GINZBURG-LANDAU FUNCTIONALS AND RENORMALIZED ENERGY: A REVISED Γ-CONVERGENCE APPROACH

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Abstract. We give short and self-contained proofs of Γ-convergence results for Ginzburg-Landau energy functionals in dimension two, in the logarithmic energetic regime. In particular, we derive the renormalized energy by Γ-convergence.

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1. Introduction

The analysis of Ginzburg-Landau functionals as the length-scale parameter tends to zero is a beautiful piece of mathematical analysis. Different ideas and techniques merged together along the last twenty years, to give a clear picture of the relevant phenomena, as concentration of energy and formation of topological singularities.

To make a long story very short, the analysis started with the study of the asymptotic behaviour of the minimizers of Ginzburg-Landau functionals in dimension two, with prescribed boundary conditions. Let $\Omega \subset \mathbb{R}^2$ be bounded and with Lipschitz boundary. The Ginzburg-Landau functionals $GL_\varepsilon : H^1(\Omega;\mathbb{R}^2) \to [0,+\infty)$, are defined as

$$GL_\varepsilon(u) := \left(\int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2} W(u) \right), \tag{1.1}$$

where $W \in C^0(\mathbb{R}^2)$ is such that $W(x) \geq 0$, $W^{-1}\{0\} = S^1$ and

$$\liminf_{|x|\to 1} \frac{W(x)}{(1-|x|)^2} > 0, \quad \liminf_{|x|\to\infty} \frac{W(x)}{|x|^2} > 0.$$

In [2] are collected the main results for the asymptotic behavior of the minimizers u_ε of GL_ε with a prescribed boundary datum $g : \partial\Omega \mapsto S^1$ with degree d. For ε small, the energy $GL_\varepsilon(u_\varepsilon)$ blows up as $d|\log\varepsilon|$, and d vortex-like singularities appear, around which u_ε looks like (a fixed rotation of) $x/|x|$. Subtracting the leading term $d|\log\varepsilon|$ from the energy, a finite quantity remains in the limit, called *renormalized energy*, depending on the position of the singularities.

After these results, much work has been devoted to understand the behavior of sequences of non minimizers, with prescribed energetic regime, in the spirit of Γ-convergence. The main issues are clearly the (zero order) Γ-convergence of $\frac{GL_\varepsilon}{|\log\varepsilon|}$ and the (first order) Γ-convergence of $GL_\varepsilon - d|\log\varepsilon|$ to the renormalized energy. The picture is nowadays well understood. For the zero order Γ-convergence [8] and [6] provide sharp lower bounds, while in [6] and [7] a Γ-convergence result is proved in $W^{1,1}(\Omega;R^2)$, and compactness of the singularities is expressed in terms of compactness properties of the Jacobians Ju_ε of u_ε in the dual of Hölder functions; in [1] the Γ-convergence result is obtained (in any dimension and codimension) with respect to the flat convergence of the Jacobians. To our knowledge a self contained proof of the first order Γ-convergence result is still missing.

The aim of this paper is to revisit these Γ-convergence results, giving short, efficient and self-contained proofs. Self-contained has to be understood in a very weak sense: we use many ideas from [1], [8], [6], [7] and (for the first order Γ-limit) the analysis for minimizers developed in [2]. Our approach is the following: we consider the *ball construction* as done in [8]: it consists in an efficient way of selecting balls where the energy concentrates. Then we plug a Dirac mass in each ball, obtaining a sequence of measures μ_ε that approximates Ju_ε, carrying all the topological information and bringing compactness and Γ-liminf inequality in the zero order Γ-convergence result. The Γ-limsup inequality is obtained by a standard construction. To prove the first order Γ-convergence result we show that, if $GL_\varepsilon(u_\varepsilon)/|\log\varepsilon|$ is bounded, then around each singularity x_i we have $GL_\varepsilon(u_\varepsilon) - \pi z_i|\log\varepsilon| \geq C$, where $z_i \in \mathbb{Z}$ is the degree of the singularity. Moreover, if u_ε are optimal in energy, then $|z_i| = 1$ and around each singularity u_ε looks like (a rotation of) $\frac{x}{|x|}$. These preliminary results allow to use the analysis done in [2] to derive the renormalized energy.

Let us conclude recalling that the case treated in this paper has been the building block for a series of important generalizations, as for the case of external magnetic field [9], the three dimensional case [1] and the study of the evolution of vortices [10]. The final goal of this paper is to rewrite this building block within the solid formalism of Γ-convergence, with efficient and self-contained short proofs.

2. Preliminary results

In this section we introduce the notion of Jacobian, current and degree. Given $u \in H^1(\Omega;\mathbb{R}^2)$, the Jacobian Ju of u is the L^1 function defined by

$$Ju := \det\nabla u.$$

Let us denote by $C^{0,1}(\Omega)$ the space of Lipschitz continuous functions on Ω endowed with the norm

$$\|\varphi\|_{C^{0,1}} := \sup_{x\in\Omega}|f(x)| + \sup_{x,\, y\in\Omega,\, x\neq y}\frac{\varphi(x)-\varphi(y)}{x-y},$$

and by $C^{0,1}_c(\Omega)$ its subspace of functions with compact support. The norm in the dual of $C^{0,1}_c(\Omega)$ will be denoted by $||\cdot||_{flat}$ and referred to as *flat norm*, while $\stackrel{flat}{\to}$ denotes the convergence in the flat norm. Finally, the norm in the dual of $C^{0,1}(\Omega)$ will be denoted by $||\cdot||_{flat(\overline{\Omega})}$ and by $\stackrel{flat(\overline{\Omega})}{\to}$ the corresponding convergence.

For every $u \in H^1(\Omega;\mathbb{R}^2)$, we can consider Ju as an element of the dual of $C^{0,1}(\Omega)$ by setting

$$< Ju, \varphi > := \int_\Omega Ju\,\varphi\,dx \qquad \text{for every } \varphi \in C^{0,1}(\Omega).$$

Notice that Ju can be written in a divergence form as $Ju = \text{div}\,(u_1(u_2)_{x_2}, -u_1(u_2)_{x_1})$, i.e., for every $\varphi \in C^{0,1}_c(\Omega)$,

$$< Ju, \varphi > = -\int_\Omega u_1(u_2)_{x_2}\varphi_{x_1} - u_1(u_2)_{x_1}\varphi_{x_2}\,dx. \tag{2.1}$$

Equivalently, we have $Ju = \text{curl}\,(u_1\nabla u_2)$ and $Ju = \frac{1}{2}\text{curl}\,j(u)$, where

$$j(u) := \frac{1}{2}(u_1\nabla u_2 + u_2\nabla u_1)$$

is the so called *current*.

Notice that if $u \in L^\infty(\Omega;\mathbb{R}^2)$, then Ju is in the dual of $H^1(\Omega)$. Let $A \subset \Omega$ open with Lipschitz boundary. Then we have

$$\int_A Ju = \frac{1}{2}\int_A \text{curl}\,j(u) := \frac{1}{2}\int_{\partial A} j(u)\cdot\tau, \tag{2.2}$$

where τ is the tangent field to ∂A, and the last integral is meant in the sense of $H^{-\frac{1}{2}}$.

Let $h \in H^{\frac{1}{2}}(\partial A;\mathbb{R}^2)$ with $|h| \geq \alpha > 0$. The degree of h is defined as follows

$$\deg(h,\partial A) := \frac{1}{2\pi}\int_{\partial A} j(h/|h|)\cdot\tau. \tag{2.3}$$

In [4], [5] it is proved that the definition is well posed, that $\deg(h,\partial A) \in \mathbb{Z}$ and, whenever $u \in H^1(A;\mathbb{R}^2)$, $|u| \geq \beta > 0$ in A, $\deg(u,\partial A) = 0$, where in the notation of degree we identify u with its trace. Finally, $\deg(u,\partial A)$ is stable with respect to the strong convergence in $H^1(A;\mathbb{R}^2)$. Notice that u can be written in polar coordinates as $u(x) = \rho(x)e^{i\theta(x)}$ on ∂A with $|\rho| \geq \alpha$, where θ is the so called *lifting* of u. By [3, Theorem 1] (see also [3, Remark 3]), if A is simply connected and $\deg(u, \partial A) = 0$, then the lifting can be selected in $H^{\frac{1}{2}}(\partial A)$ with the map $u \mapsto \theta$ continuous (but the image of a bounded subset of $H^{\frac{1}{2}}(\partial A; S^1)$ is not necessarily bounded in $H^{\frac{1}{2}}(\partial A)$). If the degree d is not zero, then the lifting can be locally selected in $H^{\frac{1}{2}}(\partial A)$ with a "jump" of order $2\pi d$.

Now we introduce some notion of modified Jacobian we will use in our Γ-convergence results. The main observation is that, for every $v := (v_1, v_2)$, $w := (w_1, w_2)$ belonging to $H^1(\Omega; \mathbb{R}^2)$ we have

$$Jv - Jw = \frac{1}{2}\big(J(v_1 - w_1, v_2 + w_2) - J(v_2 - w_2, v_1 + w_1)\big). \tag{2.4}$$

By (2.1) and (2.4) we immediately deduce the following lemma.

Lemma 2.1. *Let v_n and w_n be two sequences in $H^1(\Omega; \mathbb{R}^2)$ such that*

$$\|v_n - w_n\|_2(\|\nabla v_n\|_2 + \|\nabla w_n\|_2) \to 0.$$

Then $\|Jv_n - Jw_n\|_{flat} \to 0$.

Given $0 < \tau < 1$ and $u \in H^1(\Omega; \mathbb{R}^2)$, set

$$u_\tau := T_\tau(|u|)\frac{u}{|u|}, \qquad J_\tau u := Ju_\tau, \qquad \text{where } T_\tau(\rho) = \min\{\frac{\rho}{\tau}, 1\}. \tag{2.5}$$

By Lemma 2.1 we easily deduce the following proposition.

Proposition 2.2. *Let u_ε be a sequence in $H^1(\Omega; \mathbb{R}^2)$ such that $GL_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$, and let $0 < s_\varepsilon < 1$ be such that $\frac{s_\varepsilon}{\varepsilon|\log\varepsilon|} \to \infty$ as $\varepsilon \to 0$. Then*

$$\sup_{s_\varepsilon \le s \le 1} \|Ju_\varepsilon - J_s u_\varepsilon\|_{flat} \to 0 \qquad \text{as } \varepsilon \to 0.$$

3. Ball construction

In this section we revisit the celebrated *ball construction*, a useful machinery for providing lower bounds, following the approach by Sandier [8].

Let $\mathcal{B} = \{B_{r_1}(x_1), \dots, B_{r_m}(x_m)\}$ be a finite family of disjoint balls in $\mathbb{R}^2$, and set $\mathcal{R}ad(\mathcal{B}) := \sum r_i$. The ball construction consists in letting the balls alternatively expand and merge each other. The expansion phase consists in letting all the balls expand in such a way that at each (artificial) time the ratio $\theta(t) := r_i(t)/r_i$ is independent of i; we will parametrize the time enforcing $\theta(t) = 1 + t$. The expansion phase stops at the first time T when two balls $B_{r_i(t)}(x_i)$, $B_{r_j(t)}(x_j)$ touch each other. Then the merging phase begins. It consists in collecting the balls $B_{r_i(T)}(x_i)$ in subclasses and merging all the balls of a subclass in a larger ball $B_{R_j}(y_j)$ with the following properties.

i) R_j is not larger than the sum of all the radii of the balls $B_{r_i(T)}(x_i)$ contained in $B_{R_j}(y_j)$.
ii) The balls $B_{R_j}(y_j)$ of the new family are disjoint.

After the merging, we define in each ball $B_{R_j}(y_j)$ a seed size s_j by $R_j/s_j = \theta(T) = 1 + T$ (we set $s_j = r_j$ for $t = 0$). Then another expansion phase begins, during which we keep the seed sizes constant, and we now enforce $\theta(t) := R_j(t)/s_j = 1 + t$, where $t \ge T$. We proceed so forth alternating merging to expansion phases, until a last phase where only one ball expands. Notice that by construction, in particular by property i), $\sum s_j$ does not increase during the merging. In particular, we always have

$$\sum R_j(t) = \sum (1 + t)s_j \le (1 + t)\mathcal{R}ad(\mathcal{B}). \tag{3.1}$$

Now assume that to each ball $B_{r_i}(x_i)$ of the original family $\mathcal{B}$ corresponds some integer multiplicity $z_i \in \mathbb{Z}$, and set $\mu := \sum_i z_i \delta_{x_i}$. Let $F(\mathcal{B}, \mu, U)$ be defined as follows: if $A_{r,R}(x) := B_R(x) \setminus B_r(x)$ is an annulus that does not intersect any $B_{r_i}(x_i)$, we set

$$F(\mathcal{B}, \mu, A_{r,R}(x)) := \pi|\mu(B_r(x))| \log\left(\frac{R}{r}\right).$$

Then, for every open set $U \subset \mathbb{R}^2$ we set

$$F(\mathcal{B}, \mu, U) := \sup \sum_i F(A_i),$$

where the sup is over all finite families of disjoint annuli $A_i \subset U$ that do not intersect any $B_{r_i}(x_i)$.

Remark 3.1. The definition of F is justified by the following observation. Let $\tilde{\Omega} = \Omega \setminus \cup_{B\in\mathcal{B}} B$. Given $u \in H^1(\tilde{\Omega}; S^1)$, let $\mu := \sum_{B\in\mathcal{C}} \deg(u, \partial B)\delta_x$, where $\mathcal{C}$ denotes the family of balls in $\mathcal{B}$ that are contained in Ω, and x is the center of B. Then, by Jensen inequality we easily deduce that

$$F(\mathcal{B}, \mu, U) \le \frac{1}{2}\int_{U\cap\tilde{\Omega}} |\nabla u|^2,$$

for every open set $U \subset \Omega$ (see for instance [8]).

Set $\mathcal{B}(t)$ the family of balls at time t (with the convention $\mathcal{B}(t) = \mathcal{B}(t^-)$ if t is a merging time). By our definitions it easily follows (see [8] for more details) that for any $B \in \mathcal{B}(t)$

$$F(\mathcal{B}, \mu, B) \ge \pi|\mu(B)| \log(1 + t). \tag{3.2}$$

We introduce the modified measure $\tilde{\mu}$ as follows: let $\mathcal{C}(1)$ be the family of balls in $\mathcal{B}(1)$ that are contained in Ω. We set

$$\tilde{\mu} := \sum_{B_r(x)\in\mathcal{C}(1)} \mu(B_r(x))\delta_x. \tag{3.3}$$

Theorem 3.2. *Let $\mathcal{B}_n = \{B_{r_{i,n}}(x_{i,n})\}$ be a sequence of finite families of disjoint balls in $\mathbb{R}^2$ with $\mathcal{R}ad(\mathcal{B}_n) \to 0$, and let $\mu_n := \sum_i z_{i,n}\delta_{x_{i,n}}$, $z_{i,n} \in \mathbb{Z}$, be a given multiplicity measure. Assume*

$$F(\mathcal{B}_n, \mu_n, \Omega) \le C|\log \mathcal{R}ad(\mathcal{B}_n)|. \tag{3.4}$$

Then the following holds.

1) *Up to a subsequence $\tilde{\mu}_n \stackrel{flat}{\to} \mu$ for some $\mu = \sum_{i=1}^N z_i\delta_{x_i}$ with $z_i \in \mathbb{Z}$, $x_i \in \Omega$.*
2) *Assume $\tilde{\mu}_n \stackrel{flat}{\to} \mu = \sum_{i=1}^N z_i\delta_{x_i}$. For every i and every $\sigma \le \frac{1}{2} dist(x_i, \partial\Omega \cup_{j\neq i} x_j)$*

$$\liminf_n F(\mathcal{B}_n, \mu_n, B_\sigma(x_i)) - \pi|z_i| \log \frac{\sigma}{2\mathcal{R}ad(\mathcal{B}_n)} \ge 0.$$

Proof. We let the families $\mathcal{B}_n$ grow and merge as described above, and set $\mathcal{B}_n(t)$ the corresponding family of balls at time t. Set $\mathcal{C}_n(t) := \{B \in \mathcal{B}_n(t), B \subset \Omega\}$, $\mathcal{D}_n(t) := \{B \in \mathcal{B}_n(t), B \cap \partial\Omega \neq \emptyset\}$. Set moreover

$$t_n := \frac{1}{\sqrt{\mathcal{R}ad(\mathcal{B}_n)}} - 1, \qquad \nu_n := \sum_{B_r(x)\in\mathcal{C}_n(t_n)} \mu_n(B_r(x))\delta_x.$$

By the energy bound (3.4) and by (3.2) we have $|\tilde{\mu}_n|(\Omega) \le C|\log \mathcal{R}ad(\mathcal{B}_n)|$, $|\nu_n|(\Omega) \le C$, and hence up to a subsequence $\nu_n \stackrel{flat}{\to} \mu$ for some $\mu = \sum_{i=1}^N z_i\delta_{x_i}$. To conclude the proof of 1) it is enough to prove that $\nu_n - \tilde{\mu}_n \stackrel{flat}{\to} 0$. By construction, $(\nu_n - \tilde{\mu}_n)(B) = 0$ for every $B \in \mathcal{C}_n(t_n)$. We conclude that for every $\varphi_n \in C_c^{0,1}(\Omega)$ with $\|\varphi_n\|_{C_c^{0,1}} \le 1$ we have

$$\begin{aligned} < \nu_n - \tilde{\mu}_n, \varphi_n >= &\sum_{B\in\mathcal{C}_n(t_n)} \int_B \varphi_n \, d(\nu_n - \tilde{\mu}_n) + \sum_{B\in\mathcal{D}_n(t_n)} \int_B \varphi_n \, d(\nu_n - \tilde{\mu}_n) \le \\ &\sum_{B\in\mathcal{C}_n(t_n)} (\max_B \varphi_n - \min_B \varphi_n)\,(|\nu_n| + |\tilde{\mu}_n|)(B) + \sum_{B\in\mathcal{D}_n(t_n)} \max_B |\varphi_n|(|\nu_n| + |\tilde{\mu}_n|)(B) \le \\ &2\sum_{B\in\mathcal{B}_n(t_n)} \mathrm{diam}(B)(|\nu_n| + |\tilde{\mu}_n|)(B) \le C\sqrt{\mathcal{R}ad(\mathcal{B}_n)}|\log \mathcal{R}ad(\mathcal{B}_n)| \to 0 \qquad \text{as } n \to \infty, \end{aligned} \tag{3.5}$$

where in the last inequality we have used (see (3.1)) $\mathcal{R}ad(\mathcal{B}_n(t_n)) \le \sqrt{\mathcal{R}ad(\mathcal{B}_n)}$.

We pass to the proof of 2). Let $0 < \delta < \sigma$ be fixed. For n large enough we have that all the balls $B \in \mathcal{B}_n(t_n)$ with $\mu_n(B) \neq 0$ either have a distance from x_i smaller then δ, or larger than $2\sigma - \delta$. We call the family of balls enjoying the first property $\mathcal{G}_n$, and $\mathcal{H}_n$ the others. Notice that, for n

large enough, $\sum_{B\in\mathcal{G}_n}\mu_n(B)=z_i$. Let $\bar t_n=\frac{\sigma-\delta}{2\mathcal{R}ad(\mathcal{B}_n)}-1$. Let I_n be the family of balls $B\in\mathcal{B}_n(\bar t_n)$ which are contained in $B_\sigma(x_i)$. By (3.1) we have $\mathrm{diam}(B)\le\sigma-\delta$ for every $B\in\mathcal{B}_n(\bar t_n)$, and hence the balls in I_n contain all the balls in $\mathcal{G}_n$ (and none in $\mathcal{H}_n$). Recalling (3.2) we deduce that

$$F(\mathcal{B}_n,\mu_n,B_\sigma(x_i))\ge\sum_{B\in I_n}\pi|\mu_n(B)||\log(1+\bar t_n)|\ge\pi|z_i||\log(1+\bar t_n)|=\pi|z_i|\log\frac{\sigma-\delta}{2\mathcal{R}ad(\mathcal{B}_n)}.$$

We conclude by letting $n\to+\infty$ and $\delta\to0$. □

4. Γ-convergence of GL_ε

In this section we prove the zero order Γ-convergence result of GL_ε that collects results proved in [6], [7], [1]. We will use the standard notation $GL_\varepsilon(u,A)$ to denote the Ginzburg-Landau functional localized on the open set A, i.e., defined as in (1.1) with Ω replaced by A.

Theorem 4.1. *The following Γ-convergence result holds.*

i) *(Compactness) Let $(u_\varepsilon)\subset H^1(\Omega;\mathbb{R}^2)$ be such that $GL_\varepsilon(u_\varepsilon)\le C|\log\varepsilon|$. Then, up to a subsequence, $J(u_\varepsilon)\overset{flat}{\to}\mu$, where $\mu:=\pi\sum_{i=1}^N z_i\delta_{x_i}$ for some $x_i\in\Omega$, $z_i\in\mathbb{Z}$.*

ii) *(Γ-liminf inequality) Let $(u_\varepsilon)\subset H^1(\Omega;\mathbb{R}^2)$ be such that $J(u_\varepsilon)\overset{flat}{\to}\mu:=\pi\sum_{i=1}^N z_i\delta_{x_i}$. Then, there exists $C\in\mathbb{R}$ such that, for every i and every $\sigma\le\frac12 dist(x_i,\partial\Omega\cup_{j\ne i}x_j)$ we have*

$$\liminf_{\varepsilon} GL_\varepsilon(u_\varepsilon,B_\sigma(x_i))-\pi|z_i|\log\frac{\sigma}{\varepsilon}\ge C. \tag{4.1}$$

In particular, there exists a constant C such that

$$\liminf_{\varepsilon} GL_\varepsilon(u_\varepsilon)-|\log\varepsilon||\mu|(\Omega)\ge C. \tag{4.2}$$

iii) *(Γ-limsup inequality) For every $\mu:=\pi\sum_{i=1}^N z_i\delta_{x_i}$, there exists $(u_\varepsilon)\subset H^1(\Omega;\mathbb{R}^2)$ such that $J(u_\varepsilon)\overset{flat}{\to}\mu$, $\frac{1}{|\log\varepsilon|}GL_\varepsilon(u_\varepsilon)\to|\mu|(\Omega)$, and if $|z_i|=1$, $GL_\varepsilon(u_\varepsilon)-|\mu|(\Omega)|\log\varepsilon|\le C$ for some $C\in\mathbb{R}$.*

Proof. By standard density arguments in Γ-convergence we may assume that u_ε are smooth. Following the notations in [8], we set

$$\Omega_{\varepsilon,\tau}:=\{|u_\varepsilon|>\tau\},\quad\gamma_{\varepsilon,\tau}:=\partial\Omega_{\varepsilon,\tau}\setminus\partial\Omega,\quad\Theta_\varepsilon(\tau):=\int_{\Omega_{\varepsilon,\tau}}\left|\nabla\frac{u_\varepsilon}{|u_\varepsilon|}\right|^2,\quad n_\varepsilon(\tau):=\int_{\gamma_{\varepsilon,\tau}}|\nabla|u_\varepsilon||.$$

By Coarea Formula we have

$$GL_\varepsilon(u_\varepsilon)\ge\frac12\int_0^\infty\left(n_\varepsilon(\tau)+\frac{2W(\tau)}{\varepsilon^2}\int_{\gamma_{\varepsilon,\tau}}\frac{1}{|\nabla|u_\varepsilon||}\right)d\tau-\frac12\int_0^\infty\tau^2 d\Theta'_\varepsilon(\tau),$$

where $\Theta'_\varepsilon(\tau)$ is the distributional derivative of the decreasing function $\Theta_\varepsilon(\tau)$ and the inequality is due to the possible presence of flat regions $\{\nabla|u_\varepsilon|=0\}$ with positive measure.

Set $\tau_\varepsilon:=1-\varepsilon^{1/3}$, and $K_{\varepsilon,\tau}:=\Omega\setminus\Omega_{\varepsilon,\tau}$. Then, by the energy bounds, for every $\tau\le\tau_\varepsilon$ we have $|K_{\varepsilon,\tau}|\le C\varepsilon^{\frac43}|\log\varepsilon|\le\frac12|\Omega|$, so that, using also that Ω is Lipschitz we have

$$\mathcal{H}^1(\partial K_{\varepsilon,\tau})\le C\mathcal{H}^1(\gamma_{\varepsilon,\tau}). \tag{4.3}$$

Notice that, by definition of Hausdorff measure, being $\partial K_{\varepsilon,\tau}$ compact, it is always contained in a finite union of balls $B_{r_i}(y_i)$ such that $\sum_i r_i\le\mathcal{H}^1(\partial K_{\varepsilon,\tau})$. Moreover, after a merging procedure, we can always assume that such balls are disjoint. As a consequence, either $K_{\varepsilon,\tau}$ or $\Omega\setminus K_{\varepsilon,\tau}$ is contained in the union of such balls. In the latter case, since $|\Omega\setminus K_{\varepsilon,\tau}|\ge\frac12|\Omega|$ we have $\sum_i r_i\ge c$, and we replace these balls by one single ball containing Ω. In both cases, we have a family of balls $\mathcal{B}_{\varepsilon,\tau}$ whose union contains $K_{\varepsilon,\tau}$, such that

$$\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau})\le c\mathcal{H}^1(\partial K_{\varepsilon,\tau})\le C\mathcal{H}^1(\gamma_{\varepsilon,\tau}). \tag{4.4}$$

Let $\tilde K_{\varepsilon,\tau}$ be their union and $\tilde\Omega_{\varepsilon,\tau}:=\Omega\setminus\tilde K_{\varepsilon,\tau}$. Since $K_{\varepsilon,\tau}$ is monotone in τ (with respect to inclusion), we can always assume that $\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau})$ is increasing with respect to τ.

By Hölder inequality and (4.4), for $\tau \leq \tau_\varepsilon$ we have

$$\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau})^2 \leq C\mathcal{H}^1(\gamma_{\varepsilon,\tau})^2 \leq Cn_\varepsilon(\tau)\int_{\gamma_{\varepsilon,\tau}} \frac{1}{|\nabla|u_\varepsilon||}. \tag{4.5}$$

By (4.5), Young inequality and integration by parts of $\tau^2 d\Theta'_\varepsilon(\tau)$ we have

$$GL_\varepsilon(u_\varepsilon) \geq \frac{1}{2}\int_0^\infty n_\varepsilon(\tau) + \frac{\mathcal{H}^1(\gamma_{\varepsilon,\tau})^2 W(\tau)}{C\varepsilon^2 n_\varepsilon(\tau)}\, d\tau - \frac{1}{2}\int_0^\infty \tau^2 d\Theta'_\varepsilon(\tau) \geq$$
$$\int_0^\infty \frac{\sqrt{W(\tau)}}{C\varepsilon}\mathcal{H}^1(\gamma_{\varepsilon,\tau}) + \tau\Theta_\varepsilon(\tau)\, d\tau \geq \tag{4.6}$$
$$\int_0^{\tilde\tau_\varepsilon} \frac{\sqrt{W(\tau)}}{C\varepsilon}\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau}) + \tau\Theta_\varepsilon(\tau)\, d\tau + \int_{\tilde\tau_\varepsilon}^{\tau_\varepsilon} \frac{\sqrt{W(\tau)}}{C\varepsilon}\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau}) + \tau\Theta_\varepsilon(\tau)\, d\tau$$

where $\tilde\tau_\varepsilon = 1-2\varepsilon^{\frac{1}{3}}$. By (4.6) and the energy bound, recalling also that $\tau \mapsto \mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau})$ is increasing, it follows that

$$\mathcal{R}ad(\mathcal{B}_{\varepsilon,s}) \leq \mathcal{R}ad(\mathcal{B}_{\varepsilon,\tilde\tau_\varepsilon}) \leq C\varepsilon^{\frac{1}{3}}|\log\varepsilon| \qquad \text{for all } s \leq \tilde\tau_\varepsilon. \tag{4.7}$$

Let $\mathcal{C}_{\varepsilon,s}$ be the family of balls in $\mathcal{B}_{\varepsilon,s}$ that are contained in Ω, and set

$$\mu_{\varepsilon,s} := \sum_{B\in\mathcal{C}_{\varepsilon,s}} \deg(u_\varepsilon,\partial B)\delta_x,$$

where x is the center of B. Moreover, let $\tilde\mu_{\varepsilon,s}$ be the corresponding modified measure defined according with (3.3), i.e., letting $\mathcal{C}_{\varepsilon,s}(1)$ be the family of balls in $\mathcal{B}_{\varepsilon,s}(1)$ that are contained in Ω,

$$\tilde\mu_{\varepsilon,s} = \sum_{B\in\mathcal{C}_{\varepsilon,s}(1)} \mu_\varepsilon(B)\delta_x,$$

where x is the center of B. By (4.6) and (4.7) we have

$$\theta_\varepsilon(\tilde\tau_\varepsilon) \leq C|\log\varepsilon| \leq C|\log\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tilde\tau_\varepsilon})|.$$

Since $\theta_\varepsilon(s) \geq 2F(\mathcal{B}_{\varepsilon,s},\mu_{\varepsilon,s},\Omega)$ (see Remark 3.1), by the energy bound and Theorem 3.2 we have (up to a subsequence)

$$\pi\tilde\mu_{\varepsilon,\tilde\tau_\varepsilon} \overset{flat}{\to} \mu := \pi\sum_{i=1}^N z_i\delta_{x_i}. \tag{4.8}$$

By definition of $\tilde\mu_{\varepsilon,s}$ (see also (2.2) and (2.5)) we have that $(J_s u_\varepsilon - \pi\tilde\mu_{\varepsilon,s})(B) = 0$ for every $B \in \mathcal{C}_{\varepsilon,s}(1)$. Therefore, By Proposition (2.2) and (4.7) we have

$$\lim_{\varepsilon\to0}\sup_{\varepsilon^{\frac{1}{7}}\leq s\leq\tilde\tau_\varepsilon}\|J(u_\varepsilon)-\pi\tilde\mu_{\varepsilon,s}\|_{flat} = \lim_{\varepsilon\to0}\sup_{\varepsilon^{\frac{1}{7}}\leq s\leq\tilde\tau_\varepsilon}\|J_s(u_\varepsilon)-\pi\tilde\mu_{\varepsilon,s}\|_{flat} = \tag{4.9}$$
$$\lim_{\varepsilon\to0}\sup_{\substack{\varepsilon^{\frac{1}{7}}\leq s\leq\tilde\tau_\varepsilon\\ \|\varphi\|_{C_c^{0,1}}\leq 1}}\sum_{B\in\mathcal{B}_{\varepsilon,s}(1)}|J_s u_\varepsilon|(B)\,\mathrm{osc}_B(\varphi) \leq \lim_{\varepsilon\to0} C\varepsilon^{-\frac{2}{7}}|\log\varepsilon|\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tilde\tau_\varepsilon}(1)) = 0.$$

By (4.8) and (4.9) we deduce in particular the compactness property i).

Now we prove the Γ-liminf inequality, starting with the global version (4.2). By Theorem 3.2 and (4.9) we have

$$\liminf_{\varepsilon}\inf_{\varepsilon^{\frac{1}{7}}\leq s\leq\tilde\tau_\varepsilon}\Theta_\varepsilon(s) - 2|\mu|(\Omega)|\log\mathcal{R}ad(\mathcal{B}_{\varepsilon,s})| \geq C. \tag{4.10}$$

From (4.6) we deduce

$$GL_\varepsilon(u_\varepsilon) \geq \int_{\varepsilon^{\frac{1}{7}}}^{\tau_\varepsilon}\frac{\sqrt{W(\tau)}}{C\varepsilon}\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau}) - 2\tau|\mu|(\Omega)\log\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau})\, d\tau - C$$

Notice that the right-hand side is minimized for

$$\mathcal{R}ad(\mathcal{B}_{\varepsilon,\tau}) = \frac{2C|\mu|(\Omega)\tau\varepsilon}{\sqrt{W(\tau)}}.$$

Therefore, for ε small enough,

$$\begin{aligned} GL_\varepsilon(u_\varepsilon) &\geq \int_{\varepsilon^{\frac{1}{7}}}^{\tau_\varepsilon} 2|\mu|(\Omega)\tau - 2\tau|\mu|(\Omega)\log\frac{2C|\mu|(\Omega)\tau\varepsilon}{\sqrt{W(\tau)}}\,d\tau - C \geq \\ |\mu|(\Omega)|\log\varepsilon| &- 2|\mu|(\Omega)\int_{\varepsilon^{\frac{1}{7}}}^{\tau_\varepsilon}\log\frac{2C|\mu|(\Omega)\tau}{\sqrt{W(\tau)}}\,d\tau - C \geq |\mu|(\Omega)|\log\varepsilon| - C. \end{aligned} \tag{4.11}$$

The proof of (4.1) is identical, replacing Ω by $B_\sigma(x_i)$ and (4.10) by

$$\liminf_\varepsilon \sup_{\varepsilon^{\frac{1}{7}}\leq s\leq\tilde\tau_\varepsilon} \Theta_\varepsilon(s) - 2\pi|z_i|(\log\frac{\sigma}{\mathcal{R}ad(\mathcal{B}_{\varepsilon,s})}) \geq C.$$

Let us sketch the proof of the Γ-limsup inequality. By a standard density argument we can assume $z_i = \pm 1$. Let $u_{\varepsilon,i}(r,\theta) := e^{\pm i\theta}g(r)$, where (θ, r) are polar coordinate centered at x_i and $g(s) := \min\{\frac{s}{\varepsilon}, 1\}$. Then a recovery sequence is given by $u = \Pi_{i=1}^N u_i$, where u_i are identified with complex functions. The straightforward computations are left to the reader. □

5. First order Γ-convergence to the renormalized energy

In this section we prove the first order Γ-convergence of GL_ε to the renormalized energy, introduced in [2]. We begin by recalling the main definitions and results of [2] that we need.

Let $\mu := \sum_{i=1}^N z_i\delta x_i$, with $|z_i| = 1$, $x_i \in \Omega$. Let moreover Φ_0 be the solution of

$$\begin{cases} \Delta\Phi_0 = 2\pi\mu & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $R_0(x) = \Phi_0 - \sum d_i \log|x - x_i|$. The renormalized energy is defined as follows

$$\mathbb{W}(\mu) := -\pi\sum_{i\neq j} z_i z_j \log|x_i - x_j| - \pi\sum_i z_i R_0(x_i). \tag{5.1}$$

Let now $\sigma > 0$ be such that $B_\sigma(x_i)$ are disjoint and contained in Ω and set $\Omega_\sigma := \Omega \setminus \cup_i B_\sigma(x_i)$. Consider the following minimization problems

$$m(\sigma,\mu) := \min_{u\in H^1(\Omega_\sigma;S^1)}\left\{\frac{1}{2}\int_{\Omega_\sigma}|\nabla u|^2,\ \deg(u,\partial B_\sigma(x_i)) = z_i\right\}, \tag{5.2}$$

$$\tilde m(\sigma,\mu) := \min_{u\in H^1(\Omega_\sigma;S^1),|\alpha_i|=1}\left\{\frac{1}{2}\int_{\Omega_\sigma}|\nabla u|^2, u(z) = \frac{\alpha_i}{\sigma^{z_i}}(z-x_i)^{z_i} \text{ on } \partial B_\sigma(x_i)\right\}, \tag{5.3}$$

$$\gamma(\varepsilon,\sigma) := \min_{u\in H^1(B_\sigma;\mathbb{R}^2)}\left\{GL_\varepsilon(u,B_\sigma),\ u(x)\llcorner\partial B_\sigma = \frac{x}{|x|}\right\}. \tag{5.4}$$

Theorem 5.1 (Bethuel, Brezis, Hélein [2]). *We have*

$$\lim_{\sigma\to 0} m(\sigma,\mu) + \pi|\mu|(\Omega)\log\sigma = \lim_{\sigma\to 0}\tilde m(\sigma,\mu) + \pi|\mu|(\Omega)\log\sigma = \mathbb{W}(\mu).$$

Moreover, there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{\varepsilon\to 0}\gamma(\varepsilon,\sigma) + \pi\log\frac{\varepsilon}{\sigma} = \gamma.$$

Remark 5.2. Consider the case $\Omega = B_R$, $\mu = \delta_0$. by Jensen inequality we have that the minimizers of (5.2) are given by the one parameter family

$$\mathcal{K} := \left\{\alpha\frac{z}{|z|},\ \alpha\in\mathbb{C}, |\alpha| = 1\right\}. \tag{5.5}$$

In particular, by Theorem 5.1, for $\Omega = B_R$ we have $\mathbb{W}(\delta_0) = \pi\log R$.

Theorem 5.3. *The following Γ-convergence result holds.*

i) *(Compactness) Let $M \in \mathbb{N}$ and $(u_\varepsilon) \subset H^1(\Omega;\mathbb{R}^2)$ be such that $GL_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \le C$. Then, up to a subsequence, $Ju_\varepsilon \stackrel{flat}{\to} \mu = \sum_{i=1}^N z_i\delta_{x_i}$, with $z_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$, $\tilde N := \sum |z_i| \le M$. Moreover, if $\tilde N = M$, then $N = \tilde N = M$, i.e., $|z_i| = 1$ for every i.*

ii) *(Γ-liminf inequality) Let u_ε be such that $Ju_\varepsilon \stackrel{flat}{\to} \mu = \sum_{i=1}^M z_i\delta_{x_i}$ with $|z_i| = 1$, $x_i \in \Omega$. Then,*

$$\liminf GL_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \ge \mathbb{W}(\mu) + M\gamma.$$

iii) *(Γ-limsup inequality) Given $\mu = \sum_{i=1}^M z_i\delta_{x_i}$, $|z_i| = 1$, $x_i \in \Omega$, there exists u_ε with $Ju_\varepsilon \stackrel{flat}{\to} \mu$ such that*

$$GL_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \to \mathbb{W}(\mu) + M\gamma.$$

Proof. Proof of i). The fact that, up to a subsequence $Ju_\varepsilon \stackrel{flat}{\to} \mu = \sum_{i=1}^N z_i\delta_{x_i}$, with $\tilde N := \sum |z_i| \le M$, is a direct consequence of the zero order Γ-convergence result stated in Theorem 4.1. Assume now $\tilde N = M$, and let us prove that $|z_i| = 1$. Let $0 < \sigma_1 < \sigma_2$ be such that that $B_{\sigma_2}(x_i)$ are disjoint and contained in Ω. Then, by (4.1) in Theorem 4.1, for ε small enough we have

$$GL_\varepsilon(u_\varepsilon) \ge \sum_{i=1}^N GL_\varepsilon(u_\varepsilon, B_{\sigma_1}(x_i)) + GL_\varepsilon(u_\varepsilon, A_i) \ge \sum_{i=1}^N \pi|z_i|\log\frac{\sigma_1}{\varepsilon} + GL_\varepsilon(u_\varepsilon, A_i) - C, \tag{5.6}$$

where $A_i := B_{\sigma_2}(x_i)\setminus B_{\sigma_1}(x_i)$. By the energy bound, we deduce that $GL_\varepsilon(u_\varepsilon, A) \le C$, and hence (up to a subsequence) $u_\varepsilon \rightharpoonup u_i$ in $H^1(A_i;\mathbb{R}^2)$, for some $u_i = e^{i\theta_i(x)} \in H^1(A_i;S^1)$. Moreover, it is easy to see that $\deg(u_i, \partial B_{\sigma_2}(x_i)) = |z_i|$, for instance arguing as follows: by the energy bound and standard Fubini's arguments, for almost every $\sigma_1 < \sigma < \sigma_2$ we have that (up to a subsequence) the trace of u_ε on $\partial B_\sigma(x_i)$ is bounded in $H^1(\partial B_\sigma(x_i);\mathbb{R}^2)$ and hence weakly converge to the trace of u. The assertion follows by the very definition of degree (2.3).

For every i we have

$$\frac12\int_{A_i}|\nabla u_i|^2 = \frac12\int_{A_i}|\nabla\theta_i|^2 \ge \pi|z_i|^2(\log\sigma_2 - \log\sigma_1). \tag{5.7}$$

By (5.7) and (5.6) we conclude that, for ε small enough,

$$GL_\varepsilon(u_\varepsilon) \ge \sum_{i=1}^N \pi|z_i|\log\frac{\sigma_1}{\varepsilon} + \pi|z_i|^2(\log\sigma_2 - \log\sigma_1) - C = \pi M|\log\varepsilon| + \pi\sum_{i=1}^N (z_i^2 - |z_i|)\log\frac{\sigma_2}{\sigma_1} - C.$$

Letting $\sigma_1 \to 0$, the energy bound yields $|z_i| \equiv 1$.

Proof of ii). For every $r > 0$, by (4.1) we can assume that $GL_\varepsilon(u_\varepsilon, \Omega_r) \le C$, so that (up to a subsequence) $u_\varepsilon \rightharpoonup u$ in $H^1_{loc}(\Omega\setminus\cup x_i;\mathbb{R}^2)$. Let $\sigma > 0$ be such that $B_\sigma(x_i)$ are disjoint and contained in Ω. Let $t \le \sigma$, and consider the minimization problem (5.2) in $B_t\setminus B_{\frac t2}$ for $\mu = \delta_0$. Set

$$d_t(w,\mathcal{K}) := \min\{\|w - v\|_{H^1(B_t\setminus B_{\frac t2};\mathbb{R}^2)} : v \in \mathcal{K}\},$$

where $\mathcal{K}$ is the family of minimizers given by (5.5). It is easy to prove that for any given $\delta > 0$ there exists $c > 0$ (independent of t) such that, if $d_t(u_\varepsilon(\cdot + x_i), \mathcal{K}) \ge \delta$, then

$$\liminf_\varepsilon \int_{B_t(x_i)\setminus B_{\frac t2}(x_i)} |\nabla u_\varepsilon|^2 \ge \log 2 + c.$$

Indeed, arguing by contradiction, if there exists a subsequence u_ε such that

$$\log 2 \le \int_{B_t(x_i)\setminus B_{\frac t2}(x_i)} |\nabla u|^2 \le \lim_\varepsilon \int_{B_t(x_i)\setminus B_{\frac t2}(x_i)} |\nabla u_\varepsilon|^2 = \log 2,$$

then we would conclude that $u_\varepsilon \to u$ strongly in $H^1(B_t(x_i)\setminus B_{\frac t2}(x_i);\mathbb{R}^2)$, hence $d_t(u(\cdot + x_i), \mathcal{K}) \ge \delta$. This is in contradiction with the definition of $\mathcal{K}$ (see Remark 5.2), noticing that $\deg(u_i, \partial B_t(x_i)) = z_i = \pm 1$ and that $u(\cdot + x_i)$ has minimal energy $\log 2$. Let $m \in \mathbb{N}$ be such that $mc \ge \mathbb{W}(\mu) + M(\gamma - \log\sigma - C)$, where C is the constant in (4.1). For $l = 1, \dots, m$ set $C_l(x_i) := B_{2^{1-l}\sigma}(x_i)\setminus B_{2^{-l}\sigma}(x_i)$. We consider now two cases.

First case: for ε small enough and for every fixed $1 \le l \le m$, there exists at least one i such that $d_{2^{1-l}\sigma}(u_\varepsilon(\cdot + x_i), \mathcal{K}) \ge \delta$. Then by (4.1) we conclude (for ε small enough)

$$\begin{aligned} GL_\varepsilon(u_\varepsilon) \ge M(\log \frac{\sigma}{2^m} - \log\varepsilon + C) + \sum_{l=1}^m \sum_{i=1}^M \frac{1}{2}\int_{C_l(x_i)} |\nabla u_\varepsilon|^2 \ge \\ M(\log\sigma - m\log 2 - \log\varepsilon + C) + m\big((M\log 2 + c\big) \ge \\ M(\log\sigma - \log\varepsilon + C) + \mathbb{W}(\mu) + M(\gamma - \log\sigma - C) = M(|\log\varepsilon| + \gamma) + \mathbb{W}(\mu). \end{aligned} \tag{5.8}$$

Second case: Up to a subsequence, there exists $1 \le \bar{l} \le m$ such that for every i we have $d_{\bar\sigma}(u_\varepsilon(\cdot + x_i), \mathcal{K}) \le \delta$, where $\bar\sigma := 2^{1-\bar{l}}\sigma$. By standard Fubini's arguments there exists $2/3\bar\sigma \le \tilde\sigma \le 3/4\bar\sigma$ such that the Ginzburg-Landau energy of u_ε on $\partial B_{\tilde\sigma}(x_i)$ is uniformly bounded. We can easily modify u_ε with an infinitesimal amount of energy to enforce $|u_\varepsilon| = 1$ on $\partial B_{\tilde\sigma}(x_i)$. Let us denote by θ the lifting of $x/|x|$. By [3, Theorem 1] (see also [3, Remark 3]) there exists $r(\delta) \to 0$ as $\delta \to 0$ such that $u_\varepsilon = |u_\varepsilon| e^{i\theta_\varepsilon}$ (up to additive constant in the phase) with

$$\|\theta_\varepsilon - \theta\|_{H^{\frac12}(\partial B_{\tilde\sigma}(x_i))} \le r(\delta).$$

Let T_ε be the harmonic solution on $B_{\bar\sigma}(x_i) \setminus B_{\tilde\sigma}(x_i)$ with boundary conditions θ_ε and θ on $\partial B_{\tilde\sigma}(x_i)$ and $\partial B_{\bar\sigma}(x_i)$, respectively. We extend u_ε on $B_{\bar\sigma}(x_i) \setminus B_{\tilde\sigma}(x_i)$ setting $\tilde u_\varepsilon := e^{iT_\varepsilon}$ on $B_{\bar\sigma}(x_i) \setminus B_{\tilde\sigma}(x_i)$. Hence we have

$$GL_\varepsilon(u_\varepsilon, B_{\bar\sigma}(x_i)) \ge GL_\varepsilon(\tilde u_\varepsilon, B_{\bar\sigma}(x_i)) + r(\varepsilon,\delta) \ge \gamma(\varepsilon,\bar\sigma) + r(\varepsilon,\delta),$$

where $\lim_{\delta\to0}\lim_{\varepsilon\to0} r(\varepsilon,\delta) = 0$. By Theorem 5.1 we conclude that

$$\begin{aligned} GL_\varepsilon(u_\varepsilon) = GL_\varepsilon(u_\varepsilon, \Omega_{\bar\sigma}) + \sum_{i=1}^M GL_\varepsilon(u_\varepsilon, B_{\bar\sigma}(x_i)) \ge \\ \mathbb{W}(\mu) - M\pi\log\bar\sigma + M(\gamma - \pi\log\frac{\varepsilon}{\bar\sigma}) + r(\varepsilon,\delta) = M(|\log\varepsilon| + \gamma)) + \mathbb{W}(\mu) + r(\varepsilon,\delta). \end{aligned}$$

The proof follows by the arbitrariness of δ.

Proof of iii). Let $u_{\varepsilon,\sigma}$ be the function that agrees with a minimizer of (5.3) in Ω_σ, and with $\alpha_i w_{\varepsilon,\sigma}(x)$ in each $B_\sigma(x_i)$, where $w_{\varepsilon,\sigma}$ is a minimizer of problem (5.4), and α_i is a unit vector suitable chosen to match the boundary conditions on $\partial B_\sigma(x_i)$. By Theorem 5.1 there exists a suitable σ_ε such that $u_\varepsilon := u_{\varepsilon,\sigma_\varepsilon}$ is optimal in energy. □

6. Boundary conditions

In this section we deal with prescribed boundary conditions g on $\partial\Omega$, and prove in this context the first order Γ-convergence of GL_ε to the renormalized energy (the zero order Γ-convergence result can be deduced as a particular case). The renormalized energy $\mathbb{W}_g$ is defined according with (5.1), but now the function ϕ solves the following boundary value problem.

$$\begin{cases} \Delta\Phi = \mu & \text{in } \Omega; \\ \frac{\partial\Phi}{\partial\nu} = g \times g_\tau & \text{on } \partial\Omega. \end{cases}$$

Then, Theorem 5.1 holds true with $\mathbb{W}$ replaced by $\mathbb{W}_g$, and with $m(\sigma,\mu)$, $\tilde m(\sigma,\mu)$ replaced by $m_g(\sigma,\mu)$, $\tilde m_g(\sigma,\mu)$ defined in the obvious way. Given $g \in H^{\frac12}(\partial\Omega;\mathbb{R}^2)$ we denote by $H^1_g(\Omega;\mathbb{R}^2)$ the subspace of functions in $H^1(\Omega;\mathbb{R}^2)$ with trace g.

Theorem 6.1. *The following Γ-convergence result holds.*

i) *(Compactness) Let $M \in \mathbb{N}$, $(u_\varepsilon) \subset H^1_g(\Omega;\mathbb{R}^2)$ be such that $GL_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \le C$. Up to a subsequence, $Ju_\varepsilon \overset{flat(\overline\Omega)}{\to} \mu = \sum_{i=1}^N z_i\delta_{x_i}$, where $x_i \in \overline\Omega$, $z_i \in \mathbb{Z}\setminus\{0\}$, $\sum z_i = \deg(g,\partial\Omega)$, $\tilde N := \sum|z_i| \le M$. Moreover, if $\tilde N = M$, then $N = \tilde N = M$, i.e., $|z_i| = 1$ for every i, and all x_i's belong to Ω. In particular, if $M = \deg(g,\partial\Omega)$, then $z_i = sign(\deg(g,\partial\Omega))$.*

ii) *(Γ-liminf inequality) Let $u_\varepsilon \in H^1_g(\Omega;\mathbb{R}^2)$ be such that $Ju_\varepsilon \overset{flat}{\to} \mu = \sum_{i=1}^M z_i\delta_{x_i}$ with $|z_i| = 1$, $x_i \in \Omega$. Then,*

$$\liminf GL_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \geq \mathbb{W}_g(\mu) + M\gamma.$$

iii) *(Γ-limsup inequality) Given $\mu = \sum_{i=1}^M z_i\delta_{x_i}$ with $|z_i| = 1$, $x_i \in \Omega$, there exists $u_\varepsilon \in H^1_g(\Omega;\mathbb{R}^2)$ with $Ju_\varepsilon \overset{flat}{\to} \mu$ such that*

$$GL_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \to \mathbb{W}_g(\mu) + M\gamma.$$

Proof. The proof of ii) and iii) is exactly as the proof of the analogous statements in Theorem 5.3, so we give only the proof of i). We can assume that g is the trace of a function, still denoted by $g \in H^1(U;S^1)$, where U is a neighborhood of $\partial\Omega$. Let $\tilde\Omega := \Omega \cup U$, and let $\tilde u_\varepsilon$ be the extension of u_ε on $\tilde\Omega$ defined by $\tilde u_\varepsilon = u_\varepsilon$ in Ω and $\tilde u_\varepsilon = g$ on $U \setminus \Omega$. By construction we have

$$GL_\varepsilon(\tilde u_\varepsilon, \tilde\Omega) - M\pi|\log\varepsilon| \leq \tilde C \qquad \text{for some } \tilde C \geq 0. \tag{6.1}$$

There exists a costant $c > 0$ such that, given $\varphi \in C^{0,1}(\Omega)$, there exists an extension $\tilde\varphi \in C^{0,1}_c(\tilde\Omega)$ of φ with $\|\tilde\varphi\|_{C^{0,1}_c(\tilde\Omega)} \leq c\|\varphi\|_{C^{0,1}(\Omega)}$. By Theorem 4.1 (using also that $Jg = 0$), there exists $\mu = \sum_{i=1}^N z_i\delta_{x_i}$ with $x_i \in \overline\Omega$, $\tilde N := \sum |z_i| \leq M$, such that

$$\|Ju_\varepsilon - \mu\|_{flat(\overline\Omega)} = \sup_{\|\varphi\|_{C^{0,1}(\Omega)\leq 1}} < Ju_\varepsilon - \mu, \varphi > \leq \sup_{\|\tilde\varphi\|_{C^{0,1}_c(\tilde\Omega)\leq c}} < J\tilde u_\varepsilon - \mu, \tilde\varphi > \to 0.$$

Moreover, by (2.2) and (2.3) we have

$$\deg(g, \partial\Omega) = Ju_\varepsilon(\Omega) \to \sum_{i=1}^N z_i.$$

If now $\tilde N = M$, we deduce $|z_i| = 1$ for every i by statement i) of Theorem 5.3. Finally, the fact that x_i belong to Ω follows as in the proof of ii) of Theorem 5.3. More precisely, by (6.1) we have that (up to a subsequence) $\tilde u_\varepsilon \rightharpoonup u$ in $H^1_{loc}(\tilde\Omega \setminus \cup x_i)$. Recall that for any t such that $B_t(x_i) \subset \tilde\Omega$, given $\delta > 0$ there exists $c > 0$ (independent of t) such that, if the distance of u from $\mathcal{K}$ in $H^1(B_t(x_i) \setminus B_{\frac{t}{2}}(x_i))$ is larger than δ, then

$$\int_{B_t(x_i)\setminus B_{\frac{t}{2}}(x_i)} |\nabla u|^2 \geq \log 2 + c.$$

It follows that for any δ the distance of u from $\mathcal{K}$ is less then δ for infinitely many annuli $B_{2^{-n}}(x_i) \setminus B_{2^{-(n+1)}}(x_i)$. This is possible only if x_i belongs to Ω. □

Finally, we consider the case of varying boundary conditions g_ε. We need the following Lemma.

Lemma 6.2. *Let g_ε, $h_\varepsilon \in H^{\frac12}(\partial\Omega;\mathbb{R}^2)$ be such that $g_\varepsilon - h_\varepsilon \to 0$ in $H^{\frac12}(\partial\Omega;\mathbb{R}^2)$, and such that $|g_\varepsilon| - 1$ and $\frac{|g_\varepsilon - h_\varepsilon|}{\varepsilon}$ converge to 0 uniformly as $\varepsilon \to 0$. Finally, let $v_\varepsilon \in H^1_{g_\varepsilon}(\Omega;\mathbb{R}^2)$ with $\|v_\varepsilon\|_\infty \leq C$, $GL_\varepsilon(v_\varepsilon) \leq C|\log\varepsilon|$. Then, there exists $w_\varepsilon \in H^1_{h_\varepsilon}(\Omega;\mathbb{R}^2)$ such that*

$$GL_\varepsilon(v_\varepsilon) - GL_\varepsilon(w_\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0.$$

Proof. By [3, Theorem 1] (see also [3, Remark 3]) we can always write $v_\varepsilon = |v_\varepsilon|e^{i\theta_\varepsilon(x)}$ and $h_\varepsilon = |h_\varepsilon|e^{i(\theta_\varepsilon(x)+t_\varepsilon)}$ with $t_\varepsilon \to 0$ in $H^{\frac12}(\partial\Omega)$. Let τ_ε be the harmonic extension of t_ε in Ω, and let ρ_ε be the harmonic extension of $\frac{|h_\varepsilon|}{|g_\varepsilon|}$ in Ω. Notice that $\rho_\varepsilon e^{i\tau_\varepsilon} \to 1$ in $H^1(\Omega)$ and $(\rho_\varepsilon e^{i\tau_\varepsilon} - 1)/\varepsilon \to 0$ uniformly. The desired function w_ε is given by

$$w_\varepsilon(x) := \rho_\varepsilon(x)e^{i(\tau_\varepsilon(x))}v_\varepsilon(x).$$

□

Theorem 6.3. *Let $g \in H^{\frac12}(\partial\Omega;S^1)$, $g_\varepsilon \in H^{\frac12}(\partial\Omega;\mathbb{R}^2)$ be such that*

i) *$g_\varepsilon \to g$ in $H^{\frac12}(\partial\Omega;\mathbb{R}^2)$;*

ii) *$\frac{|g_\varepsilon - g|}{\varepsilon} \to 0$ uniformly as $\varepsilon \to 0$.*

Then, Theorem 6.1 still holds true with $u_\varepsilon \in H^1_g(\Omega;\mathbb{R}^2)$ *replaced by* $u_\varepsilon \in H^1_{g_\varepsilon}(\Omega;\mathbb{R}^2)$.

Proof. Any $u_\varepsilon \in H^1_{g_\varepsilon}(\Omega;\mathbb{R}^2)$ bounded in energy can be extended in a neighborhood $\tilde{\Omega}$ of Ω with a bounded amount of energy, so that the proof of compactness follows exactly as in the case of a fixed boundary datum. A possible extension is constructed as follows: by [3, Theorem 1] (see also [3, Remark 3]) we have $g_\varepsilon = g e^{it_\varepsilon}$ with $t_\varepsilon \to 0$ in $H^{\frac{1}{2}}(\partial\Omega)$. Then, we define $\tilde{u}_\varepsilon$ on $\tilde{\Omega}\setminus\Omega$ as $\tilde{u}_\varepsilon = \tilde{g}\rho_\varepsilon e^{i\tau_\varepsilon}$, where $\tilde{g}$ is an extension of g, ρ_ε is the harmonic extension of $|g_\varepsilon|$ with boundary datum 1 on $\partial\tilde{\Omega}$, and τ_ε is the harmonic extension of t_ε with boundary datum 0 on $\partial\tilde{\Omega}$.

Finally, by Lemma 6.2 for any sequence u_ε bounded in energy and in L^∞ one can easily switch the boundary conditions from g to g_ε and viceversa with vanishing perturbations in the energy. This is enough to deduce the Γ-liminf and Γ-limsup inequality from Theorem 6.1.

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