SPECTRAL OPTIMIZATION FOR THE STEKLOFF–LAPLACIAN: THE STABILITY ISSUE

LORENZO BRASCO, GUIDO DE PHILIPPIS, AND BERARDO RUFFINI

Abstract. We consider the problem of maximizing the first non-trivial Stekloff eigenvalue of the Laplacian, among sets with given measure. We prove that the Brock–Weinstock inequality, asserting that optimal shapes for this spectral optimization problem are balls, can be improved by means of a (sharp) quantitative stability estimate. This result is based on the analysis of a certain class of weighted isoperimetric inequalities already proved in Betta et al. (J. of Inequal. & Appl. 4: 215–240, 1999): we provide some new (sharp) quantitative versions of these, achieved by means of a suitable calibration technique.

1. Introduction

1.1. **Background.** This work is devoted to the study of some particular *spectral optimization problems*. These are shape optimization problems where the functional to be optimized is a function of the spectrum of an elliptic operator, typically the Laplacian $-\Delta$: the prototypical case is when this functional coincides with a single eigenvalue of the operator (see the book [12] or the recent survey paper [8] for the state of the art on these problems).

In order to clarify the scopes of this paper and to provide a neat framework for the results here presented, we start recalling the most famous instance of spectral optimization: the minimization of the first Dirichlet eigenvalue of the Laplacian λ_1, among sets with given measure, i.e.

$$\min\{\lambda_1(\Omega)\,:\,|\Omega|=c\}. \tag{1.1}$$

For this problem, the (unique) solution is given by a ball of measure c (see [12] for both the definition of Dirichlet eigenvalues and the proof of this result). This is the celebrated *Faber-Krahn inequality*, which can be summarized as follows

$$|\Omega|^{2/N}\,\lambda_1(\Omega)\geq|B|^{2/N}\,\lambda_1(B),$$

by noticing that the shape functional $\Omega\mapsto\lambda_1(\Omega)$ scales like a length to the power -2. Here B is any ball and equality in the previous can hold *if and only if* Ω itself is a ball.

Once the optimal shapes for such a problem have been identified, a natural question comes into play: that of *stability*. This amounts to address the following issue: suppose that Ω_0 has measure c and that $\lambda_1(\Omega_0)\simeq\min\{\lambda_1(\Omega)\,:\,|\Omega|=c\}$, is it true that Ω_0 has to "resemble" a ball? If the answer is *yes*, then one would like to quantify this stability, for example by proving a *quantitative* version of the form

$$|\Omega|^{2/N}\,\lambda_1(\Omega)\geq|B|^{2/N}\,\lambda_1(B)\,\left[1+\Phi(d(\Omega,\mathcal{B}))\right], \tag{1.2}$$

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where $\mathcal{B}$ is the set of all balls, d is a suitable distance between sets and Φ is some modulus of continuity. We say that a quantitative inequality like (1.2) is *sharp*, if there exists some family of sets $\{\Omega_\varepsilon\}_{\varepsilon\ll 1}$ approaching a ball, such that the *deficit* $|\Omega_\varepsilon|^{2/N}\,\lambda_1(\Omega_\varepsilon)-|B|^{2/N}\,\lambda_1(B)$ is converging to 0 and

$$\frac{|\Omega_\varepsilon|^{2/N}\,\lambda_1(\Omega_\varepsilon)}{|B|^{2/N}\,\lambda_1(B)}-1\simeq\Phi(d(\Omega_\varepsilon,\mathcal{B})),\qquad\text{as }\varepsilon\to 0.$$

In other words, the quantitative inequality (1.2) is sharp if it asymptotically becomes an equality, at least for particular shapes having small deficits.

In the case of problem (1.1), apparently the first ones to investigate these questions have been Hansen and Nadirashvili [11] and Melas [18], who proved an inequality like (1.2), with d being the Hausdorff distance (or a suitable variant of it) and the modulus of continuity Φ being a power function. These results are not sharp and, at least for $N\ge 3$, they apply to the case of convex sets only. Since then, other papers have tried to attack this problem: among the others, we recall the contributions (in chronological order) by Sznitman [22, Theorem A.1], Povel [21, Theorem A], Bhattacharya [3] and Fusco, Maggi and Pratelli [9]. In all of these, the Hausdorff distance is replaced by the L^1 distance of sets, i.e. the so called *Fraenkel asymmetry*

$$\mathcal{A}(\Omega):=\inf\left\{\frac{\|1_\Omega-1_B\|_{L^1(\mathbb{R}^N)}}{|\Omega|}\ :\ B\ \text{ ball with }\ |B|=|\Omega|\right\},$$

and the convexity assumption on the sets is dropped. However, in spite of a certain amount of works on this subject, we point out that a sharp quantitative version for the Faber-Krahn inequality is still missing, even for special classes of sets. Just for completeness, we mention the work [1] by Ávila, where the case of the first Dirichlet eigenvalue of the Laplace-Beltrami operator on manifolds is considered: he proved such a type of stability estimates for smooth convex sets in the hyperbolic plane and the sphere.

One may wonder what happens for the next Dirichlet eigenvalues: for example, we could consider problem (1.1) for the second Dirichlet eigenvalue of the Laplacian. This time the optimal sets are disjoint unions of two balls having measure $c/2$. Usually, this result is known as the *Krahn-Szego* or the *Hong-Krahn-Szego inequality*: some (non sharp) quantitative stability estimates for this inequality have been recently given in [5, 6], where the distance from optimal sets is still measured in the L^1 sense (using a suitable variant of the quantity $\mathcal{A}$).

Apart from the Dirichlet case, we can also consider the eigenvalues of the Laplacian with other boundary conditions, for example Neumann homogeneous ones (again, we refer to [12, Section 1] for the main definitions). In this case, problem (1.1) is no more interesting, since the first Neumann eigenvalue of a set is always zero and corresponds to constant eigenfunctions. On the contrary, now the problem of maximizing the first non-trivial eigenvalue μ_2 becomes interesting, that is

$$\max\{\mu_2(\Omega)\ :\ |\Omega|=c\}.$$

The classical *Szegő-Weinberger inequality* (see [12, Section 7]) asserts that the unique solution is given by a ball of measure c. Also in this case, some quantitative improvements are possible: apart from a paper by Xu ([25, Theorem 4]), dealing with convex sets in $\mathbb{R}^N$ and in the hyperbolic space, and a paper by Nadirashvili ([20]) concerning the case $N=2$, recently the first author and Pratelli in [6, Theorem 4.1] have succeeded to prove a *sharp* quantitative version of the Szegő-Weinberger inequality in $\mathbb{R}^N$, valid for every $N\ge 2$ and without restrictions on the geometry of the admissible sets.

1.2. **The results of this paper.** The main scope of this paper is to continue the study of stability issues for spectral optimization problems, by addressing the case of *Stekloff eigenvalues* (see Section 4 for definitions and basic properties). Here as well, like in the Neumann case, the interesting problem is that of maximization. First of all, we recall that for a set Ω, its first non-trivial eigenvalue of the Laplacian with Stekloff boundary condition is given by

$$\sigma_2(\Omega) = \inf_{u\in W^{1,2}(\Omega)\setminus\{0\}} \left\{ \frac{\displaystyle\int_\Omega |\nabla u(x)|^2\,dx}{\displaystyle\int_{\partial\Omega} u(x)^2\,d\mathcal{H}^{N-1}} \ :\ \int_{\partial\Omega} u(x)\,d\mathcal{H}^{N-1} = 0 \right\},$$

i.e. $1/\sigma_2(\Omega)$ is the best constant in the Poincaré-Wirtinger trace inequality

$$\int_{\partial\Omega} \left| u(x) - \left(\fint_{\partial\Omega} u(x) \right) \right|^2 d\mathcal{H}^{N-1} \le C_\Omega \int_\Omega |\nabla u(x)|^2\,dx, \qquad u \in W^{1,2}(\Omega). \tag{1.3}$$

The *Brock-Weinstock inequality* asserts that in the class of sets with given volume, σ_2 is *maximized* by a ball, i.e.

$$|\Omega|^{1/N}\,\sigma_2(\Omega) \le |B|^{1/N}\,\sigma_2(B), \tag{1.4}$$

where B is any ball and equality holds if and only if Ω itself is a ball: again, we used that the quantity $|\Omega|^{1/N}\,\sigma_2(\Omega)$ is scaling invariant. One of the main result of this paper is a *sharp quantitative version* of (1.4): indeed, we will show the following (Theorem 5.1 and Corollary 5.2).

Theorem A (Quantitative Brock-Weinstock inequality)**.** *For every $\Omega \subset \mathbb{R}^N$ open bounded Lipschitz set, there holds*

$$|\Omega|^{1/N}\,\sigma_2(\Omega) \le |B|^{1/N}\,\sigma_2(B)\,\left[1 - \alpha_N\,\mathcal{A}(\Omega)^2\right], \tag{1.5}$$

where α_N is an explicit dimensional constant.

Some words on the proof of this result are in order: it has to be noticed that *the maximality of the ball for σ_2 is a consequence of a further isoperimetric property of the ball.* Namely, the crucial point is that the ball centered at the origin (uniquely) minimizes the shape functional

$$\Omega \mapsto \int_{\partial\Omega} |x|^2\,d\mathcal{H}^{N-1},$$

among sets with given measure: this result is proved in [2, Theorems 2.1 and 4.2]. Here $\mathcal{H}^{N-1}$ stands for the $(N-1)-$dimensional Hausdorff measure and observe that in physical terms the latter quantity is the *moment of inertia of the boundary* $\partial\Omega$, with respect to the origin. This further isoperimetric characterization of the ball is the cornerstone of Brock's proof in [7]: then in order to derive (1.5), we are naturally lead to consider the question of stability for such a *weighted perimeter*. As a consequence, we provide the following sharp quantitative version of this isoperimetric inequality as well, which is the other main contribution of this paper (Theorem 2.2 and Corollary 2.4).

Theorem B (Quantitative weighted isoperimetric inequality)**.** *For every $\Omega \subset \mathbb{R}^N$ open bounded Lipschitz set, there holds*

$$\int_{\partial\Omega} |x|^2\,d\mathcal{H}^{N-1} \ge \int_{\partial B} |x|^2\,d\mathcal{H}^{N-1} \left[1 + \beta_N \left(\frac{|\Omega\Delta B|}{|\Omega|}\right)^2\right],$$

where β_N is an explicit dimensional constant, B is the ball centered at the origin such that $|B| = |\Omega|$ and $\Omega \Delta B$ denotes the symmetric difference.

In fact, we will see that one can also replace $|x|^2$ with other power functions or even more general weight functions, as in [2].

Concerning the *sharpness* of the exponent 2 for the Fraenkel asymmetry in (1.5), the reader could be disappointed by the fact that its proof (Theorem 6.1) is extremely much longer than the same result for weighted perimeters (Section 3). The reason is quite easy to understand: an eigenvalue does not have a straightforward geometrical meaning, like in the case of the perimeter for example, so it is much more complicate to understand how deformations of an optimal shape affects the eigenvalues. So, in principle, it is quite a difficult task even to guess what should be the sharp modulus of continuity Φ, in an inequality like (1.2). If the eigenvalue is differentiable in the sense of the *shape derivative* (see [14]) – like in the case of the first Dirichlet eigenvalue λ_1 – one can use the following argument. Any perturbation of the type $\Omega_\varepsilon := (\mathrm{Id} + \varepsilon X)(B)$, for some smooth vector field X, should provide a Taylor expansion of the form

$$|\Omega|^{2/N}\,\lambda_1(\Omega_\varepsilon) \simeq |B|^{2/N}\,\lambda_1(B) + O(\varepsilon^2), \qquad \varepsilon \ll 1, \tag{1.6}$$

since the first derivative of $|\cdot|^{2/N}\,\lambda_1(\cdot)$ has to vanish at the minimum "point" B. Then one observes that for such a family of sets, the Fraenkel asymmetry satisfies $\mathcal{A}(\Omega_\varepsilon) = O(\varepsilon)$: this explains why (1.2) is expected to hold in the (sharp) form[1]

$$|\Omega|^{2/N}\,\lambda_1(\Omega) \overset{?}{\geq} |B|^{2/N}\,\lambda_1(B)\,\left[1 + c_N\,\mathcal{A}(\Omega)^2\right].$$

For the case of the first non-trivial Stekloff eigenvalue σ_2, things are more complicate: indeed, the most basic example – nearly spherical ellipsoids – leads to an expansion with a non-trivial first order term, i.e.

$$|\Omega_\varepsilon|^{1/N}\,\sigma_2(\Omega_\varepsilon) \simeq |B|^{1/N}\,\sigma_2(B) + O(\varepsilon).$$

The same phenomenon have already been observed in [6, Section 5] for the Neumann case. A possible explanation for this fact is the following: at the maximum point, i.e. for a ball B, the eigenvalue σ_2 is multiple and thus *is not differentiable* (see [12, Section 2]). Roughly speaking, this implies that along some "directions" (i.e. for some deformations of the ball) the functional σ_2 could have a non-trivial "super-differential". In order to show that the exponent 2 in (1.5) is indeed sharp, one has to exclude that this happens for every direction: namely, one has to exhibit a particular family of deformations Ω_ε for which a correct expansion like (1.6) is guaranteed. We will achieve this by suitably refining a construction introduced in [6, Section 6], to solve the same problem in the Neumann case: in particular, a finer analyis will lead to identify a sufficient geometric condition (see equation (6.3)), ensuring that deformations of the form $\Omega_\varepsilon = (\mathrm{Id} + \varepsilon\,X)(B)$ have the sharp decay rate in (1.5). Quite interestingly, the family of nearly spherical ellipsoids – which give the sharp decay for weighted perimeters – will turn not to satisfy this condition.

1.3. **Plan of the paper.** In Section 2, we recall the definition of weighted perimeters P_V and provide a new quantitative stability estimate for the minimality of the ball under measure constraint. Then Section 3 shows that this quantitative result is indeed sharp: in order to do this, we construct a family of nearly spherical ellipsoids E_ε, whose isoperimetric deficit $P_V(E_\varepsilon) - P_V(B)$ decays to 0 as $\mathcal{A}(E_\varepsilon)^2$. We then come to the main target of the paper: to make the exposition as self contained

[1] This conjecture seems to have been first formulated in [4, Section 8]

as possible, in Section 4 we recall some basic facts about Stekloff eigenvalues, as well as the spectral optimization problems we are concerned with. Thanks to our quantitative estimates for weighted perimeters, we can finally prove that optimal shapes for these spectral problems are stable (Section 5). The corresponding stability estimates happen to be sharp as well, as shown in the (long) final Section 6.

2. Preliminaries: stability for weighted isoperimeters

Throughout the paper, we will denote by ω_N the measure of the $N-$dimensional unit ball, i.e.

$$\omega_N := \frac{\pi^{N/2}}{\Gamma(N/2+1)}.$$

The goal of this section is to establish the cornerstone of our stability estimates for Stekloff eigenvalues: a sharp quantitative version of the isoperimetric inequality

$$|\Omega|^{-\frac{N+1}{N}} \int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{N-1} \geq N \, \omega_N^{-1/N},$$

asserting that balls centered at the origin are the unique sets minimizing the *moment of inertia* (w.r.t. the origin) of the boundary, the measure being given. This is a particular instance of a general result for *weighted perimeters* (see below) proved in [2]. Actually, our method of proof adapts to cover most of the cases considered in [2], so we will give the proof under fairly more general assumptions: although we will not need this result in such generality, we believe it to be of independent interest. We also point out that for simplicity, we will work in the class of bounded open set with Lipschitz boundary. The reason is twofold: on the one hand, this is the natural setting where spectral problems for Stekloff eigenvalues can be settled (see Section 4 for more details); on the other hand, this permits to neatly present the central idea of our proof, avoiding unnecessary technicalities.

Definition 2.1 (Weighted perimeter)**.** Let $V : [0,\infty) \to [0,\infty)$ a non-negative Borel function. Then for every $\Omega \subset \mathbb{R}^N$ open bounded Lipschitz set, its *weighted perimeter* is given by

$$P_V(\Omega) = \int_{\partial\Omega} V(|x|) \, d\mathcal{H}^{N-1}(x).$$

From now on, we will assume for simplicity that $V(0) = 0$. Under the assumption

$$V \text{ strictly increasing} \qquad \text{and} \qquad v(t) := V(t^{1/N})\, t^{1-1/N}, \quad t \geq 0 \qquad \text{convex}, \tag{2.1}$$

in [2] it is proven the following sharp lower bound for the weighted perimeter

$$P_V(\Omega) \geq N \, \omega_N^{1/N} \, |\Omega|^{1-\frac{1}{N}} \, V\left(\left(\frac{|\Omega|}{\omega_N}\right)^{\frac{1}{N}}\right), \tag{2.2}$$

with equality if and only if Ω is a ball centered at the origin. This precisely implies that the ball centered at the origin is the only minimizer of P_V, under volume constraint: we now prove a quantitative stability estimate for this isoperimetric statement, slightly strengthening the assumption (2.1) (see the discussion at the end of the proof).

This is the main result of this section.

Theorem 2.2. *Let $N \geq 2$ and $V : [0, \infty) \to [0, \infty)$ be a weight function such that $V \in C^2((0, \infty))$ and satisfying the following properties:*

$$V(0) = 0 \quad \text{and} \quad W(t) := V'(t) + (N-1)\frac{V(t)}{t} \quad \text{is such that} \quad W'(t) > 0, \quad t > 0. \tag{2.3}$$

Then for every $\Omega \subset \mathbb{R}^N$ open bounded set with Lipschitz boundary, we have

$$P_V(\Omega) \geq N\, \omega_N^{1/N}\, |\Omega|^{1-\frac{1}{N}} \left[V\left(\left(\frac{|\Omega|}{\omega_N} \right)^{\frac{1}{N}} \right) + c_{N,V,|\Omega|} \left(\frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right], \tag{2.4}$$

where B is the ball centered at the origin and such that $|B| = |\Omega|$. Here $c_{N,V,|\Omega|}$ is a constant depending on N, the weight V and the measure of Ω, defined by

$$c_{N,V,|\Omega|} = \frac{1}{4} \left(\min_{t \in \left[r_\Omega,\, r_\Omega \sqrt[N]{2}\right]} W'(t) \right) \frac{\sqrt[N]{2} - 1}{N} \left(\frac{|\Omega|}{\omega_N} \right)^{\frac{2}{N}},$$

where for simplicity we set

$$r_\Omega := \left(\frac{|\Omega|}{\omega_N} \right)^{\frac{1}{N}}. \tag{2.5}$$

Proof. Let B be the ball centered at the origin and having radius r_Ω, so that $|B| = |\Omega|$. The key idea of the proof is to use a sort of *calibration technique*, adapted to the case of weighted perimeters: related ideas can be found in the recent paper [17]. Namely, we consider the following vector field

$$x \mapsto V(|x|)\, \frac{x}{|x|}, \qquad x \in \mathbb{R}^N \setminus \{0\},$$

whose divergence is given by

$$\operatorname{div}\left(V(|x|)\, \frac{x}{|x|} \right) = V'(|x|) + (N-1)\frac{V(|x|)}{|x|} = W(|x|), \quad x \in \mathbb{R}^N \setminus \{0\},$$

and this is an increasing function, by hypothesis. Integrating W on Ω and then applying the Divergence Theorem, we then obtain

$$\int_\Omega W(|x|)\, dx = \int_{\partial\Omega} V(|x|) \left\langle \frac{x}{|x|}, \nu_\Omega(x) \right\rangle d\mathcal{H}^{N-1} \leq P_V(\Omega),$$

and in the same way, integrating on B we get

$$\int_B W(|x|)\, dx = \int_{\partial B} V(|x|)\, d\mathcal{H}^{N-1} = P_V(B).$$

Subtracting $P_V(B)$ from the previous inequality, we then obtain

$$\int_\Omega W(|x|)\, dx - \int_B W(|x|)\, dx \leq P_V(\Omega) - P_V(B).$$

We now observe that thanks to the fact that $|B| = |\Omega|$, we have $|\Omega \setminus B| = |B \setminus \Omega|$ and then

$$\begin{aligned}\int_\Omega W(|x|)\,dx - \int_B W(|x|)\,dx &= \int_{\Omega\setminus B} W(|x|)\,dx - \int_{B\setminus\Omega} W(|x|)\,dx \\ &= \int_{\Omega\setminus B} \left[W(|x|) - W(r_\Omega)\right]\,dx - \int_{B\setminus\Omega} \left[W(|x|) - W(r_\Omega)\right]\,dx \\ &= \int_{\Omega\Delta B} |W(|x|) - W(r_\Omega)|\,dx =: \mathcal{R}(\Omega),\end{aligned}$$

where in the last equality we used the monotone behaviour of W. Resuming, we have obtained the following

$$P_V(\Omega) - P_V(B) \geq \mathcal{R}(\Omega), \tag{2.6}$$

and the right-hand side is just the integral of a given function over the region $\Omega\Delta B$, so very likely this gives the desired estimate (2.4). In order to make this precise, let us introduce the radius

$$r_2 = \left(r_\Omega^N + \frac{|\Omega\setminus B|}{\omega_N}\right)^{\frac{1}{N}},$$

and the annular region

$$T = \{x \in \mathbb{R}^N \,:\, r_\Omega < |x| < r_2\},$$

which by construction satisfies $|T| = |\Omega \setminus B| = |B \setminus \Omega|$: also observe that

$$r_2 \leq r_\Omega \sqrt[N]{2}.$$

Using the monotonicity of the function $t \mapsto W(t)$, we get

$$\begin{aligned}\mathcal{R}(\Omega) &= \int_{\{x\in\Omega\,:\,|x|>r_\Omega\}} \left[W(|x|) - W(r_\Omega)\right]dx + \int_{\{x\notin\Omega\,:\,|x|<r_\Omega\}} \left[W(r_\Omega) - W(|x|)\right]dx \\ &\geq \int_T \left[W(|x|) - W(r_\Omega)\right]dx,\end{aligned}$$

so that

$$\mathcal{R}(\Omega) \geq N\,\omega_N \int_{r_\Omega}^{r_2} \left[W(\varrho) - W(r_\Omega)\right]\varrho^{N-1}\,d\varrho. \tag{2.7}$$

Thanks to the hypothesis $W'(t) > 0$ if $t > 0$, if we set

$$c_1 = \min_{t\in\left[r_\Omega,\, r_\Omega \sqrt[N]{2}\right]} W'(t),$$

this is a strictly positive constant, depending on N, V and $|\Omega|$, then from (2.7) we can infer

$$\mathcal{R}(\Omega) \geq N\,\omega_N\,c_1 \int_{r_\Omega}^{r_2} (\varrho - r_\Omega)\,\varrho^{N-1}\,d\varrho.$$

We now develope the computations for this integral: keeping into account that $|\Omega| = \omega_N\,r_\Omega^N$, we have

$$\begin{aligned}\int_{r_\Omega}^{r_2} (\varrho - r_\Omega)\,\varrho^{N-1}\,d\varrho &= \frac{r_2^{N+1} - r_\Omega^{N+1}}{N+1} - r_\Omega\,\frac{r_2^N - r_\Omega^N}{N} \\ &= r_\Omega^{N+1}\left[\frac{1}{N+1}\left(\left(1 + \frac{|\Omega\setminus B|}{|\Omega|}\right)^{\frac{N+1}{N}} - 1\right) - \frac{1}{N}\,\frac{|\Omega\setminus B|}{|\Omega|}\right].\end{aligned}$$

Let us now focus on the function $\varphi(t) = (1+t)^\alpha - 1$, for $t \in [0,1]$ and with $1 < \alpha < 2$: we have the following elementary estimate

$$(1+t)^\alpha - 1 \geq \alpha\, t + c_2\, t^2, \qquad t \in [0,1],$$

with constant c_2 given by

$$c_2 = \frac{\alpha}{4}\,(2^{\alpha-1} - 1) > 0.$$

Applying this inequality with the choices $t = |\Omega \setminus B|/|\Omega|$ and $\alpha = 1 + 1/N$, we then obtain

$$\int_{r_\Omega}^{r_2} (\varrho - r_\Omega)\,\varrho^{N-1}\,d\varrho \geq r_\Omega^{N+1}\,\frac{\sqrt[N]{2}-1}{N}\,\left(\frac{|\Omega \setminus B|}{|\Omega|}\right)^2.$$

Thus, we arrive at the following estimate

$$P_V(\Omega) - P_V(B) \geq \mathcal{R}(\Omega) \geq N\,\omega_N\,r_\Omega^{N+1}\,\frac{C}{4}\,\left(\frac{|\Omega \Delta B|}{|\Omega|}\right)^2,$$

where we have set

$$C = \left(\min_{t \in \left[r_\Omega,\, r_\Omega \sqrt[N]{2}\right]} W'(t)\right)\,\frac{\sqrt[N]{2}-1}{N}.$$

This finally gives (2.4), keeping into account that

$$P_V(B) = N\,\omega_N^{1/N}\,|\Omega|^{(N-1)/N}\,V(r_\Omega).$$

and recalling the definition of r_Ω. □

Remark 2.3 (Assumptions on the weight V)**.** We point out that a possible geometric interpretation of condition (2.3) is that the *generalized mean curvature* – formally defined as the derivative of P_V with respect to the volume – is constant for a ball centered at the origin and strictly increasing, as the radius of the ball grows.

About hypothesis (2.1) used in [2]: it is not difficult to see that this is slightly more general than our (2.3), since (2.1) is equivalent to require that W is non-decreasing. Anyway, our hypothesis could be somehow relaxed: first of all, from the estimate (2.7), we easily see that our proof still characterizes the ball as the unique isoperimetric set, simply requiring that W is strictly increasing, in particular avoiding the requirement $W' > 0$ and the C^2 regularity of V. Secondly, a closer inspection of our proof reveals that it provides the stronger lower bound

$$P_V(\Omega) - P_V(B) \geq \frac{1}{2}\int_{\partial\Omega} \left|\nu_\Omega(x) - \frac{x}{|x|}\right|^2 V(|x|)\,\mathcal{H}^{N-1} + \mathcal{R}(\Omega). \tag{2.8}$$

Then a characterization of equality cases and a stability estimate seems still feasible, by simply requiring W non-decreasing (as in [2]) and exploiting the first term in the right-hand side of (2.8). A stability estimate of this type – i.e. containing the L^2 distance of the normal versors – has recently been derived in [10] for the standard isoperimetric inequality. However, in our case some additional difficulties arise, due to the presence of the weight V. The investigation of such an interesting issue would have lead us too far from the scope of this work: for these reasons, we preferred to give a concise proof under slightly stronger assumptions – covering the case which is more relevant for us, i.e. $V(t) = t^2$.

In connection with our purposes, a significant instance of function V satisfying (2.3) is given by any strictly convex power function, i.e. $V(|x|) = |x|^p$ with $p > 1$. In this case, we use the distinguished notation

$$P_p(\Omega) = \int_{\partial\Omega} |x|^p \, \mathcal{H}^{N-1},$$

and occasionally we will call $P_p(\Omega)$ the *$p-$perimeter of* Ω. We have $P_p(\lambda\,\Omega) = \lambda^{p+N-1}\,P_p(\Omega)$, for every $\lambda > 0$, which implies in particular that the shape functional

$$\Omega \mapsto |\Omega|^{(1-N-p)/N}\,P_p(\Omega),$$

is dilation invariant, then inequality (2.2) can be equivalently written in scaling invariant form as

$$|\Omega|^{\frac{1-p-N}{N}}\,P_p(\Omega) \geq N\,\omega_N^{\frac{1-p}{N}}. \tag{2.9}$$

As a corollary of the previous Theorem, we have the following quantitative improvement of (2.9).

Corollary 2.4. *Let $p > 1$, then for every set $\Omega \subset \mathbb{R}^N$ open bounded set with Lipschitz boundary, we have*

$$|\Omega|^{\frac{1-p-N}{N}}\,P_p(\Omega) \geq N\,\omega_N^{\frac{1-p}{N}}\left[1 + c_{N,p}\left(\frac{|\Omega\Delta B|}{|\Omega|}\right)^2\right], \tag{2.10}$$

where B is the ball centered at the origin such that $|\Omega| = |B|$ and $c_{N,p}$ is a constant depending only on N and p, given by

$$c_{N,p} = \frac{(N+p-1)\,(p-1)}{4}\,\frac{\sqrt[N]{2}-1}{N}\left(\min_{t\in[1,\sqrt[N]{2}]} t^{p-2}\right).$$

Proof. We start observing that if $V(t) = t^p$, then

$$W(t) = (N+p-1)\,t^{p-1} \qquad \text{and} \qquad W'(t) = (N+p-1)\,(p-1)\,t^{p-2}.$$

In particular, using the homogeneity of W' we get that

$$\min_{t\in[r_\Omega,\, r_\Omega\sqrt[N]{2}]} W'(t) = r_\Omega^{p-2}\min_{t\in[1,\sqrt[N]{2}]} W'(t) = \left(\frac{|\Omega|}{\omega_N}\right)^{\frac{p-2}{N}}(N+p-1)\,(p-1)\left(\min_{t\in[1,\sqrt[N]{2}]} t^{p-2}\right).$$

Then in order to obtain (2.10), it is sufficient to insert the previous into (2.4), to use that

$$V\left(\left(\frac{|\Omega|}{\omega_N}\right)^{\frac{1}{N}}\right) = \left(\frac{|\Omega|}{\omega_N}\right)^{\frac{p}{N}},$$

and to divide both members of (2.4) by $|\Omega|^{(p+N-1)/N}$. □

3. Nearly spherical ellipsoids

Since the main ingredient of our quantitative Brock–Weinstock inequality will be estimate (2.10), it is important to check that this is sharp. At this aim, we show that the exponent 2 for the term $|\Omega\Delta B|$ in inequality (2.4) is optimal: for this, we simply exhibit for every radius R a sequence of sets Ω_ε^R, such that $|\Omega_\varepsilon^R| = \omega_N\,R^N$ and

$$\limsup_{\varepsilon\to 0}\frac{P_V(\Omega_\varepsilon^R) - P_V(B_R)}{|B_R\Delta\Omega_\varepsilon^R|^2} \leq C, \tag{3.1}$$

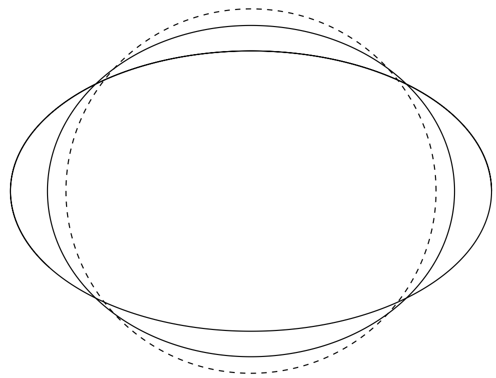

FIGURE 1. The family of ellipses E_ε.

where B_R is the ball of radius R and centered in the origin. For the sake of simplicity, we confine ourselves to consider the case $N = 2$: the very same arguments still work for every $N \geq 3$.

First of all, we aim to prove (3.1) for $R = 1$, then we will show how to obtain it for a general $R > 0$. Let us consider the following family of ellipses

$$E_\varepsilon = \left\{ \left(x\sqrt{1+\varepsilon}, \frac{y}{\sqrt{1+\varepsilon}} \right) : x^2 + y^2 < 1 \right\},$$

whose boundary can be parametrized by

$$\gamma_\varepsilon(\vartheta) = \left(\sqrt{1+\varepsilon}\,\cos\vartheta, \frac{1}{\sqrt{1+\varepsilon}}\,\sin\vartheta \right), \qquad \vartheta \in [0, 2\pi].$$

Also observe that by construction we have $|E_\varepsilon| = |B_1| = \pi$, since

$$E_\varepsilon = \mathcal{M}_\varepsilon(B_1)$$

with $\mathcal{M}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ linear application, having (with a slight abuse of notation) $\det \mathcal{M}_\varepsilon = 1$. Now, we need to expand the term

$$P_V(E_\varepsilon) = \int_0^{2\pi} V(|\gamma_\varepsilon(\vartheta)|)\,|\gamma'_\varepsilon(\vartheta)|\,d\vartheta,$$

at this aim we use the following second-order Taylor expansions for $|\gamma_\varepsilon|$, $|\gamma'_\varepsilon|$ and $V(|\gamma_\varepsilon|)$:

$$\begin{aligned} |\gamma_\varepsilon(\vartheta)| &= (1+\varepsilon)^{-1/2}\sqrt{1 + 2\varepsilon\,\cos^2\vartheta + \varepsilon^2\,\cos^2\vartheta} \\ &\simeq 1 + \varepsilon\left(\cos^2\vartheta - \frac{1}{2}\right) + \frac{\varepsilon^2}{2}\left(\frac{3}{4} - \cos^4\vartheta\right) \end{aligned}$$

and similarly

$$|\gamma'_\varepsilon(\vartheta)| \simeq 1 + \varepsilon\left(\sin^2\vartheta - \frac{1}{2}\right) + \frac{\varepsilon^2}{2}\left(\frac{3}{4} - \sin^4\vartheta\right),$$

while

$$V(|\gamma_\varepsilon(\vartheta)|) \simeq V(1) + \varepsilon\, V'(1)\, \left[\cos^2\vartheta - \frac{1}{2}\right] + \frac{\varepsilon^2}{2}\, \left[V'(1)\, \left(\frac{3}{4} - \cos^4\vartheta\right) + V''(1)\, \left(\frac{1}{2} - \cos^2\vartheta\right)^2\right].$$

Thus we have the following second-order expansion for the integrand defining $P_V(\Omega_\varepsilon)$:

$$\begin{aligned} V(|\gamma_\varepsilon(\vartheta)|)\, |\gamma_\varepsilon'(\vartheta)| \simeq V(1) &+ \varepsilon\, \left[V'(1)\, \left(\cos^2\vartheta - \frac{1}{2}\right) + V(1)\, \left(\sin^2\vartheta - \frac{1}{2}\right)\right] \\ &+ \varepsilon^2\, \left[V'(1)\, \left(\cos^2\vartheta - \frac{1}{2}\right)\, \left(\sin^2\vartheta - \frac{1}{2}\right) + \frac{V(1)}{2}\, \left(\frac{3}{4} - \sin^4\vartheta\right)\right. \\ &\left. + \frac{V''(1)}{2}\, \left(\frac{1}{2} - \cos^2\vartheta\right)^2 + \frac{V'(1)}{2}\, \left(\frac{3}{4} - \cos^4\vartheta\right)\right]. \end{aligned}$$

Finally, we observe that

$$\int_0^{2\pi} \left(\cos^2\vartheta - \frac{1}{2}\right)\, d\vartheta = \int_0^{2\pi} \left(\sin^2\vartheta - \frac{1}{2}\right)\, d\vartheta = 0,$$

and

$$\int_0^{2\pi} \left(\cos^2\vartheta - \frac{1}{2}\right)^2\, d\vartheta = -\int_0^{2\pi} \left(\cos^2\vartheta - \frac{1}{2}\right)\, \left(\sin^2\vartheta - \frac{1}{2}\right)\, d\vartheta = \frac{\pi}{4}$$

while

$$\int_0^{2\pi} \left(\frac{3}{4} - \cos^4\vartheta\right)\, d\vartheta = \int_0^{2\pi} \left(\frac{3}{4} - \sin^4\vartheta\right)\, d\vartheta = \frac{3}{4}\,\pi.$$

Summarizing, we have obtained

$$P_V(E_\varepsilon) - P_V(B_1) \simeq \frac{\pi}{8}\,\varepsilon^2\, \left[3\,V(1) + V'(1) + V''(1)\right], \tag{3.2}$$

and on the other hand it is easily seen that $|E_\varepsilon \Delta B_1| = O(\varepsilon)$, thus we get (3.1) for $R = 1$.

To obtain this result for a generic $R > 0$, we notice that for every set Ω,

$$P_V(R\,\Omega) = R\, P_{V_R}(\Omega),$$

where $V_R(t) = V(R\,t)$, $t \geq 0$. Hence, if we set $\widetilde{E}_\varepsilon := R\,E_\varepsilon$ we have

$$\begin{aligned} P_V(\widetilde{E}_\varepsilon) - P_V(B_R) &= R\, \left[P_{V_R}(E_\varepsilon) - P_{V_R}(B_1)\right] \\ &\simeq \varepsilon^2\, \frac{\pi\, R}{8}\, \left[3\,V(R) + R\,V'(R) + R^2\,V''(R)\right], \end{aligned}$$

thanks to (3.2), thus giving (3.1) also in the general case. Observe that thanks to (2.3) we easily get that

$$R^2\,V''(R) + RV'(R) > V(R),$$

and thus in particular

$$3\,V(R) + R\,V'(R) + R^2\,V''(R) > 4\,V(R) > 0.$$

4. Spectral optimization for Stekloff eigenvalues

We now arrive at the core of the paper, i.e. spectral optimization problems involving the spectrum of the Stekloff-Laplacian: to keep the exposition as self contained as possible, we start recalling some basic definitions (see also [12, Chapter 7]).

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Thanks to the compactness of the embedding of $W^{1,2}(\Omega)$ into $L^2(\partial\Omega)$, we have that the resolvant operator $\mathcal{R} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ defined by

$$\mathcal{R}g \in W^{1,2}(\Omega) \qquad \text{solves in weak sense} \qquad \left\{\begin{array}{rcll} -\Delta u &=& 0, & \text{in } \Omega,\\ \langle \nabla u, \nu_\Omega\rangle &=& g & \text{on } \partial\Omega,\end{array}\right.$$

is a compact, symmetric and positive linear operator. Hence $\mathcal{R}$ has a discrete spectrum, made only of real positive eigenvalues accumulating at 0. As a consequence, we have that the following boundary value problem for harmonic functions

$$\left\{\begin{array}{rcll} -\Delta u &=& 0, & \text{in } \Omega,\\ \langle \nabla u, \nu_\Omega\rangle &=& \sigma\, u, & \text{on } \partial\Omega,\end{array}\right.$$

has non-trivial solutions only for a discrete set of values $\sigma_1(\Omega) \le \sigma_2(\Omega) \le \sigma_3(\Omega)\dots$ accumulating at ∞: these are the so-called *Stekloff eigenvalues of* Ω. Here solutions are intended in the usual weak sense, i.e.

$$\int_\Omega \langle \nabla u(x), \nabla\varphi(x)\rangle\, dx = \sigma_k(\Omega) \int_{\partial\Omega} u(x)\,\varphi(x)\, dx, \qquad \text{for every } \varphi \in W^{1,2}(\Omega),\ k \in \mathbb{N}.$$

The corresponding solutions $\{\xi_k\}_{k\ge1}$ are called *eigenfunctions of the Stekloff-Laplacian* and they give an orthonormal basis of $L^2(\partial\Omega)$, once renormalized by $\|\xi_k\|_{L^2(\partial\Omega)} = 1$, for every $k \ge 1$. Throughout the next sections we will use the classical convention of counting the eigenvalues with their multiplicities: this means that if for a certain $k \in \mathbb{N}\setminus\{0\}$, there exist m linearly independent non-trivial solutions for $\sigma_k(\Omega)$, then we will write $\sigma_k(\Omega) = \sigma_{k+1}(\Omega) = \cdots = \sigma_{k+m-1}(\Omega)$.

Observe that if Ω has k connected components $\Omega_1,\dots,\Omega_k$, then $\sigma_1(\Omega) = \cdots = \sigma_k(\Omega) = 0$ and the corresponding renormalized eigenfunctions are constant functions, given by

$$\xi_i(x) = \frac{1_{\Omega_i}(x)}{\sqrt{\mathcal{H}^{N-1}(\partial\Omega_i)}}, \qquad i = 1,\dots,k.$$

In particular *the first Stekloff eigenvalue of a set is always trivial* and corresponds to constant functions. For this reason, given $k \in \mathbb{N}\setminus\{0\}$, we always have that

$$0 = \inf\{\sigma_k(\Omega) \,:\, |\Omega| = c\},$$

and this infimum is attained for every open set having k connected components.

Remark 4.1. For what follows, it is important to remark that the functions $\{\xi_k\}_{k\ge2}$ also give an orthogonal basis for the following closed subspace of $W^{1,2}(\Omega)$

$$\mathrm{Har}(\Omega) = \left\{u \in W^{1,2}(\Omega) \,:\, \int_{\partial\Omega} u = 0 \ \text{ and } \ \int_\Omega \langle \nabla u, \nabla\varphi\rangle = 0 \text{ for every } \varphi \in W^{1,2}_0(\Omega)\right\}, \tag{4.1}$$

on which $u \mapsto \|\nabla u\|_{L^2}$ and $u \mapsto \|u\|_{W^{1,2}}$ are equivalent norms, thanks to the Poincaré-Wirtinger inequality (1.3) and to inequality

$$\|u\|_{L^2(\Omega)} \le C_\Omega \left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}\right), \qquad u \in W^{1,2}(\Omega),$$

which can be proved by means of a standard compactness argument. Notice that for every $u \in \mathrm{Har}(\Omega)$, its Dirichlet integral can be written as

$$\int_\Omega |\nabla u(x)|^2\,dx = \sum_{k\geq 2} \alpha_k^2\,\sigma_k(\Omega), \qquad \text{where} \quad \alpha_k = \int_{\partial\Omega} \xi_k(x)\,u(x)\,d\mathcal{H}^{N-1}. \tag{4.2}$$

For any ball B of radius R, its first non-trivial Stekloff eigenvalue is given by

$$\sigma_2(B) = \frac{1}{R},$$

which corresponds to the eigenfunctions $\xi_i(x) = x_{i-1}$, with $i = 2, \dots, N+1$, i.e. the eigenvalue $\sigma_2(B)$ has multiplicity N. Also, we notice that the shape functional $\Omega \mapsto |\Omega|^{1/N}\,\sigma_2(\Omega)$ is scaling invariant, thus in particular

$$|B|^{1/N}\,\sigma_2(B) = \omega_N^{1/N},$$

for any ball B. About the first non-trivial Stekloff eigenvalue of a set Ω, we have the following sharp estimate, first derived in [24] for dimension $N = 2$ and then generalized to any dimension in [7].

Brock-Weinstock inequality. *For every $\Omega \subset \mathbb{R}^N$ open bounded set with Lipschitz boundary, we have*

$$|\Omega|^{1/N}\,\sigma_2(\Omega) \leq \omega_N^{1/N}, \tag{4.3}$$

and equality holds if and only if Ω is a ball. In other words, for every $c > 0$ the unique solution of the following spectral optimization problem

$$\max\{\sigma_2(\Omega)\,:\,|\Omega| \geq c\},$$

is given by a ball of measure c.

Remark 4.2. As already remarked in the Introduction, $1/\sigma_2(\Omega)$ can be characterized as the sharp constant in the Poincaré-Wirtinger trace inequality (1.3). We then notice that the Brock-Weinstock inequality can be extended to any set supporting such an inequality and for which the trace of a $W^{1,2}$ function is well-defined: in these cases, it is still meaningful speaking of $\sigma_2(\Omega)$, though the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ could not be compact and hence its Stekloff-Laplacian could have a continuous spectrum.

Actually, the Brock-Weinstock inequality is a straightforward consequence of a stronger estimate proved by Brock in [7], involving the first N non-trivial Stekloff eigenvalues: namely, for every $\Omega \subset \mathbb{R}^N$ bounded open set with Lipschitz boundary, we have

$$\frac{1}{|\Omega|^{1/N}} \sum_{i=2}^{N+1} \frac{1}{\sigma_i(\Omega)} \geq \frac{N}{\omega_N^{1/N}}, \tag{4.4}$$

i.e. any ball minimizes the sum of the reciprocal of the first N non-trivial Stekloff eigenvalues, among sets of given measure.

Remark 4.3. In the case of convex sets, an even stronger estimate is possible [15]: the ball maximizes the product of the first N non-trivial Stekloff eigenvalues, under measure constraint

$$|\Omega| \prod_{i=2}^{N+1} \sigma_i(\Omega) \leq \omega_N. \tag{4.5}$$

A simple application of the arithmetic-geometric mean inequality shows that the previous implies (4.4): it should be noticed that in dimension $N = 2$, the convexity assumption can be dropped (see [13]), while for higher dimensions it is still an open problem to know whether (4.5) holds for all sets or not.

5. The stability issue

The main goal of this section is to show how (4.4) and (4.3) can be improved by means of a quantitative stability estimate. At this aim, for every $\Omega \subset \mathbb{R}^N$ open set with finite measure, we recall the definition of *Fraenkel asymmetry*

$$\mathcal{A}(\Omega) := \inf \left\{ \frac{\|1_\Omega - 1_B\|_{L^1(\mathbb{R}^N)}}{|\Omega|} \, : \, B \text{ ball with } |B| = |\Omega| \right\},$$

i.e. this is the distance in the L^1 sense of a generic set Ω from the "manifold" of balls, renormalized in order to make it scaling invariant: observe that $0 \le \mathcal{A}(\Omega) < 2$. Then the main result of this section is the following quantitative improvement of (4.4).

Theorem 5.1. *For every $\Omega \subset \mathbb{R}^N$ open bounded set with Lipschitz boundary, we have*

$$\frac{1}{|\Omega|^{1/N}} \sum_{i=2}^{N+1} \frac{1}{\sigma_i(\Omega)} \ge \frac{N}{\omega_N^{1/N}} \left[1 + c_{N,2}\, \mathcal{A}(\Omega)^2\right], \tag{5.1}$$

where the dimensional constant $c_{N,2}$ is the same as in (2.10), *i.e.*

$$c_{N,2} = \frac{N+1}{N} \frac{\sqrt[N]{2} - 1}{4}.$$

Proof. We start reviewing the proof of Brock in [7]: the first step is to have a variational characterization for the sum of inverses of eigenvalues. In the case of Stekloff eigenvalues, the following formula holds (see [16, Theorem 1], for example):

$$\sum_{i=2}^{N+1} \frac{1}{\sigma_i(\Omega)} = \max_{(v_2,\dots,v_{N+1}) \in \mathcal{I}} \sum_{i=2}^{N+1} \int_{\partial\Omega} v_i(x)^2 \, d\mathcal{H}^{N-1},$$

where the set of admissible functions is given by

$$\mathcal{I} = \left\{ (v_2, \dots, v_{N+1}) \in (W^{1,2}(\Omega))^N \, : \, \int_{\partial\Omega} v_i(x)\, d\mathcal{H}^{N-1} = 0,\ \int_\Omega \langle \nabla v_i(x), \nabla v_j(x) \rangle \, dx = \delta_{ij} \right\}.$$

Observe that the quantities $\sigma_i(\Omega)$ are invariant under translations, so without loss of generality we can suppose that the barycenter of $\partial\Omega$ is in the origin, i.e.

$$\int_{\partial\Omega} x_i \, d\mathcal{H}^{N-1} = 0, \qquad i = 1, \dots, N.$$

This implies that the eigenfunctions ξ_i relative to $\sigma_2(B) = \dots = \sigma_{N+1}(B)$ are admissible in the previous maximization problem, thus as admissible functions we take

$$v_i(x) = \frac{x_{i-1}}{\sqrt{|\Omega|}}, \qquad i = 2, \dots, N+1.$$

In this way, we obtain

$$\frac{1}{|\Omega|^{1/N}}\sum_{i=2}^{N+1}\frac{1}{\sigma_i(\Omega)}\geq\frac{1}{|\Omega|^{1+1/N}}\int_{\partial\Omega}|x|^2\,d\mathcal{H}^{N-1}=|\Omega|^{-\frac{N+1}{N}}\,P_2(\Omega),$$

which implies

$$\frac{1}{|\Omega|^{1/N}}\sum_{i=2}^{N+1}\frac{1}{\sigma_i(\Omega)}-\frac{N}{\omega_N^{1/N}}\geq|\Omega|^{-\frac{N+1}{N}}\,P_2(\Omega)-\frac{N}{\omega_N^{1/N}},$$

This means that the deficit of this spectral inequality is controlling from above the deficit of the 2−perimeter. Thus it is sufficient to use the quantitative estimate (2.10) for the 2−perimeter, so to obtain

$$\frac{1}{|\Omega|^{1/N}}\sum_{i=2}^{N+1}\frac{1}{\sigma_i(\Omega)}-\frac{N}{\omega_N^{1/N}}\geq\frac{N}{\omega_N^{1/N}}\,c_{N,2}\left(\frac{|\Omega\Delta B|}{|\Omega|}\right)^2,$$

where B is the ball centered at the origin and such that $|\Omega|=|B|$. Using the definition of $\mathcal{A}(\Omega)$, we can conclude the proof. □

A straightforward consequence of the previous result is the following quantitative version of the Brock-Weinstock inequality.

Corollary 5.2. *For every $\Omega\subset\mathbb{R}^N$ open bounded set with Lipschitz boundary, we have*

$$|\Omega|^{1/N}\,\sigma_2(\Omega)\leq\omega_N^{1/N}\,\left[1-\delta_N\,\mathcal{A}(\Omega)^2\right],\tag{5.2}$$

where δ_N is a constant depending only on the dimension, given by

$$\delta_N=\frac{1}{8}\,\min\left\{1,\frac{N+1}{N}\,\left(\sqrt[N]{2}-1\right)\right\}.$$

Proof. First of all, we can suppose that

$$|\Omega|^{1/N}\,\sigma_2(\Omega)\geq\frac{1}{2}\,\omega_N^{1/N},\tag{5.3}$$

otherwise estimate (5.2) is trivially true with constant $\delta_N=1/8$, just by using the fact that $\mathcal{A}(\Omega)<2$. So, let us suppose that (5.3) holds true: since $\sigma_2(\Omega)\leq\sigma_i(\Omega)$ for every $i\geq 3$, from (5.1) we can infer

$$\frac{N}{|\Omega|^{1/N}\,\sigma_2(\Omega)}\geq\frac{N}{\omega_N^{1/N}}\,\left[1+c_{N,2}\,\mathcal{A}(\Omega)^2\right],$$

which can be rewritten as

$$|\Omega|^{1/N}\,\sigma_2(\Omega)\,\left[1+c_{N,2}\,\mathcal{A}(\Omega)^2\right]\leq\omega_N^{1/N}.$$

The previous easily implies (5.2), thanks to (5.3). □

Remark 5.3. In the next section we will prove that both the estimates derived in Theorem 5.1 and Corollary 5.2 are sharp. We point out that defining the two deficit functionals

$$\mathrm{Inv}(\Omega):=\frac{|B|^{1/N}}{N\,|\Omega|^{1/N}}\sum_{i=2}^{N+1}\frac{\sigma_2(B)}{\sigma_i(\Omega)}-1\qquad\text{and}\qquad BW(\Omega):=\frac{|B|^{1/N}\,\sigma_2(B)}{|\Omega|^{1/N}\,\sigma_2(\Omega)}-1,\tag{5.4}$$

we have that

$$c_{N,2}\,\mathcal{A}(\Omega)^2\leq\mathrm{Inv}(\Omega)\leq BW(\Omega),$$

where in the first inequality we used Theorem 5.1. Then if one can prove that the exponent 2 for $\mathcal{A}(\Omega)$ is sharp in the quantitative Brock-Weinstock inequality, this will automatically prove the optimality of the power 2 for inequality (5.1).

6. Sharpness of the quantitative Brock-Weinstock inequality

In this section, we will show the sharpness of the quantitative Brock-Weinstock inequality (5.2): as remarked, this in turn will give the sharpness of (5.1) as well. Namely, we are going to prove the following result.

Theorem 6.1. *There exists a family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of smooth sets approaching the ball B of unit radius in such a way that*

$$\mathcal{A}(\Omega_\varepsilon) \simeq \frac{|\Omega_\varepsilon \Delta B|}{|\Omega_\varepsilon|} \simeq \varepsilon \qquad \text{and} \qquad BW(\Omega_\varepsilon) \simeq \varepsilon^2, \qquad \varepsilon \ll 1, \tag{6.1}$$

where $BW(\Omega)$ is defined by (5.4).

The rest of this section is devoted to construct such a family of deformations Ω_ε. Since the whole construction is quite complicate, for the sake of readability we will divide it into 4 main steps.

6.1. Step 1: setting of the construction and basic properties.

In what follows, $B \subset \mathbb{R}^N$ stands for the open unit ball, centered at the origin. We consider a general nearly circular domain, given by

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N \, : \, x = 0 \ \text{ or } \ |x| < 1 + \varepsilon\, \psi(x/|x|)\},$$

where $\psi \in C^\infty(\partial B)$ satisfies the following assumptions.

Key assumptions. For every $a \in \mathbb{R}^N$, there holds

$$\int_{\partial B} \psi(x)\, d\mathcal{H}^{N-1} = 0, \qquad \int_{\partial B} \langle a, x\rangle\, \psi(x)\, d\mathcal{H}^{N-1} = 0, \tag{6.2}$$

and

$$\int_{\partial B} \langle a, x\rangle^2\, \psi(x)\, d\mathcal{H}^{N-1} = 0. \tag{6.3}$$

We start with a basic result of geometric type.

Lemma 6.2. *Let $\psi \in C^\infty(\partial B)$ satisfying* (6.2)*. Then*

$$|\Omega_\varepsilon| - |B| \simeq \varepsilon^2 \qquad \text{and} \qquad \mathcal{A}(\Omega_\varepsilon) \simeq \varepsilon \simeq \frac{|\Omega_\varepsilon \Delta B|}{|\Omega_\varepsilon|}. \tag{6.4}$$

Proof. Using polar coordinates, the measure $|\Omega_\varepsilon|$ can be expressed as follows

$$\begin{aligned} |\Omega_\varepsilon| &= \frac{1}{N} \int_{\partial B} (1 + \varepsilon\, \psi(x))^N\, d\mathcal{H}^{N-1} \\ &\simeq |B| + \varepsilon \int_{\partial B} \psi(x)\, d\mathcal{H}^{N-1} + \varepsilon^2\, \frac{N-1}{2} \int_{\partial B} \psi(x)^2\, d\mathcal{H}^{N-1}, \end{aligned}$$

which gives the first relation in (6.4), thanks to the fact that ψ has zero-mean on ∂B.

For the second one, we start observing that $|\Omega_\varepsilon \Delta B| \simeq \varepsilon$: still using polar coordinates, we get

$$\begin{aligned}|\Omega_\varepsilon \Delta B| = \frac{1}{N}\int_{\{x\in\partial B\,:\,\psi(x)>0\}} \left[(1+\varepsilon\,\psi(x))^N - 1\right]\, d\mathcal{H}^{N-1} \\ + \frac{1}{N}\int_{\{x\in\partial B\,:\,\psi(x)<0\}} \left[1-(1+\varepsilon\,\psi(x))^N\right]\, d\mathcal{H}^{N-1} \simeq \varepsilon \int_{\partial B} |\psi(x)|\, d\mathcal{H}^{N-1}.\end{aligned}$$

Now, let $B(x_0, r_\varepsilon)$ be a ball realizing the asymmetry, i.e. such that $\mathcal{A}(\Omega_\varepsilon)\,|\Omega_\varepsilon| = |\Omega_\varepsilon \Delta B(x_0, r_\varepsilon)|$: in particular, we have $|B(x_0, r_\varepsilon)| = |\Omega_\varepsilon|$. It is easily seen that

$$\mathcal{A}(\Omega_\varepsilon) \le c\, \frac{|\Omega_\varepsilon \Delta B|}{|\Omega_\varepsilon|}, \tag{6.5}$$

for some constant c independent of ε: indeed, by definition of $\mathcal{A}(\Omega_\varepsilon)$ and triangular inequality, we get

$$\mathcal{A}(\Omega_\varepsilon) \le \frac{|\Omega_\varepsilon \Delta B(0, r_\varepsilon)|}{|\Omega_\varepsilon|} \le \frac{|\Omega_\varepsilon \Delta B|}{|\Omega_\varepsilon|} + \frac{|B \Delta B(0, r_\varepsilon)|}{|\Omega_\varepsilon|} \le c\, \frac{|\Omega_\varepsilon \Delta B|}{|\Omega_\varepsilon|},$$

since $|B\Delta B(0, r_\varepsilon)| = |\,|B(x_0, r_\varepsilon)| - |B|\,| \simeq \varepsilon^2$, while $|\Omega_\varepsilon \Delta B| \simeq \varepsilon$.

Using the symmetries of B and (6.2), for every $a \in \mathbb{R}^N$ we can infer

$$\int_{\Omega_\varepsilon} \langle a, y\rangle\, dy = \frac{1}{N+1}\int_{\partial B} (1+\varepsilon\,\psi(x))^{N+1}\,\langle a, x\rangle\, d\mathcal{H}^{N-1} \simeq \varepsilon^2\, \frac{N}{2}\int_{\partial B} \psi(x)^2\,\langle a, x\rangle\, d\mathcal{H}^{N-1}.$$

Choosing $a = \mathbf{e}_i$, i.e. the coordinate directions, the previous implies that the barycenter of Ω_ε coincides with the origin, up to an error of order ε^2. Since the barycenter of $B(x_0, r_\varepsilon)$ is given by its center x_0, we then get

$$\begin{aligned}|B(x_0, r_\varepsilon)|\,|x_0| = \left|\int_{B(x_0,r_\varepsilon)} y\, dy\right| &\le \left|\int_{B(x_0,r_\varepsilon)} y\, dy - \int_{\Omega_\varepsilon} y\, dy\right| + \left|\int_{\Omega_\varepsilon} y\, dy\right| \\ &\le \int_{B(x_0,r_\varepsilon)\Delta\Omega_\varepsilon} |y|\, dy + \left|\int_{\Omega_\varepsilon} y\, dy\right| \le C\,|B(x_0, r_\varepsilon)\Delta\Omega_\varepsilon| + C\,\varepsilon^2,\end{aligned}$$

for some constant C independent of ε. In other words, we get $|x_0| \le C\,\mathcal{A}(\Omega_\varepsilon) + C\,\varepsilon^2$ – possibly with a different constant C, but still independent of ε – then we can estimate

$$\frac{|B\Delta\Omega_\varepsilon|}{|\Omega_\varepsilon|} \le \frac{|\Omega_\varepsilon \Delta B(x_0, r_\varepsilon)|}{|\Omega_\varepsilon|} + \frac{|B(x_0, r_\varepsilon)\Delta B|}{|\Omega_\varepsilon|} \le \mathcal{A}(\Omega_\varepsilon) + C'\,|x_0| + C''\,\varepsilon^2, \tag{6.6}$$

for some C', C'' not depending on ε: here we used that $|B(x_0, r_\varepsilon)\Delta B|$ is comparable to the distance of their centers – that is, comparable to $|x_0|$ – up to an error of order ε^2, due to the difference of the measures. It is only left to use the estimate on $|x_0|$ in (6.6), in conjunction with (6.5) and the fact that $|\Omega_\varepsilon \Delta B| \simeq \varepsilon$: we then get

$$\frac{1}{c'}\,\varepsilon \le \mathcal{A}(\Omega_\varepsilon) + C\,\varepsilon^2 \le c'\,\varepsilon,$$

for some $c' > 1$, which finally gives $\mathcal{A}(\Omega_\varepsilon) \simeq \varepsilon$, as desired. □

Remark 6.3 (Meaning of the key assumptions)**.** We point out that conditions (6.2) and (6.3) are equivalent to require that ψ is orthogonal in the $L^2(\partial B)$ sense to the first three eigenspace of the Laplace-Beltrami operator on ∂B, i.e. to spherical harmonics of order $0, 1$ and 2 respectively (see [19] for a comprehensive account on spherical harmonics). Each of these conditions will play a precise role in our construction: thanks to the previous result, the first one implies that Ω_ε has the same measure as B, up to an error of order ε^2. The second condition in (6.2) implies that Ω_ε has the same barycenter as B, still up to an error of order ε^2: then this order coincides with the magnitude of $\mathcal{A}(\Omega_\varepsilon)^2$. Eventually, recalling that every Stekloff eigenfunction ξ relative to $\sigma_2(B)$ has the form $\xi(x) = \langle a, x\rangle$, condition (6.3) implies

$$\int_{\partial B} \psi(x)\, |\xi(x)|^2\, d\mathcal{H}^{N-1} = 0 \qquad \text{and} \qquad \int_{\partial B} \psi(x)\, |\nabla_\tau \xi(x)|^2\, d\mathcal{H}^{N-1} = 0, \tag{6.7}$$

where ∇_τ is the tangential gradient. Relations (6.7) will be crucially exploited in order to prove that $\sigma_2(B) - \sigma_2(\Omega_\varepsilon) \simeq \varepsilon^2$.

Let us fix now an eigenfunction u_ε for $\sigma_2(\Omega_\varepsilon)$, normalized in such a way that

$$\int_{\partial\Omega_\varepsilon} u_\varepsilon(x)^2\, d\mathcal{H}^{N-1} = 1 \qquad \text{and} \qquad \int_{\Omega_\varepsilon} \big|\nabla u_\varepsilon(x)\big|^2\, dx = \sigma_2(\Omega_\varepsilon). \tag{6.8}$$

Remark 6.4. Thanks to the fact that $\partial\Omega_\varepsilon$ is of class C^∞, we obtain that $u_\varepsilon \in C^\infty(\overline{\Omega_\varepsilon})$. Moreover, the domains Ω_ε are uniformly of class C^k, for every $k \geq 0$, hence we can assume the functions u_ε to satisfy uniform C^k estimates, i.e.

$$\|u_\varepsilon\|_{C^k(\overline{\Omega_\varepsilon})} \leq H_k\,, \tag{6.9}$$

for some constants $H_k > 0$ depending only on $k \in \mathbb{N}$.

We now give the basic estimate of $\sigma_2(B)$ from above in terms of $\sigma_2(\Omega_\varepsilon)$: this is the cornerstone of the whole construction.

Lemma 6.5. *Let $\varepsilon_0 \ll 1$, there exist two functions $N, Q : [0, \varepsilon_0] \to \mathbb{R}$ with*

$$\lim_{\varepsilon\to 0} \left(|N(\varepsilon)| + |Q(\varepsilon)|\right) = 0,$$

and a constant $K > 0$ such that for every ε, we have

$$\sigma_2(B) \leq \frac{\sigma_2(\Omega_\varepsilon) + N(\varepsilon)}{1 + Q(\varepsilon) - K\varepsilon^2}. \tag{6.10}$$

Proof. Since we want to compare $\sigma_2(\Omega_\varepsilon)$ with $\sigma_2(B)$, we have to suitably adapt the eigenfuction u_ε, in order to let it be admissible for the Rayleigh quotient defining $\sigma_2(B)$. To do so, we start considering a C^k extension $\widetilde{u}_\varepsilon$ of u_ε with $k = [N/2] + 3$ to the larger set [2]

$$D_\varepsilon = \left\{x\,:\, |x| \leq 1 + \varepsilon\|\psi\|_{L^\infty(\partial B)}\right\} \supset \overline{B \cup \Omega_\varepsilon},$$

and we can make such an extension in such a way that

$$\|\widetilde{u}_\varepsilon\|_{C^k(D_\varepsilon)} \leq K\|u_\varepsilon\|_{C^k(\Omega_\varepsilon)}\,. \tag{6.11}$$

Then, we estimate the mean value of this extension on the boundary ∂B: we set

$$\delta := \fint_{\partial B} \widetilde{u}_\varepsilon(x)\, d\mathcal{H}^{N-1},$$

and we define the application $\phi_\varepsilon : \partial B \to \partial\Omega_\varepsilon$, given by

$$\phi_\varepsilon(x) = x + \varepsilon\, \psi(x)\, x, \qquad x \in \partial B. \tag{6.12}$$

Observe that we have

$$\widetilde{u}_\varepsilon(\phi_\varepsilon(x)) = u_\varepsilon(\phi_\varepsilon(x)), \qquad x \in \partial B,$$

[2]The choice of k will be clear in the proof of Lemma 6.11.

so that our uniform estimates (6.9) and (6.11) yield

$$\widetilde{u}_\varepsilon(x) = u_\varepsilon(\phi_\varepsilon(x)) + O(\varepsilon), \qquad x \in \partial B. \tag{6.13}$$

Using this information in the definition of δ, we get

$$\delta = \fint_{\partial B} u_\varepsilon(\phi_\varepsilon(x))\, d\mathcal{H}^{N-1} + O(\varepsilon) = \fint_{\partial B} u_\varepsilon(\phi_\varepsilon(x))\, J_\varepsilon(x)\, d\mathcal{H}^{N-1} + O(\varepsilon)$$

where in the last equality we have set

$$J_\varepsilon(x) = (1 + \varepsilon\, \psi(x))^{N-2} \sqrt{(1 + \varepsilon\, \psi(x))^2 + \varepsilon^2\, |\nabla_\tau\, \psi(x)|^2}, \qquad x \in \partial B,$$

and we used the following straightforward estimate

$$\|J_\varepsilon(y) - 1\|_{L^\infty(\partial B)} = O(\varepsilon), \tag{6.14}$$

the quantity $\nabla_\tau \psi$ being the *tangential gradient* of ψ on ∂B. With the change of variable $y = \phi_\varepsilon(x)$, we then arrive at

$$\delta = \frac{1}{\mathcal{H}^{N-1}(\partial B)} \int_{\partial\Omega_\varepsilon} u_\varepsilon(y)\, d\mathcal{H}^{N-1} + O(\varepsilon) = O(\varepsilon), \tag{6.15}$$

thanks to the fact that $\int_{\partial\Omega_\varepsilon} u_\varepsilon = 0$. We are now ready to define an admissible function for $\sigma_2(B)$: we set

$$v_\varepsilon := \widetilde{u}_\varepsilon \cdot 1_{\bar{B}} - \delta, \tag{6.16}$$

and we immediately notice that

$$\|v_\varepsilon\|_{C^k(\overline{B})} \leq K(N)\,, \tag{6.17}$$

thanks to (6.9), (6.11) and (6.15) (recall that k depends only on N). In words, v_ε is the original eigenfunction u_ε extended to the whole D_ε, then restricted to the ball B and finally vertically translated in order to satisfy the zero-mean condition on ∂B. By its very definition and using (6.15), we immediately observe that

$$\left| \int_{\partial B} v_\varepsilon^2 - \int_{\partial B} \widetilde{u}_\varepsilon^2 \right| = \left| -2\, \delta \int_{\partial B} \widetilde{u}_\varepsilon + \delta^2\, \mathcal{H}^{N-1}(\partial B) \right| = \delta^2\, \mathcal{H}^{N-1}(\partial B) \leq K\varepsilon^2. \tag{6.18}$$

Now we set

$$N(\varepsilon) := \int_{B\setminus\Omega_\varepsilon} |\nabla v_\varepsilon|^2 - \int_{\Omega_\varepsilon\setminus B} |\nabla u_\varepsilon|^2,$$

so that we can write

$$\int_B |\nabla v_\varepsilon(x)|^2 = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 + N(\varepsilon) = \sigma_2(\Omega_\varepsilon) + N(\varepsilon), \tag{6.19}$$

where we used that $\nabla v_\varepsilon = \nabla u_\varepsilon$ on $B \cap \Omega_\varepsilon$. Moreover, using (6.13) and (6.18), we have

$$\int_{\partial B} v_\varepsilon(x)^2 \geq \int_{\partial B} \widetilde{u}_\varepsilon(x)^2 - K\, \varepsilon^2 = \int_{\partial\Omega_\varepsilon} u_\varepsilon(x)^2 + Q(\varepsilon) - K\varepsilon^2 = 1 + Q(\varepsilon) - K\varepsilon^2, \tag{6.20}$$

having defined

$$Q(\varepsilon) := \int_{\partial B} \widetilde{u}_\varepsilon(x)^2 - \int_{\partial\Omega_\varepsilon} u_\varepsilon(x)^2.$$

We are now able to estimate $\sigma_2(B)$: since

$$\sigma_2(B) \leq \frac{\displaystyle\int_B |\nabla v_\varepsilon(x)|^2\, dx}{\displaystyle\int_{\partial B} v_\varepsilon(x)^2\, d\mathcal{H}^{N-1}},$$

using (6.19) and (6.20), we finally obtain (6.10). $\square$

Remark 6.6. Thanks to the uniform estimates (6.9) with $k = 0$, 1 and to (6.14), it is immediate to infer

$$\big|N(\varepsilon)\big| \leq K\,\varepsilon\,, \qquad\qquad \big|Q(\varepsilon)\big| \leq K\,\varepsilon\,, \tag{6.21}$$

which inserted in (6.10) gives the easy estimate

$$\sigma_2(B) \leq \sigma_2(\Omega_\varepsilon) + K\varepsilon,$$

possibly with a different constant $K > 0$.

The previous observation shows that in order to exhibit the sharp decay rate of the deficit along the sequence Ω_ε, we need a precise control of the decay rate of the error terms N and Q. Indeed, each estimate on them automatically translates into an estimate of the same order for $\sigma_2(B) - \sigma_2(\Omega_\varepsilon)$. Let us state precisely this observation, whose proof is immediate from (6.10).

Lemma 6.7. *There exists two constants C_1 and C_2 such that*

$$|\sigma_2(B) - \sigma_2(\Omega_\varepsilon)| \leq C_1\, (|N(\varepsilon)| + |Q(\varepsilon)|) + C_2\, \varepsilon^2, \qquad \textit{for every } \varepsilon \ll 1.$$

Keeping in mind Corollary 5.2 and (6.4), we know that

$$C_3\, \varepsilon^2 \leq BW(\Omega_\varepsilon) \leq C_4\, |\sigma_2(B) - \sigma_2(\Omega_\varepsilon)| + C_5\, \varepsilon^2, \tag{6.22}$$

hence to conclude the optimality of the exponent 2 in (5.2) one would like to enforce (6.21), proving that

$$|N(\varepsilon)| + |Q(\varepsilon)| \leq K\, \varepsilon^2.$$

6.2. **Step 2: improving the decay rate.** In order to gain this improvement, the following Lemma will be of crucial importance. This guarantees that if the distance in C^1 between v_ε and the eigenspace corresponding to $\sigma_2(B)$ has a certain rate of decaying at 0, then the decays of $N(\varepsilon)$ and $Q(\varepsilon)$ are improved of the same order. *It is precisely here*, in the proof of this result, *that the Key Assumption* (6.3) *on ψ will heavily come into play.*

Lemma 6.8. *Let $\omega : [0, 1] \to \mathbb{R}^+$ be a continuous function such that $t^2/K \leq \omega(t) \leq K\sqrt{t}$. Suppose that for every $\varepsilon \ll 1$, there exists an eigenfunction ξ_ε for $\sigma_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^1(\overline{B})} \leq C\, \omega(\varepsilon), \tag{6.23}$$

for some constant C independent of ε. Then there exists a constant C_6, still independent of ε, such that

$$|N(\varepsilon)| + |Q(\varepsilon)| \leq C_6\, \omega(\varepsilon)\, \varepsilon \qquad \textit{for every } \varepsilon \ll 1.$$

Proof. We start estimating the term $|N(\varepsilon)|$: the computations are similar to that in [6], but we have to pay attention to some extra terms, which come from the fact that we are facing a Stekloff problem.

Using the uniform estimates (6.9) and recalling the definition (6.12) of ϕ_ε, we have

$$|\nabla u_\varepsilon(x)|^2 = \left|\nabla u_\varepsilon\left(\phi_\varepsilon\left(\frac{x}{|x|}\right)\right)\right|^2 + O(\varepsilon), \qquad \text{for every } x \in \Omega_\varepsilon \setminus B,$$

and observe that, splitting the gradient in its radial and tangential components, the right-hand side can be written as

$$\begin{aligned}|\nabla u_\varepsilon\left(\phi_\varepsilon(x/|x|)\right)|^2 &= |\partial_\varrho u_\varepsilon\left(\phi_\varepsilon(x/|x|)\right)|^2 + \frac{1}{(1+\varepsilon\,\psi(x/|x|))^2}\,|\nabla_\tau u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 \\ &= |\partial_\varrho u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 + |\nabla_\tau u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 + O(\varepsilon).\end{aligned}$$

Using once again (6.9), the latter in turn can be estimated as follows

$$|\partial_\varrho u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 + |\nabla_\tau u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 = \sigma_2(\Omega_\varepsilon)^2\,|u_\varepsilon(x/|x|)|^2 + |\nabla_\tau u_\varepsilon(x/|x|)|^2 + O(\varepsilon).$$

Notice that we also used that u_ε satisfies the boundary condition

$$\langle \nabla u_\varepsilon(x), \nu_{\Omega_\varepsilon}(x)\rangle = \sigma_2(\Omega_\varepsilon)\,u_\varepsilon(x), \qquad x \in \partial\Omega_\varepsilon,$$

and that the normal vector on $\partial\Omega_\varepsilon$ is radial up to an error of order ε, since we have

$$\nu_{\Omega_\varepsilon}(x) = \frac{(1+\varepsilon\,\psi(x/|x|))\,x/|x| - \varepsilon\,\nabla_\tau\psi(x/|x|)}{\sqrt{(1+\varepsilon\,\psi(x/|x|))^2 + |\nabla_\tau\psi(x/|x|)|^2}} = \frac{x}{|x|} + O(\varepsilon), \qquad x \in \partial\Omega_\varepsilon,$$

Therefore, recalling also that $|\Omega_\varepsilon \setminus B| \simeq \varepsilon$, one obtains

$$\begin{aligned}\int_{\Omega_\varepsilon\setminus B} |\nabla u_\varepsilon(x)|^2\,dx &= \varepsilon\int_{\partial B\cap\{\psi>0\}} \psi(x)\,\left[\sigma_2(\Omega_\varepsilon)^2\,u_\varepsilon(x)^2 + |\nabla_\tau u_\varepsilon(x)|^2\right]\,d\mathcal{H}^{N-1} + O(\varepsilon^2) \\ &= \varepsilon\int_{\partial B\cap\{\psi>0\}} \psi(x)\,\left[\sigma_2(B)^2\,v_\varepsilon(x)^2 + |\nabla_\tau v_\varepsilon(x)|^2\right]\,d\mathcal{H}^{N-1} + O(\varepsilon^2)\,,\end{aligned} \tag{6.24}$$

where the last equality comes from the fact that $v_\varepsilon = u_\varepsilon$ on $\Omega_\varepsilon \cap B$ up to the additive constant δ, which is of order ε thanks to (6.15), and from the fact that $|\sigma_2(B) - \sigma_2(\Omega_\varepsilon)| \le C\,\varepsilon$. In the very same way, recalling that by definition of v_ε one has

$$\nabla v_\varepsilon(\phi_\varepsilon(x)) = \nabla u_\varepsilon(\phi_\varepsilon(x)), \qquad \text{for every } x \in \partial B \setminus \Omega_\varepsilon,$$

and that the uniform estimates holds also for v_ε by (6.17), one gets

$$\int_{B\setminus\Omega_\varepsilon} \left|\nabla v_\varepsilon(x)\right|^2 dx = -\varepsilon\int_{\partial B\cap\{\psi>0\}} \psi(x)\,\left[\sigma_2(B)^2\,v_\varepsilon(x)^2 + |\nabla_\tau v_\varepsilon(x)|^2\right]\,d\mathcal{H}^{N-1} + O(\varepsilon^2)\,. \tag{6.25}$$

Finally, recalling the definition of $N(\varepsilon)$, from (6.23), (6.24) and (6.25) one obtains

$$\begin{aligned}|N(\varepsilon)| &\le \varepsilon\,\sigma_2(B)^2\left|\int_{\partial B}\psi(x)\,v_\varepsilon(x)^2\,d\mathcal{H}^{N-1}\right|\\&+\varepsilon\left|\int_{\partial B}\psi(x)\,|\nabla_\tau v_\varepsilon(x)|^2\,d\mathcal{H}^{N-1}\right|+O(\varepsilon^2)\\&=\varepsilon\,\sigma_2(B)^2\left|\int_{\partial B}\psi(x)\,\xi_\varepsilon(x)^2\,d\mathcal{H}^{N-1}\right|\\&+\varepsilon\left|\int_{\partial B}\psi(x)\,|\nabla_\tau \xi_\varepsilon(x)|^2\,d\mathcal{H}^{N-1}\right|+C'\,\varepsilon\,\omega(\varepsilon)+O(\varepsilon^2)\le \widetilde{C}\,\varepsilon\,\omega(\varepsilon),\end{aligned}$$

where in the last estimate we used property (6.3).

We now come to the estimate of $|Q(\varepsilon)|$: remember that this is given by

$$Q(\varepsilon)=\int_{\partial B}\left[\widetilde{u}_\varepsilon(x)^2-\widetilde{u}_\varepsilon(x+\varepsilon\,\psi(x)\,x)^2\,J_\varepsilon(x)\right]\,d\mathcal{H}^{N-1},$$

i.e. this error term contains a boundary integral, then estimates are a bit different from the Neumann case treated in [6].

In order to handle this term Q, for ease of computations it could be more useful to rewrite it as follows

$$Q(\varepsilon)=Q_1(\varepsilon)+Q_2(\varepsilon),$$

where we set

$$Q_1(\varepsilon):=\int_{\partial B}\left[\widetilde{u}_\varepsilon(x)^2-\widetilde{u}_\varepsilon(\phi_\varepsilon(x))^2\right]\,d\mathcal{H}^{N-1},$$

and

$$Q_2(\varepsilon):=\int_{\partial B}\widetilde{u}_\varepsilon(\phi_\varepsilon(x/|x|))^2\,\left[1-J_\varepsilon(x)\right]\,d\mathcal{H}^{N-1}.$$

Let us start with $Q_1(\varepsilon)$: by construction $\nabla\widetilde{u}_\varepsilon(x)=\nabla v_\varepsilon(x)$, then using the uniform estimates (6.9), (6.11) and the hypothesis (6.23), we have

$$\begin{aligned}|Q_1(\varepsilon)|&=\left|\int_{\partial B}\left[\widetilde{u}_\varepsilon(x)^2-\widetilde{u}_\varepsilon\left(\phi_\varepsilon(x)\right)^2\right]\,d\mathcal{H}^{N-1}\right|\\&\le 2\,\varepsilon\left|\int_{\partial B}\widetilde{u}_\varepsilon(x)\,\partial_\varrho\widetilde{u}_\varepsilon(x)\,\psi(x)\,d\mathcal{H}^{N-1}\right|+O(\varepsilon^2)\\&\le 2\,\varepsilon\left|\int_{\partial B}\xi_\varepsilon(x)\,\partial_\varrho\xi_\varepsilon(x)\,\psi(x)\,d\mathcal{H}^{N-1}\right|+\,C\,\omega(\varepsilon)\,\varepsilon\\&=2\,\varepsilon\,\sigma_2(B)\left|\int_{\partial B}\xi_\varepsilon(x)^2\,\psi(x)\,d\mathcal{H}^{N-1}\right|+\,C\,\omega(\varepsilon)\,\varepsilon,\end{aligned}$$

which yields the estimate $|Q_1(\varepsilon)|\le C\,\omega(\varepsilon)\,\varepsilon$, again thanks to property (6.3). Observe that in the last equality we have exploited the fact that ξ_ε satisfies the Stekloff boundary condition. Finally, it is left to estimate the term $Q_2(\varepsilon)$: first of all, we have

$$1-J_\varepsilon(x)=-(N-1)\,\varepsilon\,\psi(\vartheta)+O(\varepsilon^2),$$

while using the definition of v_ε, the uniform estimates (6.9) and (6.11) and the fact that $\delta = O(\varepsilon)$, we get

$$\widetilde{u}_\varepsilon(\phi_\varepsilon(x)) = \widetilde{u}_\varepsilon(x) + O(\varepsilon) = v_\varepsilon(x) + \delta + O(\varepsilon) = v_\varepsilon(x) + O(\varepsilon), \qquad x \in \partial B.$$

Inserting these into the definition of $Q_2(\varepsilon)$ and using (6.23), we finally obtain

$$\begin{aligned} |Q_2(\varepsilon)| &\le (N-1)\,\varepsilon \left| \int_{\partial B} v_\varepsilon(x)^2\, \psi(x)\, d\mathcal{H}^{N-1} \right| + O(\varepsilon^2) \\ &\le (N-1)\,\varepsilon \left| \int_{\partial B} \xi_\varepsilon(x)^2\, \psi(x)\, d\mathcal{H}^{N-1} \right| + C\,\omega(\varepsilon)\,\varepsilon, \end{aligned}$$

which concludes the proof, again thanks to property (6.3). □

Remark 6.9. Observe that if on the contrary ψ violates condition (6.3), we can not assure that all the first-order term in the previous estimates cancel out: then we would not get any improvement on N and Q. For example, for the case of the ellipsoids E_ε considered in Section 3, their boundaries can be described as follows

$$\partial E_\varepsilon = \left\{ y = \varrho_\varepsilon(x)\, x \in \mathbb{R}^2 \,:\, x \in \partial B \quad \text{and} \quad \varrho_\varepsilon(x) = \sqrt{(1+\varepsilon)\, x_1^2 + \frac{x_2^2}{1+\varepsilon}} \right\},$$

and observe that

$$\varrho_\varepsilon(x) \simeq 1 + \varepsilon\,(x_1^2 - x_2^2), \qquad x \in \partial B.$$

It is not difficult to see that $\psi(x) = x_1^2 - x_2^2$ does not satisfy (6.3): and in fact, in analogy with the Neumann case (see [6, Section 5]), one can show that

$$\sigma_2(B) - \sigma_2(E_\varepsilon) \simeq \varepsilon,$$

i.e. ellipsoids do not exhibit the sharp decay rate for the Brock-Weinstock inequality.

6.3. **Step 3: nearness estimates.** Thanks to the previous step, we know that to improve (6.21) it is sufficient to estimate the C^1 distance of v_ε from the eigenspace relative to $\sigma_2(B)$, in terms of ε: the main point is that *we can perform such an estimation, in terms of* $|N(\varepsilon)|$ *and* $|Q(\varepsilon)|$ *themselves.* This is the content of the third step.

We start with an easy $W^{1,2}(B)$ estimate, whose proof is based on a Fourier decomposition on the basis $\{\xi_k\}_{k\ge 2}$ of Stekloff eigenfunctions for B: the idea is quite the same as in [6], but an extra difficulty arises, since we can not directly decompose v_ε in $W^{1,2}$ on the basis $\{\xi_k\}_{k\ge 2}$. Rather, we have to project it on the space of harmonic functions and to control, in terms of ε, both the Dirichlet integral of this projection and the distance between v_ε and the space of harmonic functions.

Lemma 6.10. *For every $\varepsilon \ll 1$, there exists an eigenfunction ξ_ε relative to $\sigma_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} \le C\,\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon, \qquad \text{for every } \varepsilon \ll 1, \tag{6.26}$$

for some constant C independent of ε.

Proof. First of all, let us set $f_\varepsilon := \Delta v_\varepsilon = \Delta \widetilde{u}_\varepsilon$. Thanks to the fact that $\widetilde{u}_\varepsilon$ is a C^k extension of u_ε and that the latter is harmonic on $\Omega_\varepsilon \cap B$, we get that f_ε is a C^{k-2} function on B such that

$$f_\varepsilon(x) = 0, \qquad x \in \Omega_\varepsilon \cap B.$$

Moreover, on $B \setminus \Omega_\varepsilon$ we have

$$
\begin{aligned}
|f_\varepsilon(x)| &\le \left| f_\varepsilon\left(\phi_\varepsilon\left(\frac{x}{|x|}\right)\right)\right| + \|\nabla f_\varepsilon\|_{L^\infty(B)} \left|\phi_\varepsilon\left(\frac{x}{|x|}\right) - x\right| \\
&= \|\nabla f_\varepsilon\|_{L^\infty(B)} \left|\phi_\varepsilon\left(\frac{x}{|x|}\right) - x\right| \le C \left|\phi_\varepsilon\left(\frac{x}{|x|}\right) - \frac{x}{|x|}\right| \le C\,\varepsilon\,\|\psi\|_{L^\infty}, \qquad x \in B \setminus \Omega_\varepsilon,
\end{aligned} \tag{6.27}
$$

so that in conclusion $\|f_\varepsilon\|_{L^\infty(B)} \le C\,\varepsilon$. We now introduce the *harmonic projection* φ_ε of v_ε, i.e. φ_ε solves

$$
\left\{\begin{array}{rcll} \Delta\varphi_\varepsilon &=& 0, & \text{in } B,\\ \varphi_\varepsilon &=& v_\varepsilon, & \text{on } \partial B, \end{array}\right.
$$

and observe that we have

$$
\|v_\varepsilon - \varphi_\varepsilon\|_{W^{1,2}(B)} \le C\,\|f_\varepsilon\|_{L^2(B)} \le C\,\varepsilon, \tag{6.28}
$$

where we used the previous estimate on f_ε. Since φ_ε is harmonic and $v_\varepsilon - \varphi_\varepsilon \in W^{1,2}_0(B)$, we obtain

$$
\|\nabla v_\varepsilon - \nabla\varphi_\varepsilon\|^2_{L^2(B)} = \int_B |\nabla v_\varepsilon(x)|^2\,dx - \int_B |\nabla\varphi_\varepsilon(x)|^2\,dx.
$$

Keeping into account (6.28), we finally obtain

$$
\left|\int_B |\nabla v_\varepsilon(x)|^2\,dx - \int_B |\nabla\varphi_\varepsilon(x)|^2\,dx\right| \le C\,\varepsilon^2. \tag{6.29}
$$

Since $\varphi_\varepsilon \in \mathrm{Har}(B)$ – remember the definition (4.1) – we can use a spectral decomposition for it and write

$$
\varphi_\varepsilon = \sum_{k\ge 2} \alpha_k(\varepsilon)\,\xi_k, \qquad \text{where} \qquad \alpha_k(\varepsilon) = \int_{\partial B} \varphi_\varepsilon(x)\,\xi_k(x)\,d\mathcal{H}^{N-1},\ k \ge 2,
$$

then

$$
\|\varphi_\varepsilon\|^2_{L^2(\partial B)} = \sum_{k\ge 2} \alpha_k(\varepsilon)^2 \qquad \text{and} \qquad \|\nabla\varphi_\varepsilon\|^2_{L^2(B)} = \sum_{k\ge 2} \sigma_k(B)\,\alpha_k(\varepsilon)^2,
$$

where for the second decomposition we used (4.2). By (6.18) and the definition of $Q(\varepsilon)$, we have

$$
\begin{aligned}
\left|\int_{\partial B} v_\varepsilon(x)^2 - 1\right| &\le \left|\int_{\partial B} \widetilde{u}_\varepsilon(x)^2 - \int_{\partial\Omega_\varepsilon} u_\varepsilon(x)^2\right| + \left|\int_{\partial B} v_\varepsilon(x)^2 - \int_{\partial B} \widetilde{u}_\varepsilon(x)^2\right| \\
&\le |Q(\varepsilon)| + K\,\varepsilon^2,
\end{aligned}
$$

and since $\varphi_\varepsilon = v_\varepsilon$ on ∂B, the previous implies

$$
\left|\|\varphi_\varepsilon\|^2_{L^2(\partial B)} - 1\right| \le |Q(\varepsilon)| + K\,\varepsilon^2.
$$

In particular, we get

$$
\left|\sum_{k=2}^{N+1} \alpha_k(\varepsilon)^2 - 1\right| \le \sum_{k\ge N+2} \alpha_k(\varepsilon)^2 + |Q(\varepsilon)| + K\varepsilon^2,
$$

and multiplying both members by $\sigma_2(B)$ we have

$$
\sigma_2(B)\left|\sum_{k=2}^{N+1} \alpha_k(\varepsilon)^2 - 1\right| \le \sigma_2(B) \sum_{k\ge N+2} \alpha_k(\varepsilon)^2 + c_1\,|Q(\varepsilon)| + K\varepsilon^2. \tag{6.30}
$$

On the other hand, by (6.19) and (6.29) we have

$$\begin{aligned}\left|\|\nabla\varphi_\varepsilon\|^2_{L^2(B)}-\sigma_2(B)\right| &\le \left|\|\nabla v_\varepsilon\|^2_{L^2(B)}-\sigma_2(B)\right|+\left|\|\nabla v_\varepsilon\|^2_{L^2(B)}-\|\nabla\varphi_\varepsilon\|^2_{L^2(B)}\right|\\ &\le |\sigma_2(\Omega_\varepsilon)-\sigma_2(B)|+|N(\varepsilon)|+C\,\varepsilon^2\\ &\le C\,\left(|N(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,\end{aligned}$$

which can be rewritten as

$$\left|\sigma_2(B)\left(\sum_{k=2}^{N+1}\alpha_k(\varepsilon)^2-1\right)+\sum_{k\ge N+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\right|\le c_2\,\left(|N(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,$$

and this implies

$$\sum_{k\ge N+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\le c_2\,\left(|N(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2+\sigma_2(B)\left|\sum_{k=2}^{N+1}\alpha_k(\varepsilon)^2-1\right|. \tag{6.31}$$

We can now combine (6.30) and (6.31), so to obtain

$$\sum_{k\ge N+2}\left(\sigma_k(B)-\sigma_2(B)\right)\alpha_k(\varepsilon)^2\le (c_1+c_2)\,\left(|N(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2.$$

Notice that

$$1-\frac{\sigma_2(B)}{\sigma_k(B)}>0,\qquad k\ge N+2,$$

since $\sigma_2(B)$ has multiplicty N and this forms a nondecreasing sequence, then from the previous we can infer

$$\sum_{k\ge N+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\le C\,\left(|N(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,$$

possibly with different constants C and K, depending on the spectral gap $\sigma_{N+2}(B)-\sigma_2(B)$, but not on ε. If we set

$$\xi_\varepsilon=\sum_{k=2}^{N+1}\alpha_k(\varepsilon)\xi_k,$$

we have

$$\|\varphi_\varepsilon-\xi_\varepsilon\|^2_{L^2(\partial B)}\le\sigma_{N+2}(B)\,\|\nabla v_\varepsilon-\nabla\xi_\varepsilon\|^2_{L^2(B)}$$

and

$$\|\nabla\varphi_\varepsilon-\nabla\xi_\varepsilon\|^2_{L^2(B)}=\sum_{k\ge N+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\le C\,\left(|N(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,$$

which yields

$$\|\varphi_\varepsilon-\xi_\varepsilon\|_{W^{1,2}(B)}\le C\,\sqrt{|N(\varepsilon)|+|Q(\varepsilon)|}+K\,\varepsilon,$$

thanks to the fact that $u\mapsto\|u\|_{L^2(\partial B)}+\|\nabla u\|_{L^2(B)}$ is equivalent to the standard norm of $W^{1,2}(B)$. Finally, it is only left to observe that

$$\|v_\varepsilon-\xi_\varepsilon\|_{W^{1,2}(B)}\le\|\varphi_\varepsilon-\xi_\varepsilon\|_{W^{1,2}(B)}+\|v_\varepsilon-\varphi_\varepsilon\|_{W^{1,2}(B)},$$

thus we have obtained (6.26). □

We show how the previous Sobolev estimate (6.26) can be enhanced, replacing the $W^{1,2}(B)$ norm with the C^1 one.

Lemma 6.11. *For every $\varepsilon \ll 1$, there exists an eigenfunction ξ_ε relative to $\sigma_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^1(\overline{B})} \le C_7 \sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + C_8\, \varepsilon, \qquad \text{for every } \varepsilon \ll 1, \tag{6.32}$$

for some positive constants C_7, C_8 independent of ε.

Proof. First of all, let us write down the the Neumann boundary value problems solved by v_ε and ξ_ε: these are given respectively by

$$\left\{\begin{array}{rcll} \Delta v_\varepsilon &=& f_\varepsilon, & \text{in } \partial B\\ \langle \nabla v_\varepsilon, \nu\rangle &=& \sigma_2(B)\, g_\varepsilon, & \text{on } \partial B\end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{rcll} \Delta \xi_\varepsilon &=& 0, & \text{in } \partial B\\ \langle \nabla \xi_\varepsilon, \nu\rangle &=& \sigma_2(B)\, \xi_\varepsilon, & \text{on } \partial B\end{array}\right.$$

where

$$f_\varepsilon(x) = \Delta \widetilde{u}_\varepsilon(x), \qquad x \in B,$$

and the boundary value g_ε is given by (recall that $\nabla v_\varepsilon = \nabla \widetilde{u}_\varepsilon$)

$$\begin{aligned} g_\varepsilon(x) &= v_\varepsilon(x) + \big[u_\varepsilon(\phi_\varepsilon(x)) - v_\varepsilon(x)\big]\\ &\quad + \left(\frac{\sigma_2(\Omega_\varepsilon)}{\sigma_2(B)} - 1\right) u_\varepsilon(\phi_\varepsilon(x))\\ &\quad + \frac{1}{\sigma_2(B)} \langle \nabla \widetilde{u}_\varepsilon(x) - \nabla u_\varepsilon(\phi_\varepsilon(x)), \nu_{\Omega_\varepsilon}(\phi_\varepsilon(x))\rangle\\ &\quad + \frac{1}{\sigma_2(B)} \langle \nabla \widetilde{u}_\varepsilon(x), \nu(x) - \nu_{\Omega_\varepsilon}(\phi_\varepsilon(x))\rangle =: v_\varepsilon(x) + \sum_{i=1}^4 g_{\varepsilon,i}(x) \qquad x \in \partial B. \end{aligned}$$

Thus in order to gain informations on the distance between v_ε and ξ_ε, it suffices to estimate f_ε and the boundary term $g_\varepsilon - \xi_\varepsilon$: indeed, by standard Elliptic Regularity (see [23, Proposition 7.5]) and by the triangular inequality, for every $k \ge 1$ we have

$$\begin{aligned} \|v_\varepsilon - \xi_\varepsilon\|_{W^{k,2}(B)} &\le C\left(\|v_\varepsilon - \xi_\varepsilon\|_{L^2(B)} + \|g_\varepsilon - \xi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} + \|f_\varepsilon\|_{W^{k-2,2}(B)}\right)\\ &\le C\Bigg(\|v_\varepsilon - \xi_\varepsilon\|_{L^2(B)} + \|v_\varepsilon - \xi_\varepsilon\|_{W^{k-3/2,2}(\partial B)}\\ &\quad + \sum_{i=1}^4 \|g_{\varepsilon,i}\|_{W^{k-3/2,2}(\partial B)} + \|f_\varepsilon\|_{W^{k-2,2}(B)}\Bigg). \end{aligned} \tag{6.33}$$

The first term on the right-hand side can be easily estimated as follows

$$\|v_\varepsilon - \xi_\varepsilon\|_{L^2(B)} \le \|v_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} \le C \sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + K\, \varepsilon,$$

where we used (6.26) in the second inequality: then to obtain (6.32) it suffices to prove that

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} \le C \sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + K\, \varepsilon, \tag{6.34}$$

$$\sum_{i=1}^4 \|g_{\varepsilon,i}\|_{W^{k-3/2,2}(\partial B)} \le C\, \varepsilon, \tag{6.35}$$

$$\|f_\varepsilon\|_{W^{k-2,2}(B)} \le C\, \varepsilon, \tag{6.36}$$

with $k = [N/2] + 2$. Indeed, using the Sobolev Imbedding Theorem, this would yield

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^1(\overline{B})} \le C\, \|v_\varepsilon - \xi_\varepsilon\|_{W^{[N/2]+2,2}(B)},$$

and combining (6.33) and (6.34)–(6.36), we would conclude the proof.

We now begin to estimate the terms $g_{\varepsilon,i}$: recalling that $u_\varepsilon\circ\phi_\varepsilon=\widetilde{u}_\varepsilon\circ\phi_\varepsilon$ on ∂B and using (6.13) and the uniform estimates on $\widetilde{u}_\varepsilon$, we get that

$$\|g_{\varepsilon,1}\|_{W^{k-3/2,2}(\partial B)}\le\|\widetilde{u}_\varepsilon\circ\phi_\varepsilon-\widetilde{u}_\varepsilon\|_{W^{k-3/2,2}(\partial B)}+\delta\,\left(\mathcal{H}^{N-1}(\partial B)\right)^{1/2}=O(\varepsilon).$$

For the second, we use (6.9) and Lemma 6.7, to obtain

$$\|g_{\varepsilon,2}\|_{W^{k-3/2,2}(\partial B)}\le K\,\frac{|\sigma_2(\Omega_\varepsilon)-\sigma_2(B)|}{\sigma_2(B)}\le K\,\left(|N(\varepsilon)|+|Q(\varepsilon)|\right),$$

possibly with a different constant K, still not depending on ε. For the the third term, we just use a triangular inequality and the uniform estimates (6.9), (6.11)

$$\begin{aligned}\|g_{\varepsilon,3}\|_{W^{k-3/2,2}(\partial B)}&\le C\,\|\nabla\widetilde{u}_\varepsilon-\nabla u_\varepsilon\circ\phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)}\\&\le C\,\|\nabla\widetilde{u}_\varepsilon-\nabla(u_\varepsilon\circ\phi_\varepsilon)\|_{W^{k-3/2,2}(\partial B)}\\&\quad+C\,\|\nabla(u_\varepsilon\circ\phi_\varepsilon)-\nabla u_\varepsilon\circ\phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)}\le C\,\varepsilon,\end{aligned}$$

again thanks to the fact that $\widetilde{u}_\varepsilon\circ\phi_\varepsilon=u_\varepsilon\circ\phi_\varepsilon$ on ∂B. Finally, still using the uniform estimates (6.11) and (6.9), we have

$$\|g_{\varepsilon,4}\|_{W^{k-3/2,2}(\partial B)}\le C\,\|\nu_B-\nu_{\Omega_\varepsilon}\circ\phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)}.$$

The term $\nu_{\Omega_\varepsilon}\circ\phi_\varepsilon$ can be explicitly written as

$$\nu_{\Omega_\varepsilon}(\phi_\varepsilon(x))=\frac{(1+\varepsilon\,\psi(x))\,\nu_B(x)-\varepsilon\,\nabla_\tau\psi(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2+\varepsilon^2\,|\nabla_\tau\psi(x)|^2}},\qquad x\in\partial B,$$

In this way

$$\begin{aligned}\nu_B(x)-\nu_{\Omega_\varepsilon}(\phi_\varepsilon(x))&=\nu_B(x)\left(1-\frac{1+\varepsilon\,\psi(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2+\varepsilon^2\,|\nabla_\tau\psi(x)|^2}}\right)\\&\quad-\varepsilon\,\frac{\nabla_\tau\psi(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2+\varepsilon^2\,|\nabla_\tau\psi(x)|^2}}.\end{aligned}$$

Then observe that

$$\varphi_1(x)=1-\frac{1+\varepsilon\,\psi(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2+\varepsilon^2\,|\nabla_\tau\psi(x)|^2}},\qquad x\in\partial B,$$

and

$$\varphi_2(x)=\varepsilon\,\frac{\nabla_\tau\psi(x)(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2+\varepsilon^2\,|\nabla_\tau\psi(x)|^2}},\qquad x\in\partial B,$$

are two C^∞ applications on ∂B, such that for every $m\in\mathbb{N}$

$$\|\varphi_i\|_{C^m(\partial B)}\le C_m\,\varepsilon,\qquad i=1,2,$$

where C_m is a constant depending on the $C^{m+1}(\partial B)$ norm of ψ, but not on ε. This permits to conclude the estimate on $g_{\varepsilon,4}$: we finally have

$$\|g_{\varepsilon,4}\|_{W^{k-3/2,2}(\partial B)}\le C\,\|\nu_B-\nu_{\Omega_\varepsilon}\circ\phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)}\le C\,\varepsilon,$$

so collecting all these estimates we end up with (6.35), for any k.

Concerning the term f_ε, we have already seen that $\|f_\varepsilon\|_{L^\infty(B)} \le C\,\varepsilon$: repeating the argument (6.27) for every derivative and using that the $C^{[N/2]+1}$ norm of f_ε is uniformly bounded[3], we obtain

$$\|f_\varepsilon\|_{C^{k-2}(\overline{B})} \le C\,\varepsilon,$$

for $k = [N/2] + 2$, so that the $W^{k-2,2}$ norm is estimated as follows

$$\|f_\varepsilon\|_{W^{k-2,2}(B)} \le C\,\|f_\varepsilon\|_{C^{k-2}(\overline{B})} \le C\,\varepsilon.$$

Finally, we aim to prove (6.34): by the trace inequality and (6.26) we have

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{1/2,2}(\partial B)} \le C\,\|v_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} \le C\,\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon.$$

A first application of (6.33) with $k = 2$, gives

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{2,2}(\Omega)} \le C\,\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon,$$

and applying the trace inequality we obtain

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{3/2,2}(\partial B)} \le C\,\|v_\varepsilon - \xi_\varepsilon\|_{W^{2,2}(\Omega)} \le C\,\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon,$$

thus the validity of (6.34) with $k = 3$. Finitely many repetitions of the previous argument give (6.34) with $k = [N/2] + 2$ and thus the proof is concluded. □

6.4. **Step 4: conclusion.** Thanks to Lemma 6.7, we know that

$$|\sigma_2(B) - \sigma_2(\Omega_\varepsilon)| \le C_1\,(|N(\varepsilon)| + |Q(\varepsilon)|) + C_2\,\varepsilon^2.$$

First applying Lemma 6.11 and then Lemma 6.8 with $\omega(\varepsilon) = C_7\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + C_8\,\varepsilon$, we obtain

$$|N(\varepsilon)| + |Q(\varepsilon)| \le \widetilde{C}\,\varepsilon\,\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} + \widetilde{C}\,\varepsilon^2. \tag{6.37}$$

Let us set

$$t(\varepsilon) = \frac{\varepsilon}{\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|}},$$

then from (6.37) we can infer

$$\frac{1}{\widetilde{C}} \le t(\varepsilon) + t(\varepsilon)^2,$$

which easily implies that $t(\varepsilon) \ge c$ for some costant $c > 0$, i.e.

$$\sqrt{|N(\varepsilon)| + |Q(\varepsilon)|} \le \frac{\varepsilon}{c}.$$

A further application of Lemma 6.7 finally shows that

$$|\sigma_2(B) - \sigma_2(\Omega_\varepsilon)| \le C\,\varepsilon^2,$$

possibly with a different constant C, still independent of ε. Inserting this into (6.22), we can conclude the proof of Theorem 6.1.

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[3] This is the reason why we choose $\widetilde{u}_\varepsilon$ to be a C^{k+1} extension of u_ε with $k = [N/2] + 2$.

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Laboratoire d'Analyse, Topologie, Probabilités UMR6632, Aix-Marseille Université, CMI 39, Rue Frédéric Joliot Curie, 13453 Marseille Cedex 13, France
E-mail address: brasco@cmi.univ-mrs.fr

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail address: guido.dephilippis@sns.it

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail address: berardo.ruffini@sns.it