# SHARP STABILITY INEQUALITIES
# FOR THE PLATEAU PROBLEM

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## Abstract

The validity of global quadratic stability inequalities for uniquely regular area minimizing hypersurfaces is proved to be equivalent to the uniform positivity of the second variation of the area. Concerning singular area minimizing hypersurfaces, by a "quantitative calibration" argument we prove quadratic stability inequalities with explicit constants for all the Lawson's cones, excluding six exceptional cases. As a by-product of these results, explicit lower bounds for the first eigenvalues of the second variation of the area on these cones are derived.

## 1. Introduction

**1.1. Overview.** The aim of this paper is to start the study of global stability inequalities for area minimizing surfaces, along the lines developed in recent years for isoperimetric-type problems (see, e.g. [**Fu2**, **Fu1**, **H**, **HHaW**, **FMaPr**, **Ma1**, **FiMaPr**, **CL**]). We shall focus on the codimension one case. The case of uniquely area minimizing regular hypersurfaces with positive definite second variation is addressed in sharp form, as discussed in section 1.2. This result leaves open the problem in the case of a generic area minimizing hypersurface with singularities, which may occur in (ambient space) dimension 8 or larger. However, by a "quantitative calibration" argument, we prove global quadratic stability inequalities with explicit constants for all the Lawson's cones, except for six exceptional low-dimensional cases, see section 1.3. In section 1.4 we briefly discuss the relationship between stability inequalities and foliations, while section 1.5 describes the organization of the paper.

**1.2. From infinitesimal to global stability inequalities.** We denote by

$$\mathcal{M}$$

the family of the smooth, compact, orientable hypersurfaces $M \subset \mathbb{R}^{n+1}$ with smooth boundary $\mathrm{bdry}\, M$. We say that $M \in \mathcal{M}$ is uniquely area minimizing in $\mathcal{M}$ if, denoting by $\mathcal{H}^n$ the $n$-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$,

$$\mathcal{H}^n(M') \geq \mathcal{H}^n(M)\,, \qquad \forall M' \in \mathcal{M}\,, \quad \mathrm{bdry}\, M' = \mathrm{bdry}\, M\,, \tag{1.1}$$

with $\mathcal{H}^n(M') = \mathcal{H}^n(M)$ if and only if $M' = M$. If $M, M' \in \mathcal{M}$, then there exists a Borel set $E \subset \mathbb{R}^{n+1}$ with finite Lebesgue measure $\mathcal{L}^{n+1}(E)$ bounded by $M \Delta M' = (M \setminus M') \cup (M' \setminus M)$ (see Figure 1.1 and Lemma 2.2). We thus seek necessary and sufficient conditions for the existence of a positive constant $\kappa$ (possibly depending on $M$) such that, if $\mathrm{bdry}\, M' = \mathrm{bdry}\, M$, then the "global" stability inequality

$$\mathcal{H}^n(M') - \mathcal{H}^n(M) \geq \kappa \,\min\left\{\mathcal{L}^{n+1}(E)^2\,, \mathcal{L}^{n+1}(E)^{n/(n+1)}\right\}\,, \tag{1.2}$$

holds true. The exponents $n/(n+1)$ and 2 on the right-hand side of (1.2) are motivated by the analysis of two limit regimes for the inequality, namely

$$\mathcal{H}^n(M') \to +\infty \qquad \text{and} \qquad \mathcal{H}^n(M') \to \mathcal{H}^n(M)\,.$$

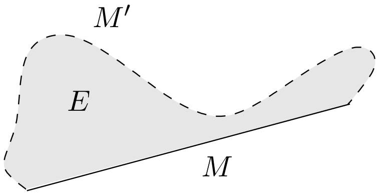

**Figure 1.1.** If $M$ and $M'$ have the same boundary, then $M\Delta M'$ is $\mathcal{H}^n$-equivalent to the boundary of a Borel set $E$. If $M$ is uniquely area minimizing, then the area minimality of $M$ should imply a control of $\mathcal{H}^n(M')-\mathcal{H}^n(M)$ on $\mathcal{L}^{n+1}(E)$. The picture refers to the planar case $n=1$.

In the first limit regime, (1.2) follows by the Euclidean isoperimetric inequality, as

$$\mathcal{H}^n(M')-\mathcal{H}^n(M)\approx\mathcal{H}^n(M')+\mathcal{H}^n(M)=\mathcal{H}^n(\partial E)\ge(n+1)\omega_{n+1}^{1/(n+1)}\,\mathcal{L}^{n+1}(E)^{n/(n+1)}\,,$$

where $\omega_k$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^k$. In the second limit regime, $\mathcal{H}^n(M')$ is very close to $\mathcal{H}^n(M)$, and, since $M$ is uniquely area minimizing, we expect $M'$ to be a small normal deformation of $M$. In other terms, if $\nu_M\in C^\infty(M;S^n)$ is a normal vector field to $M$, then we expect

$$M'\approx\Big\{x+t\,\varphi(x)\,\nu_M(x):x\in M\Big\}\,, \tag{1.3}$$

for some small $t$ and some smooth $\varphi\colon M\to\mathbb{R}$ which vanishes on $\mathrm{bdry}\,M$. In this way,

$$\mathcal{H}^n(M')-\mathcal{H}^n(M)\quad\approx\quad t^2\int_M|\nabla^M\varphi|^2-|\mathrm{II}_M|^2\varphi^2\,d\mathcal{H}^n+O(t^3)\,, \tag{1.4}$$

$$\mathcal{L}^{n+1}(E)\quad\approx\quad|t|\int_M|\varphi|\,d\mathcal{H}^n+O(t^2)\,, \tag{1.5}$$

where the first order term in (1.4) vanishes because $M$ has vanishing mean curvature. Here, $\nabla^M\varphi$ denotes the tangential gradient of $\varphi$ with respect to $M$, and $\mathrm{II}_M$ is the second fundamental form of $M$. If the first eigenvalue of the second variation of the area is strictly positive at $M$, that is, if there exists $\lambda>0$ such that

$$\int_M|\nabla^M\varphi|^2-|\mathrm{II}_M|^2\varphi^2\,d\mathcal{H}^n\ge\lambda\int_M\varphi^2\,d\mathcal{H}^n\,, \tag{1.6}$$

whenever $\varphi\in C^1_0(M)=\{\varphi\in C^1(M):\varphi=0\text{ on }\mathrm{bdry}\,M\}$, then, in view of (1.4) and (1.5), we expect $\mathcal{H}^n(M')-\mathcal{H}^n(M)$ to control $\mathcal{L}^{n+1}(E)^2$.

These considerations suggest that if $M\in\mathcal{M}$ is uniquely area minimizing in $\mathcal{M}$, then the global stability inequality (1.2) is equivalent to (1.6), the positivity of the second variation of the area at $M$. However, due to the possible presence of singular area minimizing hypersurfaces, this very natural statement may fail to be true, at least in dimension $n\ge7$. To explain what may go wrong, let us introduce the class

$$\mathcal{M}_0$$

of the bounded sets $M_0$ in $\mathbb{R}^{n+1}$ such that, for some non-empty closed set $\Sigma\subset M_0$ with $\mathcal{H}^n(\Sigma)=0$, $M_0\setminus\Sigma$ is a smooth, bounded, orientable hypersurface. Even if $M\in\mathcal{M}$ is uniquely area minimizing in $\mathcal{M}$, there could still exist some $M_0\in\mathcal{M}_0$ with $\mathcal{H}^n(M_0)=\mathcal{H}^n(M)$, such that $M_0$ and $M$ has the same boundary in the sense of Stokes Theorem

$$\int_M d\omega=\int_{M_0}d\omega\,,\qquad\forall\omega\in\mathcal{D}^{n-1}(\mathbb{R}^{n+1})\,, \tag{1.7}$$

where $\mathcal{D}^k(\mathbb{R}^{n+1})$ denotes the space of smooth $k$-forms in $\mathbb{R}^{n+1}$, and where $d\omega$ is the exterior derivative of $\omega$ (note that the integral on the right-hand side of (1.7) is unaffected by the presence of $\Sigma$, since $\mathcal{H}^n(\Sigma)=0$). If $M_0\in\mathcal{M}$, then it is possible to construct a sequence $\{M_h\}_{h\in\mathbb{N}} \subset \mathcal{M}$ with $\mathrm{bdry}\, M_h = \mathrm{bdry}\, M_0$ for every $h \in \mathbb{N}$ (in the sense of (1.7)), $\mathcal{H}^n(M_h) \to \mathcal{H}^n(M_0)$ as $h \to \infty$, and, if $F_h$ denotes the region bounded by $M_0 \Delta M_h$, with $\mathcal{L}^{n+1}(F_h) \to 0$ as $h \to \infty$. Since $\mathcal{M} \cap \mathcal{M}_0 = \emptyset$, it is necessarily $M \neq M_0$, and denoting by $E_h$ the region bounded by $M \Delta M_h$, it must be $\lim_{h\to\infty} \mathcal{L}^{n+1}(E_h) > 0$, thus contradicting inequality (1.2). In other words, even if $M$ is uniquely area minimizing in $\mathcal{M}$, nevertheless the boundary of $M$ may also span a *singular* area minimizing hypersurface $M_0$, thus breaking down the global stability inequality (1.2). In order to prove global stability inequalities we have thus to work with a stronger uniqueness assumption than being uniquely area minimizing in $\mathcal{M}$.

Our first main result, Theorem 1 below, asserts the equivalence between the infinitesimal stability inequality (1.6) and the global stability inequality (1.2), provided $M$ is assumed to be **uniquely mass minimizing as an integral $n$-current**, rather than merely uniquely area minimizing in $\mathcal{M}$. In section 2, we shall discuss this notion of minimality in detail. For the moment, it suffices to notice that it amounts in asking that

$$\mathcal{H}^n(M_0) \geq \mathcal{H}^n(M)\,, \qquad \forall M_0 \in \mathcal{M} \cup \mathcal{M}_0\,, \quad \mathrm{bdry}\, M_0 = \mathrm{bdry}\, M\,,$$

with $\mathcal{H}^n(M_0) = \mathcal{H}^n(M)$ if and only if $M_0 = M$. We also notice that if $1 \leq n \leq 6$ and $M$ is uniquely area minimizing in $\mathcal{M}$, then, by the regularity theory for integer mass minimizing currents [**Fe1**, Chapter 5], $M$ is uniquely mass minimizing as an integral $n$-current.

**Theorem 1.** *If $n \geq 1$ and $M \in \mathcal{M}$ is uniquely mass minimizing as an integral $n$-current, then the two following statements are equivalent:*

*(a) The first eigenvalue $\lambda(M)$ of the second variation of the area at $M$,*

$$\lambda(M) = \inf\left\{ \int_M |\nabla^M \varphi|^2 - |\mathrm{II}_M|^2 \varphi^2 \, d\mathcal{H}^n : \varphi \in C^1_0(M)\,, \int_M \varphi^2\, d\mathcal{H}^n = 1 \right\}, \tag{1.8}$$

*is positive.*

*(b) There exists $\kappa > 0$, depending on $M$, such that, if $M' \in \mathcal{M}$ and $\mathrm{bdry}\, M' = \mathrm{bdry}\, M$, then, for some Borel set $E \subset \mathbb{R}^{n+1}$ with $\partial E$ equivalent up to a $\mathcal{H}^n$-null set to $M \Delta M'$,*

$$\mathcal{H}^n(M') - \mathcal{H}^n(M) \geq \kappa \min\left\{ \mathcal{L}^{n+1}(E)^2, \mathcal{L}^{n+1}(E)^{n/(n+1)} \right\}. \tag{1.9}$$

**Remark 1.** We are not able to link in any explicit way $\lambda(M)$ to the constant $\kappa$ appearing in (2.3). Probably, this is not so surprising due to the level of generality allowed by the assumptions of Theorem 1 itself. The relation between these two quantities may be subtle, as shown by the example in Figure 1.2. We further notice that the positivity of $\lambda(M)$ is in fact equivalent (by a standard compactness and regularity argument) in asking that

$$\int_M |\nabla^M \varphi|^2 - |\mathrm{II}_M|^2 \varphi^2 \, d\mathcal{H}^n > 0\,, \qquad \forall \varphi \in C^1_0(M) \setminus \{0\}\,. \tag{1.10}$$

Of course, we may hope to prove inequalities like (2.3) with an explicit constants $\kappa$ on explicit examples. We shall discuss this problem in sections 1.3-1.4.

**Remark 2** (Local stability and uniform convexity)**.** The stability inequality (2.3) is easily seen to hold with respect to $C^1$-small *graph-type* variations of $M$ supported at a sufficiently small scale. Let $r_0 > 0$ be the scale, which implicitly depends on $M$, such that, in any ball of radius $r_0$, $M$ is representable as the graph of Lipschitz functions $u\colon \mathbb{R}^n \to \mathbb{R}$, with $\mathrm{Lip}(u) \leq 1$ over a disk $\mathbf{D}_{r_0} \subset \mathbb{R}^n$ of radius $r_0$. In this case, $u$ is a Lipschitz minimizer of the area functional, therefore, if $M'$ is a variation of $M$ supported in the corresponding ball of radius $r_0$, which corresponds to a Lipschitz function $v\colon \mathbb{R}^n \to \mathbb{R}$ with $v = u$ on $\mathbf{D}_{r_0}$,

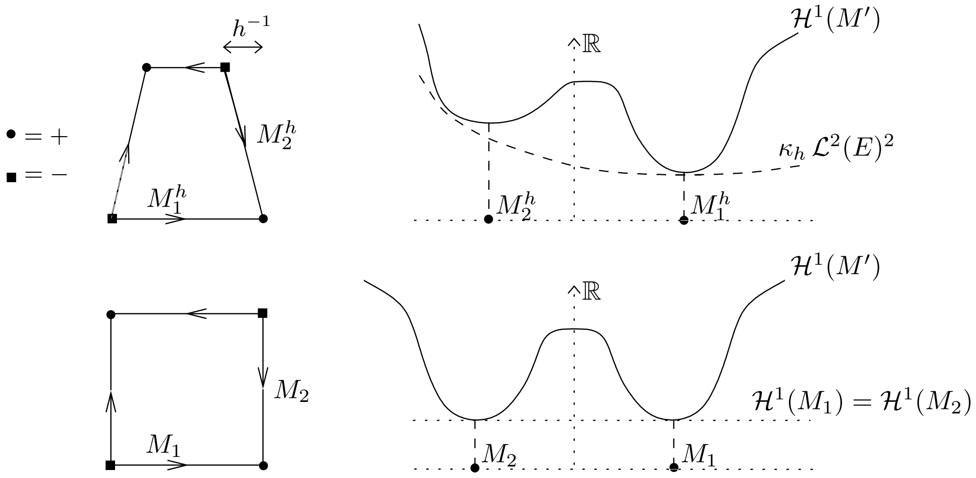

**Figure 1.2.** In the two pictures on the left, we consider the length minimizing curves spanned by a sequence of four points converging to the vertexes of a square in the plane (round points are charged positively, square points are charged negatively). For every $h \in \mathbb{N}$, let $\kappa_h$ denote the best constant for inequality (2.3). The competitors $M_2^h$ lie at uniformly positive distance from the corresponding $M_1^h$ (and are, of course, local length minimizers). Their presence forces $\kappa_h \to 0$ as $h \to \infty$. At the same time $\lambda(M_1^h) = (\mathcal{H}^1(M_1^h)/2\pi)^2$ is converging to a positive constant as $h \to \infty$. See Remark 4 for a proper reformulation of Theorem 1 in the situation considered here.

then, setting $\varphi = v - u$ and $f(\xi) = \sqrt{1 + |\xi|^2}$, $\xi \in \mathbb{R}^n$, we find,

$$\begin{aligned}\mathcal{H}^n(M') - \mathcal{H}^n(M) &= \int_{\mathbf{D}_{r_0}} f(\nabla v) - \int_{\mathbf{D}_{r_0}} f(\nabla u)\\ &= \int_{\mathbf{D}_{r_0}} \nabla f(\nabla u) \cdot \nabla\varphi + \int_{\mathbf{D}_{r_0}} \nabla^2 f(\nabla u)\Big(\nabla\varphi, \nabla\varphi\Big) + O(\|\varphi\|_{C^1}) \int_{\mathbf{D}_{r_0}} |\nabla\varphi|^2 \,.\end{aligned}$$

Now $\int_{\mathbf{D}_{r_0}} \nabla f(\nabla u) \cdot \nabla\varphi = 0$ since $u$ solves the minimal surface equation in weak form and $\varphi = 0$ on $\partial\mathbf{D}_{r_0}$, while $\nabla^2 f(\xi)$ is positive definite (depending on the dimension $n$ only), uniformly on $|\xi| \le 1$. Hence, provided $\|\varphi\|_{C^1}$ is small enough (depending on the dimension $n$ only), by the Poincaré inequality on $\mathbf{D}_{r_0}$ we find, as claimed,

$$\begin{aligned}\mathcal{H}^n(M') - \mathcal{H}^n(M) \;&\ge\; c(n) \int_{\mathbf{D}} |\nabla\varphi|^2 \ge \frac{c'(n)}{r_0^2} \int_{\mathbf{D}} |\varphi|^2\\ &\ge\; \frac{c'(n)}{\omega_n\, r_0^2} \left(\int_{\mathbf{D}} |\varphi|\right)^2 = \frac{c'(n)}{\omega_n\, r_0^2}\, \mathcal{L}^{n+1}(E)^2 \,.\end{aligned}$$

**Remark 3** (Strategy of proof)**.** It was proved by White [**Wh**] that if $M$ is a smooth hypersurface with boundary, with vanishing mean curvature and strictly positive second variation of the area, then $M$ is locally area minimizing, where “locally” means “in a small $L^\infty$-neighborhood”. Recently, Morgan and Ros [**MoR**] have extended this result, replacing $L^\infty$-neighborhoods with $L^1$-neighborhoods, at least if $n \le 6$. Hence, the main new feature of Theorem 1 is that of providing a global stability inequality (rather than a local minimality condition) starting from the strict positivity of the second variation of the area and a natural and necessary uniqueness assumption. This is achieved by developing in the context of the Plateau problem some ideas recently introduced by Cicalese and Leonardi [**CL**] in connection with the stability problem for the Euclidean isoperimetric

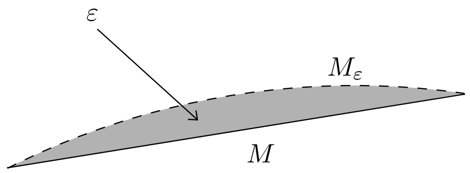

**Figure 1.3.** The selection principle allows to reduce the proof of the global stability inequality (1.2) to the case of those surfaces $M_\varepsilon$ which minimize area under the constraint of enclosing at least a volume of size $\varepsilon$ with the aid of $M$. Of course, in the planar case, $M$ is a segment, and each $M_\varepsilon$ is an arc of circle, which flattens against $M$ as $\varepsilon \to 0^+$.

inequality, and by Acerbi, Fusco and Morini [**AFM**] in the study of relative isoperimetric problems. Let us roughly explain how these ideas are employed in proving Theorem 1. One starts noticing that, given $\varepsilon_0 > 0$, up to decrease the value of $\kappa$ in correspondence to the smallness of $\varepsilon_0$ and thanks to the Euclidean isoperimetric inequality, in proving (1.2) we may directly consider surfaces $M'$ with $\operatorname{bdry} M' = \operatorname{bdry} M$ such that the $\mathcal{L}^{n+1}(E) \le \varepsilon_0$ (see Lemma 3.7 and 3.8). This said, we introduce the variational problems

$$\inf\Big\{\mathcal{H}^n(M') : \operatorname{bdry} M' = \operatorname{bdry} M\,, \mathcal{L}^{n+1}(E) = \varepsilon\Big\}\,, \tag{1.11}$$

see Figure 1.3, which we shall consider for $\varepsilon \in (0, \varepsilon_0)$ (actually, for technical reasons, we shall need to relax the constraint $\mathcal{L}^{n+1}(E) = \varepsilon$ into $\mathcal{L}^{n+1}(E) \ge \varepsilon$, see (3.1) and (3.2)). In the terminology of Cicalese and Leonardi, this will be our "selection principle". For the minimizers $M_\varepsilon$ in (1.11) (of course, in order to actually prove the existence of such minimizers we shall need to reformulate this variational problem in the language of currents), we shall see that

$$\lim_{\varepsilon \to 0^+} \mathcal{H}^n(M_\varepsilon) = \mathcal{H}^n(M)\,, \tag{1.12}$$

$$\lim_{\varepsilon \to 0^+} \mathcal{L}^{n+1}(E_\varepsilon) = 0\,, \tag{1.13}$$

for the region $E_\varepsilon$ bounded by $M \Delta M_\varepsilon$. Starting from the minimality of $M_\varepsilon$ in (1.11), and taking (1.13) into account, we reduce the proof of (1.2) to the case that $M' = M_\varepsilon$. To address this case we develop a suitable variant of a lemma by Almgren [**Al1**, Proposition VI.12] (see Lemma 3.3), which is used to prove the existence of a constant $\Lambda$, independent of $\varepsilon$, such that each $M_\varepsilon$ satisfies the $\Lambda$-mass minimality condition

$$\mathcal{H}^n(M_\varepsilon) \le \mathcal{H}^n(M') + \Lambda\, \mathcal{L}^{n+1}(E'_\varepsilon)\,, \tag{1.14}$$

whenever $\operatorname{bdry} M' = \operatorname{bdry} M_\varepsilon = \operatorname{bdry} M$, and where $E'_\varepsilon$ denotes the region bounded by $M_\varepsilon \Delta M'$. Starting from (1.14), and thanks to the interior and boundary regularity theory for $\Lambda$-minimizing currents, we finally prove the $C^1$-convergence of $M_\varepsilon$ to $M$ as $\varepsilon \to 0^+$. This will imply in particular the existence of functions $\{\varphi_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)} \subset C^1_0(M)$ such that

$$M_\varepsilon = \Big\{x + \varphi_\varepsilon(x)\, \nu_M(x) : x \in M\Big\}\,, \qquad \lim_{\varepsilon \to 0^+} \|\varphi_\varepsilon\|_{C^1(M)} = 0\,,$$

for a suitable unit normal vector field $\nu_M \in C^\infty(M; S^{n-1})$ to $M$. On this kind of competitors, by (1.4) and (1.5), the stability inequalities (1.2) and (1.6) are easily seen to coincide up to higher order terms in $\|\varphi_\varepsilon\|^2_{C^1(M)}$.

**Remark 4** (Stability inequalities and non-uniqueness)**.** As it will be evident from its proof, Theorem 1 can be immediately generalized to the following situation. We are given $N$ hypersurfaces $\{M_k\}_{k=1}^N \subset \mathcal{M}$, sharing the same boundary and minimizing mass as integral $n$-currents, so that $\gamma = \mathcal{H}^n(M_k)$ for every $k = 1, \dots, N$. This is the situation, for example, of Figure 1.2, or, in dimension three, of a catenoid spanned by two circles bounding a pair of disks with the same total area as the catenoid. In this case, one can prove that $\min\{\lambda(M_k) : 1 \le k \le N\} > 0$ if and only if there exists $\kappa > 0$ such that

$$\mathcal{H}^n(M') - \gamma \ge \kappa \min_{1\le k\le N} \Big\{\mathcal{L}^{n+1}(E_k)^2, \mathcal{L}^{n+1}(E_k)^{n/(n+1)}\Big\} \,,$$

where $E_k$ is a Borel set in $\mathbb{R}^{n+1}$, bounded by $M_k \Delta M'$, with $\mathcal{L}^{n+1}(E_k) < \infty$.

**1.3. Quantitative calibrations and Lawson's cones.** The main reason for Theorem 1 to be restricted to smooth hypersurfaces is our lack of understanding of the "close to singularities" behavior of area minimizing hypersurfaces. We would need area minimizing hypersurfaces to be locally diffeomorphic, at singular points, to their singular tangent cones. Such a result, if true, is of course far beyond the presently known regularity theory, as, for example, even the uniqueness of singular tangent cones is still conjectural. This said, an extension of Theorem 1 to generic area minimizing hypersurfaces seems problematic. We thus turn to the study of stability inequalities on explicit examples of area minimizing hypercones. We consider the Lawson's cones,

$$M_{k\,h} = \Big\{(x, y) \in \mathbb{R}^k \times \mathbb{R}^h : \frac{|x|}{\sqrt{k-1}} = \frac{|y|}{\sqrt{h-1}}\Big\} \,, \qquad 2 \le k \le h \,,$$

which are known to be area minimizing provided (see [**BDGG**, **La**, **S**, **MasMi**, **Da**, **DPP**])

$$\text{either} \qquad h + k \ge 9 \,, \tag{1.15}$$

$$\text{or} \qquad (k, h) \in \{(4, 4), (3, 5)\} \,. \tag{1.16}$$

Our second main result, Theorem 2, provides global quadratic estimates for all the Lawson's cones but for six exceptional cases. Here $B^k_R$ and $B^h_R$ denote the balls of radius $R$ and center at the origin in $\mathbb{R}^k$ and $\mathbb{R}^h$ respectively.

**Theorem 2.** *If $R > 0$, $m = h + k$, $h \ge k \ge 2$ satisfy* (1.15), (1.16), *and*

$$(k, h) \notin \{(3, 5), (2, 7), (2, 8), (2, 9), (2, 10), (2, 11)\} \,, \tag{1.17}$$

*then for every smooth, orientable hypersurface $M'$ with $M_{k\,h}\Delta M' \subset\subset H_R = B^k_R \times B^h_R$ there exists a Borel set $E$ with $\partial E$ equivalent to $M_{k\,h}\Delta M'$ up to $\mathcal{H}^{m-1}$-negligible sets, such that*

$$\left(\frac{\mathcal{L}^m(E)}{R^m}\right)^2 \le C\, \frac{\mathcal{H}^{m-1}(M' \cap H_R) - \mathcal{H}^{m-1}(M_{k\,h} \cap H_R)}{R^{m-1}} \,. \tag{1.18}$$

*Possible values for $C$ are*

$$C = \frac{2^{12}\sqrt{\omega_k\,\omega_h}}{(k-1)^{1/8}} \sqrt{\frac{hk}{m-1}} \left(\frac{h-1}{k-1}\right)^{3/2} \,, \qquad \text{if} \quad (k, h) \ne (4, 4) \,, \tag{1.19}$$

$$C = 128\,\omega_4 \,, \qquad \text{if} \quad (k, h) = (4, 4) \,. \tag{1.20}$$

In fact, as a by-product of our argument, the following explicit lower bounds on the first eigenvalues of the second variation of the area at the Lawson's cones can be deduced. These bounds show in a quantitative way that the minimality of the Simons' cones $M_{h\,h}$ is increasingly stronger as $h \to \infty$.

**Theorem 3.** *If $R, m, h, k$ are as in Theorem 2, and*

$$\lambda_{k\,h}(R) = \inf \left\{\int_{M_{k\,h}} |\nabla^{M_{k\,h}}\varphi|^2 - |\mathrm{II}_{M_{k\,h}}|^2\varphi^2 \, d\mathcal{H}^{m-1} : \int_{M_{k\,h}} \varphi^2 = 1 \,,\ \mathrm{spt}\varphi \subset\subset B^m_R\right\} \,,$$

*then,*

$$\lambda_{k\,h}(R) \ge \frac{1}{2^9\,R^2} \left(\frac{k-1}{h-1}\right)^{9/4} \frac{(m-2)^{1/2}}{(h-1)^{1/4}} \,, \qquad \text{if} \quad (k, h) \ne (4, 4) \,, \tag{1.21}$$

$$\lambda_{4\,4}(R) \ge \frac{\sqrt{2}}{16\,R^2} \,. \tag{1.22}$$

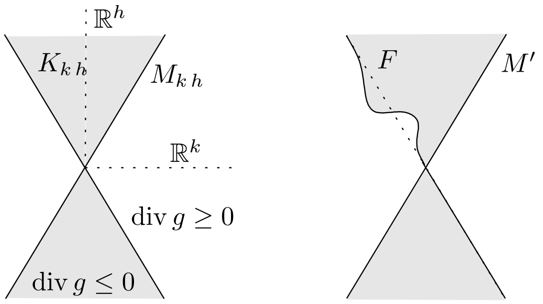

**Figure 1.4.** If $M'\Delta M_{k\,h} \subset\subset B_R$, then we may find a set $F$ with $K_{k\,h}\Delta F \subset\subset B_R$ such that (1.24) takes the form (1.28).

**Remark 5** (Strategy of proof)**.** The proof of Theorem 2 and Theorem 3 is based on a "quantitative calibration" argument, which we are now going to describe. We shall regard the Lawson's cone $M_{k\,h}$ as the topological boundary of the open cone

$$K_{k\,h} = \left\{(x,y) \in \mathbb{R}^k \times \mathbb{R}^h : \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}}\right\}. \tag{1.23}$$

The area minimizing property of the Lawson's cone $M_{k\,h}$ implies that

$$\mathcal{H}^{m-1}\Big(B_R \cap M_{k\,h}\Big) \le \mathcal{H}^{m-1}\Big(B_R \cap M'\Big)\,, \tag{1.24}$$

whenever $R > 0$ and $M'$ is a smooth, orientable hypersurface such that $M'\Delta M_{k\,h} \subset\subset B_R$. The validity of (1.24) is usually proved by the calibration method, that consists in in showing the existence of a (suitably regular) vector field $g\colon \mathbb{R}^n \to \mathbb{R}^n$ with

$$g = \nu_{K_{k\,h}}\,, \qquad \text{on } M_{k\,h}\,, \tag{1.25}$$

$$|g| \le 1\,, \qquad \text{on } \mathbb{R}^n\,, \tag{1.26}$$

$$\operatorname{div} g = 0\,, \qquad \text{on } \mathbb{R}^n\,, \tag{1.27}$$

where $\nu_{K_{k\,h}}$ is the outer unit normal to $K_{k\,h}$. Indeed, if $M'$ is a smooth, orientable hypersurface such that $M'\Delta M_{k\,h} \subset\subset B_R$, then we may construct a Borel set $F$ such that $K_{k\,h}\Delta F \subset\subset B_R$, and (1.24) takes the equivalent form

$$\mathcal{H}^{m-1}\Big(B_R \cap \partial K_{k\,h}\Big) \le \mathcal{H}^{m-1}\Big(B_R \cap \partial F\Big)\,, \tag{1.28}$$

see Figure 1.4. By (formally) applying the divergence theorem to the vector-field $g$ over the set $K_{k\,h}\Delta F$, and by taking (1.27) into account, we find

$$\begin{aligned}
& 0 && = \int_{K_{k\,h}\Delta F} \operatorname{div} g = \int_{B_R\cap\partial F} g\cdot\nu_F\, d\mathcal{H}^{m-1} - \int_{B_R\cap\partial K_{k\,h}} g\cdot\nu_{K_{k\,h}}\, d\mathcal{H}^{m-1}\\
&\text{by (1.25)} && = \int_{B_R\cap\partial F} g\cdot\nu_F\, d\mathcal{H}^{m-1} - \mathcal{H}^{m-1}\Big(B_R\cap\partial K_{k\,h}\Big)\\
&\text{by (1.26)} && \le \mathcal{H}^{m-1}\Big(B_R\cap\partial F\Big) - \mathcal{H}^{m-1}\Big(B_R\cap\partial K_{k\,h}\Big)\,,
\end{aligned}$$

that is (1.28). A major difficulty in constructing such a calibration is achieving the divergence-free constraint (1.27). In the present situation, however, the considered hypersurfaces are actually boundaries, and (1.27) can be replaced by the two softer requirements

$$\operatorname{div} g \ge 0\,, \qquad \text{on } \mathbb{R}^n \setminus K_{k\,h}\,, \tag{1.29}$$

$$\operatorname{div} g \le 0\,, \qquad \text{on } K_{k\,h}\,. \tag{1.30}$$

Indeed, if these conditions hold in place of (1.27), then by (again, formally) applying the

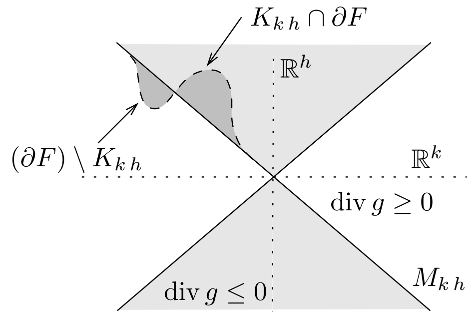

**Figure 1.5.** Quantitative calibrations and the proof of (1.33).

divergence theorem to $g$ on $K_{k\,h}\setminus F$, and thanks to (1.29), we find (see Figure 1.5)

$$\begin{aligned}
0 &\ge \int_{K_{k\,h}\setminus F}\operatorname{div} g=\int_{(B_R\cap\partial K_{k\,h})\setminus F} g\cdot\nu_{K_{k\,h}}\,d\mathcal{H}^{m-1}-\int_{B_R\cap K_{k\,h}\cap\partial F} g\cdot\nu_F\,d\mathcal{H}^{m-1}\\
&= \mathcal{H}^{m-1}\Big((B_R\cap\partial K_{k\,h})\setminus F\Big)-\int_{B_R\cap K_{k\,h}\cap\partial F} g\cdot\nu_F\,d\mathcal{H}^{m-1}\\
&\ge \mathcal{H}^{m-1}\Big((B_R\cap\partial K_{k\,h})\setminus F\Big)-\mathcal{H}^{m-1}\Big(B_R\cap K_{k\,h}\cap\partial F\Big)\,. \qquad (1.31)
\end{aligned}$$

The divergence theorem applied to $g$ on $F\setminus K_{k\,h}$ and (1.30) similarly imply

$$\mathcal{H}^{m-1}\Big((B_R\cap\partial F)\setminus K_{k\,h}\Big)\le\mathcal{H}^{m-1}\Big(\partial K_{k\,h}\cap B_R\cap F\Big)\,. \tag{1.32}$$

Adding up (1.31) and (1.32), we come to (1.28). Replacing condition (1.27) with (1.29) and (1.30) not only reduces (ideally speaking) the difficulty of proving the area minimizing property of $M_{k\,h}$: it also provides a first term on the right-hand side of the identity

$$\mathcal{H}^{m-1}\Big(B_R\cap\partial F\Big)-\mathcal{H}^{m-1}\Big(B_R\cap\partial K_{k\,h}\Big)=\int_{K_{k\,h}\Delta F}|\operatorname{div} g|+\int_{B_R\cap\partial F}1-(g\cdot\nu_F)\,d\mathcal{H}^{m-1}\,, \tag{1.33}$$

which, if the signs in (1.29) and (1.30) are strict, may be used to control $\mathcal{L}^m(K_{k\,h}\Delta F)^2$. Indeed we shall prove that the vector fields $g=\nabla f/|\nabla f|$ corresponding to the functions $f\colon\mathbb{R}^k\times\mathbb{R}^h\to\mathbb{R}$, defined at $(x,y)\in\mathbb{R}^k\times\mathbb{R}^h$ as

$$f(x,y)=\frac14\left(\frac{|x|}{\sqrt{k-1}}\right)^4-\frac14\left(\frac{|y|}{\sqrt{h-1}}\right)^4,\qquad \text{if } (k,h)\ne(4,4)\,, \tag{1.34}$$

$$f(x,y)=\frac27\Big(|x|^{7/2}-|y|^{7/2}\Big),\qquad \text{if } (k,h)=(4,4)\,, \tag{1.35}$$

are such that

$$|\operatorname{div}\, g(z)|\ge c\,\frac{\operatorname{dist}(z,M_{k\,h})}{|z|^2}\,,\qquad\forall z\in\mathbb{R}^m\,. \tag{1.36}$$

Combining (1.33) and (1.36) we shall then deduce Theorem 2 and Theorem 3.

**1.4. Minimal foliations and stability inequalities.** We close this introduction with a brief, heuristic discussion about the connection between minimal foliations and stability inequalities. This is done with a twofold aim. On the one hand, we roughly indicate how the boundary term in (1.33) could be used in proving stability inequalities. On the other hand, we provide some insight on how the constant $\kappa$ appearing in (2.3) is related to some basic analytic properties of a given minimal foliation of $M$. In particular, these considerations may be of some help in proving global stability inequalities with explicit constants on some specific example of area minimizing hypersurfaces. We now come to describe our argument. Let $M$ be a smooth, compact hypersurface with boundary in $\mathbb{R}^{n+1}$,

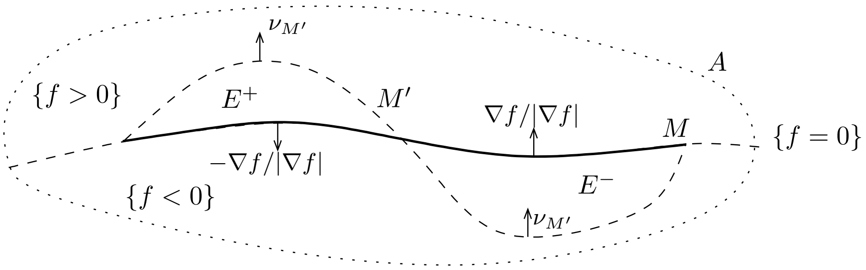

**Figure 1.6.** The situation in the proof of (1.39).

and, given a bounded open neighborhood $A$ of $M$, let $f \in C^2(A)$ be a foliation of $M$ in $A$, that is, let us assume that $M \subset \{f=0\}$, and that

$$0 < a \le |\nabla f| \le b < \infty\,, \qquad \text{on } A\,, \tag{1.37}$$

$$\operatorname{div} \frac{\nabla f}{|\nabla f|} = 0\,, \qquad \text{on } A\,. \tag{1.38}$$

The divergence theorem, combined with (1.38) only, implies $M$ to be area minimizing in $A$ (this is, again, the calibration method). In fact, by the argument sketched below, the validity of (1.37) implies the following global stability inequality to hold true,

$$\mathcal{H}^n(M')\Big(\mathcal{H}^n(M') - \mathcal{H}^n(M)\Big) \ge \frac{n^2}{4\,\mathrm{diam}(A)^2}\left(\frac{a}{b}\right)^2 \mathcal{L}^{n+1}(E)^2\,, \tag{1.39}$$

whenever $M' \subset A$ is a smooth, compact hypersurface with $\mathrm{bdry}\, M' = \mathrm{bdry}\, M$, that bounds, together with $M$, an open set $E$ contained in $A$. Indeed, let $E^+ = E \cap \{f>0\}$, $E^- = E \cap \{f<0\}$, and assume there exists a normal unit vector field $\nu_{M'}$ to $M'$, with

$$\begin{aligned}
\nu_{E^+}\,\mathcal{H}^n \llcorner \partial E^+ &= \nu_{M'}\,\mathcal{H}^n \llcorner \Big(M' \cap \{f>0\}\Big) - \frac{\nabla f}{|\nabla f|}\,\mathcal{H}^n \llcorner M^+\,,\\
\nu_{E^-}\,\mathcal{H}^n \llcorner \partial E^- &= -\nu_{M'}\,\mathcal{H}^n \llcorner \Big(M' \cap \{f<0\}\Big) + \frac{\nabla f}{|\nabla f|}\,\mathcal{H}^n \llcorner M^-\,,
\end{aligned}$$

where $\{M^+, M^-\}$ is a suitable partition of $M$, see Figure 1.6. Let us now compare the area of $M'$ in $\{f>0\}$ with that of $M^+$,

$$\begin{aligned}
\mathcal{H}^n\Big(M' \cap \{f>0\}\Big) - \mathcal{H}^n(M^+) \quad &= \quad \int_{M'\cap\{f>0\}} 1 - \left(\frac{\nabla f}{|\nabla f|}\cdot \nu_{M'}\right) d\mathcal{H}^n\\
&\quad + \int_{M'\cap\{f>0\}} \frac{\nabla f}{|\nabla f|}\cdot \nu_{M'} - \int_{M^+} \frac{\nabla f}{|\nabla f|}\cdot \nu_{E^+}\, d\mathcal{H}^n\,.
\end{aligned}$$

The second term vanishes, by the divergence theorem (applied on $E^+$) and by (1.38),

$$\begin{aligned}
\int_{M'\cap\{f>0\}} \frac{\nabla f}{|\nabla f|}\cdot \nu_{M'} - \int_{M^+} \frac{\nabla f}{|\nabla f|}\cdot \nu_{E^+}\, d\mathcal{H}^n\,, \quad &= \quad \int_{\partial E^+} \frac{\nabla f}{|\nabla f|}\cdot \nu_{E^+}\, d\mathcal{H}^n\\
&= \quad \int_{E^+} \operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) d\mathcal{L}^{n+1} = 0\,.
\end{aligned}$$

By (1.37), and recalling that $\nabla^M f = \nabla f - (\nabla f\cdot\nu_M)\nu_M$, we find

$$\begin{aligned}
\mathcal{H}^n\Big(M'\cap\{f>0\}\Big)-\mathcal{H}^n(M^+) &= \int_{M'\cap\{f>0\}} 1-\Big(\frac{\nabla f}{|\nabla f|}\cdot\nu_{M'}\Big)\,d\mathcal{H}^n\\
&\ge \frac{1}{b}\int_{M'\cap\{f>0\}} |\nabla f|-(\nabla f\cdot\nu_{M'})\,d\mathcal{H}^n\\
&= \frac{1}{b}\int_{M'\cap\{f>0\}} \frac{|\nabla^{M'} f|^2}{|\nabla f|+(\nabla f\cdot\nu_{M'})}\,d\mathcal{H}^n\\
&\ge \frac{1}{2b^2}\int_{M'\cap\{f>0\}} |\nabla^{M'} f|^2\,d\mathcal{H}^n\,,
\end{aligned}$$

so that, by Hölder inequality,
(1.40)

$$2\,b^2\,\mathcal{H}^n\Big(M'\cap\{f>0\}\Big)\left(\mathcal{H}^n\Big(M'\cap\{f>0\}\Big)-\mathcal{H}^n(M^+)\right)\ge\left(\int_{M'\cap\{f>0\}}|\nabla^{M'}f|\,d\mathcal{H}^n\right)^2.$$

On the one hand, by the coarea formula on hypersurfaces,

$$\int_{M'\cap\{f>0\}}|\nabla^{M'}f|\,d\mathcal{H}^n=\int_0^\infty\mathcal{H}^{n-1}\Big(M'\cap\{f=s\}\Big)\,ds\,. \tag{1.41}$$

On the other hand, by (1.38), for a.e. every $s\in\mathbb{R}$, $E^+\cap\{f=s\}$ is a minimal hypersurface in $\mathbb{R}^{n+1}$, having $M'\cap\{f=s\}$ as its boundary. If $\nu\in C^\infty(M'\cap\{f=s\};S^n)$ denotes the orientation of $M'\cap\{f=s\}$ induced by $E^+\cap\{f=s\}$, then by the divergence theorem for hypersurfaces and since $E^+\cap\{f=s\}$ has vanishing mean curvature (see [**Si**, (7.1)]),

$$\int_{M'\cap\{f=s\}} g\cdot\nu\,d\mathcal{H}^{n-1}=\int_{E^+\cap\{f=s\}}\mathrm{div}^{\{f=s\}}g\,d\mathcal{H}^n\,,$$

for every $g\in C^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. In particular, by plugging in the test field $g(x)=x-x_0$, and optimizing in $x_0\in M$,

$$\mathrm{diam}(A)\,\mathcal{H}^{n-1}\Big(M'\cap\{f=s\}\Big)\ge n\,\mathcal{H}^n(E^+\cap\{f=s\})\,. \tag{1.42}$$

Combining (1.40), (1.41), (1.42), (1.37) with the coarea formula (applied to $f$ on $E^+$),

$$\begin{aligned}
&2\,b^2\,\mathrm{diam}(A)\,\mathcal{H}^n\Big(M'\cap\{f>0\}\Big)\Big(\mathcal{H}^n(M'\cap\{f>0\})-\mathcal{H}^n(M^+)\Big)\\
\ge\ &\left(n\int_0^\infty\mathcal{H}^n\Big(E^+\cap\{f=s\}\Big)\,ds\right)^2=\left(n\int_{E^+}|\nabla f|\,d\mathcal{L}^{n+1}\right)^2\ge a^2n^2\mathcal{L}^{n+1}(E^+)^2\,,
\end{aligned}$$

that is,

$$\mathcal{L}^{n+1}(E^+)^2\le 2\left(\frac{\mathrm{diam}(A)}{n}\,\frac{b}{a}\right)^2\mathcal{H}^n\Big(M'\cap\{f>0\}\Big)\Big(\mathcal{H}^n(M'\cap\{f>0\})-\mathcal{H}^n(M^+)\Big)\,.$$

Finally, we repeat this argument on $E^-$ and we sum the two inequalities obtained in this way to prove (1.39).

**1.5. Organization of the paper.** The paper is structured in three sections. In section 2, we recall some basic definitions and facts about currents and sets of finite perimeter. In particular, we generalize Theorems 1 and 2 in this setting, see Theorems 4 and 5 and show how this generalized statements imply Theorems 1 and 2 respectively. In section 3 we prove Theorem 4, while in section 4 we prove Theorem 5, together with Theorem 3.

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## 2. Currents and sets of finite perimeter

*Rectifiable sets*: A Borel measurable set $M \subset \mathbb{R}^{n+1}$ is locally $k$-rectifiable if $\mathcal{H}^k(M \cap B_R) < \infty$ for every $R > 0$, and if there exists a Borel set $N_0 \subset \mathbb{R}^{n+1}$, open sets $\{A_h\}_{h\in\mathbb{N}}$ in $\mathbb{R}^k$ and $C^1$-embeddings $\{f_h\}_{h\in\mathbb{N}}$ of $\mathbb{R}^k$ into $\mathbb{R}^{n+1}$, with

$$M = N_0 \cup \bigcup_{h\in\mathbb{N}} f_h(A_h)\,, \qquad \mathcal{H}^k(N_0) = 0\,.$$

If $\mathcal{H}^k(M) < \infty$, then $M$ is said $k$-rectifiable. For $\mathcal{H}^k$-a.e. $x \in M$ there exists a unique $k$-dimensional plane $T_xM$ in $\mathbb{R}^{n+1}$, the approximate tangent space of $M$ at $x$, such that

$$\lim_{r\to 0^+} \frac{1}{r^k} \int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) = \int_{T_xM} \varphi\, d\mathcal{H}^k\,, \qquad \forall \varphi \in C^0_c(\mathbb{R}^n)\,.$$

Moreover, if $M$ is a $k$-dimensional surface in $\mathbb{R}^{n+1}$, then $M$ is locally $k$-rectifiable and $T_xM$ agrees with the classical tangent space of $M$ at $x$. Denote by $\Lambda_k(\mathbb{R}^{n+1})$ the space of $k$-vectors in $\mathbb{R}^{n+1}$, and, if $\tau \in \Lambda_k(\mathbb{R}^{n+1})$ is simple, then let $\langle\tau\rangle$ denote the oriented $k$-dimensional plane in $\mathbb{R}^{n+1}$ associated to $\tau$. An orientation of a locally $k$-rectifiable set is a Borel map $\tau_M\colon M \to \Lambda_k(\mathbb{R}^{n+1})$ with $\tau_M(x)$ a unit simple $k$-vector such that $\langle\tau_M(x)\rangle = T_xM$ for $\mathcal{H}^k$-a.e. $x \in M$. If $M$ is a $k$-dimensional orientable surface of class $C^1$, then every orientation $\tau_M$ of $M$ is tacitly assumed to be a continuous map.

*Spaces of currents* [**Fe1**, **Si**, **Mo**, **KPa**]: We denote by $\Lambda^k(\mathbb{R}^{n+1})$ the space of $k$-covectors in $\mathbb{R}^{n+1}$, and by $\mathcal{D}^k(\mathbb{R}^{n+1}) = C^\infty_c(\mathbb{R}^{n+1};\Lambda^k(\mathbb{R}^{n+1}))$ the space of smooth, compactly supported $k$-forms on $\mathbb{R}^{n+1}$. A $k$-current in $\mathbb{R}^{n+1}$ is a continuous linear functional on $\mathcal{D}^k(\mathbb{R}^{n+1})$. If $T$ is a $k$-current, then the boundary $\partial T$ of $T$ is the $(k-1)$-current defined by

$$\langle \partial T, \omega\rangle = \langle T, d\omega\rangle\,, \qquad \forall \omega \in \mathcal{D}^{k-1}(\mathbb{R}^{n+1})\,. \tag{2.1}$$

The support $\operatorname{spt} T$ of $T$ is the smallest closed set $C$ such that $\omega \in \mathcal{D}^k(\mathbb{R}^{n+1})$, $C \cap \operatorname{spt}\omega = \emptyset$, implies $\langle T, \omega\rangle = 0$. The mass of $T$ is defined as

$$\mathbf{M}(T) = \sup\Big\{|T(\omega)| : \omega \in \mathcal{D}^k(\mathbb{R}^{n+1})\,, \sup_{x\in\mathbb{R}^{n+1}} |\omega(x)| \le 1\Big\}\,.$$

If $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^m$ is smooth and proper, then the push-forward $f_\# T$ of $T$ through $f$ is the $k$-current on $\mathbb{R}^m$ defined by

$$\langle f_\# T, \omega\rangle = \Big\langle T, f^\#\omega\Big\rangle\,, \qquad \forall \omega \in \mathcal{D}^k(\mathbb{R}^m)\,,$$

where $f^\#\omega$ denotes the pull-back through $f$ of $\omega$. A $k$-current $T$ is $k$-rectifiable, $T \in \mathcal{R}_k(\mathbb{R}^{n+1})$, if there exists a $k$-rectifiable set $M$ in $\mathbb{R}^{n+1}$, a Borel measurable orientation $\tau_M$ of $M$, and a Borel function $\theta \in L^1(\mathcal{H}^k \llcorner M, \mathbb{Z})$ (called the density of $T$), such that

$$\langle T, \omega\rangle = \int_M \langle \omega(x), \tau_M(x)\rangle \theta(x)\, d\mathcal{H}^k(x)\,, \qquad \forall \omega \in \mathcal{D}^k(\mathbb{R}^{n+1})\,.$$

In this case, we set $T = [\![M, \tau_M, \theta]\!]$. If $\theta = 1$, then we simply set $T = [\![M, \tau_M]\!]$, or even $T = [\![M]\!]$, provided we don't need to specify the choice of the orientation $\tau_M$ of $M$. The variation measure $\|T\|$ of $T$ is the Radon measure on $\mathbb{R}^{n+1}$ defined by

$$\|T\|(E) = \int_{M\cap E} |\theta|\, d\mathcal{H}^k\,,$$

whenever $E \subset \mathbb{R}^{n+1}$ is a Borel set. In this way, of course,

$$\mathbf{M}(T) = \|T\|(\mathbb{R}^{n+1}) = \int_M |\theta|\, d\mathcal{H}^k\,.$$

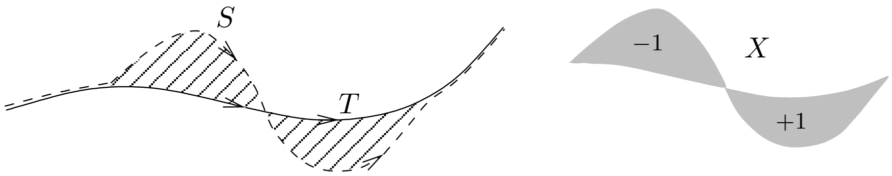

**Figure 2.1.** The distance $d(S,T)$ defined in (2.4) is the mass of the unique $(n+1)$-dimensional filling of $S-T$ in $\mathbb{R}^{n+1}$. In this example, $d(S,T)$ agrees with the area of the dashed region.

We finally consider the space of $k$-integral currents

$$\mathcal{I}_k(\mathbb{R}^{n+1}) = \Big\{T \in \mathcal{R}_k(\mathbb{R}^{n+1}) : \partial T \in \mathcal{R}_{k-1}(\mathbb{R}^{n+1})\Big\}\,,$$

which naturally contains the family of $k$-dimensional smooth, compact, orientable manifolds with boundary. For example, let us consider the family $\mathcal{M}$ of the smooth, compact, orientable hypersurfaces with smooth boundary in $\mathbb{R}^{n+1}$. If we fix a (smooth) orientation $\tau_M$ of $M \in \mathcal{M}$, then $T = [\![M, \tau_M, 1]\!]$ defines a $n$-rectifiable current in $\mathbb{R}^{n+1}$. Moreover, the orientation $\tau_\Gamma$ induced on $\Gamma = \mathrm{bdry}\, M$ by Stokes theorem is such that

$$\partial\, [\![M, \tau_M, 1]\!] = [\![\Gamma, \tau_\Gamma, 1]\!]\,,$$

that is, the boundary of $T$ in the sense of currents is the current identified by boundary of $M$ as a classical hypersurface, with the natural orientation induced by $M$ through Stokes theorem. In the following, given $M \in \mathcal{M}$, we shall always given for granted that a smooth orientation of $M$ has been fixed, and simply write

$$T = [\![M]\!]\,, \qquad \partial T = [\![\Gamma]\!]\,, \qquad \Gamma = \mathrm{bdry}\, M\,,$$

to realize $M$ as an integral $n$-current $T$, with $\mathbf{M}(T) = \mathcal{H}^n(M)$. If now $M$ is area minimizing in $\mathcal{M}$ (as specified in (1.1)), then $T = [\![M]\!]$ is mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$, that is

$$\mathbf{M}(S) \ge \mathbf{M}(T)\,, \qquad \forall S \in \mathcal{I}_n(\mathbb{R}^{n+1})\,, \quad \partial S = \partial T\,. \tag{2.2}$$

Indeed, if $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$, $\partial S = \partial T$ and $\mathbf{M}(S) = \mathbf{M}(T)$, then it is possible to construct a sequence $\{M_h\}_{h\in\mathbb{N}} \subset \mathcal{M}$ such that $\partial[\![M_h]\!] = \partial T$ and $\mathcal{H}^n(M_h) \to \mathbf{M}(S)$ as $h \to \infty$. Therefore, there is no difference in assuming that $M$ is area minimizing in $\mathcal{M}$ or that $T = [\![M]\!]$ is mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$. The situation is different when we come to discuss uniqueness. We shall say that $M \in \mathcal{M}$ is *uniquely mass minimizing as an $n$-integral current* if $T = [\![M]\!]$ is mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$, and if $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$, $\mathbf{M}(S) = \mathbf{M}(T)$, $\partial S = \partial T$, implies $S = T$. We are now in the position to state the following theorem, which we claim to imply Theorem 1 as a particular case.

**Theorem 4.** *If $n \ge 1$, $M \in \mathcal{M}$, and $T = [\![M]\!]$ is uniquely mass minimizing as an integral $n$-current, then, equivalently:*

*(a) $\lambda(M)$, as defined in* (1.8)*, is positive;*

*(b) there exists $\kappa > 0$, depending on $M$, such that, for every integral $n$-current $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$ such that $\partial S = \partial T$, one has*

$$\mathbf{M}(S) - \mathbf{M}(T) \ge \kappa\, \min\Big\{d(S,T)^2, d(S,T)^{n/(n+1)}\Big\}\,, \tag{2.3}$$

*where we have set (see Figure* 2.1*),*

$$d(S,T) = \inf\Big\{\mathbf{M}(X) : X \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})\,, \partial X = S - T\Big\}\,. \tag{2.4}$$

Theorem 4 is proved in section 3. Before proving it implies Theorem 1, we need to introduce some further terminology from the theory of sets of finite perimeter.

*Sets of finite perimeter and functions of bounded variation* [**AmFP**, **G**, **Ma2**]: A function $u \in L^1_{loc}(\mathbb{R}^{n+1})$ is of locally bounded variation, $u \in BV_{loc}(\mathbb{R}^{n+1})$, provided

$$\langle Du, \varphi\rangle = -\int_{\mathbb{R}^{n+1}} u\,\nabla\varphi\,d\mathcal{L}^{n+1}\,, \qquad \varphi \in C^\infty_c(\mathbb{R}^{n+1})\,, \tag{2.5}$$

defines a $\mathbb{R}^{n+1}$-valued Radon measure $Du$ on $\mathbb{R}^{n+1}$. If this is the case, the total variation $|Du|$ of $Du$ defines a Radon measure on $\mathbb{R}^{n+1}$, which satisfies

$$|Du|(A) = \sup\left\{\int_{\mathbb{R}^{n+1}} u\,\mathrm{div}\,g\,d\mathcal{L}^{n+1} : g \in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})\,, \sup_{x\in\mathbb{R}^{n+1}}|g(x)| \le 1\right\}\,, \tag{2.6}$$

whenever $A \subset \mathbb{R}^{n+1}$ is open. We say that $u$ has bounded variation, $u \in BV(\mathbb{R}^{n+1})$, if $u \in L^1(\mathbb{R}^{n+1})$ and $|Du|(\mathbb{R}^{n+1}) < \infty$. A Borel set $E \subset \mathbb{R}^{n+1}$ is of locally finite perimeter if $1_E \in BV_{loc}(\mathbb{R}^{n+1})$, it is of finite perimeter if $1_E \in BV(\mathbb{R}^{n+1})$. The relative perimeter of $E$ in the Borel set $F \subset \mathbb{R}^{n+1}$ is defined as $P(E;F) = |D1_E|(F)$, while $P(E) = P(E;\mathbb{R}^{n+1})$ is called the perimeter of $E$. If $E$ is of locally finite perimeter in $\mathbb{R}^{n+1}$, then we call $\mu_E = -D1_E$ the Gauss-Green measure of $E$, and (2.5) becomes

$$\int_E \nabla\varphi\,d\mathcal{L}^{n+1} = \int_{\mathbb{R}^{n+1}} \varphi\,d\mu_E\,, \qquad \forall\varphi \in C^\infty_c(\mathbb{R}^{n+1})\,. \tag{2.7}$$

In particular, if $E$ is an open set with $C^1$-boundary, then $E$ is of locally finite perimeter and $\mu_E = \nu_E\,\mathcal{H}^n \llcorner \partial E$, where $\nu_E$ denotes the outer unit normal to $E$. Let us now consider the set of points of density $t \in [0,1]$ of $E$, namely

$$E^{(t)} = \left\{x \in \mathbb{R}^{n+1} : \lim_{r\to 0^+}\frac{|E\cap B(x,r)|}{\omega_n r^n} = t\right\}\,,$$

and let $\partial_{1/2}E = E^{(1/2)}$ denote the set of points of density $1/2$ of $E$. The structure theory for sets of locally finite perimeter asserts that, for $\mathcal{H}^n$-a.e. $x \in \partial_{1/2}E$, the limit

$$\nu_E(x) = \lim_{r\to 0^+}\frac{\mu_E(B(x,r)}{|\mu_E(B(x,r))|}\,,$$

exists, belongs to $S^n$, and thus defines a Borel measurable vector-field $\nu_E\colon \partial_{1/2}E \to S^n$, called the measure theoretic outer unit normal to $E$. Moreover,

$$\mu_E = \nu_E\,\mathcal{H}^n \llcorner \partial_{1/2}E\,,$$

and $\nu_E(x)^\perp$ is the approximate tangent space to the locally $n$-rectifiable set $\partial_{1/2}E$ for $\mathcal{H}^n$-a.e. $x \in \mathbb{R}^{n+1}$. In particular, we have

$$P(E;F) = \mathcal{H}^n(F\cap\partial_{1/2}E)\,, \qquad \text{for every Borel set } F \subset \mathbb{R}^{n+1}\,,$$

$$\int_E \mathrm{div}\,g\,d\mathcal{L}^{n+1} = \int_{\partial_{1/2}E} g\cdot\nu_E\,d\mathcal{H}^n\,, \qquad \forall g \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})\,. \tag{2.8}$$

We are now in the position to state the generalized form of Theorem 2.

**Theorem 5.** *If $R > 0$, $m = h + k$, $h \ge k \ge 2$ satisfy* (1.15), (1.16), *and*

$$(k,h) \notin \{(3,5),(2,7),(2,8),(2,9),(2,10),(2,11)\}\,, \tag{2.9}$$

*then*

$$\left(\frac{\mathcal{L}^m(K_{k\,h}\Delta F)}{R^m}\right)^2 \le C\,\frac{P(F;H_R) - P(K_{k\,h};H_R)}{R^{m-1}} \tag{2.10}$$

*whenever $F$ is a set of locally finite perimeter with $K_{k\,h}\Delta F \subset\subset B^k_R \times B^h_R$. The values of $C$ in* (2.10) *are the same as in Theorem* 2.

Theorem 5 is proved in section 4. Later on in this section, we are going to show that it implies Theorem 2 as a particular case.

*The spaces* $\mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ *and* $BV(\mathbb{R}^{n+1};\mathbb{Z})$: By [**Fe1**, 4.5.7], $X \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ if and only if there exists $u \in BV(\mathbb{R}^{n+1};\mathbb{Z})$ with $X = \mathbf{E}^{n+1} \llcorner u$, which means

$$\langle X, \omega \rangle = \int_{\mathbb{R}^{n+1}} f\, u \, d\mathcal{L}^n \,, \qquad \forall\, \omega = f\, dx^1 \wedge \cdots \wedge dx^{n+1} \in \mathcal{D}^{n+1}(\mathbb{R}^{n+1}) \,.$$

If now $\widehat{dx_i} = (-1)^{i+1} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_i \wedge \dots dx_{n+1}$, then $\omega \in \mathcal{D}^n(\mathbb{R}^{n+1})$ if and only if $\omega = \sum_{i=1}^{n+1} f_i \widehat{dx_i}$. In this way, if $f = (f_1, \dots, f_{n+1})$ denotes the vector field associated to $\omega$, then $d\omega = \operatorname{div} f\, dx_1 \wedge \dots dx_n$, and, by (2.1) and (2.5),

$$\langle \partial X, \omega \rangle = \int_{\mathbb{R}^{n+1}} f \cdot Du \,, \qquad \forall \omega \in \mathcal{D}^n(\mathbb{R}^{n+1}) \,. \tag{2.11}$$

We thus have, for every open set $A \subset \mathbb{R}^{n+1}$,

$$\|X\|(A) = \int_A |u| \,, \qquad \|\partial X\|(A) = |Du|(A) \,.$$

We shall frequently consider the two subsets $\mathcal{I}^+_{n+1}(\mathbb{R}^{n+1})$ and $\mathcal{I}^-_{n+1}(\mathbb{R}^{n+1})$ of $\mathcal{I}_{n+1}(\mathbb{R}^{n+1})$,

$$\mathcal{I}^+_{n+1}(\mathbb{R}^{n+1}) = \Big\{ X \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1}) : X = \mathbf{E}^{n+1} \llcorner u \,, u \geq 0 \Big\} \,, \tag{2.12}$$

$$\mathcal{I}^-_{n+1}(\mathbb{R}^{n+1}) = \Big\{ X \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1}) : X = \mathbf{E}^{n+1} \llcorner u \,, u \leq 0 \Big\} \,, \tag{2.13}$$

see, in particular, the variational problems (3.1) and (3.2).

**Lemma 2.1.** *If* $T, S \in \mathcal{I}_n(\mathbb{R}^{n+1})$ *with* $\partial T = \partial S$, *then there exists a unique* $X \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ *with* $\mathbf{M}(X) < \infty$ *such that* $\partial X = S - T$. *In particular,* $d(S,T) = \mathbf{M}(X) < \infty$.

*Proof.* The existence of $X \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ with $\mathbf{M}(X) < \infty$ such that $\partial X = S - T$ follows from the isoperimetric inequality [**Fe1**, 4.2.10]. If $\partial X_1 = \partial X_2 = S - T$, then $\partial(X_1 - X_2) = 0$. By the constancy theorem [**Fe1**, 4.1.7], $X_1 = X_2 + c\, [\![\mathbb{R}^{n+1}]\!]$ for some $c \in \mathbb{Z}$. Thus, if $X_1$ has finite mass, then $\mathbf{M}(X_2) = \infty$. q.e.d.

**Lemma 2.2.** *If* $M, M' \in \mathcal{M}$ *with* $\operatorname{bdry} M' = \operatorname{bdry} M$ *and* $T = [\![M]\!]$, $S = [\![M']\!]$, *then* $\partial T = \partial S$ *and there exists a set of finite perimeter* $E \subset \mathbb{R}^{n+1}$ *with* $\partial E$ *equivalent up to* $\mathcal{H}^n$*-negligible sets to* $M \Delta M'$, *such that* $d(S,T) = \mathcal{L}^{n+1}(E) < \infty$.

*Proof.* Let $\tau_M$ and $\tau_{M'}$ denote orientations, respectively, of $M$ and $M'$. Setting $T = [\![M, \tau_M]\!]$ and either $S = [\![M', \tau_{M'}]\!]$ or $S = [\![M', -\tau_{M'}]\!]$, we achieve $\partial T = \partial S$. Moreover, the Hodge-star operation allows to define a smooth unit normal vector field $\nu_M$ to $M$ such that $\langle \omega, \tau_M \rangle = f \cdot \nu_M$ if $\omega = \sum_{i=1}^{n+1} f_i\, \widehat{dx_i} \in \mathcal{D}^n(\mathbb{R}^{n+1})$ and $f = (f_1, \dots, f_{n+1})$. In this way,

$$\langle T, \omega \rangle = \int_M f \cdot \nu_M \, d\mathcal{H}^n \,, \qquad \forall \omega \in \mathcal{D}^n(\mathbb{R}^{n+1}) \,.$$

We similarly define an outer unit normal vector field $\nu_{M'}$ to $M'$ starting from $S$. By Lemma 2.1, there exists $X \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ with $\partial X = S - T$ and $\mathbf{M}(X) < \infty$. In particular,

$$\langle \partial X, \omega \rangle = \int_{M'} f \cdot \nu_{M'} \, d\mathcal{H}^n - \int_M f \cdot \nu_M \, d\mathcal{H}^n \,, \qquad \forall \omega \in \mathcal{D}^n(\mathbb{R}^{n+1}) \,,$$

and, moreover, $\nu_M = \pm \nu_{M'}$ at $\mathcal{H}^n$-a.e. $x \in M \cap M'$. By (2.11), we find

$$Du = \nu_{M'}\, \mathcal{H}^n \llcorner M' - \nu_M\, \mathcal{H}^n \llcorner M \,.$$

Hence, by the structure theorem for functions of bounded variation [**AmFP**, Section 3.9], and since $\int_{\mathbb{R}^{n+1}} |u| = \mathbf{M}(X) < \infty$, there exists a set of finite perimeter $E \subset \mathbb{R}^{n+1}$ such that $|u| = 1_E$, $\mathcal{L}^{n+1}(E) < \infty$ and $\operatorname{spt} \mu_E$ is equivalent, up to $\mathcal{H}^n$-negligible sets, to $M \Delta M'$.

Finally, up to modify $E$ on and by a set of Lebesgue measure zero, we may assume the topological boundary $\partial E$ of $E$ to agree with $\operatorname{spt} D1_E$ [**Ma2**, Proposition 12.19]. q.e.d.

*Theorem* 4 *implies Theorem* 1. Immediate from Lemma 2.1 and Lemma 2.2. q.e.d.

*Theorem* 5 *implies Theorem* 2. Let $M'$ be a smooth, orientable hypersurface in $\mathbb{R}^m$, with $M_{k\,h}\Delta M' \subset\subset H_R$ for some $R > 0$. Arguing as in Lemma 2.1 and Lemma 2.2 we show the existence of a set of finite perimeter $E \subset \mathbb{R}^m$ with topological boundary $\partial E = \operatorname{spt}\mu_E$ which is $\mathcal{H}^{m-1}$-equivalent to $M_{k\,h}\Delta M'$. The set $F = K_{k\,h}\Delta E$ is of locally finite perimeter, with $K_{k\,h}\Delta F = E \subset\subset H_R$ and

$$|\mu_F| = \mathcal{H}^{m-1} \llcorner \Big(M' \setminus M_{k\,h}\Big) + \mathcal{H}^{m-1} \llcorner \Big(M_{k\,h} \cap M'\Big) .$$

Since $P(K_{k\,h}; H_R) = \mathcal{H}^{m-1}(M_{k\,h} \cap H_R)$, we thus find

$$\begin{aligned} P(F; H_R) - P(K_{k\,h}; H_R) &= \mathcal{H}^{m-1}\Big((M' \setminus M_{k\,h}) \cap H_R\Big) - \mathcal{H}^{m-1}\Big((M_{k\,h} \setminus M') \cap H_R\Big) \\ &= \mathcal{H}^{m-1}(M' \cap H_R) - \mathcal{H}^{m-1}(M_{k\,h} \cap H_R) . \end{aligned}$$

By applying Theorem 5 to $F$ we prove Theorem 2 on $M'$. q.e.d.

*Generalized divergence theorem*: We conclude this section with a generalized form of the divergence theorem which we shall use to justify some technical aspects of the proof of Theorem 5 (see, in particular, Proposition 4.1). If $u \in W^{1,1}_{loc}(\mathbb{R}^{n+1})$, and $M$ is a locally $n$-rectifiable set in $\mathbb{R}^{n+1}$, then every orientation of $M$ defines a trace operator on $W^{1,1}_{loc}(\mathbb{R}^{n+1})$ with values in $L^1(\mathcal{H}^n \llcorner M)$, see [**AmFP**, Theorem 3.87]. In this way, the values of $u$ are unambiguously defined at $\mathcal{H}^n$-a.e. point of $M$, and the divergence theorem

$$\int_E \operatorname{div}\, g(x)\, dx = \int_{\partial_{1/2} E} g \cdot \nu_E \, d\mathcal{H}^n , \tag{2.14}$$

holds true for every $g \in W^{1,1}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ and set of locally finite perimeter $E$.

## 3. From infinitesimal to global stability inequalities

**3.1. Theorem 4, scheme of proof.** We start by briefly introducing the scheme of the proof of Theorem 4. In section 3.2 we derive the Taylor expansion of $\mathbf{M}(S)$ and $d(S,T)$ when $T = [\![M]\!]$ for an area minimizing $M \in \mathcal{M}$, and $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$ is a small $C^1$-perturbation of $M$ with $\partial S = \partial T$. Starting from these results we immediately deduce that (b) implies (a). We then turn to the proof of the reverse implication. In section 3.3 we prove a lemma which will provide us the major technical tool in subsequent proofs. This lemma is a sort of generator of "inclusion preserving and volume fixing variations", modeled after [**Al1**, Proposition VI.12]. In section 3.4 we introduce the variational problems

$$\inf\Big\{\mathbf{M}(S) : X \in \mathcal{I}^+_{n+1}(\mathbb{R}^{n+1}) \,, \partial X = S - T \,, \mathbf{M}(X) \geq \varepsilon\Big\} , \tag{3.1}$$

$$\inf\Big\{\mathbf{M}(S) : X \in \mathcal{I}^-_{n+1}(\mathbb{R}^{n+1}) \,, \partial X = S - T \,, \mathbf{M}(X) \geq \varepsilon\Big\} , \tag{3.2}$$

and prove the existence of minimizers $S_\varepsilon$ for $\varepsilon$ small enough. These $\varepsilon$-approximating currents are crucial in our argument. They provide a sort of *asymptotically worst* test sets for the global stability inequality, and indeed, as we show in section 3.5, we may deduce that (a) implies (b) in the general case provided we are able to prove the validity of the global stability inequality on those $S_\varepsilon$. To this end, in section 3.6 we start proving that they are all $\Lambda$-minimizers of the mass, with $\Lambda$ independent on $\varepsilon$. From this information we deduce in section 3.7 that they converge in $C^1$ towards $T$. In particular, these sets are small $C^1$-perturbation of the limit area minimizing hypersurface, so that, as discussed in section 3.8, the global stability inequality on them follows from the results of section 3.2.

### 3.2. Small $C^1$-perturbations.

**Lemma 3.1.** *If $M \in \mathcal{M}$ and $\nu_M$ is a smooth unit normal vector field to $M$, then there exists a positive constant $\varepsilon_0(M)$ such that, for every $\varphi \in C^1(M)$ with $\|\varphi\|_{C^0(M)} \le \varepsilon_0$,*

$$M_\varphi = \Big\{ x + \varphi(x)\nu_M(x) : x \in M \Big\} \in \mathcal{M}\,, \tag{3.3}$$

$$\mathcal{H}^n(M_\varphi) = \int_M \sqrt{1 + \sum_{i=1}^n \Big(\frac{\partial_i \varphi}{1+\lambda_i\varphi}\Big)^2} \prod_{j=1}^n |1+\lambda_j\varphi|\, d\mathcal{H}^n \tag{3.4}$$

$$d([\![M_\varphi]\!],[\![M]\!]) = \int_0^1 ds \int_M |\varphi| \prod_{j=1}^n |1+s\lambda_j\varphi|\, d\mathcal{H}^n\,. \tag{3.5}$$

*Here, $\{\lambda_i\}_{i=1}^n$ are the principal curvatures of $M$, corresponding to the principal directions $\{\tau_i\}_{i=1}^n$, and $\partial_i$ denotes differentiation with respect to $\tau_i$.*

*Proof.* Define $f : M \to \mathbb{R}^{n+1}$ by $f(x) = x + \varphi(x)\nu_M(x)$, $x \in M$. Since $\partial_i\nu_M = \lambda_i\tau_i$, we find that $\partial_i f = (1+\lambda_i\varphi)\,\tau_i + \partial_i\,\varphi\,\nu_M$. Therefore,

$$|\partial_1 f \wedge \cdots \wedge \partial_n f| = \sqrt{1 + \sum_{i=1}^n \Big(\frac{\partial_i \varphi}{1+\lambda_i\varphi}\Big)^2} \prod_{j=1}^n |1+\lambda_j\varphi|\,. \tag{3.6}$$

In particular, if $\sup_M |\varphi| \le \min_{1\le i\le n} \min_M 1/|\lambda_i|$, then $f$ is locally injective. By compactness of $M$ and by the explicit formula for $f$, we easily see that, up to further decrease the value of $\varepsilon$, then $f$ is globally injective, and thus, that $M_\varphi \in \mathcal{M}$. By (3.6), and the area formula between rectifiable sets [**Fe1**, Corollary 3.2.20],

$$\mathcal{H}^n(M_\varphi) = \int_M |\partial_1 f \wedge \cdots \wedge \partial_n f|\, d\mathcal{H}^n\,,$$

so that (3.4) immediately follows. If we now consider the map $H : M \times [0,1] \to \mathbb{R}^n$,

$$H(x,t) = x + t\,\varphi(x)\nu_M(x) \qquad (x,t) \in M \times [0,1]\,,$$

then $X = H_\#\Big([\![M]\!] \times [\![0,1]\!]\Big)$ satisfies $\partial X = [\![M_\varphi]\!] - [\![M]\!]$. Therefore (denoting with $J^{M\times[0,1]}H$ the tangential Jacobian, see [**Fe1**, Corollary 3.2.20]),

$$d([\![M]\!],[\![M_\varphi]\!]) = \mathbf{M}\Big(H_\#([\![M]\!] \times [\![0,1]\!])\Big) = \int_{M\times[0,1]} J^{M\times[0,1]}H\, d\mathcal{H}^{n+1}\,,$$

from which (3.5) immediately follows. q.e.d.

We shall also need the following classical remark; see e.g. [**GH**, Theorem 1-(v),p.272]. We include the proof for the sake of clarity.

**Lemma 3.2.** *If $M \in \mathcal{M}$, $\lambda > 0$, and*

$$\int_M |\nabla^M\varphi|^2 - |\mathrm{II}_M|^2\varphi^2\, d\mathcal{H}^n \ge \lambda \int_M \varphi^2\, d\mathcal{H}^n\,, \qquad \forall \varphi \in C^1_0(M)\,, \tag{3.7}$$

*then there exists $\mu > 0$ such that*

$$\int_M |\nabla^M\varphi|^2 - |\mathrm{II}_M|^2\varphi^2\, d\mathcal{H}^n \ge \mu \int_M \varphi^2 + |\nabla^M\varphi|^2\, d\mathcal{H}^n\,, \qquad \forall \varphi \in C^1_0(M)\,. \tag{3.8}$$

*Proof.* By contradiction: consider a sequence $\{\varphi_h\}_{h\in\mathbb{N}} \subset C^1_0(M)$ such that

$$\int_M \varphi_h^2 + |\nabla^M\varphi_h|^2\, d\mathcal{H}^n = 1\,, \qquad \lim_{h\to\infty} \int_M |\nabla^M\varphi_h|^2 - |\mathrm{II}_M|^2\varphi_h^2\, d\mathcal{H}^n = 0\,. \tag{3.9}$$

By (3.7), we know that $\int_M \varphi_h^2 \, d\mathcal{H}^n \to 0$ as $h \to \infty$; hence, $\int_M |\nabla^M \varphi_h|^2 \, d\mathcal{H}^n \to 1$, and, in particular

$$\liminf_{h\to\infty} \int_M |\nabla^M \varphi_h|^2 - |\mathrm{II}_M|^2 \varphi_h^2 \, d\mathcal{H}^n \geq 1 \,,$$

since $M$ is compact and thus $\sup_M |\mathrm{II}_M| < \infty$. This contradicts the second equation on (3.9) and concludes the proof. q.e.d.

**Theorem 6.** *If $M \in \mathcal{M}$ has vanishing mean curvature and for some $\lambda > 0$,*

$$\int_M |\nabla^M \varphi|^2 - |\mathrm{II}_M|^2 \varphi^2 \, d\mathcal{H}^n \geq \lambda \int_M \varphi^2 \, d\mathcal{H}^n \,, \qquad \forall \varphi \in C^1_0(M), \tag{3.10}$$

*then there exist positive constants $\varepsilon_0(M)$ and $\kappa_0(M)$ depending only on $M$ such that*

$$\mathcal{H}^n(M_\varphi) - \mathcal{H}^n(M) \geq \kappa_0 \, d([\![M_\varphi]\!], [\![M]\!])^2 \,, \tag{3.11}$$

*for every $\varphi \in C^1_0(M)$ with $\|\varphi\|_{C^1(M)} \leq \varepsilon_0$ and $M_\varphi$ as in (3.3).*

*Proof.* By (3.4) we have

$$\mathcal{H}^n(M_\varphi) - \mathcal{H}^n(M) = \int_M \left( \sqrt{1 + \sum_{i=1}^n \Big( \frac{\partial_i \varphi}{1 + \lambda_i \varphi} \Big)^2} \prod_{j=1}^n |1 + \lambda_j \varphi| - 1 \right) d\mathcal{H}^n \,. \tag{3.12}$$

Taking into account that $H_M = \sum_{i=1}^n \lambda_i = 0$, by Taylor's formula we find,

$$\begin{aligned}
&\sqrt{1 + \sum_{i=1}^n \Big( \frac{\partial_i \varphi}{1 + \lambda_i \varphi} \Big)^2} \prod_{j=1}^n |1 + \lambda_j \varphi| \\
&= \Big( 1 + \frac{1}{2} \sum_{i=1}^n (\partial_i \varphi)^2 \big( 1 + O(\|\varphi\|_{C^1}) \big) \Big) \Big( 1 + \varphi \sum_{i=1}^n \lambda_i + \varphi^2 \sum_{i<j} \lambda_i \lambda_j + O(\|\varphi\|^3_{C^0}) \Big) \\
&= \Big( 1 + \frac{1}{2} \sum_{i=1}^n (\partial_i \varphi)^2 \big( 1 + O(\|\varphi\|_{C^1}) \big) \Big) \Big( 1 + \varphi^2 \sum_{i<j} \lambda_i \lambda_j + O(\|\varphi\|^3_{C^0}) \Big) \,.
\end{aligned}$$

From the identity

$$0 = (\lambda_1 + \cdots + \lambda_n)^2 = \sum_{i=1}^n \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j \,,$$

and since $|\mathrm{II}_M|^2 = \sum_{i=1}^n \lambda_i^2$, we finally conclude that

$$\begin{aligned}
&\sqrt{1 + \sum_{i=1}^n \Big( \frac{\partial_i \varphi}{1 + \lambda_i \varphi} \Big)^2} \prod_{j=1}^n |1 + \lambda_j \varphi| - 1 \\
&= \frac{1}{2} |\nabla^M \varphi|^2 + \varphi^2 \sum_{i<j} \lambda_i \lambda_j + \Big( |\nabla^M \varphi|^2 + \varphi^2 \Big) O(\|\varphi\|_{C^1}) \\
&= \frac{1}{2} \Big( |\nabla^M \varphi|^2 - |\mathrm{II}_M|^2 \varphi^2 \Big) + \Big( |\nabla^M \varphi|^2 + \varphi^2 \Big) O(\|\varphi\|_{C^1}) \,.
\end{aligned}$$

By (3.10), Lemma 3.2, (3.12), and provided $\|\varphi\|_{C^1(M)}$ is suitably small, we thus conclude that

$$\begin{aligned}\mathcal{H}^n(M_\varphi)-\mathcal{H}^n(M) =&\frac{1}{2}\int_M (|\nabla^M\varphi|^2-|\mathrm{II}_M|^2\varphi^2)\,d\mathcal{H}^n+O(\|\varphi\|_{C^1})\int_M(|\nabla^M\varphi|^2+\varphi^2)\,d\mathcal{H}^n\\ &\ge\Big(\frac{\mu}{2}+O(\|\varphi\|_{C^1})\Big)\int_M(|\nabla^M\varphi|^2+\varphi^2)\,d\mathcal{H}^n\\ &\ge\frac{\mu}{4}\int_M\varphi^2\,d\mathcal{H}^n\\ &\ge\frac{\mu}{4\mathcal{H}^n(M)}\Big(\int_M|\varphi|\,d\mathcal{H}^n\Big)^2\\ &\ge\kappa\Big(\int_0^1ds\int_M|\varphi|\prod_{j=1}^n|1+s\lambda_j\varphi|\,d\mathcal{H}^n\Big)^2=\kappa\,d([\![M]\!],[\![M_\varphi]\!])^2\,.\end{aligned}$$

q.e.d.

*Proof of Theorem* 4*, (b) implies (a).* For $\varphi\in C^1_0(M)$, $t\in[0,1]$, define $H_t:M\to\mathbb{R}^n$ as

$$H_t(x)=x+t\,\varphi(x)\nu_M(x)\,,\qquad x\in M\,,$$

and set $S_t=(H_t)_\# T$, $T=[\![M]\!]$. Clearly $\partial S_t=\partial T$ so that, by assumption,

$$\mathbf{M}(S_t)-\mathcal{H}^n(M)\ge\kappa\,d(S_t,T)^2\,. \tag{3.13}$$

Now, by the Taylor expansion in the proof of Theorem 6, we have

$$\mathbf{M}(S_t)\;=\;\mathcal{H}^n(M)+\frac{t^2}{2}\int_M\Big(|\nabla^M\varphi|^2-|\mathrm{II}_M|^2\varphi^2\Big)\,d\mathcal{H}^n+O(t^3)\,, \tag{3.14}$$

$$d(S_t,T)\;=\;t\int_M|\varphi|\,d\mathcal{H}^n+O(t^2)\,. \tag{3.15}$$

so that (3.13) immediately implies

$$\int_M|\nabla^M\varphi|^2-|\mathrm{II}_M|^2\,\varphi^2\,d\mathcal{H}^n\ge2\kappa\left(\int_M|\varphi|d\mathcal{H}^n\right)^2\,. \tag{3.16}$$

By Nash's inequality, for every $\psi\in W^{1,2}_0(M)$ we have

$$\int_M\psi^2\le c_1\,\varepsilon\int_M|\nabla^M\psi|^2+\frac{c_2}{\varepsilon}\left(\int_M\psi\right)^2\,,$$

(where $c_1$ and $c_2$ may be taken independent from $M$, just on the dimension $n$, thanks to the vanishing mean curvature condition $H_M=0$ see [**Si**, Section 18]). We apply this inequality to $\psi=|\varphi|$, and combine it with (3.16) to find that

$$\int_M\varphi^2\le\left(c_1\varepsilon+\frac{2c_2}{\varepsilon\kappa}\right)\left(\int_M|\nabla\varphi|^2-|\mathrm{II}_M|^2\varphi^2\right)+c_1\varepsilon\int_M|\mathrm{II}_M|^2\varphi^2\,.$$

Taking into account that $M\in\mathcal{M}$, so that $\sup_M|\mathrm{II}_M|<\infty$, we conclude

$$\left(1-c_1\,\varepsilon\sup_M|\mathrm{II}_M|\right)\int_M\varphi^2\le\left(c_1\,\varepsilon+\frac{2c_2}{\varepsilon\kappa}\right)\left(\int_M|\nabla\varphi|^2-|\mathrm{II}_M|^2\varphi^2\right)\,.$$

By suitably choosing $\varepsilon$, we prove (1.8). q.e.d.

**3.3. Almgren-type lemma.** In the following lemma we adapt to our needs a construction originally introduced by Almgren in the proof of the existence of minimal clusters [**Al1**, VI]. The idea behind the lemma is easily explained in the simplified framework of sets of finite perimeter. We are given two sets of finite perimeter $E$ and $F$ with $E \subset F$, and we seek a way to modify $F$ inside a small ball so to obtain a new set $G$ which still contains $E$ and such that $|G \setminus E|$ is increased with respect to $|F \setminus E|$ by a given (but sufficiently small) amount. Roughly speaking, we construct a one-parameter family of diffeomorphisms $\{f_t\}_{|t|<\varepsilon}$ such that $f_t(x) - x \neq 0$ only inside a small ball centered at a regular point $x_0$ of $E$, and with the property that $f_t$ pushes $E$ in the direction $\nu_E(x_0)$. We may arrange things carefully, so that $E \subset f_t(E)$, $|f_t(E)|$ is increasing for $t \in (0, \varepsilon)$, and $(d/dt)P(f_t(E))$ is bounded. The sets $f_t(F)$ provides a suitable choice for $G$. Indeed, it turns out that $E \subset f_t(E) \subset f_t(F)$ and that $|f_t(F) \setminus E| \geq |f_t(E) \setminus E| \geq c\,t$ for some positive $c$ and provided $|F \setminus E|$ is sufficiently small. In the framework of currents, the inclusion property $E \subset F$ is replaced by the requirement that $S = T + \partial X$ for some $X \in \mathcal{I}^+_{n+1}(\mathbb{R}^{n+1})$.

**Notation 3.1.** We introduce the following useful notation. Decomposing $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$, we let $\mathbf{p}\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $\mathbf{q}\colon \mathbb{R}^{n+1} \to \mathbb{R}$ denote the corresponding orthogonal projections. Moreover, given $r > 0$, $z \in \mathbb{R}^n$ and $x \in \mathbb{R}^{n+1}$, we set

$$\mathbf{D}(z,r) = \{y \in \mathbb{R}^n : |z - y| < r\}\,, \qquad \mathbf{C}(x,r) = \mathbf{D}(\mathbf{p}x, r) \times (\mathbf{q}x - r, \mathbf{q}x + r)\,,$$

for the $n$-dimensional disk of center $z$ and radius $r$ in $\mathbb{R}^n$, and for the cylinder of height $2r$ and radius $r$ centered at $x$ in $\mathbb{R}^{n+1}$.

**Lemma 3.3.** *If $M \in \mathcal{M}$ and $T = [\![M]\!]$, then there exist positive constants $\delta_0, t_0, c_0$ and $C_0$ (all depending only on $M$) with the following property. If $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$ with*

$$S = T + \partial X\,, \tag{3.17}$$

$$X = \mathbf{E}^{n+1} \llcorner u\,, \tag{3.18}$$

$$u \in BV(\mathbb{R}^{n+1}; \mathbb{N})\,, \tag{3.19}$$

$$\mathbf{M}(X) \leq \delta_0\,, \tag{3.20}$$

$$\mathbf{M}(S) \leq 2\mathbf{M}(T)\,, \tag{3.21}$$

*then for every $t \in (0, t_0)$, there exists $S_t \in \mathcal{I}_n(\mathbb{R}^{n+1})$ such that*

$$S_t = T + \partial X_t\,, \tag{3.22}$$

$$X_t = \mathbf{E}^{n+1} \llcorner u_t\,, \tag{3.23}$$

$$u_t \in BV(\mathbb{R}^{n+1}; \mathbb{N})\,, \tag{3.24}$$

$$\mathbf{M}(X_t) \geq \mathbf{M}(X) + c_0\,t\,, \tag{3.25}$$

$$\mathbf{M}(S_t) \leq \mathbf{M}(S) + C_0\,t\,. \tag{3.26}$$

*Proof.* Given $x_0 \in M$, there exist $r_0$ and $u : \mathbb{R}^n \to \mathbb{R}$ such that, up to a rotation,

$$\mathbf{C}(x_0, 2r_0) \cap M = \Big\{z + v(z)\,e_{n+1} : z \in \mathbf{D}(\mathbf{p}x_0, 2r_0)\Big\}\,, \tag{3.27}$$

and, moreover, $\mathbf{C}(x_0, 2r_0) \cap \Gamma = \emptyset$, $\Gamma = \operatorname{bdry} M$. We now fix $\varphi \in C^1_c(\mathbf{C}(x_0, 2r_0))$, $\varphi \geq 0$, $\varphi = 1$ on $\mathbf{C}(x_0, r_0)$, and then define

$$H(x,t) = H_t(x) = x + t\varphi(x)e_{n+1}\,, \qquad x \in \mathbb{R}^{n+1}, t \geq 0\,.$$

Clearly, there exists $t_0 > 0$ such that $\{H_t\}_{|t|<t_0}$ is a family of smooth diffeomorphism of $\mathbb{R}^{n+1}$ into itself, with

$$\{x : H_t(x) \neq x\} \subset\subset \mathbf{C}(x_0, 2r_0)\,. \tag{3.28}$$

Moreover, up to restrict the value of $r_0$ we may find $\{\tau_h\}_{h=1}^n \subset C^1(\mathbf{C}(x_0,2r_0);\mathbb{R}^{n+1})$ such that $\{\tau_h(x)\}_{h=1}^n$ is an orthonormal basis of $T_xM$ for every $x\in\mathbf{C}(x_0,2r_0)\cap M$ and

$$\langle dx^1\wedge\cdots\wedge dx^{n+1}, e_{n+1}\wedge\tau_1\wedge\cdots\wedge\tau_n\rangle\ge c\,,\qquad\text{on } \mathbf{C}(x_0,2r_0)\,,\tag{3.29}$$

for some positive constant $c>0$, where we have also used (3.27). We now set,

$$T_t=(H_t)_\# T\qquad Z_t=H_\#([\![0,t]\!]\times T),\tag{3.30}$$

for $t\in(0,t_0)$. Clearly $\partial Z_t=T_t-T$, and $Z_t=\mathbf{E}^{n+1}\llcorner z_t$ for some $z_t\in BV(\mathbb{R}^{n+1};\mathbb{Z})$ with $\mathrm{spt} z_t\subset\subset\mathbf{C}(x_0,2r_0)$. In fact, $z_t\ge 0$: indeed, by the homotopy formula, if $f\ge 0$, then

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} z_t(x)f(x)dx &= \langle Z_t, f\,dx^1\wedge\cdots\wedge dx^{n+1}\rangle\\ &= \int_{[0,t]\times M}\varphi\, f\,\langle dx^1\wedge\cdots\wedge dx^{n+1}, e_{n+1}\wedge\tau_1\wedge\cdots\wedge\tau_n\rangle\, d\mathcal{H}^{n+1}\ge 0\,,\end{aligned}$$

as desired. In particular, thanks to (3.29),

$$\int_{\mathbb{R}^{n+1}} z_t=\mathbf{M}(Z_t)=\langle Z_t, dy^1\wedge\cdots\wedge dy^n\rangle\ge c\,\mathcal{H}^n(M\cap\mathbf{C}(x_0,r_0))\,t=c'\,t\,.\tag{3.31}$$

We now consider $S_t=(H_t)_\# S$ and $Y_t=(H_t)_\# X$, so that $S_t\in\mathcal{I}_n(\mathbb{R}^{n+1})$, with

$$S_t=\partial Y_t+T_t=T+\partial(Y_t+Z_t)\,.$$

Moreover, since $H_t$ is an orientation preserving diffeomorphisms, by (3.18) and (3.19) we have $Y_t=\mathbf{E}^{n+1}\llcorner y_t$, with $y_t\in BV(\mathbb{R}^{n+1};\mathbb{N})$. Therefore, if we set,

$$X_t=Y_t+Z_t\,,$$

then we have $X_t=\mathbf{E}^{n+1}\llcorner u_t$ for $u_t=y_t+z_t\in BV(\mathbb{R}^{n+1};\mathbb{N})$, with

$$\begin{aligned}\mathbf{M}(X_t) &= \int_{\mathbb{R}^{n+1}}|y_t+z_t|=\int_{\mathbb{R}^{n+1}}y_t+\int_{\mathbb{R}^{n+1}}z_t=\mathbf{M}(Y_t)+\mathbf{M}(Z_t)\,,\\ \mathbf{M}(Y_t) &= \int_{\mathbb{R}^{n+1}}u\circ H_t^{-1}=\int_{\mathbb{R}^{n+1}}u\det DH_t=\int_{\mathbb{R}^{n+1}}u\Big(1+t\,\mathrm{div}\,g+o(t)\Big)\\ &\ge \mathbf{M}(X)-C\,\mathbf{M}(X)\,t\ge\mathbf{M}(X)-C\delta_0 t\,.\end{aligned}$$

where we have set $g(x)=\varphi(x)e_{n+1}$. By (3.31) we thus find that, provided $\delta_0$ is small enough,

$$\mathbf{M}(X_t)\ge\mathbf{M}(X)+(c'-C\delta_0)t\ge\mathbf{M}(X)+c_0\,t\,.$$

By the area formula between rectifiable sets, denoting by $M_S$ and $\theta_S$ the $n$-rectifiable set carrying $S$ and the density of $S$, we find

$$\mathbf{M}(S_t)=\mathbf{M}((H_t)_\# S)=\int_{M_S}J^{M_S}H_t\,\theta_S\,d\mathcal{H}^n=\int_{M_S}\Big(1+t\,\mathrm{div}^{M_S}g+o(t)\Big)\,\theta_S\,d\mathcal{H}^n\,,$$

where, again, $g(x)=\varphi(x)e_{n+1}$. Hence, by (3.21),

$$\frac{d}{dt}\mathbf{M}(S_t)\Big|_{t=0}=\int_{M_S}\mathrm{div}^{M_S}g\,\theta_S\,d\mathcal{H}^n\le\sup_{\mathbb{R}^n}|\nabla g|\,\mathbf{M}(S)\le 2\sup_{\mathbb{R}^n}|\nabla g|\,\mathbf{M}(T)\,.$$

Therefore, up to further decrease the value of $t_0$, we certainly have

$$\mathbf{M}(S_t)\le\mathbf{M}(S)+2\|Dg\|_\infty\mathbf{M}(T)\,t=\mathbf{M}(S)+C_0\,t\,,\qquad\forall t\in(0,t_0)\,.$$

q.e.d.

**3.4. Existence of the $\varepsilon$-approximating currents.** We now prove the existence of minimizers in the variational problems (3.1) and (3.2).

**Lemma 3.4.** *If $M \in \mathcal{M}$ and $T = [\![M]\!]$ is mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$, then there exists a positive constant $\varepsilon_0$ (depending on $M$) such that the variational problem*

$$\inf\left\{\mathbf{M}(S) : S = T + \partial\Big(\mathbf{E}^{n+1} \llcorner u\Big), u \in BV(\mathbb{R}^{n+1};\mathbb{N}), \int_{\mathbb{R}^{n+1}} u \ge \varepsilon\right\}, \tag{3.32}$$

*admits at least a minimizer $S_\varepsilon$, provided $\varepsilon \in (0, \varepsilon_0)$. Moreover,*

$$\lim_{\varepsilon\to 0^+} \mathbf{M}(S_\varepsilon) = \mathbf{M}(T),$$

*for every family $\{S_\varepsilon\}_{\varepsilon>0}$ of such minimizers.*

We first need to prove the following lemma.

**Lemma 3.5.** *If $u \in BV(\mathbb{R}^{n+1};\mathbb{N})$, then for almost every $r > 0$ it holds*

$$\int_{\partial B_r} u \, d\mathcal{H}^n \le |Du|(B_r^c).$$

*In particular, if $S = T + \partial(\mathbf{E}^{n+1} \llcorner u)$ with $u \in BV(\mathbb{R}^{n+1}, \mathbb{N})$ and $\operatorname{spt} T \subset\subset B_r$, then*

$$\int_{\partial B_r} u \, d\mathcal{H}^n \le \|S\|(B_r^c).$$

*Proof of Lemma* 3.5. Since $\operatorname{div}(x/|x|) = n/|x|$ for $x \ne 0$, then, by applying the divergence theorem on $B_s \setminus B_r$ to the vector field $u\, g$ for $g(x) = x/|x|$, we find that, for a.e. $r, s > 0$,

$$0 \le \int_{B_s\setminus B_r} u \operatorname{div} g = -\int_{B_s\setminus B_r} g \cdot Du - \int_{\partial B_r} u \, d\mathcal{H}^n + \int_{\partial B_s} u \, d\mathcal{H}^n.$$

In particular,

$$\int_{\partial B_r} u \, d\mathcal{H}^n \le |Du|(B_r^c) + \int_{\partial B_s} u \, d\mathcal{H}^n. \tag{3.33}$$

Since $\int_{\mathbb{R}^{n+1}} u = \int_0^\infty ds \int_{\partial B_s} u \, d\mathcal{H}^n$ is finite, we can find $s = s_h \to \infty$ as $h \to \infty$ such that $\int_{\partial B_{s_h}} u \, d\mathcal{H}^n \to 0$ as $h \to \infty$ and (3.33) holds true. q.e.d.

*Proof of Lemma* 3.4. We let $\delta_0, t_0, c_0$ and $C_0$ be as in Lemma 3.3.

*Step one:* We claim that, if $\gamma(\varepsilon)$ denotes the infimum in (3.32), then

$$\mathbf{M}(T) \le \gamma(\varepsilon) \le \mathbf{M}(T) + \Lambda\, \varepsilon, \qquad \forall \varepsilon \in (0, \varepsilon_0), \tag{3.34}$$

where $\Lambda = C_0/c_0$ and $\varepsilon_0 = c_0\, t_0$. Indeed, applying Lemma 3.3 to $X = 0$, we find that, for every $t \in (0, t_0)$, there exists $X_t \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ such that $X_t = \mathbf{E}^{n+1} \llcorner u_t$ for $u_t \in BV(\mathbb{R}^{n+1};\mathbb{N})$ and

$$\mathbf{M}(X_t) \ge c_0\, t, \qquad \mathbf{M}(T + \partial X_t) \le \mathbf{M}(T) + C_0\, t. \tag{3.35}$$

In particular, if $\varepsilon < c_0\, t_0$, then $t(\varepsilon) = \varepsilon/c_0 \in (0, t_0)$ and, setting (with a slight abuse of notation) $X_\varepsilon = X_{t(\varepsilon)}$ we find $\mathbf{M}(X_\varepsilon) \ge \varepsilon$. Therefore,

$$\gamma(\varepsilon) \le \mathbf{M}(T + \partial X_\varepsilon) \le \mathbf{M}(T) + C_0\, t(\varepsilon) = \mathbf{M}(T) + \frac{C_0}{c_0}\, \varepsilon,$$

which is (3.34) (the fact that $\gamma(\varepsilon) \ge \mathbf{M}(T)$ being trivial since $M$ is area minimizing).

*Step two:* Let $\varepsilon \in (0, \varepsilon_0)$ and let $\{S_h^\varepsilon\}_{h\in\mathbb{N}}$ be a minimizing sequence for (3.32), with

$$S_h^\varepsilon = T + \partial(\mathbf{E}^{n+1} \llcorner u_h^\varepsilon), \qquad u_h^\varepsilon \in BV(\mathbb{R}^{n+1};\mathbb{N}).$$

By (3.34), $\sup_{h\in\mathbb{N}}|Du^\varepsilon_h|(\mathbb{R}^{n+1})<\infty$. By the compactness theorem for $BV$ functions, there exists $u^\varepsilon\in L^1_{\rm loc}(\mathbb{R}^{n+1};\mathbb{N})$, with $|Du^\varepsilon|(\mathbb{R}^{n+1})<\infty$, such that, up to extracting a not-relabeled subsequence,

$$u^\varepsilon_h\to u^\varepsilon\qquad\text{in } L^1_{\rm loc}(\mathbb{R}^{n+1})\,.$$

In particular, if $S^\varepsilon=T+\partial(\mathbf{E}^{n+1}\llcorner u^\varepsilon)$, then $S^\varepsilon_h\rightharpoonup S^\varepsilon$. The problem now is that $u^\varepsilon$ may fail to satisfy the constraint

$$\int_{\mathbb{R}^{n+1}}u^\varepsilon\ge\varepsilon\,. \tag{3.36}$$

The next steps of the proof are devoted to show how to find a minimizing sequence $\hat S^\varepsilon_h$ such that the convergence of the associated $\hat u^\varepsilon_h$ to $\hat u$ is actually in $L^1(\mathbb{R}^n)$. This will suffice to guarantee that $\hat u$ satisfies (3.36), and hence that $\hat S^\varepsilon$ is a minimizer in (3.32).

*Step three:* We show that, if $\varepsilon\in(0,\varepsilon_0)$, $\{S^\varepsilon_h\}_{h\in\mathbb{N}}$ and $S^\varepsilon$ are as in step two, then

$$\limsup_{R\to\infty}\limsup_{h\to\infty}\int_{B_R^c}u^\varepsilon_h\le(2\Lambda\,\varepsilon)^{(n+1)/n}\,. \tag{3.37}$$

Indeed, by (3.34) and by the lower semi-continuity of the variation measure,

$$\begin{aligned}\mathbf{M}(T)+\Lambda\,\varepsilon&\ge\limsup_{h\to\infty}\Big(\|S^\varepsilon_h\|(B_R^c)+\|S^\varepsilon_h\|(B_R)\Big)\\&\ge\limsup_{h\to\infty}\|S^\varepsilon_h\|(B_R^c)+\liminf_{h\to\infty}\|S^\varepsilon_h\|(B_R)\\&\ge\limsup_{h\to\infty}\|S^\varepsilon_h\|(B_R^c)+\|S^\varepsilon\|(B_R)\,.\end{aligned}$$

Letting $R\to\infty$, and taking also into account that $\mathbf{M}(S^\varepsilon)\ge\mathbf{M}(T)$ by minimality of $M$,

$$\limsup_{R\to\infty}\limsup_{h\to\infty}\|S^\varepsilon_h\|(B_R^c)\le\mathbf{M}(T)+\Lambda\,\varepsilon-\mathbf{M}(S^\varepsilon)\le\Lambda\,\varepsilon\,. \tag{3.38}$$

Thanks to Lemma 3.5 we have that for a.e. $R>R_0$ (where $R_0$ is such that $M\subset B_{R_0}$),

$$\|S^\varepsilon_h\|(B_R^c)\ge\int_{\partial B_R}u^\varepsilon_h\,,\qquad\mathbf{M}(S^\varepsilon_h\llcorner B_R^c)=\|S^\varepsilon_h\|(B_R^c)+\int_{\partial B_R}u^\varepsilon_h\,.$$

By the Sobolev inequality on $BV$-functions, and since $u^\varepsilon_h\ge1$ on its support,

$$\begin{aligned}\|S^\varepsilon_h\|(B_R^c)&\ge\frac12\,\mathbf{M}(S^\varepsilon_h\llcorner B_R^c)\\&\ge\frac12\left(\int_{B_R^c}(u^\varepsilon_h)^{(n+1)/n}\right)^{n/(n+1)}\ge\frac12\left(\int_{B_R^c}u^\varepsilon_h\right)^{n/(n+1)}\end{aligned} \tag{3.39}$$

which immediately implies (3.37).

*Step four:* For $\varepsilon_1\le\varepsilon_0$ to be chosen later, let $\varepsilon\in(0,\varepsilon_1)$, and let $\{S^\varepsilon_h\}_{h\in\mathbb{N}}$ be as in step two. By (3.37), we can find $R_1\ge R_0$ such that, up to subsequences,

$$\sup_{h\in\mathbb{N}}\int_{B_R^c}u^\varepsilon_h\le(3\Lambda\varepsilon_1)^{(n+1)/n}\,, \tag{3.40}$$

for every $R\ge R_1$. We now claim that, if we define

$$R_2=R_1+4(n+1)(3\Lambda\varepsilon_1)^{1/n}\,,$$

then, for every $h\in\mathbb{N}$ there exists $\hat S^\varepsilon_h\in\mathcal{I}_n(\mathbb{R}^{n+1})$, admissible in (3.32), such that

$$\mathbf{M}(\hat S^\varepsilon_h)\le\mathbf{M}(S^\varepsilon_h)\,,\qquad{\rm spt}S^\varepsilon_h\subset\overline{B_{R_2}}\,. \tag{3.41}$$

Indeed, let us fix $h \in \mathbb{N}$ and let $S_h^\varepsilon = T + \partial(\mathbf{E}^{n+1} \llcorner u_h^\varepsilon)$. If we have

$$\int_{B_{R_2}} u_h^\varepsilon \geq \varepsilon$$

then it suffices to set $\hat{S}_h^\varepsilon = T + \partial(\mathbf{E}^{n+1} \llcorner 1_{B_{R_2}} u_h^\varepsilon)$ and apply (3.46) below in order to achieve (3.41). Therefore we may directly assume that, for the considered values of $\varepsilon$ and $h$, we have

$$\int_{B_R} u_h^\varepsilon < \varepsilon\,, \qquad \forall R \leq R_2\,. \tag{3.42}$$

In order to prove (3.41) in this case, we shall preliminary show the existence of $I_h \subset (R_1, R_2)$ with $\mathcal{L}^1(I_h) > 0$, such that

$$\|S_h^\varepsilon\|(B_R^c) \geq \int_{\partial B_R} u_h^\varepsilon + \Lambda \int_{B_R^c} u_h^\varepsilon\,, \qquad \forall R \in I_h\,. \tag{3.43}$$

Indeed suppose that it holds

$$\|S_h^\varepsilon\|(B_R^c) \leq \int_{\partial B_R} u_h^\varepsilon + \Lambda \int_{B_R^c} u_h^\varepsilon \tag{3.44}$$

for almost all $R \in (R_1, R_2)$. If we introduce the non-increasing function,

$$m_h^\varepsilon(R) = \int_{B_R^c} u_h^\varepsilon\,, \qquad R > 0\,,$$

which, for a.e. $R > 0$, satisfies $(m_h^\varepsilon)'(R) = -\int_{\partial B_R} u_h^\varepsilon$, then by (3.39) we find

$$m_h^\varepsilon(R)^{n/(n+1)} \leq -2(m_h^\varepsilon)'(R) + 2\Lambda\, m_h^\varepsilon(R)\,. \tag{3.45}$$

Choosing $\varepsilon_1 \leq 1/(3 \cdot 4^n \Lambda^{n+1})$, from (3.40) we find that

$$2\Lambda\, m_h^\varepsilon(R) \leq \frac{(m_h^\varepsilon(R))^{n/(n+1)}}{2}\,, \qquad \forall R \geq R_1\,,$$

which, combined with (3.45), implies

$$\frac{m_h^\varepsilon(R)^{n/(n+1)}}{2} \leq -2(m_h^\varepsilon)'(R)\,, \qquad \text{for a.e. } R \in (R_1, R_2)\,.$$

In other words,

$$\frac{d}{dR}\Big(m_h^\varepsilon(R)\Big)^{1/(n+1)} \leq -\frac{1}{4(n+1)}\,, \qquad \text{for a.e. } R \in (R_1, R_2).$$

Integrating this differential inequality between $R_1$ and $R_2$, and taking into account equation (3.40), we finally obtain

$$m_h^\varepsilon(R_2)^{1/(n+1)} \leq (3\Lambda\varepsilon_1)^{1/(n+1)} - \frac{R_2 - R_1}{4(n+1)}\,,$$

which, by the choice of $R_2$, gives $0 = m_h^\varepsilon(R_2) = \int_{B_{R_2}^c} u_h^\varepsilon$, against $\int_{\mathbb{R}^{n+1}} u_h^\varepsilon \geq \varepsilon$ and (3.42).

Having proved (3.43), we are now in the position of construct $\hat{S}_h^\varepsilon$ satisfying (3.41) also in the case (3.42) holds true. Indeed, by suitably choosing a radii $R \in I_h$, we shall construct $\hat{S}_h^\varepsilon$ by modifying

$$T + \partial(\mathbf{E}^{n+1} \llcorner 1_{B_R} u_h^\varepsilon)\,,$$

through the use of Lemma 3.3. First of all, notice that, setting $X_h^\varepsilon = \mathbf{E}^{n+1} \llcorner 1_{B_R} u_h^\varepsilon$ and taking into account Lemma 3.5, we have

$$\mathbf{M}(T + \partial X_h^\varepsilon) = \mathbf{M}(T + \partial(\mathbf{E}^{n+1} \llcorner 1_{B_R} u_h^\varepsilon)) = \|S_h^\varepsilon\|(B_R) + \int_{\partial B_R} u_h^\varepsilon \leq \mathbf{M}(S_h^\varepsilon)\,. \tag{3.46}$$

Moreover, since

$$\mathbf{M}(X_h^\varepsilon) = \int_{B_R} u_h^\varepsilon < \varepsilon\,, \tag{3.47}$$

we can apply Lemma 3.3 to $X_h^\varepsilon$ with

$$t = \frac{\varepsilon - \int_{B_R} u_h^\varepsilon}{c_0} = \frac{\varepsilon - \mathbf{M}(X_h^\varepsilon)}{c_0}\,,$$

provided

$$\varepsilon_1 \le \min\left\{\varepsilon_0, \delta_0\,, \frac{\mathbf{M}(T)}{2\Lambda}\right\}.$$

Indeed $\varepsilon_0 = c_0\,t_0$, while, by (3.47), $\mathbf{M}(X_h^\varepsilon) \le \varepsilon \le \varepsilon_1$. Moreover,

$$\mathbf{M}(T + \partial X_h^\varepsilon) \le \mathbf{M}(S_h^\varepsilon) \le \mathbf{M}(T) + 2\Lambda\,\varepsilon \le 2\,\mathbf{M}(T)\,,$$

by equations (3.34), (3.46), and up to extracting a subsequence. Thus, by Lemma 3.3, there exists $Y_h^\varepsilon \in \mathcal{I}_{n+1}^+(\mathbb{R}^{n+1})$ such that

$$\mathbf{M}(Y_h^\varepsilon) \ge \mathbf{M}(X_h^\varepsilon) + c_0\,t = \varepsilon$$

and

$$\begin{aligned}
\mathbf{M}(T + \partial Y_h^\varepsilon) &\le \mathbf{M}(T + \partial X_h^\varepsilon) + \Lambda\left(\varepsilon - \int_{B_R} u_h^\varepsilon\right)\\
&\le \mathbf{M}(T + \partial X_h^\varepsilon) + \Lambda\left(\int_{\mathbb{R}^{n+1}} u_h^\varepsilon - \int_{B_R} u_h^\varepsilon\right) = \mathbf{M}(T + \partial X_h^\varepsilon) + \Lambda\int_{B_R^c} u_h^\varepsilon\,.
\end{aligned}$$

Moreover, as it is evident from the proof of Lemma 3.3, we may safely assume that $\mathrm{spt} Y_h^\varepsilon \subset B_R \subset B_{R_2}$. Now the previous equation, together with equations (3.43) and (3.46), implies

$$\begin{aligned}
\mathbf{M}(T + \partial Y_h^\varepsilon) &\le \mathbf{M}(T + \partial X_h^\varepsilon) + \Lambda\int_{B_R^c} u_h^\varepsilon\\
&= \|S_h^\varepsilon\|(B_R) + \int_{\partial B_R} u_h^\varepsilon + \Lambda\int_{B_R^c} u_h^\varepsilon\\
&\le \|S_h^\varepsilon\|(B_R) + \|S_h^\varepsilon\|(B_R^c) = \mathbf{M}(S_h^\varepsilon).
\end{aligned}$$

In this case we set $\hat{S}_h^\varepsilon = T + \partial Y_h^\varepsilon$. We have thus provided a minimizing sequence $\{\hat{S}_h^\varepsilon\}_{h\in\mathbb{N}}$ in (3.32), with $\mathrm{spt}\,\hat{S}_h^\varepsilon \subset B_{R_2}$. Hence, the corresponding functions $\hat{u}_h^\varepsilon \in BV(\mathbb{R}^{n+1};\mathbb{N})$ converge in $L^1$ to a function $\hat{u}^\varepsilon$, which satisfies $\int_{\mathbb{R}^{n+1}} \hat{u}^\varepsilon \ge \varepsilon$. The current $\hat{S}^\varepsilon = T + \partial(\mathbf{E}^{n+1} \llcorner \hat{u}^\varepsilon)$ is thus a minimizer in (3.32). q.e.d.

**3.5. Reduction to the $\varepsilon$-approximating currents.** We now show that in proving the global stability inequality (2.3) of Theorem 1, one may directly reduce to consider the inequality on the minimizers of (3.32). Notice that, in proving this fact, we do not need to assume that $T = [\![M]\!]$ for some $M \in \mathcal{M}$.

**Theorem 7.** *If $T \in \mathcal{I}_n(\mathbb{R}^{n+1})$ is a uniquely mass minimizing integral $n$-current with multiplicity one, and if there exist positive constants $\varepsilon_0$ and $\kappa_0$ such that*

$$\mathbf{M}(S_\varepsilon) - \mathbf{M}(T) \ge \kappa_0\,d(S_\varepsilon, T)^2\,,$$

*whenever $\varepsilon \in (0, \varepsilon_0)$, and $S_\varepsilon$ denotes a minimizer in one of the variational problems* (3.1) *or* (3.2)*, then there exists a positive constant $\kappa$ such that*

$$\mathbf{M}(S) - \mathbf{M}(T) \ge \kappa\left\{d(S,T)^2, d(S,T)^{n/(n+1)}\right\}\,,$$

*whenever $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$ with $\partial S = \partial T$.*

**Lemma 3.6.** *If* $T \in \mathcal{I}_n(\mathbb{R}^{n+1})$, $S = T + \partial X$, $X = \mathbf{E}^{n+1} \llcorner u$, $u \in BV(\mathbb{R}^{n+1};\mathbb{Z})$, *then*

$$\mathbf{M}(S) \geq \mathbf{M}(S^+) + \mathbf{M}(S^-) - \mathbf{M}(T)\,,$$

*where* $S^+ = T + \partial\Big(\mathbf{E}^{n+1} \llcorner u^+\Big)$ *and* $S^- = T - \partial\Big(\mathbf{E}^{n+1} \llcorner u^-\Big)$.

*Proof.* From the theory of functions of bounded variation we know that if $u \in BV(\mathbb{R}^{n+1};\mathbb{Z})$ then there exists a locally $n$-rectifiable set $J$, two Borel functions $a, b : J \to \mathbb{Z}$ with $b > a$ on $J$ and a unit $n$-vector-field $\tau_J : J \to \Lambda_n(\mathbb{R}^{n+1})$ such that

$$\partial\Big(\mathbf{E}^{n+1} \llcorner u\Big) = (b-a)\,\tau_J\,\mathcal{H}^n \llcorner J\,,$$

where $\tau_J(x)$ provides an orientation to $T_x J$ for every $x \in J$. Correspondingly, we have

$$\begin{aligned}\partial\Big(\mathbf{E}^{n+1} \llcorner u^+\Big) &= (b^+ - a^+)\,\tau_J\,\mathcal{H}^n \llcorner J\,,\\ \partial\Big(\mathbf{E}^{n+1} \llcorner u^-\Big) &= (b^- - a^-)\,\tau_J\,\mathcal{H}^n \llcorner J\,.\end{aligned}$$

Taking now into account that $T = \theta\,\tau_M\,\mathcal{H}^n \llcorner M$, with $\theta : M \to \mathbb{N}$ and $\tau_M(x)$ which provides an orientation of $T_x M$, denoting for the sake of brevity

$$\begin{aligned}\{\tau_M = \tau_J\} &= \{x \in M \cap J : \tau_M(x) = \tau_J(x)\}\,,\\ \{\tau_M = -\tau_J\} &= \{x \in M \cap J : \tau_M(x) = -\tau_J(x)\}\,,\end{aligned}$$

and recalling that $\mathcal{H}^n((M \cap J) \setminus \{\tau_M = \pm\tau_J\}) = 0$, we thus find that

$$\begin{aligned}S^+ &= \theta\,\tau_M\,\mathcal{H}^n \llcorner (M \setminus J)\\ &\quad + (b^+ - a^+)\,\tau_J\,\mathcal{H}^n \llcorner (J \setminus M)\\ &\quad + (b^+ - a^+ + \theta)\mathcal{H}^n \llcorner \{\tau_M = \tau_J\}\\ &\quad + (b^+ - a^+ - \theta)\mathcal{H}^n \llcorner \{\tau_M = -\tau_J\}\,,\\ S^- &= \theta\,\tau_M\,\mathcal{H}^n \llcorner (M \setminus J)\\ &\quad + (a^- - b^-)\,\tau_J\,\mathcal{H}^n \llcorner (J \setminus M)\\ &\quad + (a^- - b^- + \theta)\mathcal{H}^n \llcorner \{\tau_M = \tau_J\}\\ &\quad + (a^- - b^- - \theta)\mathcal{H}^n \llcorner \{\tau_M = -\tau_J\}\,,\\ S &= \theta\,\tau_M\,\mathcal{H}^n \llcorner (M \setminus J)\\ &\quad + (b - a)\,\tau_J\,\mathcal{H}^n \llcorner (J \setminus M)\\ &\quad + (b - a + \theta)\mathcal{H}^n \llcorner \{\tau_M = \tau_J\}\\ &\quad + (b - a - \theta)\mathcal{H}^n \llcorner \{\tau_M = -\tau_J\}\,.\end{aligned}$$

We may thus compute,

$$\begin{aligned}&\mathbf{M}(S) + \mathbf{M}(T) - \mathbf{M}(S^+) - \mathbf{M}(S^-)\\ &= \int_{J \setminus M} \Big((b-a) - (b^+ - a^+) - (a^- - b^-)\Big)\, d\mathcal{H}^n\\ &+ \int_{\{\tau_M = \tau_J\}} \Big(|b - a + \theta| + \theta - |b^+ - a^+ + \theta| - |a^- - b^- + \theta|\Big)\, d\mathcal{H}^n\\ &+ \int_{\{\tau_M = -\tau_J\}} \Big(|b - a - \theta| + \theta - |b^+ - a^+ - \theta| - |a^- - b^- - \theta|\Big)\, d\mathcal{H}^n\,.\end{aligned}$$

The first integrand is identically zero, while the second and the third integrand are non-negative, as it may easily checked. q.e.d.

In proving Theorem 7 we shall first rule out the case in which the mass of $S$ is not close to the mass of $T$. To this end, it is convenient to introduce the mass deficit of $S$ with respect to $T$, defined as

$$\delta(S;T)=\frac{\mathbf{M}(S)}{\mathbf{M}(T)}-1\,.$$

If $T$ is uniquely mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$, then $\delta(S;T)\ge 0$ for every $S\in\mathcal{I}_n(\mathbb{R}^{n+1})$, with $\delta(S;T)=0$ if and only if $S=T$. We now prove two simple preparatory lemmas.

**Lemma 3.7.** *Let $T$ be uniquely mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$. If $\delta(S;T)\ge\delta>0$, then there exists $c(n,\delta)>0$ such that*

$$\mathbf{M}(S)-\mathbf{M}(T)\ge c(n,\delta)\,d(S,T)^{n/(n+1)}\,.$$

*Proof.* By Lemma 2.1, there exists $X\in\mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ with $\mathbf{M}(X)=d(S,T)$, $\partial X=S-T$ and $c(n)\mathbf{M}(X)^{n/(n+1)}\le\mathbf{M}(\partial X)\le\mathbf{M}(S)+\mathbf{M}(T)$. We conclude since $\delta(S;T)\ge\delta$ implies

$$\mathbf{M}(S)-\mathbf{M}(T)\ge\frac{\mathbf{M}(S)+\mathbf{M}(T)}{C(\delta)}\,,\qquad C(\delta)=1+\frac{2}{\delta}\,.$$

q.e.d.

**Lemma 3.8.** *If $T$ is a uniquely mass minimizing integer $n$-current, then for every $\varepsilon>0$ there exists $\delta>0$ such that if $\delta(S;T)\le\delta$ then $d(S,T)\le\varepsilon$.*

*Proof.* By contradiction, there exist $\varepsilon_0>0$ and $\{u_h\}_{h\in\mathbb{N}}\subset BV(\mathbb{R}^{n+1};\mathbb{Z})$ such that, if we set $S_h=T+\partial(\mathbf{E}^{n+1}\llcorner u_h)$, then $\mathbf{M}(S_h)\to\mathbf{M}(T)$ as $h\to\infty$, with $d(S_h,T)\ge\varepsilon_0$ for every $h\in\mathbb{N}$. Exploiting the decomposition $S_h=S_h^+ + S_h^-$ of Lemma 3.6, since $T$ is uniquely minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$, we find that both $\mathbf{M}(S_h^+)$ and $\mathbf{M}(S_h^-)$ converge to $\mathbf{M}(T)$ as $h\to\infty$, with

$$d(S_h,T)=\int_{\mathbb{R}^{n+1}}|u_h|=\int_{\mathbb{R}^{n+1}}u_h^+ +\int_{\mathbb{R}^{n+1}}u_h^- = d(S_h^+,T)+d(S_h^-,T)\,.$$

In other words, we may have assumed from the beginning that $u\in BV(\mathbb{R}^{n+1};\mathbb{N})$. This said, repeating the compactness argument in the proof of Lemma 3.4 we may construct a $\{\tilde{S}_h\}_{h\in\mathbb{N}}\subset\mathcal{I}_n(\mathbb{R}^{n+1})$ and $S\in\mathcal{I}_n(\mathbb{R}^{n+1})$, such that $\tilde{S}_h\rightharpoonup S$, $\partial S=\partial T$, $\mathbf{M}(\tilde{S}_h)\to\mathbf{M}(T)$ as $h\to\infty$, and $d(S,T)\ge\varepsilon_0$, against the fact that $T$ is uniquely mass minimizing. q.e.d.

*Proof of Theorem 7.* By Lemma 3.7 and Lemma 3.8, we may assume $d(S,T)\le\varepsilon_0$. Decomposing $S$ as $S=S^+ + S^-$, we have $d(S^+,T)\le\varepsilon_0$ and $d(S^-,T)\le\varepsilon_0$. Let us now consider the minimizers $S_{\varepsilon^+}$ and $S_{\varepsilon^-}$ in (3.1) and (3.2), corresponding to the choices $\varepsilon^+=d(S^+,T)$ and $\varepsilon^-=d(S^-,T)$. By construction,

$$\mathbf{M}(S^+)-\mathbf{M}(T)\ge\mathbf{M}(S_{\varepsilon^+})-\mathbf{M}(T)\ge\kappa_0\,d(S_{\varepsilon^+},T)^2\ge\kappa_0\,d(S^+,T)^2\,,$$

and, in the same way, $\mathbf{M}(S^-)-\mathbf{M}(T)\ge\kappa_0\,d(S^-,T)^2$. By adding up these inequalities, by Lemma 3.6, and since $d(S,T)=d(S^+,T)+d(S^-,T)$, we conclude the proof. q.e.d.

### 3.6. Properties of the $\varepsilon$-approximating currents.

**Lemma 3.9.** *If $M\in\mathcal{M}$ and $T=[\![M]\!]$ is mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$, then there exist positive constants $\Lambda$ and $\varepsilon_0$ (depending on $M$ only) such that every minimizer $S_\varepsilon$ in (3.32) with $\varepsilon\in(0,\varepsilon_0)$ is $\Lambda$-mass minimizing, in the sense that*

$$\mathbf{M}(S_\varepsilon)\le\mathbf{M}(S_\varepsilon+\partial Y)+\Lambda\,\mathbf{M}(Y)\,,\qquad\forall Y\in\mathcal{I}_{n+1}(\mathbb{R}^{n+1})\,.\tag{3.48}$$

*Proof.* By construction, $S_\varepsilon=T+\partial\Big(\mathbf{E}^{n+1}\llcorner u_\varepsilon\Big)$, with $u_\varepsilon\in BV(\mathbb{R}^{n+1};\mathbb{N})$, $\int_{\mathbb{R}^{n+1}}u_\varepsilon\ge\varepsilon$. Moreover, by Lemma 3.4, we may also assume that

$$\mathbf{M}(S_\varepsilon)\le 2\,\mathbf{M}(T)\,,\tag{3.49}$$

for every $\varepsilon\in(0,\varepsilon_0)$. We now divide the argument in two steps.

*Step one:* Let $Y\in\mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ so that $Y=\mathbf{E}^{n+1}\llcorner v$ for some $v\in BV(\mathbb{R}^{n+1};\mathbb{Z})$. We claim the existence of $Z\in\mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ such that $Z=\mathbf{E}^{n+1}\llcorner w$ for some $w\in BV(\mathbb{R}^{n+1};\mathbb{Z})$ with

$$w\ge -u_\varepsilon\,,\qquad \mathbf{M}(S_\varepsilon+\partial Z)\le\mathbf{M}(S_\varepsilon+\partial Y)\,,\qquad \mathbf{M}(Z)\le\mathbf{M}(Y)\,.$$

Indeed it suffices to set $w=(u_\varepsilon+v)^+-u_\varepsilon$. Clearly, $w\ge -u_\varepsilon$. By Lemma 3.6, and since $T$ is mass minimizing in $\mathcal{I}_n(\mathbb{R}^{n+1})$, we have

$$\begin{aligned}\mathbf{M}(S_\varepsilon+\partial Y) &= \mathbf{M}\left(T+\partial(\mathbf{E}^{n+1}\llcorner(u_\varepsilon+v))\right)\\ &\ge \mathbf{M}\Big(T+\partial(\mathbf{E}^{n+1}\llcorner(u_\varepsilon+v)^+)\Big)+\mathbf{M}\Big(T-\partial(\mathbf{E}^{n+1}\llcorner(u_\varepsilon+v)^-)\Big)-\mathbf{M}(T)\\ &\ge \mathbf{M}\Big(T+\partial(\mathbf{E}^{n+1}\llcorner(u_\varepsilon+v)^+)\Big)=\mathbf{M}(S_\varepsilon+\partial Z)\,.\end{aligned}$$

At the same time, since $u_\varepsilon\ge 0$,

$$\mathbf{M}(Y)-\mathbf{M}(Z)=\int_{\mathbb{R}^{n+1}}|v|-|w|=\int_{\mathbb{R}^{n+1}}|v|-|(u_\varepsilon+v)^+-u_\varepsilon|=\int_{\mathbb{R}^{n+1}}(u_\varepsilon+v)^-\ge 0\,.$$

*Step two:* We are left to prove (3.48) for $Y\in\mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ such that $Y=\mathbf{E}^{n+1}\llcorner v$ for some $v\in BV(\mathbb{R}^{n+1};\mathbb{Z})$ with $u_\varepsilon+v\ge 0$. If we set $X=\mathbf{E}^{n+1}\llcorner(u_\varepsilon+v)$, then

$$S_\varepsilon+\partial Y=T+\partial X\,.$$

and, in particular,

$$\mathbf{M}(X)=\int_{\mathbb{R}^{n+1}}u_\varepsilon+v\,.$$

If $\mathbf{M}(X)\ge\varepsilon$, then, by minimality of $S_\varepsilon$ in (3.32), we trivially have $\mathbf{M}(S_\varepsilon)\le\mathbf{M}(S_\varepsilon+\partial Y)$, and (3.48) is proved. We may thus assume that

$$\mathbf{M}(X)\le\varepsilon\,. \tag{3.50}$$

We may also assume that

$$\mathbf{M}(T+\partial X)\le 2\,\mathbf{M}(T)\,. \tag{3.51}$$

Indeed, if this were not the case, then, this time by (3.49), we would have, as required,

$$\mathbf{M}(S_\varepsilon)\le 2\,\mathbf{M}(T)\le\mathbf{M}(T+\partial X)=\mathbf{M}(S_\varepsilon+\partial Y)\,,$$

If now $t_0$, $c_0$, $\delta_0$ and $C_0$ are the constants appearing in Lemma 3.3, then by (3.50) and (3.51), and provided $\varepsilon_0\le\delta_0$, for every $t\in(0,t_0)$ there exist $X_t\in\mathcal{I}^+_{n+1}(\mathbb{R}^{n+1})$ such that

$$\mathbf{M}(X_t)\ge\mathbf{M}(X)+c_0\,t\,,\qquad \mathbf{M}(T+\partial X_t)\le\mathbf{M}(T+\partial X)+C_0\,t\,.$$

If we further assume that $\varepsilon_0\le c_0\,t_0$, then the following value of $t$ is admissible in this construction,

$$t=\frac{\varepsilon-\mathbf{M}(X)}{c_0}\,,$$

and, correspondingly, we find that $\mathbf{M}(X_t)\ge\varepsilon$ with $X_t\in\mathcal{I}^+_{n+1}(\mathbb{R}^{n+1})$. Exploiting the minimality property of $S_\varepsilon$, we conclude that

$$\begin{aligned}\mathbf{M}(S_\varepsilon) &\le \mathbf{M}(T+\partial X_t)\le\mathbf{M}(T+\partial X)+C_0\,t=\mathbf{M}(T+\partial X)+\frac{C_0}{c_0}\Big(\varepsilon-\mathbf{M}(X)\Big)\\ &\le \mathbf{M}(T+\partial X)+\frac{C_0}{c_0}\left(\int_{\mathbb{R}^{n+1}}u_\varepsilon-\int_{\mathbb{R}^{n+1}}(u_\varepsilon+v)\right)\\ &\le \mathbf{M}(T+\partial X)+\frac{C_0}{c_0}\int_{\mathbb{R}^{n+1}}|v|=\mathbf{M}(S_\varepsilon+\partial Y)+\Lambda\,\mathbf{M}(Y)\,,\end{aligned}$$

provided we set $\Lambda=C_0/c_0$. q.e.d.

**3.7. $C^1$-convergence of the $\varepsilon$-approximating currents.** The following theorem provides a standard application of the regularity theory for $\Lambda$-mass minimizing currents. In the proof, which is briefly sketched for the reader's convenience, we shall use Notation 3.1.

**Theorem 8.** *If $M \in \mathcal{M}$, $T = [\![M]\!]$, $\{S_h\}_{h\in\mathbb{N}} \subset \mathcal{I}_n(\mathbb{R}^{n+1})$ are $\Lambda$-mass minimizing, i.e.*

$$\mathbf{M}(S_h) \le \mathbf{M}(S_h + \partial Y) + \Lambda\, \mathbf{M}(Y)\,, \qquad \forall Y \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})\,,$$

*with $\partial S_h = \partial T$ and $d(S_h, T) \to 0$ as $h \to \infty$, then there exists $\{\varphi_h\}_{h\in\mathbb{N}} \subset C^1_0(M)$ with*

$$S_h = (\mathrm{Id} + \varphi_h\, \nu_M)_\#\, T\,, \tag{3.52}$$

*where $\nu_M$ is a smooth unit normal vector field to $M$, and*

$$\lim_{h\to\infty} \|\varphi_h\|_{C^1(\overline{M})} = 0\,. \tag{3.53}$$

*Proof.* If $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$ and $x \in \mathrm{spt} S$, then the excess of $S$ in $\mathbf{C}(x,r)$ is defined as

$$\mathbf{e}(S,x,r) = \frac{\mathbf{M}(S \llcorner \mathbf{C}(x,r)) - \mathbf{M}(\mathbf{p}_\#(S \llcorner \mathbf{C}(x,r)))}{r^n}\,,$$

If $u : \mathbb{R}^n \to \mathbb{R}$, then we set $\mathrm{Id} \times u : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $\mathrm{Id} \times u(x) = (x, u(x))$ for $x \in \mathbb{R}^n$.

*Step one:* First of all notice that from the density estimates for $\Lambda$-mass minimizing currents (see [**DSt1**, Lemma 2.2, Lemma 2.3]) and classical arguments (see for instance [**Fe1**, Theorem 5.4.2] and [**Ma2**, Chapter 3]) the following two properties hold true:

- (i) $\|S_h\| \overset{*}{\rightharpoonup} \|T\|$ as Radon measure in $\mathbb{R}^n$, and $\|S_h\|(\mathbb{R}^n) \to \|T\|(\mathbb{R}^n)$.
- (ii) $\mathrm{spt}\, S_h$ converges to $\mathrm{spt} T$ in the Kuratowski sense, that is
  - (a) for every $x \in \mathrm{spt}\, S$ there exists $\{x_h\}_{h\in\mathbb{N}} \subset \mathrm{spt}\, T_h$ converging to $x$;
  - (b) if $x_h \to x$ and $x_h \in \mathrm{spt}\, S_h$, then $x \in \mathrm{spt}\, T$.

*Step two:* Let $\Gamma = \mathrm{bdry}\, M$. Given $x \in M \setminus \Gamma$ and $\varepsilon > 0$, there exists $r_0 > 0$ and a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{Lip}(u) \le 1/2$, such that, up to a rotation of $T$,

$$\Gamma \cap \mathbf{C}(x, 2\, r_0) \;=\; \emptyset\,, \tag{3.54}$$

$$T \llcorner \mathbf{C}(x, 2\, r_0) \;=\; (\mathrm{Id} \times u)_\#\Big(\mathbf{E}^n \llcorner \mathbf{D}(\mathbf{p}x, 2\, r_0)\Big)\,, \tag{3.55}$$

and, for every $r \in (0, r_0)$,

$$\mathbf{p}_\#(T \llcorner \mathbf{C}(x, 2\, r)) \;=\; \mathbf{E}^n \llcorner \mathbf{D}(\mathbf{p}x, 2\, r)\,, \tag{3.56}$$

$$\mathbf{e}(T, x, 2\, r) \;\le\; \varepsilon\,. \tag{3.57}$$

We now claim that if $\{x_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^{n+1}$, $x_h \to x$ and $x_h \in \mathrm{spt}\, S_h$, then there exists $s \in (r_0, 2\, r_0)$ such that, for $h$ large enough

$$\partial S_h \llcorner \mathbf{C}(x_h, s) \;=\; 0\,, \tag{3.58}$$

$$\mathbf{p}_\#(S_h \llcorner \mathbf{C}(x_h, s)) \;=\; \mathbf{E}^n \llcorner \mathbf{D}(\mathbf{p}x_h, s)\,, \tag{3.59}$$

$$\mathbf{e}(S_h, x_h, s) \;\le\; 2\varepsilon\,. \tag{3.60}$$

Defining $f : \mathbb{R}^{n+1} \to [0, \infty)$ as

$$f(y) = \max\{|\mathbf{p}y|, |\mathbf{q}y|\}\,, \qquad y \in \mathbb{R}^{n+1}\,, \tag{3.61}$$

so that $\mathbf{C}(x_h, s) = \{y : f(y - x_h) < s\}$, we select $s \in (r_0, 2r_0)$ such that

$$\mathbf{M}(S_h \llcorner \{y : |f(y - x_h)| = s\}) = 0\,, \qquad \forall h \in \mathbb{N}\,. \tag{3.62}$$

Since $\partial S_h = \partial T$, (3.58) immediately follows from (3.54). By (3.58) and, thanks to (3.62), by slicing of currents (see [**Fe1**, Section 4.2.1], [**Si**, Section 28]),

$$\mathrm{spt}\Big(\partial\Big(S_h \llcorner \mathbf{C}(x_h, s)\Big)\Big) = \mathrm{spt}\, \langle S_h, f(\cdot - x_h), s\rangle \subset \Big(\mathrm{spt} S_h\Big) \cap \{y : f(y - x_h) = s\}\,.$$

Moreover, by (3.55), $x \in \operatorname{spt} T$, $x_h \to x$, and $\operatorname{Lip}(u) \le 1/2$,

$$\Big(\operatorname{spt} T\Big) \cap \{y : f(y-x_h) = s\} \subset \left\{ y : \mathbf{p}(y-x_h) = s\,, |\mathbf{q}(y-x_h)| < \frac{3}{4}\,s \right\}.$$

Combining the two previous inclusions with the Kuratowski convergence of $\operatorname{spt} S_h$ to $\operatorname{spt} T$,

$$\operatorname{spt}\Big(\partial\Big(S_h \llcorner \mathbf{C}(x_h,s)\Big)\Big) \subset \left\{ y : \mathbf{p}(y-x_h) = s\,, |\mathbf{q}(y-x_h)| < \frac{4}{5}\,s \right\}.$$

By the constancy theorem [**Fe1**, 4.1.7], there exists $m_h \in \mathbb{Z}$ such that

$$\mathbf{p}_\#\Big(S_h \llcorner \mathbf{C}(x_h,s)\Big) = m_h\,\mathbf{E}^n \llcorner \mathbf{D}(\mathbf{p}x_h,s)\,.$$

Again by Kuratowski convergence of $\operatorname{spt} S_h$ to $\operatorname{spt} T$, we easily see that

$$\mathbf{p}_\#\Big(S_h \llcorner \mathbf{C}(x_h,s)\Big) \rightharpoonup \mathbf{p}_\#\Big(T \llcorner \mathbf{C}(x,s)\Big)\,.$$

In particular, by (3.56), it must be $m_h \to 1$ as $h \to \infty$, so that $m_h = 1$ for every $h$ large enough. This proves (3.59). Finally from $\|S_h\| \overset{*}{\rightharpoonup} \|T\|$, (3.56) and (3.59) we deduce (3.60).

*Step three:* Given $x \in \Gamma$ and $\varepsilon > 0$, there exists $r_0 > 0$, a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{Lip}(u) \le 1/2$, an open set $E \subset \mathbb{R}^n$ with smooth boundary, such that, up to a rotation, $T$ satisfies the following properties:

$$\Gamma \cap \mathbf{C}(x,2r_0) = \{(y,u(y)) : y \in \mathbf{D}(\mathbf{p}x,2r_0) \cap \partial E\}\,, \tag{3.63}$$

$$|\nu_E(y) - \nu_E(z)| \le \varepsilon|y-z|\,, \qquad \forall y,z \in \mathbf{D}(\mathbf{p}x,2r_0) \cap \partial E\,, \tag{3.64}$$

$$T \llcorner \mathbf{C}(x,2\,r_0) = (\mathrm{Id} \times u)_\#\Big(\mathbf{E}^n \llcorner (\mathbf{D}(\mathbf{p}x,2\,r_0) \cap E)\Big)\,, \tag{3.65}$$

and, for every $r \in (0,r_0)$,

$$\mathbf{p}_\#(T \llcorner \mathbf{C}(x,2\,r)) = \mathbf{E}^n \llcorner (\mathbf{D}(\mathbf{p}x,2\,r) \cap E)\,, \tag{3.66}$$

$$\mathbf{e}(T,x,2\,r) \le \varepsilon\,. \tag{3.67}$$

We now claim the existence of $s \in (r_0, 2\,r_0)$ such that, for $h$ large enough

$$\partial S_h \llcorner \mathbf{C}(x,s) = [\![\Gamma]\!] \llcorner \mathbf{C}(x,s)\,, \tag{3.68}$$

$$\mathbf{p}_\#(S_h \llcorner \mathbf{C}(x,s)) = \mathbf{E}^n \llcorner (\mathbf{D}(\mathbf{p}x,s) \cap E)\,, \tag{3.69}$$

$$\mathbf{e}(S_h,x,s) \le 2\varepsilon\,. \tag{3.70}$$

We select $s \in (r_0, 2r_0)$ such that, for every $h \in \mathbb{N}$,

$$\mathbf{M}(S_h \llcorner \{y : |f(y-x_h)| = s\}) = \mathbf{M}\Big((\partial S_h) \llcorner \{y : |f(y-x_h)| = s\}\Big) = 0\,, \tag{3.71}$$

where $f$ is defined in (3.61) and $x_h \to x$. Since $\partial S_h = [\![\Gamma]\!]$, (3.68) is trivial. Repeating the argument of step two, we now see that, for $h$ large enough,

$$\operatorname{spt}\Big(\partial\Big((S_h - T) \llcorner \mathbf{C}(x,s)\Big)\Big) \subset \left\{ y : \mathbf{p}(y-x) = s\,, |\mathbf{q}(y-x)| < \frac{4}{5}\,s \right\},$$

so that, by the constancy theorem, there exists $m_h \in \mathbb{Z}$ such that

$$\mathbf{p}_\#\Big((S_h - T) \llcorner \mathbf{C}(x,s)\Big) = m_h\,\mathbf{E}^n \llcorner \mathbf{D}(\mathbf{p}x,s)\,.$$

However, by (3.66) and since

$$\mathbf{p}_\#\Big(S_h \llcorner \mathbf{C}(x_h,s)\Big) \rightharpoonup \mathbf{p}_\#\Big(T \llcorner \mathbf{C}(x,s)\Big)\,,$$

we easily infer that $m_h = 0$ for $h$ large enough. In particular, (3.69) follows. The proof of (3.70) is again consequence of (3.67), (3.69) and the fact that $\|S_h\| \overset{*}{\rightharpoonup} \|T\|$.

*Step four:* By compactness, we can cover $\operatorname{spt} T$ with finitely many cylinders $\mathbf{C}(x_i, s_0)$ such that (3.58), (3.59), (3.60) (if $x_i \in M \setminus \Gamma$) or (3.68), (3.69), (3.70) (if $x_i \in \Gamma$) hold true. By the interior and boundary regularity theory for $\Lambda$-mass minimizing integral currents (see, e.g., [**DSt2**, Theorem 6.1, Theorem 6.4]), if $\varepsilon < \varepsilon_0(\Lambda, s_0)$, $h \ge h_0 \in \mathbb{N}$, and $\gamma \in (0, 1/2)$, then there exist $N \in \mathbb{N}$ (depending on $T$) and $\{u^i_h : 1 \le i \le N\}_{h \ge h_0} \subset C^{1,\gamma}(\mathbb{R}^n)$ with $\max_{1\le i\le N, h\ge h_0} \|u^i_h\|_{C^{1,\gamma}(\mathbb{R}^n)} \le C_0$, such that, up to a rotation which depends on $i$, and for $h \ge h_0$,

$$S_h \llcorner C(x_i, s_0) = (\mathrm{Id} \times u)_\# \Big(\mathbf{E}^n \llcorner \mathbf{D}(\mathbf{p}x_i, s_0)\Big)\,,$$

if $x_i \in M \setminus \Gamma$, and

$$\begin{aligned} S_h \llcorner \mathbf{C}(x_i, s_0) &= (\mathrm{Id} \times u)_\# \Big(\mathbf{E}^n \llcorner(\mathbf{D}(\mathbf{p}x_i, s_0) \cap E)\Big)\,, \\ [\![\Gamma]\!] &= (\mathrm{Id} \times u)_\# \Big([\![\partial E]\!] \llcorner(\mathbf{D}(\mathbf{p}x_i, s_0))\Big)\,, \end{aligned}$$

if $x_i \in \Gamma$. We thus conclude that (3.52) holds true. Moreover, by the Kuratowski convergence of $\mathrm{spt} S_h$ to $\mathrm{spt} S$, and by the uniform $C^{1,\gamma}$-bound on the $u^i_h$, we obtain (3.53). q.e.d.

**3.8. Proof of Theorem 4, (a) implies (b).** By Theorem 7 it is enough to show that there exists $\varepsilon_0 > 0$ such that, if $\varepsilon < \varepsilon_0$ and $S_\varepsilon$ is a minimizer in (3.32), then

$$\mathbf{M}(S_\varepsilon) - \mathbf{M}(T) \ge \kappa_0\, d(S_\varepsilon, T)^2\,. \tag{3.72}$$

By Lemma 3.9, there exists $\Lambda$ (independent from $\varepsilon$), such that each $S_\varepsilon$ is $\Lambda$-mass minimizing. Since $T$ is uniquely mass minimizing, by arguing as in Lemma 3.8 we see that $d(S_\varepsilon, T) \to 0$ as $\varepsilon \to 0$. By Theorem 8, $S_\varepsilon = (\mathrm{Id} + \varphi_h\, \nu_\varepsilon)_\# T$, for $\varphi_\varepsilon \in C^1_0(M)$, with $\|\varphi_\varepsilon\|_{C^1(\overline{M})} \to 0$ as $\varepsilon \to 0$. By Theorem 6, we finally prove (3.72).

## 4. Stability inequalities for Lawson's cones

In this section we present the proof of Theorem 5, introduced in section 1.3, and of Theorem 3. We start with a proposition which allows to make rigorous the argument based on the divergence theorem from section 1.3 (recall that Sobolev functions on $\mathbb{R}^m$ are unambiguously defined $\mathcal{H}^{m-1}$-a.e. on $(m-1)$-rectifiable sets, and that the the generalized divergence theorem (2.14) holds true).

**Proposition 4.1.** *If $m \ge 2$, $E$ is of locally finite perimeter in $\mathbb{R}^m$, and $g \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m; \mathbb{R}^m)$,*

$$|g| \le 1\,, \qquad \text{on } \mathbb{R}^m\,, \tag{4.1}$$

$$\mathrm{div}\, g \ge 0\,, \qquad \text{a.e. in } E^c\,, \tag{4.2}$$

$$\mathrm{div}\, g \le 0\,, \qquad \text{a.e. in } E\,, \tag{4.3}$$

$$g = \nu_E\,, \qquad \mathcal{H}^{m-1}\text{-a.e. on } \partial_{1/2} E\,, \tag{4.4}$$

*then $E$ is a local minimizer of the perimeter in $\mathbb{R}^m$, with*

$$P(F; A) - P(E; A) = \int_{E\Delta F} |\mathrm{div} g| + \int_{A\cap\partial_{1/2} F} 1 - (g \cdot \nu_F)\, d\mathcal{H}^{m-1}\,. \tag{4.5}$$

*whenever $A$ is a bounded open set with $E\Delta F \subset\subset A \subset \mathbb{R}^m$.*

The second tool used in the proof of Theorem 5 are the "quantitative calibrations" for the Lawson's cones constructed in the following lemma.

**Lemma 4.1.** *For $h \ge k \ge 2$, $m = k + h$, set*

$$M_{k\,h} = \Big\{(x, y) \in \mathbb{R}^m : \frac{|x|}{\sqrt{k-1}} = \frac{|y|}{\sqrt{h-1}}\Big\}\,,$$

$$K_{k\,h} = \Big\{(x, y) \in \mathbb{R}^m : \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}}\Big\}\,.$$

*For $p \geq 2$, set*

$$f(z) = f(x,y) = \frac{1}{p}\left(\frac{|x|^p}{(k-1)^{p/2}} - \frac{|y|^p}{(h-1)^{p/2}}\right), \qquad z=(x,y)\in\mathbb{R}^m, \tag{4.6}$$

*and define $g\colon \mathbb{R}^m \setminus M_{k\,h} \to \mathbb{R}^m$ by $g = \nabla f/|\nabla f|$. Then $K_{k\,h} = \{f<0\}$, $M_{k\,h} = \{f=0\}$, $g \in W^{1,1}(\mathbb{R}^m;\mathbb{R}^m)$ and*

$$\begin{aligned} g &= \nu_{K_{k\,h}}, &\qquad &\text{on} \quad \partial_{1/2}K_{k\,h},\\ |g| &\leq 1, & &\text{on} \quad \mathbb{R}^m. \end{aligned}$$

*Moreover, if either*

$$p=4, \qquad h+k\geq 9, \qquad \text{and} \quad h\geq 12 \quad \text{if} \quad k=2, \tag{4.7}$$

$$\text{or} \qquad p=\frac{7}{2}, \qquad k=h=4, \tag{4.8}$$

*then*

$$\begin{aligned} \operatorname{div} g(z) &\geq \frac{c}{|z|^2}\left|\frac{|x|}{\sqrt{k-1}} - \frac{|y|}{\sqrt{h-1}}\right|, &\qquad &\text{if} \quad \frac{|x|}{\sqrt{k-1}} > \frac{|y|}{\sqrt{h-1}},\\ \operatorname{div} g(z) &\leq -\frac{c}{|z|^2}\left|\frac{|x|}{\sqrt{k-1}} - \frac{|y|}{\sqrt{h-1}}\right|, & &\text{if} \quad \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}}. \end{aligned}$$

*where*

$$c = \frac{1}{512}\left(\frac{k-1}{h-1}\right)^2 (k-1)^{3/4}, \qquad \text{if} \quad (k,h)\neq(4,4), \tag{4.9}$$

$$c = \frac{\sqrt{3}}{16}, \qquad \text{if} \quad (k,h)=(4,4). \tag{4.10}$$

We now prove, in order, Theorem 5 and Theorem 3, Proposition 4.1 and Lemma 4.1.

*Proof of Theorem* 5 *and Theorem* 3. Let $R>0$. If $F\subset\mathbb{R}^m$ with $K_{k\,h}\Delta F \subset\subset H_R$, we set

$$\delta = \frac{P(F;H_R) - P(K_{k\,h};H_R)}{R^{m-1}}, \qquad \alpha = \frac{|K_{k\,h}\Delta F|}{R^m}.$$

We have $\alpha \leq R^{-m}\mathcal{L}^m(H_R) = \mathcal{L}^m(H_1) = \omega_k\,\omega_h$. If $\delta \geq \omega_k\,\omega_h$, then we trivially find

$$\alpha \leq \omega_k\,\omega_h \leq \sqrt{\omega_k\,\omega_h}\,\sqrt{\delta}. \tag{4.11}$$

We thus assume $\delta < \omega_k\,\omega_h$. By applying Proposition 4.1 to the vector field $g$ associated to $k$ and $h$ through Lemma 4.1, we find that

$$P(F;H_R) - P(K_{k\,h};H_R) \quad \geq \quad \int_{K_{k\,h}\Delta F} |\operatorname{div} g|, \tag{4.12}$$

$$|\operatorname{div} g(z)| \quad \geq \quad c\,\frac{p(z)}{|z|^2}, \qquad \forall z\in\mathbb{R}^m\setminus\{0\}, \tag{4.13}$$

where we have set, for the sake of brevity,

$$p(z) = p(x,y) = \left|\frac{|x|}{\sqrt{k-1}} - \frac{|y|}{\sqrt{h-1}}\right|, \qquad z=(x,y)\in\mathbb{R}^k\times\mathbb{R}^h,$$

and where $c$ is defined in (4.9) if $(k,h)\neq(4,4)$, $c=\sqrt{3}/16$ if $(k,h)=(4,4)$. We now divide the argument in two steps.

*Step one:* We prove Theorem 3. By (4.12) and (4.13) we find that, if $E\Delta F\subset\subset H_R$, then

$$P(F;H_R) - P(K_{k\,h};H_R) \geq \frac{c}{R^2}\int_{K_{k\,h}\Delta F} p(z)\,dz. \tag{4.14}$$

If now $\varphi \in C^1(M_{k\,h})$, $\mathrm{spt}\varphi \cap \{0\} = \emptyset$, and $\mathrm{spt}\varphi \subset\subset B^m_R$, then there exists $t_0 > 0$ such that for every $t < t_0$ we may define an open set $F \subset \mathbb{R}^m$ with $\partial F \setminus \{0\}$ a smooth hypersurface and $E\Delta F \subset\subset H_R$, such that

$$\partial F \setminus \{0\} = \Big\{ z + t\,\varphi(z)\,\nu_{K_{k\,h}}(z) : z \in M_{k\,h} \setminus \{0\} \Big\}\,.$$

By Taylor expansion, and since

$$\begin{aligned} p(z) &= \ell\,\mathrm{dist}(z, M_{k\,h})\,, \qquad \forall z \in \mathbb{R}^m\,, \\ \ell &= \sqrt{\frac{1}{k-1} + \frac{1}{h-1}}\,, \end{aligned}$$

we see that

$$\begin{aligned} P(F; H_R) - P(K_{k\,h}; H_R) &= \frac{t^2}{2}\int_{M_{k\,h}} |\nabla^{M_{k\,h}}\varphi|^2 - |\mathrm{II}_{M_{k\,h}}|^2\varphi^2\,d\mathcal{H}^{m-1} + O(t^3)\,, \\ \int_{K_{k\,h}\Delta F} p(z)\,dz &= \ell\,(1+O(t))\int_{M_{k\,h}} d\mathcal{H}^{m-1}(z)\int_0^{t|\varphi(z)|} s\,ds \\ &= \frac{\ell\,t^2}{2}\int_{M_{k\,h}} \varphi^2\,d\mathcal{H}^{m-1} + O(t^3)\,. \end{aligned}$$

By (4.14) we thus conclude that

$$\int_{M_{k\,h}} |\nabla^{M_{k\,h}}\varphi|^2 - |\mathrm{II}_{M_{k\,h}}|^2\varphi^2\,d\mathcal{H}^{m-1} \ge \frac{c\,\ell}{R^2}\int_{M_{k\,h}} \varphi^2\,d\mathcal{H}^{m-1}\,, \tag{4.15}$$

for every $\varphi \in C^1(M_{k\,h})$ such that $\mathrm{spt}\,\varphi \cap \{0\} = \emptyset$ and $\mathrm{spt}\varphi \subset\subset B^m_R$. To extend (4.15) to every $\varphi \in C^1(M_{k\,h})$ with $\mathrm{spt}\varphi \subset\subset B^m_R$, we consider a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset C^\infty(\mathbb{R}^m)$ with $\mathrm{Lip}\,\psi_j \le 2\,j$, $\psi_j = 0$ on $B^m_{1/j}$, and $\psi_j = 1$ on $\mathbb{R}^m \setminus B^m_{2/j}$. By standard density estimates, $\mathcal{H}^m(M_{k\,h} \cap B^m_r) \le c(m) r^m$, while $|\mathrm{II}_{M_{k\,h}}(x)| \le C_{k\,h}/|x|$ for every $x \ne 0$ since $M_{k\,h}$ is a cone: hence, we may pass to the limit as $j \to \infty$ in (4.15) applied to $\psi_j\,\varphi$, to deduce (4.15) on $\varphi$, as required.

*Step two:* We prove Theorem 5, i.e. we prove (2.10). By (4.12) and (4.13),

$$\begin{aligned} \Big|K_{k\,h}\Delta F\Big| &\le \Big|\Big(K_{k\,h}\Delta F\Big) \cap \{p > \varepsilon\}\Big| + \Big|\{p < \varepsilon\} \cap H_R\Big| \\ &\le \frac{2\,R^2}{\varepsilon}\int_{(K_{k\,h}\Delta F)\cap\{p>\varepsilon\}} \frac{p(z)}{|z|^2}\,dz + \Big|\{p < \varepsilon\} \cap H_R\Big| \\ &\le \frac{2\,R^2}{c\,\varepsilon}\Big(P(F; H_R) - P(K_{k\,h}; H_R)\Big) + \Big|\{p < \varepsilon\} \cap H_R\Big|\,. \end{aligned} \tag{4.16}$$

We now claim that

$$\Big|\{p < \varepsilon\} \cap H_R\Big| \le \gamma\,R^{m-1}\,\varepsilon\,, \qquad \text{whenever} \quad \varepsilon < \frac{R}{2\sqrt{h-1}}\,, \tag{4.17}$$

where we have set

$$\gamma = 2\,\omega_k\,\omega_h\,\frac{k\,h}{m-1}\left(\frac{k-1}{h-1}\right)^{k/2}\sqrt{h-1}\,. \tag{4.18}$$

Indeed, we hve

$$\begin{aligned} \Big|\{p < \varepsilon\} \cap H_R\Big| &= \int_{B^h_R} \mathcal{H}^k\left(\left\{x \in B^k_R : \frac{|y|}{\sqrt{h-1}} - \varepsilon < \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}} + \varepsilon\right\}\right) dy \\ &\le \omega_k (k-1)^{k/2}\int_{B^h_R} \left(\frac{|y|}{\sqrt{h-1}} + \varepsilon\right)^k - \left(\frac{|y|}{\sqrt{h-1}} - \varepsilon\right)^k_+ dy\,. \end{aligned}$$

On the one hand,

$$\begin{aligned}
&\int_{B^h_{\varepsilon\sqrt{h-1}}}\left(\frac{|y|}{\sqrt{h-1}}+\varepsilon\right)^k-\left(\frac{|y|}{\sqrt{h-1}}-\varepsilon\right)_+^k\,dy\\
=\;&\int_{B^h_{\varepsilon\sqrt{h-1}}}\left(\frac{|y|}{\sqrt{h-1}}+\varepsilon\right)^k\,dy\le\omega_h\,(h-1)^{h/2}2^k\varepsilon^{h+k}
\end{aligned}$$

On the other hand, since $(1+t)^k-(1-t)^k\le kt$ for every $t\in(0,1)$,

$$\begin{aligned}
&\int_{B^h_R\setminus B^h_{\varepsilon\sqrt{h-1}}}\left(\frac{|y|}{\sqrt{h-1}}+\varepsilon\right)^k-\left(\frac{|y|}{\sqrt{h-1}}-\varepsilon\right)^k\,dy\\
=\;&\frac{1}{(h-1)^{k/2}}\int_{B^h_R\setminus B^h_{\varepsilon\sqrt{h-1}}}|y|^k\left\{\left(1+\frac{\varepsilon\sqrt{h-1}}{|y|}\right)^k-\left(1-\frac{\varepsilon\sqrt{h-1}}{|y|}\right)^k\right\}dy\\
\le\;&\frac{k}{(h-1)^{k/2}}\int_{B^h_R\setminus B^h_{\varepsilon\sqrt{h-1}}}|y|^k\,\frac{\varepsilon\sqrt{h-1}}{|y|}\,dy\\
=\;&\frac{h\,k\,\omega_h\,\varepsilon}{(h-1)^{(k-1)/2}}\int_{\varepsilon\sqrt{h-1}}^R r^{h+k-2}\,dr\le\omega_h\,\frac{h\,k}{m-1}\frac{R^{m-1}}{(h-1)^{(k-1)/2}}\,\varepsilon\,,
\end{aligned}$$

where, recall, $m=k+h$. We thus find

$$\Big|\{p<\varepsilon\}\cap H_R\Big|\le\omega_h\,\omega_k(h-1)^{h/2}\,(k-1)^{k/2}\left(2^k\varepsilon^{m-1}+\frac{h\,k}{m-1}\frac{R^{m-1}}{(h-1)^{(m-1)/2}}\right)\varepsilon$$

which implies (4.17) since $hk/(m-1)>1$ and $\varepsilon<R/(2\sqrt{h-1})$, so that

$$2^k\varepsilon^{m-1}\le\frac{2^k}{2^{m-1}}\,\frac{R^{m-1}}{(h-1)^{(m-1)/2}}\le\frac{h\,k}{m-1}\,\frac{R^{m-1}}{(h-1)^{(m-1)/2}}$$

We may thus combine (4.16) and (4.17) to find

$$\alpha\le\frac{2\,R\,\delta}{c\,\varepsilon}+\frac{\gamma\,\varepsilon}{R}\,,\qquad\text{whenever}\qquad\varepsilon<\frac{R}{2\sqrt{h-1}}\,.\tag{4.19}$$

If $\varphi(\varepsilon)$ denotes the right-hand side of (4.19), then $\varphi$ attains its minimum on $\varepsilon>0$ at $\varepsilon_0$,

$$\varepsilon_0=\sqrt{\frac{2\,\delta}{c\,\gamma}}\,R\,.$$

If $\varepsilon_0<R/(2\sqrt{h-1})$, then by (4.19)

$$\alpha\le\varphi(\varepsilon_0)=\frac{2\,\gamma\,\varepsilon_0}{R}=\sqrt{\frac{8\,\gamma}{c}}\,\sqrt{\delta}\,.\tag{4.20}$$

Otherwise, $1/(2\sqrt{h-1})<\sqrt{2\delta/c\,\gamma}$. Hence, by $\delta<\omega_k\,\omega_h$, and setting $\gamma_0=\gamma/\omega_k\,\omega_h$,

$$\begin{aligned}
\alpha\;&\le\;\varphi\left(\frac{R}{2\sqrt{h-1}}\right)=\frac{4\sqrt{h-1}}{c}\,\delta+\frac{\gamma}{R}\,\frac{R}{2\sqrt{h-1}}\\
&\le\;\frac{4\sqrt{h-1}}{c}\,\delta+\gamma\,\sqrt{\frac{2\delta}{c\gamma}}\le\sqrt{\omega_k\,\omega_h}\left(\frac{4\,\sqrt{h-1}}{c}+\sqrt{\frac{2\gamma_0}{c}}\right)\sqrt{\delta}\,,
\end{aligned}\tag{4.21}$$

Combining (4.11), (4.20) and (4.21), we thus find

$$\alpha\le\sqrt{\omega_k\,\omega_h}\,\max\left\{1,\frac{8\,\sqrt{h-1}}{c},\sqrt{\frac{8\gamma_0}{c}}\right\}\sqrt{\delta}\,.$$

If $(k,h)\neq(4,4)$, then by (4.9)

$$\begin{aligned}
\frac{8\sqrt{h-1}}{c} &= 2^{12}\left(\frac{h-1}{k-1}\right)^{3/2}\frac{1}{(k-1)^{1/4}}\,,\\
\sqrt{\frac{8\gamma_0}{c}} &= \sqrt{2^{13}\left(\frac{h-1}{k-1}\right)^{(5-k)/2}\frac{hk}{m-1}\,\frac{1}{(k-1)^{1/4}}}\,.
\end{aligned}$$

Since $hk\geq m-1$ and $(5-k)/4\leq 3/2$, we have

$$\max\left\{\frac{8\sqrt{h-1}}{c},\sqrt{\frac{8\gamma_0}{c}}\right\}\leq\frac{2^{12}}{(k-1)^{1/8}}\sqrt{\frac{hk}{m-1}}\,\left(\frac{h-1}{k-1}\right)^{3/2}\,,$$

where the right-hand side of this inequality is larger than one since $k/2\leq hk/(m-1)$. This proves that $\alpha\leq C\sqrt{\delta}$, with $C$ as in (1.19), when $(k,h)\neq(4,4)$. If $k=h=4$, then $c$ satisfies (4.10), and

$$\begin{aligned}
\frac{8\sqrt{h-1}}{c} &= \frac{8\sqrt{3}\,16}{\sqrt{3}}=128\,,\\
\sqrt{\frac{8\gamma_0}{c}} &= \sqrt{\frac{2^8}{7}\sqrt{3}\,\frac{16}{\sqrt{3}}}=\sqrt{\frac{2^{12}}{7}}<64\,.
\end{aligned}$$

Thus $\alpha\leq C\sqrt{\delta}$, with $C$ as in (1.20). q.e.d.

*Proof of Proposition* 4.1. Let $F$ be a set of locally finite perimeter with $E\Delta F\subset\subset A\subset\mathbb{R}^m$. By [**Ma2**, Theorem 16.3],

$$\mu_{E\setminus F}=\mu_E\llcorner\Big(F^{(0)}\cup\{\nu_E=-\nu_F\}\Big)-\mu_F\llcorner E^{(1)}\,,$$

where $\{\nu_E=-\nu_F\}=\{x\in\partial_{1/2}E\cap\partial_{1/2}F:\nu_E(x)=-\nu_F(x)\}$. Thus, by applying the divergence theorem (2.14) to $g$ on $E\setminus F$, and denoting for the sake of simplicity by $g$ the trace of $g$ along $\partial_{1/2}E$ (oriented by $\nu_E$) and along $\partial_{1/2}F$ (oriented by $\nu_F$), we find that

$$\begin{aligned}
\int_{E\setminus F}\mathrm{div}g &= \int_{\mathbb{R}^n}g\cdot d\mu_{E\setminus F}\\
&= P\Big(E;F^{(0)}\cup\{\nu_E=-\nu_F\}\Big)-\int_{E^{(1)}\cap\partial_{1/2}F}g\cdot\nu_F\,d\mathcal{H}^{m-1}\,,
\end{aligned}$$

where (4.1) was taken into account. Since $E\setminus F\subset E$, by (4.3),

$$\begin{aligned}
P\Big(E;F^{(0)}\cup\{\nu_E=-\nu_F\}\Big)+\int_{E\setminus F}|\mathrm{div}g| &= \int_{E^{(1)}\cap\partial_{1/2}F}g\cdot\nu_F\,d\mathcal{H}^{m-1}\\
&= P(F;E^{(1)})-\int_{E^{(1)}\cap\partial_{1/2}F}\Big(1-(g\cdot\nu_F)\Big)\,d\mathcal{H}^{m-1}\,. \qquad (4.22)
\end{aligned}$$

Similarly, again by [**Ma2**, Section II.5.1],

$$\mu_{F\setminus E}=\mu_F\llcorner\Big(E^{(0)}\cup\{\nu_E=-\nu_F\}\Big)-\mu_E\llcorner F^{(1)}\,,$$

by applying the divergence theorem (2.14) to $g$ on $F\setminus E$ and by (4.1),

$$\begin{aligned}
\int_{F\setminus E}\mathrm{div}g &= \int_{\mathbb{R}^n}g\cdot d\mu_{F\setminus E}\\
&= \int_{(E^{(0)}\cup\{\nu_E=-\nu_F\})\cap\partial_{1/2}F}g\cdot\nu_F\,d\mathcal{H}^{m-1}-P(E;F^{(1)})\,.
\end{aligned}$$

Since $F \setminus E \subset E^0$, by (4.2) we find

$$\begin{aligned} P(E;F^{(1)}) + \int_{F\setminus E} |\mathrm{div} g| &= \int_{(E^{(0)}\cup\{\nu_E=-\nu_F\})\cap\partial_{1/2}F} g\cdot\nu_F \, d\mathcal{H}^{m-1} \\ &= P\Big(F; E^{(0)}\cup\{\nu_E=\nu_F\}\Big) - \int_{(E^{(0)}\cup\{\nu_E=-\nu_F\})\cap\partial_{1/2}F} \Big(1-(g\cdot\nu_F)\Big)\, d\mathcal{H}^{m-1}\,. \end{aligned} \tag{4.23}$$

Since $\partial_{1/2}E\setminus\overline{A} = \partial_{1/2}F\setminus\overline{A} = \{\nu_E=\nu_F\}\setminus\overline{A}$, by (4.22) and (4.23) we find (4.5). q.e.d.

We finally prove Lemma 4.1. We recall the following elementary inequalities,

$$(a^q+b^q)^{1/q} \le (a^2+b^2)^{1/2} \le a+b\,, \tag{4.24}$$

$$(a^2+b^2)^{1/2} \le \sqrt{2}\max\{a,b\}\,, \tag{4.25}$$

$a,b\ge 0$, $q\ge 2$ and we premise the following remark.

**Remark 6.** In proving Lemma 4.1-(2), we shall make use of the following inequality,

$$3(1+s+s^2+s^{10}+s^{11}+s^{12})-\frac{5}{2}(s^3+s^4+s^5+s^6+s^7+s^8+s^9)\ge\frac{1}{4}\,,\qquad \forall s\in[0,1]\,. \tag{4.26}$$

One can prove this inequality by dividing $[0,1]$ into suitable subintervals, and by exploiting the resulting inequalities on $s$ to prove the non-negativity of suitably regrouped differences of positive and negative terms. We omit the details of this rather elementary and lengthy argument, as it is uninteresting.

*Proof of Lemma* 4.1. *Step one:* Since $f\in C^\infty(\mathbb{R}^m)$, with $|\nabla f|>0$ on $\mathbb{R}^m\setminus\{0\}$, it turns out that $g\in C^\infty(\mathbb{R}^m\setminus\{0\};\mathbb{R}^m)$, with

$$\mathrm{div}\ g = \frac{|\nabla f|^2\,\Delta f - \nabla^2 f(\nabla f,\nabla f)}{|\nabla f|^3}\,,\qquad \text{on } \mathbb{R}^m\setminus\{0\}. \tag{4.27}$$

Setting $\nabla f = (\nabla_x f, \nabla_y f)$, we now compute from (4.6) that

$$\nabla_x f = \frac{|x|^{p-2}x}{(k-1)^{p/2}}\,,\qquad \nabla_y f = -\frac{|y|^{p-2}y}{(h-1)^{p/2}}\,.$$

We easily deduce that $g\in W^{1,1}(\mathbb{R}^m;\mathbb{R}^m)$. Moreover,

$$\nabla^2 f = \begin{pmatrix} \nabla^2_{xx}f & 0 \\ 0 & \nabla^2_{yy}f \end{pmatrix}\,,$$

where

$$\begin{aligned} \nabla^2_{xx}f &= \frac{|x|^{p-2}}{(k-1)^{p/2}}\left(\mathrm{Id}_x + (p-2)\frac{x\otimes x}{|x|^2}\right)\,, \\ \nabla^2_{yy}f &= -\frac{|y|^{p-2}}{(h-1)^{p/2}}\left(\mathrm{Id}_y + (p-2)\frac{y\otimes y}{|y|^2}\right)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} |\nabla f| &= \sqrt{\frac{|x|^{2(p-1)}}{(k-1)^p}+\frac{|y|^{2(p-1)}}{(h-1)^p}}\,, \\ \Delta f &= \frac{(k+p-2)}{(k-1)^{p/2}}\,|x|^{p-2} - \frac{(h+p-2)}{(h-1)^{p/2}}\,|y|^{p-2}\,, \\ \nabla^2 f(\nabla f) &= (p-1)\left(\frac{|x|^{2p-4}x}{(k-1)^p},\frac{|y|^{2p-4}y}{(h-1)^p}\right)\,, \\ \nabla^2 f(\nabla f,\nabla f) &= (p-1)\left(\frac{|x|^{3p-4}}{(k-1)^{3p/2}}-\frac{|y|^{3p-4}}{(h-1)^{3p/2}}\right)\,. \end{aligned}$$

By combining these identities with (4.27), we thus find

$$\operatorname{div} g(z) = \frac{N(z)}{D(z)}\,, \tag{4.28}$$

where we have set

$$\begin{aligned} N(z) &= \left(\frac{|x|^{2(p-1)}}{(k-1)^p} + \frac{|y|^{2(p-1)}}{(h-1)^p}\right)\left(\frac{(k+p-2)}{(k-1)^{p/2}}\,|x|^{p-2} - \frac{(h+p-2)}{(h-1)^{p/2}}\,|y|^{p-2}\right) \\ &\quad -(p-1)\left(\frac{|x|^{3p-4}}{(k-1)^{3p/2}} - \frac{|y|^{3p-4}}{(h-1)^{3p/2}}\right)\,, \qquad (4.29) \\ D(z) &= \left(\sqrt{\frac{|x|^{2(p-1)}}{(k-1)^p} + \frac{|y|^{2(p-1)}}{(h-1)^p}}\right)^3\,. \qquad (4.30) \end{aligned}$$

*Step two:* We let $p=4$ and prove assertion (i). For the sake of brevity, we set

$$\alpha = k-1\,, \qquad \beta = h-1\,.$$

We start noticing that,

$$\begin{aligned} N(z) &= \left(\frac{|x|^6}{\alpha^4} + \frac{|y|^6}{\beta^4}\right)\left((k+2)\,\frac{|x|^2}{\alpha^2} - (h+2)\,\frac{|y|^2}{\beta^2}\right) - 3\left(\frac{|x|^8}{\alpha^6} - \frac{|y|^8}{\beta^6}\right) \\ &= \left(\frac{|x|^2}{\alpha} - \frac{|y|^2}{\beta}\right)\frac{\alpha^4|y|^6 + \beta^4|x|^6 - 3\alpha^2\beta|x|^2|y|^4 - 3\alpha\beta^2|x|^4|y|^2}{\alpha^4\beta^4}\,. \qquad (4.31) \end{aligned}$$

We now notice that

$$\begin{aligned} \frac{1}{D(z)}\left(\frac{|x|}{\sqrt{\alpha}} + \frac{|y|}{\sqrt{\beta}}\right) &= \frac{\alpha^6\beta^6}{(\beta^4|x|^6 + \alpha^4|x|^6)^{3/2}}\,\frac{\sqrt{\beta}|x| + \sqrt{\alpha}|y|}{\sqrt{\alpha\beta}} \\ \text{as } \beta \geq \alpha \quad &\geq \frac{\alpha^6\beta^6}{\beta^{3/2}\Big((\sqrt{\beta}|x|)^6 + (\sqrt{\alpha}|y|)^6\Big)^{3/2}}\,\frac{\sqrt{\beta}|x| + \sqrt{\alpha}|y|}{\sqrt{\alpha\beta}} \\ &= \frac{\alpha^{11/2}\beta^4}{\Big((\sqrt{\beta}|x|)^6 + (\sqrt{\alpha}|y|)^6\Big)^{4/3}}\,\frac{\sqrt{\beta}|x| + \sqrt{\alpha}|y|}{\Big((\sqrt{\beta}|x|)^6 + (\sqrt{\alpha}|y|)^6\Big)^{1/6}} \\ \text{by (4.24)} \quad &\geq \frac{\alpha^{11/2}\beta^4}{\Big((\sqrt{\beta}|x|)^6 + (\sqrt{\alpha}|y|)^6\Big)^{4/3}} \\ \text{as } \beta \geq \alpha \text{ and by (4.24)} \quad &\geq \frac{\alpha^{11/2}}{(|x|^6 + |y|^6)^{4/3}} \geq \frac{\alpha^{11/2}}{|z|^8}\,. \end{aligned}$$

Hence, by (4.28) and (4.31), we conclude that

$$\operatorname{div} g(z) = A(z)\,B(z)\left(\frac{|x|}{\sqrt{\alpha}} - \frac{|y|}{\sqrt{\beta}}\right)\,, \tag{4.32}$$

where

$$\begin{aligned} A(z) &\geq \frac{\alpha^{11/2}}{|z|^8}\,\frac{1}{\alpha^4\beta^4} = \frac{\alpha^{3/2}}{\beta^4}\,\frac{1}{|z|^8}\,, \qquad (4.33) \\ B(z) &= \alpha^4|y|^6 + \beta^4|x|^6 - 3\alpha^2\beta|x|^2|y|^4 - 3\alpha\beta^2|x|^4|y|^2\,. \end{aligned}$$

If $|y| \geq |x|$ and $t = (|x|/|y|)^2 \in [0,1]$, then

$$B(z) = \max\{|x|,|y|\}^6\Big(\beta^4\,t^3 - 3\alpha\beta^2t^2 - 3\alpha^2\beta t + \alpha^4\Big)\,. \tag{4.34}$$

If $|x| \geq |y|$ and $t = (|y|/|x|)^2 \in [0,1]$, then

$$B(z) = \max\{|x|,|y|\}^6 \Big(\alpha^4\, t^3 - 3\alpha^2\beta t^2 - 3\alpha\beta^2 t + \beta^4\Big) . \tag{4.35}$$

By Lemma 4.2 and Lemma 4.3 below, provided $k+h \geq 9$ and, if $k=2$, $h \geq 12$, we have

$$\beta^4\, t^3 - 3\alpha\beta^2 t^2 - 3\alpha^2\beta t + \alpha^4 \geq \frac{\alpha^4}{64} , \tag{4.36}$$

$$\alpha^4\, t^3 - 3\alpha^2\beta t^2 - 3\alpha\beta^2 t + \beta^4 \geq \frac{\beta^4}{64} , \tag{4.37}$$

for every $t \in [0,1]$. Combining (4.34), (4.35), (4.36) and (4.37) with (4.25), we thus find

$$B(z) \geq \frac{\max\{|x|,|y|\}^6 \alpha^4}{64} \geq \frac{|z|^6 \alpha^4}{512} . \tag{4.38}$$

We finally combine (4.32), (4.33) and (4.38) to deduce that

$$\operatorname{div} g(z) \geq \frac{\alpha^{11/2}}{512\,\beta^4} \frac{1}{|z|^2} \left| \frac{|x|}{\sqrt{\alpha}} - \frac{|y|}{\sqrt{\beta}} \right| , \qquad \text{on } \mathbb{R}^m \setminus K_{k\,h} , \tag{4.39}$$

$$\operatorname{div} g(z) \leq -\frac{\alpha^{11/2}}{512\,\beta^4} \frac{1}{|z|^2} \left| \frac{|x|}{\sqrt{\alpha}} - \frac{|y|}{\sqrt{\beta}} \right| , \qquad \text{on } K_{k\,h} . \tag{4.40}$$

*Step three:* We let $k = h = 4$ and $p = 7/2$. In this way (4.28), (4.29) and (4.30) give

$$\operatorname{div} g = \frac{\frac{11}{2}\Big(|x|^{3/2} - |y|^{3/2}\Big)\Big(|x|^5 + |y|^5\Big) - \frac{5}{2}\Big(|x|^{13/2} - |y|^{13/2}\Big)}{\Big(|x|^5 + |y|^5\Big)^{3/2}} .$$

Since $(a^{3/2} - b^{3/2})(a^5 + b^5) = (a^{13/2} - b^{13/2}) - a^{3/2} b^{3/2} (a^{7/2} - b^{7/2})$ and $(a^5 + b^5)^{1/5} \leq (a^2 + b^2)^{1/2}$, we find that

$$\Big|\text{div } g\Big| \geq \frac{3\Big(|x|^{13/2} - |y|^{13/2}\Big) - \frac{11}{2}\,|x|^{3/2}|y|^{3/2}(|x|^{7/2} - |y|^{7/2})}{|z|^{15/2}} .$$

We now notice that

$$\begin{aligned} |x|^{13/2} - |y|^{13/2} &= \Big(|x|^{1/2} - |y|^{1/2}\Big) \sum_{k=0}^{12} |x|^{k/2} |y|^{(12-k)/2} , \\ |x|^{7/2} - |y|^{7/2} &= \Big(|x|^{1/2} - |y|^{1/2}\Big) \sum_{k=0}^{6} |x|^{k/2} |y|^{(6-k)/2} , \end{aligned}$$

so that,

$$\Big|\text{div } g\Big| \geq \sqrt{2}\, \frac{\big|\,|x|^{1/2} - |y|^{1/2}\big|}{|z|^{15/2}} \Big( 3 \sum_{k=0}^{12} |x|^{k/2} |y|^{(12-k)/2} - \frac{11}{2}\,|x|^{3/2}|y|^{3/2} \sum_{k=0}^{6} |x|^{k/2}|y|^{(6-k)/2} \Big) .$$

If $|y| < |x|$ and we set $t = |x|/|y| \geq 1$, then

$$\begin{aligned} \Big|\,|x|^{1/2} - |y|^{1/2}\Big| &= |y|^{1/2}(t^{1/2} - 1) \geq |y|^{1/2}(t-1) \\ &= \frac{|x| - |y|}{|y|^{1/2}} = \frac{\Big||x| - |y|\Big|}{|z|^{1/2}} . \end{aligned}$$

By symmetry, we thus find,

$$\Big|\text{div } g\Big| \geq \frac{\Big||x| - |y|\Big|}{|z|^8} \left( 3 \sum_{k=0}^{12} |x|^{k/2} |y|^{(12-k)/2} - \frac{11}{2}\,|x|^{3/2}|y|^{3/2} \sum_{k=0}^{6} |x|^{k/2}|y|^{(6-k)/2} \right) .$$

Let us now notice that if $b \geq a > 0$, then $s = a/b \in (0,1]$ and

$$\begin{aligned} h(a,b) &= 3\sum_{k=0}^{12} a^k b^{12-k} - \frac{11}{2}\, a^3 b^3 \sum_{k=0}^{6} a^k b^{6-k} \\ &= b^{12}\left(3\sum_{k=0}^{12} s^k - \frac{11}{2}\, s^3 \sum_{k=0}^{6} s^k\right) = b^{12}\left(3\sum_{k=0}^{12} s^k - \frac{11}{2} \sum_{k=3}^{9} s^k\right) \\ &= b^{12}\left(3(1+s+s^2+s^{10}+s^{11}+s^{12}) - \frac{5}{2}(s^3+s^4+s^5+s^6+s^7+s^8+s^9)\right) \\ &\geq \frac{b^{12}}{4}\,, \end{aligned}$$

by (4.26). Hence, by (4.25),

$$\begin{aligned} & 3\sum_{k=0}^{12} |x|^{k/2}|y|^{(12-k)/2} - \frac{11}{2}\,|x|^{3/2}|y|^{3/2}\sum_{k=0}^{6}|x|^{k/2}|y|^{(6-k)/2} \\ & \geq \frac{1}{4}\,\max\{|x|,|y|\}^6 \geq \frac{1}{2}\left(\frac{|z|}{\sqrt{2}}\right)^6 = \frac{|z|^6}{16}\,. \end{aligned}$$

In conclusion, $|\operatorname{div} g(z)| \geq |\,|x|-|y|\,|/16\,|z|^2$, as required. q.e.d.

We conclude this section with the proof of the two lemmas used in step two of the proof of Lemma 4.1. Note that the restriction (1.17) in Theorem 5 arise in discussing the sign of $B^4 t^3 - 3AB^2t^2 - 3A^2Bt + A^4$, $B = \sqrt{h-1}$, $A = \sqrt{k-1}$ on $t \in [0,1]$ (Lemma 4.2), see also Remark 7. The sign of $A^4 t^3 - 3A^2Bt^2 - 3AB^2t + B^4$ on $t \in [0,1]$ (Lemma 4.3) does not require to put any further assumption than (1.15) and (1.16) on $k$ and $h$. In particular, this fact can be used to prove stability inequalities for *all* the Lawson's cones with respect to compact variations $F$ with $K_{k\,h}\Delta F \subset\subset H_R$ and $E \subset F$.

**Lemma 4.2.** *If*

$$B \geq A \geq 1\,, \qquad A + B \geq 6\,, \tag{4.41}$$

*and, moreover,*

$$B \geq 4\,, \qquad \text{if } A \geq 3\,, \tag{4.42}$$

$$B \geq 5\,, \qquad \text{if } A = 2\,, \tag{4.43}$$

$$B \geq 11\,, \qquad \text{if } A = 1\,, \tag{4.44}$$

*then*

$$p(t) = B^4 t^3 - 3AB^2t^2 - 3A^2Bt + A^4 \geq \frac{A^4}{64}\,, \qquad \forall t \in [0,1]\,.$$

*Proof.* Clearly we have

$$p'(t) = 3B\Big(B^3t^2 - 2ABt - A^2\Big)\,,$$

so that $p'(t) = 0$ if and only if

$$t = \frac{A}{B^2}\left(1 \pm \sqrt{1+B}\right)\,.$$

Thus $p'(t) < 0$ for $t > 0$ if and only if

$$t < \frac{A}{B^2}\left(1 + \sqrt{1+B}\right) =: t_{A\,B}\,.$$

In particular,

$$p(t) \geq p(t_{A\,B}) = \frac{A^3}{B^2}\Big(AB^2 - 2(1+B)^{3/2} - 2 - 3B\Big)\,, \qquad \forall t \geq 0\,.$$

Let us now set

$$q(A,B) = AB^2 - 2(1+B)^{3/2} - 2 - 3B\,.$$

We now claim that, under the assumptions of the lemma on $A$ and $B$, we have

$$q(A,B) \geq \frac{AB^2}{64}\,. \tag{4.45}$$

Indeed, let us set

$$q_0(A,B) = \frac{63}{64}\,AB^2 - 2(1+B)^{3/2} - 2 - 3B\,,$$

and prove $q_0 > 0$. Let us notice that,

$$\frac{\partial q_0}{\partial B} = \frac{63}{32}\,AB - 3\Big(1+\sqrt{1+B}\Big)\,. \tag{4.46}$$

*Case* $A = 1$*:* By (4.41) and $A = 1$, we have $B \geq 5$ and

$$\frac{\partial q_0}{\partial B} = \frac{63}{32}\,B - 3\Big(1+\sqrt{1+B}\Big)\,.$$

In particular, $q_0(1,B)$ is increasing on $B \in [6,\infty)$. By direct computation, $q_0(1,10) < -6$, while $q_0(1,11) > 0.9$. Hence,

$$q_0(1,B) \geq q_0(1,11) > 0\,, \qquad \forall B \geq 11\,,$$

and (4.45) follows under (4.44).

*Case* $A = 2$*:* By (4.41) and $A = 2$, we have $B \geq 4$ and

$$\frac{\partial q_0}{\partial B} = \frac{126}{32}\,B - 3\Big(1+\sqrt{1+B}\Big)\,.$$

In particular, $q_0(1,B)$ is increasing on $B \in [4,\infty)$. By direct computation, $q_0(2,4) < -4$, while $q_0(2,5) > 2$. Hence,

$$q_0(2,B) \geq q_0(2,5) > 0\,, \qquad \forall B \geq 5\,,$$

and (4.45) follows under (4.43).

*Case* $A \geq 3$*:* By (4.41) and $A \geq 3$, we have $B \geq 3$, and

$$\frac{\partial q_0}{\partial B} \geq \frac{189}{32}\,B - 3\Big(1+\sqrt{1+B}\Big)\,.$$

In particular, $q_0(A,B)$ is increasing on $B \in [3,\infty)$ for every $A \geq 3$. A direct computation shows that $q_0(3,3) < -0.4$, while $q_0(3,4) > 10$. Since $q_0$ is increasing on $A \in \mathbb{R}$, we find

$$q_0(A,B) \geq q_0(3,B) \geq q_0(3,4) > 0\,, \qquad \forall A \geq 3, B \geq 4\,,$$

and (4.45) is proved under assumption (4.42) too. q.e.d.

**Remark 7.** If one is not interested in the sharp behavior of $\min_{[0,1]} p$ in the limits $A \to \infty$ or $B \to \infty$, then it would suffice to prove $q(A,B) > 0$ (4.45) in order to prove stability inequalities for the cone corresponding to a given $(A,B)$. In this way, one could hope to recover some of the cases which are admissible for (4.41), but that are not covered by (4.42), (4.43) and (4.44), namely: $(A,B) \in \{(3,3),(2,4),(1,C) : 5 \leq C \leq 10\}$. However, as it can be directly checked, even the condition $q(A,B) > 0$, is characterized by (4.42), (4.43) and (4.44).

**Lemma 4.3.** *If*

$$B \geq A \geq 1\,, \qquad A+B \geq 6\,, \tag{4.47}$$

*then*

$$p(t) = A^4\,t^3 - 3A^2Bt^2 - 3AB^2t + B^4 \geq \frac{B^4}{64}\,, \qquad \forall t \in [0,1]\,.$$

*Proof.* This time we have $p'(t) = 3A\Big(A^3t^2 - 2ABt - B^2\Big)$, so that $p'(t) = 0$ if and only if

$$t = \frac{B}{A^2}\left(1 \pm \sqrt{1+A}\right)\,.$$

In particular, $p'(t) < 0$ for $t > 0$ if and only if

$$t < \frac{B}{A^2}\left(1 + \sqrt{1+A}\right) =: t_{A\,B}\,.$$

*Case $A = 1$:* In the case $A = 1$ we see that $t_{1\,B} = B(1+\sqrt{2}) > 1$. Hence we find

$$p(t) \geq p(1) = 1 - 3B - 3B^2 + B^4\,, \qquad \forall t \in [0,1]\,.$$

By (4.47), $B \geq 5$, hence $p(t) \geq B^4/4$ for $t \in [0,1]$, since

$$\frac{3}{4}\,B^4 \geq \frac{3}{4}\Big(25B^2\Big) \geq \frac{3}{4}\Big(20B^2 + 25B\Big) \geq 15B^2 + 21B > 3B^2 + 3B\,.$$

*Case $A = 2$:* In this case (4.47) implies $B \geq 4$, which gives $t_{2\,B} > 1$. Once again,

$$p(t) \geq p(1) = 16 - 12B - 6B^2 + B^4\,, \qquad \forall t \in [0,1]\,.$$

Using again $B \geq 4$ we find that

$$\frac{3}{4}\,B^4 \geq \frac{3}{4}\Big(16B^2\Big) \geq \frac{3}{4}\Big(8B^2 + 24B\Big) \geq 6B^2 + 24B > 6B^2 + 12B\,,$$

that is, $p(t) \geq B^4/4$ if $t \in [0,1]$.

*Case $A = 3$, $B \geq 4$:* Also in this case we have $t_{3\,B} \geq 1$, hence

$$p(t) \geq p(1) = B^4 - 9B^2 - 27B + 81\,, \qquad \forall t \in [0,1]\,.$$

Since $B \geq 4$ we have

$$\frac{63}{64}\,B^4 - 9B^2 - 27B \geq \left(\frac{63}{4} - 9\right)\,B^2 - 27B = \frac{27}{4}\,B^2 - 27\,B \geq 0\,.$$

Hence, $p(t) \geq B^4/64$ if $t \in [0,1]$.

*Case $A \geq 4$:* By $p'(t) = 3A\Big(A^3t^2 - 2ABt - B^2\Big)$ we find that $p'(t) = 0$ if and only if

$$t = \frac{B}{A^2}\left(1 \pm \sqrt{1+A}\right)\,.$$

Thus $p'(t) < 0$ for $t > 0$ if and only if

$$t < \frac{B}{A^2}\left(1 + \sqrt{1+A}\right) =: t_{A\,B}\,,$$

and, in particular,

$$p(t) \geq p(t_{A\,B}) = \frac{B^3}{A^2}\Big(A^2B - 2(1+A)^{3/2} - 2 - 3A\Big)\,, \qquad \forall t \geq 0\,.$$

We now set

$$q_1(A,B) = A^2B - 2(1+A)^{3/2} - 2 - 3A\,,$$

and aim to prove that $q_1(A,B) \geq A^2B/4$. Indeed,

$$\frac{3}{4}\,A^2B - 2(1+A)^{3/2} - 2 - 3A \geq \frac{3}{4}\,A^3 - 2(1+A)^{3/2} - 2 - 3A\,,$$

where

$$\frac{\partial}{\partial A}\Big(\frac{3}{4}\,A^3 - 2(1+A)^{3/2} - 2 - 3A\Big) \;=\; \frac{9}{4}\,A^2 - 3(1+A)^{1/2} - 3$$
$$=\; 9A - 3(1+A)^{1/2} - 3 \geq 0\,,$$

if and only if $3A-1 \geq (1+A)^{1/2}$, if and only if $9A^2 \geq 7A$, i.e. $A \geq 7/9$. q.e.d.

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