MAXWELL'S EQUATIONS IN ANISOTROPIC MEDIA AND MAXWELL'S EQUATIONS IN CARNOT GROUPS AS VARIATIONAL LIMITS

ANNALISA BALDI
BRUNO FRANCHI

ABSTRACT. Let $\mathbb{G}$ be a free Carnot group (i.e. a connected simply connected nilpotent stratified free Lie group) of step 2. In this paper, we prove that the variational functional generated by "intrinsic" Maxwell's equations in $\mathbb{G}$ is the Γ-limit of a sequence of classical (i.e. Euclidean) variational functionals associated with strongly anistropic dielectric permittivity and magnetic permeability in the Euclidean space.

1. INTRODUCTION

Classical Maxwell's equations in the vacuum (neither charges nor currents) in $\mathbb{R}_s \times \mathbb{R}^3_x$ take the form

$$\frac{\partial \vec{H}}{\partial s} = \operatorname{curl} \vec{E} \quad , \quad \operatorname{div} \vec{H} = 0 \tag{1}$$

and

$$\frac{\partial \vec{E}}{\partial s} = -\operatorname{curl} \vec{H} \quad , \quad \operatorname{div} \vec{E} = 0, \tag{2}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields, respectively.

Recently, in a series of papers ([5], [16], [15]), the authors introduced the notion of "intrinsic" Maxwell's equations in Carnot groups, that are prototypes of non-Riemannian metric structures.

We recall that a connected and simply connected Lie group $(\mathbb{G}, \cdot)$ (in general non-commutative) is said a *Carnot group of step* κ if its Lie algebra $\mathfrak{g}$ admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. The first layer V_1, the so-called *horizontal layer*, plays a key role in the theory, since it generates the all of $\mathfrak{g}$ by commutation.

The Carnot group $\mathbb{G}$ is said to be free if its Lie algebra is free, i.e, if the commutators satisfy no linear relations other than antisymmetry and the Jacobi identity.

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A Carnot group $\mathbb{G}$ can be always identified, through exponential coordinates, with the Euclidean space $\mathbb{R}^n$, where n is the dimension of $\mathfrak{g}$, endowed with a suitable group operation.

In addition, the stratification of the Lie algebra induces a family of anisotropic dilations δ_λ ($\lambda > 0$) on $\mathfrak{g}$ and therefore, through exponential map, on $\mathbb{G}$. We refer to [12] or [6] for an exhaustive introduction.

The Euclidean space $(\mathbb{R}^n, +)$ provides a trivial example of (Abelian) Carnot group of step 1. The simplest example of non-Abelian Carnot group of step 2 is given by the first Heisenberg group $\mathbb{H}^1 \equiv \mathbb{R}^3$, with variables (x, y, t) and product

$$(x, y, t) \cdot (\xi, \eta, \tau) = (x + \xi,\, y + \eta,\, t + \tau - \frac{1}{2}(y\xi - x\eta).$$

Indeed, let us set $X := \partial_x - \frac{1}{2}y\partial_t$, $Y := \partial_y + \frac{1}{2}x\partial_t$, $T := \partial_t$. Since $T = [X, Y]$, the stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X, Y\}$ and $V_2 = \text{span}\,\{T\}$.

From now on, we use the word "intrinsic" when we want to stress a privileged role played by the horizontal layer and by group translations and dilations. On the contrary, the word "Euclidean" is used when dealing with the special commutative group $(\mathbb{R}^n, +)$. In fact, this distinction will be crucial through this paper, since we have to deal with two different superposed structures in $\mathbb{R}^n$: a non-Abelian Carnot group $(\mathbb{G}, \cdot)$ and the Abelian group $(\mathbb{R}^n, +)$.

Let us recall the definition of Maxwell's equation in a Carnot group. We start from classical Euclidean Maxwell's equations in the free space $\mathbb{R}_s \times \mathbb{R}^3_x$, written in terms of differential forms. Denote by (Ω^*, d) the de Rham's complex of differential forms in $\mathbb{R}^3$ associated with the exterior differential d. If $E, H \in \Omega^1$ are differential 1-forms, dual of the electric and of the magnetic fields, respectively, δ is the formal adjoint in L^2 of d, and $*$ is the standard Hodge-star operator in $\mathbb{R}^3$, then equations (1) and (2) become

$$\frac{\partial H}{\partial s} = *dE \quad , \quad \delta H = 0 \tag{3}$$

and

$$-\frac{\partial E}{\partial s} = *dH \quad , \quad \delta E = 0. \tag{4}$$

Clearly, the above equations make perfectly sense in $\mathbb{R}^n$ with $n \geq 3$, provided $*$ is meant to be the Hodge duality operator in $\mathbb{R}^n$, and $E \in \Omega^1$, $H \in \Omega^{n-2}$.

If we want to define intrinsic Maxwell's equations in a Carnot group, we need preliminarily and intrinsic notion of differential form in a group. Unfortunately, De Rham's complex (Ω^*, d) of differential forms, endowed with the usual exterior differential, does not fit our aim of an intrinsic theory, since it is not invariant under group dilations, basically since it mixes derivatives along all the layers of the stratification. Consider for instance the Heisenberg group $\mathbb{H}^1$, and let dx, dy and $\theta = dt + \frac{1}{2}ydx - \frac{1}{2}xdy$ (the contact form of $\mathbb{H}^1$) be the (left invariant) dual covectors of X, Y, T, respectively. If we write the exterior differential df of a smooth function f in terms of dx, dy and θ, we obtain

$$df = (Xf)dx + (Yf)dy + (Tf)\theta,$$

that is not invariant under group dilations, since dx, dy are homogeneous of degree 1 with respect to group dilations, whereas θ is homogeneous of degree 2.

One could think of replacing d by the "horizontal differantial" d_H defined on functions by cutting out the non-horizontal part $(Tf)\theta$, i.e., by setting

$$d_H f = (Xf)dx + (Yf)dy$$

that is homogeneous of degree 1. Unfortunately, d_H does not act correctly on 1-forms, since $d_H^2 f \neq 0$, because of the lack of commutativy of $\mathfrak{g}$, and therefore d_H fails to be a good differential for the construction of an intrinsic complex.

On the other hand, an intrinsic complex of differential forms in a Carnot group satisying our invariance requirements has been defined in the last few years by M. Rumin in a series of papers ([24], [26], [25]). We refer to Appendix B for a description of Rumin's complex, as well as to [3].

Roughly speaking, Rumin defines a kind of "minimal" class E_0^* of intrinsic forms as well as an exterior intrinsic differential d_c that makes (E_0^*, d_c) a sub-complex of De Rham's complex. The differential d_c is defined through a suitable (non-orthogonal) projection Π_E of (E_0^*, d_c) on a sub-complex $(E^*, d) \subset (\Omega^*, d)$ of "lifted forms", and then by projecting orthogonally on E_0^* through the map Π_{E_0}. The following diagram gives a synopsis of the construction

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots \\
 & & \Pi_E \big\downarrow & & \big\uparrow \Pi_{E_0} & & \\
\cdots & \xrightarrow{d} & E^h & \xrightarrow{d} & E^{h+1} & \xrightarrow{d} & \cdots \\
 & & i \big\downarrow & & \big\downarrow i & & \\
\cdots & \xrightarrow{d} & \Omega^h & \xrightarrow{d} & \Omega^{h+1} & \xrightarrow{d} & \cdots
\end{array}$$

A crucial property of d_c, that will affect all the subsequence theory, is that, in general, it is a differential operator in the horizontal derivatives of order greater than 1.

The complex (E_0^*, d_c) being given, it is easy to define formally a set of intrinsic Maxwell's equations in a Carnot group $\mathbb{G} = \mathbb{R}^n$ as follows: given $E \in E_0^1$ and $H \in E_0^{n-2}$, we say that they satisfy our Maxwell's equations if

(5) $$\frac{\partial H}{\partial s} = *d_c E \quad , \quad \delta_c H = 0$$

and

(6) $$-\frac{\partial E}{\partial s} = *d_c H \quad , \quad \delta_c E = 0,$$

where δ_c and $*$ are associated with suitable left-invariant scalar product. Notice that intrinsic Maxwell's equations are no more of order 1, depending on the order of d_c.

So far, this construction is purely formal. On the other hand, it has been shown in [15] that equations (5) and (6) are invariant under the action of a suitable class of contact Lorentz transformations (contact maps are algebra endomorphisms that preserve the stratification). In addition, in $\mathbb{H}^1$ equations (5) and (6) arise in the study of quasiconformal or quasiregular maps in $\mathbb{H}^1$, precisely as classical Maxwell's equations appear in quasiconformal map theory in the Euclidean setting (see [19], [1]).

The aim of the present paper is to show that intrinsic Maxwell's equation can be seen as (variational) limits of "true" Maxwell's equations in anisotropic media. To this end, we state preliminarily classical Maxwell equations in matter for time-harmonic vector fields. In bounded regions of the space, the study of Maxwell's equations, together with (say) relative boundary conditions, can be reduced to the study of stationary points of variational functionals ([7], [21], [20]). Thus, in some sense, these functionals contain all the structure of Maxwell's equations. In this paper, we prove precisely that the variational functional in the group generated by intrinsic Maxwell's equations is the Γ-limit of a sequence of the variational functionals associated with Euclidean strongly anistropic dielectric permittivity and magnetic permeability. We notice that the correct formulation of the relative boundary condition in the group is not straightforward, and the Γ-limit result depends strongly on it.

We want to state explicitly that the Γ-convergence is not meant to derive existence results of solutions of the group Maxwell's equation, because of the lack of several well-known crucial property in the limit functional, like coerciveness (tink for instance of Gaffney's inequality) and higher order regularity properties of the stationary points. Instead, it is meant to show that the "structure" of the intrinsic system (5), (6) is the limit of the corresponding structure of a sequence of "true" equations in the matter.

The paper is organized as follows. In Section 2 we recall some basic facts about Carnot groups, but most of the property and definitions about Carnot groups and multilinear algebra in a Carnot group are contained in the Appendices A and B. In Section 3, we write Maxwell's equations in the matter in the Euclidean space $\mathbb{R}^n$. Starting from Section 4, we suppose $\mathbb{G}$ to be a free Carnot group of step two. We recall the notion of "'intrinsic" Maxwell's equations and we relate them to an intrinsic variational functional. The properties of the "intrinsic" complex of differential forms can be found in the Appendix B. Section 5 contains our main results. In particular we prove that if $\mathbb{G}$ is a free Carnot group of step two, then the variational functional in the group, generated by intrinsic Maxwell's equations, is a Γ-limit of a class of variational functional in $\mathbb{R}^n$ with strongly anistropic dielectric permittivity and magnetic permeability. Finally, in Section 6, we give explicit computations of the dielectric permittivity and magnetic permeability in the first Heisenberg group $\mathbb{H}^1$.

In addition, the paper contains three Appendices. Appendix A is meant to establish a few notations of multilinear algebra. Appendix B contains some basic notions and results concerning Carnot group, as well as a self-contained sketch of Rumin's theory. Finally, in Appendix C, we give a gist of the variational approach to Maxwell's equations.

2. Notations and preliminary results: two scalar products

Let $(\mathbb{G}, \cdot)$ be a *Carnot group of step* κ identified to $\mathbb{R}^n$ through exponential coordinates, as defined in the Introduction.

As customary, we denote by $T\mathbb{G}$ the tangent fiber bundle of $\mathbb{G}$, and by $T\mathbb{G}_x$ its fiber over $x \in \mathbb{G}$.

By definition, $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$ (i.e. the Lie algebra of the left-invariant vector fields on $\mathbb{G}$) can be written as

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa. \tag{7}$$

Set $m_i = \dim(V_i)$, for $i = 1, \dots, \kappa$ and $h_i = m_1 + \cdots + m_i$ with $h_0 = 0$. Clearly, $h_\kappa = n$. We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(V_i).$$

If e is the unit element of $(\mathbb{G}, \cdot)$, we remind that the map $X \to X(e)$, that associate with a left-invariant vector field X its value at e, is an isomorphism from $\mathfrak{g}$ to $T\mathbb{G}_e$, in turn identified with $\mathbb{R}^n$. From now on, we shall use systematically these identications.

We choose now a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ adapted to the stratification of $\mathfrak{g}$, i.e. such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \text{ is a basis of } V_j \ \text{ for each } j = 1, \dots, \kappa.$$

Then, we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathfrak{g}$ making the adapted basis $\{e_1, \dots, e_n\}$ orthonormal. Moreover, let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(e) = e_i$, $i = 1, \dots, n$. Clearly, X is othonormal with respect to $\langle \cdot, \cdot \rangle$.

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1, \cdots, X_n$, is the family of covectors $\{\theta_1, \cdots, \theta_n\}$. We indicate by $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\theta_1, \cdots, \theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1, \cdots, \theta_n$ may have polynomial (hence variable) coefficients.

Following Federer (see [10] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge_h \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge^h \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \leq h \leq n$,

$$\begin{aligned} \textstyle\bigwedge_h \mathfrak{g} &:= \mathrm{span}\{X_{i_1} \wedge \cdots \wedge X_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}, \\ \textstyle\bigwedge^h \mathfrak{g} &:= \mathrm{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}. \end{aligned}$$

The elements of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ are called *h-vectors* and *h-covectors*.

The dual space $\bigwedge^1(\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$.

Definition 2.1. We denote by Θ^h the basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}$ of $\bigwedge^h \mathfrak{g}$. The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_h \mathfrak{g}$ and to $\bigwedge^h \mathfrak{g}$ making the bases $X_{i_1} \wedge \cdots \wedge X_{i_h}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ orthonormal.

We can define now, as usual, two families of vector bundles, denoted by $\bigwedge_* (T\mathbb{G})$ (the exterior power of the tangent bundle) and $\bigwedge^* (T\mathbb{G})$ (the exterior power of the cotangent bundle) over $\mathbb{G}$.

Sections of $\bigwedge^h (T\mathbb{G})$ are differential h-forms, and sections of $\bigwedge_h (T\mathbb{G})$ are h-vector fields, $1 \le h \le n$. If $\mathcal{U} \subset \mathbb{G}$ is an open set, we denote by $\Omega^h(\mathcal{U})$ and $\Omega_h(\mathcal{U})$ the spaces of h-forms and of h-vectors in $\mathcal{U}$. If $\mathcal{U} = \mathbb{G}$, we write simply Ω^h and Ω_h.

With the notations of [17], Chapter 2, Section 2.1, if V, W are finite dimensional linear vector spaces and $L : V \to W$ is a linear map, we denote by

$$\Lambda^h L : \bigwedge^h W \to \bigwedge^h V$$

the linear map induced by L between exterior powers (see Definition 7.2 in Appendix A).

We stress now that the aim of this paper is to compare Maxwell's equation in $\mathbb{G}$ with usual Maxwell's equations in the Euclidean space $\mathbb{R}^n$. Since different scalar products are involved, we must proceed carefully and to establish precise definitions to take into account the two different (superposed) structures. Indeed, all our equations are written on the exterior algebra of the cotangent bundle of $\mathbb{G}$, but they are associated with different scalar products, coming ultimately, from two different group structures on $\mathbb{G}$, i.e. $(\mathbb{G}, \cdot)$ and the Abelian Euclidean group $(\mathbb{G}, +)$.

In particular for any $x \in \mathbb{G}$ two families of (left) translations $\tau_x : \mathbb{G} \to \mathbb{G}$ and $t_x : \mathbb{G} \to \mathbb{G}$ are defined respectively by

$$\tau_x p := x \cdot p\,, \quad t_x p := x + p\;.$$

Thanks to the identification of $T\mathbb{G}_e$ with $\mathfrak{g}$, the maps

$$(x, \alpha) \longrightarrow \big(x, \Lambda^h(d\tau_x(e))\alpha\big) \quad \text{from} \quad \bigwedge^h (T\mathbb{G}) \quad \text{to} \quad \mathbb{G} \times \bigwedge^h \mathfrak{g}$$

and

$$(x, \alpha) \longrightarrow \big(x, \Lambda^h(dt_x(e))\alpha\big) \quad \text{from} \quad \bigwedge^h (T\mathbb{G}) \quad \text{to} \quad \mathbb{G} \times \bigwedge^h \mathfrak{g}$$

define two canonical (global) trivializations of the vector bundle $\bigwedge^h (T\mathbb{G})$ associated with the two families τ_x and t_x of left translations in $\mathbb{G}$.

Starting from the scalar product defined in $\bigwedge^h \mathfrak{g}$, each trivialization defines on the fiber $\bigwedge^h (T\mathbb{G})_p$ over $p \in \mathbb{G}$ two scalar products

$$\langle \Lambda^h d\tau_p(\alpha), \Lambda^h d\tau_p(\beta)\rangle_p := \langle \alpha, \beta\rangle$$

and

$$\langle \Lambda^h dt_p(\alpha), \Lambda^h dt_p(\beta)\rangle_p := \langle \alpha, \beta\rangle_{\mathrm{Euc}}.$$

Analogously, τ and t define on $\bigwedge^h (T\mathbb{G})_p$ two (orthonormal) bases Θ^h_p and $\Theta^{h,\mathrm{Euc}}_p$ through the identities

$$\Theta^h_p := \Lambda^h d\tau_{p^{-1}}(e)\Theta^h \quad \text{and} \quad \Theta^{h,\mathrm{Euc}}_p := \Lambda^h dt_{-p}(e)\Theta^h.$$

In particular,

$$\Theta^{h,\mathrm{Euc}}_p = \Lambda^h(dt_{-p}(p))\Lambda^h(d\tau_p(e))\Theta^h_p.$$

Obviously, a parallel argument can be carried out for h-vectors in $\bigwedge_h (T\mathbb{G})$.

From now on, if $\alpha \in \bigwedge^h \mathfrak{g}$, we denote by $\alpha_{\mathfrak{E}} = (\alpha_{\mathfrak{E},1}, \alpha_{\mathfrak{E},2}, \dots)$ and by $\alpha_{\mathfrak{g}} = (\alpha_{\mathfrak{g},1}, \alpha_{\mathfrak{g},2}, \dots)$ the vector of its coordinates with respect to the bases $\Theta^{h,\mathrm{Euc}}$ and Θ^h, respectively.

It is easy to see that

$$\langle \alpha, \beta \rangle = \alpha_{\mathfrak{g}} \bullet \beta_{\mathfrak{g}} \quad \text{and } \langle \alpha, \beta \rangle_{\mathrm{Euc}} = \alpha_{\mathfrak{E}} \bullet \beta_{\mathfrak{E}}, \tag{8}$$

where $\bullet$ denotes the Euclinean scalar product in $\mathbb{R}^N$, with $N = \binom{n}{h}$.

Definition 2.2. The dual space $\bigwedge^1(\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$.

The action of a h-covector φ on a h-vector v is denoted as $\langle \varphi | v \rangle$. The same notation is used for the duality between forms and vector fields.

If $v \in \Omega_h$ we define $v^\natural \in \Omega^h$ by the identity

$$\langle v^\natural | w \rangle := \langle v, w \rangle, \quad \langle v^{\natural,\mathrm{Euc}} | w \rangle := \langle v, w \rangle_{\mathrm{Euc}}$$

and analogously we define $\varphi^\natural \in \Omega_h$ and $\varphi^{\natural,\mathrm{Euc}} \in \Omega_h$ for $\varphi \in \Omega^h$.

However, to avoid cumbersome notations, in the sequel we write also $v^\natural$ and $\phi^\natural$ with respect to the Euclidean scalar product, if this is evident from the context.

3. Maxwell's equations in matter

Classical Maxwell's equations in a bounded smooth connected region $\mathcal{U} \subset \mathbb{R}^3$, for time-harmonic vector fields in $\mathbb{R}_s \times \mathbb{R}^3_x$

$$e^{i\omega s}\vec{E},\ e^{i\omega s}\vec{H},\ e^{i\omega s}\vec{B},\ e^{i\omega s}\vec{D},$$

with no charges and no currents, read as follows:

$$\begin{aligned} \operatorname{curl} \vec{H} - i\omega \vec{D} = 0 \quad \text{in } \mathcal{U}, \\ \operatorname{curl} \vec{E} + i\omega \vec{B} = 0 \quad \text{in } \mathcal{U}, \\ \operatorname{div} \vec{D} = 0, \quad \operatorname{div} \vec{B} = 0 \quad \text{in } \mathcal{U}, \end{aligned}$$

together with the constitutive relations

$$\vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H},$$

and the boundary condition

$$\vec{E}_{\tan}\Big|_{\partial\mathcal{U}} = \vec{F} \quad \text{on } \partial\mathcal{U}. \tag{9}$$

Here $\varepsilon = \varepsilon(x), \mu = \mu(x) \in GL(\mathbb{R}^3)$ are invertible linear maps depending (say) smoothly on x and $\vec{F}$ is a tangent vector field on $\partial\mathcal{U}$. Usually, the matrices $[\varepsilon], [\mu]$ are called respectively the dielectric permittivity and the magnetic permeability.

From previous equations we get the following differential equation stisfied by $\vec{D}$.

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \varepsilon^{-1} \vec{D} = \omega^2 \vec{D}. \tag{10}$$

Consider now the 1-forms $E = \vec{E}^\natural$, $H = \vec{H}^\natural$ and the 1-foms $*D := -\vec{D}^\natural$ and $*B := -\vec{B}^\natural$. The duality in the previous expression is meant with respect to the Euclidean scalar product $\langle \cdot, \cdot \rangle_{\mathrm{Euc}}$, as stated in Definition 2.2. Since, trough all this section, we shall refer only to the "Euclidean" duality between forms and vector fields, in the sequel, we shall write e.g. $D^\natural$ instead of $D^{\natural,\mathrm{Euc}}$.

On the other hand, keeping in mind that, if $\vec{V}$ is a vector field in $\mathbb{R}^3$, then curl $\vec{V} = \big(* d\vec{V}^\natural\big)^\natural$. Using Proposition 7.3 of Appendix A, from the equations above we obtain

$$*dE = i\omega * B\,, \quad , \quad \delta * B = 0, \tag{11}$$

and

$$*dH = -i\omega * D\,, \quad , \quad \delta * D = 0, \tag{12}$$

together with the constitutive relations

$$*B = (\Lambda^1\mu^*)H, \qquad *D = (\Lambda^1\varepsilon^*)E. \tag{13}$$

where in the last equations we use Proposition 7.3 - i) of Appendix A. If we sobstitute equation (11) in equation (12), taking also into account the constitutive relations (13), we get

$$*d(\Lambda^1\mu^*)^{-1} * d(\Lambda^1\varepsilon^*)^{-1} * D = \omega^2 * D. \tag{14}$$

Thus, we have the following result.

Proposition 3.1. *Let $\vec{D} \in \Omega_1(\mathcal{U})$ be a solution of the differential equation* (10)*. Then the* 1*-form $\alpha = *D \in \Omega^1(\mathcal{U})$ satisfies the differential equation*

$$\delta M d N\alpha - \omega^2\alpha = 0, \quad \delta\alpha = 0 \tag{15}$$

where $M := (\det[\mu])^{-1}(\Lambda^2\mu)$, $N := \Lambda^1(\varepsilon^*)^{-1}$. Moreover the boundary condition (9) *becomes*

$$t(N\alpha) = \phi,$$

where ϕ is a 1-form on $\partial\mathcal{U}$ and $t(N\alpha)$ is the tangential part of $N\alpha$ (see [17]*, Section 2.6, and Proposition 4.6 below).*

Proof. As shown above, the equation in (10) is equivalent to equation (14).

Since on a 1-form

$$*d = \delta*\,,$$

(see (62) in the Appendix A), from equation (14) we get

$$\delta * (\Lambda^1\mu^*)^{-1} * d(\Lambda^1\varepsilon^*)^{-1} * D - \omega^2 * D = 0.$$

Notice now that, by Lemma 7.4 in Appendix A,

$$*(\Lambda^1(\mu^*)^{-1})* = * * (\det[\mu])^{-1}(\Lambda^2\mu) = (\det[\mu])^{-1}(\Lambda^2\mu).$$

Hnece, we get that $*D$ satisfies (15), and we are done. □

Notice that, by (15), $*D$ is a stationary point of the functional

$$J^{\mu,\varepsilon}(\alpha) := \int_{\mathcal{U}} \langle M dN\alpha, dN\alpha\rangle_{\mathrm{Euc}}\, dV - \omega^2 \int_{\mathcal{U}} \langle N\alpha, \alpha\rangle_{\mathrm{Euc}}\, dV,$$

in $\{\alpha \in W^{1,2}(\mathcal{U}, \bigwedge^1 T\mathbb{R}^3),\ t(N\alpha) = \phi \text{ in } \partial\mathcal{U}\} =: W^{1,2}_{DN}(\mathcal{U}, \bigwedge^1 T\mathbb{R}^3)$.

A standard approach (see Appendix C) to the Dirichlet problem with relative boundary conditions in a bounded open set $\mathcal{U}$ for system (15) relies on a variational argument for the functional

$$\begin{aligned}\tilde{J}^{\mu,\varepsilon}(\alpha) := \int_{\mathcal{U}} \langle M dN\alpha, dN\alpha\rangle_{\text{Euc}}\, dV + \sigma \int_{\mathcal{U}} |\delta\alpha|^2\, dV \\ + C \int_{\mathcal{U}} \langle N\alpha, \alpha\rangle_{\text{Euc}}\, dV\end{aligned} \tag{16}$$

in $W^{1,2}_{DN}(\mathcal{U}, \bigwedge^1 \mathbb{R}^3)$. Here, $\sigma > 0$ is a positive parameter, and $C > 0$ is a large constant.

We stress that all the peculiarities of the structure of Maxwell's equations are all contained in the functional (16). Indeed, the further steps of the technique (Steps 3–5 of Appendix C) rely on standard compactness arguments and on regularity properties of solutions of elliptic pde's.

Finally, we notice that (15) and all the previous arguments make perfectly sense in $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Indeed, let $\mathcal{U} \subset \mathbb{R}^n$ be a smooth connected region. Let $\varepsilon = \varepsilon(x), \mu = \mu(x) \in GL(\mathbb{R}^n)$ be invertible linear maps depending (say) smoothly on x. Consider two differential forms E and H in $\mathcal{U}$, with $E \in \Omega^1(\mathcal{U})$ and $H \in \Omega^{n-2}(\mathcal{U})$. If $*$ denotes the Hodge-star operator in $\mathbb{R}^n$, the n-dimensional Maxwell's equations for time-harmonic forms in an anisotropic medium read as follows,

$$*dE = i\omega\, (\Lambda^{n-2}\mu^*)H\,, \quad , \quad \delta(\Lambda^{n-2}\mu^*)H = 0 \tag{17}$$

and

$$*dH = -i\omega\, (\Lambda^1\varepsilon^*)E\,, \quad , \quad \delta(\Lambda^1\varepsilon^*)E = 0. \tag{18}$$

Arguing as above, the 1-form $\alpha = (\Lambda^1\varepsilon^*)E$ satisfies equation (15). Again, the study of equation (15) can be reduced to the study of a functional akin to (16).

4. Maxwell's equations in group and related functionals

From now on, we denote by (E_0^*, d_c) the complex of intrinsic differential forms that is described in detail in Appendix B.

If we consider the time-harmonic differential forms $e^{i\omega s}E, e^{i\omega s}H$ in the "free" space $\mathbb{R}_s \times \mathbb{G}_x$, where $E \in E_0^1(\mathbb{G})$ and $H \in E_0^{n-2}(\mathbb{G})$, equations (5) and (6) became

$$*d_cE = i\omega H \qquad *\, d_cH = -i\omega E, \tag{19}$$

$$\delta_c H = 0 \qquad \delta_c E = 0. \tag{20}$$

Hence, $\alpha = E$ satisfy

$$\delta_c\; d_c\alpha - \omega^2\alpha = 0 \qquad \delta_c\alpha = 0. \tag{21}$$

Clearly, if we consider equations (19) and (20) in the open set $\mathcal{U}$, we must couple equation (21) with suitable boundary conditions (see below). Then

α is also a stationary point of the functional

$$\int_{\mathcal{U}} |d_c\alpha|^2\, dV - \omega^2 \int_{\mathcal{U}} |\alpha|^2\, dV.$$

Mimicking the Euclidean approach we have to consider the functional

$$\int_{\mathcal{U}} |d_c\alpha|^2\, dV + \sigma \int_{\mathcal{U}} |\delta_c\alpha|^2\, dV + C \int_{\mathcal{U}} |\alpha|^2\, dV, \tag{22}$$

which still hides all the peculiarities of the structure of our intrinsic Maxwell's equations.

So far, our argument are rather vague. To make it accurate, we need to set precisely the functions spaces involved in our problem, as well as the boundary condition and to prove a formula of integration by parts for d_c and δ_c (see Theorem 4.19 below).

First of all, since from now on we focus our interest in the functionals instead of the equation, without loss of generality we take $\sigma = 1$. Nhe natural setting for the functional (22) is provided by Folland-Stein function spaces that are left-invariant and group-homogeneous.

Since here we are dealing only with integer order spaces, we can give this simpler definition (for a general presentation, see e.g. [11]).

Definition 4.1. Let $\mathcal{U} \subset \mathbb{G}$ be an open set. If $1 < s < \infty$ and $m \in \mathbb{N}$, then the space $W^{m,s}_{\mathbb{G}}(\mathcal{U})$ is the space of all $u \in L^s(\mathcal{U})$ such that

$$X^I u \in L^s(\mathcal{U}) \quad \text{for all multi-index } I \text{ with } d(I) = m,$$

endowed with the natural norm.

We remind that

Proposition 4.2. *Let $\mathcal{U} \subset \mathbb{G}$ be an open set. If $1 < s < \infty$ and $m \in \mathbb{N}$, then the space $W^{m,s}_{\mathbb{G}}(\mathcal{U})$ is independent of the choice of $X_1, \dots, X_{m_1}$.*

Proposition 4.3. *Let $\mathcal{U} \subset \mathbb{G}$ be an open set. If $1 < s < \infty$ and $m \in \mathbb{N}$, then $\mathcal{E}(\mathcal{U}) \cap W^{m,s}_{\mathbb{G}}(\mathcal{U})$ is dense in $W^{m,s}_{\mathbb{G}}(\mathcal{U})$.*

The following result can be derived from Theorems 10.4 and 13.5 in [9], thanks also to [22]

Theorem 4.4. *Let $\mathcal{U} \subset \mathbb{G}$ be an open connected smooth region, locally lying on one side of its boundary $\partial\mathcal{U}$. Then*

i) *$\mathcal{E}(\bar{\mathcal{U}})$ is dense in $W^{m,s}_{\mathbb{G}}(\mathcal{U})$;*
ii) *there exists a bounded linear map*

$$\mathcal{P} : W^{1,2}_{\mathbb{G}}(\mathcal{U}) \longrightarrow W^{1,2}_{\mathbb{G}}(\mathbb{G}) \quad \textit{such that} \quad \mathcal{P}u_{\big|\mathcal{U}} = u$$

for any $u \in W^{1,2}_{\mathbb{G}}(\mathcal{U})$;
iii) *the linear map*

$$\gamma : \mathcal{E}(\bar{\mathcal{U}}) \longrightarrow \mathcal{E}(\partial\mathcal{U}), \quad \gamma u = u_{\big|\partial\mathcal{U}}$$

can be continued as a bounded map

$$\gamma : W^{1,2}_{\mathbb{G}}(\mathcal{U}) \longrightarrow L^2(\partial\mathcal{U}, d\sigma),$$

where

$$d\sigma = \Big(\sum_{j=1}^m \langle X_j, \nu\rangle^2\Big)^{1/2} d\mathcal{H}^{n-1}, \tag{23}$$

ν being the outward unit normal to $\partial\mathcal{U}$.

Definition 4.5. Let $\mathcal{U} \subset \mathbb{G}$ be an open set. If $0 \le h \le n$, $1 \le s \le \infty$ and $m \ge 0$, we denote by $W^{m,s}_{\mathbb{G}}(\mathcal{U}, \bigwedge^h \mathfrak{g})$ the space of all sections of $\bigwedge^h \mathfrak{g}$ such that their components with respect to the basis Θ^h belong to $W^{m,s}_{\mathbb{G}}(\mathcal{U})$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself.

Moreover, if E_0^h is the class of the intrinsic h-forms, then the space $W^{m,s}_{\mathbb{G}}(\mathcal{U}, E_0^h)$ are defined analogously, replacing Θ^h by Ξ^h (see Remark 8.9 in Appendix B).

Obviously, Propositions 4.2, 4.3 and Theorem 4.4 still hold for form-valued spaces.

If $\alpha \in \mathbf{C}^\infty(\partial\mathcal{U}, \bigwedge^h \mathfrak{g})$, as in [17], Section 2.6, we denote by $t(\alpha)$ and $n(\alpha)$ the tangential and the normal parts of α, respectively. We stress that both t and n can be extended to continuous linear maps from $L^2(\partial\mathcal{U}, \bigwedge^h \mathfrak{g})$ to $L^2(\partial\mathcal{U}, \bigwedge^h \mathfrak{g})$, since for any $x \in \partial\mathcal{U}$ $|\alpha(x)|^2 = |t(\alpha)(x)|^2 + |n(\alpha)(x)|^2$. It is important to stress that here and in the sequel, the space $L^2(\partial\mathcal{U}, \bigwedge^h \mathfrak{g})$ has to be understood with respect to the measure $d\sigma$.

Thus, the following proposition follows by Theorem 4.4.

Proposition 4.6. *If $\alpha \in \mathbf{C}^\infty(\bar{\mathcal{U}}, \bigwedge^h \mathfrak{g})$ and we denote by γ the trace operator*

$$\gamma : \mathbf{C}^\infty(\bar{\mathcal{U}}, \bigwedge^h \mathfrak{g}) \to \mathbf{C}^\infty(\partial\mathcal{U}, \bigwedge^h \mathfrak{g}), \quad \gamma(\alpha) = \alpha_{\big|\partial\mathcal{U}},$$

the maps

$$\alpha \longrightarrow t(\gamma(\alpha)), \quad \alpha \longrightarrow n(\gamma(\alpha))$$

can be extended as linear continuous maps from $W^{1,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^h \mathfrak{g})$ to $L^2(\partial\mathcal{U}, \bigwedge^h \mathfrak{g})$.

For sake of simplicity, from now on, if $\alpha \in W^{1,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^h \mathfrak{g})$, we write $t(\alpha)$ and $n(\alpha)$ instead of $t(\gamma(\alpha))$ and $n(\gamma(\alpha))$, respectively. This notation, though usually established in the literature, may yield ambiguities in our setting. A point must be stressed: we apply first the trace operator, and then we decompose the trace in his tangential and normal parts. In fact, it might be possible to decompose first the form in a neighorhood of the boundary, and to apply later the trace operator. This distinction would not have any influence in the Euclidean setting, but here is crucial. Indeed the decomposition is not compatible with the Folland-Sobolev spaces, since it mixes variable of different layers of the stratification.

Assumption. *From now on, we assume $\mathbb{G}$ is a free Carnot group of step 2.*

We remind that, roughly speaking, the group $\mathbb{G}$ is said to be free if its Lie algebra is free, i.e. the commutators satisfy no linear relationships other than antisymmetry and the Jacobi identity. This is a large and relevant class of Carnot groups. We remind also that Carnot groups can always be "lifted" to free groups (see [23] and [6], Chapter 17). For our purposes, the main property of free Carnot groups relies on the fact that intrinsic 1-forms *and 2-forms on free groups have all the same weight (see Theorem 8.16). This helps at several steps of the proofs.*

We stress that several statements below and, in particular, the main result Theorem 5.7 still hold in larger classes of Carnot groups (see, for instance, Remark 6.1). However, for sake of simplicity, we do not make our intermediate statements as sharp as possible.

Proposition 4.7. *Let $\mathbb{G}$ be a free Carnot group of step 2. If Π_E denotes the lifting operator from the complex (E_0^*, d_c) to (E, d) defined in Appendix B, Theorem 8.11, the linear map*

$$\alpha \longrightarrow t(\Pi_E \alpha)$$

is well defined and continuous from $W^{2,2}_{\mathbb{G}}(\mathcal{U}, E_0^h)$ to $L^2(\partial\mathcal{U}, \bigwedge^h \mathfrak{g})$.

Proof. The assertion follows from Proposition 4.6, since Π_E is an operator of order 1 in the horizontal derivatives, by Theorem 8.11 in Appendix B. □

Now, we can define precisely our intrinsic functional.

Definition 4.8. We define a functional in $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ as follows:

$$J(\alpha) = \begin{cases} \displaystyle\int_{\mathcal{U}} |d_c\alpha|^2\, dV + \int_{\mathcal{U}} |\delta_c\alpha|^2\, dV + C\int_{\mathcal{U}} |\alpha|^2\, dV \\ \qquad\qquad \text{if } \alpha \in W^{2,2}_{\mathbb{G}}(\mathcal{U}, E_0^1) \text{ and } t(\Pi_E\alpha) = 0 \\ +\infty \quad \text{otherwise.} \end{cases}$$

Remark 4.9. The boundary condition $t(\Pi_E\alpha) = 0$ il motivated by Green formula of Theorem 4.19 at the end of this Section.

Maxwell's equations, as well the associated functional (16), depend on the scalar product on the fibers of $\bigwedge^* T\mathbb{G}$ (as for the equations, through the Hodge operator $*$). In fact, they are invariant under the action of Lorentz transformations for the Minkowskian metric associated with the scalar product. But it is important to point out that classical Maxwell's equations are associated with the Euclidean scalar product on the fibers $\langle\cdot,\cdot\rangle_{\mathrm{Euc}}$, whereas group Maxwell's equations are associated with $\langle\cdot,\cdot\rangle$.

Thus, if we want to see intrinsic Maxwell's equations as a limit of classical Maxwell's equations in strongly anisotropic media, we have to take into account the change of the scalar product through a change of coordinates in the tangent bundle $\bigwedge_1 T\mathbb{G} = T\mathbb{G}$.

Remark 4.10. In particular, though the exterior differential does not depend on the coordinates, the codifferential does depend on the scalar product. Thus, from now on, we denote by δ the codifferential associated with the Euclidean scalar product $\langle\cdot,\cdot\rangle_{\mathrm{Euc}}$, and by $\delta_{\mathfrak{g}}$ the codifferential associated with the intrinsic scalar product $\langle\cdot,\cdot\rangle$. We remind that δ_c denotes instead the intrinsic codifferential, associated obviously with the intrinsic scalar product.

Definition 4.11. We define the linear map

$$R : \bigwedge\nolimits_1 T\mathbb{G} \longrightarrow \bigwedge\nolimits_1 T\mathbb{G} \quad \langle \Lambda_1 R u, v\rangle_{\mathrm{Euc}} = \langle u, v\rangle,$$

for any $u, v \in \Omega_1$. Obviously, if we set $R^h := \Lambda^h R$, then

$$\langle R^h\alpha, \beta\rangle_{\mathrm{Euc}} = \langle\alpha, \beta\rangle$$

for any $\alpha, \beta \in \Omega^h$.

Denote now by $[T]_p = (T_{ij}(p))_{ij}$ the matrix of the identity map in $\mathbb{R}^n$ with respect to the bases $\{e_1, \dots, e_n\}$ and $\{X_1, \dots, X_n\}$, i.e. $[T]_p$ satisfies

$$X_j(p) = \sum_i T_{ij}(p) e_i, \quad j = 1, \dots, n.$$

Remark 4.12. $[T]_p$ coincides with $[(dt_{-p})(p) \circ (d\tau_p)(e)]$, the matrix of $(dt_{-p})(p) \circ (d\tau_p)(e) : \bigwedge_1 T\mathbb{G} \to \bigwedge_1 T\mathbb{G}$ with respect to the basis $\{e_1, \dots, e_n\}$.

By Proposition 8.2 in Appendix B, $\det[T] = 1$.

As customary, from now on we drop the index p when there is no ambiguity.

Proposition 4.13. *If $\alpha \in \Omega^h$, then*

$$\alpha_{\mathfrak{E}} = [\Lambda^h T] \alpha_{\mathfrak{g}}. \tag{24}$$

An analogous assertion follows if we replace covectors by vectors.

Moreover, the matrix of R with respect to the Euclidean coordinates of $\mathfrak{g}$ is given by

$$[R]^{-1} = [T] \cdot [T]^t.$$

In particular, by Remark 4.12, $\det[R] = 1$.

Proof. Denote by θ_p^j and $\theta_p^{j,\mathrm{Euc}}$ the j-element of the basis Θ_p^h and of Θ_p^{Euc}, respectively. By Remark 4.12, we have

$$\begin{aligned} \alpha_{\mathfrak{g},j}(p) &= \langle \alpha, \theta_p^j \rangle_p = \langle \Lambda^h d\tau_p(e) \alpha, \theta_e^j \rangle_e \\ &= \langle \Lambda^h dt_{-p}(p) \Lambda^h d\tau_p(e) \alpha, \theta_p^{j,\mathrm{Euc}} \rangle_{\mathrm{Euc},p} \\ &= \big(\Lambda^h dt_{-p}(p) \Lambda^h d\tau_p(e) \alpha \big)_{\mathfrak{E},j}(p). \end{aligned}$$

Then assertion (24) follows.

□

The identification of $\mathfrak{g}$ and $T\mathbb{G}_e$ induces a stratification of $T\mathbb{G}_e = \bigwedge_1 T\mathbb{G}_e$ given by $\bigwedge_1 T\mathbb{G}_e = \bigwedge_1 V_1 \oplus \bigwedge_1 V_2$. Hence, by left translation, we define also two fibre bundles on $\mathbb{G}$, $\bigwedge_1 V_i$, $i = 1, 2$.

Definition 4.14. If $r > 0$, we denote by $C_r : \bigwedge_1 T\mathbb{G} \to \bigwedge_1 T\mathbb{G}$ the linear bundle map defined on a generic fiber by

$$C_r(X_\ell) := r^j X_\ell \quad \text{if } X_\ell \in \bigwedge\nolimits_1 V_j, \quad j = 1, 2. \tag{25}$$

Notice $(\Lambda^2 C_r) \theta^i \wedge \theta^j = r^{w(i)+w(j)} \theta^i \wedge \theta^j$ if $w(i), w(j)$ are the weights of the $\theta^i, \theta^j \in \bigwedge^1 \mathfrak{g}$, respectively.

We denote by $[C_r]_{\mathfrak{g}}$ the (diagonal) matrix associate with C_r with respect to the basis $\{X_1, \dots, X_n\}$.

Lemma 4.15. *If $r > 0$, let d_r be the "weighted exterior differential"*

$$d_r = d_0 + r d_1 + r^2 d_2.$$

Then

$$d_r \alpha = (\Lambda^2 C_r) d (\Lambda^1 C_r^{-1}) \alpha$$

for any $\alpha \in \Omega^1$.

Proof. Choose $\alpha = \alpha_i \theta^i$. Then

$$d_r\alpha = \sum_{j=0}^{2} r^j d_j(\alpha_i\theta^i) = \sum_{j=0}^{2} r^{j+w(i)} d_j(r^{-w(i)}\alpha_i\theta^i) = (\Lambda^2 C_r) d(\Lambda^1 C_r^{-1})\alpha.$$

□

Proposition 4.16. *Let $r > 0$ be given. Let us choose*

$$\mu_r = r^{-2(Q-3)/(n-2)} C_r^* R C_r,$$

and

$$\varepsilon_r = r^{-1}(RC_r)^*,$$

where C_r^* and $(RC_r)^*$ are the adjoint maps of C_r and RC_r, respectively, with respect to the Euclidean scalar product.

Then, if $\alpha \in W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g}) \Big(\subset W^{1,2}(\mathcal{U}, \bigwedge^1 \mathfrak{g}) \Big)$,

$$r^{-4}|d_r\alpha|^2 = \langle M_r dN_r R\alpha, dN_r R\alpha\rangle_{\text{Euc}} \tag{26}$$

and

$$r\langle C_r^{-1}\alpha, \alpha\rangle = \langle N_r \Lambda^1 R\alpha, \Lambda^1 R\alpha\rangle_{\text{Euc}}, \tag{27}$$

where $M_r := (\det[\mu_r])^{-1} \cdot \Lambda^2\mu_r$, and $N_r := \Lambda^1(\varepsilon_r^*)^{-1}$.

Proof. Keeping in mind 4.15, we have:

$$\begin{aligned}
|d_r\alpha|^2 &= \langle \Lambda^2(RC_r) d(\Lambda^1 C_r^{-1})\alpha, (\Lambda^2 C_r) d(\Lambda^1 C_r^{-1})\alpha\rangle_{\text{Euc}} \\
&= \langle (\Lambda^2(RC_r)) d\Lambda^1(RC_r)^{-1} R\alpha, (\Lambda^2(C_r)) d\Lambda^1(RC_r)^{-1} R\alpha\rangle_{\text{Euc}} \\
&= \langle \Lambda^2(C_r^* RC_r) d\Lambda^1(RC_r)^{-1} R\alpha, d\Lambda^1(RC_r)^{-1} R\alpha\rangle_{\text{Euc}}.
\end{aligned}$$

Then assertion (26) follows since $(\Lambda^2\sigma L) = \sigma^2(\Lambda^2 L)$ for any linear map L and for any $\sigma \in \mathbb{R}$, and

$$\det \mu_r = r^{2(3n-2Q)/(n-2)}.$$

This proves (26). Identity (27) follows analogously. □

If in the "real world" functional $\tilde{J}^{\mu,\varepsilon}$ in (16) we choose $\mu = \mu_r$, $\varepsilon = \varepsilon_r$ as above, we obtain a sequence of functionals $(\tilde{J}^{\mu_r,\varepsilon_r})_{r>0}$ with boundary conditions $t(N_r\alpha) = 0$.

As already pointed out, the functionals $\tilde{J}^{\mu_r,\varepsilon_r}$ contain all the informations about Maxwell's equations in the matter. Now it is clear in what sense we can think of intrinsic Maxwell's equations in Carnot groups as limits of usual Maxwell's equations for anisotropic media: the Euclidean "energy" functionals $\tilde{J}^{\mu_r,\varepsilon_r}$ Γ-converge to "energy" functional associated with the intrinsic Maxwell's equations in the group as in Definition 4.8.

However, at this point it is important to notice that this convergence has a meaning provided the functionals are all written in the same coordinates, i.e. through the same trivialization. Because of the privileged role in our approach of the limit functional, we choose to write all functionals in an intrinsic way. Therefore, through the map R^1, we write $\tilde{J}^{\mu_r,\varepsilon_r}$ in the trivialization associated with group translations, even though its "physical" meaning appears when we write $\tilde{J}^{\mu_r,\varepsilon_r}$ through the usual Euclidean trivialization.

Definition 4.17. With the notations of Proposition 4.16, if $r > 0$, we denote by J^r the functional in $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$

$$J^r(\alpha) := \begin{cases} J^{\mu_r,\varepsilon_r}(R^1\alpha) & \text{if } t(N_r R^1\alpha) = 0 \quad \text{on } \partial\mathcal{U} \\ +\infty & \text{otherwise.} \end{cases} \tag{28}$$

In other words, J^r is the variational functional associated with the classical Maxwell's equations in $\mathbb{R}^n$ for an anisotropic medium with magnetic permeability $[\mu_r]$ and dielettric permittivity $[\varepsilon_r]$ given by

$$[\mu_r] = r^{-2(Q-3)/(n-2)}\, [T^{-1}]^t \cdot [C_r]^2_{\mathfrak{g}} \cdot [T^{-1}] \tag{29}$$

and

$$[\varepsilon_r] = r^{-1}\, [T^{-1}]^t \cdot [C_r]_{\mathfrak{g}} \cdot [T^{-1}]. \tag{30}$$

Proposition 4.18. *If $\alpha \in W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$, we have*

i) $\delta_{\mathfrak{g}}(\alpha) = \delta(R^1\alpha)$;

ii) $J^r(\alpha) = r^{-4}\int_{\mathcal{U}} |d_r\alpha|^2\, dV + \int_{\mathcal{U}} |\delta_{\mathfrak{g}}\alpha|^2\, dV + Cr\int_{\mathcal{U}}\langle C_r^{-1}\alpha, \alpha\rangle\, dV$ *when* $J^r(\alpha) < \infty$.

Proof. Assertion i) is a trivial consequence of the definition of R. Assertion ii) follows by Proposition 4.16,. □

We conclude this Section going back to the condition $t(\Pi_E\alpha) = 0$ that is justified by the following Green formula.

Theorem 4.19 (Green formula)**.** *Let $\mathbb{G}$ be a Carnot group of arbitrary step. If $\alpha \in C^\infty(\bar{\mathcal{U}}, E_0^{h+1})$, $0 \le h \le n-1$, and $\beta \in C^\infty(\bar{\mathcal{U}}, E_0^h)$, we have*

$$\int_{\mathcal{U}} \langle \alpha, d_c\beta\rangle\, dV = \int_{\mathcal{U}} \langle \delta_c\alpha, \beta\rangle\, dV + \int_{\partial\mathcal{U}} t(\Pi_E\beta) \wedge *n(\alpha). \tag{31}$$

Proof. First of all, we notice the following identity holds:

$$\Pi_{E_0}\Pi_E + \Pi_F(I - \Pi_{E_0}\Pi_E) = I. \tag{32}$$

Indeed, by Theorem 8.11, $I = \Pi_E + \Pi_F = \Pi_E\Pi_{E_0}\Pi_E + \Pi_F = \Pi_{E_0}\Pi_E - \Pi_F\Pi_{E_0}\Pi_E + \Pi_F = \Pi_{E_0}\Pi_E + \Pi_F(I - \Pi_{E_0}\Pi_E)$.

Then, by classical Green formula (see e.g. [17], Chapter 5, Section 2.6, Proposition 2),

$$
\begin{aligned}
&\int_{\mathcal U}\langle\alpha, d_c\beta\rangle\, dV - \int_{\partial\mathcal U} t(\Pi_E\beta)\wedge *n(\alpha)\\
&\quad= \int_{\mathcal U}\langle\alpha, \Pi_{E_0} d\Pi_E\beta\rangle\, dV - \int_{\partial\mathcal U} t(\Pi_E\beta)\wedge *n(\alpha)\\
&\quad= \int_{\mathcal U}\langle\alpha, d\Pi_E\beta\rangle\, dV - \int_{\partial\mathcal U} t(\Pi_E\beta)\wedge *n(\alpha) \quad (\text{since } \alpha\in E_0^{h+1})\\
&\quad= \int_{\mathcal U}\langle\delta\alpha, \Pi_E\beta\rangle\, dV\\
&\quad= \int_{\mathcal U}\langle\delta\alpha, \Pi_E\beta\rangle\, dV = (-1)^{h(n-h)}\int_{\mathcal U}\langle * * \delta\alpha, \Pi_E\beta\rangle\, dV\\
&\quad= (-1)^{h(n-h)}\int_{\mathcal U}\langle *\Pi_{E_0}\Pi_E * \delta\alpha, \Pi_E\beta\rangle\, dV\\
&\quad+ (-1)^{h(n-h)}\int_{\mathcal U}\langle *\Pi_F(I-\Pi_{E_0}\Pi_E) * \delta\alpha, \Pi_E\beta\rangle\, dV \quad (\text{by } (32))\\
&\quad= (-1)^{h(n-h)}\int_{\mathcal U}\langle *\Pi_{E_0}\Pi_E * \delta\alpha, \Pi_E\beta\rangle\, dV \quad (\text{by Lemma 8.14})\\
&\quad= (-1)^{h(n-h+1)+1}\int_{\mathcal U}\langle *\Pi_{E_0}\Pi_E d * \alpha, \Pi_E\beta\rangle\, dV\\
&\quad= (-1)^{h(n-h+1)+1}\int_{\mathcal U}\langle *\Pi_{E_0} d\Pi_E * \alpha, \Pi_E\beta\rangle\, dV\\
&\quad= (-1)^{h(n-h+1)+1}\int_{\mathcal U}\langle * d_c * \alpha, \Pi_E\beta\rangle\, dV\\
&\quad= \int_{\mathcal U}\langle\delta_c\alpha, \Pi_E\beta\rangle\, dV = \int_{\mathcal U}\langle\Pi_{E_0}\delta_c\alpha, \Pi_E\beta\rangle\, dV \quad (\text{since } \delta_c\alpha\in E_0^h)\\
&\quad= \int_{\mathcal U}\langle\delta_c\alpha, \Pi_{E_0}\Pi_E\Pi_{E_0}\beta\rangle\, dV \quad (\text{since } \beta\in E_0^h)\\
&\quad= \int_{\mathcal U}\langle\delta_c\alpha, \Pi_{E_0}\beta\rangle\, dV\\
&\quad= \int_{\mathcal U}\langle\delta_c\alpha, \beta\rangle\, dV \quad (\text{again since } \beta\in E_0^h).
\end{aligned}
$$

Thus (31) is proved. □

5. Main results

In this section we state our main convergence result. We briefly recall the definition of Γ-convergence.

Definition 5.1 (see, e.g. [8], Chapter 4)**.** Let X be a topological space, and denote by $\mathcal N(x)$ the family of all open neighborhoods of x in X. If $h\in\mathbb N$, let

$$J_h:\ X\longrightarrow[-\infty,+\infty]$$

be functionals on X. We set

$$(\Gamma-\liminf_{h\to\infty} J_h)(x) = \sup_{V\in\mathcal N(x)}\ \liminf_{h\to\infty}\ \inf_{y\in V} J_h(y),$$

and

$$(\Gamma - \limsup_{h\to\infty} J_h)(x) = \sup_{V\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{y\in V} J_h(y).$$

If there exists $J : X \longrightarrow [-\infty, +\infty]$ such that

$$\Gamma - \liminf_{h\to\infty} J_h = \Gamma - \limsup_{h\to\infty} J_h = J,$$

then we say that $\{J_h\}_{h\in\mathbb{N}}$ Γ-converges to J in X, and we write

$$J = \Gamma - \lim_{h\to\infty} J_h.$$

In metric spaces we have the following characterization of Γ-convergence.

Proposition 5.2 (see, e.g., [8], Proposition 8.1)**.** *Let X be a metric space, and let*

$$J_r,\ J :\ X \longrightarrow [-\infty, +\infty]$$

with $r > 0$ be functionals on X. Then $\{J_r\}_{r>0}$ Γ-converges to J on X as r goes to zero if and only if the following two conditions hold:

1) *for every $u \in X$ and for every sequence $\{u_{r_k}\}_{k\in\mathbb{N}}$ with $r_k \to 0$ as $k \to \infty$, which converges to u in X, there holds*

$$\liminf_{k\to\infty} J_{r_k}(u_{r_k}) \geq J(u); \tag{33}$$

2) *for every $u \in X$ and for every sequence $\{r_k\}_{k\in\mathbb{N}}$ with $r_k \to 0$ as $k \to \infty$ there exists a subsequence (still denoted by $\{r_k\}_{k\in\mathbb{N}}$) such that $\{u_{r_k}\}_{k\in\mathbb{N}}$ converges to u in X and*

$$\limsup_{k\to\infty} J_{r_k}(u_{r_k}) \leq J(u) \tag{34}$$

In a metric space, to avoid cumbersome notations, from now on we write systematically $\lim_{r\to 0}$ to mean a limit with $r = r_k$, where $\{r_k\}_{k\in\mathbb{N}}$ is any sequence with $r_k \to 0$ as $k \to \infty$.

For a deep and detailed survey on Γ-convergence, we refer to the monograph [8].

Proposition 5.3. *Suppose $\alpha^r \to \alpha$ as $r \to 0$ in $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$. Then*

$$J(\alpha) \leq \liminf_{r\to 0} J_r(\alpha^r). \tag{35}$$

Proof. As usual, without loss of generality, we may assume $\liminf_{r\to 0} J_r(\alpha^r) < \infty$.

We can argue now following the guidelines of Theorem 5.1 in [2].

First of all, we show that

$$\int_{\mathcal{U}} |d_c\alpha|^2\, dV \leq \liminf_{r\to 0} r^{-4} \int_{\mathcal{U}} |d_r\alpha^r|^2\, dV. \tag{36}$$

To this end, keeping in mind (69), we write

$$\alpha^r = \alpha_1^r + \alpha_2^r,$$

with $\alpha_i^r \in \Omega^{1,i}$, $i = 1, 2$. Reordering the terms of $d_r\alpha^r$ according to their weights, we have the following orthogonal decomposition:

$$d_r\alpha^r = \big(d_0\alpha_2^r + rd_1\alpha_1^r\big) + \big(rd_1\alpha_2^r + r^2 d_2\alpha_1^r\big) + r^2 d_2\alpha_2^r. \tag{37}$$

Therefore we can write

$$
\begin{aligned}
r^{-4}\int_{\mathcal U} |d_r\alpha^r|^2\,dV = \; & r^{-4}\int_{\mathcal U} |d_0\alpha_2^r + r d_1\alpha_1^r|^2\,dV \\
& + r^{-2}\int_{\mathcal U} |d_1\alpha_2^r + r d_2\alpha_1^r|^2\,dV \\
& + \int_{\mathcal U} |d_2\alpha_2^r|^2\,dV.
\end{aligned}
\tag{38}
$$

Since $\liminf_{r\to 0} J_r(\alpha^r) < \infty$, we have

- if $r \in (0,1)$,

$$
r^{-1}\big(d_1\alpha_2^r + r d_2\alpha_1^r\big) \quad \text{is uniformly bounded in } L^2(\mathcal U, \textstyle\bigwedge^2 \mathfrak g)); \tag{39}
$$

- if $r \to 0$, then

$$
d_0\alpha_2^r + r d_1\alpha_1^r = O(r^2) \quad \text{in } L^2(\mathcal U, \textstyle\bigwedge^2 \mathfrak g)). \tag{40}
$$

Now, (40) yields eventually that

$$
d_0\alpha_2 = 0, \tag{41}
$$

since d_0 is algebraic and $\{\alpha^r\}_{0<r<1}$ is bounded in $W^{2,2}_{\mathbb G}(\mathcal U, \bigwedge^1 \mathfrak g)$. Thus, keeping in mind that $d_0(\bigwedge^{1,1} \mathfrak g) = \{0\}$, we can conclude that $\alpha \in \ker d_0 = E_0^1$, and therefore $\alpha = \alpha_1$.

Because of convergence of $(\alpha^r)_{r>0}$ in $W^{2,2}_{\mathbb G}(\mathcal U, \bigwedge^1 \mathfrak g)$ when $r \to 0$, it follows that

$$
\alpha_1^r \to \alpha_1 := (\Pi_E \alpha)_1 \tag{42}
$$

in $L^2(\mathcal U, \bigwedge^1 \mathfrak g)$ as $r \to 0$ as well as

$$
d_0^{-1} d_1 \alpha_1^r \to d_0^{-1} d_1 \alpha_1 := -(\Pi_E\alpha)_2 \quad \text{in } L^2(\mathcal U, \textstyle\bigwedge^1 \mathfrak g), \tag{43}
$$

since d_1 is an homogeneous differential operator in the horizontal derivatives of order 1 and d_0^{-1} is algebraic. By (40), we have now

$$
d_0\alpha_2^r + r d_1\alpha_1^r = O(r^2) \quad \text{in } L^2(\mathcal U, \textstyle\bigwedge^2 \mathfrak g). \tag{44}
$$

On the other hand, by Lemma 8.10, ii), $d_0^{-1} d_0 \alpha_2^r = \alpha_2^r$, since α_2^r has weight 2 and hence is orthogonal to $\Omega^{1,1} = \ker d_0$. Thus, keeping in mind that d_0^{-1} is algebraic, it follows from (44) that

$$
\frac{1}{r}\alpha_2^r + d_0^{-1} d_1 \alpha_1^r = O(r) \quad \text{in } L^2(\mathcal U, \textstyle\bigwedge^1 \mathfrak g), \tag{45}
$$

and therefore

$$
\frac{1}{r}\alpha_2^r \to (\Pi_E\alpha)_2 \quad \text{in } L^2(\mathcal U, \textstyle\bigwedge^1 \mathfrak g). \tag{46}
$$

Now, by (42) and (43), we obtain

$$
\frac{1}{r}\Big(d_1\alpha_2^r + r d_2 \alpha_1^r\Big) \longrightarrow d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1 \tag{47}
$$

as $r \to 0$ in the sense of distributions. On the other hand, the limit $d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1$ belongs to $L^2(\mathcal{U}, \bigwedge^2 \mathfrak{g})$ (since $d_\ell(\Pi_E\omega)_{3-\ell}$ is an homogeneous differential operator in the horizontal derivatives of order 2, by Theorem 8.11, i) and Definition 8.6), and

$$\left\{\frac{1}{r}\big(d_1\alpha_2^r + rd_2\alpha_1^r\big)\right\}_{r>0} \quad \text{is equibounded in } L^2(\mathcal{U}, \textstyle\bigwedge^2 \mathfrak{g}), \tag{48}$$

as $r \to 0$, by (39). Combining (48) and (47) we obtain that the limit in (47) is in fact a weak limit in $L^2(\mathcal{U}, \bigwedge^2 \mathfrak{g})$

Thus, if we keep in mind Theorem 8.16, we obtain

$$\begin{aligned}\int_{\mathcal{U}} |d_c\alpha|^2\, dV &= \int_{\mathcal{U}} |\Pi_{E_0}\big(d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1\big)|^2\, dV \\ &\le \int_{\mathcal{U}} |\big(d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1\big)|^2\, dV \\ &\le \liminf_{r\to 0} r^{-2} \int_{\mathcal{U}} |d_1\alpha_2^r + rd_2\alpha_1^r|^2\, dV \le \liminf_{r\to 0} r^{-4} \int_{\mathcal{U}} |d_r\alpha^r|^2\, dV.\end{aligned}$$

This proves (36).

Let us consider now the divergence term in $J_r(\alpha^r)$. Since $\liminf_{r\to 0} J_r(\alpha^r) < \infty$, we have that

$$(\delta_g\alpha^r)_{r>0} \quad \text{is uniformly bounded in } L^2(\mathcal{U}). \tag{49}$$

On the other hand, we can write $\delta_g\alpha^r$ as

$$\delta_g\alpha^r = \delta_g\alpha_1^r + \delta_g\alpha_2^r, \tag{50}$$

where, for $i = 1, 2$, $\delta_g\alpha_i^r$ is a sum of terms of the form $X_\ell(\alpha^r)_\ell$, where $(\alpha^r)_\ell$ is the ℓ-th component of α^r with respect to the basis $\theta_1, \dots, \theta_n$, and $X_\ell \in V_i$. But $\delta_g\alpha_1^r \to \delta_g\alpha_1 = \delta_c\alpha$ in $L^2(\mathcal{U})$ (since $\delta_g\alpha_1^r$ contains only first order horizontal derivatives), and therefore $\delta_g\alpha_2^r$ weakly converge in $L^2(\mathcal{U})$. On the other hand, $\delta_g\alpha_2^r \to \delta_g\alpha_2 = 0$ in the sense of distributions, since $\alpha^r \to \alpha$ in $L^2(\mathcal{U})$, and α is horizontal (as seen in (42)). Thus, eventually,

$$\delta_g\alpha^r \to \delta_c\alpha \quad \text{weakly in } L^2(\mathcal{U}), \tag{51}$$

and therefore

$$\int_{\mathcal{U}} |\delta_c\alpha|^2 dV \le \liminf_{r\to 0} \int_{\mathcal{U}} |\delta_g\alpha^r|^2 dV. \tag{52}$$

On the other hand, if we split again α^r gathering the terms by their weights (i.e. $\alpha^r = \alpha_1^r + \alpha_2^r$, with $\alpha_j^r \in \Omega^{1,j}$), keeping in mind that $\alpha_1^r \to \alpha_1$ in L^2, we have

$$\begin{aligned}\liminf_{r\to 0} \int_{\mathcal{U}} r\langle C_r^{-1}\alpha^r, \alpha^r\rangle\, dV &= \liminf_{r\to 0} \Big(\int_{\mathcal{U}} |\alpha_1^r|^2\, dV + \frac{1}{r}\int_{\mathcal{U}} |\alpha_2^r|^2\, dV\Big) \\ &\ge \liminf_{r\to 0} \int_{\mathcal{U}} |\alpha_1^r|^2\, dV = \int_{\mathcal{U}} |\alpha_1|^2\, dV = \int_{\mathcal{U}} |\alpha|^2\, dV,\end{aligned} \tag{53}$$

since $\alpha \in E_0^1$.

Summing up (36), (52), and (53), in order to obtain equation (35), we have but to prove that $t(\Pi_E\alpha) = 0$. This follows straightforwardly since $0 = t(N_r R^1\alpha^r) = r\, t(C_{1/r}\alpha^r) = t(\alpha_1^r + r^{-1}\alpha_2^r)$, and then $t(\alpha_1^r + r^{-1}\alpha_2^r) = 0$. But, by Proposition 4.7, $t(\alpha_1^r + r^{-1}\alpha_2^r)$ weakly converge to $t(\Pi_E\alpha)$ in the space of 1-forms on $\partial\mathcal{U}$, endowed with the L^2-norm with respect to the measure $d\sigma$, as defined in (23).

This achieves the proof of (35).

□

Definition 5.4. If $\alpha \in \Omega^1(\mathcal{U})$, keeping in mind (69), we write

$$\alpha = \alpha_1 + \alpha_2,$$

with $\alpha_i \in \Omega^{1,i}(\mathcal{U})$, $i = 1, 2$. If $m \geq 2$, we say that

$$\alpha \in \widehat{W}^{m,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g}) \quad \text{iff} \quad \alpha_i \in W^{m+1-i,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g}), \quad i = 1, 2. \tag{54}$$

The space $\widehat{W}^{m,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ is endowed with its natural norm.

Remark 5.5. Any horizontal form in $\widehat{W}^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ belongs to $W^{2,2}_{\mathbb{G}}(\mathcal{U}, E_0^1)$. In addition, the map $\alpha \to \Pi_E \alpha$ is continuous from $W^{2,2}_{\mathbb{G}}(\mathcal{U}, E_0^1)$ to $\widehat{W}^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$.

Proposition 5.6. *Notice that* $\widehat{W}^{3,2}(\mathcal{U}, \bigwedge^1 \mathfrak{g}) \cap L^2(\mathcal{U}, E_0^1) = W^{3,2}_{\mathbb{G}}(\mathcal{U}, E_0^1)$, *and let* $\alpha \in W^{3,2}_{\mathbb{G}}(\mathcal{U}, E_0^1)$ *be such that* $t(\Pi_E \alpha) = 0$. *Then there exists a sequence* $(\alpha^r)_{r>0}$ *in* $\widehat{W}^{3,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ *such that*

i) $\alpha^r \to \alpha$ *in* $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$;
ii) $J_r(\alpha^r) \to J(\alpha)$ *as* $r \to 0$.

Proof. Arguing as in [2] (dropping however the reduction argument), we choose

$$\alpha^r = \alpha + r(\Pi_E \alpha)_2. \tag{55}$$

First of all, $(\Pi_E \alpha)_2 \in W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$, since Π_E is an operator of order 1 in the horizontal derivatives, and hence i) follows trivially. Moreover

$$t(N_r R^1 \alpha^r) = r\, t(C_{1/r} \alpha^r) = t(\alpha + (\Pi_E \alpha)_2) = t(\Pi_E \alpha) = 0.$$

Thus, in order to prove ii), we have but to show that

$$J_r(\alpha^r) \to J(\alpha) \quad \text{as } r \to 0. \tag{56}$$

Arguing as in [2], we can write

$$\begin{aligned} d_r \alpha^r &= r\big(d_0(\Pi_E \alpha)_2 + d_1(\Pi_E \alpha)_1\big) + r^2\big(d_1(\Pi_E \alpha)_2 + d_2(\Pi_E \alpha)_1\big) \\ &\quad + r^3 d_2(\Pi_E \alpha)_2. \end{aligned}$$

Notice also that

$$d_0(\Pi_E \alpha)_2 + d_1(\Pi_E \alpha)_1 = -d_0 d_0^{-1} d_1 \alpha + d_1 \alpha = 0,$$

since $d_0^{-1} d_0 = Id$ on $\mathcal{R}(d_0)$, and $d_1 \alpha \in \mathcal{R}(d_0)$, by Lemma 8.17 - (1) in Appendix B.

Let us suppose for a while we know that

$$d_1(\Pi_E \alpha)_2 + d_2(\Pi_E \alpha)_1 \in E_0^2, \tag{57}$$

in particular $\Pi_{E_0^\perp}(d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1) = 0$. Thus, we can write

$$\begin{aligned} r^{-4} &\int_{\mathcal{U}} |d_r\alpha^r|^2\, dV \\ &= \int_{\mathcal{U}} |d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1|^2\, dV + r^2 \int_{\mathcal{U}} |d_2(\Pi_E\alpha)_2|^2\, dV \\ &= \int_{\mathcal{U}} |(\Pi_{E_0}(d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1)|^2\, dV + r^2 \int_{\mathcal{U}} |d_2(\Pi_E\alpha)_2|^2\, dV \\ &= \int_{\mathcal{U}} |d_c\alpha|^2\, dV + r^2 \int_{\mathcal{U}} |d_2(\Pi_E\alpha)_2|^2\, dV \to \int_{\mathcal{U}} |d_c\alpha|^2\, dV, \end{aligned}$$

as $r \to 0$, since $d_2(\Pi_E\alpha)_2 \in L^2(\mathcal{U}, \bigwedge^2 \mathfrak{g})$.

On the other hand, it is obvious that $\delta_g\alpha^r \to \delta_c\alpha$ in $L^2(\mathcal{U})$, again since the coefficients of $(\Pi_E\alpha)_2$ belong to $W^{2,2}_{\mathbb{G}}(\mathcal{U})$.

Finally

$$\int_\Omega r\langle C_r^{-1}\alpha^r, \alpha^r\rangle\, dV = \int_\Omega |\alpha|^2\, dV + r\int_\Omega |(\Pi_E\alpha)_2|^2\, dV \to \int_\Omega |\alpha|^2\, dV$$

as $r \to 0$. This achieves the proof of the Proposition, once (57) is proved. Indeed, (57) follows straitforwardly by the identity $d^2 = 0$. In fact, gathering all terms of the same weight, we get

$$d_0^2 = 0, \qquad d_0 d_1 = -d_1 d_0, \qquad d_0 d_2 = -d_2 d_0 - d_1^2.$$

Hence

$$d_0\big(d_1(\Pi_E\alpha)_2 + d_2(\Pi_E\alpha)_1\big) = d_1 d_0 d_0^{-1} d_1\alpha - d_1^2\alpha = 0,$$

since, again by Lemma 8.17 in Appendix B, $d_1\alpha \in \mathcal{R}(d_0)$ and $d_0^{-1}d_0 = Id$ on $\mathcal{R}(d_0)$.

□

Denote respictively by j_r and j the restrictions of J_r and J to $\widehat{W}^{3,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$. We see below that $(j_r)_{r>0}$ Γ-converges (with respect to the topology induced by $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$) to j. However, in general, the Γ-limit of the restriction of a sequence of functionals is only greater or equal than the restrictions of both $\Gamma - \limsup$ and $\Gamma - \liminf$ of the same sequence (for further details, see Proposition 6.14 in [8] and the remarks therein).

On the contrary, in our case Propositions 5.3 and 5.6 yield a more precise result.

Theorem 5.7. *Let $\mathbb{G}$ be a free step 2 Carnot group. If $r > 0$, denote respectively by j_r and j the restrictions of J_r and J to $\widehat{W}^{3,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$. Then*

i) *$(j_r)_{r>0}$ Γ-converges to j in $\widehat{W}^{3,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ with respect to the topology induced by $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$;*

ii) *in $\widehat{W}^{3,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ we have*

$$\Gamma - \liminf_{r\to 0} J^r \equiv \Gamma - \limsup_{r\to 0} J^r \equiv j \equiv J\big|_{\widehat{W}^{3,2}_{\mathbb{G}}(\mathcal{U}, E_0^1)}$$

where the Γ-limits must be meant with respect to the topology induced by $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$.

Proof. Assertion i) follows from Propositions 5.3 and 5.6.

As for ii), let now $\alpha \in \widehat{W}^{3,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ be given. By definition ([8], Proposition 8.1), there exists a sequence $(\alpha^r)_{r>0}$ in $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$, converging to α in the topology of $W^{2,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$, such that $J^r(\alpha^r) \to \big(\Gamma - \liminf_{r\to 0} J^r\big)(\alpha)$ as $r \to 0$. Suppose now $\big(\Gamma - \liminf_{r\to 0} J^r\big)(\alpha) < \infty$. Then, by Proposition 5.3,

$$J(\alpha) \le \liminf_{r\to 0} J^r(\alpha^r) = \big(\Gamma - \liminf_{r\to 0} J^r\big)(\alpha). \tag{58}$$

Clearly, inequality (58) still holds trivially if $\big(\Gamma - \liminf_{r\to 0} J^r\big)(\alpha) = \infty$.

On the other hand, by [8], Proposition 6.14,

$$J(\alpha) \le \limsup_{r\to 0} J^r(\alpha^r) \le (\alpha) \le \limsup_{r\to 0} J^r(\alpha^r) = j(\alpha) = J(\alpha). \tag{59}$$

Combining (58) and (59) we achieve the proof of the theorem. □

6. An example: Maxwell's equation in $\mathbb{H}^1$

Consider now in particular the first Heisenberg group $\mathbb{G} = \mathbb{H}^1$, with variables x, y, t. Set $X := \partial_x - \frac{1}{2} y \partial_t$, $Y := \partial_y + \frac{1}{2} x \partial_t$, $T := \partial_t$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X, Y\}$ and $V_2 = \text{span}\,\{T\}$. We have $X^\natural = dx$, $Y^\natural = dy$, $T^\natural = \theta := dt + \frac{1}{2}(ydx - xdy)$ (the contact form of $\mathbb{H}^1$). In this case

$$\begin{aligned} E_0^1 &= \text{span}\,\{dx, dy\}; \\ E_0^2 &= \text{span}\,\{dx \wedge \theta, dy \wedge \theta\}; \\ E_0^3 &= \text{span}\,\{dx \wedge dy \wedge \theta\}. \end{aligned}$$

The action of d_c on E_0^1 is given by ([24], [14], [4])

$$\begin{aligned} &d_c(\alpha_1 dx + \alpha_2 dy) \\ &\quad = (X^2\alpha_2 - 2XY\alpha_1 + YX\alpha_1) dx \wedge \theta + (2YX\alpha_2 - Y^2\alpha_1 - XY\alpha_2) dy \wedge \theta \\ &\quad := P_1(\alpha_1, \alpha_2) dx \wedge \theta + P_2(\alpha_1, \alpha_2) dy \wedge \theta. \end{aligned}$$

We see that d_c is a homogeneous operator of order 2 in the horizontal derivatives.

On the other hand

$$\delta_c \alpha = X\alpha_1 + Y\alpha_2.$$

Thus, if $E = E_1 dx + E_2 dy$ and $H = H_1 dx + H_2 dy$, equations (19) and (20) read as

$$\begin{aligned} P_2(E_1, E_2)\, dx - P_1(E_1, E_2)\, dy &= i\omega(H_1\, dx + H_2\, dy) \\ P_2(H_1, H_2)\, dx - P_1(H_1, H_2)\, dy &= -i\omega(E_1\, dx + E_2\, dy) \end{aligned}$$

and

$$XH_1 + YH_2 = 0, \qquad XE_1 + YE_2 = 0.$$

In this case

$$[T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix}, \qquad [C_r]_{\mathfrak{g}} = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^2 \end{pmatrix}.$$

Thus, by (29) and (30), the magnetic permeability $[\mu_r]$ and dielettric permittivity $[\varepsilon_r]$ of the "approximating equations" take the forms

$$[\mu_r] = [\varepsilon_r] = \begin{pmatrix} 1+\frac{r}{4}y^2 & -\frac{r}{4}xy & \frac{r}{2}y \\ -\frac{r}{4}xy & 1+\frac{r}{4}x^2 & -\frac{r}{2}x \\ \frac{r}{2}y & -\frac{r}{2}x & r \end{pmatrix}.$$

Remark 6.1. Heisenberg groups $\mathbb{H}^N$ with $N > 1$ are still step 2 groups, though not free. Nevertheless, intrinsic 2-forms on $\mathbb{H}^N$ have weight 2, and therefore all our arguments can be carried out with the following choice of μ_r and ε_r:

$$\mu_r = r^{-4N/(2N-1)} C_r^* R C_r,$$

and

$$\varepsilon_r = r^{-1}(RC_r)^*,$$

where, again, C_r^* and $(RC_r)^*$ are the adjoint maps of C_r and RC_r, respectively, with respect to the Euclidean scalar product.

Remark 6.2. With the notations of (50), both in Proposition 5.6 and Theorem 5.7 we can replace the assumption $\alpha \in W^{3,2}(\Omega, E_0^1)$ by

$$\alpha \in W^{2,2}(\Omega, E_0^1), \quad \delta_g d_0^{-1} d_1 \alpha \in L^2(\Omega) \quad \text{and} \quad d_2 d_0^{-1} d_1 \alpha \in L^2(\Omega, \textstyle\bigwedge^2 \mathfrak{g}).$$

In particular, if $\mathbb{G} = \mathbb{H}^1$, $d_0^{-1} d_1 \alpha$ is a 1-form of weight 2, and then $d_2 d_0^{-1} d_1 \alpha = 0$, since there are no 2-forms of weight 4.

7. Appendix A: Multilinear algebra

Basically to establish our notations, we remind some known definitions and results of multilinear algebra. For undefined notations, we refer to Section 2 and to [17] and [10].

Definition 7.1. We define linear isomorphisms (Hodge duality: see [10] 1.7.8)

$$* : \textstyle\bigwedge_h \mathfrak{g} \longleftrightarrow \bigwedge_{n-h} \mathfrak{g} \quad \text{and} \quad * : \bigwedge^h \mathfrak{g} \longleftrightarrow \bigwedge^{n-h} \mathfrak{g},$$

for $1 \le h \le n$, putting, for $v, w \in \bigwedge_h \mathfrak{g}$ and $\varphi, \psi \in \bigwedge^h \mathfrak{g}$

$$v \wedge *w = \langle v, w \rangle X_1 \wedge \cdots \wedge X_n, \qquad \varphi \wedge *\psi = \langle \varphi, \psi \rangle \theta_1 \wedge \cdots \wedge \theta_n.$$

It is easy to see that

$$\begin{aligned} * * v &= (-1)^{h(n-h)} v, \qquad * * \varphi = (-1)^{h(n-h)} \varphi, \\ \langle *\varphi | *v \rangle &= \langle \varphi | v \rangle. \end{aligned} \tag{60}$$

From now on, we refer to the n-form

$$dV := \theta_1 \wedge \cdots \wedge \theta_n$$

as to the canonical volume form in $\mathbb{G}$.

If d is the usual De Rham's exterior differential in $\mathbb{G}$ identified with $\mathbb{R}^n$, we denote by $\delta = d^*$ its formal adjoint in $L^2(\mathbb{G}, \Omega^*)$. We remind that, when acting on h-forms

$$\delta = (-1)^{n(h+1)+1} * d * . \tag{61}$$

On a h-form ω

$$(62)\qquad *\delta\omega = (-1)^h d*\omega, \quad *d\omega = (-1)^{h+1}\delta*\omega.$$

If $v \in \bigwedge_h \mathfrak{g}$ we define $v^\natural \in \bigwedge^h \mathfrak{g}$ by the identity $\langle v^\natural | w\rangle := \langle v, w\rangle$, and analogously we define $\varphi^\natural \in \bigwedge_h \mathfrak{g}$ for $\varphi \in \bigwedge^h \mathfrak{g}$.

To fix our notations, we remind the following definition (see e.g. [17], Section 2.1).

Definition 7.2. If V, W are finite dimensional linear vector spaces and $L : V \to W$ is a linear map, we define

$$\Lambda_h L : \bigwedge_h V \to \bigwedge_h W$$

as the linear map defined by

$$(\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) = L(v_1) \wedge \cdots \wedge L(v_h)$$

for any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$, and

$$\Lambda^h L : \bigwedge^h W \to \bigwedge^h V$$

as the linear map defined by

$$\langle (\Lambda^h L)(\alpha) | v_1 \wedge \cdots \wedge v_h\rangle = \langle \alpha | (\Lambda_h L)(v_1 \wedge \cdots \wedge v_h)\rangle$$

for any $\alpha \in \bigwedge^h W$ and any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$.

Proposition 7.3 (see,e.g., [17], Chapter 2, Section 2.1)**.** *If V, W are finite dimensional linear vector spaces endowed with a scalar product that is naturally extended to the associated graded algebras. Let $L : V \to W$ be a linear map, then*

i) *if $v \in \bigwedge_1 V$ and $\alpha \in \bigwedge^1 W$, then $(\Lambda_1 L)v = Lv$ and*
$$L\alpha^\natural = (\Lambda_1 L)\alpha^\natural = \big((\Lambda^1 L^*)\alpha\big)^\natural;$$
ii) *if $\alpha \in \bigwedge^k W$ and $\beta \in \bigwedge^h W$, then $(\Lambda^{k+h} L)(\alpha \wedge \beta) = (\Lambda^k L)\alpha \wedge (\Lambda^h L)\beta$;*
iii) *if $v \in \bigwedge_k V$ and $w \in \bigwedge_h V$, then $(\Lambda_{k+h} L)(v \wedge w) = (\Lambda_k L)v \wedge (\Lambda_h L)w$;*
iv) *$(\Lambda_h L)^* = \Lambda_h(L^*)$ and $(\Lambda^h L)^* = \Lambda^h(L^*)$;*
v) *if H is another finite dimensional linear vector space and $G : H \to V$ is a linear map, then $\Lambda_h(L \circ G) = (\Lambda_h L) \circ (\Lambda_h G)$ and $\Lambda^h(L \circ G) = (\Lambda^h G) \circ (\Lambda^h L)$;*

Lemma 7.4. *With the notations of Proposition 7.3, if $W = V$ and $L \in GL(V)$, then*

$$*(\Lambda^h L)* = (\det L) \cdot (\Lambda^{n-h}(L^*)^{-1}) * .$$

Proof. If $\alpha, \beta \in \bigwedge^h V$, we have:

$$\begin{aligned}
\beta \wedge *(\Lambda^h L)\alpha &= \langle \beta, (\Lambda^h L)\alpha\rangle dV = \langle (\Lambda^h L^*)\beta, \alpha\rangle dV = \langle *(\Lambda^h L^*)\beta, *\alpha\rangle dV \\
&= *\alpha \wedge (\Lambda^h L^*)\beta = (\det L)(\Lambda^{n-h}(L^*)^{-1}) * \alpha \wedge \beta \\
&= (-1)^{h(n-h)} \cdot (\det L)\beta \wedge (\Lambda^{n-h}(L^*)^{-1}) * \alpha.
\end{aligned}$$

□

8. Appendix B: Carnot groups and Rumin's complex

With the notations of Section 1, let $(\mathbb{G}, \cdot)$ be a Carnot group of step κ identified to $\mathbb{R}^n$ through exponential coordinates. As above, $\{e_1, \cdots, e_n\}$ is a basis of $\mathbb{R}^n$ adapeted to the stratification of $\mathfrak{g}$. Moreover, let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$, $i = 1, \dots, n$. Since $\mathbb{G}$ is written in exponential coordinates, a point $p \in \mathbb{G}$ is identified with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we we can identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{d_1} x_1, ..., \lambda^{d_n} x_n), \tag{63}$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [12] Chapter 1).

The dilations δ_λ are group automorphisms, since

$$\delta_\lambda x \cdot \delta_\lambda y = \delta_\lambda(x \cdot y).$$

We remind that the generating vector fields $X_1, \dots, X_m$ are homogeneous of degree 1 with respect to group dilations.

Proposition 8.1 (see, e.g.[13], Proposition 2.1)**.** *The group product has the form*

$$x \cdot y = x + y + \mathcal{Q}(x, y), \qquad \textit{for all } x, y \in \mathbb{R}^n \tag{64}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and each* $\mathcal{Q}_i$ *is a homogeneous polynomial of degree* d_i *with respect to the intrinsic dilations of* $\mathbb{G}$ *defined in (63), that is*

$$\mathcal{Q}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{d_i} \mathcal{Q}_i(x, y), \quad \textit{for all } x, y \in \mathbb{G}.$$

Moreover, again for all $x, y \in \mathbb{G}$

$$\mathcal{Q}_1(x, y) = ... = \mathcal{Q}_{m_1}(x, y) = 0,$$

$$\mathcal{Q}_j(x, 0) = \mathcal{Q}_j(0, y) = 0, \quad \textit{for } m_1 < j \le n, \tag{65}$$

(66)

$$\mathcal{Q}_j(x, y) = \mathcal{Q}_j(x_1, \dots, x_{h_{i-1}}, y_1, \dots, y_{h_{i-1}}), \quad \textit{if} \quad 1 < i \le \kappa \quad \textit{and} \quad j \le h_i.$$

Proposition 8.2 (see, e.g.[13], Proposition 2.2)**.** *The vector fields* X_j *have polynomial coefficients and have the form*

$$X_j(x) = \partial_j + \sum_{i > h_l}^{n} q_{i,j}(x) \partial_i, \quad \textit{for } j = 1, \dots, n \textit{ and } j \le h_l, \tag{67}$$

where $q_{i,j}(x) = \frac{\partial \mathcal{Q}_i}{\partial y_j}(x, y)|_{y=0}$ *so that if* $j \le h_l$ *then* $q_{i,j}(x) = q_{i,j}(x_1, ..., x_{h_{l-1}})$ *and* $q_{i,j}(0) = 0$.

From now on, if $\mathcal{U} \subset \mathbb{G}$ is an open set and $h = 0, 1, \dots, n$ we denote by $\Omega_h(\mathcal{U})$ and $\Omega^h(\mathcal{U})$ the sets of all sections of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, respectively. If

$\mathcal{U} = \mathbb{G}$ we write only Ω_h and Ω^h. We refer to elements of Ω_h as to fields of h-vectors and to elements of Ω^h as to h-forms.

We recall the notion of weight of a form that is necessary to introduce the Rumin's complex.

Definition 8.3. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight* k, and we write $w(\alpha) = k$, if $\alpha^\natural \in V_k$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

Remark 8.4. If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$. Indeed, it is enough to notice that, if $w(\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}) \neq w(\theta_{J_1} \wedge \cdots \wedge \theta_{J_h})$, with $i_1 < i_2 < \cdots < i_h$ and $J_1 < J_2 < \cdots < J_h$, then for at least one of the indices $\ell = 1, \dots, h$, $i_\ell \neq J_\ell$, and hence $\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}, \theta_{J_1} \wedge \cdots \wedge \theta_{J_h} \rangle = 0$.

We have ([3], formula (16))

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{68}$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h–covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of left-invariant h-covectors.

Keeping in mind the decomposition (68), we can define in the same way several left invariant fiber bundles over $\mathbb{G}$, that we still denote with the same symbol $\bigwedge^{h,p} \mathfrak{g}$.

We notice also that the fiber $\bigwedge_x^h \mathfrak{g}$ (and hence the fiber $\bigwedge_x^{h,p} \mathfrak{g}$) can be endowed with a natural scalar product $\langle \cdot, \cdot \rangle_x$.

We denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{69}$$

The following crucial property of the weight follows from Cartan identintity: see [26], Section 2.1:

Lemma 8.5. *We have* $d(\bigwedge^{h,p} \mathfrak{g}) \subset \bigwedge^{h+1,p} \mathfrak{g}$, *i.e., if* $\alpha \in \bigwedge^{h,p} \mathfrak{g}$ *is a left invariant h-form of weight p with* $d\alpha \neq 0$, *then* $w(d\alpha) = w(\alpha)$.

Definition 8.6. Let now $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, \theta_i^h \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. Then we can write

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$$

does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{m_1} (X_j\alpha_i)\theta_j \wedge \theta_i^h$$

increases the weight by 1, and, more generally,

$$d_i\alpha = \sum_{\theta_i^h\in\Theta^{h,p}} \sum_{X_j\in V_i} (X_j\alpha_i)\theta_j\wedge\theta_i^h,$$

increases the weight by i, when $i = 0, 1, \dots, \kappa$. In particular, d_0 is an algebraic operator. We denote by δ_0 its adjoint.

Lemma 8.7. *$d_0^2 = 0$, i.e. (Ω^*, d_0) is a complex.*

Proof. Take $\alpha \in \Omega^{h,p}$, and write the identity $d^2\alpha = 0$, gathering all terms according their weights. Since terms with different weights are orthogonal, this yields that all groups of given weight vanish. But the group of weight p is precisely $d_0^2\alpha$, and we are done.

□

The following definition of intrinsic covectors (and therefore of intrinsic forms) is due to M. Rumin ([26], [25]).

Definition 8.8. If $0 \le h \le n$ we set

$$E_0^h := \ker d_0 \cap \ker\delta_0 = \ker d_0 \cap (\mathrm{Im}\, d_0)^\perp \subset \Omega^h$$

We refer to the elements of E_0^h as to *intrinsic h-forms on $\mathbb{G}$*. Since the construction of E_0^h is left invariant, this space of forms can be seen as the space of sections of a fiber subbundle of $\bigwedge^h\mathfrak{g}$, generated by left translation and still denoted by E_0^h. In particular E_0^h inherits from $\bigwedge^h\mathfrak{g}$ the scalar product on the fibers.

Remark 8.9. There exists a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h that is adapted to the filtration (68).

Since it is easy to see that $E_0^1 = \mathrm{span}\,\{\theta_1, \dots, \theta_m\}$, without loss of generality, we can take $\xi_j = \theta_j$ for $j = 1, \dots, m$.

Finally, we denote by $N_h^{\min}$ and $N_h^{\max}$ respectively the lowest and highest weight of forms in E_0^h.

We define now a (pseudo) inverse of d_0 as follows (see [3], Lemma 2.11):

Lemma 8.10. *If $\beta \in \bigwedge^{h+1}\mathfrak{g}$, then there exists a unique $\alpha \in \bigwedge^h\mathfrak{g}\cap(\ker d_0)^\perp$ such that*

$$\delta_0 d_0\alpha = \delta_0\beta. \quad \textit{We set} \quad \alpha := d_0^{-1}\beta.$$

In particular

$$\alpha = d_0^{-1}\beta \quad \textit{if and only if} \quad d_0\alpha - \beta \in \ker\delta_0 = \mathcal{R}(d_0)^\perp.$$

In addition, d_0^{-1} preserves the weights.

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [26] and [3], Section 2) .

Theorem 8.11. *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1}d) \quad \textit{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(dd_0^{-1}).$$

We have

i) *Let* Π_E *be the projection on* E *along* F *(that is not an orthogonal projection). Then for any* $\alpha \in E_0^{h,p}$*, if we denote by* $(\Pi_E\alpha)_j$ *the component of* $\Pi_E\alpha$ *of weight* j*, then*

$$
\begin{aligned}
&(\Pi_E\alpha)_p = \alpha \\
(70)\qquad &(\Pi_E\alpha)_{p+k+1} = -d_0^{-1}\Big(\sum_{1\le \ell\le k+1} d_\ell(\Pi_E\alpha)_{p+k+1-\ell}\Big).
\end{aligned}
$$

Notice that $\alpha \to (\Pi_E\alpha)_{p+k+1}$ *is an homogeneous differential operator of order* $k+1$ *in the horizontal derivatives.*

ii) Π_E *is a chain map, i.e.*

$$d\Pi_E = \Pi_E d.$$

iii) *Let* Π_{E_0} *be the orthogonal projection from* Ω^* *on* E_0^**, then*

$$\Pi_{E_0} = Id - d_0^{-1}d_0 - d_0 d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1}d_0 + d_0 d_0^{-1}. \tag{71}$$

Notice that, since d_0 *and* d_0^{-1} *are algebraic, then formulas* (71) *hold also for covectors.*

iv) $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ *and* $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$.

Set now

$$d_c = \Pi_{E_0}\, d\, \Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \dots, n-1.$$

We have:

v) $d_c^2 = 0$*;*

vi) *the complex* $E_0 := (E_0^*, d_c)$ *is exact;*

vii) *with respect to the bases* Ξ^**, the intrinsic differential* d_c *can be seen as a matrix-valued operator such that, if* α *has weight* p*, then the component of weight* q *of* $d_c\alpha$ *is given by an homogeneous differential operator in the horizontal derivatives of order* $q-p\ge 1$*, acting on the components of* α*.*

Remark 8.12. We have also $E = \ker\delta_0 + \ker(\delta_0 d)$ and $F = \mathrm{Im}\ \delta_0 + \mathrm{Im}\ (d\delta_0)$.

Proposition 8.13. *Denote by* $\delta_c = d_c^*$ *the formal adjoint of* d_c *in* $L^2(\mathbb{G}, E_0^*)$*.*
Then assertions (62) *still hold if we replace* d *and* δ *by* d_c *and* δ_c*, respectively, or by* d_0*,* δ_0*, respectively.*

Lemma 8.14. *If* $\omega \in E^h$ *and* $t(\omega) = 0$*, then* $\omega \in *F^\perp$ *(with respect to the* L^2*-product).*

Proof. By [27], identity (2.27), $n(*\omega) = 0$. First of all, we can show that

$$*\omega \in \ker d_0 \cap \ker(d_0\delta).$$

Indeed, by Proposition 8.13, $d_0(*\omega) = (-1)^h * \delta_0\omega = 0$ and $d_0\delta * \omega = (-1)^{h+1} d_0 * d\omega = *\delta_0 d\omega = 0$. Take now $\alpha = \delta_0\xi + d\delta_0\eta \in F$. Clearly

$$\int_{\mathcal U} \langle *\omega, \delta_0\xi\rangle\, dV = \int_{\mathcal U} \langle d_0 * \omega, \xi\rangle\, dV = 0.$$

On the other hand, by classical Green's formula

$$\begin{aligned}
&\int_{\mathcal{U}} \langle *\omega, d\delta_0\eta\rangle \, dV \\
&\quad = \int_{\mathcal{U}} \langle \delta * \omega, \delta_0\eta\rangle \, dV + \int_{\partial\mathcal{U}} t(\delta_0\eta)\wedge *n(*\omega) \\
&\quad = \int_{\mathcal{U}} \langle \delta * \omega, \delta_0\eta\rangle \, dV \quad (\text{since } n(*\omega) = *t(\omega)) \\
&\quad = \int_{\mathcal{U}} \langle d_0\delta * \omega, \eta\rangle \, dV = 0.
\end{aligned}$$

This proves that $*\omega \in F^\perp$, and then the assertion follows. □

Finally, we remind the definition of free Carnot group (see, for instance [6], Section 14.1).

Definition 8.15. Let $m \geq 2$ and $\kappa \geq 1$ be fixed integers. We say that $\mathfrak{f}_{m,\kappa}$ is the free Lie algebra with m generators $x_1, \dots, x_m$ and nilpotent of step κ if:

- i) $\mathfrak{f}_{m,\kappa}$ is a Lie algebra generated by its elements $x_1, \dots, x_m$, i.e. $\mathfrak{f}_{m,\kappa} = \mathrm{Lie}(x_1, \dots, x_m)$;
- ii) $\mathfrak{f}_{m,\kappa}$ is nilpotent of step κ;
- iii) for every Lie algebra $\mathfrak{n}$ nilpotent of step κ and for every map ϕ from the set $\{x_1, \dots, x_m\}$ to $\mathfrak{n}$, there exists a (unique) homomorphism of Lie algebras Φ from $\mathfrak{f}_{m,\kappa}$ to $\mathfrak{n}$ which extends ϕ.

The Carnot group $\mathbb{G}$ is said free if its Lie algebra $\mathfrak{g}$ is isomorphic to a free Lie algebra.

When $\mathbb{G}$ is a free group, we can assume $\{X_1, \dots, X_n\}$ a Grayson-Grossman-Hall basis of $\mathfrak{g}$ (see [18], [6], Theorem 14.1.10). This makes several computations much simpler. In particular, $\{[X_i, X_j], X_i, X_j \in V_1, i < j\}$ provides an orthonormal basis of V_2.

Theorem 8.16 ([15], Theorem 5.9)**.** *Let $\mathbb{G}$ be a free group of step κ. Then all forms in E_0^1 have weight 1 and all forms in E_0^2 have weight $\kappa + 1$.*

In particular, the differential $d_c : E_0^1 \to E_0^2$ can be identified, with respect to the adapted bases Ξ_0^1 and Ξ_0^2, with a homogeneous matrix-valued differential operator of degree κ in the horizontal derivatives.

Moreover, since Π_{E_0} preserves the weights, if $\xi \in \bigwedge^{2,p}\mathfrak{g}$ with $p \neq \kappa+1$, then $\Pi_{E_0}\xi = 0$. Indeed, $\Pi_{E_0}\xi$ has weight p, and therefore has to be zero, since $\Pi_{E_0}\xi \in \bigwedge^{2,\kappa+1}\mathfrak{g}$.

Lemma 8.17 ([15], Lemma 6.3)**.** *If $\mathfrak{g}$ is a free algebra of step 2, then*

1. $d_0(\bigwedge^1 \mathfrak{g}) = \bigwedge^{2,2}\mathfrak{g}$;
2. *if $\theta_i \wedge \theta_j \in \bigwedge^{2,2}\mathfrak{g}$, then $d_0^{-1}(\theta_i \wedge \theta_j) = -[X_i, X_j]^\natural$;*
3. *if $\theta_i \wedge \theta_j \in \bigwedge^{2,2}\mathfrak{g}$, then $d_0 d_0^{-1}(\theta_i \wedge \theta_j) = \theta_i \wedge \theta_j$;*
4. *if $\theta_i \wedge \theta_j \in \bigwedge^{2,3}\mathfrak{g}$ or $\theta_i \wedge \theta_j \in \bigwedge^{2,4}\mathfrak{g}$ then $d_0^{-1}(\theta_i \wedge \theta_j) = 0$, so that again $d_0^{-1}(\theta_i \wedge \theta_j) = -[X_i, X_j]^\natural$.*

9. Appendix C: A variational approach to Maxwell's equations

A standard approach to the Dirichlet problem with relative boundary conditions in a bounded open set $\mathcal{U}$ for system (15) relies on a variational approach combined with a compactness argument (see, for instance, [7] and Appendix A of [20]).

Step 1. We replace the two equations of (15) by the single equation

$$\delta M d N\alpha + \sigma (N^{-1})^* d\delta\alpha - \omega^2\, \alpha = 0 \tag{72}$$

in $W^{1,2}_{DN}(\mathcal{U}, \bigwedge^1 \mathbb{R}^3)$, where $\sigma > 0$. The condition $\delta\alpha = 0$ will follow later from the equation

$$\sigma\delta\big((N^{-1})^* d\delta\alpha\big) - \omega^2\, \delta\alpha = 0,$$

which is obtained by applying δ to both sides of (72).

Step 2. The second step consists in proving, via Lax-Milgram theorem applied to the quadratic form

$$\begin{aligned}\tilde{J}^{\mu,\varepsilon}(\alpha) := &\int_{\mathcal{U}} \langle M d N\alpha, dN\alpha\rangle_{\mathrm{Euc}}\, dV + \sigma \int_{\mathcal{U}} |\delta\alpha|^2\, dV \\ &+ C \int_{\mathcal{U}} \langle N\alpha, \alpha\rangle_{\mathrm{Euc}}\, dV\end{aligned} \tag{73}$$

in $W^{1,2}_{DN}(\mathcal{U}, \bigwedge^1 \mathbb{R}^3)$, the existence of a (unique) solution of the equation

$$\delta M d N\alpha + \sigma d(\delta\alpha) + C\,\alpha = f \in W^{1,2}_{DN}(\mathcal{U}, \bigwedge^1 T\mathbb{R}^3)^*$$

in $W^{1,2}_{DN}(\mathcal{U}, \bigwedge^1 T\mathbb{R}^3)$, where $C > 0$. Indeed, it is easy to see that $\tilde{J}$ is coercive in $W^{1,2}_{DN}(\mathcal{U}, \bigwedge^1 T\mathbb{R}^3)$. Notice also that Green formula together with the boundary condition $t(N\alpha) = 0$ shows an hidden (for the moment only formal) property of the minimizers of $\tilde{J}^{\mu,\varepsilon}$: the fact that $\delta\alpha = 0$ on $\partial\mathcal{U}$.

Step 3. By a compactness argument, it follows from Step 2 that equation (72) has a unique solution $\alpha_0 \in W^{1,2}_D(\mathcal{U}, \bigwedge^1 T\mathbb{R}^3)$, provided ω does not belong to a suitable discrete set Σ_σ.

Step 4. If $f \in L^2(\mathcal{U}, \bigwedge^1 \mathbb{R}^3)$, a regularization argument shows now that $\alpha_0 \in W^{2,2}(\mathcal{U}, \bigwedge^1 T\mathbb{R}^3)$ and that $\delta\alpha_0 = 0$ on $\partial\mathcal{U}$ in the usual trace sense.

Step 5. Applying δ to both sides of (72), and keeping in mind that $\delta\alpha_0$ vanishes on $\partial\mathcal{U}$, we conclude that eventually we can drop the artificial term $(N^{-1})^* d\delta\alpha_0$ in (72).

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Annalisa Baldi and Bruno Franchi
Dipartimento di Matematica
Piazza di Porta S. Donato 5, 40126 Bologna, Italy.
e-mail: annalisa.baldi2@unibo.it, bruno.franchi@unibo.it.