By (3.9) and (3.10) we easily deduce that

$$\left|\int_{\bar\tau}^{\tau} f_\varepsilon(k_\varepsilon(s))k'_\varepsilon(s)\cosh^{n-1}s\,ds\right| \leq C \qquad \text{for a constant } C \text{ independent of } \tau. \tag{3.11}$$

A simple integration by parts gives

$$\int_{\bar\tau}^{\tau} f_\varepsilon(k_\varepsilon(s))k'_\varepsilon(s)\cosh^{n-1}s\,ds = \\ = \int_{\bar\tau}^{\tau}(n-1)W_\varepsilon(k_\varepsilon(s))\cosh^{n-1}s\tanh s - \frac{d}{ds}(W_\varepsilon(k_\varepsilon(s))\cosh^{n-1}s)\,ds = \\ = \int_{\bar\tau}^{\tau}(n-1)W_\varepsilon(k_\varepsilon(s))\cosh^{n-1}s\tanh s\,ds + W_\varepsilon(k_\varepsilon(\bar\tau))\cosh^{n-1}\bar\tau - W_\varepsilon(k_\varepsilon(\tau))\cosh^{n-1}\tau \tag{3.12}$$

Taking into account the exponential decay of k'_ε given by the second inequality in (3.10), a simple integration yields the exponential decay of $1 - k_\varepsilon^2(\tau)$, so that $(1-k_\varepsilon^2(\tau))^2\cosh^{n-1}\tau \leq C < \infty$ for every τ. By Taylor expansion around the minima of W we also get

$$W_\varepsilon(k_\varepsilon(\tau))\cosh^{n-1}\tau \leq C < \infty \qquad \text{for every } \tau. \tag{3.13}$$

By (3.11), (3.12) and (3.13) we deduce that

$$\int_{\mathbb{R}} W_\varepsilon(k_\varepsilon)\cosh^{n-1}(\tau)\,d\tau < \infty,$$

which together with (3.10) yields $E_\varepsilon(k_\varepsilon) < \infty$. □

Proposition 3.3. *Every solution k_ε to* (3.2) *is strictly increasing and minimizes the energy E_ε in* (3.3) *among all smooth functions h satisfying $h(\pm\infty) = \pm 1$.*

As a consequence, the solution h_ε provided by Proposition 3.1 is the unique solution to (3.2) *vanishing at zero.*

Proof. In view of Lemma 3.2 we have that k_ε is strictly increasing and it has finite energy. Now we show that k_ε is the unique energy minimizer in every compact interval $I \subset \mathbb{R}$, with respect to its own boundary values. We will use a contradiction argument similar to the one in the proof of [27, Theorem 2.3]. Assume that there exists an energy minimizer $j_\varepsilon \neq k_\varepsilon$ (with the same boundary values), set $A_1 := \{j_\varepsilon > k_\varepsilon\}$, $A_2 := \{j_\varepsilon < k_\varepsilon\}$, and let us show that both these open sets are empty. We show only that $A_1 = \emptyset$, since $A_2 = \emptyset$ can be proved in the same way. If $A_1 \neq \emptyset$, then there exists a maximal interval $I_1 := (\tau_1, \tau_2) \subseteq A_1 \subseteq I$. Since I_1

is maximal, we clearly have $j_\varepsilon(\tau_i) = k_\varepsilon(\tau_i)$ for $i = 1, 2$. By construction, and in view also of ODE uniqueness, we have

$$j_\varepsilon'(\tau_1) > k_\varepsilon'(\tau_1) > 0, \qquad j_\varepsilon'(\tau_2) < k_\varepsilon'(\tau_2). \tag{3.14}$$

Set $\tau_{min} = \tau_2$ if $j_\varepsilon' > 0$ in (τ_1, τ_2), and otherwise we set τ_{min} to be the minimal $\tau \in (\tau_1, \tau_2)$ such that $j_\varepsilon'(\tau) = 0$. Both in the case $\tau_{min} < \tau_2$ and $\tau_{min} = \tau_2$, in view of (3.14) we deduce that the trajectories corresponding to j_ε and k_ε cross each other in the phase space $(h, h') \in \mathbb{R}^2$, i.e., there exists t_1, $t_2 \in (\tau_1, \tau_{min})$ with

$$(j_\varepsilon(t_1), j_\varepsilon'(t_1)) = (k_\varepsilon(t_2), k_\varepsilon'(t_2)), \tag{3.15}$$

and we may assume that t_1 and t_2 are the minimal times such that (3.15) holds. By construction j_ε and k_ε are strictly increasing in (τ_1, τ_{min}), so that we can consider their inverse, and we have $j_\varepsilon'(j_\varepsilon^{-1}(h)) > k_\varepsilon'(k_\varepsilon^{-1}(h))$ for all $h \in (j_\varepsilon(\tau_1), j_\varepsilon(t_1)) = (k_\varepsilon(\tau_1), k_\varepsilon(\tau_2))$. Since

$$t_1 - \tau_1 = \int_{j_\varepsilon(\tau_1)}^{j_\varepsilon(t_1)} \frac{dh}{j_\varepsilon'(j_\varepsilon^{-1}(h))}, \qquad t_2 - \tau_1 = \int_{k_\varepsilon(\tau_1)}^{k_\varepsilon(t_2)} \frac{dh}{k_\varepsilon'(k_\varepsilon^{-1}(h))},$$

we deduce that $t_1 < t_2$. In addition, by construction we have

$$\frac{j_\varepsilon''(t_1)}{j_\varepsilon'(t_1)} = \frac{d}{dh} j_\varepsilon'(j_\varepsilon^{-1}(h))|_{j_\varepsilon(t_1)} \leq \frac{d}{dh} k_\varepsilon'(k_\varepsilon^{-1}(h))|_{k_\varepsilon(t_2)} = \frac{k_\varepsilon''(t_2)}{k_\varepsilon'(t_2)} = \frac{k_\varepsilon''(t_2)}{j_\varepsilon'(t_1)},$$

and hence $j_\varepsilon''(t_1) \leq k_\varepsilon''(t_2)$. On the other hand, equation (3.2) implies that

$$j_\varepsilon''(t_1) = -(n-1)\tanh t_1\, j_\varepsilon'(t_1) - f_\varepsilon(j_\varepsilon(t_1)) > -(n-1)\tanh t_2\, k_\varepsilon'(t_2) - f_\varepsilon(k_\varepsilon(t_2)) = k_\varepsilon''(t_2),$$

that together with $j_\varepsilon''(t_1) \leq k_\varepsilon''(t_2)$ provides a contradiction. This shows that $j_\varepsilon = k_\varepsilon$, and hence concludes the proof that k_ε is the only energy minimizer in I with respect to its own boundary values.

Now we show that k_ε is an energy minimizer among all smooth functions h such that $h(\pm\infty) = \pm 1$. To this purpose, let $\varphi_m(\tau) := \varphi(\tau/m)$ be a sequence of standard smooth cut-off functions, i.e., $\varphi \in C_0^\infty(\mathbb{R})$, $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ for $|\tau| \geq 1$, $\varphi \equiv 1$ for $|\tau| \leq 1/2$. Given any smooth h_ε with finite energy such that $h_\varepsilon^{\pm\infty} = \pm 1$, we set $h_{\varepsilon,m} := \varphi_m k_\varepsilon + (1 - \varphi_m) h_\varepsilon$. Since k_ε minimizes the energy in any interval $I_m := (-m, m)$, we have

$$E_\varepsilon(k_\varepsilon, I_m) \leq E_\varepsilon(h_{\varepsilon,m}, I_m) \leq E_\varepsilon(h_{\varepsilon,m}, I_m \setminus I_{m/2}) + E_\varepsilon(h_\varepsilon), \tag{3.16}$$

where $E_\varepsilon(h, J)$ denotes the integral on the set J of the energy density of h defined in (3.3). It is easy to check that, as $m \to \infty$, $E_\varepsilon(h_{\varepsilon,m}, I_m \setminus I_{m/2}) \to 0$ and, in view of (3.16), we easily conclude $E_\varepsilon(k_\varepsilon) \leq E_\varepsilon(h_\varepsilon)$, i.e., k_ε is a minimizer.

Finally, we pass to the proof of the uniqueness of h_ε as given by Proposition 3.1. To this purpose let k_ε be a solution to (3.2) vanishing at zero, and let us prove that $h_\varepsilon = k_\varepsilon$. Notice that, in view of the previous part, both h_ε and k_ε are energy minimizer, hence, by standard odd reflection arguments, we have

$$E_\varepsilon(h_\varepsilon, \mathbb{R}^-) = E_\varepsilon(h_\varepsilon, \mathbb{R}_+) = E_\varepsilon(k_\varepsilon, \mathbb{R}^-) = E_\varepsilon(h_\varepsilon, \mathbb{R}_+).$$

Therefore also the function j_ε defined as h_ε in $\mathbb{R}^-$ and as k_ε on $\mathbb{R}_+$ is an energy minimizer. Thus, j_ε satisfies (3.2) and by standard ODE regularity we deduce that h_ε and k_ε have same derivative at the origin, therefore they coincide by ODE uniqueness. □

3.3. **Existence and uniqueness of elementary solutions of the PDE.** We are in a position to prove Theorem 1.1. It is clear by our construction that, for $\Sigma_0 = \{x_1 = 0\}$ in the half-space model, the function $U_\varepsilon(x) := h_\varepsilon(\tilde{d}(x, \Sigma_0))$ is a solution to equation (1.2) with boundary conditions as $x_n \searrow 0$ given by $U_\varepsilon(x) = \operatorname{sgn}(x_1)$. Clearly, such solution can be viewed in the Poincaré ball model, and the corresponding boundary conditions are given by $U_\varepsilon(x) = \pm 1$ on two disjoint half spheres of the sphere at infinity $S^{n-1}(\infty)$.

In the general case, we set $U_\varepsilon(x) = h_\varepsilon(\tilde{d}(x, \Sigma))$ and we may assume $\Sigma = T(\Sigma_0)$ for some hyperbolic isometry T. By definition of U_ε we have

$$U_\varepsilon(T(x)) = h_\varepsilon(\tilde{d}(T(x), \Sigma)) = h_\varepsilon(\tilde{d}(T(x), T(\Sigma_0))) = h_\varepsilon(\tilde{d}(x, \Sigma_0)).$$

Since equation (1.2) is invariant under isometries, we conclude that $U_\varepsilon(x)$ is a solution of equation (1.2), and by construction it clearly satisfies the desired boundary conditions. Moreover, as a direct consequence of Proposition 3.1 and of the previous equalities, we also deduce that $U_\varepsilon \equiv 0$ on Σ, and $U_\varepsilon(x) \to \operatorname{sgn}(\tilde{d}(x, \Sigma))$ locally uniformly in $B_1 \setminus \Sigma$ as $\varepsilon \to 0$.

Let us pass to the proof of the minimality property of U_ε. We will work in the half space model, and since the local minimality property also is invariant under isometries, we may assume without loss of generality $\Sigma = \Sigma_0 = \{x_1 = 0\}$. Our proof is based on a uniqueness argument and a sliding technique, inspired by the two dimensional analysis done in [11]

To prove the minimality of U_ε in any regular open set $A \subset\subset \mathbb{R}^n_+$, we will prove indeed that any solution $\tilde{U}_\varepsilon$ of equation (1.2) in A, with $-1 \le \tilde{U}_\varepsilon \le 1$ and coinciding with U_ε on ∂A, is in fact equal to U_ε. This is enough to conclude since in an open set A any minimizer is clearly a solution.

Note that, by standard truncation arguments, any local minimizer $\tilde{U}_\varepsilon$ satisfies $|U_\varepsilon| \le 1$, and indeed $|\tilde{U}_\varepsilon| < 1$ in $\overline{A}$ by standard maximum principles. Therefore, it remains to prove that $\tilde{U}_\varepsilon = U_\varepsilon$. We will prove the inequality $\tilde{U}_\varepsilon \le U_\varepsilon$, the other inequality being analogous.

Since h_ε is increasing, we deduce by construction that also U_ε is increasing with respect to x_1. As a consequence, we have that the functions $U_{\varepsilon,\tau}(\,\cdot\,) := U_\varepsilon(\,\cdot\, + \tau e_1)$ are well ordered, i.e., $\tau_1 < \tau_2 \Rightarrow U_{\varepsilon,\tau_1} < U_{\varepsilon,\tau_2}$, and $U_{\varepsilon,\tau} \to \pm 1$ uniformly in $\overline{A}$ as $\tau \to \pm\infty$. Since $-1 < \tilde{U}_\varepsilon < 1$ in $\overline{A}$, we have that $\tilde{U}_\varepsilon < U_{\varepsilon,\tau}$ for τ large enough. By continuity there exists a minimum $\tau \in \mathbb{R}$, denoted by τ_{min}, satisfying $\tilde{U}_\varepsilon \le U_{\varepsilon,\tau}$ in $\overline{A}$. Clearly we have $\tau_{min} \ge 0$ because of the values at the boundary. If $\tau_{min} > 0$, then there exists $x \in A$ with $\tilde{U}_\varepsilon(x) = U_{\varepsilon,\tau_{min}}(x)$, but since $U_{\varepsilon,\tau_{min}}$ is also a solutions, this is in contradiction with standard maximum principles (see for instance [11, Lemma 2.3]). Therefore we have $\tau_{min} = 0$, and hence $\tilde{U}_\varepsilon \le U_\varepsilon$. Arguing similarly we also get $U_\varepsilon \ge \tilde{U}_\varepsilon$, whence $U_\varepsilon \equiv \tilde{U}_\varepsilon$ and U_ε is a local minimizer.

Finally, let u_ε be a solution to (1.2) satisfying the same boundary conditions of U_ε on $S^{n-1}(\infty)$. According to [9, Theorem 3.5], u_ε is indeed one-dimensional, i.e., $u_\varepsilon(x) = k_\varepsilon(\tilde{d}(x, \Sigma))$ for a suitable k_ε solving problem 3.2. Since $u_\varepsilon \equiv 0$ on Σ, we have $k_\varepsilon(0) = 0$. By Proposition 3.3 we infer $h_\varepsilon = k_\varepsilon$, so that $u_\varepsilon = U_\varepsilon$, that concludes the proof of Theorem 1.1.

Remark 3.4. It would be interesting to know whether the uniqueness statement in Theorem 1.1 still holds without the assumption $U_\varepsilon \equiv 0$ on Σ. In light of [9, Theorem 3.5], this uniqueness property is indeed equivalent to the uniqueness of the solution h of (3.2). We have proved this uniqueness property in Proposition 3.3 only under the additional assumption $h(0) = 0$ which corresponds to $U_\varepsilon \equiv 0$ on Σ. Finally, we notice that the uniqueness property is know to fail in the Euclidean context because of the translation invariance of the equation. On the other hand, in the hyperbolic space, due to the presence of the weight $\cosh^{n-1} \tau$ in the energy functional (3.3) there is no translation invariance, and since the weight is increasing in $|\tau|$, it seems very likely that the minimizer vanishes at zero.

4. Multidimensional phase transitions

In this section we will construct our minimizing phase transitions in hyperbolic space with prescribed boundary value at infinity. More precisely, we will construct global solutions for the equation (1.2) that are local minimizer of the energy functional (1.1), and satisfying the prescribed boundary conditions $u = \pm 1$ on given open subsets Ω^+, $\Omega^- \subset S^{n-1}(\infty)$.

As a building block, we will use the one dimensional solutions obtained in Section 3 to construct barriers $\underline{\psi}_\varepsilon$, $\overline{\psi}_\varepsilon$, defined as the supremum and the infimum, respectively, of suitable one dimensional solutions. Such barriers, in view of the inequality $\underline{\psi}_\varepsilon \le u_\varepsilon \le \overline{\psi}_\varepsilon$, will be used to control the behaviour of the solution u_ε at infinity. We adopt a strategy similar to the one suggested in [15] and used in [20] in dimension two. Thus, we construct u_ε as the limit of energy minimizers $u_{\varepsilon,R}$ defined on a family of exhausting subdomains, and with free boundary value between $\underline{\psi}_\varepsilon$ and $\overline{\psi}_\varepsilon$. In view of comparison principles, we show that the inequality $\underline{\psi}_\varepsilon \le u_{\varepsilon,R} \le \overline{\psi}_\varepsilon$ holds also in the interior of each subdomain, and it yields in the limit $\underline{\psi}_\varepsilon \le u_\varepsilon \le \overline{\psi}_\varepsilon$ in the whole $\mathbb{H}^n$, ensuring in this way that u_ε attains the desired boundary values at infinity.

As in Theorem 1.1, let C^+ and C^- be disjoint open spherical caps in $S^{n-1}(\infty)$ (in the Poincaré ball model) with common boundary L. The sets C^+ and C^- can be equivalently described as $C^+ = I_{r^+}(p^+)$, $C^- = I_{r^-}(p^-)$, for suitable antipodal point $p^\pm \in S^{n-1}(\infty)$ and suitable radii $r^\pm$ with $r^+ + r^- = \pi$, where $I_r(p)$ denotes the ball of radius r and center $p \in S^{n-1}(\infty)$ with respect to the standard Riemannian distance on the sphere. Moreover, whenever $r^+ \ne r^- \ne \pi/2$, the sets C^+ and C^- uniquely determine (and at the same time they are determined by) a unique Euclidean ball B (actually a half space in the limiting case $r^+ = r^- = \pi/2$). Indeed, let C_{min} be the smallest spherical cap between C^+ and C^-. Then there exists a unique Euclidean ball B such that $B \cap S^{n-1}(\infty) = C_{min}$ and $\Sigma := \partial B \cap B_1$ touches ∂B_1 orthogonally along $L = \partial\Sigma$.

In our construction of the solution we will use the signed distance function $\tilde{d}(x, \Sigma)$ from the set Σ defined above, with the convention $\tilde{d}(x, \Sigma) \to \pm\infty$ as $x \to C^\pm$ respectively.

Note that when $r^+ = r^- = \pi/2$, the corresponding Σ is a $(n-1)$-dimensional disk; e.g., $\Sigma := \Sigma_0 = \{x_n = 0\}$. Moreover, all the sets Σ's are isometrically equivalent, and hence in particular there they are isometrically equivalent to Σ_0.

4.1. Sub-solutions and super-solutions.

In this part we will define suitable barriers for the solution to equation (1.2) which we will construct in the next paragraph. The idea here is to combine one dimensional solutions provided in Theorem 1.1, corresponding to two families of spherical caps, exhausting the open sets Ω^+ and Ω^- respectively.

The following lemma establishes a monotonicity property for the family of one dimensional solutions.

Lemma 4.1. *Let $\{C_1^+, C_1^-\}$ and $\{C_2^+, C_2^-\}$ be two pairs of spherical caps in $S^{n-1}(\infty)$, and let U_ε^1 and U_ε^2 be the corresponding one dimensional solutions given by Theorem 1.1.*

Then we have $U_\varepsilon^1 \le U_\varepsilon^2$ if and only if $C_1^+ \subseteq C_2^+$. Moreover we have strict inequality $U_\varepsilon^1(x) < U_\varepsilon^2(x)$ for every $x \in \mathbb{H}^n$ whenever the inclusion $C_1^+ \subset C_2^+$ is strict.

Proof. Assume $U^1_\varepsilon \le U^2_\varepsilon$ and let $p \in C_1^+$. Since $U^2_\varepsilon \le 1$, we have $1 \le \lim_{x\to p} U^1_\varepsilon \le \lim_{x\to p} U^2_\varepsilon \le 1$, i.e., $x \in C_2^+$. We conclude that $C_1^+ \subseteq C_2^+$.

Now, Let Σ_1 and Σ_2 be the zero level sets of U_1 and U_2 respectively, so that $U^1_\varepsilon(x) = h_\varepsilon(\tilde d(x,\Sigma_1))$, $U^2_\varepsilon(x) = h_\varepsilon(\tilde d(x,\Sigma_2))$.

If $C_1^+ \subseteq C_2^+$, then $d(\cdot,\Sigma_1) \le d(\cdot,\Sigma_2)$, and the inequality is strict whenever the inclusion is strict. Since h_ε is strictly increasing the conclusion follows. □

Another useful property of one dimensional solutions is that they are essentially closed under uniform convergence of compact sets. More precisely we have the following lemma

Lemma 4.2. *Let U^m_ε be one dimensional solutions, corresponding to pairs of spherical caps C^+_m, C^-_m with common boundary. Up to a subsequence, we have $U^m_\varepsilon \to U_\varepsilon$ locally uniformly, for some solution U_ε of equation* (1.2). *Moreover, either $U_\varepsilon \equiv \pm 1$ or it is a one dimensional solution corresponding to some spherical caps C^+, C^-.*

Proof. Since the compactness property of U^m_ε is clearly invariant by composing U^m_ε with a convergent sequence of isometries, we may assume without loss of generality that C^+_m and C^-_m are concentric. Let Σ^m_ε be the spherical caps corresponding to the zero level sets of U^m_ε. Let T_m be the hyperbolic isometries mapping Σ_m into $\Sigma_0 := \{x_n = 0\}$, corresponding to pure dilations in the half-space model. Up to a subsequence, we have that either T_m converge locally uniformly to some limit isometry T, or T_m converges locally uniformly to the constant map $T(x) \equiv p$, where p is a center of the concentric caps C^+_m, C^-_m.

In the first case, set $\Sigma := T^{-1}(\Sigma_0)$,

$$U_\varepsilon(x) := h_\varepsilon(\tilde d(x,\Sigma)) = h_\varepsilon(\tilde d(Tx,T\Sigma)) = h_\varepsilon(\tilde d(Tx,\Sigma_0)),$$

and C^+ and C^- the corresponding spherical caps. Since h_ε is continuous and T_m converges to T locally uniformly and

$$U^m_\varepsilon(x) = h_\varepsilon(\tilde d(x,\Sigma_m)) = h_\varepsilon(\tilde d(x,T_m^{-1}\Sigma_0)) = h_\varepsilon(\tilde d(T_m x,\Sigma_0)),$$

we deduce that U^m_ε converges locally uniformly to the function U_ε, that has all the desired properties.

Finally, in the second case we have $\Sigma_m \cap K = \emptyset$ for every compact set $K \subset \mathbb{H}^n$ and m large enough. Then, it is easy to see that $\tilde d(x,\Sigma_m) \to \pm\infty$ locally uniformly in $\mathbb{H}^n$, hence $U^m_\varepsilon \to \pm 1$ locally uniformly. □

Now we will construct the barriers $\underline{\psi}_\varepsilon$, $\overline{\psi}_\varepsilon$. Let Ω^+, Ω^- be disjoint open subset of $S^{n-1}(\infty)$, and let (C^+,C^-) denote any pair of disjoint spherical caps in $S^{n-1}(\infty)$ with common boundary. We set

$$\mathcal{F}^+ := \{(C^+,C^-) : C^+ \subset \Omega^+\}, \qquad \mathcal{F}^- := \{(C^+,C^-) : C^- \subset \Omega^-\}. \tag{4.1}$$

Given a pair (C^+,C^-), the corresponding one dimensional solution provided by Theorem 1.1 will be denoted by U^{C^+,C^-}_ε. Finally, for every $x \in \mathbb{H}^n$ we set

$$\underline{\psi}_\varepsilon(x) := \sup\{U^{C^+,C^-}_\varepsilon(x),\, (C^+,C^-) \in \mathcal{F}^+\}, \quad \overline{\psi}_\varepsilon(x) := \inf\{U^{C^+,C^-}_\varepsilon(x),\, (C^+,C^-) \in \mathcal{F}^-\}. \tag{4.2}$$

In the next proposition we summarize some properties satisfied by the barriers just introduced.

Proposition 4.3. *The barriers $\underline{\psi}_\varepsilon$, $\overline{\psi}_\varepsilon$ defined in* (4.2) *are Lipschitz in $\mathbb{H}^n$ with respect to the hyperbolic metric. In addition, we have $-1 < \underline{\psi}_\varepsilon(x) \le \overline{\psi}_\varepsilon(x) < 1$ for all $x \in \mathbb{H}^n$. Moreover for every $p^\pm \in \Omega^\pm$ we have $\lim_{x\to p^+} \underline{\psi}_\varepsilon(x) = 1$ and $\lim_{x\to p^-} \overline{\psi}_\varepsilon(x) = -1$.*

Finally, either $\underline{\psi}_\varepsilon(x) < \overline{\psi}_\varepsilon(x)$ *for every* $x \in \mathbb{H}^n$*, or* $\underline{\psi}_\varepsilon \equiv \overline{\psi}_\varepsilon = U_\varepsilon$ *for some one-dimensional solution* U_ε*, and* Ω^+ *and* Ω^- *are open disjoint spherical caps with common boundary.*

Proof. Given $\Sigma \subset \mathbb{H}^n$, we clearly have that the signed distance $\tilde{d}(x,\Sigma)$ is 1-Lipschitz with respect to the hyperbolic distance. Therefore, since the function h_ε is Lipschitz, we deduce that the one dimensional solutions $U_\varepsilon(x) = h_\varepsilon(\tilde{d}(x,\Sigma))$ are Lipschitz in the hyperbolic space, with Lipschitz constant independent of Σ. Therefore, passing to the supremum and the infimum respectively, we deduce that the same property is inherited by $\underline{\psi}_\varepsilon$ and $\overline{\psi}_\varepsilon$.

Note that given two pairs $(C_1^+, C_1^-) \in \mathcal{F}^+$, $(C_2^+, C_2^-) \in \mathcal{F}^-$, we always have $C_1^+ \subseteq C_2^+$. Thus the corresponding one-dimensional solutions U_ε^1 and U_ε^2 satisfy $-1 < U_\varepsilon^1 \le U_\varepsilon^2 < 1$, with strict inequality unless (see Lemma 4.1) $(C_1^+, C_1^-) = (C_2^+, C_2^-)$. Taking the supremum and the infimum respectively on $\mathcal{F}^+$ and $\mathcal{F}^-$ we obtain $-1 < \underline{\psi}_\varepsilon(x) \le \overline{\psi}_\varepsilon(x) < 1$.

Since the families of spherical caps in $\mathcal{F}^\pm$ give coverings of $\Omega^\pm$, then the limits $\lim_{x\to p^+} \underline{\psi}_\varepsilon(x) = 1$ and $\lim_{x\to p^-} \overline{\psi}_\varepsilon(x) = -1$ follow easily by construction of $\underline{\psi}_\varepsilon$, $\overline{\psi}_\varepsilon$, and in particular by the asymptotic behavior of one-dimensional solutions, stated in Theorem 1.1.

Finally, assume that equality holds at some point $x \in \mathbb{H}^n$. Then there exists sequences $\{(C_{1,m}^+, C_{1,m}^-)\} \subset \mathcal{F}^+$, $\{(C_{2,m}^+, C_{2,m}^-)\} \subset \mathcal{F}^-$, $C_{1,m}^+ \subseteq C_{1,m}^+$, such that the corresponding solutions $U_{\varepsilon,m}^1$, $U_{\varepsilon,m}^2$ satisfy $U_{\varepsilon,m}^1 \le U_{\varepsilon,m}^2$ and $\lim_{m\to\infty} U_{\varepsilon,m}^1(x) = \lim_{m\to\infty} U_{\varepsilon,m}^2 = l \in (-1,1)$. Since $l \in (-1,1)$, by Lemma 4.2 we deduce that (up to a subsequence) $U_{\varepsilon,m}^i \to U_\varepsilon^i$ locally uniformly, for some one dimensional solutions U_ε^i, corresponding to some spherical caps C_i^+, for $i = 1, 2$. Clearly $U_\varepsilon^1 \le U_\varepsilon^2$ in $\mathbb{H}^n$, hence $C_1^+ \subseteq C_2^+$ by Lemma 4.1. On the other hand, $U_\varepsilon^1(x) = U_\varepsilon^2(x) = l$, so that, in view of Lemma 4.1, $U_\varepsilon^1 \equiv \underline{\psi}_\varepsilon \equiv \overline{\psi}_\varepsilon \equiv U_\varepsilon^2$ and the proof is complete. □

4.2. **Construction of solutions of the PDE.** Here we will prove Theorem 1.2. In particular, we will construct global solutions for the equation (1.2) that are local minimizer of the energy functional (1.1), and satisfying the prescribed boundary conditions $u = \pm 1$ on given open subsets Ω^+, $\Omega^- \subset S^{n-1}(\infty)$.

First we consider an increasing sequence $r_k \to 1$, and construct energy minimizers $u_{\varepsilon,k}$ defined on B_{r_k}, with $\underline{\psi}_\varepsilon \le u_{\varepsilon,k} \le \overline{\psi}_\varepsilon$. Then, letting $k \to \infty$, we obtain by compactness a limit solution $u_\varepsilon := \lim_k u_{\varepsilon,k}$ such that $\underline{\psi}_\varepsilon \le u_\varepsilon \le \overline{\psi}_\varepsilon$, in the whole hyperbolic space $\mathbb{H}^n$.

Let $r_k \in (0,1)$ be fixed. The solution $u_{\varepsilon,k}$ in B_{r_k} is defined as a minimum point of the following minimization problem,

$$\min\{\mathcal{E}_\varepsilon(u, B_{r_k}), u \in H^1_{\Psi_\varepsilon}(B_{r_k})\} \tag{4.3}$$

where $\mathcal{E}_\varepsilon$ is defined in (2.3) and $H^1_{\Psi_\varepsilon}(B_{r_k})$ denotes the set of H^1 functions with traces on ∂B_{r_k} between $\underline{\psi}_\varepsilon$ and $\overline{\psi}_\varepsilon$.

Proposition 4.4. *The minimum problem* (4.3) *admits a minimizer* $u_{\varepsilon,k} \in H^1_{\Psi_\varepsilon}(B_{r_k})$*, satisfying* $\underline{\psi}_\varepsilon \le u_{\varepsilon,k} \le \overline{\psi}_\varepsilon$ *in* B_{r_k}*. Moreover* $u_{\varepsilon,k} \in C^2(B_{r_k})$ *is a classical solution of* (1.2)*.*

Proof. Assume first $\underline{\psi}_\varepsilon(x) = \overline{\psi}_\varepsilon(x)$ for some $x \in \mathbb{H}^n$. In view of Proposition 4.3 we have $\underline{\psi}_\varepsilon = \overline{\psi}_\varepsilon = U_\varepsilon$ for some one dimensional solution U_ε. Since U_ε is a minimizer of (4.3) with respect to his own boundary conditions (indeed, following the proof of Theorem 1.1, the unique minimizer), we conclude that $u_{\varepsilon,k}^m \equiv U_\varepsilon$ is a solution of (4.3).

Now assume $\underline{\psi}_\varepsilon < \overline{\psi}_\varepsilon$ in $\overline{B}_1$. Let $u^m_{\varepsilon,k}$ be a minimizing sequence for problem (4.3). Because of the strict inequality, by standard truncation and approximation arguments we may assume that $u^m_{\varepsilon,k}$ are smooth up to the boundary and less than or equal to one in modulus. Notice that the energy functional $\mathcal{E}_\varepsilon$ in (4.3) is sequentially weakly lower semi-continuous on H^1, so that, following the direct method of calculus of variations, the energy $\mathcal{E}_\varepsilon$ admits a minimizer among all $u \in H^1(B_{r_k})$ with $u = g$ on ∂B_{r_k}, where $g \in H^{1/2}(\partial B_{r_k})$ is a given boundary condition. Therefore, by further minimization we may assume that $u^m_{\varepsilon,k}$ minimize (4.3) with respect to their own boundary conditions $g^m_{\varepsilon,k} := Tr(u^m_{\varepsilon,k})$, where Tr denotes the trace operator. In particular $u^m_{\varepsilon,k}$ solve equation (2.4) with smooth boundary conditions $g^m_{\varepsilon,k}$, so that they are smooth (say C^2) up to the boundary by standard regularity theory for elliptic equations.

Now we aim to prove the inequality $\underline{\psi}_\varepsilon \le u^m_{\varepsilon,k} \le \overline{\psi}_\varepsilon$ in $\overline{B}_{r_k}$. We will prove just the inequality $\underline{\psi}_\varepsilon \le u^m_{\varepsilon,k}$, the proof of the other one being entirely similar. By definition of $\underline{\psi}_\varepsilon$, it is enough to prove the inequality $U_\varepsilon \le u^m_{\varepsilon,k}$ in $\overline{B}_{r_k}$ for every one-dimensional solution U_ε corresponding to some spherical caps $(C^+, C^-) \in \mathcal{F}^+$, according with (4.1). Since $U_\varepsilon \le u^m_{\varepsilon,k}$ holds on ∂B_{r_k} and since the solutions $u^m_{\varepsilon,k}$ belong to $C^2(B_{r_k}) \cap C^0(\overline{B}_{r_k})$, we can repeat the sliding argument used in the proof of Theorem 1.1 to obtain the inequality $U_\varepsilon \le u^m_{\varepsilon,k}$ in $\overline{B_{r_k}}$. This concludes the proof of

$$\underline{\psi}_\varepsilon \le u^m_{\varepsilon,k} \le \overline{\psi}_\varepsilon \qquad \text{in } \overline{B}_{r_k}. \tag{4.4}$$

Now, letting $m \to \infty$, up to a subsequence we have $u^m_{\varepsilon,k} \rightharpoonup u_{\varepsilon,k}$ for some $u_{\varepsilon,k} \in H^1_\Psi(B_{r_k})$. Since $u^m_{\varepsilon,k}$ is a minimizing sequence, by lower semi-continuity we conclude that $u_{\varepsilon,k}$ is a minimum for the variational problem (4.3). Clearly, $u_{\varepsilon,k}$ is a solution of the corresponding Euler-Lagrange equation (2.4), and hence it is C^2 in B_{r_k} by standard regularity theory. Finally, as $m \to \infty$, by (4.4) we deduce $\underline{\psi}_\varepsilon \le u_{\varepsilon,k} \le \overline{\psi}_\varepsilon$ in B_{r_k} as desired. □

We are in a position to complete the proof of Theorem 1.2. As already explained, the last step consists in taking the limit of the solutions $u_{\varepsilon,k}$ given by Proposition 4.4, as $r_k \to 1$.

Proof of Theorem 1.2. Let $u_{\varepsilon,k}$ be the solutions in B_{r_k} given by Proposition 4.4. Since they are equi-bounded and smooth, by standard elliptic regularity theory they are bounded in $C^{2,\alpha}_{loc}$, and hence they are precompact in $C^2_{loc}(B_1)$. Hence, up to a subsequence we may assume that $u_{\varepsilon,k}$ converge to some u_ε in $C^2_{loc}(B_1)$. Clearly u_ε, being limit of locally minimizing solutions, is itself a local minimizer of the energy $\mathcal{E}_\varepsilon$ in (1.1), and $u_\varepsilon \in C^2(\mathbb{H}^n)$ is a classical solution of (1.2). Finally, since $\underline{\psi}_\varepsilon \le u_{\varepsilon,k} \le \overline{\psi}_\varepsilon$ in B_{r_k}, letting $r_k \to 1$ we get $\underline{\psi}_\varepsilon \le u_\varepsilon \le \overline{\psi}_\varepsilon$ in $\mathbb{H}^n$. In view of Proposition 4.3 we conclude that u_ε satisfies the desired boundary conditions, i.e. $u_\varepsilon \in C^0(\mathbb{H} \cup \Omega^+ \cup \Omega^-)$, and $u_\varepsilon(x) = \pm 1$ on $\Omega^\pm$.

Next, we prove the inclusion $\Sigma_\varepsilon \subset \overline{conv(F)}$. First we recall that by closed half-spaces we mean the closure in $\mathbb{H}^n$ of any connected component of $\mathbb{H}^n \setminus \Sigma$, where Σ is (in the ball model of $\mathbb{H}^n$) a spherical cap touching $S^{n-1}(\infty)$ orthogonally. Notice that by Theorem 1.1, we can identify the half spaces with the sets of positivity of elementary solutions, and such correspondence is bijective. Now we consider the family of all closed half spaces such that their Euclidean closure in $\overline{B}_1$ contains F. Then it is a standard fact that $\overline{conv(F)} \cap B_1$ coincides with the intersection of all such closed half spaces.

By (4.2) it follows that if $U_\varepsilon = U_\varepsilon^{C^+,C^-}$ with $(C^+, C^-) \in \mathcal{F}^+$, then the set of negativity of u_ε is contained in the set of negativity of U_ε, and an analogous inclusion relation holds for the set of positivity of u_ε and for any elementary solution $U_\varepsilon = U_\varepsilon^{C^+,C^-}$, with $(C^+,C^-) \in \mathcal{F}^-$. Thus, varying (C^+,C^-) in $\mathcal{F}^+$ and $\mathcal{F}^-$ respectively, we deduce

$$\{u_\varepsilon \le 0\} \subset \bigcap_{(C^+,C^-)\in\mathcal{F}^+} \{U_\varepsilon^{C^+,C^-} \le 0\}, \qquad \{u_\varepsilon \ge 0\} \subset \bigcap_{(C^+,C^-)\in\mathcal{F}^-} \{U_\varepsilon^{C^+,C^-} \ge 0\},$$

and therefore

$$\{u_\varepsilon = 0\} \subset \bigcap_{(C^+,C^-)\in\mathcal{F}^+} \{U_\varepsilon^{C^+,C^-} \le 0\} \bigcap \bigcap_{(C^+,C^-)\in\mathcal{F}^-} \{U_\varepsilon^{C^+,C^-} \ge 0\}. \tag{4.5}$$

Since to each closed half space with Euclidean closure in $\overline{B}_1$ containing F corresponds (either the positivity or the negativity set of) an elementary solutions $U_\varepsilon = U_\varepsilon^{C^+,C^-}$ (with (C^+,C^-) either in $\mathcal{F}^-$ or in $\mathcal{F}^+$, respectively) the inclusion (4.5) is equivalent to $\Sigma_\varepsilon \subset \overline{conv(F)}$.

The inclusion $\overline{\Sigma_\varepsilon} \cap S^{n-1}(\infty) \subset F$ is now a direct consequence of $\Sigma_\varepsilon \subset \overline{conv(F)}$ and $\overline{conv(F)} \cap S^{n-1}(\infty) = F$, so we pass to prove the inclusion $\partial\Omega^+ \cap \partial\Omega^- \subseteq \overline{\Sigma_\varepsilon} \cap S^{n-1}(\infty)$. To this purpose, it is enough to notice that if $x \in \partial\Omega^+ \cap \partial\Omega^-$, then for every positive ρ there exists a continuous path $\gamma \subset B_\rho(x)$ joining a point of Ω^+ with a point of Ω^-. By continuity of u_ε, we deduce that there exists $\overline{x} \in B_\rho(x)$ with $u_\varepsilon(\bar{x}) = 0$, i.e., $\bar{x} \in \Sigma_\varepsilon$. By the arbitrariness of ρ, we deduce $x \in \overline{\Sigma_\varepsilon} \cap S^{n-1}(\infty)$ for all $x \in \partial\Omega^+ \cap \partial\Omega^-$, that is $\partial\Omega^+ \cap \partial\Omega^- \subseteq \overline{\Sigma_\varepsilon} \cap S^{n-1}(\infty)$.

Finally we prove that Σ_ε is always a C^2 hypersurface for $n \le 7$ and $\varepsilon \le \varepsilon_0$ sufficiently small. We work in the Poincaré ball model and we prove the claim arguing by contradiction. Indeed, if by contradiction Σ_ε are not smooth, then by the implicit function theorem we have that, for some sequence $\varepsilon_m \to 0$, there are locally energy minimizing solutions $\{u_{\varepsilon_m}\}$ of (2.4) such that (up to hyperbolic isometries) $u_{\varepsilon_m}(0) = 0$ and $\nabla u_{\varepsilon_m}(0) = 0$. We introduce the scaled functions $\tilde{u}_m \in C^2(2{\varepsilon_m}^{-1}B_1)$ as $\tilde{u}_m(x) = u_{\varepsilon_m}(\frac{\varepsilon_m}{2}x)$, so that for each $m \ge 1$ each $\tilde{u}_m$ solves

$$(1 - |\varepsilon_m x|^2)^2 \Delta\tilde{u}_m + (n-2)(1 - |\varepsilon_m x|^2)\varepsilon_m x \cdot \nabla\tilde{u}_m + f(u_{\varepsilon_m}) = 0. \tag{4.6}$$

According to the standard elliptic regularity theory for (4.6) the sequence $\{\tilde{u}_m\}$ is compact in C^2_{loc}, so, up to subsequences, there exists $\tilde{u} \in C^2(\mathbb{R}^n)$ such that as $m \to \infty$ we have $\tilde{u}_m \to \tilde{u}$ in C^2_{loc}, $\tilde{u}$ is an entire solution of $\Delta\tilde{u} + f(\tilde{u}) = 0$, $\tilde{u}(0) = 0$ and $\nabla\tilde{u}(0) = 0$. Since local energy minimality passes to the limit under smooth convergence, it's easy to check that the limiting function $\tilde{u}$ is also a local energy minimizer of the energy functional (1.1) on $\mathbb{R}^n$ with the standard metric. Since $\tilde{u} \not\equiv \pm 1$, according to [28], Theorem 2.3, for $n \le 7$ we have $\tilde{u}(x) = g(a \cdot x)$ for some unit vector $a \in \mathbb{R}^n$ and some strictly increasing function $g \in C^2(\mathbb{R})$ vanishing at the origin which solves the ODE $g'' + f(g) = 0$ on the real line. On the other hand, since $g'(0) = \nabla\tilde{u}(0) \cdot a = 0$ and f is C^1 and odd we conclude $g \equiv 0$ by ODE uniqueness, which is a contradiction because g is strictly increasing. □

4.3. **Asymptotic behavior and fine properties of solutions.** In this paragraph we study the asymptotic behaviour of the solution constructed in Theorem 1.2 under the assumption that $L := \partial\Omega^\pm$ is a C^1 hypersurface in the sphere at infinity $S^{n-1}(\infty)$. First, in Proposition 4.5 we show that blowing up the solution u_ε around a point of L the sets $\Omega^\pm$ converge (under scaling) to a pair of $(n-1)$-dimensional half spaces, while u_ε converges to the corresponding one dimensional solution given by Theorem 1.1. As a consequence, we will be in a position to prove Theorem 1.3, showing that the zero level set of Σ touches orthogonally the sphere at infinity along L, and proving the asymptotic expansion (1.4) for u_ε near L. In the following we set $\nu_L(p)$ the inner unit normal to $\partial\Omega^+$ at p. Finally, we define g_ε as the solution to problem (3.1) vanishing at zero, corresponding to the unique solution h_ε to (3.2) vanishing at zero (see Proposition 3.1 and Proposition 3.3).

Proposition 4.5. *Let Ω^+ and Ω^- be disjoint open subsets of $S^{n-1}(\infty)$ with common boundary L, and assume that $L \subset S^{n-1}(\infty)$ is a smooth hypersurface of class C^1. Let moreover $\{p_k\} \subset L$ converging to some $p \in L$ and $\lambda_k \searrow 0$ as $k \to \infty$. Finally, let u_ε be a local minimizer of the energy $\mathcal{E}_\varepsilon$ in (1.1), such that $\underline{\psi}_\varepsilon \le u_\varepsilon \le \overline{\psi}_\varepsilon$, where $\underline{\psi}_\varepsilon$ and $\overline{\psi}_\varepsilon$ are defined in (4.2).*

Then, in the half space model $\mathbb{H}^n \simeq \mathbb{R}^{n-1} \times (0,\infty)$ we have

$$u_\varepsilon(p_k + \lambda_k R_k y) \to g_\varepsilon\left(\frac{\nu_L(p)\cdot y}{y_n}\right) \qquad \text{as } k \to \infty \tag{4.7}$$

in $C^2_{loc}(\mathbb{R}^{n-1} \times (0,\infty))$, for a suitable sequence $\{R_k\} \subset O(\mathbb{R}^{n-1})$ converging to the identity, with $R_k \nu_L(p) = \nu_L(p_k)$.

Finally, for every $\{q_k\} \subset \mathbb{H}^n$ with $q_k \to q_\infty \in \{\nu_L(p)\cdot y \neq 0,\, y_n = 0\} \subset \mathbb{R}^{n-1} \times \{0\}$ we have $u_\varepsilon(p_k + \lambda_k R_k q_k) \to$ sgn $\nu_L(p)\cdot q_\infty$ as $k \to \infty$.

Proof. Up to a translation we can always assume $p = 0 \in \mathbb{R}^{n-1} \times \{0\}$. Moreover, up to a rotation $R \in O(\mathbb{R}^{n-1})$ we can assume that $\nu_L(p) = (1,0,\dots,0)$, so that locally around p we have $L = \{(f(x_2,\dots,x_{n-1}), x_2,\dots,x_{n-1})\}$ for some C^1 function f such that $f(0,\dots,0) = 0$, $\nabla f(0,\dots 0) = 0$, and

$$\Omega^+ = \{(x_1,\dots,x_{n-1}) : x_1 > f(x_2,\dots,x_{n-1})\}, \quad \Omega^- = \{(x_1,\dots,x_{n-1}) : x_1 < f(x_2,\dots,x_{n-1})\}.$$

Since $p_k \to p$ as $k \to \infty$ and f is C^1 we can choose rotations $R_k \in O(\mathbb{R}^{n-1}) \subset O(\mathbb{R}^n)$ with $R_k \to Id$ such that $R_k(1,0,\dots,0)$ is the inner unit normal to $\partial\Omega^+$ at p_k. Let us set

$$v_{\varepsilon,k}(y) := u_\varepsilon(p_k + \lambda_k(R_k y)),$$

so that $v_{\varepsilon,k}$ are smooth solutions to equation (2.2) in $\mathbb{R}^{n-1} \times (0,\infty)$, and let us prove that

$$v_{\varepsilon,k}(y) \to g_\varepsilon\left(\frac{y_1}{y_n}\right) \qquad \text{locally uniformly as } k \to \infty. \tag{4.8}$$

To this purpose, let B_1, B_2 two given balls in $\mathbb{R}^{n-1}$ with $B_1 \subset \{y_1 > 0\}$ and $B_2 \subset \{y_1 < 0\}$. Since L is C^1 we clearly have that, for k large enough,

$$B_1 \subset \lambda_k^{-1} R_k^{-1}(\Omega^+ - p_k), \qquad B_2 \subset \lambda_k^{-1} R_k^{-1}(\Omega^- - p_k),$$

or, equivalently,

$$C^+_{1,k} := p_k + \lambda_k R_k B_1 \subset \Omega^+, \qquad C^-_{2,k} := p_k + \lambda_k R_k B_2 \subset \Omega^-. \tag{4.9}$$

Let us consider the elementary solution $U^{1,k}_\varepsilon$ corresponding to the spherical cap $C^+_{1,k}$ (and to its complementary $C^-_{1,k}$ in the sphere at infinity), and analogously let $U^{2,k}_\varepsilon$ be the elementary solution corresponding to $C^-_{2,k}$ (and $C^+_{2,k}$). By (4.2) and the assumption on u_ε we have

$$U^{1,k}_\varepsilon(x) \le \underline{\psi}_\varepsilon(x) < u_\varepsilon(x) < \overline{\psi}_\varepsilon(x) \le U^{2,k}_\varepsilon(x) \qquad \text{for all } x \text{ in } \mathbb{H}^n.$$

Changing variables in the previous inequality, we get

$$U^1_\varepsilon(y) < v_{\varepsilon,k}(y) < U^2_\varepsilon(y), \qquad \text{for all } y \text{ in } \mathbb{R}^{n-1} \times (0,\infty)\,, \tag{4.10}$$

where U^1_ε and U^2_ε are the elementary solutions corresponding to B_1, B_2 and their complements. Since $v_{\varepsilon,k}(y)$ are uniformly bounded solutions of (2.2), by standard a priori estimates we have that, up to subsequences, $v_{\varepsilon,k}(y)$ converges in $C^2_{loc}(\mathbb{R}^{n-1}\times(0,\infty))$ to some function $v_{\varepsilon,\infty}\in C^2(\mathbb{R}^{n-1}\times(0,\infty))$ which solves (2.2). Clearly, inequality (4.10) yields

$$U^1_\varepsilon(y) < v_{\varepsilon,\infty}(y) < U^2_\varepsilon(y)\,, \qquad \text{for all } y \text{ in } \mathbb{R}^{n-1}\times(0,\infty).$$

Since $B_1\subset\{y_1>0\}$ and $B_2\subset\{y_1<0\}$ can be chosen arbitrarily, taking the supremum and the infimum respectively in the previous inequality, in view also of Proposition 4.3 we deduce that $v_{\varepsilon,\infty}(y)=g_\varepsilon\left(\frac{y_1}{y_n}\right)$, i.e., $v_{\varepsilon,\infty}$ is the elementary solution corresponding to the half spaces $C^+=\{y_1>0\}$ and $C^-=\{y_1<0\}$. By the uniqueness of the limit we conclude that the whole sequence $v_{\varepsilon,k}(y)$ converges to $g_\varepsilon(y_1/y_n)$ in $C^2_{loc}(\mathbb{R}^{n-1}\times(0,\infty))$, i.e. (4.7) holds.

Finally, we can always assume that $q_\infty\in B_1\cup B_2$, so that the last statement of the proposition easily follows from (4.10), choosing $y=q_k$ and letting $k\to\infty$. □

We are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. The existence of an entire solution $u_\varepsilon\in C^2(\mathbb{H}^n)\cap C^0(\mathbb{H}^n\cup S^{n-1}(\infty)\setminus L)$ to equation (1.2) satisfying the prescribed boundary conditions, that is a local minimizer of the energy $\mathcal{E}_\varepsilon$ in (1.1), and with $\Sigma_\varepsilon\subset\overline{conv(L)}$ is provided by Theorem 1.2.

Now we pass to the proof of the regularity property of $\Sigma_\varepsilon:=u_\varepsilon^{-1}(0)$ and its orthogonality to $S^{n-1}(\infty)$, using a blow-up argument based on Proposition 4.5. Let $\{P_k\}\subset\mathbb{H}^n$ be a sequence of points converging to some limit $P_\infty\in L$, and denote by p_k a projection of P_k on L, i.e., a point in L of minimal Euclidean distance from P_k in the half space model (with origin in P_∞), so that $\lambda_k:=|p_k-P_k|=\mathrm{dist}_E(P_k,L)$ is the Euclidean distance between P_k and L and $p_k\to P_\infty=0$ as $k\to\infty$. By Proposition 4.5 we have that, for suitable rotations $R_k\in O(\mathbb{R}^{n-1})$ converging to the identity

$$u_\varepsilon(p_k+\lambda_k R_k y)\to g_\varepsilon\left(\frac{\nu_L(P_\infty)\cdot y}{y_n}\right) \qquad \text{in } C^2_{loc}\big(\mathbb{R}^{n-1}\times(0,+\infty)\big), \tag{4.11}$$

as $k\to\infty$. By construction of p_k, we have that P_k belongs to the plane generated by $e_n:=(0,\dots,1)$ and $\nu_L(p_k)$ and passing through p_k. Clearly $P_k=p_k+\lambda_k R_k y_k$ for some $y_k\in\mathbb{R}^{n-1}\times(0,\infty)$ with $|y_k|=1$. Up to subsequence we have $y_k\to y_\infty$ for some $y_\infty\in\mathbb{R}^{n-1}\times[0,\infty)$ with $|y_\infty|=1$. Now we assume that $u_\varepsilon(P_k)=0$, i.e., $P_k\in\Sigma_\varepsilon$ for all k, and we wish to show that $y_\infty=e_n$. First, we claim that $y_\infty\cdot e_n\neq 0$. Indeed, if by contradiction, $y_\infty\cdot e_n=0$, then we would have $y_k\cdot e_n\to 0$, so that

$$1=\lim_k|\nu_L(p_k)\cdot R_k y_k|=\lim_k|\nu_L(P_\infty)\cdot y_k|=|\nu_L(P_\infty)\cdot y_\infty|.$$

In particular, we would have $\nu_L(P_\infty)\cdot y_\infty\neq 0$, $y_\infty\in\mathbb{R}^{n-1}\times\{0\}$ and $y_\infty=\pm\nu_L(P_\infty)$. Since $P_k\in\Sigma_\varepsilon$, the last statement in Proposition 4.5 would give a contradiction, and this proves the claim. Now $y_\infty\cdot e_n\neq 0$, hence (4.11) yields

$$g_\varepsilon\left(\frac{y_\infty\cdot\nu_L(P_\infty)}{y_\infty\cdot e_n}\right)=\lim_k u_\varepsilon(p_k+\lambda_k R_k y_k)=0,$$

which gives $y_\infty\cdot\nu_L(P_\infty)=0$. Since y_∞ belongs to the vector space generated by e_n and $\nu_L(P_\infty)$ and it has unit length, we conclude that $y_\infty=e_n$, and the whole sequence y_k converges to e_n as $k\to\infty$.

We are in the position to conclude the proof of the regularity of Σ_ε near $S^{n-1}(\infty)$, and its orthogonality property. Indeed, since

$$\nabla g_\varepsilon\left(\frac{y\cdot\nu_L(P_\infty)}{y\cdot e_n}\right)\neq 0 \qquad \text{for } y=y_\infty=e_n,$$

by (4.11), we deduce that also $\nabla u_\varepsilon(P_k)\neq 0$ for k large enough. Thus, as the sequence $\{P_k\}$ can be chosen arbitrarily, we conclude that Σ_ε is smooth near the sphere at infinity by the implicit function theorem, with a well defined normal vector field $\nu_{\Sigma_\varepsilon}(P)=\frac{\nabla u_\varepsilon(P)}{|\nabla u_\varepsilon(P)|}$.

Now, since $y_k\to e_n$, by (4.11) we deduce,

$$\nu_{\Sigma_\varepsilon}(P_k)=\frac{\nabla u_\varepsilon(P_k)}{|\nabla u_\varepsilon(P_k)|}\to\nu_L(P_\infty), \tag{4.12}$$

i.e., the normal vector field extends continuously up to the boundary, and this is enough to conclude that $\Sigma_\varepsilon\cup L$ is a C^1 hypersurface with boundary.

Finally, we prove the asymptotic expansion (1.4), using a blow-up argument analogous to that used to prove (4.11). Let $\{P_k\}\in\mathbb{H}^n$ converging to some $P_\infty\in S^{n-1}(\infty)$. If $P_\infty\notin L$ the proof is straightforward, since for $P_\infty\in\Omega^\pm$ we have that $u_\varepsilon(P_k)\to\pm1$ and $\tilde d(P_k,K(L))\to\pm\infty$.

Now, we consider the case $P_\infty\in L$, working as above in the half space model with origin in P_∞, so that $K(L)$ is the cone over L from the point e_n. Let $p_k\in L$ be points of minimal Euclidean distance from P_k, let $\lambda_k=|p_k-P_k|$, and let $R_k\in O(\mathbb{R}^{n-1})$ such that $R_k\nu_L(P_\infty)=\nu_L(p_k)$ and $R_k\to Id$, as in (4.11). Again, $P_k=p_k+\lambda_kR_ky_k$ for some $y_k\in\mathbb{R}^{n-1}\times(0,\infty)$ with $|y_k|=1$ and (up to a subsequence) $y_k\to y_\infty$ for some $y_\infty\in\mathbb{R}^{n-1}\times[0,\infty)$ with $|y_\infty|=1$.

Now we distinguish two cases, corresponding to $y_\infty\cdot e_n=0$ and $y_\infty\cdot e_n\neq0$. If $y_\infty\cdot e_n=0$, then, arguing as above, we have $y_\infty=\pm\nu_L(P_\infty)$, and hence by the last statement in Proposition 4.5 we have $u_\varepsilon(P_k)\to sgn(\nu_L(P_\infty)\cdot y_\infty)$. Thus, we have to prove that also $h_\varepsilon(\tilde d(P_k,K(L)))\to sgn(\nu_L(P_\infty)\cdot y_\infty)$. To this purpose, it is enough to notice that P_k lies always (for k large enough) on the same side of $K(L)$ and that $d(P_k,K(L))=d(R_ky_k,\lambda_k^{-1}(K(L)-p_k))\to\infty$ as $k\to\infty$, since $R_ky_k\to y_\infty$ while $\lambda_k^{-1}(K(L)-p_k)$ approaches the vertical half plane passing through the origin and orthogonal to y_∞.

We pass to consider the case $y_\infty\cdot e_n\neq0$. Since

$$g_\varepsilon\left(\frac{\nu_L(p_k)\cdot R_ky_k}{R_ky_k\cdot e_n}\right)\to g_\varepsilon\left(\frac{\nu_L(P_\infty)\cdot y_\infty}{y_\infty\cdot e_n}\right)\qquad \text{as } k\to\infty,$$

thanks to the blow-up formula given by (4.11) for $y=y_\infty$, it is clearly enough to prove that

$$g_\varepsilon\left(\frac{\nu_L(p_k)\cdot R_ky_k}{R_ky_k\cdot e_n}\right)-h_\varepsilon\left(\tilde d\big(R_ky_k,\lambda_k^{-1}(K(L)-p_k)\big)\right)\to0\qquad\text{as } k\to\infty. \tag{4.13}$$

Let us set $K^\infty(L)$ the cone over K from the point at infinity, i.e., $K^\infty(L)=L\times(0\times\infty)$. Then, it is easily seen that by construction of p_k, we have $\tilde d(P_k,K^\infty(L))=\tilde d(P_k,T_{p_k}L\times(0,\infty))$. Therefore,

$$g_\varepsilon\left(\frac{\nu_L(p_k)\cdot R_ky_k}{R_ky_k\cdot e_n}\right)=h_\varepsilon\left(\tilde d(P_k,T_{p_k}L\times(0,\infty))\right)=h_\varepsilon\left(\tilde d(P_k,K^\infty(L))\right).$$

Since $\tilde d(P_k,K^\infty(L))=\tilde d\big(R_ky_k,\lambda_k^{-1}(K^\infty(L)-p_k)\big)$, (4.13) is equivalent to $\square$

$$h_\varepsilon\left(\tilde{d}\big(R_k y_k, \lambda_k^{-1}(K^\infty(L)-p_k)\big)\right) - h_\varepsilon\left(\tilde{d}\big(R_k y_k, \lambda_k^{-1}(K(L)-p_k)\big)\right) \to 0 \tag{4.14}$$

as $k \to \infty$. In order to prove (4.14), it is enough to check the Hausdorff convergence on compact sets (usually referred to as Kuratowsky convergence) in $\mathbb{R}^{n-1} \times (0,\infty)$ of $\lambda_k^{-1}(K^\infty(L)-p_k)$ and $\lambda_k^{-1}(K(L)-p_k)$ to $T_{P_\infty}L \times (0,\infty)$. Finally, this Hausdorff convergence is indeed a direct consequence of the fact that $K^\infty(L)$ and $K(L)$ are tangent along L, since they both touch the sphere at infinity orthogonally along the smooth hypersurface L; for sake of brevity we skip the details which are standard.

5. Minimal hypersurfaces

In this final section we study the limit when ε tends to zero. First we investigate the behaviour of the energy functionals $\mathcal{E}_\varepsilon$ using Γ-convergence and we prove Theorem 1.4. Then we apply this result to the local minimizers u_ε to construct entire minimal hypersurfaces Σ_ε with prescribed boundary at infinity and we prove Theorem 1.5.

5.1. **Proof of the Γ-convergence result.** Here we prove the Γ-convergence result given by Theorem 1.4. The proof relies on the very well known arguments in the Euclidean setting [25], with some care in order to treat the boundary conditions $v_\varepsilon = w_\varepsilon$ on ∂B_R. We divide the proof in several steps, using the same notations defined in the Introduction.

Step 1 (Compactness.) Since the metric on compact subsets of $\mathbb{H}^n$ is equivalent to the Euclidean one, clearly we may assume that

$$\varepsilon_m \int_{B_R} \frac{1}{2}|\nabla v_{\varepsilon_m}|^2 + W_{\varepsilon_m}(v_{\varepsilon_m})dx \le C\,,$$

where $C>0$ depends only on R. Since $|v_{\varepsilon_m}| \le 1$, then arguing as in [25], Proposition 3, up to subsequence we have $v_{\varepsilon_m} \to v^*$ in $L^1(B_R)$, where $v^* \in BV(B_R;\{-1,+1\})$.

Step 2 (Γ-liminf.) By *Step 1* we may assume $v \in BV(B_R;\{-1,+1\})$, i.e., $\mathcal{F}(v;w^*,B_R)<\infty$. Moreover, we may assume that each v_{ε_m} has finite energy in B_R and $|v_{\varepsilon_m}| \le 1$ a.e., because energy decreases under truncation and truncation keeps the boundary conditions $v_{\varepsilon_m} = w_{\varepsilon_m}$ on ∂B_R. Now we essentially follows [25] but with some extra care because of the possible jump between v and w^* along ∂B_R.

Let $A \subset\subset B_1$ be an open set with compact closure such that $\overline{B_R} \subset A$ and let $\Psi(t) = \int_0^t \sqrt{W(s)}ds$, so that $\Psi \in C^1(\mathbb{R})$ and it is an odd function. We consider $\hat{v}_{\varepsilon_m} \in H^1(A)$ as v_{ε_m} extended as w_{ε_m} outside B_R. Since $\hat{v}_{\varepsilon_m} \in H^1(A)$, by the chain rule in $H^1(A)$ the functions $\Psi(\hat{v}_{\varepsilon_m})$ satisfy $\Psi(\hat{v}_{\varepsilon_m}) \in W^{1,1}(A) \subset BV(A)$ and

$$2|\Psi(\hat{v}_{\varepsilon_m})|_{BV_g(A)} = 2\int_A \sqrt{W(\hat{v}_{\varepsilon_m})}\|\nabla_g \hat{v}_{\varepsilon_m}\| dVol_g \le \sqrt{2}\varepsilon_m \mathcal{E}(\hat{v}_{\varepsilon_m},A) = \mathcal{F}_{\varepsilon_m}(\hat{v}_{\varepsilon_m};w_{\varepsilon_m},A)\,.$$

Taking (1.5) into account we have $\hat{v}_{\varepsilon_m} \to \tilde{v} = \tilde{v}_{w^*}$ in $L^1(A)$ as $m\to\infty$. By lower semicontinuity of the total variation and using the pointwise equality $2\Psi(\tilde{v}) = C_W\tilde{v}$ we obtain

$$C_W|\tilde{v}|_{BV_g(\overline{B_R})} \le C_W|\tilde{v}|_{BV_g(A)} = 2|\Psi(\tilde{v})|_{BV_g(A)} \le \liminf_{\varepsilon_m} \mathcal{F}_{\varepsilon_m}(v_{\varepsilon_m};w_{\varepsilon_m},A). \tag{5.1}$$

Finally, since $\mathcal{F}_{\varepsilon_m}(v_{\varepsilon_m};w_{\varepsilon_m},A) = \mathcal{F}_{\varepsilon_m}(v_{\varepsilon_m};w_{\varepsilon_m},B_R) + \mu_{\varepsilon_m}(A\setminus B_R))$ and $\mu^*(\partial B_R)=0$ the conclusion follows from (5.1) and (1.5) when $A = B_\rho$ and $\rho \searrow R$.

Step 3 (Γ-limsup without boundary conditions.) Here we show that, for any given function $v \in BV(B_R; \{-1,+1\})$, there exists a sequence $v_{\varepsilon_m} \in H^1(B_R)$ with $|v_{\varepsilon_m}| \le 1$, $v_{\varepsilon_m} \to v$ in $L^1(B_R)$ and such that

$$\sqrt{2}\varepsilon_m \mathcal{E}_{\varepsilon_m}(v_{\varepsilon_m}, B_R) \to C_W |v|_{BV_g(B_R)} \qquad \text{as } \varepsilon_m \to 0. \tag{5.2}$$

This Γ-limsup inequality is well understood in the Euclidean setting. The proof in the present case could be obtained by standard localization arguments, freezing the x dependence in the energy density functionals. Here for the reader convenience we sketch the original proof in [25], adapting it to the hyperbolic setting. By standard density arguments in the Euclidean setting and formulas (2.5) and (2.6), the class of functions in $BV_g(B_R; \{-1,+1\})$ with smooth jump set are actually dense in L^1 and in energy. Therefore, by diagonal arguments in Γ-convergence we can prove (5.2) assuming S_v smooth. In this case, following [25, Proposition 2] it turns out that a recovery sequence is given by $v_{\varepsilon_m}(x) = h_{\varepsilon_m}(\tilde{d}(x, S_v))$, where h_{ε_m} is the optimal one-dimensional profile given by Proposition 3.1, and $\tilde{d}(x, S_v) = v(x)d(x, S_v)$ is the hyperbolic signed distance from S_v (unique up to the sign).

Step 4 (Γ-limsup with boundary conditions). In this step we construct a recovery sequence taking into account the boundary conditions. To this purpose let $v \in BV(B_R; \{-1,+1\})$. First we show that the class of functions coinciding with w^* in a neighborhood of ∂B_R are dense in energy and in $L^1(B_R)$. Indeed, let $0 < \lambda < 1$ and set

$$v_\lambda(x) := \begin{cases} v(x) & \text{if } |x| \le \lambda R; \\ w^*(x) & \text{otherwise.} \end{cases}$$

Then we have $v_\lambda \equiv w^*$ near ∂B_R and $v_\lambda \to v$ in $L^1(B_R)$ as $\lambda \nearrow 1$. Moreover, since $\mu^*(\partial B_R) = 0$ it is easy to prove that $|D_g v_\lambda|(\partial B_{\lambda R}) \to |D_g \tilde{v}_{w^*}|(\partial B_R)$, so that $\mathcal{F}(v_\lambda; w^*, B_R) \to \mathcal{F}(v; w^*, B_R)$ as $\lambda \nearrow 1$. Therefore, up to a further diagonal argument, without loss of generality we may assume $v \equiv w^*$ in a neighborhood of ∂B_R, so that $\mathcal{F}(v; w^*, B_R) = C_W |v|_{BV_g(B_R)}$.

Now we aim to glue together the recovery sequence $v_{\varepsilon_m} \to v$ constructed in *Step 3* with w_{ε_m}, in order to obtain a recovery sequence which takes into account the boundary conditions. To this purpose, for any fixed $\eta > 0$ we construct an approximated recovery sequence $\hat{v}_{\varepsilon_m} \in H^1_{w_{\varepsilon_m}}(B_R)$ (depending on η), with $\hat{v}_{\varepsilon_m} \to v$ in $L^1(B_R)$ as $m \to \infty$ and satisfying

$$\limsup_m \mathcal{F}_{\varepsilon_m}(\hat{v}_{\varepsilon_m}; w_{\varepsilon_m}, B_R) \le \mathcal{F}(v; w^*, B_R) + C\eta. \tag{5.3}$$

Then, the Γ-limsup inequality follows from (5.3) by a standard diagonal argument as $\eta \to 0$.

To prove (5.3) let $\delta = \delta(\eta) > 0$ be so small such that the following holds.

i) $v = w^*$ in $C_\delta := B_R \setminus \overline{B_{R-\delta}}$;
ii) $\mathcal{E}_{\varepsilon_m}(w_{\varepsilon_m}, C_\delta) \le \eta {\varepsilon_m}^{-1}$ for every m;
iii) $\mathcal{E}_{\varepsilon_m}(v_{\varepsilon_m}, C_\delta) \le \eta {\varepsilon_m}^{-1}$ for every m.

Notice that, for δ suitably small ii) holds since $\mu^*(\partial B_R) = 0$, and iii) is true since v_{ε_m} is a recovery sequence for v in B_R, and therefore also in C_δ, and $|v|_{BV(C_\delta)} \to 0$ as $\delta \to 0$. For each m we divide the annulus C_δ in $M_m := [\frac{\delta}{\eta\varepsilon_m}]$ (where $[\cdot]$ is the integer part) concentric annuli of thickness $\tilde{\varepsilon}_m := \frac{\delta}{M_m}$. In this way we clearly have $\tilde{\varepsilon}_m = \eta_m \varepsilon_m$ with $\eta_m \to \eta$ as $m \to \infty$. Since $v_{\varepsilon_m} \to v = w^*$ and $w_{\varepsilon_m} \to w^*$ in $L^1(C_\delta)$ we have $v_{\varepsilon_m} - w_{\varepsilon_m} \to 0$ in $L^1(C_\delta)$. Therefore, by the mean value theorem we can choose $k_m \in \{1, \dots M_m\}$ such that

$$\frac{1}{\tilde{\varepsilon}_m} \int_{\tilde{C}_{k_m}} |v_{\varepsilon_m} - w_{\varepsilon_m}|\, dx \to 0 \qquad \text{as } m \to \infty, \tag{5.4}$$

where $\tilde{C}_{k_m} = \{R'_m < |x| < R''_m\}$ and $R'_m := R - \delta + (k_m - 1)\tilde{\varepsilon}_m$, $R''_m := R - \delta + k_m \tilde{\varepsilon}_m$. Let φ_{ε_m} be a radial Lipschitz cut-off function such that $\varphi_{\varepsilon_m}(x) \equiv 0$ for $|x| \geq R''_m$, $\varphi_{\varepsilon_m}(x) \equiv 1$ for $|x| \leq R'_m$, and decreases linearly along the rays in $\tilde{C}_{k_m}$. For all $x \in B_R$ we set

$$\hat{v}_{\varepsilon_m}(x) := v_{\varepsilon_m}(x)\varphi_{\varepsilon_m}(x) + w_{\varepsilon_m}(x)(1 - \varphi_{\varepsilon_m}(x)).$$

By construction we have for a.e. $x \in B_R$

$$|\nabla \hat{v}_{\varepsilon_m}(x)| \leq |\nabla v_{\varepsilon_m}| + |\nabla w_{\varepsilon_m}| + \frac{1}{\tilde{\varepsilon}_m}|v_{\varepsilon_m} - w_{\varepsilon_m}|.$$

Then, by ii) and iii) above, by Young inequality, and the bounds $|w_{\varepsilon_m}| \leq 1$, $|v_{\varepsilon_m}| \leq 1$ in B_R we obtain

$$\begin{aligned}\mathcal{E}_{\varepsilon_m}(\hat{v}_{\varepsilon_m}, \tilde{C}_{k_m}) &\leq C \int_{\tilde{C}_{k_m}} |\nabla \hat{v}_{\varepsilon_m}|^2 + W_{\varepsilon_m}(\hat{v}_{\varepsilon_m})\, dx \leq C \int_{C_\delta} |\nabla v_{\varepsilon_m}|^2 + |\nabla w_{\varepsilon_m}|^2\, dx + \\ &+ C \int_{\tilde{C}_{k_m}} \frac{1}{(\tilde{\varepsilon}_m)^2}|v_{\varepsilon_m} - w_{\varepsilon_m}|^2 + \frac{1}{(\varepsilon_m)^2}\, dx \leq C\eta {\varepsilon_m}^{-1} + \frac{C{\varepsilon_m}^{-1}}{\eta} \int_{\tilde{C}_{k_m}} \frac{1}{\tilde{\varepsilon}_m}|v_{\varepsilon_m} - w_{\varepsilon_m}|\, dx.\end{aligned}$$

Therefore, by (5.4), for m large enough (depending only on η) we have

$$\varepsilon_m \mathcal{E}_{\varepsilon_m}(\hat{v}_{\varepsilon_m}, \tilde{C}_{k_m}) \leq C\eta \tag{5.5}$$

By (5.5), in view of ii) above we have

$$\begin{aligned}\varepsilon_m \mathcal{E}_{\varepsilon_m}(\hat{v}_{\varepsilon_m}, B_R) = \varepsilon_m \mathcal{E}_{\varepsilon_m}(v_{\varepsilon_m}, B_{R'_m}) + \varepsilon_m \mathcal{E}_{\varepsilon_m}(\hat{v}_{\varepsilon_m}, \tilde{C}_{k_m}) + \\ + \varepsilon_m \mathcal{E}_{\varepsilon_m}(w_{\varepsilon_m}, B_R \setminus \overline{B_{R''_m}}) \leq \varepsilon_m \mathcal{E}_{\varepsilon_m}(v_{\varepsilon_m}, B_R) + C\eta\end{aligned}$$

Passing to the limit for $m \to \infty$, we obtain

$$\limsup_m \mathcal{F}_{\varepsilon_m}(\hat{v}_{\varepsilon_m}; w_{\varepsilon_m}, B_R) \leq \limsup_m \sqrt{2}\varepsilon_m \mathcal{E}_{\varepsilon_m}(v_{\varepsilon_m}, B_R) + C\eta = \mathcal{F}(v; w^*, B_R) + C\eta,$$

so that (5.3) holds, and this concludes the proof of the Γ-limsup inequality.

Remark 5.1. The assumption $\mu^*(\partial B_R) = 0$ is essential in order to identify the boundary term in the Γ-limit $\mathcal{F}$. Indeed, for w^* equal to 1 in B_R and $w^* = -1$ in $B_1 \setminus B_R$, it is very easy to construct two approximating sequences $w^{\pm}_{\varepsilon_m}$ for w^* satisfying (1.5), with traces on ∂B_R equal to ± 1, respectively. Therefore, the corresponding Γ-limit is clearly given by (1.7) with w^* replaced by ± 1 on $B_1 \setminus B_R$, respectively. More generally, given $\tilde{w}$ and w^* it is always possible to construct an approximating sequences w_{ε_m} for w^* such that the corresponding Γ-limit is given by (1.7) with w^* replaced by $\tilde{w}$. Thus we see that, removing the assumption $\mu^*(\partial B_R) = 0$, the Γ-limit may depend on the whole sequence w_{ε_m} and not only on w^*.

5.2. **Existence and asymptotic behavior of minimal hypersurfaces.** In this final part, we prove the existence of an entire minimal hypersurface with prescribed behaviour at infinity. First we give a local energy bound for the minimizers u_ε which allows to obtain a limiting function $u^* \in BV_{loc}(B_1; \{-1, 1\})$ with the desired behaviour at infinity. Then, we can apply the Γ-convergence result in the previous subsection to get the area-minimizing property of the jump set S_{u^*} and to conclude the proof of Theorem 1.5.

Lemma 5.2. *Let $0 < R < 1$, let $\varepsilon < R/2$ and let u_ε be a local minimizer of* (1.1)*. Then we have $\mathcal{E}_\varepsilon(u_\varepsilon, B_R) \le C\varepsilon^{-1}$, where C is a constant depending only on R.*

Proof. Since u_ε are uniformly bounded, by equation (2.4) and by standard elliptic regularity we have that $|\nabla u_\varepsilon| \le c\varepsilon^{-1}$, where c depends only on R. We deduce that

$$\|\nabla u_\varepsilon\|^2 + W_\varepsilon(u_\varepsilon) \le \frac{c}{\varepsilon^2} \qquad \text{in } B_R,$$

where c depends only on R. Let φ_ε be a radial cut-off function, equal to 1 for $|x| \le R - \varepsilon$, and decreasing linearly to zero along rays for $R-\varepsilon \le |x| \le R$. Let us set $v_\varepsilon := \varphi + (1-\varphi)u_\varepsilon$. By construction we have that $v_\varepsilon = u_\varepsilon$ on ∂B_R, and $|\nabla v_\varepsilon| \le c\varepsilon^{-1}$, with c depending only on R. Thus, by local energy minimality of u_ε we have

$$\mathcal{E}_\varepsilon(u_\varepsilon, B_R) \le \mathcal{E}_\varepsilon(v_\varepsilon, B_R) \le \mathcal{E}_\varepsilon(v_\varepsilon, B_R \setminus B_{R-\varepsilon}) \le \frac{C}{\varepsilon},$$

where C is a constant depending only on R. □

Proof of Theorem 1.5. We will prove claims i) and ii) of the theorem separately, using the Poincaré ball model. Claim iii) is well known and it has been already discussed in the Introduction.

Proof of i). By Lemma 5.2 we have $\mathcal{F}_{\varepsilon_m}(u_{\varepsilon_m}; u_{\varepsilon_m}, B_R) \le C$, and hence by Theorem 1.4, i), passing to a subsequence we have $u_{\varepsilon_m} \to u^*$ in $L^1(B_R)$ for some $u^* \in BV(B_R; \{-1,1\})$. Thus a simple diagonal argument yields $u_{\varepsilon_m} \to u^*$ in $L^1_{loc}(B_1)$ for some $u^* \in BV_{loc}(B_1; \{-1,1\})$ as $m \to \infty$.

We pass to the proof of $S_{u^*} \subset \overline{\mathrm{conv}(F)}$. To this purpose, notice that if $x \in B_1 \setminus \overline{\mathrm{conv}(F)}$, then there is a neighborhood N_x of x compactly contained in $B_1 \setminus \overline{\mathrm{conv}(F)}$, and there exists an elementary solution U_ε such that either $0 < U_\varepsilon \le u_\varepsilon \le 1$ with $U_\varepsilon \to 1$ uniformly in N_x, or $-1 \le u_\varepsilon \le U_\varepsilon < 0$ with $U_\varepsilon \to -1$ uniformly in N_x. In both cases we deduce that u^* is constant in N_x, so that in particular $S_{u^*} \cap N_x = \emptyset$. By the arbitrariness of $x \in B_1 \setminus \overline{\mathrm{conv}(F)}$ we conclude $S_{u^*} \subset \overline{\mathrm{conv}(F)}$, which clearly implies $\overline{S_{u^*}} \cap S^{n-1}(\infty) \subset F$.

It remains only to prove the inclusion $\partial\Omega^+ \cap \partial\Omega^- \subseteq \overline{S_{u^*}}$. Let $p \in \partial\Omega^+ \cap \partial\Omega^-$, and for any given $\delta > 0$ let us fix two points $q^\pm \in \Omega^\pm$ such that $q^\pm \in I_\delta(p)$. Moreover, let $0 < \rho < \delta$ be such that $B_\rho(q^+) \cap B_\rho(q^-) = \emptyset$ and $B_\rho(q^\pm) \cap B_1 \cap \overline{\mathrm{conv}(F)} = \emptyset$. Set $B^\pm := B_\rho(q^\pm) \cap S^{n-1}(\infty) \subset \Omega^\pm$ and consider the tube $T_\rho := \overline{\mathrm{conv}(B^+ \cup B^-)}$. Then, since $u^* = \pm 1$ on $B_\rho(q^\pm) \cap T_\rho$, we have that u^* takes both values $+1$ and -1 on sets of positive measure in T_ρ, that clearly implies $|Du^*|(T_\rho) = \mathcal{H}^{n-1}(S_{u^*} \cap T_\rho) > 0$. Therefore $B_{C\delta}(p) \cap S_{u^*} \ne \emptyset$ where C is a constant independent of δ, and this, by the arbitrariness of δ, yields $p \in \overline{S_{u^*}}$, which concludes the proof of property *i).*

Proof of ii). Let u^* as given by part *i)* and let $v^* \in BV_{\mathrm{loc}}(B_1; \{-1,1\})$ such that the support of $u^* - v^*$ is compactly contained in B_R for some $R \in (0,1)$. Note that, since $u_{\varepsilon_m} \to u^*$ in $L^1_{loc}(B_1)$ and Lemma 5.2 holds, passing to a subsequence if necessary, we may assume that $\{u_{\varepsilon_m}\}$ satisfies assumption (1.5) (with $w_{\varepsilon_m} = u_{\varepsilon_m}$). Thus, changing R slightly if necessary, we may also assume that $\mu^*(\partial B_R) = 0$, because such condition may fail for at most countably many radii. Let $v_{\varepsilon_m} \to v^*$ in $L^1(B_R)$ be a recovery sequence for v^* in B_R with u_{ε_m} as boundary data as given by Theorem 1.4. Thus, combining the Γ-convergence result given by Theorem 1.4 and the energy minimality of each u_{ε_m} we obtain

$$\mathcal{F}(u^*; u^*, B_R) \le \liminf_{\varepsilon_m} \mathcal{F}_{\varepsilon_m}(u_{\varepsilon_m}; u_{\varepsilon_m}, B_R) \le \limsup_{\varepsilon_m} \mathcal{F}_{\varepsilon_m}(v_{\varepsilon_m}; u_{\varepsilon_m}, B_R) \le \mathcal{F}(v; u^*, B_R)\,,$$

i.e. $|u^*|_{BV_g(B_R)} \leq |v^*|_{BV_g(B_R)}$ as claimed. □

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