ON THE BOUNDARY OF THE ATTAINABLE SET OF THE DIRICHLET SPECTRUM

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Abstract. Denoting by $\mathcal{E}\subseteq\mathbb{R}^2$ the set of the pairs $\big(\lambda_1(\Omega),\,\lambda_2(\Omega)\big)$ for all the open sets $\Omega\subseteq\mathbb{R}^N$ with unit measure, and by $\Theta\subseteq\mathbb{R}^N$ the union of two disjoint balls of half measure, we give an elementary proof of the fact that $\partial\mathcal{E}$ has horizontal tangent at its lowest point $\big(\lambda_1(\Theta),\,\lambda_2(\Theta)\big)$.

1. Introduction

Given an open set $\Omega\subseteq\mathbb{R}^N$ with finite measure, its Dirichlet-Laplacian spectrum is given by the numbers $\lambda>0$ such that the boundary value problem

$$-\Delta u=\lambda\,u\ \ \text{in }\Omega,\qquad u=0\ \ \text{on }\partial\Omega,$$

has non trivial solutions. Such numbers λ are called *eigenvalues of the Dirichlet-Laplacian in* Ω, and form a discrete increasing sequence $0<\lambda_1(\Omega)\le\lambda_2(\Omega)\le\lambda_3(\Omega)\dots$, diverging to $+\infty$ (see [4], for example). In this paper, we will work with the first two eigenvalues λ_1 and λ_2, for which we briefly recall the variational characterization: introducing the *Rayleigh quotient* as

$$\mathcal{R}_\Omega(u)=\frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}},\qquad u\in H^1(\Omega)\,,$$

the first two eigenvalues of the Dirichlet-Laplacian satisfy

$$\lambda_1(\Omega)=\min\Big\{\mathcal{R}_\Omega(u):\ u\in H^1_0(\Omega)\setminus\{0\}\Big\}\,,$$

$$\lambda_2(\Omega)=\min\left\{\mathcal{R}_\Omega(u)\ :\ u\in H^1_0(\Omega)\setminus\{0\},\ \int_\Omega u(x)\,u_1(x)\,dx=0\right\}\,,$$

where u_1 is a first eigenfunction.

We are concerned about the *attainable set* of the first two eigenvalues λ_1 and λ_2, that is,

$$\mathcal{E}:=\Big\{\big(\lambda_1(\Omega),\lambda_2(\Omega)\big)\in\mathbb{R}^2\ :\ \big|\Omega\big|=\omega_N\Big\}\,,$$

where ω_N is the volume of the ball of unit radius in $\mathbb{R}^N$. Of course, the set $\mathcal{E}$ depends on the dimension N of the ambient space. The set $\mathcal{E}$ has been deeply studied (see for instance [1, 3, 6]); an approximate plot is shown in Figure 1. Let us recall now some of the most important known facts. In what follows, we will always denote by B a ball of unit radius (then, of volume ω_N), and by Θ a disjoint union of two balls of volume $\omega_N/2$.

Basic properties of $\mathcal{E}$. *The attainable set $\mathcal{E}$ has the following properties:*

(i) *for every $(\lambda_1,\lambda_2)\in\mathcal{E}$ and every $t\ge1$, one has $(t\,\lambda_1,t\,\lambda_2)\in\mathcal{E}$;*

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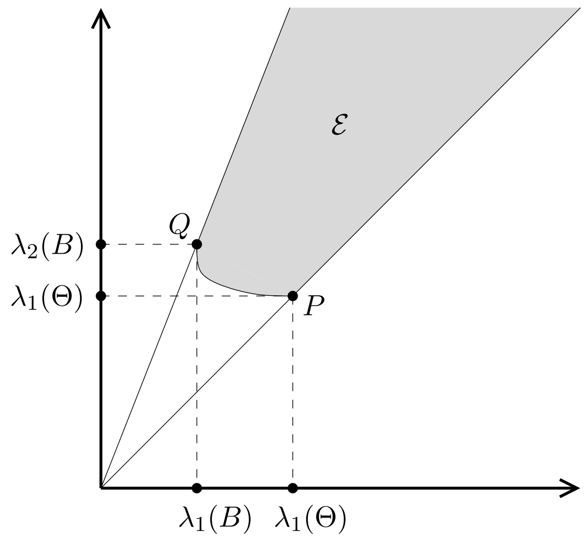

FIGURE 1. The attainable set $\mathcal{E}$

(ii)
$$\mathcal{E} \subseteq \left\{ x \geq \lambda_1(B),\, y \geq \lambda_2(\Theta),\, 1 \leq \frac{y}{x} \leq \frac{\lambda_2(B)}{\lambda_1(B)} \right\};$$

(iii) $\mathcal{E}$ *is* horizontally *and* vertically convex, *i.e., for every* $0 \leq t \leq 1$
$$(x_0, y), (x_1, y) \in \mathcal{E} \Longrightarrow \big((1-t)x_0 + tx_1, y\big) \in \mathcal{E}\,,$$
$$(x, y_0), (x, y_1) \in \mathcal{E} \Longrightarrow \big(x, (1-t)y_0 + ty_1\big) \in \mathcal{E}\,.$$

The first property is a simple consequence of the scaling property $\lambda_i(t\,\Omega) = t^{-2}\lambda_i(\Omega)$, valid for any open set $\Omega \subseteq \mathbb{R}^N$ and any $t > 0$. The second property is true because, for every open set Ω of unit measure, the Faber–Krahn inequality ensures $\lambda_1(\Omega) \geq \lambda_1(B)$, the Krahn–Szego inequality (see [5, 7, 8]) ensures $\lambda_2(\Omega) \geq \lambda_2(\Theta) = \lambda_1(\Theta)$, and a celebrated result by Ashbaugh and Benguria (see [2]) ensures
$$1 \leq \frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B)}{\lambda_1(B)}.$$
Finally, the third property is proven in [3]. It has been conjectured also that the set $\mathcal{E}$ is convex, as it seems reasonable by a numerical plot, but a proof for this fact is still not known.

Thanks to the above listed properties, the set $\mathcal{E}$ is completely known once one knows its "lower boundary"
$$\mathcal{C} := \Big\{ \big(\lambda_1,\, \lambda_2\big) \in \overline{\mathcal{E}} :\, \forall\, t < 1,\, \big(t\lambda_1,\, t\lambda_2\big) \notin \mathcal{E} \Big\},$$
therefore studying $\mathcal{E}$ is equivalent to study $\mathcal{C}$. Notice in particular that $\partial\mathcal{E}$ consists of the union of $\mathcal{C}$ with the two half-lines
$$\big\{(t,t) :\, t \geq \lambda_1(\Theta)\big\} \qquad \text{and} \qquad \left\{ \left(t, \frac{\lambda_2(B)}{\lambda_1(B)}\, t\right) :\, t \geq \lambda_1(B) \right\}.$$
Let us call for brevity P and Q the endpoints of $\mathcal{C}$, that is, $P \equiv \big(\lambda_1(\Theta), \lambda_2(\Theta)\big)$ and $Q \equiv \big(\lambda_1(B), \lambda_2(B)\big)$.

The plot of the set $\mathcal{E}$ seems to suggest that the curve $\mathcal{C}$ reaches the point Q with vertical tangent, and the point P with horizontal tangent. In fact, Wolf and Keller in [6, Section 5] proved the first fact, and they also suggested that the second fact should be true, providing a numerical evidence. The aim of the present paper is to give a short proof of this fact.

Theorem. *For every dimension $N \geq 2$, the curve $\mathcal{C}$ reaches the point P with horizontal tangent.*

The rest of the paper is devoted to prove this result: the proof will be achieved by exhibiting a suitable family $\{\widetilde{\Omega}_\varepsilon\}_{\varepsilon>0}$ of deformations of Θ having measure ω_N and such that

$$\lim_{\varepsilon\to 0} \frac{\lambda_2(\widetilde{\Omega}_\varepsilon) - \lambda_2(\Theta)}{\lambda_1(\Theta) - \lambda_1(\widetilde{\Omega}_\varepsilon)} = 0\,. \tag{1.1}$$

2. Proof of the Theorem

Throughout this section, for any given $x = (x_1, ..., x_N) \in \mathbb{R}^N$, we will write $x = (x_1, x')$ where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$.

We will make use of the sets $\{\Omega_\varepsilon\} \subseteq \mathbb{R}^N$, shown in Figure 2, defined by

$$\begin{aligned}\Omega_\varepsilon :=& \Big\{(x_1, x') \in \mathbb{R}^+ \times \mathbb{R}^{N-1} : \, (x_1 - 1 + \varepsilon)^2 + |x'|^2 < 1\Big\}\\ &\cup \Big\{(x_1, x') \in \mathbb{R}^- \times \mathbb{R}^{N-1} : \, (x_1 + 1 - \varepsilon)^2 + |x'|^2 < 1\Big\}\\ =:& \Omega_\varepsilon^+ \cup \Omega_\varepsilon^- \,.\end{aligned}$$

for every $\varepsilon > 0$ sufficiently small. The sets $\widetilde{\Omega}_\varepsilon$ for which we will eventually prove (1.1) will be rescaled copies of Ω_ε, in order to have measure ω_N.

To get our thesis, we need to provide an upper bound to $\lambda_1(\Omega_\varepsilon)$ and an upper bound to $\lambda_2(\Omega_\varepsilon)$; this will be the content of Lemmas 2.1 and 2.2 respectively.

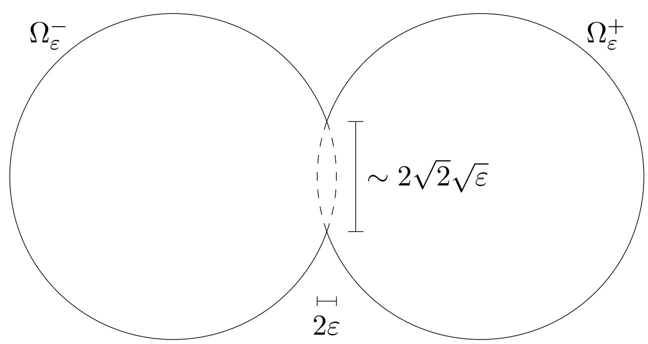

Figure 2. The sets $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$

Lemma 2.1. *There exists a constant $\gamma_1 > 0$ such that for every $\varepsilon \ll 1$ it is*

$$\lambda_1(\Omega_\varepsilon) \leq \lambda_1(B) - \gamma_1\, \varepsilon^{N/2}. \tag{2.1}$$

Proof. Let B_ε be the ball of unit radius centered at $(1-\varepsilon, 0)$, so that $B_\varepsilon \subseteq \Omega_\varepsilon$ and in particular $\Omega_\varepsilon^+ = B_\varepsilon \cap \{x_1 > 0\}$. Let also u be a first Dirichlet eigenfunction of B_ε with unit L^2 norm, and denote by $\mathbb{T}$ the region (shaded in Figure 3) bounded by the right circular conical surface $\{\sqrt{2\varepsilon - \varepsilon^2} - x_1 - |x'| = 0\}$ and by the plane $\{x_1 = 0\}$.

Since the normal derivative of u is constantly κ on ∂B_ε^+, we know that

$$Du(x_1, x') = Du(0, x') + O\big(\sqrt{\varepsilon}\big) = \big(\kappa, 0\big) + O\big(\sqrt{\varepsilon}\big) \quad \text{on } \mathbb{T}\,. \tag{2.2}$$

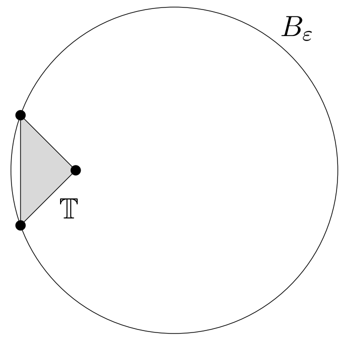

FIGURE 3. The ball B_ε and the cone $\mathbb{T}$ (shaded) in the proof of Lemma 2.1

Let us now define the function $\tilde u : \Omega_\varepsilon^+ \to \mathbb{R}$ as

$$\tilde u(x_1,x') := \begin{cases} u(x_1,x') & \text{if } (x_1,x') \notin \mathbb{T}\,, \\ u(x_1,x') + \dfrac{\kappa}{2}\Big(\sqrt{2\varepsilon-\varepsilon^2} - x_1 - |x'|\Big) & \text{if } (x_1,x') \in \mathbb{T}\,. \end{cases}$$

It is immediate to observe that $\tilde u = u$ on the surface $\Big\{\sqrt{2\varepsilon-\varepsilon^2} - x_1 - |x'| = 0\Big\} \cap \{x_1 > 0\}$, so that $\tilde u \in H^1(\Omega_\varepsilon^+)$. Notice that $\tilde u \notin H^1_0(\Omega_\varepsilon^+)$ since $\tilde u$ does not vanish on $\{x_1 = 0\} \cap \partial\Omega_\varepsilon^+$. By construction and recalling (2.2),

$$D\tilde u(x_1,x') = Du(x_1,x') + \Big(-\frac{\kappa}{2}, -\frac{\kappa}{2}\,\frac{x'}{|x'|}\Big) = \Big(\frac{\kappa}{2}, -\frac{\kappa}{2}\,\frac{x'}{|x'|}\Big) + O\big(\sqrt{\varepsilon}\big) \quad \text{on } \mathbb{T}\,. \tag{2.3}$$

Since $\tilde u \geq u$ on Ω_ε^+, and recalling that $u \in H^1_0(B_\varepsilon^+)$, one clearly has

$$\int_{\Omega_\varepsilon^+} \tilde u^2 dx \geq \int_{\Omega_\varepsilon^+} u^2 dx = \int_{B_\varepsilon^+} u^2 dx + O(\varepsilon^{(N+5)/2}) = 1 + O(\varepsilon^{(N+5)/2})\,, \tag{2.4}$$

since the small region $B_\varepsilon \setminus \Omega_\varepsilon^+$ has volume $O(\varepsilon^{(N+1)/2})$, and on this region $u = O(\varepsilon)$.

On the other hand, comparing (2.2) and (2.3), one has

$$\big|D\tilde u\big|^2 = \big|Du\big|^2 - \frac{\kappa^2}{2} + O\big(\sqrt{\varepsilon}\big) \quad \text{on } \mathbb{T}\,,$$

and since the volume of $\mathbb{T}$ is $\frac{\omega_{N-1}}{N}\big(2\varepsilon-\varepsilon^2\big)^{N/2}$ we deduce

$$\begin{aligned} \int_{\Omega_\varepsilon^+} \big|D\tilde u\big|^2 dx &= \int_{\Omega_\varepsilon^+} \big|Du\big|^2 dx - \frac{\omega_{N-1}}{N}\big(2\varepsilon-\varepsilon^2\big)^{N/2}\Big(\frac{\kappa^2}{2} + O(\sqrt{\varepsilon})\Big) \\ &= \int_{\Omega_\varepsilon^+} \big|Du\big|^2 dx - \frac{\omega_{N-1}}{N}\kappa^2 2^{(N/2-1)}\varepsilon^{N/2} + O(\varepsilon^{(N+1)/2}) \\ &= \int_{B_\varepsilon^+} \big|Du\big|^2 dx - C_N\kappa^2\varepsilon^{N/2} + O(\varepsilon^{(N+1)/2})\,, \end{aligned} \tag{2.5}$$

where $C_N = \frac{\omega_{N-1}}{N}2^{(N/2-1)}$.

Therefore, by (2.4) and (2.5) we obtain

$$\begin{aligned} \mathcal{R}_{\Omega_\varepsilon^+}(\tilde u) = \frac{\displaystyle\int_{\Omega_\varepsilon^+} \big|D\tilde u\big|^2 dx}{\displaystyle\int_{\Omega_\varepsilon^+} \tilde u^2\, dx} &\leq \mathcal{R}_{B_\varepsilon^+}(u) - C_N\kappa^2\varepsilon^{N/2} + O(\varepsilon^{(N+1)/2}) \\ &= \lambda_1(B) - C_N\kappa^2\varepsilon^{N/2} + O(\varepsilon^{(N+1)/2})\,. \end{aligned}$$

We can finally extend $\tilde u$ to the whole Ω_ε, simply defining $\tilde u(x_1,x') = \tilde u(|x_1|,x')$ on Ω_ε^-. By construction, $\tilde u \in H^1_0(\Omega_\varepsilon)$, and

$$\lambda_1(\Omega_\varepsilon) \le \mathcal{R}_{\Omega_\varepsilon}(\tilde u) = \mathcal{R}_{\Omega_\varepsilon^+}(\tilde u) \le \lambda_1(B) - C_N\kappa^2\varepsilon^{N/2} + O(\varepsilon^{(N+1)/2})\,,$$

so that (2.1) follows and the proof is concluded. □

Lemma 2.2. *There exists a constant $\gamma_2 > 0$ such that for every $\varepsilon \ll 1$, it is*

$$\lambda_2(\Omega_\varepsilon) \le \lambda_1(B) + \gamma_2\,\varepsilon^{(N+1)/2}. \tag{2.6}$$

Proof. First of all, we start underlining that

$$\lambda_2(\Omega_\varepsilon) \le \lambda_1(\Omega_\varepsilon^+)\,; \tag{2.7}$$

in fact if we define

$$\tilde u(x_1,x') := \begin{cases} u_\varepsilon(x_1,x')\,, & \text{if } x_1 \in \Omega_\varepsilon^+\,,\\ -u_\varepsilon(-x_1,x')\,, & \text{if } x_1 \in \Omega_\varepsilon^-\,, \end{cases}$$

then by construction it readily follows that $-\Delta\tilde u = \lambda_1(\Omega_\varepsilon^+)\tilde u$. As a consequance $\lambda_1(\Omega_\varepsilon^+)$ is an eigenvalue of Ω_ε, say $\lambda_1(\Omega_\varepsilon^+) = \lambda_\ell(\Omega_\varepsilon)$. Since Ω_ε is connected and $\tilde u$ changes sign, it is not possible $\ell = 1$, hence

$$\lambda_2(\Omega_\varepsilon) \le \lambda_\ell(\Omega_\varepsilon) = \lambda_1(\Omega_\varepsilon^+)\,.$$

It is then enough for us to estimate $\lambda_1(\Omega_\varepsilon^+)$. To this aim, define the set

$$\mathcal{O}_\varepsilon := \big\{(x_1,x') \in \Omega_\varepsilon^+ \,:\, x_1 \ge \varepsilon\big\},$$

and take a Lipschitz cut-off function $\xi_\varepsilon \in W^{1,\infty}(\Omega_\varepsilon^+)$ such that

$$0 \le \xi_\varepsilon \le 1 \text{ on } \Omega_\varepsilon^+\,, \qquad \xi_\varepsilon \equiv 1 \text{ on } \mathcal{O}_\varepsilon\,, \qquad \xi_\varepsilon \equiv 0 \text{ on } \partial\Omega_\varepsilon^+ \cap \{x_1 = 0\}\,, \qquad \|\nabla\xi_\varepsilon\|_\infty \le L\,\varepsilon^{-1}\,.$$

As in Lemma 2.1, let again u be a first eigenfunction of the ball B_ε of radius 1 centered at $(1-\varepsilon,0)$ having unit L^2 norm, and define on Ω_ε the function $\varphi = u\,\xi_\varepsilon$. Since by construction φ belongs to $H^1_0(\Omega_\varepsilon)$, we obtain

$$\lambda_1(\Omega_\varepsilon^+) \le \mathcal{R}(\varphi,\Omega_\varepsilon^+) = \frac{\displaystyle\int_{\Omega_\varepsilon^+}\Big[|\nabla u|^2\,\xi_\varepsilon^2 + |\nabla\xi_\varepsilon|^2\,u^2 + 2\,u\,\xi_\varepsilon\,\langle\nabla u,\nabla\xi_\varepsilon\rangle\Big]\,dx}{\displaystyle\int_{\Omega_\varepsilon^+} u^2\,\xi_\varepsilon^2\,dx}\,. \tag{2.8}$$

We can start estimating the denominator very similarly to what already done in (2.4). Indeed, recalling that $\big|\Omega_\varepsilon^+ \setminus \mathcal{O}_\varepsilon\big| = O(\varepsilon^{(N+1)/2})$ and that in that small region $u = O(\varepsilon)$, we have

$$\int_{\Omega_\varepsilon^+} u^2\,\xi_\varepsilon^2\,dx = \int_{B_\varepsilon} u^2\,dx - \int_{B_\varepsilon\setminus\Omega_\varepsilon^+} u^2\,dx - \int_{\Omega_\varepsilon^+\setminus\mathcal{O}_\varepsilon} u^2(1-\xi_\varepsilon^2)\,dx = 1 + O(\varepsilon^{(N+5)/2})\,.$$

Let us pass to study the numerator: first of all, being $0 \le \xi_\varepsilon \le 1$ we have

$$\int_{\Omega_\varepsilon^+} |\nabla u|^2\,\xi_\varepsilon^2\,dx \le \int_{B_\varepsilon} |\nabla u|^2\,dx = \lambda_1(B)\,.$$

Moreover,

$$\int_{\Omega_\varepsilon^+} |\nabla\xi_\varepsilon|^2\,u^2\,dx = \int_{\Omega_\varepsilon^+\setminus\mathcal{O}_\varepsilon} |\nabla\xi_\varepsilon|^2\,u^2\,dx \le \frac{L^2}{\varepsilon^2}\,|\Omega_\varepsilon^+ \setminus \mathcal{O}_\varepsilon|\,\|u\|^2_{L^\infty(\Omega_\varepsilon^+\setminus\mathcal{O}_\varepsilon)} = O(\varepsilon^{(N+1)/2})\,,$$

and in the same way

$$\int_{\Omega_\varepsilon^+} u\,\xi_\varepsilon\,\langle \nabla u, \nabla \xi_\varepsilon\rangle\,dx \le \int_{\Omega_\varepsilon^+\setminus \mathcal{O}_\varepsilon} |u|\,|\nabla u|\,|\nabla \xi_\varepsilon|\,dx = O(\varepsilon^{(N+1)/2})\,.$$

Summarizing, by (2.8) we deduce

$$\lambda_1(\Omega_\varepsilon^+) \le \lambda_1(B) + O(\varepsilon^{(N+1)/2})\,,$$

thus by (2.7) we get the thesis. □

We are now ready to conclude the paper by giving the proof of the Theorem.

Proof of the Theorem. For any small $\varepsilon > 0$, we define $\widetilde{\Omega}_\varepsilon = t_\varepsilon\,\Omega_\varepsilon$, where $t_\varepsilon = \sqrt[N]{\omega_N/|\Omega_\varepsilon|}$ so that $|\widetilde{\Omega}_\varepsilon| = \omega_N$. Notice that

$$|\Omega_\varepsilon| = 2\omega_N + O(\varepsilon^{(N+1)/2})\,,$$

thus $t_\varepsilon = 2^{-1/N} + O(\varepsilon^{(N+1)/2})$. Recalling the trivial rescaling formula $\lambda_i(t\Omega) = t^{-2}\lambda_i(\Omega)$, valid for any natural i, any positive t and any open set Ω, we can then estimate by Lemma 2.1 and Lemma 2.2

$$\lambda_1(\widetilde{\Omega}_\varepsilon) = \left(\frac{|\Omega_\varepsilon|}{\omega_N}\right)^{2/N}\lambda_1(\Omega_\varepsilon) \le 2^{2/N}\lambda_1(B) - 2^{2/N}\gamma_1\varepsilon^{N/2} + O(\varepsilon^{(N+1)/2})\,,$$

$$\lambda_2(\widetilde{\Omega}_\varepsilon) = \left(\frac{|\Omega_\varepsilon|}{\omega_N}\right)^{2/N}\lambda_2(\Omega_\varepsilon) \le 2^{2/N}\lambda_1(B) + O(\varepsilon^{(N+1)/2})\,.$$

Since $\lambda_1(\Theta) = \lambda_2(\Theta) = 2^{2/N}\lambda_1(B)$, the two above estimates give

$$\lim_{\varepsilon\to 0}\frac{\lambda_2(\widetilde{\Omega}_\varepsilon) - \lambda_2(\Theta)}{\lambda_1(\Theta) - \lambda_1(\widetilde{\Omega}_\varepsilon)} = 0\,,$$

which as already noticed in (1.1) implies the thesis. □

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