GENERALISED FUNCTIONS OF BOUNDED DEFORMATION

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Abstract. We introduce the space GBD of generalized functions of bounded deformation and study the structure properties of these functions: the rectifiability and the slicing properties of their jump sets, and the existence of their approximate symmetric gradients. We conclude by proving a compactness results for GBD, which leads to a compactness result for the space $GSBD$ of generalized special functions of bounded deformation. The latter is connected to the existence of solutions to a weak formulation of some variational problems arising in fracture mechanics in the framework of linearized elasticity.

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1. Introduction

The space $BD(\Omega)$ of *functions of bounded deformation* was investigated in [25, 31, 32, 24, 30] to study mathematical models of small strain elasto-plasticity (see also [22, 29, 8, 7]). If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\mathbb{M}^{n\times n}$ denotes the space of $n\times n$-matrices, $BD(\Omega)$ is the space of functions $u \in L^1(\Omega;\mathbb{R}^n)$ such that the $\mathbb{M}^{n\times n}$-valued distribution Eu, defined by $(Eu)_{ij} := \frac{1}{2}(D_i u_j + D_j u_i)$, is a bounded Radon measure.

The fine structure of the functions $u \in BD(\Omega)$ was investigated in [23, 5]. In particular it can be proved that the jump set J_u of u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and that the measure Eu can be written as the sum of three measures:

$$Eu = E^a u + E^c u + E^j u\,,$$

where $E^a u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, $E^c u$ is singular with respect to $\mathcal{L}^n$ and satisfies $|E^c u|(B) = 0$ for every Borel set $B \subset \Omega$ with $\mathcal{H}^{n-1}(B) < +\infty$, while $E^j u$ is concentrated on the jump set J_u. Moreover, if $\mathcal{E}u \in L^1(\Omega;\mathbb{M}^{n\times n})$ is the density of $E^a u$ with respect to $\mathcal{L}^n$, then for $\mathcal{L}^n$-a.e $x \in \Omega$ we have (see [5, Theorem 4.3])

$$\lim_{\rho\to 0+} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{\big|\big(u(y) - u(x) - \mathcal{E}u(x)(y-x)\big)\cdot(y-x)\big|}{|y-x|^2}\, dy = 0\,, \tag{1.1}$$

where $B_\rho(x)$ denotes the open ball with centre x and radius ρ, while the dot denotes the scalar product in $\mathbb{R}^n$. Finally, $\mathcal{E}u$ and J_u can be reconstructed from the derivatives and the jump sets of the one-dimensional slices of the function u (see [5, Theorem 4.5]).

The space $SBD(\Omega)$ of *special functions of bounded deformation* was introduced in [5] and is defined as the space of all functions $u \in BD(\Omega)$ with $E^c u = 0$. In the framework of linearized elasticity the variational models for fracture mechanics originated by the seminal

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paper [20] have a sound mathematical formulation in the space $SBD(\Omega)$ (see, e.g., [27, 10, 13, 28, 12]). The common feature of these models is that the main energy term has the form

$$F_Q(u) := \int_\Omega Q(\mathcal{E}u)\,dx + \mathcal{H}^{n-1}(J_u)\,, \tag{1.2}$$

where Q is a positive definite quadratic form, which gives the stored elastic energy density as a function of the strain $\mathcal{E}u$.

To prove the existence of solutions to minimum problems related to (1.2) one can use a compactness result proved in [9, Theorem 1.1]: if u_k is a sequence in $SBD(\Omega)$ such that $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^n)}$ and $F_Q(u_k)$ are bounded uniformly with respect to k, then there exist a subsequence, still denoted by u_k, and a function $u \in SBD(\Omega)$, such that $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. on Ω, $\mathcal{E}u_k \rightharpoonup \mathcal{E}u$ weakly in $L^1(\Omega;\mathbb{M}^{n\times n})$, and $\mathcal{H}^{n-1}(J_u) \le \liminf_k \mathcal{H}^{n-1}(J_{u_k})$. The drawback of this result is that it is difficult to obtain *a priori* bounds of $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^n)}$ for a minimizing sequence, even if lower order terms are present.

A similar difficulty appears also in the study of variational models of fracture mechanics in the framework of finite elasticity (see [16, 17]), whose mathematical formulation uses the function space $SBV(\Omega;\mathbb{R}^n)$, for which we refer to [6, Chapter 4]. In these models $\mathcal{E}u$ is replaced by $\nabla u \in L^1(\Omega;\mathbb{M}^{n\times n})$, defined for every $u \in SBV(\Omega;\mathbb{R}^n)$ as the density of the absolutely continuous part of the measure Du with respect to $\mathcal{L}^n$, and the main energy term has the form

$$F_W(u) := \int_\Omega W(\nabla u)\,dx + \mathcal{H}^{n-1}(J_u)\,, \tag{1.3}$$

where W is polyconvex and satisfies $W(A) \ge |A|^2$ for every $A \in \mathbb{M}^{n\times n}$. The basic compactness theorem for SBV (see [2, 4] and [6, Theorem 4.8]) requires that $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^n)}$ and $F_W(u_k)$ are bounded, and an L^∞ bound for the minimizing sequences is problematic also in this setting.

In the antiplane case (see [19]) u is a scalar function on Ω and the L^∞ bound is obtained by truncation, assuming that the prescribed boundary values are bounded in L^∞. In the vector case, the solution adopted in [16, 17] is to formulate the problems in the larger space $GSBV(\Omega;\mathbb{R}^n)$, defined as the set of all $\mathcal{L}^n$-measurable functions $u\colon \Omega \to \mathbb{R}^n$ such that $\psi(u) \in BV_{loc}(\Omega;\mathbb{R}^n)$ for every $\psi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ such that $\nabla\psi$ has compact support. For every $u \in GSBV(\Omega;\mathbb{R}^n)$ one can define a unique $\mathcal{L}^n$-measurable function $\nabla u\colon \Omega \to \mathbb{M}^{n\times n}$ such that $\nabla\big(\psi(u)\big) = \nabla\psi(u)\nabla u$ $\mathcal{L}^n$-a.e. in Ω for every ψ considered above, so that the functional F_W can be defined on $GSBV(\Omega;\mathbb{R}^n)$.

In this new setting one can rely on the compactness result for $GSBV$ proved in [3] (see also [6, Theorem 4.36]): if u_k is a sequence in $GSBV(\Omega;\mathbb{R}^n)$ such that $\|u_k\|_{L^1(\Omega;\mathbb{R}^n)}$ and $F_W(u_k)$ are bounded uniformly with respect to k, then there exist a subsequence, still denoted by u_k, and a function $u \in GSBV(\Omega;\mathbb{R}^n)$, such that $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. on Ω, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^1(\Omega;\mathbb{M}^{n\times n})$, and $\mathcal{H}^{n-1}(J_u) \le \liminf_k \mathcal{H}^{n-1}(J_{u_k})$. An L^1 bound for a minimizing sequence can be easily obtained from the lower order terms that are usually present in the minimum problems for (1.3).

One may think that the same strategy can be used to formulate and solve the minimum problems for (1.2). The first difficulty in this approach comes from the fact that, if $u \in SBD(\Omega)$, then, in general, the composite function $\psi(u)$ does not belong to $SBD(\Omega)$ (it does not even belong to $BD(\Omega)$), unless $\psi(y) = y_0 + \lambda y$ for some $y_0 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Therefore a definition of $GSBD(\Omega)$ that mimics the definition of $GSBV(\Omega;\mathbb{R}^n)$ is doomed to failure, since it would not lead to a space containing $SBD(\Omega)$.

In this paper we propose a different definition of the space $GSBD(\Omega)$ of *generalized special functions of bounded deformation* and of the larger space $GBD(\Omega)$ of *generalized functions of bounded deformation*. The definition is given by slicing. For every $\xi \in \mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ let $\Pi^\xi := \{y \in \mathbb{R}^n : y\cdot\xi = 0\}$ be the hyperplane orthogonal to ξ passing through the origin. For every set $B \subset \mathbb{R}^n$ and for every $y \in \Pi^\xi$ we define

$$B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\}.$$

Moreover, for every function $u\colon B \to \mathbb{R}^n$ we define the function $\hat{u}^\xi_y\colon B^\xi_y \to \mathbb{R}$ by

$$\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi.$$

If $u\colon B \to \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the jump set of $\hat{u}^\xi_y$ is denoted by $J_{\hat{u}^\xi_y}$. Moreover we set

$$J^1_{\hat{u}^\xi_y} := \{t \in J_{\hat{u}^\xi_y} : |(\hat{u}^\xi_y)^+(t) - (\hat{u}^\xi_y)^-(t)| \geq 1\},$$

where $(\hat{u}^\xi_y)^-(t)$ and $(\hat{u}^\xi_y)^+(t)$ are the approximate left and right limits of $\hat{u}^\xi_y$ at t.

The space $GBD(\Omega)$ is defined (see Definition 4.1) as the space of all $\mathcal{L}^n$-measurable functions $u\colon \Omega \to \mathbb{R}^n$ such that there exists a bounded Radon measure λ on Ω with the following property: for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat{u}^\xi_y$ belongs to $BV_{loc}(\Omega^\xi_y)$ and

$$\int_{\Pi^\xi} \Big(|D\hat{u}^\xi_y|(B^\xi_y \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat{u}^\xi_y}) \Big)\, d\mathcal{H}^{n-1}(y) \leq \lambda(B) \tag{1.4}$$

for every Borel set $B \subset \Omega$. If we replace $BV_{loc}(\Omega^\xi_y)$ by $SBV_{loc}(\Omega^\xi_y)$, we obtain the definition of the space $GSBD(\Omega)$ (see Definition 4.2).

The inclusion $BD(\Omega) \subset GBD(\Omega)$ follows from the structure theorem for BD functions (see [5, Theorem 4.5]), while the inclusion $SBD(\Omega) \subset GSBD(\Omega)$ follows from [5, Proposition 4.7]. Example 12.3 shows that these inclusions are strict.

We prove (see Theorem 6.2) that for every $u \in GBD(\Omega)$ the *approximate jump set* J_u (see Definition 2.4) is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable according to [21, Section 3.2.14] and can be reconstructed from the jump sets of the one-dimensional slices $\hat{u}^\xi_y$ (see Theorem 8.1). More precisely, if $[u] := u^+ - u^-$ is the jump of u on J_u (see Definition 2.4) and $J^\xi_u := \{x \in J_u : [u](x) \cdot \xi \neq 0\}$, then $(J^\xi_u)^\xi_y = J_{\hat{u}^\xi_y}$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

To prove these results we first study the traces of a function $u \in GBD(\Omega)$ on a C^1 submanifold M of Ω of dimension $n-1$. In this analysis we use the fact that the directional derivative $D_\xi\big(\tau(u\cdot\xi)\big)$ is a bounded Radon measure for every $u \in GBD(\Omega)$, for every $\xi \in \mathbb{S}^{n-1}$, and for every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} < \tau < \frac{1}{2}$ and $0 < \tau' < 1$ (see Theorem 3.5). Then we can apply the result proved in [31, Lemma 1.1] on the traces on M of functions $v \in L^1(\Omega)$ such that a single directional derivative $D_\xi v$ is a bounded Radon measure on Ω, provided that ξ is transversal to M. Inverting τ we obtain that the trace of $u\cdot\xi$ is well defined for a set of vectors ξ forming a basis of $\mathbb{R}^n$, and this allows us to define the trace of u (see Theorem 5.2).

In the proof of the rectifiability of J_u the measure $|\mathcal{E}u|$ used in [5] is replaced by the measure $\hat{\mu}_u$ defined for every Borel set $B \subset \Omega$ by

$$\hat{\mu}_u(B) := \sup_k \; \sup \sum_{i=1}^k \hat{\mu}^{\xi_i}_u(B_i),$$

where $\hat{\mu}^\xi_u(B)$ is defined as the left-hand side of (1.4) and the second supremum is over all families $\xi_1, \dots, \xi_k$ of elements of $\mathbb{S}^{n-1}$ and over all families $B_1, \dots, B_k$ of pairwise disjoint Borel subsets of B. We first prove (see Theorem 6.1) that the set

$$\Theta_u := \Big\{x \in \Omega : \limsup_{\rho \to 0+} \frac{\hat{\mu}_u(B_\rho(x))}{\rho^{n-1}} > 0\Big\}$$

is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, following an argument developed in [23]. Then we prove (see Theorem 6.2) that $J_u \subset \Theta_u$ and $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$, using the results on the traces of GBD functions on C^1 manifolds.

A crucial step in the proof of the slicing result for J_u is a difficult technical result (see Theorem 7.1) concerning the jump points of the restriction to hyperplanes of a GBD function. The proof of this result follows the lines of the analogous result for BD functions proved in [5, Theorem 5.1], with $|Eu|$ replaced again by $\hat\mu_u$.

Another result of this paper is the existence, for every $u \in GBD(\Omega)$, of a *symmetric approximate gradient*. This is a function $\mathcal{E}u \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$, where $\mathbb{M}^{n\times n}_{sym}$ is the space of symmetric $n{\times}n$ matrices, such that the following variant of (1.1) holds (see Theorem 9.1 and Remark 2.2):

$$\lim_{\rho\to 0+}\frac{1}{\rho^n}\int_{B_\rho(x)}\psi\Big(\frac{\big|\big(u(y)-u(x)-\mathcal{E}u(x)(y-x)\big)\cdot(y-x)\big|}{|y-x|^2}\Big)\,dy=0$$

for $\mathcal{L}^n$-a.e $x\in\Omega$ and for every bounded increasing continuous function $\psi\colon\mathbb{R}\to\mathbb{R}$. Moreover we prove that $\mathcal{E}u$ can be reconstructed from the approximate gradients $\nabla\hat u^\xi_y$ of the one-dimensional slices $\hat u^\xi_y$ (see Theorem 9.1): for every $\xi\in\mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ we have $(\mathcal{E}u)^\xi_y\,\xi\cdot\xi=\nabla\hat u^\xi_y$ $\mathcal{L}^1$-a.e. on Ω^ξ_y.

In the last section we prove the following analogue of the compact embedding of $BD(\Omega)$ into $L^1(\Omega;\mathbb{R}^n)$ (see Theorem 11.1): every sequence u_k in $GBD(\Omega)$ satisfying uniform bounds for $\|u_k\|_{L^1(\Omega;\mathbb{R}^n)}$ and for the measures $\hat\mu^\xi_{u_k}$ has a subsequence that converges pointwise $\mathcal{L}^n$-a.e. on Ω. A slightly stronger bound implies that the limit function belongs to $GBD(\Omega)$ (see Corollary 11.2).

For the proof we have to modify the well-known Fréchet-Kolmogorov compactness criterion in L^1 and to find a new version, based on the behaviour of the one-dimensional slices (see Lemma 10.7). The proof follows the lines of [1, Theorem 6.6]. The main difference is that our assumptions concern only the components $u\cdot\xi$ of u and the corresponding slices in the same direction ξ.

Arguing as in the proof of [9, Theorem 1.1], we deduce from these results on $GBD(\Omega)$ the following compactness property for $GSBD(\Omega)$ (see Theorem 11.3): if u_k is a sequence in $GSBD(\Omega)$ such that $\|u_k\|_{L^1(\Omega;\mathbb{R}^n)}$, $\|\mathcal{E}u_k\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})}$, and $\mathcal{H}^{n-1}(J_{u_k})$ are bounded uniformly with respect to k, then there exist a subsequence, still denoted by u_k, and a function $u\in GSBD(\Omega)$, such that $u_k\to u$ pointwise $\mathcal{L}^n$-a.e. on Ω, $\mathcal{E}u_k\rightharpoonup\mathcal{E}u$ weakly in $L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $\mathcal{H}^{n-1}(J_u)\le\liminf_k\mathcal{H}^{n-1}(J_{u_k})$.

Finally, Example 12.3 shows that there exists a sequence in $SBD(\Omega)$, satisfying the hypotheses of the compactness theorem for $GSBD(\Omega)$, such that the limit function, which necessarily belongs to $GSBD(\Omega)$, does not belong to $BD(\Omega)$.

2. Notation and preliminary results

For every $x\in\mathbb{R}^n$ the open ball of centre x and radius ρ is denoted by $B_\rho(x)$. For every $x,y\in\mathbb{R}^n$, we use the notation $x\cdot y$ for the scalar product and $|x|$ for the norm. The n-dimensional Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, while $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. We use the standard notation $\mathbb{S}^{n-1}:=\{\xi\in\mathbb{R}^n:|\xi|=1\}$ and $\omega_n:=\mathcal{L}^n(B_1(0))$, so that $\mathcal{H}^{n-1}(\mathbb{S}^{n-1})=n\,\omega_n$.

If μ is a Borel measure on a Borel set $E\subset\mathbb{R}^n$, its total variation is denoted by $|\mu|$. If $A\subset E$ is a Borel set, the Borel measure $\mu\llcorner A$ is defined by $(\mu\llcorner A)(B):=\mu(A\cap B)$ for every Borel set $B\subset E$. If $U\subset\mathbb{R}^n$ is an open set, $\mathcal{M}(U)$ is the space of all Radon measures on U, $\mathcal{M}_b(U):=\{\mu\in\mathcal{M}(U):|\mu|(U)<+\infty\}$ is the space of all bounded Radon measures on U, and $\mathcal{M}^+_b(U):=\{\mu\in\mathcal{M}_b(U):\mu(B)\ge 0$ for every Borel set $B\subset U\}$ is the space of all nonnegative bounded Radon measures on U.

Definition 2.1. Let A be a subset of $\mathbb{R}^n$, let $v\colon A\to\mathbb{R}^m$ be an $\mathcal{L}^n$-measurable function, let $x\in\mathbb{R}^n$ be such that

$$\limsup_{\rho\to 0+}\frac{\mathcal{L}^n\big(A\cap B_\rho(x)\big)}{\rho^n}>0\,,$$

and let $a \in \mathbb{R}^m$. We say that a is the *approximate limit* of v as y tends to x, and write

$$\operatorname*{ap\,lim}_{y \to x} v(y) = a \tag{2.1}$$

if

$$\lim_{\rho \to 0+} \frac{\mathcal{L}^n(\{y \in A \cap B_\rho(x) : |v(y) - a| > \varepsilon\})}{\rho^n} = 0$$

for every $\varepsilon > 0$.

Remark 2.2. Let A, v, x, and a be as in Definition 2.1 and let ψ be a homeomorphism between $\mathbb{R}^m$ and a bounded open subset of $\mathbb{R}^m$. It is easy to prove that (2.1) holds if and only if

$$\lim_{\rho \to 0+} \frac{1}{\rho^n} \int_{A \cap B_\rho(x)} |\psi(v(y)) - \psi(a)|\, dy = 0\,.$$

Definition 2.3. Let U be an open subset of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $v\colon U \to \mathbb{R}^m$ we define the *approximate continuity set* as the set of points $x \in U$ for which there exists $a \in \mathbb{R}^m$ such that

$$\operatorname*{ap\,lim}_{y \to x} v(y) = a\,.$$

The vector a is uniquely determined and is denoted by $\tilde{v}(x)$. The *approximate discontinuity set* S_v is defined as the complement in U of the approximate continuity set.

Definition 2.4. Let U be an open subset of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $v\colon U \to \mathbb{R}^m$ we define the *approximate jump set* J_v as the set of points $x \in U$ for which there exist a, $b \in \mathbb{R}^m$, with $a \neq b$, and $\nu \in \mathbb{S}^{n-1}$ such that

$$\operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu > 0\\ y \to x}} v(y) = a \quad \text{and} \quad \operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu < 0\\ y \to x}} v(y) = b\,. \tag{2.2}$$

The triplet (a, b, ν) is uniquely determined up to a permutation of (a, b) and a change of sign of ν, and is denoted by $(v^+(x), v^-(x), \nu_v(x))$. The *jump* of v is the function $[v]\colon J_v \to \mathbb{R}^m$ defined by $[v](x) := v^+(x) - v^-(x)$ for every $x \in J_v$. Finally, we define

$$J_v^1 := \{x \in J_v : |[v](x)| \geq 1\}\,. \tag{2.3}$$

Remark 2.5. It follows easily from the definitions that $J_v \subset S_v$ for every $\mathcal{L}^n$-measurable function $v\colon U \to \mathbb{R}^m$. Moreover, $\mathcal{L}^n(S_v) = 0$ and $v = \tilde{v}$ $\mathcal{L}^n$-a.e. in U by Remark 2.2 and by Lebesgue's differentiation theorem.

Thanks to Remark 2.2 the next proposition follows from [6, Proposition 3.69].

Proposition 2.6. *Let U be an open subset of $\mathbb{R}^n$ and let $v\colon U \to \mathbb{R}^m$ be an $\mathcal{L}^n$-measurable function. Then S_v, J_v, and J_v^1 are Borel sets and $\tilde{v}\colon U \setminus S_u \to \mathbb{R}^m$ is a Borel function. Moreover, for every $x \in J_v$ we can choose the sign of $\nu_v(x)$ so that $v^+\colon J_v \to \mathbb{R}^m$, $v^-\colon J_v \to \mathbb{R}^m$, and $\nu_v\colon J_v \to \mathbb{S}^{n-1}$ are Borel functions.*

If $U \subset \mathbb{R}^n$ is an open set and $v \in L^1_{loc}(U)$, the gradient Dv of v in the sense of distributions is the $\mathbb{R}^n$-valued distribution defined by $Dv = (D_1 v, \dots, D_n v)$. For every $\xi \in \mathbb{R}^n$ the directional derivative $D_\xi v$ is the distribution $D_\xi v := Dv \cdot \xi = \sum_i \xi_i D_i v$. The space $BV(U)$ of *functions of bounded variation* is defined as the space of functions $v \in L^1(U)$ such that $D_i v \in \mathcal{M}_b(U)$ for $i = 1, \dots, n$, while $BV_{loc}(U)$ is the space of functions $v \in L^1_{loc}(U)$ such that $D_i v \in \mathcal{M}(U)$ for $i = 1, \dots, n$. For the properties of BV functions we refer to [18] and [6].

3. Slicing of directional derivatives

For every $\xi \in \mathbb{R}^n \setminus \{0\}$, for every $y \in \mathbb{R}^n$, and for every set $B \subset \mathbb{R}^n$ and we define

$$B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\}\,.$$

Moreover, for every function $v\colon B \to \mathbb{R}^m$ we define the function $v^\xi_y\colon B^\xi_y \to \mathbb{R}^m$ by

$$v^\xi_y(t) := v(y + t\xi)\,.$$

When $m = n$, we consider also the function $\hat{v}^\xi_y\colon B^\xi_y \to \mathbb{R}$ defined by

$$\hat{v}^\xi_y(t) := v(y + t\xi) \cdot \xi = v^\xi_y(t) \cdot \xi\,.$$

The hyperplane orthogonal to ξ passing through the origin is denoted by $\Pi^\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\}$ and the orthogonal projection from $\mathbb{R}^n$ onto Π^ξ is denoted by $\pi^\xi\colon \mathbb{R}^n \to \Pi^\xi$.

Throughout the paper Ω is a fixed bounded open subset of $\mathbb{R}^n$. The following proposition is proved in [6, Theorem 3.103] (see also [26]).

Proposition 3.1. *Let $v \in L^1(\Omega)$ and let $\xi \in \mathbb{R}^n \setminus \{0\}$. The following conditions are equivalent:*

(a) $D_\xi v \in \mathcal{M}_b(\Omega)$*;*

(b) *For $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function v^ξ_y belongs to $BV(\Omega^\xi_y)$ and*

$$\int_{\Pi^\xi} |Dv^\xi_y|(\Omega^\xi_y)\, d\mathcal{H}^{n-1}(y) < +\infty\,. \tag{3.1}$$

If these conditions are satisfied, then for every Borel function $g\colon \Omega \to \mathbb{R}^+$ the function

$$y \mapsto \int_{\Omega^\xi_y} g^\xi_y\, d|Dv^\xi_y| \tag{3.2}$$

is $\mathcal{H}^{n-1}$-measurable on Π^ξ and

$$\int_\Omega g\, d|D_\xi v| = \int_{\Pi^\xi} \Big(\int_{\Omega^\xi_y} g^\xi_y\, d|Dv^\xi_y| \Big)\, d\mathcal{H}^{n-1}(y)\,. \tag{3.3}$$

Given an open set $U \subset \mathbb{R}$, let $E \subset U$ be $\mathcal{L}^1$-measurable with $\mathcal{L}^1(U \setminus E) = 0$, and let $v\colon E \to \mathbb{R}^m$ be an integrable function. As v is defined $\mathcal{L}^1$-a.e. in U, it can be considered as a distribution on U, whose derivative is denoted by Dv. The *pointwise variation* $(Vv)(I)$ of v on an open interval $I \subset U$ is defined by

$$(Vv)(I) := \sup\Big\{ \sum_{i=1}^k |v(t_i) - v(t_{i-1})| : t_0 < t_1 < \cdots < t_k,\ t_i \in E \cap I \Big\}\,. \tag{3.4}$$

We observe that Vv, unlike Dv, is sensitive to changes of v (or of the domain of v) on sets of Lebesgue measure zero. If $(Vv)(I)$ is finite, then Dv can be represented by a bounded measure on I with $|Dv|(I) \le (Vv)(I)$. Moreover, if $(Vv)(I) < +\infty$ for every open interval $I \subset U$, then Vv can be extended to a non-negative Radon measure, still denoted by Vv, defined on all Borel subsets of U. Indeed, if $A \subset U$ is open, we define $(Vv)(A)$ as the sum of $(Vv)(I)$ over all connected components I of A. Then $A \mapsto (Vv)(A)$, defined now for all open subsets of U, is increasing, inner regular, subadditive and additive on disjoint open sets. Therefore the set function defined for every Borel set $B \subset U$ by

$$(Vv)(B) := \inf\{(Vv)(A) : A \text{ open},\ B \subset A \subset U\}$$

is a Radon measure on U, which coincides with (3.4) on all open intervals $I \subset U$ (see, e.g., [15, Theorem 14.23]).

Let $v\colon \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable. By Definition 2.3 for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and every $y \in \Pi^\xi$ the function $\tilde{v}^\xi_y$ is defined on $\Omega^\xi_y \setminus (S_u)^\xi_y$.

Proposition 3.2. *Let $v \in L^1(\Omega)$ and let $\xi \in \mathbb{R}^n \setminus \{0\}$. Assume that $D_\xi v \in \mathcal{M}_b(\Omega)$. Then the following conditions are satisfied for $\mathcal{H}^{n-1}$-a.e $y \in \Pi^\xi$:*

(a) *$\tilde{v}^\xi_y$ is defined and coincides with v^ξ_y $\mathcal{L}^1$-a.e. on Ω^ξ_y;*

(b) *$v^\xi_y \in BV(\Omega^\xi_y)$ and $|Dv^\xi_y|(B) = (V\tilde{v}^\xi_y)(B)$ for every Borel set $B \subset \Omega^\xi_y$.*

Proof. It is enough to repeat the proof of [5, Proposition 3.2]. □

We now investigate the behaviour of truncations of scalar functions, and the combined effect of truncation and slicing. The following definition introduces the relevant truncation functions.

Definition 3.3. Let $\mathcal{T}$ be the set of all functions $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \le \tau \le \frac{1}{2}$ and $0 \le \tau' \le 1$.

The following proposition deals with the one-dimensional case. It provides a bound on the distributional derivative of a function starting from a uniform bound of its truncations.

Proposition 3.4. *Let U be a bounded open subset of $\mathbb{R}$, let $v\colon U \to \mathbb{R}$ be $\mathcal{L}^1$-measurable, and let $\lambda \in \mathcal{M}^+_b(U)$. Suppose that for every $\tau \in \mathcal{T}$ we have $\tau(v) \in BV(U)$ and*

$$\big|D\big(\tau(v)\big)\big|(B) \le \lambda(B) \tag{3.5}$$

for every Borel set $B \subset U$. Then $v \in BV_{loc}(U)$ and

$$|Dv|(B \setminus J^1_v) + \mathcal{H}^0(B \cap J^1_v) \le \lambda(B) \tag{3.6}$$

for every Borel set $B \subset U$. If U has a finite number of connected components, then $v \in BV(U)$.

Proof. It is enough to prove the result when U is a bounded open interval. In this case we have to prove that $v \in BV(U)$ and that (3.6) holds. Let us fix $\tau_0 \in \mathcal{T}$ with $\tau_0'(t) > 0$ and $\tau_0(-t) = -\tau_0(t)$ for every $t \in \mathbb{R}$. Then the function $v_0 := \tau_0(v)$ belongs to $BV(U)$. Since τ_0^{-1} is continuous, we have $J_v = J_{v_0}$.

For every $a > 0$ let σ_a be the truncation function defined by $\sigma_a(t) = -a$ for $t \le -a$, $\sigma_a(t) = t$ for $-a \le t \le a$, and $\sigma_a(t) = a$ for $t \ge a$. Let us fix an integer $m > 0$ and let $v_m := \sigma_m(v)$. We claim that $v_m \in BV(U)$ and

$$|Dv_m|(B \setminus J^1_{v_m}) + \mathcal{H}^0(B \cap J^1_{v_m}) \le \lambda(B) \tag{3.7}$$

for every Borel set $B \subset U$. Indeed, since $v_0 \in BV(U)$, $v_m = \sigma_m(\tau_0^{-1}(v_0))$, and $\sigma_m(\tau_0^{-1}) = \tau_0^{-1}(\tau_{\tau_0(m)})$ is Lipschitz continuous on $\mathbb{R}$, we deduce that $v_m \in BV(U)$. By (3.5) and by Vol'pert's chain rule in BV (see [33] and [6, Theorem 3.96]) for every $\tau \in \mathcal{T}$ and for every Borel set $B \subset U$ we have

$$\int_{B\setminus J_{v_m}} \tau'(\tilde{v}_m)\, d|Dv_m| = \big|D\big(\tau(v_m)\big)\big|(B \setminus J_{v_m}) \le \big|D\big(\tau(v)\big)\big|(B \setminus J_{v_m}) \le \lambda(B \setminus J_{v_m}), \tag{3.8}$$

where $\tilde{v}_m$ is the precise representative introduced in Definition 2.3. Note that $\tilde{v}_m(t)$ is defined for every $t \in U \setminus J_{v_m}$ by well known properties of BV functions in dimension one.

For every integer k there exists a function $\tau_k \in \mathcal{T}$ such that $\tau_k(t) = t - \frac{k}{2}$ for $\frac{k}{2} - \frac{1}{4} \le t \le \frac{k}{2} + \frac{1}{4}$. Thus (3.8) gives

$$|Dv_m|(B_k) \le \lambda(B_k),$$

where $B_k := \{t \in B \setminus J_{v_m} : \frac{k}{2} - \frac{1}{4} < \tilde{v}_m(t) \le \frac{k}{2} + \frac{1}{4}\}$. Summing over k we obtain

$$|Dv_m|(B \setminus J_{v_m}) \le \lambda(B \setminus J_{v_m}). \tag{3.9}$$

Let us fix $t \in J_{v_m} \subset J_v$. By (3.5) for every $\tau \in \mathcal{T}$ we have

$$|\tau(v^+(t)) - \tau(v^-(t))| \le \lambda(\{t\}).$$

If $t \in J_{v_m} \setminus J^1_{v_m}$, there exists $\tau \in \mathcal{T}$ such that $|[v_m](t)| = |\tau(v^+_m(t)) - \tau(v^-_m(t))| \le |\tau(v^+(t)) - \tau(v^-(t))|$. This implies $|[v_m](t)| \le \lambda(\{t\})$, hence

$$|Dv_m|(\{t\}) \le \lambda(\{t\}) \tag{3.10}$$

for every $t \in J_{v_m} \setminus J^1_{v_m}$. If $t \in J^1_{v_m}$, for every $\varepsilon > 0$ there exists $\tau \in \mathcal{T}$ such that $1 - \varepsilon \le |\tau(v^+_m(t)) - \tau(v^-_m(t))| \le |\tau(v^+(t)) - \tau(v^-(t))|$, which gives

$$1 \le \lambda(\{t\}) \tag{3.11}$$

for every $t \in J^1_{v_m}$. Inequality (3.7) follows now from (3.9), (3.10), and (3.11).

Let $J^{1/2}_{v_m} := \{t \in J_{v_m} : |[v_m](t)| \ge \frac{1}{2}\}$. By (3.10) and (3.11) we have $\mathcal{H}^0(J^{1/2}_{v_m}) \le 2\,\lambda(J^{1/2}_{v_m}) \le 2\,\lambda(U) < +\infty$. Since J^1_v is contained in the union of the increasing sequence $J^{1/2}_{v_m}$, we obtain $\mathcal{H}^0(J^1_v) \le 2\,\lambda(U) < +\infty$. Using (3.7) we obtain

$$|Dv_m|(U) \le \lambda(U) + \int_{J^1_{v_m}} |[v_m]|\, d\mathcal{H}^0 \le \lambda(U) + \int_{J^1_v} |[v]|\, d\mathcal{H}^0 < +\infty\,. \tag{3.12}$$

Let us fix $t_0 \in U \setminus J_v \subset U \setminus J_{v_m}$. Since U is an interval, we have

$$|\tilde{v}_m(t)| \le |\tilde{v}_m(t) - \tilde{v}_m(t_0)| + |\tilde{v}_m(t_0)| \le |Dv_m|(U) + |\tilde{v}(t_0)|$$

for every m and for every $t \in U \setminus J_v \subset U \setminus J_{v_m}$. By (3.12) this inequality implies

$$\|v_m\|_{L^\infty(U)} \le \lambda(U) + \int_{J^1_v} |[v]|\, d\mathcal{H}^0 + |\tilde{v}(t_0)| < +\infty\,.$$

Since the right-hand side does not depend on m, there exists m_0 such that $\|v_{m_0}\|_{L^\infty(U)} < m_0$. This implies that $v = v_{m_0}$, hence $v \in BV(U)$ and (3.6) follows from (3.7). □

The following theorem is the main result of this section. It connects a uniform estimate on the directional derivatives of the truncations with an estimate on the one-dimensional slices. The equivalence proved in the theorem will be used in the definition of the space $GBD(\Omega)$.

Theorem 3.5. *Let $v\colon \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable, let $\xi \in \mathbb{R}^n \setminus \{0\}$, and let $\lambda \in \mathcal{M}^+_b(\Omega)$. The following conditions are equivalent:*

(a) *for every $\tau \in \mathcal{T}$ the partial derivative $D_\xi\big(\tau(v)\big)$ belongs to $\mathcal{M}_b(\Omega)$ and its total variation satisfies*

$$\big|D_\xi\big(\tau(v)\big)\big|(B) \le \lambda(B) \tag{3.13}$$

for every Borel set $B \subset \Omega$;

(b) *for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function v^ξ_y belongs to $BV_{loc}(\Omega^\xi_y)$ and*

$$\int_{\Pi^\xi} \big(|Dv^\xi_y|(B^\xi_y \setminus J^1_{v^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{v^\xi_y})\big)\, d\mathcal{H}^{n-1}(y) \le \lambda(B) \tag{3.14}$$

for every Borel set $B \subset \Omega$.

The following lemma justifies the integral in (3.14).

Lemma 3.6. *Let $v\colon \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable and let $\xi \in \mathbb{R}^n \setminus \{0\}$. Assume that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function v^ξ_y belongs to $BV_{loc}(\Omega^\xi_y)$. Then for every Borel set $B \subset \Omega$ the function*

$$y \mapsto |Dv^\xi_y|(B^\xi_y \setminus J^1_{v^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{v^\xi_y}) \tag{3.15}$$

is $\mathcal{H}^{n-1}$-measurable on Π^ξ.

Proof. By modifying v on a set of Lebesgue measure zero, we may assume that v is a Borel function on Ω and that $v^\xi_y \in BV_{loc}(\Omega^\xi_y)$ for every $y \in \Pi^\xi$. For every $x \in \Omega$ we define

$$v^\xi_+(x) := \limsup_{\rho\to 0+} \frac{1}{\rho}\int_0^\rho v(x+s\xi)\,ds \quad \text{and} \quad v^\xi_-(x) := \limsup_{\rho\to 0+} \frac{1}{\rho}\int_{-\rho}^0 v(x+s\xi)\,ds. \tag{3.16}$$

By Fubini's theorem v^ξ_+ and v^ξ_- are Borel functions on Ω. Therefore $F := \{x \in \Omega : |v^\xi_+(x) - v^\xi_-(x)| \geq 1\}$ is a Borel set. For every $y \in \Pi^\xi$ we have $(v^\xi_+)^\xi_y = (v^\xi_-)^\xi_y = v^\xi_y$ $\mathcal{L}^1$-a.e. in Ω^ξ_y thanks to Lebesgue's differentiation theorem. By elementary properties of BV functions in dimension one, this implies $J^1_{v^\xi_y} = F^\xi_y$ for every $y \in \Pi^\xi$. Therefore

$$|Dv^\xi_y|(B^\xi_y \setminus J^1_{v^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{v^\xi_y}) = |Dv^\xi_y|(B^\xi_y \setminus F^\xi_y) + \mathcal{H}^0(B^\xi_y \cap F^\xi_y)$$

for every Borel set $B \subset \Omega$ and for every $y \in \Pi^\xi$. The measurability of (3.15) follows now from (3.2) and from the measurable projection theorem (see, e.g., [14, Proposition 8.4.4]). □

Definition 3.7. If Condition (b) of Theorem 3.5 is satisfied, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we can define a measure $\mu^\xi_y \in \mathcal{M}^+_b(\Omega^\xi_y)$ by setting

$$\mu^\xi_y(B) := |Dv^\xi_y|(B \setminus J^1_{v^\xi_y}) + \mathcal{H}^0(B \cap J^1_{v^\xi_y}) \tag{3.17}$$

for every Borel set $B \subset \Omega^\xi_y$. Moreover, by Lemma 3.6 and by (3.14) we can define a measure $\mu^\xi \in \mathcal{M}^+_b(\Omega)$ by setting

$$\mu^\xi(B) := \int_{\Pi^\xi} \mu^\xi_y(B^\xi_y)\,d\mathcal{H}^{n-1}(y) \tag{3.18}$$

for every Borel set $B \subset \Omega$.

It follows from Condition (b) of Theorem 3.5 that

$$\mu^\xi(B) \leq \lambda(B) \tag{3.19}$$

for every Borel set $B \subset \Omega$.

Proof of Theorem 3.5. Assume (a). Let $\hat{\mathcal{T}}$ be a countable subset of $\mathcal{T}$ such that for every $\tau \in \mathcal{T}$ there exists a sequence τ_k in $\hat{\mathcal{T}}$ converging to τ pointwise on $\mathbb{R}$. Let us fix $\tau \in \hat{\mathcal{T}}$ and let $w := \tau(v)$ and $\omega := \pi^\xi(\lambda)$, where π^ξ is the orthogonal projection onto Π^ξ. Let N be a Borel subset of Π^ξ, with $\mathcal{H}^{n-1}(N) = 0$, such that the singular part of ω with respect to $\mathcal{H}^{n-1} \llcorner \Pi^\xi$ is concentrated on N, and let $g\colon \Pi^\xi \to \mathbb{R}^+$ be the density of the absolutely continuous part of ω with respect to $\mathcal{H}^{n-1} \llcorner \Pi^\xi$. By the disintegration theorem (see, e.g., [6, Theorem 2.28]) there exists a Borel measurable family $(\lambda^\xi_y)_{y\in\Pi^\xi}$ of Radon measures, with $\lambda^\xi_y \in \mathcal{M}^+_b(\Omega^\xi_y)$, such that

$$\lambda(B) = \int_{\Pi^\xi} \lambda^\xi_y(B^\xi_y)\,d\omega(y) \tag{3.20}$$

for every Borel set $B \subset \Omega$.

By (a) and by Proposition 3.1 the function w^ξ_y belongs to $BV(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Given a Borel set $A \subset \Pi^\xi$ and an interval $I \subset \mathbb{R}$, by (3.3), (3.13), and (3.20), applied to the set $\{y + t\xi : y \in A,\ t \in I \cap \Omega^\xi_y\}$, we have

$$\int_A |Dw^\xi_y|(I\cap\Omega^\xi_y)\,d\mathcal{H}^{n-1}(y) = \int_{A\setminus N} |Dw^\xi_y|(I\cap\Omega^\xi_y)\,d\mathcal{H}^{n-1}(y) \leq \int_A \lambda^\xi_y(I\cap\Omega^\xi_y)\,g(y)\,d\mathcal{H}^{n-1}(y)\,.$$

It follows that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi_y$ we have

$$|Dw^\xi_y|(B) \leq g(y)\,\lambda^\xi_y(B) \tag{3.21}$$

for every Borel set $B \subset \Omega^\xi_y$.

Since $w^\xi_y = \tau(v^\xi_y)$, from (3.21) we deduce that

$$\big|D\big(\tau(v^\xi_y)\big)\big|(B) \le g(y)\,\lambda^\xi_y(B) \tag{3.22}$$

for every Borel set $B\subset\Omega^\xi_y$, hence

$$\int_{\Omega^\xi_y} \tau(v^\xi_y)\, D\varphi\, dt \le g(y)\,\lambda^\xi_y(\operatorname{supp}\varphi) \tag{3.23}$$

for every $\varphi\in C^1_c(\Omega^\xi_y)$ with $|\varphi|\le 1$ on Ω^ξ_y. Since $\hat{\mathcal T}$ is countable, for $\mathcal H^{n-1}$-a.e. $y\in\Pi^\xi_y$ inequality (3.23) holds for all $\tau\in\hat{\mathcal T}$. From the density property of $\hat{\mathcal T}$ we conclude that for $\mathcal H^{n-1}$-a.e. $y\in\Pi^\xi_y$ inequality (3.23), and hence (3.22), holds for every $\tau\in\mathcal T$. Therefore Proposition 3.4 implies that $v^\xi_y\in BV_{loc}(\Omega^\xi_y)$ and

$$|Dv^\xi_y|(B\setminus J^1_{v^\xi_y}) + \mathcal H^0(B\cap J^1_{v^\xi_y}) \le g(y)\,\lambda^\xi_y(B) \tag{3.24}$$

for $\mathcal H^{n-1}$-a.e. $y\in\Pi^\xi$ and for every Borel set $B\subset\Omega^\xi_y$.

Integrating (3.24) over Π^ξ we obtain

$$\begin{aligned}\int_{\Pi^\xi}\big(|Dv^\xi_y|(B^\xi_y\setminus J^1_{v^\xi_y}) + \mathcal H^0(B^\xi_y\cap J^1_{v^\xi_y})\big)\, d\mathcal H^{n-1}(y) \le\\ \le \int_{\Pi^\xi} g(y)\,\lambda^\xi_y(B^\xi_y)\, d\mathcal H^{n-1}(y) \le \lambda(B)\,,\end{aligned}$$

where we have used (3.20) in the last line. This concludes the proof of (3.14) and of the implication (a) $\Rightarrow$ (b).

Assume now (b) and let μ^ξ_y and μ^ξ be the measures introduced in Definition 3.7. We fix $\tau\in\mathcal T$ and we set $w:=\tau(v)$. Then for $\mathcal H^{n-1}$-a.e. $y\in\Pi^\xi$ the function w^ξ_y belongs to $BV(\Omega^\xi_y)$. Since $0\le\tau'\le 1$ we have $|Dw^\xi_y|(B)\le|Dv^\xi_y|(B)$ for every Borel set $B\subset\Omega^\xi_y$. Since $|\tau\big((v^\xi_y)^+\big)-\tau\big((v^\xi_y)^-\big)|\le 1$ we have also $|Dw^\xi_y|(B)\le\mathcal H^0(B)$ for every Borel set $B\subset J_{v^\xi_y}$. Using (3.17) we obtain that

$$\begin{aligned}|Dw^\xi_y|(B\setminus J^1_{v^\xi_y}) &\le |Dv^\xi_y|(B\setminus J^1_{v^\xi_y}) = \mu^\xi_y(B\setminus J^1_{v^\xi_y})\,,\\ |Dw^\xi_y|(B\cap J^1_{v^\xi_y}) &\le \mathcal H^0(B\cap J^1_{v^\xi_y}) = \mu^\xi_y(B\cap J^1_{v^\xi_y})\end{aligned}$$

for every Borel set $B\subset\Omega^\xi_y$. It follows that $|Dw^\xi_y|(B)\le\mu^\xi_y(B)<+\infty$ for $\mathcal H^{n-1}$-a.e. $y\in\Pi^\xi$ and for every Borel set $B\subset\Omega^\xi_y$. By Proposition 3.1 and by (3.18) and (3.19) we have $D_\xi w\in\mathcal M_b(\Omega)$ and $|D_\xi w|(B)\le\mu^\xi(B)\le\lambda(B)$ for every Borel set $B\subset\Omega$. This proves (3.13) and concludes the proof of the implication (b) $\Rightarrow$ (a). ☐

The following theorem shows that the measure μ^ξ, which was defined by slicing, can also be obtained from the measures $D_\xi\big(\tau(v)\big)$ with $\tau\in\mathcal T$.

Theorem 3.8. *Let $v\colon\Omega\to\mathbb R$ be $\mathcal L^n$-measurable and let $\xi\in\mathbb R^n\setminus\{0\}$. Assume that Conditions (a) and (b) of Theorem 3.5 are satisfied, and let μ^ξ be the measure introduced in Definition 3.7. Then for every open set $U\subset\Omega$ we have*

$$\mu^\xi(U) := \sup_k\ \sup\sum_{i=1}^k |D_\xi\big(\tau_i(v)\big)|(U_i)\,, \tag{3.25}$$

where the second supremum is over all families $\tau_1,\dots,\tau_k$ of elements of $\mathcal T$ and all families $U_1,\dots,U_k$ of pairwise disjoint open subsets of U. In other words, μ^ξ coincides with the smallest measure λ such that (3.13) *holds for every $\tau\in\mathcal T$ and for every Borel set $B\subset\Omega$.*

Proof. In the proof of the implication (b) $\Rightarrow$ (a) in Theorem 3.5 we have already shown that $|D_\xi\big(\tau(v)\big)|(B)\le\mu^\xi(B)$ for every Borel set $B\subset\Omega$. This implies that the right-hand side of (3.25) is less than or equal to $\mu^\xi(U)$.

To prove the opposite inequality we fix an open set $U \subset \Omega$. By modifying v on a set of Lebesgue measure zero, we may assume that v is a Borel function on Ω and that $v^\xi_y \in BV_{loc}(\Omega^\xi_y)$ for every $y \in \Pi^\xi$. Let v^ξ_+ and v^ξ_- be the Borel functions defined by (3.16) and let $E := \{x \in \Omega : v^\xi_+(x) \neq v^\xi_-(x)\}$ and $F := \{x \in E : |v^\xi_+(x) - v^\xi_-(x)| \geq 1\}$. For every $y \in \Pi^\xi$ we have $(v^\xi_+)^\xi_y = (v^\xi_-)^\xi_y = v^\xi_y$ $\mathcal{L}^1$-a.e. in Ω^ξ_y by Lebesgue's differentiation theorem. By elementary properties of BV functions in dimension one, for every $y \in \Pi^\xi$ this implies

$$\lim_{\rho \to 0+} \frac{1}{2\rho} \int_{t-\rho}^{t+\rho} |v^\xi_y(s) - (v^\xi_+)^\xi_y(t)|\, ds = 0\,, \tag{3.26}$$

$$(v^\xi_y)^+(t) = (v^\xi_+)^\xi_y(t) \quad \text{and} \quad (v^\xi_y)^-(t) = (v^\xi_-)^\xi_y(t) \tag{3.27}$$

for every $t \in \Omega^\xi_y \setminus E^\xi_y$. This implies that $J_{v^\xi_y} = E^\xi_y$ and $J^1_{v^\xi_y} = F^\xi_y$ for every $y \in \Pi^\xi$.

For every $0 < \varepsilon < 1$ we can find three sequences of pairwise disjoint Borel sets A_i, B_i, C_i and six sequences of real numbers a^1_i, a^2_i, b^1_i, b^2_i, c^1_i, c^2_i such that

$$U \setminus E = \textstyle\bigcup_i A_i\,, \quad U \cap E \setminus F = \bigcup_i B_i\,, \quad U \cap F = \bigcup_i C_i \tag{3.28}$$

$$a^1_i < v^\xi_+(x) < a^2_i \quad \text{for every } x \in A_i\,, \tag{3.29}$$

$$b^1_i < \min\{v^\xi_+(x), v^\xi_-(x)\} < \max\{v^\xi_+(x), v^\xi_-(x)\} < b^2_i \quad \text{for every } x \in B_i\,, \tag{3.30}$$

$$\min\{v^\xi_+(x), v^\xi_-(x)\} < c^1_i < c^2_i < \max\{v^\xi_+(x), v^\xi_-(x)\} \quad \text{for every } x \in C_i\,, \tag{3.31}$$

$$a^1_i < a^2_i < a^1_i + 1\,, \quad b^1_i < b^2_i < b^1_i + 1\,, \quad c^1_i + 1 < c^2_i + \varepsilon\,. \tag{3.32}$$

By (3.28) we have

$$\mu^\xi(U) = \sum_{i=1}^\infty \mu^\xi(A_i) + \sum_{i=1}^\infty \mu^\xi(B_i) + \sum_{i=1}^\infty \mu^\xi(C_i)\,.$$

Let us fix a constant $\alpha < \mu^\xi(U)$. Then there exists an integer $k > 0$ such that

$$\alpha < \sum_{i=1}^k \mu^\xi(A_i) + \sum_{i=1}^k \mu^\xi(B_i) + \sum_{i=1}^k \mu^\xi(C_i)\,.$$

By standard approximation properties there exist pairwise disjoint compact sets $\hat{A}_1, \dots, \hat{A}_k$, $\hat{B}_1, \dots, \hat{B}_k$, $\hat{C}_1, \dots, \hat{C}_k$, with $\hat{A}_i \subset A_i$, $\hat{B}_i \subset B_i$, and $\hat{C}_i \subset C_i$, such that

$$\alpha < \sum_{i=1}^k \mu^\xi(\hat{A}_i) + \sum_{i=1}^k \mu^\xi(\hat{B}_i) + \sum_{i=1}^k \mu^\xi(\hat{C}_i)\,. \tag{3.33}$$

Let $\tilde{A}_1, \dots, \tilde{A}_k$, $\tilde{B}_1, \dots, \tilde{B}_k$, $\tilde{C}_1, \dots, \tilde{C}_k$ be pairwise disjoint open subsets of U with $\hat{A}_i \subset \tilde{A}_i$, $\hat{B}_i \subset \tilde{B}_i$, and $\hat{C}_i \subset \tilde{C}_i$.

By (3.32) for every i there exists $\rho_i \in \mathcal{T}$ such that $\rho'_i(s) = 1$ for every $s \in [a^1_i, a^2_i]$. Since $\hat{A}_i \subset A_i$ and $(A_i)^\xi_y \cap J_{v^\xi_y} = \emptyset$, from (3.17) and (3.29) we obtain, using Vol'pert's chain rule in BV (see [6, Theorem 3.96]) and (3.26),

$$\mu^\xi_y\big((\hat{A}_i)^\xi_y\big) = |Dv^\xi_y|\big((\hat{A}_i)^\xi_y\big) = \int_{(\hat{A}_i)^\xi_y} \rho'_i\big((v^\xi_+)^\xi_y\big)\, d|Dv^\xi_y| = \big|D\big(\rho_i(v^\xi_y)\big)\big|\big((\hat{A}_i)^\xi_y\big)$$

for every i and for every $y \in \Pi^\xi$. Integrating over Π^ξ and using Proposition 3.1 and (3.18) we obtain

$$\mu^\xi(\hat{A}_i) = \big|D_\xi\big(\rho_i(v)\big)\big|(\hat{A}_i) \leq \big|D_\xi\big(\rho_i(v)\big)\big|(\tilde{A}_i)\,. \tag{3.34}$$

By (3.32) for every i there exists $\sigma_i \in \mathcal{T}$ such that $|\sigma_i(s_2) - \sigma_i(s_1)| = |s_2 - s_1|$ for every s_1, $s_2 \in [b^1_i, b^2_i]$. Since $\hat{B}_i \subset B_i$ and $(B_i)^\xi_y \subset J_{v^\xi_y} \setminus J^1_{v^\xi_y}$, from (3.17) and (3.30) we obtain, using Vol'pert's chain rule in BV (see [6, Theorem 3.96]) and (3.27),

$$\mu^\xi_y\big((\hat B_i)^\xi_y\big) = \int_{(\hat B_i)^\xi_y} \big|(v^\xi_y)^+ - (v^\xi_y)^-\big|\, d\mathcal{H}^0 =$$

$$= \int_{(\hat B_i)^\xi_y} \big|\sigma_i\big((v^\xi_y)^+\big) - \sigma_i\big((v^\xi_y)^-\big)\big|\, d\mathcal{H}^0 = \big|D\big(\sigma_i(v^\xi_y)\big)\big|\big((\hat B_i)^\xi_y\big)$$

for every i and for every $y \in \Pi^\xi$. Integrating over Π^ξ and using Proposition 3.1 and (3.18) we obtain

$$\mu^\xi(\hat B_i) = \big|D_\xi\big(\sigma_i(v)\big)\big|(\hat B_i) \le \big|D_\xi\big(\sigma_i(v)\big)\big|(\tilde B_i)\,. \tag{3.35}$$

By (3.32) for every i there exists $\tau_i \in \mathcal{T}$ such that $\tau_i(s_2) - \tau_i(s_1) \ge 1 - \varepsilon$ for every $s_1 < c^1_i < c^2_i < s_2$. Since $\hat C_i \subset C_i$ and $(C_i)^\xi_y \subset J^1_{v^\xi_y}$, from (3.17) and (3.31) we obtain, using Vol'pert's chain rule in BV (see [6, Theorem 3.96]) and (3.27),

$$(1-\varepsilon)\, \mu^\xi_y\big((\hat C_i)^\xi_y\big) = (1-\varepsilon)\, \mathcal{H}^0\big((\hat C_i)^\xi_y\big) \le$$

$$\le \int_{(\hat C_i)^\xi_y} \big|\tau_i\big((v^\xi_y)^+\big) - \tau_i\big((v^\xi_y)^-\big)\big|\, d\mathcal{H}^0 = \big|D\big(\tau_i(v^\xi_y)\big)\big|\big((\hat C_i)^\xi_y\big)$$

for every i and for every $y \in \Pi^\xi$. Integrating over Π^ξ and using Proposition 3.1 and (3.18) we obtain

$$(1-\varepsilon)\, \mu^\xi(\hat C_i) = \big|D_\xi\big(\tau_i(v)\big)\big|(\hat C_i) \le \big|D_\xi\big(\tau_i(v)\big)\big|(\tilde C_i)\,. \tag{3.36}$$

By (3.33) and (3.34)-(3.36) we obtain

$$(1-\varepsilon)\, \alpha < \sum_{i=1}^k \big|D_\xi\big(\rho_i(v)\big)\big|(\tilde A_i) + \sum_{i=1}^k \big|D_\xi\big(\sigma_i(v)\big)\big|(\tilde B_i) + \sum_{i=1}^k \big|D_\xi\big(\tau_i(v)\big)\big|(\tilde C_i)\,.$$

This concludes the proof of (3.28), since $\alpha < \mu^\xi(U)$ and $0 < \varepsilon < 1$ are arbitrary. □

4. Definition and first properties

In this section we define the space $GBD(\Omega)$ of generalised functions of bounded deformation and the space $GSBD(\Omega)$ of generalised special functions of bounded deformation.

Definition 4.1. The space $GBD(\Omega)$ of *generalised functions of bounded deformation* is the space of all $\mathcal{L}^n$-measurable functions $u\colon \Omega \to \mathbb{R}^n$ with the following property: there exists $\lambda \in \mathcal{M}^+_b(\Omega)$ such that the following equivalent (see Theorem 3.5) conditions hold for every $\xi \in \mathbb{S}^{n-1}$:

(a) for every $\tau \in \mathcal{T}$ the partial derivative $D_\xi\big(\tau(u\cdot\xi)\big)$ belongs to $\mathcal{M}_b(\Omega)$ and its total variation satisfies

$$\big|D_\xi\big(\tau(u\cdot\xi)\big)\big|(B) \le \lambda(B) \tag{4.1}$$

for every Borel set $B \subset \Omega$;

(b) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat u^\xi_y := u^\xi_y \cdot \xi$ belongs to $BV_{loc}(\Omega^\xi_y)$ and

$$\int_{\Pi^\xi} \big(\, |D\hat u^\xi_y|(B^\xi_y \setminus J^1_{\hat u^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat u^\xi_y})\big)\, d\mathcal{H}^{n-1}(y) \le \lambda(B) \tag{4.2}$$

for every Borel set $B \subset \Omega$.

Definition 4.2. The space $GSBD(\Omega)$ of *generalised special functions of bounded deformation* is the set of all functions $u \in GBD(\Omega)$ such that for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat u^\xi_y := u^\xi_y \cdot \xi$ belongs to $SBV_{loc}(\Omega^\xi_y)$ (see [6, Section 4.1] for the definition of this space).

Remark 4.3. Arguing as in the proof of Proposition 3.4 we can prove that Condition (b) of Definition 4.1 is equivalent to (b′) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat{u}^\xi_y := u^\xi_y \cdot \xi$ belongs to $GBV(\Omega^\xi_y)$ and

$$\int_{\Pi^\xi} \big(|D\big(\sigma_a(\hat{u}^\xi_y)\big)|(B^\xi_y \setminus J^1_{\sigma_a(\hat{u}^\xi_y)}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\sigma_a(\hat{u}^\xi_y)})\big)\, d\mathcal{H}^{n-1}(y) \le \lambda(B)$$

for every Borel set $B \subset \Omega$ and for every $a > 0$,

where σ_a be the truncation function defined by $\sigma_a(t) = -a$ for $t \le -a$, $\sigma_a(t) = t$ for $-a \le t \le a$, and $\sigma_a(t) = a$ for $t \ge a$. For the same reason, Definition 4.2 does not change if $\hat{u}^\xi_y \in SBV_{loc}(\Omega^\xi_y)$ is replaced by $\hat{u}^\xi_y \in GSBV(\Omega^\xi_y)$ (see [6, Section ??] for the definition of this space).

Remark 4.4. When $n = 1$ the space $GBD(\Omega)$ reduces to $\{u \in BV_{loc}(\Omega) : |Du|(\Omega) < +\infty\}$ and $GSBD(\Omega)$ reduces to $\{u \in SBV_{loc}(\Omega) : |Du|(\Omega) < +\infty\}$. In the case $n > 1$, using the slicing theory for BV functions developed in [6, Section 3.11], we can prove that if $u \in [GBV(\Omega)]^n$ (see [6, Definition 4.26]) and u satisfies the natural estimate considered in [6, Theorem 4.40], then $u \in GBD(\Omega)$. A similar result holds for $[GSBV(\Omega)]^n$ and $GSBD(\Omega)$.

Remark 4.5. Let $u \in BD(\Omega)$. By the structure theorem for BD functions (see [5, Theorem 4.5]) for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have $\hat{u}^\xi_y \in BV(\Omega^\xi_y)$ and

$$\int_{\Pi^\xi} \big(|D\hat{u}^\xi_y|(B^\xi_y \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat{u}^\xi_y})\big)\, d\mathcal{H}^{n-1}(y) \le |Eu|(B)\,,$$

where Eu is the matrix-valued Radon measure defined by $(Eu)_{ij} := \frac{1}{2}(D_i u_j + D_j u_i)$. It follows that $BD(\Omega) \subset GBD(\Omega)$. Using [5, Proposition 4.7] we can also prove that $SBD(\Omega) \subset GSBD(\Omega)$. These inclusions are strict, as shown in Example 12.3.

Remark 4.6. Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be the truncation function defined by $\sigma(s) := \min\{|s|, 1\}$. Since

$$|D\hat{u}^\xi_y|(B \cap J_{\hat{u}^\xi_y}) = \int_{B \cap J_{\hat{u}^\xi_y}} |[\hat{u}^\xi_y]|\, d\mathcal{H}^0$$

for every Borel set $B \subset \Omega^\xi_y$, inequality (4.2) is equivalent to

$$\int_{\Pi^\xi} \Big(|D\hat{u}^\xi_y|(B^\xi_y \setminus J_{\hat{u}^\xi_y}) + \int_{B^\xi_y \cap J_{\hat{u}^\xi_y}} \sigma\big([\hat{u}^\xi_y]\big)\, d\mathcal{H}^0 \Big)\, d\mathcal{H}^{n-1}(y) \le \lambda(B)$$

for every Borel set $B \subset \Omega$. Using the fact that $|D\hat{u}^\xi_y|(\{t\}) = 0$ for every $t \in \Omega^\xi_y \setminus J_{\hat{u}^\xi_y}$, we can write the previous inequality as

$$\int_{\Pi^\xi} \Big(|D\hat{u}^\xi_y|(B^\xi_y \setminus J_{\xi,y}) + \int_{B^\xi_y \cap J_{\xi,y}} \sigma\big([\hat{u}^\xi_y]\big)\, d\mathcal{H}^0 \Big)\, d\mathcal{H}^{n-1}(y) \le \lambda(B)\,, \tag{4.3}$$

where $J_{\xi,y}$ is an arbitrary countable set containing $J_{\hat{u}^\xi_y}$ and $[\hat{u}^\xi_y](t) := 0$ for every $t \in \Omega^\xi_y \setminus J_{\hat{u}^\xi_y}$.

Since $\sigma(s + t) \le \sigma(s) + \sigma(t)$ and $\sigma(\rho s) \le \max\{|\rho|, 1\}\, \sigma(s)$ for every $s,\ t,\ \rho \in \mathbb{R}$, we deduce from Condition (b) of Definition 4.1 and from (4.3) that $GBD(\Omega)$ and $GSBD(\Omega)$ are vector subspaces of the vector space of all $\mathcal{L}^n$-measurable functions from Ω to $\mathbb{R}^n$.

Remark 4.7. For every $B \subset \Omega$, for every $\rho \in \mathbb{R}$, and for every $\xi \in \mathbb{R}^n \setminus \{0\}$ we have $\rho B^{\rho\xi}_y = B^\xi_y$. Moreover, for every $u\colon \Omega \to \mathbb{R}^n$ and for every $t \in \Omega^{\rho\xi}_y$ we have $\hat{u}^{\rho\xi}_y(t) = \rho\, \hat{u}^\xi_y(\rho t)$. It follows that, if $u \in GBD(\Omega)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, then $u^\xi_y \in BV_{loc}(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and the left-hand side of (4.2) is finite.

Definition 4.8. Let $u \in GBD(\Omega)$, let $\xi \in \mathbb{S}^{n-1}$, and let $y \in \mathbb{R}^n$ with $\hat{u}^\xi_y \in BV(\Omega^\xi_y)$, $|D\hat{u}^\xi_y|(\Omega^\xi_y) < +\infty$, and $\mathcal{H}^0(J^1_{\hat{u}^\xi_y}) < +\infty$. The measure $\hat{\mu}^\xi_y \in \mathcal{M}^+_b(\Omega^\xi_y)$ is defined by

$$\hat{\mu}^\xi_y(B) := |D\hat{u}^\xi_y|(B \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B \cap J^1_{\hat{u}^\xi_y}) \tag{4.4}$$

for every Borel set $B \subset \Omega^\xi_y$.

Remark 4.9. By Condition (b) of Definition 4.1 the measure $\hat\mu^\xi_y$ is defined for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. More in general, an easy change of variables shows that $\hat\mu^\xi_y$ is defined for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\eta$ for every $\eta \in \mathbb{S}^{n-1}$ with $\eta \cdot \xi \neq 0$.

Definition 4.10. Let $u \in GBD(\Omega)$ and let $\xi \in \mathbb{S}^{n-1}$. The measure $\hat\mu^\xi \in \mathcal{M}^+_b(\Omega)$ is defined by

$$\hat\mu^\xi(B) := \int_{\Pi^\xi} \hat\mu^\xi_y(B^\xi_y)\, d\mathcal{H}^{n-1}(y) \tag{4.5}$$

for every Borel set $B \subset \Omega$. We use the notation $(\hat\mu_u)^\xi_y$ and $\hat\mu^\xi_u$ when we want to underline the dependence on u.

Remark 4.11. If $\eta \in \mathbb{S}^{n-1}$ and $\eta \cdot \xi \neq 0$, an obvious change of variables shows that

$$\hat\mu^\xi(B) = |\eta \cdot \xi| \int_{\Pi^\eta} \hat\mu^\xi_y(B^\xi_y)\, d\mathcal{H}^{n-1}(y)$$

for every Borel set $B \subset \Omega$.

Remark 4.12. The measures $\hat\mu^\xi_y$ and $\hat\mu^\xi$ corresponding to $u \in GBD(\Omega)$ coincide with the measures μ^ξ_y and μ^ξ introduced in Definition 3.7 for the scalar function $v := u \cdot \xi$. By (4.2), (4.4), and (4.5) we have

$$\hat\mu^\xi(B) \leq \lambda(B) \tag{4.6}$$

for every Borel set $B \subset \Omega$.

Remark 4.13. Let $u \in BD(\Omega)$. By [5, Theorem 4.5] and by the area formula (see, e.g., [6, Theorem 2.71]) for every $\xi \in \mathbb{S}^{n-1}$ and for every Borel set $B \subset \Omega$ we have

$$\hat\mu^\xi_u(B) = |Eu\,\xi \cdot \xi|(B \setminus J_u) + \int_{B \cap J_u} \sigma([u]\cdot\xi)\, |\nu_u \cdot \xi|\, d\mathcal{H}^{n-1} \leq |Eu\,\xi \cdot \xi|(B)$$

where σ is the function introduced in Remark 4.6.

Remark 4.14. Let $u \in GBD(\Omega)$. For every $\tau \in \mathcal{T}$, for every open set $U \subset \Omega$, and for every $\varphi \in C^1_c(\Omega)$ the function

$$\xi \mapsto \int_U \tau(u\cdot\xi)\, \nabla\varphi \cdot \xi\, dx$$

is continuous on $\mathbb{S}^{n-1}$. Since

$$|D_\xi\big(\tau(u\cdot\xi)\big)|(U) = \sup_{\substack{\varphi \in C^1_c(\Omega)\\ |\varphi| \leq 1}} \int_U \tau(u\cdot\xi)\, \nabla\varphi \cdot \xi\, dx\,, \tag{4.7}$$

the function $\xi \mapsto |D_\xi\big(\tau(u\cdot\xi)\big)|(U)$ is lower semicontinuous on $\mathbb{S}^{n-1}$. By Theorem 3.8 and Remark 4.12 it follows that $\xi \mapsto \hat\mu^\xi(U)$ is lower semicontinuous on $\mathbb{S}^{n-1}$.

Remark 4.15. By standard properties of bounded measures, it is enough to check (4.1) and (4.2) when $B \in \mathcal{B}$, where $\mathcal{B}$ is a base for the topology of Ω and $\mathcal{B}$ is stable under finite unions and intersections. By the lower semicontinuity of $\xi \mapsto |D_\xi\big(\tau(u\cdot\xi)\big)|(U)$ when U is open (Remark 4.14), it is enough to check (4.1) for every ξ in a dense subset Ξ of $\mathbb{S}^{n-1}$. Since Conditions (a) and (b) of Definition 4.1 are equivalent for every ξ by Theorem 3.5, it is enough that one of them is satisfied for every $B \in \mathcal{B}$ and every $\xi \in \Xi$.

Definition 4.16. For every $u \in GBD(\Omega)$ let $\hat\mu_u \in \mathcal{M}^+_b(\Omega)$ be the measure defined by setting, for every Borel set $B \subset \Omega$,

$$\hat\mu_u(B) := \sup_k\, \sup \sum_{i=1}^k \hat\mu^{\xi_i}_u(B_i)\,, \tag{4.8}$$

where the second supremum is over all families $\xi_1,\dots,\xi_k$ of elements of $\mathbb{S}^{n-1}$ and over all families $B_1,\dots,B_k$ of pairwise disjoint Borel subsets of B.

By (4.4) and (4.5) for every $u\in GBD(\Omega)$ the measure $\hat\mu_u$ is the smallest measure λ that satisfies Condition (b) of Definition 4.1.

Proposition 4.17. *Let $u\in GBD(\Omega)$ and let $\lambda\in\mathcal{M}_b^+(\Omega)$ be the measure considered in Definition 4.1. Then for every Borel set $B\subset\Omega$ we have*

$$\hat\mu_u(B)\le\lambda(B)\,. \tag{4.9}$$

Moreover, if $\mathcal{H}^{n-1}(\pi^\xi(B))=0$ for $\mathcal{H}^{n-1}$-a.e. $\xi\in\mathbb{S}^{n-1}$, where π^ξ is the orthogonal projection onto Π^ξ, then $\hat\mu_u(B)=0$.

Proof. Inequality (4.9) follows from (4.6). To prove the second statement, we fix a Borel set $B_0\subset\Omega$. We consider the set $S_0:=\{\xi\in\mathbb{S}^{n-1}:\mathcal{H}^{n-1}(\pi^\xi(B_0))=0\}$ and we assume that $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}\setminus S_0)=0$. Let $\tilde\mu_u$ be the measure defined as in (4.8), with the constraint that $\xi_1,\dots,\xi_k$ are now elements of S_0. By (4.6) we have

$$\tilde\mu_u(B)\le\lambda(B) \tag{4.10}$$

for every Borel set $B\subset\Omega$.

Let $\tilde\lambda$ be the absolutely continuous part of λ with respect to $\tilde\mu_u$. From (4.10) we deduce that $\tilde\mu_u(B)\le\tilde\lambda(B)$ for every Borel set $B\subset\Omega$. Therefore the definition of $\tilde\mu_u$ gives $\mu_u^\xi(U)\le\tilde\lambda(U)$ for every $\xi\in S_0$ and for every open set $U\subset\Omega$. Since $\xi\mapsto\hat\mu_u^\xi(U)$ is lower semicontinuous on $\mathbb{S}^{n-1}$ by Remark 4.14 and S_0 is dense in $\mathbb{S}^{n-1}$, we conclude that $\mu_u^\xi(U)\le\tilde\lambda(U)$ for every $\xi\in\mathbb{S}^{n-1}$ and for every open set $U\subset\Omega$. It follows that $\mu_u^\xi(B)\le\tilde\lambda(B)$ for every $\xi\in\mathbb{S}^{n-1}$ and for every Borel set $B\subset\Omega$, which implies

$$\hat\mu_u(B)\le\tilde\lambda(B) \tag{4.11}$$

for every Borel set $B\subset\Omega$.

Since by (4.5)

$$\hat\mu^\xi(B):=\int_{\pi^\xi(B)}\hat\mu_y^\xi(B_y^\xi)\,d\mathcal{H}^{n-1}(y)\,,$$

we have $\hat\mu^\xi(B_0)=0$ for every $\xi\in S_0$. It follows that $\tilde\mu_u(B_0)=0$. As $\tilde\lambda$ is absolutely continuous with respect to $\tilde\mu_u$, we have also $\tilde\lambda(B_0)=0$. By (4.11) this gives $\hat\mu_u(B_0)=0$, which concludes the proof. ☐

In the proof of the rectifiability of J_u we need the following semicontinuity result.

Lemma 4.18. *Let u_k be a sequence in $GBD(\Omega)$ converging in $\mathcal{L}^n$-measure to a function $u\in GBD(\Omega)$. Then*

$$\hat\mu_u^\xi(U)\le\liminf_{k\to\infty}\hat\mu_{u_k}^\xi(U) \tag{4.12}$$

for every $\xi\in\mathbb{S}^{n-1}$ and for every open set $U\subset\Omega$.

Proof. For every $\varphi\in C_c^1(\Omega)$ the function

$$u\mapsto\int_U\tau(u\cdot\xi)\,\nabla\varphi\cdot\xi\,dx$$

is lower semicontinuous with respect to convergence in $\mathcal{L}^n$-measure. By (4.7) the function $u\mapsto|D_\xi\big(\tau(u\cdot\xi)\big)|(U)$ is lower semicontinuous. The conclusion follows now from Theorem 3.8 and Remark 4.12. ☐

The following theorem concerns k-dimensional slices. For every linear subspace V of $\mathbb{R}^n$ of dimension $k>0$ and for every bounded open set Ω_V in the relative topology of V, the space $GBD(\Omega_V)$ is defined as in Definition 4.1, with Ω replaced by Ω_V, $\mathbb{R}^n$ replaced by V, $\mathbb{S}^{n-1}$ replaced by $\mathbb{S}_V^{k-1}:=\mathbb{S}^{n-1}\cap V$, Π^ξ replaced by $\Pi_V^\xi:=\Pi^\xi\cap V$, $\mathcal{L}^n$ replaced by $\mathcal{H}^k\llcorner V$, and $\mathcal{H}^{n-1}$ replaced by $\mathcal{H}^{k-1}$.

Theorem 4.19. *Let V be a linear subspace of $\mathbb{R}^n$ of dimension $k > 0$, let $V^\perp$ be its orthogonal subspace, and let π_V be the orthogonal projection from $\mathbb{R}^n$ onto V. Given a function $u \in GBD(\Omega)$, for every $y \in V^\perp$ let $\Omega_y := \{z \in V : y + z \in \Omega\}$ and let $u_y \colon \Omega_y \to V$ be the function defined by $u_y(z) := \pi_V(u(y + z))$. Then $u_y \in GBD(\Omega_y)$ for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$.*

Proof. By Fubini's theorem the function $u_y \colon \Omega_y \to V$ is $\mathcal{H}^k$-measurable on Ω_y for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$. Let us prove that for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$ there exists $\hat\lambda_y \in \mathcal{M}_b^+(\Omega_y)$ such that u_y satisfies Condition (b) of Definition 4.1 on $\Omega_y \subset V$ for every $\xi \in V$.

We begin by observing that, if $\xi \in V$, then the hyperplane Π^ξ is the sum of the orthogonal subspaces $V^\perp$ and $\Pi^\xi_V := \Pi^\xi \cap V$, of dimension $n-k$ and $k-1$, respectively. Since $\xi \in V$, we have $u \cdot \xi = \pi_V(u) \cdot \xi$. This implies that $\hat u^\xi_{y+z} = \widehat{(u_y)}^\xi_z$ on $\Omega^\xi_{y+z} = (\Omega_y)^\xi_z$ for every $y \in V^\perp$ and for every $z \in \Pi^\xi_V$.

For every Borel set $B \subset \Omega$ and for every $y \in V^\perp$ we define $B_y := \{z \in V : y+z \in B\}$, so that $B^\xi_{y+z} = (B_y)^\xi_z$ for every $z \in V$. Let $\omega = \pi_V(\lambda)$, let N be a Borel subset of $V^\perp$, with $\mathcal{H}^{n-k}(N) = 0$, such that the singular part of ω with respect to $\mathcal{H}^{n-k} \llcorner V^\perp$ is concentrated on N, and let $g \colon V^\perp \to \mathbb{R}^+$ be the density of the absolutely continuous part of ω with respect to $\mathcal{H}^{n-k} \llcorner V^\perp$. By the disintegration theorem (see, e.g., [6, Theorem 2.28]) there exists a Borel measurable family $(\lambda_y)_{y \in V^\perp}$ of Radon measures, with $\lambda_y \in \mathcal{M}_b^+(\Omega_y)$, such that

$$\lambda(B) = \int_{V^\perp} \lambda_y(B_y)\, d\omega(y)\,. \tag{4.13}$$

for every Borel set $B \subset \Omega$.

Let us fix a countable dense subset Ξ of $\mathbb{S}^{k-1}_V := \mathbb{S}^{n-1} \cap V$. By Condition (b) of Definition 4.1 and by Fubini's theorem for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$ the functions $\hat u^\xi_{y+z} = (\hat u_y)^\xi_z$ belong to $BV_{loc}(\Omega^\xi_{y+z}) = BV_{loc}\big((\Omega_y)^\xi_z\big)$ for every $\xi \in \Xi$ and for $\mathcal{H}^{k-1}$-a.e. $z \in \Pi^\xi_V$.

Let $\mathcal{B}$ be a countable base for the topology of V such that $U_1 \cap U_2 \in \mathcal{B}$ for every U_1, $U_2 \in \mathcal{B}$. By Remark 4.15 to conclude the proof it is enough to show that for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$ the function u_y satisfies the analogue of (4.2) in V, with λ replaced by $\hat\lambda_y := g(y)\,\lambda_y$.

Given a Borel set $A \subset V^\perp$ and a open set $U \in \mathcal{B}$, we consider the Borel set $B \subset \Omega$ defined by $B := \{y+z : y \in A,\ z \in U \cap \Omega_y\}$. Let $\tilde A := A \setminus N$ and $\tilde B := \{y+z : y \in \tilde A,\ z \in U \cap \Omega_y\}$ By Fubini's theorem and by (4.4)-(4.6) and (4.13) we have

$$\begin{aligned}
&\int_A \Big(\int_{\Pi^\xi_V} \hat\mu^\xi_{y+z}\big(U^\xi_z \cap (\Omega_y)^\xi_z\big)\, d\mathcal{H}^{k-1}(z) \Big)\, d\mathcal{H}^{n-k}(y) = \\
&= \int_{\tilde A} \Big(\int_{\Pi^\xi_V} \hat\mu^\xi_{y+z}\big(U^\xi_z \cap (\Omega_y)^\xi_z\big)\, d\mathcal{H}^{k-1}(z) \Big)\, d\mathcal{H}^{n-k}(y) = \\
&= \int_{\Pi^\xi} \hat\mu^\xi_x(\tilde B^\xi_x)\, d\mathcal{H}^{n-1}(x) = \hat\mu^\xi(\tilde B) \le \lambda(\tilde B) = \\
&= \int_{\tilde A} \lambda_y(U \cap \Omega_y)\, d\omega(y) = \int_A \lambda_y(U \cap \Omega_y)\, g(y)\, \mathcal{H}^{n-k}(y)\,.
\end{aligned}$$

Since this inequality holds for every Borel set $A \subset V^\perp$ we conclude that for every $U \in \mathcal{B}$ we have

$$\int_{\Pi^\xi_V} \hat\mu^\xi_{y+z}\big(U^\xi_z \cap (\Omega_y)^\xi_z\big)\, d\mathcal{H}^{k-1}(z) \le g(y)\, \lambda_y(U \cap \Omega_y) \tag{4.14}$$

for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$. Since $\mathcal{B}$ is countable, we conclude that for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$ inequality (4.14) holds for every $U \in \mathcal{B}$. This shows that for $\mathcal{H}^{n-k}$-a.e. $y \in V^\perp$ the function u_y satisfies Condition (b) of Definition 4.1 on V for every $\xi \in \Xi$ and for every $B = U \in \mathcal{B}$, hence $u_y \in GBD(\Omega_V)$ by Remark 4.15. $\square$

5. Traces on regular submanifolds and on the boundary

The following theorem summarizes the known results on the traces of functions $v \in L^1(\Omega)$ satisfying $D_\xi v \in \mathcal{M}_b(\Omega)$ for some vector $\xi \in \mathbb{S}^{n-1}$.

Theorem 5.1. *Let U and V be open subsets of $\mathbb{R}^n$ of the form*

$$U := \{y + t\xi : y \in B,\ a < t < \psi(y)\} \quad \text{and} \quad V := \{y + t\xi : y \in B,\ a < t < b\}\,, \tag{5.1}$$

where $\xi \in \mathbb{S}^{n-1}$, B is a relatively open ball in Π^ξ, a, $b \in \mathbb{R}$, with $a < b$, and $\psi\colon \overline{B} \to (a,b)$ is Lipschitz continuous. Let $v \in L^1(\Omega)$ with $D_\xi v \in \mathcal{M}_b(\Omega)$, let

$$M := \{y + \psi(y)\xi : y \in B\}\,, \tag{5.2}$$

and let ν be the outer unit normal to M. Then there exists a functions $v_M \in L^1_{\mathcal{H}^{n-1}}(M)$ such that

$$\int_U v\, D_\xi\varphi\, dx + \int_U \varphi\, d(D_\xi v) = \int_M \varphi\, v_M\, \xi\cdot\nu\, d\mathcal{H}^{n-1} \tag{5.3}$$

for every $\varphi \in C^1_c(V)$. Moreover

$$\lim_{\rho\to 0+} \frac{1}{\rho^n} \int_{B_\rho(x)\cap U} |v(z) - v_M(x)|\, dz = 0 \tag{5.4}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in M$. Finally for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$v_M(y + \psi(y)\xi) = \operatorname*{ap\,lim}_{t\to\psi(y)-} v^\xi_y(t)\,. \tag{5.5}$$

Proof. The existence of $v_M \in L^1_{\mathcal{H}^{n-1}}(M)$ satisfying (5.3) follows from [31, Lemma 1.1]. The proof of (5.4) can be obtained by slight modifications of the arguments of [18, Theorem 5.3.2], where the use of the coarea formula can be avoided. Equality (5.5) can be easily deduced from [31, formula (1.17)]. ☐

We are now in a position to prove the main result about traces of functions $u \in GBD(\Omega)$ on a regular submanifold.

Theorem 5.2. *Let $u \in GBD(\Omega)$ and let $M \subset \Omega$ be a C^1 submanifold of dimension $n-1$ with unit normal ν. Then for $\mathcal{H}^{n-1}$-a.e. $x \in M$ there exist $u^+_M(x)$, $u^-_M(x) \in \mathbb{R}^n$ such that*

$$\operatorname*{ap\,lim}_{\substack{\pm(y-x)\cdot\nu(x)>0\\ y\to x}} u(y) = u^\pm_M(x)\,. \tag{5.6}$$

Moreover for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$u^\pm_M(y + t\xi)\cdot\xi = \operatorname*{ap\,lim}_{\substack{\sigma^\xi_y(t)(s-t)>0\\ s\to t}} \hat{u}^\xi_y(s) \quad \text{for every } t \in M^\xi_y\,, \tag{5.7}$$

where $\sigma\colon M \to \{-1,1\}$ is defined by $\sigma(x) := \operatorname{sign}(\xi\cdot\nu(x))$. Finally, the functions $u^\pm_M\colon M \to \mathbb{R}^n$ are $\mathcal{H}^{n-1}$-measurable.

Proof. It is enough to prove (5.6) in a neighbourhood of each point. For every $x_0 \in M$ there exist an open neighbourhood A of x_0, a vector $\xi_0 \in \mathbb{S}^{n-1}$, and a constant $0 < \varepsilon < 1$ such that for every $\xi \in \mathbb{S}^{n-1}$ with $|\xi - \xi_0| < \varepsilon$ we can represent $M \cap A$ as a Lipschitz graph in the direction determined by ξ:

$$M \cap A \subset \{y + \psi(y)\xi : y \in B\} \subset M\,,$$

where ψ, B, a, and b are as in Theorem 5.1. We may also assume that the set V defined by (5.1) is contained in Ω and that $\nu(x)\cdot\xi > 0$ for every $x \in M \cap V$. We set

$$A^- := A \cap U \qquad \text{and} \qquad A^+ := A \setminus (M \cup A^-)\,,$$

where U is defined in (5.1).

Given $\tau \in \mathcal{T}$ with $\tau'(t) > 0$ for every $t \in \mathbb{R}$, we define $v := \tau(u \cdot \xi)$. By Condition (a) of Definition 4.1 and by Theorem 5.1 for $\mathcal{H}^{n-1}$-a.e. $x \in M \cap A$ there exist two real numbers $v^+_M(x)$ and $v^-_M(x)$ such that (5.4) holds with $v_M(x)$ replaced by $v^\pm_M(x)$ and U replaced by $A^\pm$. This implies that

$$\operatorname*{ap\,lim}_{\substack{\pm(y-x)\cdot\nu(x)>0\\ y\to x}} \tau(u(y)\cdot\xi) = v^\pm_M(x)\,. \tag{5.8}$$

Moreover, by Theorem 5.1 the functions $v^\pm_M \colon M \cap A \to \mathbb{R}$ are $\mathcal{H}^{n-1}$-measurable.

By (5.5) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and for every $t \in M^\xi_y \cap A^\xi_y$ we have

$$v^\pm_M(y + t\xi) = \operatorname*{ap\,lim}_{s\to t\pm} v^\xi_y(s) = \tau\Big(\operatorname*{ap\,lim}_{s\to t\pm} \hat u^\xi_y(s)\Big)\,, \tag{5.9}$$

where the existence of the approximate limit of $\hat u^\xi_y$ follows from the fact that $\hat u^\xi_y \in BV_{loc}(\Omega^\xi_y)$ by Condition (b) of Definition 4.1. By (5.9) we have $v^\pm_M(x) \in \tau(\mathbb{R})$ for $\mathcal{H}^{n-1}$-a.e. $x \in M\cap A$. By inverting the function τ we obtain from (5.8) that for $\mathcal{H}^{n-1}$-a.e. $x \in M \cap A$ there exist two real numbers $u^+_{\xi,M}(x)$ and $u^-_{\xi,M}(x)$ such that

$$\operatorname*{ap\,lim}_{\substack{\pm(y-x)\cdot\nu(x)>0\\ y\to x}} u(y)\cdot\xi = u^\pm_{\xi,M}(x)\,. \tag{5.10}$$

Moreover, the functions $u^\pm_{\xi,M} \colon M \cap A \to \mathbb{R}$ are $\mathcal{H}^{n-1}$-measurable. Since there exists a basis of $\mathbb{R}^n$ composed of vectors $\xi \in \mathbb{S}^{n-1}$ with $|\xi - \xi_0| < \varepsilon$, equality (5.10) implies that for $\mathcal{H}^{n-1}$-a.e. $x \in M \cap A$ there exist two vectors $u^+_M(x)$ and $u^-_M(x) \in \mathbb{R}^n$ satisfying (5.6) and that the functions $u^\pm_M \colon M \cap A \to \mathbb{R}^n$ are $\mathcal{H}^{n-1}$-measurable.

Let us prove (5.7) for an arbitrary $\xi \in \mathbb{S}^{n-1}$. Since $\mathcal{H}^{n-1}\big(\pi^\xi(\{x \in M : \xi\cdot\nu(x) = 0\})\big) = 0$ by the area formula (see, e.g., [6, Theorem 2.91]), by localization we may assume that M can be represented as in (5.2) and that $\xi\cdot\nu(x) > 0$ for every $x \in M$. Let τ be as in the first part of the proof and let $U^\pm := \{y + t\xi : y \in B,\ a < t < b,\ \pm(t - \psi(y)) > 0\}$. By (5.6) we have

$$\operatorname*{ap\,lim}_{\substack{y\in U^\pm\\ y\to x}} \tau(u(y)\cdot\xi) = \tau(u^\pm_M(x)\cdot\xi)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in M$. Since $\tau(u\cdot\xi)$ is bounded, this implies that

$$\lim_{\rho\to0+} \frac{1}{\rho^n}\int_{B_\rho(x)\cap U^\pm} |\tau(u(y)\cdot\xi) - \tau(u^\pm_M(x)\cdot\xi)|\,dy = 0$$

By Theorem 5.1, applied to $v := \tau(u\cdot\xi)$, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$\tau\big(u^\pm_M(y + t\xi)\cdot\xi\big) = \operatorname*{ap\,lim}_{s\to t\pm} \tau\big(\hat u^\xi_y(s)\big) \quad \text{for every } t \in M^\xi_y\,.$$

By inverting the function τ we obtain that (5.7) holds for $\mathcal{H}^{n-1}$-a.e. $x \in M$. □

Definition 5.3. Let $u \in GBD(\Omega)$ and let $M \subset \Omega$ be a C^1 submanifold of dimension $n-1$ with normal ν. The $\mathbb{R}^n$-valued $\mathcal{H}^{n-1}$-measurable functions u^+_M and u^-_M, defined $\mathcal{H}^{n-1}$-a.e. on M and satisfying (5.6), are called the *traces* of u on the two sides of M.

Remark 5.4. Let $u \in GBD(\Omega)$ and let $M \subset \Omega$ be a C^1-manifold of dimension $n-1$ with normal ν. By (5.6) we have $\{x \in M : u^+_M(x) \neq u^-_M(x)\} \subset J_u \cap M$ and

$$\mathcal{H}^{n-1}\big(J_u \cap M \setminus \{x \in M : u^+_M(x) \neq u^-_M(x)\}\big) = 0\,.$$

Moreover

$$(u^+(x), u^-(x), \nu_u(x)) = (u^+_M(x), u^-_M(x), \nu(x)) \quad \text{or} \quad = (u^-_M(x), u^+_M(x), -\nu(x))$$

for $\mathcal{H}^{n-1}$-a.e. $x \in J_u \cap M$.

When Ω has a Lipschitz boundary we can consider also traces on the boundary.

Theorem 5.5. *Assume that* Ω *has a Lipschitz boundary and let* ν *be the outward unit normal. Then for every* $u \in GBD(\Omega)$ *and for* $\mathcal{H}^{n-1}$*-a.e.* $x \in \partial\Omega$ *there exist* $u_{\partial\Omega}(x) \in \mathbb{R}^n$ *such that*

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in\Omega}} u(y) = u_{\partial\Omega}(x)\,. \tag{5.11}$$

Moreover for every $\xi \in \mathbb{S}^{n-1}$ *and for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \Pi^\xi$ *we have*

$$u_{\partial\Omega}(y+t\xi)\cdot\xi = \operatorname*{ap\,lim}_{\substack{\sigma^\xi_y(t)(s-t)>0\\ s\to t}} \hat{u}^\xi_y(s) \quad \text{for every } t \in (\partial\Omega)^\xi_y\,, \tag{5.12}$$

where $\sigma\colon \partial\Omega \to \{-1,1\}$ *is given by* $\sigma(x) := \operatorname{sign}(\xi\cdot\nu(x))$*. Finally, the function* $u_{\partial\Omega}\colon \partial\Omega \to \mathbb{R}^n$ *is* $\mathcal{H}^{n-1}$*-measurable.*

Proof. The proof is similar to the proof of Theorem 5.2, and therefore is omitted. ☐

Definition 5.6. Assume that Ω has a Lipschitz boundary. For every $u \in GBD(\Omega)$ the $\mathbb{R}^n$-valued $\mathcal{H}^{n-1}$-measurable function $u_{\partial\Omega}$, defined $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$ and satisfying (5.11), is called the *trace* of u on $\partial\Omega$.

6. Rectifiability of the jump set

In this section we prove that for every $u \in GBD(\Omega)$ the jump set J_u introduced in Definition 2.4 is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable according to [21, Section 3.2.14]. We recall that, by [21, Theorem 3.2.29], a set $E \subset \mathbb{R}^n$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable if and only if $\mathcal{H}^{n-1}$-almost all of E is contained in the union of a countable family of $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$ of class C^1.

To prove the rectifiability of J_u, for every $u \in GBD(\Omega)$ we consider the set

$$\Theta_u := \Big\{ x \in \Omega : \limsup_{\rho\to 0+} \frac{\hat{\mu}_u(B_\rho(x))}{\rho^{n-1}} > 0 \Big\}\,, \tag{6.1}$$

where $\hat{\mu}_u$ is the measure introduced in Definition 4.16.

Proposition 6.1. *Let* $u \in GBD(\Omega)$*. Then* Θ_u *is a countably* $(\mathcal{H}^{n-1}, n-1)$*-rectifiable Borel set.*

Proof. The proof is a variant of the proof of [23, Part II, Theorem 4.18]. By Fatou's lemma for every $\rho > 0$ the function $x \mapsto \hat{\mu}_u(B_\rho(x)\cap\Omega)$ is lower semicontinuous on Ω. Since the limsup can be computed by considering only rational numbers $\rho > 0$, we deduce that Θ_u is a Borel set.

To prove the rectifiability, for every $\varepsilon > 0$ we consider the Borel set

$$\Theta^\varepsilon_u := \Big\{ x \in \Omega : \limsup_{\rho\to 0+} \frac{\hat{\mu}_u(B_\rho(x))}{\rho^{n-1}} > \varepsilon \Big\}\,. \tag{6.2}$$

It is enough to show that Θ^ε_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable. By [21, Theorem 2.10.19] we have that

$$\varepsilon\,\mathcal{H}^{n-1}(B) \le \omega_n\,\hat{\mu}_u(B) \tag{6.3}$$

for every Borel set $B \subset \Theta^\varepsilon_u$. In particular $\mathcal{H}^{n-1}(\Theta^\varepsilon_u) < +\infty$ and we can apply Federer's structure theorem [21, Theorems 3.3.13 and 2.10.15] to obtain a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable Borel set $R \subset \Theta^\varepsilon_u$ such that

$$\mathcal{H}^{n-1}\big(\pi^\xi\big(\Theta^\varepsilon_u \setminus R\big)\big) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}\,.$$

By Proposition 4.17 we have $\hat{\mu}_u(\Theta^\varepsilon_u \setminus R) = 0$. Choosing $B = \Theta^\varepsilon_u \setminus R$ in (6.3) we obtain $\mathcal{H}^{n-1}(\Theta^\varepsilon_u \setminus R) = 0$. This proves that Θ^ε_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and concludes the proof of the proposition. ☐

We are now in a position to prove that the jump set of a function of $GBD(\Omega)$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable.

Theorem 6.2. *Let $u \in GBD(\Omega)$, let J_u be the jump set introduced in Definition 2.4, and let Θ_u be the set defined in* (6.1). *Then J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, $J_u \subset \Theta_u$, and $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$.*

Proof. To prove that $J_u \subset \Theta_u$, let us fix $x_0 \in J_u$. Up to a translation, we may assume that $x_0 = 0$ and that $u^-(0) = 0$. By Definition 2.4 there exist $a \in \mathbb{R}^n$, with $a \neq 0$, and $\nu \in \mathbb{S}^{n-1}$ such that

$$\operatorname*{ap\,lim}_{\substack{x\cdot\nu>0\\ x\to 0}} u(x) = a \quad\text{and}\quad \operatorname*{ap\,lim}_{\substack{x\cdot\nu<0\\ x\to 0}} u(x) = 0\,. \tag{6.4}$$

Let $r > 0$ such that $B_r(0) \subset \Omega$. For every $0 < \rho < r$ we define $u_\rho\colon B_1(0) \to \mathbb{R}^n$ by setting $u_\rho(y) := u(\rho y)$ for every $y \in B_1(0)$. By a change of variables is easy to see that $u_\rho \in GBD(B_1(0))$ and that

$$\hat\mu^\xi_{u_\rho}(B_1(0)) = \frac{\hat\mu^\xi_u(B_\rho(0))}{\rho^{n-1}} \tag{6.5}$$

for every $\xi \in \mathbb{S}^{n-1}$ and for every $0 < \rho < r$. By (6.4) $u_\rho \to u_0$ in $\mathcal{L}^n$-measure on $B_1(0)$, where $u_0(x) = a$ for $x\cdot\nu > 0$ and $u_0(x) = 0$ for $x \cdot \nu < 0$. Let us fix $\xi \in \mathbb{S}^{n-1}$ such that $\nu\cdot\xi \neq 0$ and $0 < |a\cdot\xi| < 1$. By Remark 4.13 we have $\hat\mu^\xi_{u_0}(B_1(0)) = \omega_{n-1}|\nu\cdot\xi|\,|a\cdot\xi|$. Therefore (4.8) and (6.5), together with Lemma 4.18, give that

$$0 < \omega_{n-1}|\nu\cdot\xi|\,|a\cdot\xi| \le \liminf_{\rho\to 0+} \frac{\hat\mu^\xi_u(B_\rho(0))}{\rho^{n-1}} \le \liminf_{\rho\to 0+} \frac{\hat\mu_u(B_\rho(0))}{\rho^{n-1}}\,.$$

This implies that $0 \in \Theta_u$ by (6.1), and concludes the proof of the inclusion $J_u \subset \Theta_u$.

Since Θ_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable by Proposition 6.1, the rectifiability of J_u follows from the inclusion $J_u \subset \Theta_u$.

Let us prove that $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$. It suffices to show that $\mathcal{H}^{n-1}(\Theta^\varepsilon_u \setminus J_u) = 0$ for every $\varepsilon > 0$, where Θ^ε_u is the set defined in (6.2). By (6.3) it is enough to prove that $\hat\mu_u(\Theta^\varepsilon_u \setminus J_u) = 0$, and by (4.8) we have to show that

$$\hat\mu^\xi_u(\Theta^\varepsilon_u \setminus J_u) = 0 \tag{6.6}$$

for every $\xi \in \mathbb{S}^{n-1}$.

Let us fix $\xi \in \mathbb{S}^{n-1}$. Since Θ^ε_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, we can write

$$\Theta^\varepsilon_u \setminus J_u = N_0 \cup \bigcup_{i=1}^\infty N_i\,, \tag{6.7}$$

with $\mathcal{H}^{n-1}(N_0) = 0$ and $N_i \subset M_i$ for every $i \ge 1$, where each M_i is a C^1 manifold of dimension $n-1$ with normal unit vector ν_i. We define

$$M^\pm_i := \{x \in M_i : \pm\xi\cdot\nu_i(x) > 0\} \quad\text{and}\quad M^0_i := \{x \in M_i : \xi\cdot\nu_i(x) = 0\}\,. \tag{6.8}$$

Therefore $N_i = N^+_i \cup N^-_i \cup N^0_i$, where $N^\pm_i := N_i \cap M^\pm_i$ and $N^0_i := N_i \cap M^0_i$. Since $\mathcal{H}^{n-1}(N_0) = 0$ we have $\mathcal{H}^{n-1}(\pi^\xi(N_0)) = 0$. By the area formula (see, e.g., [6, Theorem 2.91]) and by (6.8) we have that $\mathcal{H}^{n-1}(\pi^\xi(N^0_i)) = 0$ for every i. Therefore (4.5) implies that $\hat\mu^\xi_u(N_0) = 0$ and $\hat\mu^\xi_u(N^0_i) = 0$ for every i. It follows from (6.7) that

$$\hat\mu^\xi_u(\Theta^\varepsilon_u \setminus J_u) \le \sum_{i=1}^\infty \hat\mu^\xi(N^+_i) + \sum_{i=1}^\infty \hat\mu^\xi(N^-_i)\,. \tag{6.9}$$

To prove (6.6) it is enough to show that for every i we have

$$\hat\mu^\xi_u(N^+_i) = 0 \quad\text{and}\quad \hat\mu^\xi_u(N^-_i) = 0\,. \tag{6.10}$$

Let us fix i and let u^+_i and u^-_i be the traces of u on M_i, oriented by ν_i. Splitting M^+_i into a countable number of pieces, we may assume that there exist an open set A in the relative topology of Π^ξ and a function $\psi \in C^1(A)$ such that $M^+_i = \{y + \psi(y)\xi : y \in A\}$.

By (6.8) we have $\xi\cdot\nu_i(x)>0$ for every $x\in M_i^+$. By Theorem 5.2 for $\mathcal{H}^{n-1}$-a.e. $y\in A$ we have

$$u_i^-(y+\psi(y)\xi)\cdot\xi=\operatorname*{ap\,lim}_{\substack{t\to\psi(y)\\ t<\psi(y)}}\hat{u}_y^\xi(t)\quad\text{and}\quad u_i^+(y+\psi(y)\xi)\cdot\xi=\operatorname*{ap\,lim}_{\substack{t\to\psi(y)\\ t>\psi(y)}}\hat{u}_y^\xi(t)\,. \tag{6.11}$$

Since $N_i^+\cap J_u=\emptyset$, we have $u_i^+(x)=u_i^-(x)$ for every $x\in N_i^+$ by Remark 5.4. By (6.11) we have

$$\operatorname*{ap\,lim}_{\substack{t\to\psi(y)\\ t<\psi(y)}}\hat{u}_y^\xi(t)=\operatorname*{ap\,lim}_{\substack{t\to\psi(y)\\ t>\psi(y)}}\hat{u}_y^\xi(t)$$

for $\mathcal{H}^{n-1}$-a.e. $y\in B:=\pi^\xi(N_i^+)$, hence $\psi(y)\notin J_{\hat{u}_y^\xi\cdot\xi}$ for $\mathcal{H}^{n-1}$-a.e. $y\in B$. Since $N_i^+=\{y+\psi(y)\xi:y\in B\}$ we have $(N_i^+)_y^\xi=\{\psi(y)\}$ for $y\in B$ and $(N_i^+)_y^\xi=\emptyset$ for $y\in\Pi^\xi\setminus B$. Therefore (4.4) and (4.5) give

$$\hat{\mu}_u^\xi(N_i^+)=\int_B|D\hat{u}_y^\xi|(\{\psi(y)\})\,d\mathcal{H}^{n-1}(y)=0\,,$$

since $\psi(y)\notin J_{\hat{u}_y^\xi}$ for $\mathcal{H}^{n-1}$-a.e. $y\in B$. A similar argument shows that $\hat{\mu}_u^\xi(N_i^-)=0$. This proves (6.10) and concludes the proof of the equality $\mathcal{H}^{n-1}(\Theta_u\setminus J_u)=0$. □

7. The jump points of the restriction to hyperplanes

In this section we prove a technical result that will play a crucial role in the proof of the slicing theorem for the jump set of a GBD function u: all jump points of the restriction of the function $\pi^\eta(u)$ to the hyperplane $x_0+\Pi^\eta$ belong to the set Θ_u introduced in (6.1), provided that $\mathcal{H}^{n-1}\big(S_u\cap(x_0+\Pi^\eta)\big)=0$.

A key tool in the proof is the following parallelogram identity, which holds for every function $v\colon\Omega\to\mathbb{R}^n$:

$$\begin{aligned}&v(x+h\xi)\cdot(\xi+\eta)-v(x-h\eta)\cdot(\xi+\eta)+\\&+v(x+h\eta)\cdot(\xi+\eta)-v(x-h\xi)\cdot(\xi+\eta)+\\&+v(x+h\xi)\cdot(\xi-\eta)-v(x+h\eta)\cdot(\xi-\eta)+\\&+v(x-h\eta)\cdot(\xi-\eta)-v(x-h\xi)\cdot(\xi-\eta)=\\&=2v(x+h\xi)\cdot\xi-2v(x-h\xi)\cdot\xi+\\&+2v(x+h\eta)\cdot\eta-2v(x-h\eta)\cdot\eta\end{aligned} \tag{7.1}$$

for every $x\in\Omega$, for every ξ, $\eta\in\mathbb{R}^n$, and for every $h>0$ such that $x\pm h\xi$, $x\pm h\eta\in\Omega$.

Theorem 7.1. *Let $u\in GBD(\Omega)$, let $x_0\in\Omega$, and let $\eta\in\mathbb{S}^{n-1}$. Assume that*

$$\mathcal{H}^{n-1}\big(S_u\cap(x_0+\Pi^\eta)\big)=0\,. \tag{7.2}$$

Let $v\colon(-x_0+\Omega)\cap\Pi^\eta\to\Pi^\eta$ be the function defined by $v(y):=\pi^\eta(\tilde{u}(x_0+y))$ for $\mathcal{H}^{n-1}$-a.e. $y\in(-x_0+\Omega)\cap\Pi^\eta$. Suppose that there exists $\nu\in\mathbb{S}^{n-1}\cap\Pi^\eta$ and b^+, $b^-\in\Pi^\eta$ such that

$$\lim_{\rho\to0}\frac{\mathcal{H}^{n-1}\big(\{y\in B_\rho(0)\cap\Pi^\eta:\pm y\cdot\nu>0,\ |v(y)-b^\pm|>\varepsilon\}\big)}{\rho^{n-1}}\,. \tag{7.3}$$

If $b^+\neq b^-$, then $x_0\in\Theta_u$.

Proof. It is not restrictive to consider only the case $x_0=0$. We assume, by contradiction, that $b^+\neq b^-$ and $0\notin\Theta_u$, and we fix $\xi\in\mathbb{S}^{n-1}\cap\Pi^\eta$ such that

$$|(b^+-b^-)\cdot\xi|\geq\tfrac12|b^+-b^-|\quad\text{and}\quad\nu\cdot\xi>0\,. \tag{7.4}$$

Let S be the set of all $s\in\mathbb{R}^+$ such that $y+s\eta\notin S_u$ and $y-s\eta\notin S_u$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\eta$. Then $0\in S$ and $\mathcal{L}^1(\mathbb{R}^+\setminus S)=0$ by Fubini's theorem, since $\mathcal{L}^n(S_u)=0$ by Remark 2.5.

For every $\rho > 0$ we set $B_\rho := B_\rho(0)$, $B^0_\rho := B_\rho \cap \Pi^\eta$, $B^{0\pm}_\rho := \{y \in B^0_\rho : \pm y \cdot \nu > 0\}$, and $A_\rho := B^0_{c\rho} = B_{c\rho} \cap \Pi^\eta$, with $0 < c < \nu \cdot \xi \leq 1$. It follows that

$$\rho\xi + A_\rho \subset B^{0+}_{2\rho} \quad \text{and} \quad -\rho\xi + A_\rho \subset B^{0-}_{2\rho} . \tag{7.5}$$

Since $0 \notin \Theta_u$, by (6.1) we have

$$\lim_{\rho \to 0+} \frac{\hat{\mu}_u(B_\rho)}{\rho^{n-1}} = 0 . \tag{7.6}$$

Let us fix $\varepsilon > 0$ such that $3\varepsilon < \frac{1}{2}|b^+ - b^-|$. By (7.4) for every $\rho > 0$ and for every $y \in A_\rho$ we have

$$\begin{aligned} 3\varepsilon &< |(b^+ - b^-) \cdot \xi| \leq \\ &\leq |b^+ \cdot \xi - v(y + \rho\xi) \cdot \xi| + |(v(y + \rho\xi) - v(y - \rho\xi)) \cdot \xi| + |v(y - \rho\xi) \cdot \xi - b^- \cdot \xi| \leq \\ &\leq |b^+ - v(y + \rho\xi)| + |(v(y + \rho\xi) - v(y - \rho\xi)) \cdot \xi| + |v(y - \rho\xi) - b^-| . \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{H}^{n-1}(A_\rho) \leq\, & \mathcal{H}^{n-1}\big(\{y \in A_\rho : |b^+ - v(y + \rho\xi)| > \varepsilon\}\big) + \\ &+ \mathcal{H}^{n-1}\big(\{y \in A_\rho : |(v(y + \rho\xi) - v(y - \rho\xi)) \cdot \xi| > \varepsilon\}\big) + \\ &+ \mathcal{H}^{n-1}\big(\{y \in A_\rho : |v(y - \rho\xi) - b^-| > \varepsilon\}\big) . \end{aligned} \tag{7.7}$$

To conclude the proof of the theorem it is enough to show that

$$\mathcal{H}^{n-1}(A_\rho) = o(\rho^{n-1}) \tag{7.8}$$

along a sequence converging to zero. Indeed, the definition of A_ρ gives $\mathcal{H}^{n-1}(A_\rho) = \omega_{n-1} c^{n-1} \rho^{n-1}$, which contradicts (7.8) and shows that the relations $b^+ \neq b^-$ and $0 \notin \Theta_u$ cannot be true simultaneously.

To estimate the first term of the right-hand side of (7.7), we use (7.5) and we obtain

$$\begin{aligned} &\mathcal{H}^{n-1}\big(\{y \in A_\rho : |b^+ - v(y + \rho\xi)| > \varepsilon\}\big) = \\ &= \mathcal{H}^{n-1}\big(\{x \in \rho\xi + A_\rho : |b^+ - v(x)| > \varepsilon\}\big) \leq \\ &\leq \mathcal{H}^{n-1}\big(\{x \in B^{0+}_{2\rho} : |b^+ - v(x)| > \varepsilon\}\big) . \end{aligned}$$

By (7.3) the last term is $o(\rho^{n-1})$, so that

$$\mathcal{H}^{n-1}\big(\{y \in A_\rho : |b^+ - v(y + \rho\xi)| > \varepsilon\}\big) = o(\rho^{n-1}) . \tag{7.9}$$

In the same way we prove that

$$\mathcal{H}^{n-1}\big(\{y \in A_\rho : |v(y - \rho\xi) - b^-| > \varepsilon\}\big) = o(\rho^{n-1}) . \tag{7.10}$$

It remains to estimate $\mathcal{H}^{n-1}\big(\{y \in A_\rho : |(v(y + \rho\xi) - v(y - \rho\xi)) \cdot \xi| > \varepsilon\}\big)$. Since $\xi \in \Pi^\eta$, we have $v(y) \cdot \xi = \tilde{u}(y) \cdot \xi$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\eta$, hence

$$\begin{aligned} &\mathcal{H}^{n-1}\big(\{y \in A_\rho : |(v(y + \rho\xi) - v(y - \rho\xi)) \cdot \xi| > \varepsilon\}\big) = \\ &= \mathcal{H}^{n-1}\big(\{y \in A_\rho : |(\tilde{u}(y + \rho\xi) - \tilde{u}(y - \rho\xi)) \cdot \xi| > \varepsilon\}\big) . \end{aligned} \tag{7.11}$$

By the parallelogram identity (7.1) we have

$$\begin{aligned} &\mathcal{H}^{n-1}\big(\{y \in A_\rho : |(\tilde{u}(y + \rho\xi) - \tilde{u}(y - \rho\xi)) \cdot \xi| > \varepsilon\}\big) \leq \\ &\quad \leq \mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y + \rho\eta) \cdot \eta - \tilde{u}(y - \rho\eta) \cdot \eta| > \tfrac{\varepsilon}{5}\}\big) + \\ &\quad + \mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y + \rho\xi) \cdot (\xi + \eta) - \tilde{u}(y - \rho\eta) \cdot (\xi + \eta)| > \tfrac{2\varepsilon}{5}\}\big) + \\ &\quad + \mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y + \rho\eta) \cdot (\xi + \eta) - \tilde{u}(y - \rho\xi) \cdot (\xi + \eta)| > \tfrac{2\varepsilon}{5}\}\big) + \\ &\quad + \mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y + \rho\xi) \cdot (\xi - \eta) - \tilde{u}(y + \rho\eta) \cdot (\xi - \eta)| > \tfrac{2\varepsilon}{5}\}\big) + \\ &\quad + \mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y - \rho\eta) \cdot (\xi - \eta) - \tilde{u}(y - \rho\xi) \cdot (\xi - \eta)| > \tfrac{2\varepsilon}{5}\}\big) . \end{aligned} \tag{7.12}$$

To estimate the first term in the right-hand side we fix $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} < \tau(t) < \frac{1}{2}$, $0 < \tau'(t) < 1$, and $\tau(-t) = -\tau(t)$ for every $t \in \mathbb{R}$. Since τ is increasing, we have

$$\begin{aligned} &\mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y+\rho\eta)\cdot\eta - \tilde{u}(y-\rho\eta)\cdot\eta| > \tfrac{\varepsilon}{5}\}\big) = \\ &= \mathcal{H}^{n-1}\big(\{y \in A_\rho : \big|\tau\big(\tilde{u}(y+\rho\eta)\cdot\eta - \tilde{u}(y-\rho\eta)\cdot\eta\big)\big| > \tau(\tfrac{\varepsilon}{5})\}\big)\,. \end{aligned} \tag{7.13}$$

Let $r\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection about Π^η:

$$r(x) := x - 2\,(x\cdot\eta)\,\eta$$

for every $x \in \mathbb{R}^n$. Let $\hat{\Omega} := \Omega \cap r(\Omega)$ and let $\varphi \in L^\infty(\hat{\Omega})$ be the function defined by

$$\varphi(x) := \tau\big(u(x)\cdot\eta - u(r(x))\cdot\eta\big)\,.$$

If $x \in \hat{\Omega} \setminus S_u$ and $r(x) \notin S_u$, then $x \notin S_\varphi$ and

$$\tilde{\varphi}(x) = \tau\big(\tilde{u}(x)\cdot\eta - \tilde{u}(r(x))\cdot\eta\big)\,. \tag{7.14}$$

For every $y \in \Pi^\eta$ we have

$$\varphi^\eta_y = \tau\big(\hat{u}^\eta_y - \check{u}^\eta_y\big) \quad \text{on } \hat{\Omega}^\eta_y\,, \tag{7.15}$$

where $\check{u}^\eta_y(t) := \hat{u}^\eta_y(-t) = u(y - t\eta)\cdot\eta$.

By Condition (b) of Definition 4.1 we have that $\hat{u}^\eta_y \in BV_{loc}\big(\hat{\Omega}^\eta_y\big)$ and $\check{u}^\eta_y \in BV_{loc}\big(\hat{\Omega}^\eta_y\big)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\eta$. By (7.15) this implies that $\varphi^\eta_y \in BV_{loc}\big(\hat{\Omega}^\eta_y\big)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\eta$. Since $0 < \tau' < 1$ and $-\frac{1}{2} < \tau < \frac{1}{2}$, arguing as in the proof of Proposition 3.4 we obtain from Vol'pert's chain rule in BV (see [6, Theorem 3.96]) that

$$\begin{aligned} |D\varphi^\eta_y|(B \setminus J_{\varphi^\eta_y}) &\le |D\hat{u}^\eta_y|(B \setminus J_{\hat{u}^\eta_y}) + |D\check{u}^\eta_y|(B \setminus J_{\check{u}^\eta_y})\,, \\ |D\varphi^\eta_y|(B \cap J_{\varphi^\eta_y}) &\le |D\hat{u}^\eta_y|(B \cap J_{\hat{u}^\eta_y} \setminus J^1_{\hat{u}^\eta_y}) + \mathcal{H}^0(B \cap J^1_{\hat{u}^\eta_y}) + \\ &\quad + |D\check{u}^\eta_y|(B \cap J_{\check{u}^\eta_y} \setminus J^1_{\check{u}^\eta_y}) + \mathcal{H}^0(B \cap J^1_{\check{u}^\eta_y}) \end{aligned}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\eta$ and for every Borel set $B \subset \hat{\Omega}^\eta_y$. By an easy change of variables we obtain from the previous inequalities and from (4.4)

$$\begin{aligned} |D\varphi^\eta_y|(B) &\le |D\hat{u}^\eta_y|(B \setminus J^1_{\hat{u}^\eta_y}) + \mathcal{H}^0(B \cap J_{\hat{u}^\eta_y}) + \\ &\quad + |D\hat{u}^\eta_y|((-B) \setminus J^1_{\hat{u}^\eta_y}) + \mathcal{H}^0((-B) \cap J_{\hat{u}^\eta_y}) \le 2\,\mu^\eta_y(B \cup (-B))\,. \end{aligned} \tag{7.16}$$

Integrating on Π^η we get

$$\int_{\Pi^\eta} |D\varphi^\eta_y|(\hat{\Omega}^\eta_y)\, d\mathcal{H}^{n-1}(y) \le 2\int_{\Pi^\eta} \mu^\eta_y(\hat{\Omega}^\eta_y)\, d\mathcal{H}^{n-1}(y) = 2\,\mu^\eta(\hat{\Omega}) < +\infty\,,$$

so that $D_\eta\varphi \in \mathcal{M}_b(\hat{\Omega})$ by Proposition 3.1.

Let us fix $\rho \in S$ with $B_{2\rho} \subset \Omega$. Since $y \notin S_u$ and $y \pm \rho\eta \notin S_u$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\eta$, while $y \pm \rho\eta \in B_{2\rho} \subset \Omega$ for every $y \in A_\rho$, we have $y \in \hat{\Omega} \setminus S_\varphi$ and $y \pm \rho\eta \in \hat{\Omega} \setminus S_\varphi$ for $\mathcal{H}^{n-1}$-a.e. $y \in A_\rho$. Moreover $\tilde{\varphi}(y) = 0$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega \cap \Pi^\eta$ by (7.14). We can now apply Proposition 3.2 and we obtain

$$|\tilde{\varphi}(y+\rho\eta)| = |\tilde{\varphi}(y+\rho\eta) - \tilde{\varphi}(y)| \le (V\tilde{\varphi}^\eta_y)([0,\rho]) = |D\varphi^\eta_y|([0,\rho])$$

for $\mathcal{H}^{n-1}$-a.e. $y \in A_\rho$. Therefore (7.16) yields

$$|\tilde{\varphi}(y+\rho\eta)| \le 2\,\mu^\eta_y([-\rho,\rho]) \tag{7.17}$$

Integrating over A_ρ we get

$$\int_{A_\rho} |\tilde{\varphi}(y+\rho\eta)|\, d\mathcal{H}^{n-1}(y) \le 2\int_{A_\rho} \mu^\eta_y([-\rho,\rho])\, d\mathcal{H}^{n-1}(y)\,. \tag{7.18}$$

Since $y + t\eta \in B_{2\rho}$ for every $y \in A_\rho$ and for every $t \in [-\rho,\rho]$, by (4.5) we get

$$\int_{A_\rho} \mu^\eta_y([-\rho,\rho])\, d\mathcal{H}^{n-1}(y) \le \mu^\eta(B_{2\rho})\,. \tag{7.19}$$

From (4.8), (7.18), and (7.19) we deduce that

$$\int_{A_\rho} |\tilde\varphi(y+\rho\eta)|\, d\mathcal{H}^{n-1}(y) \le 2\,\mu^\eta(B_{2\rho}) \le 2\,\mu_u(B_{2\rho})\,. \tag{7.20}$$

Since $y\pm\rho\eta \in \Omega\setminus S_u$ for $\mathcal{H}^{n-1}$-a.e. $y\in A_\rho$, by (7.14) we have $\tau\big(\tilde u(y+\rho\eta)\cdot\eta - \tilde u(y-\rho\eta)\cdot\eta\big) = \tilde\varphi(y+\rho\eta)$ for $\mathcal{H}^{n-1}$-a.e. $y\in A_\rho$. Therefore (7.13), (7.20), and Chebyshev's inequality give

$$\begin{aligned}\mathcal{H}^{n-1}\big(\{y\in A_\rho : |\tilde u(y+\rho\eta)\cdot\eta - \tilde u(y-\rho\eta)\cdot\eta| > \tfrac{\varepsilon}{5}\}\big) = \\ = \mathcal{H}^{n-1}\big(\{y\in A_\rho : |\tilde\varphi(y+\rho\eta)| > \tau(\tfrac{\varepsilon}{5})\}\big) \le \\ \le \frac{1}{\tau(\frac{\varepsilon}{5})}\int_{A_\rho} |\tilde\varphi(y+\rho\eta)|\, d\mathcal{H}^{n-1}(y) \le \frac{2}{\tau(\frac{\varepsilon}{5})}\,\mu_u(B_{2\rho})\,.\end{aligned}$$

By (7.6) this implies that

$$\mathcal{H}^{n-1}\big(\{y\in A_\rho : |\tilde u(y+\rho\eta)\cdot\eta - \tilde u(y-\rho\eta)\cdot\eta| > \tfrac{\varepsilon}{5}\}\big) = o(\rho^{n-1}) \quad \text{for } \rho\in S\,.$$

To estimate the second term in the right-hand side of (7.12) we set $\omega := (\xi+\eta)/\sqrt2$ and we replace the reflection r by the involution

$$q(x) := x - 2\sqrt2\,(x\cdot\eta)\,\omega - \sqrt2\,\rho\,\omega = x - 2\,(x\cdot\eta)\,(\xi+\eta) - \rho\,(\xi+\eta)\,,$$

which leaves the hyperplane $-\frac{\rho}{2}\eta+\Pi^\eta$ fixed and moves all points in the direction determined by ω. We now define $\tilde\Omega := \Omega\cap q(\Omega)$ and $\psi\in L^\infty(\tilde\Omega)$ by

$$\psi(x) := \tau\big(u(x)\cdot(\xi+\eta) - u(q(x))\cdot(\xi+\eta)\big)\,.$$

If $x\in\tilde\Omega\setminus S_u$ and $q(x)\notin S_u$, then $x\notin S_\psi$ and

$$\tilde\psi(x) = \tau\big(\tilde u(x)\cdot(\xi+\eta) - \tilde u(q(x))\cdot(\xi+\eta)\big)\,. \tag{7.21}$$

For every $y\in -\frac{\rho}{2}\eta+\Pi^\eta$ we have $\psi^\omega_y = \tau\big(\sqrt2\,\hat u^\omega_y - \sqrt2\,\check u^\omega_y\big)$ on $\tilde\Omega^\omega_y$, where $\check u^\omega_y(t) := \hat u^\omega_y(-t) = u(y-t\omega)\cdot\omega$.

Arguing as in the previous step we obtain now

$$|D\psi^\omega_y|(B) \le 2\sqrt2\,\mu^\omega_y(B\cup(-B)) \tag{7.22}$$

for $\mathcal{H}^{n-1}$-a.e. $y\in -\frac{\rho}{2}\eta+\Pi^\eta$ and for every Borel set $B\subset\tilde\Omega^\omega_y$. Integrating on Π^ω and $-\frac{\rho}{2}\eta+\Pi^\eta$ we obtain, thanks to Remark 4.11,

$$\begin{aligned}\int_{\Pi^\omega} |D\psi^\omega_y|(\tilde\Omega^\omega_y)\, d\mathcal{H}^{n-1}(y) = \frac{1}{\sqrt2}\int_{-\frac{\rho}{2}\eta+\Pi^\eta} |D\psi^\omega_y|(\tilde\Omega^\omega_y)\, d\mathcal{H}^{n-1}(y) \le \\ \le 2\int_{-\frac{\rho}{2}\eta+\Pi^\eta} \mu^\omega_y(\tilde\Omega^\omega_y)\, d\mathcal{H}^{n-1}(y) = 2\sqrt2\,\mu^\omega(\tilde\Omega) < +\infty\,,\end{aligned}$$

which gives $D_\omega\psi\in\mathcal{M}_b(\tilde\Omega)$ thanks to Proposition 3.1.

Let us fix $\rho\in S$ with $\frac{\rho}{2}\in S$ and $B_{2\rho}\subset\Omega$. For every $y\in\Pi^\eta$ we define $a(y) := y+\frac{\rho}{2}(\xi-\eta)\in -\frac{\rho}{2}\eta+\Pi^\eta$, so that $y+\rho\xi = a(y)+\frac{\rho}{2}(\xi+\eta) = a(y)+\frac{\rho}{\sqrt2}\omega$ and $y-\rho\eta = a(y)-\frac{\rho}{2}(\xi+\eta) = a(y)-\frac{\rho}{\sqrt2}\omega$. Since $y+\rho\xi\notin S_u$, $y-\rho\eta\notin S_u$, and $a(y)\notin S_u$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\eta$, while $y+\rho\xi\in B_{2\rho}\subset\Omega$, $y-\rho\eta\in B_{2\rho}\subset\Omega$, and $a(y)\in B_{2\rho}\subset\Omega$ for every $y\in A_\rho$, we have $y+\rho\xi\in\tilde\Omega\setminus S_\psi$, $y-\rho\eta\in\tilde\Omega\setminus S_\psi$, and $a(y)\in\tilde\Omega\setminus S_\psi$ for $\mathcal{H}^{n-1}$-a.e. $y\in A_\rho$. Moreover $\tilde\psi(a(y)) = 0$ for $\mathcal{H}^{n-1}$-a.e. $y\in A_\rho$ by (7.21), since $q(a(y)) = a(y)$ for every $y\in\Pi^\eta$. We can now apply Proposition 3.2 and we obtain

$$|\tilde\psi(y+\rho\xi)| = |\tilde\psi(a(y)+\tfrac{\rho}{\sqrt2}\omega) - \tilde\psi(a(y))| \le (V\tilde\psi^\omega_{a(y)})([0,\tfrac{\rho}{\sqrt2}]) = |D\psi^\omega_{a(y)}|([0,\tfrac{\rho}{\sqrt2}])$$

for $\mathcal{H}^{n-1}$-a.e. $y\in A_\rho$. Therefore (7.22) yields

$$|\tilde\psi(y+\rho\xi)| \le 2\sqrt2\,\mu^\omega_{a(y)}([-\tfrac{\rho}{\sqrt2},\tfrac{\rho}{\sqrt2}]) \tag{7.23}$$

Integrating over A_ρ we get

$$\int_{A_\rho} |\tilde{\psi}(y+\rho\xi))|\, d\mathcal{H}^{n-1}(y) \le 2\sqrt{2}\int_{A_\rho} \mu^\omega_{a(y)}([-\tfrac{\rho}{\sqrt{2}}, \tfrac{\rho}{\sqrt{2}}])\, d\mathcal{H}^{n-1}(y)\,. \tag{7.24}$$

Since $a(y)+t\omega \in B_{2\rho}$ for every $y \in A_\rho$ and for every $t \in [-\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}]$, by Remark 4.11 we get

$$\int_{A_\rho} \mu^\omega_{a(y)}([-\tfrac{\rho}{\sqrt{2}}, \tfrac{\rho}{\sqrt{2}}])|\, d\mathcal{H}^{n-1}(y) \le \sqrt{2}\,\mu^\omega(B_{2\rho})\,. \tag{7.25}$$

From (4.8), (7.24), and (7.25) we deduce that

$$\int_{A_\rho} |\tilde{\psi}(y+\rho\xi)|\, d\mathcal{H}^{n-1}(y) \le 4\,\mu^\omega(B_{2\rho}) \le 4\,\mu_u(B_{2\rho})\,. \tag{7.26}$$

Since $y+\rho\xi \in \Omega\setminus S_u$ and $q(y+\rho\xi) = y-\rho\eta \in \Omega\setminus S_u$ for $\mathcal{H}^{n-1}$-a.e. $y \in A_\rho$, by (7.21) we have $\tau\big(\tilde{u}(y+\rho\xi)\cdot(\xi+\eta) - \tilde{u}(y-\rho\eta)\cdot(\xi+\eta)\big) = \tilde{\psi}(y+\rho\xi)$ for $\mathcal{H}^{n-1}$-a.e. $y \in A_\rho$. Therefore (7.26) and Chebyshev's inequality give

$$\mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y+\rho\xi)\cdot(\xi+\eta) - \tilde{u}(y-\rho\eta)\cdot(\xi+\eta)| > \tfrac{2\varepsilon}{5}\}\big) =$$

$$= \mathcal{H}^{n-1}\big(\{y \in A_\rho : \big|\tau\big(\tilde{u}(y+\rho\xi)\cdot(\xi+\eta) - \tilde{u}(y-\rho\eta)\cdot(\xi+\eta)\big)\big| > \tau(\tfrac{2\varepsilon}{5}\})\big) =$$

$$= \mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{\psi}(y+\rho\xi)| > \tau(\tfrac{2\varepsilon}{5})\}\big) \le$$

$$\le \frac{1}{\tau(\frac{2\varepsilon}{5})}\int_{A_\rho} |\tilde{\psi}(y+\rho\xi)|\, d\mathcal{H}^{n-1}(y) \le \frac{4}{\tau(\frac{\varepsilon}{5})}\,\mu_u(B_{2\rho})\,.$$

By (7.6) this impies that

$$\mathcal{H}^{n-1}\big(\{y \in A_\rho : |\tilde{u}(y+\rho\xi)\cdot(\xi+\eta) - \tilde{u}(y-\rho\eta)\cdot(\xi+\eta)| > \tfrac{\varepsilon}{5}\}\big) = o(\rho^{n-1})$$

for $\rho \in S$ with $\frac{\rho}{2} \in S$.

The other terms in the right-hand side of (7.12) can be estimated in a similar way. This proves (7.8) and concludes the proof of the theorem. □

8. Slicing of the jump set

In this section we prove that for every $u \in GBD(\Omega)$ the jump set J_u introduced in Definition 2.4 can be reconstructed from the jump sets of the one-dimensional slices $\hat{u}^\xi_y$.

Theorem 8.1. *Let $u \in GBD(\Omega)$, let $\xi \in \mathbb{S}^{n-1}$, and let*

$$J^\xi_u := \{x \in J_u : [u](x)\cdot\xi \ne 0\}\,. \tag{8.1}$$

Then for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$(J^\xi_u)^\xi_y = J_{\hat{u}^\xi_y}\,, \tag{8.2}$$

$$u^\pm(y+t\xi)\cdot\xi = (\hat{u}^\xi_y)^\pm(t) \quad \textit{for every } t \in (J_u)^\xi_y\,, \tag{8.3}$$

where the normals to J_u and $J_{\hat{u}^\xi_y}$ are oriented so that $\xi\cdot\nu_u \ge 0$ and $\nu_{\hat{u}^\xi_y} = 1$.

Proof. Let us prove that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$(J^\xi_u)^\xi_y \subset J_{\hat{u}^\xi_y}\,. \tag{8.4}$$

Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable by Theorem 6.2, we can write

$$J_u = N_0 \cup \bigcup_{i=1}^\infty N_i\,, \tag{8.5}$$

with $\mathcal{H}^{n-1}(N_0) = 0$ and $N_i \subset M_i$ for every $i \ge 1$, where each M_i is a C^1 manifold of dimension $n-1$ with normal unit vector ν_i. By Remark 5.4 we have $\nu_u = \pm\nu_i$ $\mathcal{H}^{n-1}$-a.e. on

N_i for every $i \geq 1$. Removing an $\mathcal{H}^{n-1}$-negligible set, we may assume that these equalities hold everywhere on N_i. Splitting, if needed, each N_i into two parts, we may also assume that the sign is constant in each N_i, and we may reorient the manifold M_i so that $\nu_u = \nu_i$ on N_i for every $i \geq 1$.

Let $M_i^\pm$ and M_i^0 be the sets defined in (6.8). Since $\xi \cdot \nu_i = \xi \cdot \nu_u \geq 0$ on N_i, we have $N_i = N_i^+ \cup N_i^0$, where $N_i^+ := N_i \cap M_i^+$ and $N_i^0. := N_i \cap M_i^0$. Since $\mathcal{H}^{n-1}(N_0) = 0$ we have $\mathcal{H}^{n-1}\big(\pi^\xi(N_0)\big) = 0$. By the area formula (see, e.g., [6, Theorem 2.91]) and by (6.8) we have that $\mathcal{H}^{n-1}\big(\pi^\xi(N_i^0)\big) = 0$ for every i. Let E_0 be the union of the sets $\pi^\xi(N_0)$ and $\pi^\xi(N_i^0)$ for $i \geq 1$. Then $\mathcal{H}^{n-1}(E_0) = 0$ and it is enough to prove (8.4) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi \setminus E_0$.

To obtain this result it suffices to show that for every $i \geq 1$ we have

$$(N_i^+)^\xi_y \cap (J_u^\xi)^\xi_y \subset J_{\hat{u}^\xi_y} \tag{8.6}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Let us fix $i \geq 1$ and let u_i^+ and u_i^- be the traces of u on M_i, oriented by ν_i. Splitting M_i^+ into a countable number of pieces, we may assume that there exist an open set A in the relative topology of Π^ξ and a function $\psi \in C^1(A)$ such that $M_i^+ = \{y + \psi(y)\xi : y \in A\}$. By (6.8) we have $\xi \cdot \nu_i(x) > 0$ for every $x \in M_i^+$. By Theorem 5.2 for $\mathcal{H}^{n-1}$-a.e. $y \in A$ we have

$$u_i^-(y + \psi(y)\xi) \cdot \xi = \operatorname*{ap\,lim}_{\substack{t \to \psi(y) \\ t < \psi(y)}} \hat{u}^\xi_y(t) \quad \text{and} \quad u_i^+(y + \psi(y)\xi) \cdot \xi = \operatorname*{ap\,lim}_{\substack{t \to \psi(y) \\ t > \psi(y)}} \hat{u}^\xi_y(t)\,. \tag{8.7}$$

By Remark 5.4 we have $u_i^+(x) \cdot \xi = u^+(x) \cdot \xi \neq u^-(x) \cdot \xi = u_i^-(x) \cdot \xi$ for $\mathcal{H}^{n-1}$-a.e. $x \in N_i^+ \cap J_u^\xi$. This inequality, together with (8.7), gives

$$\operatorname*{ap\,lim}_{\substack{t \to \psi(y) \\ t < \psi(y)}} \hat{u}^\xi_y(t) \neq \operatorname*{ap\,lim}_{\substack{t \to \psi(y) \\ t > \psi(y)}} \hat{u}^\xi_y(t)$$

for $\mathcal{H}^{n-1}$-a.e. $y \in B := \pi^\xi(N_i^+ \cap J_u^\xi)$, hence $\psi(y) \in J_{\hat{u}^\xi_y}$ for $\mathcal{H}^{n-1}$-a.e. $y \in B$. Since $N_i^+ \cap J_u^\xi = \{y + \psi(y)\xi : y \in B\}$ we have $(N_i^+)^\xi_y \cap (J_u^\xi)^\xi_y = \{\psi(y)\}$ for $y \in B$ and $(N_i^+)^\xi_y \cap (J_u^\xi)^\xi_y = \emptyset$ for $y \in \Pi^\xi \setminus B$. Therefore we have $(N_i^+)^\xi_y \cap (J_u^\xi)^\xi_y \subset J_{\hat{u}^\xi_y}$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. This proves (8.6) and concludes the proof of (8.4). Moreover (8.7), together with the equality $u_i^+(x) = u^+(x)$ and $u_i^-(x) = u^-(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in N_i^+$ (see Remark 5.4), proves (8.3) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

Let us prove that

$$J_{\hat{u}^\xi_y} \subset (J_u)^\xi_y \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi\,. \tag{8.8}$$

This inclusion is trivial for $n = 1$. We prove it by induction on the dimension n. By changing u on a set of Lebesgue measure zero, we may assume that u is a Borel function and that $\hat{u}^\xi_y \in BV_{loc}(\Omega^\xi_y)$ for every $y \in \Pi^\xi$. Since $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$ by Theorem 6.2, to prove (8.8) it is enough to show that

$$J_{\hat{u}^\xi_y} \subset (\Theta_u)^\xi_y \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi\,. \tag{8.9}$$

Let $n \geq 2$ and assume that (8.8) is true in dimension $n-1$. We fix $\eta \in \mathbb{S}^{n-1}$ with $\eta \cdot \xi = 0$. For every $s \in \mathbb{R}$ and for every $B \subset \Omega$ let $B_s := \{z \in \Pi^\eta : z + s\eta \in B\}$ and let $u_s \colon \Omega_s \to \Pi^\eta$ be the function defined by $u_s(z) := \pi^\eta\big(u(z + s\eta)\big)$. Then $(s, z) \mapsto u_s(z)$ is a Borel function on the open set $\tilde{\Omega} := \{(s, z) : s \in \mathbb{R}, z \in \Omega_s\} \subset \mathbb{R} \times \Pi^\eta$.

Let $\tilde{F} := \{(s, z) : s \in \mathbb{R},\ z \in J_{u_s}\} \subset \tilde{\Omega}$ and let $F := \{z + s\eta : s \in \mathbb{R},\ z \in J_{u_s}\} = \{z + s\eta : (s, z) \in \tilde{F}\} \subset \Omega$, so that

$$J_{u_s} = F_s := \{z \in \Pi^\eta : z + s\eta \in F\}\,. \tag{8.10}$$

Arguing as in the proof of [6, Proposition 3.69] and using Remark 2.2 we find that $\tilde{F}$ is a Borel subset of $\tilde{\Omega}$, hence F is a Borel subset of Ω.

Let $\Pi^{\eta\xi} := \Pi^\eta \cap \Pi^\xi$. Since $\eta\cdot\xi = 0$, we have that

$$\Pi^\xi = \{a + s\eta : a \in \Pi^{\eta\xi},\ s \in \mathbb{R}\} \tag{8.11}$$

and

$$B^\xi_{a+s\eta} = (B_s)^\xi_a \tag{8.12}$$

for every $a \in \Pi^{\eta\xi}$, for every $s \in \mathbb{R}$, and for every $B \subset \Omega$. Since $u\cdot\xi = \pi^\eta(u)\cdot\xi$, we have that

$$\hat{u}^\xi_{a+s\eta} = (\widehat{u_s})^\xi_a \quad \text{on} \quad \Omega^\xi_{a+s\eta} = (\Omega_s)^\xi_a \tag{8.13}$$

for every $a \in \Pi^{\eta\xi}$ and for every $s \in \mathbb{R}$.

For every $x \in \Omega$ we can define

$$\hat{u}^\xi_+(x) := \limsup_{\rho\to 0+} \frac{1}{\rho}\int_0^\rho u(x+s\xi)\cdot\xi\, ds \quad \text{and} \quad \hat{u}^\xi_-(x) := \limsup_{\rho\to 0+} \frac{1}{\rho}\int_{-\rho}^0 u(x+s\xi)\cdot\xi\, ds\,.$$

Since we are assuming that u is a Borel function, by Fubini's theorem $\hat{u}^\xi_+$ and $\hat{u}^\xi_-$ are Borel functions on Ω. Therefore $E := \{x \in \Omega : \hat{u}^\xi_+(x) \neq \hat{u}^\xi_-(x)\}$ is a Borel set. For every $y \in \Pi^\xi$ we have $(\hat{u}^\xi_+)^\xi_y = (\hat{u}^\xi_-)^\xi_y = \hat{u}^\xi_y$ $\mathcal{L}^1$-a.e. in Ω^ξ_y thanks to Lebesgue's differentiation theorem. By elementary properties of BV functions in dimension one, this implies that

$$J_{\hat{u}^\xi_y} = E^\xi_y \tag{8.14}$$

for every $y \in \Pi^\xi$.

By Theorem 4.19 there exists a Borel set $N_1 \subset \mathbb{R}$ such that for every $s \in \mathbb{R}\setminus N_1$ the function u_s belongs to $GBD(\Omega_s)$. Moreover, since $\mathcal{L}^n(S_u) = 0$ and $u = \tilde{u}$ $\mathcal{L}^n$-a.e. in Ω by Remark 2.5, using Fubini's theorem we find a Borel set $N_2 \subset \mathbb{R}$, with $\mathcal{L}^1(N_2) = 0$, such that for every $s \in \mathbb{R}\setminus N_2$ we have $\mathcal{H}^{n-1}\big(S_u \cap (s\eta + \Pi^\eta)\big) = 0$ and $u = \tilde{u}$ $\mathcal{H}^{n-1}$-a.e. in $s\eta + \Pi^\eta$. Let $N_0 := N_1 \cup N_2$.

By the inductive hypothesis for every $s \in \mathbb{R}\setminus N_0$ we have $J_{(\widehat{u_s})^\xi_a} \subset (J_{u_s})^\xi_a$ for $\mathcal{H}^{n-2}$-a.e. $a \in \Pi^{\eta\xi} := \Pi^\eta \cap \Pi^\xi$. By (8.10) and (8.12)-(8.14) we have

$$E^\xi_{a+s\eta} = J_{\hat{u}^\xi_{a+s\eta}} = J_{(\widehat{u_s})^\xi_a} \subset (J_{u_s})^\xi_a = (F_s)^\xi_a = F^\xi_{a+s\eta}$$

for every $s \in \mathbb{R}\setminus N_0$ and for $\mathcal{H}^{n-2}$-a.e. $a \in \Pi^{\eta\xi}$. By (8.11) and by Fubini's theorem there exists a Borel set $N \subset \Pi^\xi$, with $\mathcal{H}^{n-1}(N) = 0$, such that for every $y \in \Pi^\xi \setminus N$ we have $E^\xi_y \subset F^\xi_y$ and $y\cdot\eta \notin N_0$.

Let us fix $y \in \Pi^\xi \setminus N$ and let $t \in J_{\hat{u}^\xi_y}$. Then $y = a + s\eta$ with $a \in \Pi^{\eta,\xi}$ and $s \in \mathbb{R}\setminus N_0$. Therefore

$$J_{\hat{u}^\xi_y} = E^\xi_y \subset F^\xi_{a+s\eta} = (F_s)^\xi_a = (J_{u_s})^\xi_a$$

by (8.10), (8.12), and (8.14), so that $t \in (J_{u_s})^\xi_a$, hence $a + t\xi \in J_{u_s}$. Let $x_0 := y + t\xi = a + s\eta + t\xi$. Since $x_0 + \Pi^\eta = s\eta + \Pi^\eta$ and $s \notin N_2$, we have $\mathcal{H}^{n-1}\big(S_u \cap (s\eta + \Pi^\eta)\big) = 0$ and $u = \tilde{u}$ $\mathcal{H}^{n-1}$-a.e. in $s\eta + \Pi^\eta$. Therefore the function v considered in Theorem 7.1 satisfies $v(z) = u_s(z + a + t\xi)$ for $\mathcal{H}^{n-1}$-a.e. $z \in \Pi^\eta$. Since $a + t\xi \in J_{u_s}$, hypothesis (7.3) is satisfied with $b^+ \neq b^-$. Therefore Theorem 7.1 implies that $x_0 \in \Theta_u$. Since $x_0 := y + t\xi$ we have $t \in (\Theta_u)^\xi_y$. This proves (8.9) and concludes the proof of (8.8).

Let us prove that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$J_{\hat{u}^\xi_y} \subset (J^\xi_u)^\xi_y\,. \tag{8.15}$$

By (8.3) and (8.8) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and for every $t \in J_{\hat{u}^\xi_y}$ we have that $y + t\xi \in J_u$ and

$$(u^+(y+t\xi) - u^-(y+t\xi))\cdot\xi = [\hat{u}^\xi_y](t) \neq 0\,,$$

hence $y + t\xi \in J^\xi_u$ by (8.1). This proves (8.15) and concludes the proof of the theorem. □

9. Approximate symmetric differentiability

In this section we prove that every $u \in GBD(\Omega)$ has an *approximate symmetric gradient* $\mathcal{L}^n$-a.e. in Ω. This means that for $\mathcal{L}^n$-a.e. $x \in \Omega$ there exists a symmetric matrix, denoted by $\mathcal{E}u(x)$, such that

$$\operatorname*{ap\,lim}_{y\to x} \frac{\big(u(y)-u(x)-\mathcal{E}u(x)(y-x)\big)\cdot(y-x)}{|y-x|^2} = 0\,. \tag{9.1}$$

We also prove that the function $x \mapsto \mathcal{E}u(x)$, defined $\mathcal{L}^n$-a.e. in Ω, belongs to $L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$, where $\mathbb{M}^{n\times n}_{sym}$ is the space of symmetric $n\times n$ matrices. We also show that the one-dimensional slices of $\mathcal{E}u$ are related with the density $\nabla \hat u^\xi_y$ of the absolutely continuous part $D^a\hat u^\xi_y$ of the measure $D\hat u^\xi_y$ with respect to $\mathcal{L}^1$.

Theorem 9.1. *Let $u \in GBD(\Omega)$. Then there exists a function $\mathcal{E}u \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ such that* (9.1) *holds for $\mathcal{L}^n$-a.e. $x\in\Omega$. Moreover for every $\xi \in \mathbb{R}^n\setminus\{0\}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have*

$$(\mathcal{E}u)^\xi_y\,\xi\cdot\xi = \nabla \hat u^\xi_y \tag{9.2}$$

$\mathcal{L}^1$-a.e. on Ω^ξ_y.

Proof. Since the problem is local, we may assume that u has compact support in Ω. Let us fix $\xi \in \mathbb{R}^n\setminus\{0\}$. By modifying u on a set of Lebesgue measure zero, we may assume that u is a Borel function on Ω and $\hat u^\xi_y \in BV_{loc}(\Omega^\xi_y)$ for every $y\in\Pi^\xi$. For every $x\in\Omega$ we define

$$\hat u^\xi(x) := \limsup_{\rho\to 0+} \frac{1}{2\rho}\int_{-\rho}^{\rho} u(x+s\xi)\cdot\xi\,ds\,, \tag{9.3}$$

$$e^\xi(x) := \limsup_{\rho\to 0+} \frac{1}{\rho}\int_0^{\rho} \frac{\hat u^\xi(x+s\xi)-\hat u^\xi(x)}{s}\,ds\,. \tag{9.4}$$

Then u^ξ and e^ξ are Borel functions and have compact support on Ω. By an easy change of variables we can prove that

$$e^{\rho\xi}(x) = \rho^2 e^\xi(x) \tag{9.5}$$

for every $\rho>0$ and for every $x\in\Omega$

By the Lebesgue Differentiation Theorem for every $y\in\Pi^\xi$ we have

$$(\hat u^\xi)^\xi_y = \hat u^\xi_y \quad \mathcal{L}^1\text{-a.e. in } \Omega^\xi_y\,. \tag{9.6}$$

Since $\hat u^\xi_y \in BV(\Omega^\xi_y)$ and $(\hat u^\xi)^\xi_y$ is a good representative of $\hat u^\xi_y$ by (9.3), using well known properties of BV functions in dimension one (see, e.g., [6, Section 3.2]) we deduce that

$$(\nabla\hat u^\xi_y)(t) = \lim_{s\to 0}\frac{(\hat u^\xi)^\xi_y(t+s)-(\hat u^\xi)^\xi_y(t)}{s} = (e^\xi)^\xi_y(t) \tag{9.7}$$

for every $y\in\Pi^\xi$ and for $\mathcal{L}^1$-a.e. $t\in\Omega^\xi_y$.

Let $g\colon\mathbb{R}\to[0,1)$ be an even continuous function, with $g(0)=0$, such that g is strictly increasing and concave on $\mathbb{R}^+$. It is easy to prove that g satisfies the triangle inequality

$$g(s+t)\le g(s)+g(t) \tag{9.8}$$

for every s, $t\in\mathbb{R}$. By (9.6) and (9.7) we have

$$\lim_{\rho\to 0+}\frac{1}{\rho}\int_0^\rho g\Big(\frac{\hat u^\xi_y(t+s)-\hat u^\xi_y(t)}{s} - (e^\xi)^\xi_y(t)\Big)\,ds = 0$$

for every $y\in\Pi^\xi$ and for $\mathcal{L}^1$-a.e. $t\in\Omega^\xi_y$. By Fubini's theorem this implies that

$$\lim_{\rho\to 0+}\frac{1}{\rho}\int_0^\rho g\Big(\frac{u(x+s\xi)\cdot\xi - u(x)\cdot\xi}{s} - e^\xi(x)\Big)\,ds = 0 \tag{9.9}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$. Integrating over Ω and exchanging the order of integration we obtain

$$\lim_{\rho\to 0+} \frac{1}{\rho} \int_0^\rho \Big[\int_\Omega g\Big(\frac{u(x+s\xi)\cdot\xi - u(x)\cdot\xi}{s} - e^\xi(x) \Big)\, dx \Big]\, ds = 0 \tag{9.10}$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$. We define $e^\xi(x) = 0$ for $\xi = 0$. Note that (9.10) holds also in this case.

Let us fix $\eta \in \mathbb{R}^n$. By the triangle inequality (9.8) for every $s > 0$ small enough we have

$$\begin{aligned} &\int_\Omega g\Big(\frac{u(x+s\eta+s\xi)\cdot\xi - u(x+s\eta)\cdot\xi}{s} - e^\xi(x) \Big) dx \leq \\ &\leq \int_\Omega g\Big(\frac{u(x+s\eta+s\xi)\cdot\xi - u(x+s\eta)\cdot\xi}{s} - e^\xi(x+s\eta) \Big) dx + \int_\Omega g\big(e^\xi(x+s\eta) - e^\xi(x)\big) dx = \\ &= \int_\Omega g\Big(\frac{u(x+s\xi)\cdot\xi - u(x)\cdot\xi}{s} - e^\xi(x) \Big) dx + \int_\Omega g\big(e^\xi(x+s\eta) - e^\xi(x)\big) dx\,, \end{aligned} \tag{9.11}$$

where, in the last equality, we have used the fact that u and e^ξ have compact support in Ω.

Let us prove that

$$\lim_{s\to 0+} \int_\Omega g\big(e^\xi(x+s\eta) - e^\xi(x)\big)\, dx = 0\,. \tag{9.12}$$

Let us fix $\tau \in \mathcal{T}$ with $\tau'(t) > 0$ for every $t \in \mathbb{R}$. By the continuity of translations in $L^1(\Omega)$ we have

$$\lim_{s\to 0+} \int_\Omega \big|\tau\big(e^\xi(x+s\eta)\big) - \tau\big(e^\xi(x)\big)\big|\, dx = 0\,.$$

This implies that for every sequence $s_k \to 0$ there exists a subsequence s_{k_j} such that $\tau\big(e^\xi(x+s_{k_j}\eta)\big) \to \tau\big(e^\xi(x)\big)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$. Since τ is invertible and the inverse function is continuous, we deduce that $e^\xi(x+s_{k_j}\eta) \to e^\xi(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$. This implies that

$$\lim_{j\to\infty} \int_\Omega g\big(e^\xi(x+s_{k_j}\eta) - e^\xi(x)\big)\, dx = 0\,,$$

by the Dominated Convergence Theorem. Since the sequence $s_k \to 0$ is arbitrary, we obtain (9.12). That equality, together with (9.10) and (9.11), gives

$$\lim_{\rho\to 0+} \frac{1}{\rho} \int_0^\rho \Big[\int_\Omega g\Big(\frac{u(x+s\eta+s\xi)\cdot\xi - u(x+s\eta)\cdot\xi}{s} - e^\xi(x) \Big) dx \Big]\, ds = 0\,. \tag{9.13}$$

Let us fix ξ, $\eta \in \mathbb{R}^n$. We want to prove that the following parallelogram identity holds:

$$e^{\xi+\eta}(x) + e^{\xi-\eta}(x) = 2e^\xi(x) + 2e^\xi(x) \tag{9.14}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$. By the parallelogram identity (7.1), by the homogeneity condition (9.5), and by the triangle inequality (9.10) for every $s > 0$ small enough we have

$$\begin{aligned} &g(2e^{\xi+\eta}(x) + 2e^{\xi-\eta}(x) - 4e^\xi(x) - 4e^\xi(x)) \leq \\ &\leq g\big(e^{\xi+\eta}(x) - \frac{u(x+s\xi)\cdot(\xi+\eta) - u(x-s\eta)\cdot(\xi+\eta)}{s}\big) + \\ &\quad + g\big(e^{\xi+\eta}(x) - \frac{u(x+s\eta)\cdot(\xi+\eta) - u(x-s\xi)\cdot(\xi+\eta)}{s}\big) + \\ &\quad + g\big(e^{\xi-\eta}(x) - \frac{u(x+s\xi)\cdot(\xi-\eta) - u(x+s\eta)\cdot(\xi-\eta)}{s}\big) + \\ &\quad + g\big(e^{\xi-\eta}(x) - \frac{u(x-s\eta)\cdot(\xi-\eta) - u(x-s\xi)\cdot(\xi-\eta)}{s}\big) + \\ &\quad + g\big(e^{2\xi}(x) - \frac{u(x+s\xi)\cdot(2\xi) - u(x-s\xi)\cdot(2\xi)}{s}\big) + \\ &\quad + g\big(e^{2\eta}(x) - \frac{u(x+s\eta)\cdot(2\eta) - u(x-s\eta)\cdot(2\eta)}{s}\big)\,. \end{aligned}$$

Using (9.13) we obtain

$$\lim_{\rho\to 0+}\frac{1}{\rho}\int_0^\rho\Big[\int_\Omega g(2e^{\xi+\eta}(x)+2e^{\xi-\eta}(x)-4e^\xi(x)-4e^\xi(x))\,dx\Big]\,ds=0\,,$$

which implies

$$\int_\Omega g(2e^{\xi+\eta}(x)+2e^{\xi-\eta}(x)-4e^\xi(x)-4e^\xi(x))\,dx=0\,.$$

Since $g(s)=0$ if and only if $s=0$, we obtain (9.14).

Let $\mathbb{Q}$ be the field of rational numbers. By (9.14) there exists a Borel set $N\subset\Omega$, with $\mathcal{L}^n(N)=0$, such that for every $x\in\Omega\setminus N$ the paralleogram identity

$$e^{\xi+\eta}(x)+e^{\xi-\eta}(x)=2e^\xi(x)+2e^\xi(x) \tag{9.15}$$

holds for every ξ, $\eta\in\mathbb{Q}^n$. Since $e^\xi(x)$ is also positively homogeneous of degree 2 by (9.5), arguing as in the proof of [15, Proposition 11.9] we deduce that for every $x\in\Omega\setminus N$ there exists a symmetric $\mathbb{Q}$-bilinear form $B_x\colon\mathbb{Q}^n\times\mathbb{Q}^n\to\mathbb{R}$ such that

$$e^\xi(x)=B_x(\xi,\xi)$$

for every $\xi\in\mathbb{Q}^n$. This implies that for every $x\in\Omega\setminus N$ there exists a symmetric matrix $\mathcal{E}u(x)\in\mathbb{M}^{n\times n}_{sym}$ such that

$$e^\xi(x)=\mathcal{E}u(x)\,\xi\cdot\xi \tag{9.16}$$

for every $\xi\in\mathbb{Q}^n$.

Let us fix $\xi_0\in\mathbb{R}^n$. We want to prove that (9.16) holds for $\xi=\xi_0$ and for $\mathcal{L}^n$-a.e. $x\in\Omega$. Let Ξ be the vector subspace over $\mathbb{Q}$ generated by $\mathbb{Q}^n\cup\{\xi_0\}$. Since Ξ is countable, there exists a Borel set $N_0\subset\mathbb{R}^n$, with $N_0\supset N$ and $\mathcal{L}^n(N_0)=0$, such that (9.15) holds for every $x\in\Omega\setminus N_0$ and for every ξ, $\eta\in\Xi$. Arguing as before we prove that for every $x\in\Omega\setminus N_0$ there exists a symmetric matrix $A(x)\in\mathbb{M}^{n\times n}_{sym}$ such that

$$e^\xi(x)=A(x)\,\xi\cdot\xi \tag{9.17}$$

for every $\xi\in\Xi$. Since $\mathbb{Q}^n\subset\Xi$ and $N\subset N_0$, equalities (9.16) and (9.17) hold for every $x\in\Omega\setminus N_0$ and for every $\xi\in\mathbb{Q}^n$. This implies that $A(x)=\mathcal{E}u(x)$ for every $x\in\Omega\setminus N_0$. Since (9.17) holds for every $x\in\Omega\setminus N_0$ and for every $\xi\in\Xi$, we deduce that the same is true for (9.16). Since $\xi_0\in\Xi$, we conclude that (9.16) holds for $\xi=\xi_0$ and for every $x\in\Omega\setminus N_0$.

Since ξ_0 is arbitrary, we have shown that for every $\xi\in\mathbb{R}^n$ we have

$$e^\xi(x)=\mathcal{E}u(x)\,\xi\cdot\xi\quad \mathcal{L}^n\text{-a.e. in }\Omega\,. \tag{9.18}$$

By Fubini's theorem (9.18) gives

$$(e^\xi)^\xi_y(t)=(\mathcal{E}u)^\xi_y(t)\,\xi\cdot\xi$$

for every $\xi\in\mathbb{R}^n\setminus\{0\}$, for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$, and for $\mathcal{L}^1$-a.e. $t\in\Omega^\xi_y$. Together with (9.7), this property implies (9.2) for every $\xi\in\mathbb{R}^n\setminus\{0\}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$.

By (9.9) and (9.18) for every $\xi\in\mathbb{S}^{n-1}$ and for $\mathcal{L}^n$-a.e. $x\in\Omega$ we have

$$\lim_{\rho\to 0+}\frac{1}{\rho}\int_0^\rho g\Big(\frac{u(x+s\xi)\cdot(s\xi)-u(x)\cdot(s\xi)-\mathcal{E}u(x)\,(s\xi)\cdot(s\xi)}{s^2}\Big)\,ds=0\,.$$

This implies

$$\lim_{\rho\to 0+}\frac{1}{\rho^n}\int_0^\rho g\Big(\frac{u(x+s\xi)\cdot(s\xi)-u(x)\cdot(s\xi)-\mathcal{E}u(x)\,(s\xi)\cdot(s\xi)}{s^2}\Big)\,s^{n-1}\,ds=0\,.$$

Integrating over $\mathbb{S}^{n-1}$ and using the formula for polar coordinates we obtain

$$\begin{aligned}0&=\lim_{\rho\to 0+}\frac{1}{\rho^n}\int_{B_\rho(0)} g\Big(\frac{u(x+y)\cdot y-u(x)\cdot y-\mathcal{E}u(x)\,y\cdot y}{|y|^2}\Big)\,dy=\\&=\lim_{\rho\to 0+}\frac{1}{\rho^n}\int_{B_\rho(x)} g\Big(\frac{\big(u(y)-u(x)-\mathcal{E}u(x)\,(y-x)\big)\cdot(y-x)}{|y-x|^2}\Big)\,dy\,,\end{aligned}$$

which implies (9.1) by Chebyshev's inequality and by the properties of g.

It remains to prove that $\mathcal{E}u \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$. Let $\hat\mu^\xi_y$ and $\hat\mu^\xi$ be the measures introduced in Definitions 4.8 and 4.10. By (9.2) we have

$$\int_{\Omega^\xi_y} |(\mathcal{E}u)^\xi_y\,\xi\cdot\xi|\,dt = |D^a\hat u^\xi_y|(\Omega^\xi_y) \le \hat\mu^\xi_y(\Omega^\xi_y)$$

for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$. Integrating over Π^ξ and using Fubini's theorem and (4.6) for every $\xi\in\mathbb{S}^{n-1}$ we obtain

$$\int_\Omega |\mathcal{E}u\,\xi\cdot\xi|\,dx \le \hat\mu^\xi(\Omega) \le \hat\mu_u(\Omega) < +\infty\,,$$

where $\hat\mu_u$ is the measure introduced in Definition 4.16. This implies that $\mathcal{E}u\in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ and concludes the proof of the theorem. □

Remark 9.2. By the structure theorem for BD functions (see [5, Theorem 4.5]) and by Theorem 9.1, for every $u\in BD(\Omega)$ the function $\mathcal{E}u$ coincides with the density of the absolutely continuous part of Eu with respect to $\mathcal{L}^n$.

Remark 9.3. Let $\sigma\colon\mathbb{R}\to\mathbb{R}$ be the truncation function defined by $\sigma(s) := \min\{|s|,1\}$. By Theorems 8.1 and 9.1 for every $u\in GBD(\Omega)$ we have

$$|D^a\hat u^\xi_y|(B^\xi_y) + \int_{B^\xi_y\cap J_{\hat u^\xi_y}} \sigma\big([\hat u^\xi_y]\big)\,d\mathcal{H}^0 = \int_{B^\xi_y} |(\mathcal{E}u)^\xi_y\,\xi\cdot\xi|\,dt + \int_{B^\xi_y\cap(J^\xi_u)^\xi_y} \sigma\big([u]^\xi_y\cdot\xi\big)\,d\mathcal{H}^0$$

for every $\xi\in\mathbb{S}^{n-1}$, for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$, and for every Borel set $B\subset\Omega$. By the area formula (see, e.g., [6, Theorem 2.71]) and by Fubini's theorem it follows that for every $u\in GSBD(\Omega)$ the measure $\hat\mu^\xi_u$ defined by (4.5) satisfies

$$\hat\mu^\xi_u(B) = \int_B |\mathcal{E}u\,\xi\cdot\xi|\,dx + \int_{B\cap J_u} \sigma([u]\cdot\xi)\,|\nu_u\cdot\xi|\,d\mathcal{H}^{n-1} \le \int_B |\mathcal{E}u|\,dx + \mathcal{H}^{n-1}(B\cap J_u)$$

for every $\xi\in\mathbb{S}^{n-1}$ and for every Borel set $B\subset\Omega$. Therefore for every $u\in GSBD(\Omega)$ the measure $\hat\mu_u$ introduced in Definition 4.16 satisfies the estimate

$$\hat\mu_u(B) \le \int_B |\mathcal{E}u|\,dx + \mathcal{H}^{n-1}(B\cap J_u) \tag{9.19}$$

for every Borel set $B\subset\Omega$.

10. Compactness and slicing

In this section we prove some extensions of the well-known Fréchet-Kolmogorov compactness criterion in L^1. In particular we are interested in some conditions that imply sequential compactness with respect to $\mathcal{L}^n$-a.e. pointwise convergence. The main result is obtained by assuming suitable properties of the one-dimensional slices.

To simplify the exposition, in this section every function u defined on Ω is always extended to $\mathbb{R}^n$ by setting $u(x)=0$ for every $x\in\mathbb{R}^n\setminus\Omega$. These results are based on the notion of modulus of continuity, made precise by the following definition.

Definition 10.1. A *modulus of continuity* is an increasing continuous function $\omega\colon\mathbb{R}^+\to\mathbb{R}^+$ such that $\omega(0)=0$.

The first lemma provides a compactness result with respect to pointwise $\mathcal{L}^n$-a.e. convergence. Note that the usual L^1 bound is replaced by (10.3).

Lemma 10.2. *Let $\mathcal{U}$ be a set of $\mathcal{L}^n$-measurable functions from Ω into $\mathbb{R}^n$, let $g\colon\mathbb{R}^+\to\mathbb{R}^+$ be a nondecreasing continuous function satisfying*

$$g(0)=0 \quad\textit{and}\quad \liminf_{s\to0+}\frac{g(s)}{s} > 0\,, \tag{10.1}$$

and let $\psi_0\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing continuous function with*

$$\lim_{s\to+\infty} \psi_0(s) = +\infty\,. \tag{10.2}$$

Assume that there exist a constant $M \in \mathbb{R}^+$ *and a modulus of continuity* ω *such that*

$$\int_\Omega \psi_0(|u|)\,dx \le M\,, \tag{10.3}$$

$$\int_{\mathbb{R}^n} g\big(|u(x+h\xi)-u(x)|\big)\,dx \le \omega(h) \tag{10.4}$$

for every $u\in\mathcal{U}$, *for every* $\xi\in\mathbb{S}^{n-1}$, *and for every* $0<h<1$. *Then every sequence in* $\mathcal{U}$ *has a subsequence that converges pointwise* $\mathcal{L}^n$*-a.e. on* Ω *to an* $\mathcal{L}^n$*-measurable function* $u\colon\Omega\to\mathbb{R}^n$.

Proof. By (10.1) there exist $a>0$ and $r>0$ such that $as\le g(s)$ for every $0\le s\le 2r$. Let $\varphi\colon\mathbb{R}^n\to B_r(0)$ be the homeomorphism defined by

$$\varphi(z)=\frac{rz}{\sqrt{r^2+|z|^2}}\,. \tag{10.5}$$

There exists $c\in\mathbb{R}^+$ such that $|\varphi(z_2)-\varphi(z_1)|\le c|z_2-z_1|$ for every $z_1,z_2\in\mathbb{R}^n$. Therefore we have

$$\begin{aligned}\frac{a}{c}\int_{\mathbb{R}^n}|\varphi(u(x+h\xi))-\varphi(u(x))|\,dx &\le \int_{\mathbb{R}^n} g\big(|\varphi(u(x+h\xi))-\varphi(u(x))|\big)\,dx \le\\ &\le \int_{\mathbb{R}^n} g\big(c|u(x+h\xi)-u(x)|\big)\,dx\le\omega(ch)\end{aligned}$$

for every $u\in\mathcal{U}$, for every $\xi\in\mathbb{S}^{n-1}$, and for every $0<h<1$. By the Fréchet-Kolmogorov compactness criterion every sequence u_k in $\mathcal{U}$ has a subsequence, not relabelled, such that $v_k:=\varphi(u_k)$ converges strongly in $L^1(\Omega;\mathbb{R}^n)$ to a function $v\colon\Omega\to\overline{B}_r(0)$. Passing to a further subsequence we may assume that v_k converges to v pointwise $\mathcal{L}^n$-a.e. on Ω.

Let us prove that $|v(x)|<r$ for $\mathcal{L}^n$-a.e. $x\in\Omega$. Let $A:=\{x\in\Omega: |v(x)|=r\}$. By (10.5) we have

$$u_k=\frac{rv_k}{\sqrt{r^2-v_k^2}}\,, \tag{10.6}$$

so that $|u_k(x)|\to+\infty$ for $\mathcal{L}^n$-a.e. $x\in A$. By (10.2) this implies $\psi_0(|u_k(x)|)\to+\infty$ for $\mathcal{L}^n$-a.e. $x\in A$. By (10.3) and by Fatou's lemma we conclude that $\mathcal{L}^n(A)=0$, hence $|v(x)|<r$ for $\mathcal{L}^n$-a.e. $x\in\Omega$. By (10.6) we deduce that u_k converges pointwise $\mathcal{L}^n$-a.e. on Ω to the function

$$u=\frac{rv}{\sqrt{r^2-v^2}}\,.$$

This concludes the proof. ☐

The next lemma shows that in the Fréchet-Kolmogorov condition it is enough to consider only the components of a vector function along the translation vectors.

Lemma 10.3. *Let* $\mathcal{U}$ *be a set of* $\mathcal{L}^n$*-measurable functions from* Ω *into* $\mathbb{R}^n$, *let* $g\colon\mathbb{R}^+\to\mathbb{R}^+$ *be a nondecreasing continuous function such that*

$$g(0)=0 \quad\textit{and}\quad g(s+t)\le g(s)+g(t) \tag{10.7}$$

for every s, $t\in\mathbb{R}^+$, *and let* $\psi_0\colon\mathbb{R}^+\to\mathbb{R}^+$ *be an increasing continuous function satisfying* (10.2). *Assume that there exist a constant* $M\in\mathbb{R}^+$ *and two moduli of continuity* ω *and* $\hat\omega$ *such that* (10.3) *holds and*

$$\int_{\mathbb{R}^n} g\big(|u(x+h\xi)\cdot\xi-u(x)\cdot\xi|\big)\,dx\le\omega(h)\,, \tag{10.8}$$

$$g(hs)\le\hat\omega(h)\,\psi_0(s) \tag{10.9}$$

for every $u \in \mathcal{U}$, *for every* $\xi \in \mathbb{S}^{n-1}$, *for every* $0 < h < 1$, *and for every* $s \in \mathbb{R}^+$. *Then there exists a modulus of continuity* $\tilde{\omega}$ *such that*

$$\int_{\mathbb{R}^n} g\big(|u(x+h\xi) - u(x)|\big)\, dx \le \tilde{\omega}(h) \tag{10.10}$$

for every $u \in \mathcal{U}$, *for every* $\xi \in \mathbb{S}^{n-1}$, *and for every* $0 < h < 1$.

Proof. Let us fix $\xi \in \mathbb{S}^{n-1}$ and $0 < h < \frac{1}{2}$. There exist $\eta^1, \dots, \eta^{n-1} \in \mathbb{S}^{n-1}$ such that $\xi, \eta^1, \dots, \eta^{n-1}$ form an orthonormal basis. Then

$$|u(x+h\xi) - u(x)| \le |u(x+h\xi)\cdot\xi - u(x)\cdot\xi| + \sum_{i=1}^{n-1} |u(x+h\xi)\cdot\eta^i - u(x)\cdot\eta^i| \tag{10.11}$$

for every $x \in \mathbb{R}^n$. By the triangle inequality we have

$$|u(x+h\xi)\cdot\eta^i - u(x)\cdot\eta^i| \le |u(x+h\xi)\cdot\eta^i - u(x+\sqrt{h}\eta^i)\cdot\eta^i| + |u(x+\sqrt{h}\eta^i)\cdot\eta^i - u(x)\cdot\eta^i|\,. \tag{10.12}$$

Let

$$\eta^i_h := \frac{\sqrt{h}}{\sqrt{h+h^2}}\eta^i - \frac{h}{\sqrt{h+h^2}}\xi\,,$$

so that

$$|\eta^i_h| = 1\,, \quad h\xi - \sqrt{h}\eta^i = -s_h\eta^i_h\,, \text{ and } \quad |\eta^i_h - \eta^i| \le \sqrt{2}\sqrt{h}\,, \tag{10.13}$$

with $s_h := \sqrt{h+h^2}$. By the triangle inequality and by (10.13) we have

$$\begin{aligned} |u(x+h\xi)\cdot\eta^i - u(x+\sqrt{h}\eta^i)\cdot\eta^i| &\le |u(x+h\xi)|\,|\eta^i - \eta^i_h| + \\ + |u(x+h\xi)\cdot\eta^i_h - u(x+\sqrt{h}\eta^i)\cdot\eta^i_h| &+ |u(x+\sqrt{h}\eta^i_h)|\,|\eta^i_h - \eta^i| \le \\ \le |u(x+\sqrt{h}\eta^i - s_h\eta^i_h)\cdot\eta^i_h &- u(x+\sqrt{h}\eta^i)\cdot\eta^i_h| + \\ + |u(x+h\xi)|\sqrt{2}\sqrt{h} &+ |u(x+\sqrt{h}\eta^i_h)|\sqrt{2}\sqrt{h}\,. \end{aligned} \tag{10.14}$$

From (10.7), (10.8), (10.11), (10.12), and (10.14) for every $u \in \mathcal{U}$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} g\big(|u(x+h\xi) - u(x)|\big)\, dx \le \\ &\quad \le \int_{\mathbb{R}^n} g\big(|u(x+h\xi)\cdot\xi - u(x)\cdot\xi|\big)\, dx + \sum_{i=1}^{n-1}\int_{\mathbb{R}^n} g\big(|u(x-s_h\eta^i_h)\cdot\eta^i_h - u(x)\cdot\eta^i_h|\big)\, dx + \\ &\quad + 2(n-1)\int_{\mathbb{R}^n} g\big(|u(x)|\sqrt{2}\sqrt{h}\big)\, dx + \sum_{i=1}^{n-1}\int_{\mathbb{R}^n} g\big(|u(x+\sqrt{h}\eta^i)\cdot\eta^i - u(x)\cdot\eta^i|\big)\, dx \le \\ &\quad \le \omega(h) + (n-1)\,\omega(\sqrt{h+h^2}) + 2(n-1)\int_{\mathbb{R}^n} g\big(|u(x)|\sqrt{2}\sqrt{h}\big)\, dx + (n-1)\,\omega(\sqrt{h})\,. \end{aligned}$$

By (10.3) and (10.9) we have

$$\int_{\mathbb{R}^n} g\big(|u(x)|\sqrt{2}\sqrt{h}\big)\, dx \le \hat{\omega}(\sqrt{2}\sqrt{h})\int_{\mathbb{R}^n} \psi_0(|u(x)|)\, dx \le M\,\hat{\omega}(\sqrt{2}\sqrt{h})\,,$$

which, together with the previous inequality, gives

$$\begin{aligned} &\int_{\mathbb{R}^n} g\big(|u(x+h\xi) - u(x)|\big)\, dx \le \\ &\quad \le \omega(h) + (n-1)\,\omega(\sqrt{h+h^2}) + 2(n-1)M\,\hat{\omega}(\sqrt{2}\sqrt{h}) + (n-1)\,\omega(\sqrt{h})\,. \end{aligned}$$

for every $0 < h < \frac{1}{2}$. By the triangle inequality (10.7) this implies that (10.10) holds for every $0 < h < 1$ with $\tilde{\omega}(h) := 2\omega(h/2) + 2(n-1)\,\omega(\sqrt{2h+h^2}/2) + 4(n-1)M\,\hat{\omega}(\sqrt{h}) + 2(n-1)\,\omega(\sqrt{h}/\sqrt{2})$. □

Remark 10.4. Inequality (10.9) is satisfied if

$$g(s) \le \psi_0(s) \quad \text{and} \quad \psi_0(hs) \le \hat\omega(h)\,\psi_0(s) \tag{10.15}$$

for every $s \in \mathbb{R}^+$ and for every $0 < h < 1$. Note that the second inequality in (10.15) holds when $\psi_0(s) := s^p$ with $p > 0$. In particular, Lemma 10.8 can be applied with $g(s) = \psi_0(s) = s$ for every $s \in \mathbb{R}^+$.

Remark 10.5. Inequality (10.9) is satisfied if

$$\psi_0(0) > 0 \quad \text{and} \quad \lim_{s\to+\infty} \frac{\psi_0(s)}{g(s)} = +\infty\,. \tag{10.16}$$

Indeed, (10.9) is equivalent to

$$\lim_{h\to 0+} \sup_{s\in\mathbb{R}^+} \frac{g(hs)}{\psi_0(s)} = 0\,. \tag{10.17}$$

By (10.16) the supremum is attained at a point s_h. Let h_j be a sequence in $\mathbb{R}^+$ converging to 0. If s_{h_j} is bounded, then $g(h_j s_{h_j})/\psi_0(s_{h_j}) \le g(h_j s_{h_j})/\psi_0(0) \to 0$ since g is continuous and $g(0) = 0$ by (10.7). It $s_{h_j} \to +\infty$, then $g(h_j s_{h_j})/\psi_0(s_{h_j}) \le g(s_{h_j})/\psi_0(s_{h_j}) \to 0$ by (10.16). Since every sequence s_{h_j} has either a bounded subsequence or a subsequence diverging to $+\infty$, we obtain (10.17). Since (10.7) gives $g(s) \le 2sg(1)$ for every $s \ge 1$, (10.16) is always satisfied if $\psi_0(0) > 0$ and $\psi_0(s)/s \to +\infty$ as $s \to +\infty$.

Remark 10.6. When Ω has a Lipschitz boundary, Lemma 10.8 provides a quick proof of the compactness of the embedding of $BD(\Omega)$ into $L^1(\Omega;\mathbb{R}^n)$. For every $u \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, for every $\xi \in \mathbb{S}^{n-1}$, and for every $h > 0$ we have

$$|u(x+h\xi)\cdot\xi - u(x)\cdot\xi| \le \int_0^h |Du(x+t\xi)\xi\cdot\xi|\,dt = \int_0^h |Eu(x+t\xi)\xi\cdot\xi|\,dt$$

where $(Eu)_{ij} = \frac{1}{2}(D_i u_j + D_j u_i)$. It follows that

$$\int_{\mathbb{R}^n} |u(x+h\xi)\cdot\xi - u(x)\cdot\xi|\,dx \le \int_0^h \Big(\int_{\mathbb{R}^n} |Eu(x+t\xi)\xi\cdot\xi|\,dx\Big)dt \le h\int_{\mathbb{R}^n} |Eu(x)|\,dx\,.$$

If $u \in BD(\Omega)$, we can approximate by convolutions its extension, which belongs to $BD(\mathbb{R}^n)$ by the regularity of the boundary, and we get

$$\int_{\mathbb{R}^n} |u(x+h\xi)\cdot\xi - u(x)\cdot\xi|\,dx \le h|Eu|(\mathbb{R}^n)\,.$$

If $\mathcal{U}$ is a bounded subset of $BD(\Omega)$, we can apply Lemma 10.3 with $g(s) = \psi_0(s) = s$ (see Remark 10.4) and we obtain that there exists a modulus of continuity $\tilde\omega$ such that

$$\int_{\mathbb{R}^n} |u(x+h\xi) - u(x)|\,dx \le \tilde\omega(h)$$

for every $u \in \mathcal{U}$ and for every $0 < h < 1$. By the Fréchet-Kolmogorov compactness criterion $\mathcal{U}$ is relatively compact in $L^1(\Omega;\mathbb{R}^n)$.

In the next lemma we obtain the relative compactness with respect to pointwise $\mathcal{L}^n$-a.e. convergence from the behaviour of the one-dimensional slices. The proof follows the lines of [1, Theorem 6.6]. The main difference is that our assumptions concern only the components $u\cdot\xi$ of u and the corresponding slices in the same direction ξ. Moreover we cannot assume L^∞ bounds in view of the application to Theorem 11.1. This makes the statement of the lemma quite involved.

Lemma 10.7. *Let $\mathcal{U}$ be a set of $\mathcal{L}^n$-measurable functions from Ω into $\mathbb{R}^n$, let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing continuous function satisfying* (10.1), (10.7), *and*

$$g(s) \le s \qquad \textit{for every } s \in \mathbb{R}^+\,, \tag{10.18}$$

and let $\psi_0\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing continuous function satisfying* (10.2)*. Assume that there exist* $M \in \mathbb{R}^+$ *such that* (10.3) *holds for every* $u \in \mathcal{U}$ *and a modulus of continuity* $\hat\omega$ *such that* (10.9) *holds for every* $0 < h < 1$ *and for every* $s \in \mathbb{R}^+$*. Assume also that for every* $\delta > 0$ *we can find a modulus of continuity* ω_δ *such that for every* $\xi \in \mathbb{S}^{n-1}$ *there exists a set* $\mathcal{V}^\xi_\delta$ *of* $\mathcal{L}^n$*-measurable functions from* Ω *into* $\mathbb{R}$ *with the following properties:*

(a) *for every* $u \in \mathcal{U}$ *there exists* $v \in \mathcal{V}^\xi_\delta$ *with*

$$\int_{\mathbb{R}^n} g(|u(x)\cdot\xi - v(x)|)\,dx \le \delta\,; \tag{10.19}$$

(b) *for every* $v \in \mathcal{V}^\xi_\delta$ *and for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \Pi^\xi$ *we have*

$$\int_{\mathbb{R}} |v^\xi_y(t+h) - v^\xi_y(t)|\,dt \le \omega_\delta(h) \tag{10.20}$$

for every $0 < h < 1$.

Then every sequence in $\mathcal{U}$ *has a subsequence that converges pointwise* $\mathcal{L}^n$*-a.e. on* Ω *to an* $\mathcal{L}^n$*-measurable function* $u\colon \Omega \to \mathbb{R}^n$*. If, in addition,* $g(s) = s$ *for every* $s \in \mathbb{R}^+$*, then* $\mathcal{U} \subset L^1(\Omega;\mathbb{R}^n)$ *and every sequence in* $\mathcal{U}$ *has a subsequence that converges strongly in* $L^1(\Omega;\mathbb{R}^n)$*.*

Note that the modulus of continuity in (10.20) does not depend on y, nor on ξ.

Proof of Lemma 10.7. Let us fix $u \in \mathcal{U}$, $\delta > 0$, and $\xi \in \mathbb{S}^{n-1}$. Then there exists $v \in \mathcal{V}^\xi_\delta$ satisfying (10.19). By (10.7), (10.18), (10.20) for every $0 < h < 1$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} g\big(|u(x+h\xi)\cdot\xi - u(x)\cdot\xi|\big)\,dx &\le 2\delta + \int_{\mathbb{R}^n} |v(x+h\xi) - v(x)|\,dx = \\ = 2\delta + \int_{\pi^\xi(\Omega)} \Big(\int_{\mathbb{R}} |v^\xi_y(t+h) - v^\xi_y(t)|\,dt\Big)\,d\mathcal{H}^{n-1}(y) &\le 2\delta + c_\Omega\,\omega_\delta(h)\,,\end{aligned} \tag{10.21}$$

where $c_\Omega := \omega_{n-1}\mathrm{diam}(\Omega)^{n-1}$. Let

$$\omega(h) := \inf_{\delta>0} \big(2\delta + c_\Omega\,\omega_\delta(h)\big)$$

By (10.21) we have (10.8) for every $u \in \mathcal{U}$, for every $\xi \in \mathbb{S}^{n-1}$, and for every $0 < h < 1$. Since $\omega(h) \to 0$ as $h \to 0+$, by Lemma 10.3 there exists a modulus of continuity $\tilde\omega$ such that (10.10) holds. The main conclusion follows now from Lemma 10.2.

If $g(s) = s$, then $s \le \hat\omega(1)\,\psi_0(s)$ for every $s \in \mathbb{R}^+$ by (10.9). Therefore (10.3) implies that $\mathcal{U} \subset L^1(\Omega;\mathbb{R}^n)$ and that $\mathcal{U}$ is bounded in $L^1(\Omega;\mathbb{R}^n)$. The relative compactness in $L^1(\Omega;\mathbb{R}^n)$ follows now from the Fréchet-Kolmogorov criterion. □

In the proof of the compactness theorem for $GBD(\Omega)$ we need the following estimate of the modulus of continuity in L^1 of the translations of BV functions of one real variable.

Lemma 10.8. *Let* $z \in BV(\mathbb{R})$*. Assume that there exist two constants* $a > 0$ *and* $b > 0$ *such that*

$$|Dz|(\mathbb{R}\setminus J^1_z) + \mathcal{H}^0(J^1_z) \le a \quad \text{and} \quad \|z\|_{L^\infty(\mathbb{R})} \le b\,. \tag{10.22}$$

Then

$$\int_{\mathbb{R}} |z(t+h) - z(t)|\,dt \le (a + 2ab)h \tag{10.23}$$

for every $h > 0$.

Proof. By (10.22) $\mathcal{H}^0(J^1_z) \le a < +\infty$ and for every $t \in J^1_z$ we have $|Dz|(\{t\}) = |[z](t)| \le 2b$, so that $|Dz|(J^1_z) \le 2ab$. Using (10.22) again, we obtain $|Dz|(\mathbb{R}) \le a + 2ab$. Regularizing z by convolutions, we find a sequence z_k in $C^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ such that

$$z_k \to z \quad \text{strongly in } L^1(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} |z'_k|\,dt \le a + 2ab \quad \text{for every } k\,. \tag{10.24}$$

For every $t \in \mathbb{R}$ and for every $h > 0$ we have

$$|z_k(t+h) - z_k(t)| \le \int_0^h |z_k'(t+s)|\, ds\,.$$

By (10.24), integrating over $\mathbb{R}$ and interchanging the order of integration we get

$$\int_{\mathbb{R}} |z_k(t+h) - z_k(t)|\, dt \le \int_{\mathbb{R}} \Big(\int_0^h |z_k'(t+s)|\, ds \Big) dt =$$
$$= \int_0^h \Big(\int_{\mathbb{R}} |z_k'(t+s)|\, dt \Big) ds = h \int_{\mathbb{R}} |z_k'(t)|\, dt \le (a + 2ab) h\,.$$

Passing to the limit se $k \to \infty$ and using (10.24) we obtain (10.23). □

11. Two compactness results

In this section we prove the following analogue of the compact embedding of $BD(\Omega)$ into $L^1(\Omega;\mathbb{R}^n)$: every subset of $GBD(\Omega)$ satisfying uniform bounds for the measures $\hat\mu_u^\xi$ and for suitable integrals involving u has a subsequence that converges pointwise $\mathcal{L}^n$-a.e. on Ω. This allows us to obtain a compactness result for $GSBD(\Omega)$, following the proof of the analogous result for $SBD(\Omega)$ developed in [9]. As in the previous section, every function u defined on Ω is always extended to $\mathbb{R}^n$ by setting $u(x) = 0$ for every $x \in \mathbb{R}^n \setminus \Omega$.

Theorem 11.1. *Let $\mathcal{U}$ be a subset of $GBD(\Omega)$. Suppose that there exist a constant $M \in \mathbb{R}^+$ and an increasing continuous function $\psi_0\colon \mathbb{R}^+ \to \mathbb{R}^+$, with*

$$\lim_{s\to+\infty} \psi_0(s) = +\infty\,, \tag{11.1}$$

such that for every $u \in \mathcal{U}$ and for every $\xi \in \mathbb{S}^{n-1}$ we have

$$\int_\Omega \psi_0(|u|)\, dx \le M \qquad \text{and} \qquad \hat\mu_u^\xi(\Omega) \le M\,, \tag{11.2}$$

where $\hat\mu_u^\xi$ is the measure introduced in Definition 4.10. Then every sequence in $\mathcal{U}$ has a subsequence that converges pointwise $\mathcal{L}^n$-a.e. on Ω to an $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}^n$. If, in addition

$$\lim_{s\to+\infty} \frac{\psi_0(s)}{s} = +\infty\,, \tag{11.3}$$

then $\mathcal{U} \subset L^1(\Omega;\mathbb{R}^n)$ and every sequence in $\mathcal{U}$ has a subsequence that converges strongly in $L^1(\Omega;\mathbb{R}^n)$.

Proof. It is enough to prove the result for every relatively compact open subset of Ω. Therefore it is not restrictive to assume that Ω is the union of a finite number of open rectangles. This implies, in particular, that $\mathcal{H}^0\big(\partial(\Omega_y^\xi)\big) < +\infty$ for every $\xi \in \mathbb{S}^{n-1}$ and for every $y \in \Pi^\xi$, so that for every $u \in \mathcal{U}$ the slice $\hat u_y^\xi$ belongs to $BV(\Omega_y^\xi)$ (see Proposition 3.4), and hence to $BV(\mathbb{R})$.

To prove tha main assertion, it is enough to show that $\mathcal{U}$ satisfies the hypotheses of Lemma 10.7. For every $u \in \mathcal{U}$, $\xi \in \mathbb{S}^{n-1}$, and $a > 0$ we define

$$\hat A_u^{\xi,a} := \{y \in \Pi^\xi : \hat u_y^\xi \in BV(\mathbb{R})\,,\ |D\hat u_y^\xi|(\mathbb{R} \setminus J^1_{\hat u_y^\xi}) + \mathcal{H}^0(J^1_{\hat u_y^\xi}) \le a\}\,. \tag{11.4}$$

Moreover we set $\hat B_u^{\xi,a} := \Pi^\xi \setminus \hat A_u^{\xi,a}$ and we define

$$A_u^{\xi,a} := \{x \in \Omega : \pi^\xi(x) \in \hat A_u^{\xi,a}\} \quad \text{and} \quad B_u^{\xi,a} := \{x \in \Omega : \pi^\xi(x) \in \hat B_u^{\xi,a}\}\,. \tag{11.5}$$

Since for $\mathcal{H}^{n-1}$-a.e. $y \in \hat B_u^{\xi,a}$ we have $\hat u_y^\xi \in BV(\mathbb{R})$ and

$$|D\hat u_y^\xi|(\Omega_y^\xi \setminus J^1_{\hat u_y^\xi}) + \mathcal{H}^0(\Omega_y^\xi \cap J^1_{\hat u_y^\xi}) + \mathcal{H}^0\big(\partial(\Omega_y^\xi)\big) > a,$$

by Chebyshev's inequality and by (4.4), (4.5), and (11.2) we have

$$\mathcal{H}^{n-1}(\hat{B}^{\xi,a}_u) \le \frac{M+\mathcal{H}^{n-1}(\partial\Omega)}{a}\,,$$

hence by Fubini's theorem

$$\mathcal{L}^n(B^{\xi,a}_u) \le \frac{M+\mathcal{H}^{n-1}(\partial\Omega)}{a}\,\mathrm{diam}(\Omega)\,. \tag{11.6}$$

For every $b>0$ let σ_b be the truncation function defined by $\sigma_b(t)=-b$ for $t\le -b$, $\sigma_b(t)=t$ for $-b\le t\le b$, and $\sigma_b(t)=b$ for $t\ge b$. We define $v^{\xi,a}_{u,b}\in L^1(\Omega)$ by setting

$$v^{\xi,a}_{u,b} := \begin{cases} \sigma_b(u\cdot\xi) & \text{in } A^{\xi,a}_u\,, \\ 0 & \text{in } B^{\xi,a}_u\,. \end{cases} \tag{11.7}$$

Let g be a function satisfying all assumptions of Lemma 10.7 and such that

$$\lim_{s\to+\infty}\frac{\psi_0(s)}{g(s)} = +\infty\,.$$

For every $\delta>0$ there exists $b_\delta>0$ such that $g(s)\le \frac{\delta}{4M}\psi_0(s)$ for every $s\ge b_\delta$. By (11.6) there exists $a_\delta>0$ such that $g(b)\,\mathcal{L}^n(B^{\xi,a}_u)\le\frac{\delta}{2}$. Therefore (11.2) gives

$$\begin{aligned} &\int_\Omega g\big(|u\cdot\xi - v^{\xi,a}_{u,b_\delta}|\big)\,dx = \int_{A^{\xi,a}_u} g\big(|u\cdot\xi-\sigma_{b_\delta}(u\cdot\xi)|\big)\,dx + \int_{B^{\xi,a}_u} g\big(|u\cdot\xi|\big)\,dx \le \\ &\le 2\int_{\{|u\cdot\xi|>b\}} g(|u|)\,dx + g(b)\,\mathcal{L}^n(B^{\xi,a}_u) \le \frac{\delta}{2M}\int_\Omega \psi_0(|u|)\,dx + g(b)\,\mathcal{L}^n(B^{\xi,a}_u)\le\delta\,. \end{aligned} \tag{11.8}$$

Then we define $\mathcal{V}^\xi_\delta := \{v^{\xi,a_\delta}_{u,b_\delta} : u\in\mathcal{U}\}$, so that Condition (a) of Lemma 10.7 is satisfied.

As for Condition (b), we observe that, if $v=v^{\xi,a_\delta}_{u,b_\delta}\in\mathcal{V}^\xi_\delta$, then by (11.7) the slices v^ξ_y satisfy $v^\xi_y=\sigma_{b_\delta}(\hat{u}^\xi_y)$ for every $y\in\hat{A}^{\xi,a_\delta}_u$ and $v^\xi_y=0$ for every $y\in\hat{B}^{\xi,a_\delta}_u$. Since, by definition, $\hat{u}^\xi_y\in BV(\mathbb{R})$ for every $y\in\hat{A}^{\xi,a}_u$, it follows that $v^\xi_y\in BV(\mathbb{R})$ and $\|v^\xi_y\|_{L^\infty(\mathbb{R})}\le b_\delta$ for every $y\in\Pi^\xi$. Moreover, $|Dv^\xi_y|(B)\le|D\hat{u}^\xi_y|(B)$ for every Borel set $B\subset\mathbb{R}$, $J_{v^\xi_y}\subset J_{\hat{u}^\xi_y}$, and $|[v^\xi_y]|\le|[\hat{u}^\xi_y]|$ on $J_{v^\xi_y}$, hence $J^1_{v^\xi_y}\subset J^1_{\hat{u}^\xi_y}$. Therefore (11.4) implies that $|Dv^\xi_y|(\mathbb{R}\setminus J^1_{v^\xi_y})+\mathcal{H}^0(J^1_{v^\xi_y}) = |Dv^\xi_y|(\mathbb{R}\setminus J^1_{\hat{u}^\xi_y})+|Dv^\xi_y|(J^1_{\hat{u}^\xi_y}\setminus J^1_{v^\xi_y})+\mathcal{H}^0(J^1_{v^\xi_y})\le|D\hat{u}^\xi_y|(\mathbb{R}\setminus J^1_{\hat{u}^\xi_y})+\mathcal{H}^0(J^1_{\hat{u}^\xi_y}\setminus J^1_{v^\xi_y})+\mathcal{H}^0(J^1_{v^\xi_y})\le a_\delta$ for every $y\in\hat{A}^{\xi,a_\delta}_u$. Since $|Dv^\xi_y|(\mathbb{R}\setminus J^1_{v^\xi_y})+\mathcal{H}^0(J^1_{v^\xi_y})=0$ for every $y\in\hat{B}^{\xi,a_\delta}_u$, using Lemma 10.8 we obtain (10.20) with $\omega_\delta(h):=(a_\delta+2a_\delta b_\delta)h$. Therefore Condition (b) of Lemma 10.7 is satisfied and the proof of the main assertion is complete.

If (11.3) holds, the $\mathcal{U}\subset L^1(\Omega;\mathbb{R}^n)$ by (11.2) and we can take $g(s)=s$ in the proof, thanks to Remark 10.5. The convergence in $L^1(\Omega;\mathbb{R}^n)$ follows now from the last part of Lemma 10.7. □

The following corollary is an easy consequence of Theorem 11.1 and of the arguments used in the proof of Lemma 4.18.

Corollary 11.2. *Let u_k be sequence in $GBD(\Omega)$. Suppose that there exist an increasing continuous function $\psi_0\colon\mathbb{R}^+\to\mathbb{R}^+$ satisfying (11.1) and a constant $M\in\mathbb{R}^+$ such that*

$$\int_\Omega \psi_0(|u_k|)\,dx\le M \qquad \text{and} \qquad \hat{\mu}_{u_k}(\Omega)\le M \tag{11.9}$$

for every k, where $\hat{\mu}_{u_k}$ is the measure introduced in Definition 4.16. Then there exist a subsequence, still denoted by u_k, and a function $u\in GBD(\Omega)$, such that $u_k\to u$ pointwise $\mathcal{L}^n$-a.e. on Ω. If, in addition, (11.3) holds, then $u_k\in L^1(\Omega;\mathbb{R}^n)$ for every k, $u\in L^1(\Omega;\mathbb{R}^n)$, and the subsequence converges strongly in $L^1(\Omega;\mathbb{R}^n)$.

Proof. Since $\hat{\mu}^\xi_{u_k} \le \hat{\mu}_{u_k}$ for every $\xi \in \mathbb{S}^{n-1}$ and for every k, by Theorem 11.1 there exist a subsequence, still denoted by u_k, and an $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}^n$, such that $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. on Ω. We want to prove that $u \in GBD(\Omega)$.

By (11.9) there exist a subsequence, still denoted by $\hat{\mu}_{u_k}$, and a measure $\lambda \in \mathcal{M}_b^+(\Omega)$, such that $\hat{\mu}_{u_k} \rightharpoonup \lambda$ weakly* in $\mathcal{M}_b(\Omega)$. By Theorem 3.5 and Definitions 4.8, 4.10, and 4.16 for every $\tau \in \mathcal{T}$, for every $\xi \in \mathbb{S}^{n-1}$, and for every $\varphi \in C^1_c(\Omega)$, with $|\varphi| \le 1$ in Ω, we have

$$\int_\Omega \tau(u_k\cdot\xi)\,\nabla\varphi\cdot\xi\,dx \le \int_\Omega |\varphi|\,d|D_\xi\big(\tau(u_k\cdot\xi)\big)| \le \int_\Omega |\varphi|\,d\hat{\mu}^\xi_{u_k} \le \int_\Omega |\varphi|\,d\hat{\mu}_{u_k}\,.$$

Passing to the limit as $k \to \infty$ we get

$$\int_\Omega \tau(u\cdot\xi)\,\nabla\varphi\cdot\xi\,dx \le \int_\Omega |\varphi|\,d\lambda\,.$$

This implies that for every $\xi \in \mathbb{S}^{n-1}$ and for every $\tau \in \mathcal{T}$ the partial derivative $D_\xi\big(\tau(u\cdot\xi)\big)$ belongs to $\mathcal{M}_b(\Omega)$ and its total variation satisfies

$$\big|D_\xi\big(\tau(u\cdot\xi)\big)\big|(B) \le \lambda(B)$$

for every Borel set $B \subset \Omega$. Therefore u satisfies Condition (a) of Definition 4.1 for every $\xi \in \mathbb{S}^{n-1}$, hence $u \in GBD(\Omega)$.

If (11.3) holds, then $u_k \in L^1(\Omega;\mathbb{R}^n)$ for every k by (11.9). The other assertions follow from the last part of Theorem 11.1. □

We are now in a position to prove the compactness result for $GSBD(\Omega)$.

Theorem 11.3. *Let u_k be a sequence in $GSBD(\Omega)$. Suppose that there exist a constant $M \in \mathbb{R}^+$ and two increasing continuous functions $\psi_0\colon \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi_1\colon \mathbb{R}^+ \to \mathbb{R}^+$, with*

$$\lim_{s\to+\infty}\psi_0(s) = +\infty \qquad \textit{and} \qquad \lim_{s\to+\infty}\frac{\psi_1(s)}{s} = +\infty\,, \tag{11.10}$$

such that

$$\int_\Omega \psi_0(|u_k|)\,dx + \int_\Omega \psi_1(|\mathcal{E}u_k|)\,dx + \mathcal{H}^{n-1}(J_{u_k}) \le M \tag{11.11}$$

for every k. Then there exist a subsequence, still denoted by u_k, and a function $u \in GSBD(\Omega)$, such that

$$u_k \to u \quad \textit{pointwise } \mathcal{L}^n\textit{-a.e. on } \Omega\,, \tag{11.12}$$

$$\mathcal{E}u_k \rightharpoonup \mathcal{E}u \quad \textit{weakly in } L^1(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{11.13}$$

$$\mathcal{H}^{n-1}(J_u) \le \liminf_{k\to\infty}\mathcal{H}^{n-1}(J_{u_k})\,. \tag{11.14}$$

If, in addition, (11.3) *holds, then $u_k \in L^1(\Omega;\mathbb{R}^n)$ for every k, $u \in L^1(\Omega;\mathbb{R}^n)$, and the subsequence converges strongly in $L^1(\Omega;\mathbb{R}^n)$.*

Proof. By (11.10) and (11.11) there exists a constant $M_1 \in \mathbb{R}^+$ such that

$$\int_\Omega |\mathcal{E}u_k|\,dx + \mathcal{H}^{n-1}(J_{u_k}) \le M_1$$

for every k. Therefore (9.19) implies that $\hat{\mu}_{u_k}(\Omega) \le M_1$ for every k. By Corollary 11.2 there exist a subsequence, still denoted by u_k, and a function $u \in GBD(\Omega)$, such that $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. on Ω.

Taking into account Remark 4.3, to prove that $u \in GSBD(\Omega)$ it is enough to show that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat{u}^\xi_y := u^\xi_y\cdot\xi$ belongs to $GSBV(\Omega^\xi_y)$. This property, as well as (11.12)-(11.14), can be obtained as in the proof of [9, Theorem 1.1]. We have just to redefine the function $\mathrm{II}_{y,\xi}(u_k)$ introduced on page 342 of that paper by

$$\mathrm{II}_{y,\xi}(u_k) := \int_{\Omega^\xi_y} \psi_0\big(|(\hat{u}_k)^\xi_y|\big)\,dt\,,$$

and to modify the proof accordingly. In particular we cannot use the boundedness in $BD(\Omega)$ and we apply the compactness theorem for $GSBV(\Omega^\xi_y)$ (see [6, ???]) to the one dimensional slices in order to obtain (2.13) and the formula after (2.18) of [9].

If (11.3) holds, the assertions at the end of the theorem follow from the last part of Corollary 11.2. □

12. Two examples

In this section we give two examples that show that the compactness result for $GSBD(\Omega)$ (Theorem 11.3) cannot be easily improved. The first example shows that we cannot expect convergence in $L^1(\Omega;\mathbb{R}^n)$ if we remove (11.3). More precisely, we show that, if we take $\psi_0(s) = s$ for every $s \in \mathbb{R}^+$, then, in general, we have only pointwise $\mathcal{L}^n$-a.e. convergence. Note that in this case (11.11) gives that u_k is bounded in $L^1(\Omega;\mathbb{R}^n)$ and that the pointwise limit u belongs to $L^1(\Omega;\mathbb{R}^n)$ by the Fatou lemma.

Example 12.1. Let $x_0 \in \Omega$, let $\xi \in \mathbb{S}^{n-1}$, and, for every k, let $u_k \in SBV(\Omega;\mathbb{R}^n)$ be defined by $u_k(x) = k^n\xi$ for $|x - x_0| < \frac{1}{k}$ and $u_k(x) = 0$ for $|x - x_0| \geq \frac{1}{k}$. Then $\mathcal{E}u_k = 0$ $\mathcal{L}^n$-a.e. on Ω, $\mathcal{H}^{n-1}(J_{u_k}) = n\omega_n/k^{n-1}$, and $\int_\Omega |u_k|\,dx = \omega_n$ for k large enough. Therefore the hypotheses of Theorem 11.3 are satisfied with $\psi_0(s) := s$ and $\psi_1(s) := s^2$ for every $s \in \mathbb{R}^+$. The sequence u_k converges to 0 pointwise $\mathcal{L}^n$-a.e. on Ω, but u_k does not converge to 0 in $L^1(\Omega;\mathbb{R}^n)$.

In the rest of this section we construct a sequence u_j in $SBD(\Omega)$ that satisfies all hypotheses of the compactness result for $GSBD(\Omega)$ (Theorem 11.3), but the limit function u does not belong to $BD(\Omega)$. Since u belongs to $GSBD(\Omega)$, this shows also that $GSBD(\Omega) \neq SBD(\Omega)$ and $GBD(\Omega) \neq BD(\Omega)$. For the construction we need the elementary result contained in the following lemma.

Lemma 12.2. *Let ρ_k be a sequence in $\mathbb{R}^+$ such that*

$$\sum_{k=1}^{\infty} \rho_k^n < +\infty \,. \tag{12.1}$$

Then there exists a sequence x_k in $\mathbb{R}^n$ converging to 0 such that the balls $B_{\rho_k}(x_k)$ are pairwise disjoint.

Proof. It is not restrictive to assume that $0 < \rho_{k+1} \leq \rho_k \leq \frac{1}{2}$ for every k. For every integer $i \geq 1$ let k_i be the smallest index k such that $\rho_k \leq 2^{-i-1}$. Then $k_1 = 1$ and for every $k_i \leq k < k_{i+1}$ we have

$$2^{-i-1} < 2\rho_k \leq 2^{-i} \,, \tag{12.2}$$

which, together with (12.1), gives

$$\sum_{i=1}^{\infty} (k_{i+1} - k_i)\, 2^{-ni} < +\infty \,. \tag{12.3}$$

Since $\rho_k > 0$ for every k, we have

$$\lim_{i\to\infty} k_i = +\infty \tag{12.4}$$

Let a_i be the largest integer such that $(k_{i+1} - k_i + 1)\, 2^{-ni} \leq 2^{-na_i}$. By definition we have

$$a_i \leq i \quad \text{and} \quad 2^{-na_i - n} < (k_{i+1} - k_i + 1)\, 2^{-ni} \leq 2^{-na_i} \,. \tag{12.5}$$

By (12.3) and (12.5) we have

$$\sum_{i=1}^{\infty} 2^{-na_i} < +\infty \,. \tag{12.6}$$

Let

$$\beta_j := \sum_{i=j}^{\infty} 2^{-na_i} . \tag{12.7}$$

By (12.6) the sequence β_j is decreasing and tends to 0 as $j \to \infty$. Let j_0 be the largest integer such that $\beta_1 \leq 2^{-nj_0}$. For every integer $j \geq j_0$ let m_j be the smallest integer i such that $\beta_i \leq 2^{-nj}$. Then $m_{j_0} = 1$ and for every $m_j \leq i < m_{j+1}$ we have

$$2^{-nj-n} < \beta_i \leq 2^{-nj} . \tag{12.8}$$

Since $\beta_i > 0$ for every i, we have

$$\lim_{j \to \infty} m_j = +\infty . \tag{12.9}$$

Moreover by (12.7) and (12.8) we have

$$\sum_{m_j \leq i < m_{j+1}} 2^{-na_i} < \beta_{m_j} \leq 2^{-nj}, . \tag{12.10}$$

If $i \geq m_j$, by (12.7) we have

$$2^{-na_i} < \beta_i \leq \beta_{m_j} \leq 2^{-nj} , \tag{12.11}$$

so that by (12.5)

$$j < a_i \leq i . \tag{12.12}$$

Let $Q = [0,1)^n$ and let $Q_j := 2^{-j}Q$. By (12.12) for $m_j \leq i < m_{j+1}$ the set $Q_{j-1} \setminus Q_j$ is the union of disjoint cubes of the form $z + Q_i$, where $z \in 2^{-i}\mathbb{Z}^n$ and $\mathbb{Z}$ is the set of integers. We start with $i = m_j$ and observe that $(k_i - k_{i+1})\, 2^{-ni} < 2^{-na_i} < 2^{-nj} \leq 2^{-n(j-1)} - 2^{-nj} = \mathcal{L}^n(Q_{j-1} \setminus Q_j)$ by (12.5) and (12.11). Therefore we can find a family Q_k^{j,m_j}, $k_{m_j} \leq k < k_{m_j+1}$, of pairwise disjoint cubes of the form described above and contained in $Q_{j-1} \setminus Q_j$. Suppose now that $i = m_j + 1 < m_{j+1}$ and let A be the union (with respect to k) of the cubes Q_k^{j,m_j}, $k_{m_j} \leq k < k_{m_j+1}$. By (12.5) we have $\mathcal{L}^n(A) < 2^{-na_{m_j}}$, so that $\mathcal{L}^n\big((Q_{j-1} \setminus Q_j) \setminus A\big) > 2^{-nj}(2^n - 1) - 2^{-na_{m_j}} \geq 2^{-na_i}$ by (12.10). Since the set $(Q_{j-1} \setminus Q_j) \setminus A$ is the union of disjoint cubes of the form $z + Q_i$, where $z \in 2^{-i}\mathbb{Z}^n$ and $i = m_j + 1$, there exists a family $Q_k^{j,i}$, $k_i \leq k < k_{i+1}$ of pairwise disjoint cubes of this form and contained in $(Q_{j-1} \setminus Q_j) \setminus A$. Continuing in the same way for every $m_j \leq i < m_{j+1}$, we construct a family $Q_k^{j,i}$, $k_i \leq k < k_{i+1}$ of pairwise disjoint cubes of side 2^{-i} and contained in $Q_{j-1} \setminus Q_j$, such that the cubes of two different families have empty intersection.

Let $x_k^{j,i}$, $j \geq j_0$, $m_j \leq i < m_{j+1}$, $k_i \leq k < k_{i+1}$, be the centres of the cubes $Q_k^{j,i}$. By (12.2) we have

$$B_{\rho_k}(x_k^{j,i}) \subset Q_k^{j,i} . \tag{12.13}$$

Let x_k be the sequence defined by $x_k := x_k^{j,i}$ for $k_i \leq k < k_{i+1}$, $m_j \leq i < m_{j+1}$, and $j \geq j_0$. By (12.4) and (12.9) we have $x_k \to 0$. Since the cubes $Q_k^{j,i}$ are pairwise disjoint, the balls $B_{\rho_k}(x_k)$ are pairwise disjoint by (12.13). □

The following example shows that $GSBD(\Omega) \neq SBD(\Omega)$ and $GBD(\Omega) \neq BD(\Omega)$. Moreover it shows that, if a sequence in $SBD(\Omega)$ satisfies the assumptions of Theorem 11.3, but does not satisfy the assumptions of [9, Theorem 1.1], then the limit of a subsequence may not belong to $SBD(\Omega)$.

Example 12.3. Assume that $n \geq 2$. Let $p > 1$ and let ρ_k be a sequence of positive real numbers such that

$$\sum_{k=1}^{\infty} \rho_k^{n-1} < +\infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} \rho_k^{n-1-\frac{1}{p}} = +\infty , \tag{12.14}$$

for instance $\rho_k := k^{-1/(n-1-\frac{1}{p})}$. By Lemma 12.2 there exist a sequence x_k and a point $x_0 \in \Omega$ such that the balls $B_k := B_{\rho_k}(x_k)$ are pairwise disjoint and $x_k \to x_0$ as $k \to \infty$. Let A_k be a sequence of antisymmetric $n\times n$ matrices such that

$$|A_k| = \rho_k^{-1-\frac{1}{p}} \tag{12.15}$$

for every k. From (12.14) and (12.15) we obtain

$$\sum_{k=1}^{\infty} |A_k|^p \rho_k^{p+n} < +\infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} |A_k|\, \rho_k^n = +\infty\,. \tag{12.16}$$

For every k let w_k be the function defined by $w_k(x) = A_k(x-x_k)$ if $x \in B_k$ and $w_k(x) = 0$ if $x \in \Omega \setminus B_k$. Finally, let

$$u_j := \sum_{k=1}^{j} w_k \qquad \text{and} \qquad u := \sum_{k=1}^{\infty} w_k\,.$$

For every j the function u_j belongs to $SBV(\Omega;\mathbb{R}^n) \subset SBD(\Omega) \subset GSBD(\Omega)$. Moreover $u_j \in L^\infty(\Omega;\mathbb{R}^n)$ and $\mathcal{E}u_j = 0$ $\mathcal{L}^n$-a.e. in Ω, since each matrix A_k is antisymmetric. As $|w_k| \le |A_k|\,\rho_k$, using the inequalities in (12.14) and (12.16) we find a constant $M \in \mathbb{R}^+$ such that (11.11) holds with $\psi_0(s) = \psi_1(s) := s^p$. The inequality in (12.16) implies also that $u \in L^p(\Omega;\mathbb{R}^n)$ and $u_j \to u$ strongly in $L^p(\Omega;\mathbb{R}^n)$ as $j \to \infty$. By Theorem 11.3 we have $u \in GSBD(\Omega)$. This follows also from Condition (b) of Definition 4.1, using the fact that for every Borel set $B \subset \Omega$, for every $\xi \in \mathbb{S}^{n-1}$, and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have $|D\hat{u}^\xi_y|(B^\xi_y \setminus J_{\hat{u}^\xi_y}) = 0$ and

$$\begin{aligned}\int_{\Pi^\xi} \big(|D\hat{u}^\xi_y|(B^\xi_y \cap J_{\hat{u}^\xi_y} \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat{u}^\xi_y})\big)\, d\mathcal{H}^{n-1}(y) &\le \int_{\Pi^\xi} \mathcal{H}^0(B^\xi_y \cap J_{\hat{u}^\xi_y})\, d\mathcal{H}^{n-1}(y) \le \\ \le \sum_{k=1}^{\infty} \int_{\Pi^\xi} \mathcal{H}^0(B^\xi_y \cap (\partial B_k)^\xi_y)\, d\mathcal{H}^{n-1}(y) &\le \sum_{k=1}^{\infty} \mathcal{H}^{n-1}(B \cap \partial B_k) =: \lambda(B) < +\infty\,,\end{aligned}$$

where the last inequality follows from (12.14).

Let Eu be the matrix-valued Radon measure considered in Remark 4.5. For every $\varepsilon > 0$ we have

$$|Eu|(\Omega \setminus B_\varepsilon(x_0)) \ge \frac{1}{\sqrt{2}} \sum_{k=1}^{k_\varepsilon} \int_{B_k} |A_k(x-x_k)|\, d\mathcal{H}^{n-1}(x) \ge c_n \sum_{k=1}^{k_\varepsilon} |A_k|\, \rho_k^n\,, \tag{12.17}$$

where k_ε is the largest index such that $B_k \cap B_\varepsilon(x_0) = \emptyset$ for every $k \le k_\varepsilon$, and c_n is a constant depending only on the dimension n. If $u \in BD(\Omega)$, then $|Eu|(\Omega \setminus B_\varepsilon(x_0)) \le |Eu|(\Omega) < +\infty$ for every $\varepsilon > 0$. By (12.17) this contradicts the equality in (12.16). Therefore $u \notin BD(\Omega)$.

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