

A geometric approach for convexity in some variational problem in the Gauss space

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Abstract

In this short note we prove the convexity of minimizers of some variational problem in the Gauss space. This proof is based on a geometric version of an older argument due to Korevaar.

1 Introduction

In the paper [5] we prove together with A. Chambolle and M. Novaga, the convexity of the minimizers of the following variational problem in the Wiener space X :

$$\min \int_X F(\nabla u) + \frac{(u - g)^2}{2} d\gamma \quad (1)$$

under the general hypothesis that F and g are convex functions. The idea is to approximate the infinite dimensional problem by a finite dimensional one, to show convexity of the minimizers of the finite dimensional problems and prove convergence of these minimizers towards the minimum of (1). In [5], we followed the approach of Alvarez Lasry and Lions [1] to prove convexity in finite dimension. The aim of this note is to show an alternative proof based on ideas of Korevaar [10] when F is the total variation (which was our main motivation in [5]). More precisely, we will show that for $g \in L^2_\gamma(\mathbb{R}^m)$ a convex function then the solution of

$$\min_{BV_\gamma(\mathbb{R}^m) \cap L^2_\gamma(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_\gamma u| + \frac{1}{2} \int_{\mathbb{R}^m} |u - g|^2 d\gamma \quad (2)$$

is convex. As a by-product of our analysis we will also get that the minimizers of the Ornstein-Uhlenbeck functional

$$\min_{H^1_\gamma(\mathbb{R}^m)} \int_{\mathbb{R}^m} \frac{|\nabla u|^2}{2} d\gamma + \frac{1}{2} \int_{\mathbb{R}^m} |u - g|^2 d\gamma$$

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are convex if g is convex.

The plan of the note is the following. In Section 2 we recall some notation about functions of bounded variation and in Section 3 we show the convexity of the minima of (2).

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2 Notation and preliminary results

Let $m \in \mathbb{N}$ be fixed and let γ be the standard Gaussian measure on \mathbb{R}^m . Let us denote by $L^2_\gamma(\mathbb{R}^m) := L^2(\mathbb{R}^m, \gamma)$.

We now give the definitions of Sobolev spaces and functions of bounded variation in the Gauss space. For a smooth function $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$, we define

$$\operatorname{div}_\gamma \Phi(x) := \operatorname{div} \Phi - \Phi \cdot x.$$

The operator $\operatorname{div}_\gamma$ is the adjoint of the gradient for the $L^2_\gamma(\mathbb{R}^m)$ duality so that for every $u \in C_c^1(\mathbb{R}^m)$ and every $\Phi \in C_c^1(\mathbb{R}^m, \mathbb{R}^m)$, the following integration by parts holds:

$$\int_{\mathbb{R}^m} u \operatorname{div}_\gamma \Phi d\gamma = - \int_{\mathbb{R}^m} \nabla u \cdot \Phi d\gamma. \quad (3)$$

The gradient is a closable operator and we will denote by $H_\gamma^1(\mathbb{R}^m)$ the domain of its closure in $L^2_\gamma(\mathbb{R}^m)$. From this, formula (3) still holds for $u \in H_\gamma^1(\mathbb{R}^m)$ and $\Phi \in C_c^1(\mathbb{R}^m, \mathbb{R}^m)$.

Given $u \in L^1_\gamma(\mathbb{R}^m)$ we say that $u \in BV_\gamma(\mathbb{R}^m)$ if

$$\int_{\mathbb{R}^m} |D_\gamma u| = \sup \left\{ \int_{\mathbb{R}^m} u \operatorname{div}_\gamma \Phi d\gamma; \Phi \in C_c^1(\mathbb{R}^m, \mathbb{R}^m), |\Phi(x)| \leq 1 \forall x \in \mathbb{R}^m \right\} < +\infty.$$

We see that functions in $BV_\gamma(\mathbb{R}^m)$ are in $BV_{\text{loc}}(\mathbb{R}^m)$ and that $D_\gamma u = \gamma Du$ so that most of the properties of classical BV functions extends to the function in $BV_\gamma(\mathbb{R}^m)$ (see [2]). In particular for every set E of finite Gaussian perimeter, the reduced boundary $\partial^* E$ of E is rectifiable and every point of this reduced boundary has an outward normal ν^E . Defining the sets $E^{(s)}$ by

$$E^{(s)} = \left\{ x \in \mathbb{R}^m : \lim_{r \rightarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = s \right\}$$

then we have $\mathcal{H}^{m-1}((E^{(0)} \cup E^{(1)} \cup \partial^* E)^c) = 0$ for every set of finite Gaussian perimeter. Here we denoted by \mathcal{H}^{m-1} the $m - 1$ dimensional Hausdorff measure.

We finally recall some facts about pairings between measures and bounded functions (see [4] for more details).

We define the space X_2 to be the space of bounded functions z with $\operatorname{div}_\gamma z \in L^2_\gamma(\mathbb{R}^m)$ (where the divergence is intended in the distributional sense). For every smooth open set Ω , the trace $[z \cdot \nu]$ can be defined in such a way that the integration by part formula

$$\int_\Omega (z, D_\gamma u) + \int_\Omega u \operatorname{div}_\gamma z d\gamma = \int_{\partial\Omega} [z \cdot \nu] u \gamma(x) \mathcal{H}^{m-1}$$

holds for $z \in X_2$ and $u \in BV_\gamma \cap L^2_\gamma(\mathbb{R}^m)$ where as usual $(z, D_\gamma u)$ is the measure defined by

$$\int_{\mathbb{R}^m} \varphi(z, D_\gamma u) = - \int_{\mathbb{R}^m} u \varphi \operatorname{div}_\gamma z d\gamma - \int_{\mathbb{R}^m} u z \cdot \nabla \varphi d\gamma$$

for every $\varphi \in \mathcal{C}_c^1(\mathbb{R}^m, \mathbb{R}^m)$.

3 Convexity of the minimizer

In this section we are going to prove the following result: if $g \in L^2_\gamma(\mathbb{R}^m)$ be a convex function then the minimizer of

$$\min_{BV_\gamma \cap L^2_\gamma(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_\gamma u| + \frac{1}{2} \int_{\mathbb{R}^m} |u - g|^2 d\gamma \quad (4)$$

is a convex function.

As in many other papers involving the total variation, we are going to study first the regularized problem associated to the functional:

$$\min_{BV_\gamma \cap L^2_\gamma(\mathbb{R}^m)} J_\varepsilon(u) = \int_{\mathbb{R}^m} \sqrt{\varepsilon^2 + |D_\gamma u|^2} + \frac{1}{2} \int_{\mathbb{R}^m} |u - g|^2 d\gamma \quad (5)$$

where as usual, if the Radon-Nikodym decomposition of $D_\gamma u$ is given by $D_\gamma u = \nabla u d\gamma + D_\gamma^s u$ we let

$$\int_{\mathbb{R}^m} \sqrt{\varepsilon^2 + |D_\gamma u|^2} = \int_{\mathbb{R}^m} \sqrt{\varepsilon^2 + |\nabla u|^2} d\gamma + |D_\gamma^s u|(\mathbb{R}^m).$$

As a simple consequence of the Reshetnyak's continuity Theorem we have that J_ε is lower semicontinuous for the $L^2_\gamma(\mathbb{R}^m)$ convergence (see [2]).

We start by studying the Dirichlet problem on balls, namely

$$\min_{BV_\gamma(B_R) \cap L^2_\gamma(\mathbb{R}^m)} \int_{B_R} \sqrt{\varepsilon^2 + |D_\gamma u|^2} + \frac{1}{2} \int_{B_R} |u - g|^2 d\gamma + \int_{\partial B_R} |u - M| \gamma d\mathcal{H}^{m-1}. \quad (6)$$

Here B_R is the open ball of radius R centered in the origin and M is a constant to be chosen later. The term $\int_{\partial B_R} |u - M| \gamma d\mathcal{H}^{m-1}$ can be seen as a Dirichlet term (see [6] and [3]). In the following we will denote by $F(p) = \sqrt{\varepsilon^2 + |p|^2}$. On bounded domains, by Theorem 6.7 in [3] we can give a characterization of the minimizers of (6).

Theorem 3.1 (Characterization of the minima). *A function $u \in BV_\gamma(B_R)$ minimizes (6) if and only if $\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \in X_2$ and*

$$- \operatorname{div}_\gamma \left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \right) + u = g, \quad \frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \cdot D_\gamma^s u = |D_\gamma^s u| \quad |D_\gamma^s u| - a.e.$$

$$\text{and} \quad \left[\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \cdot \nu \right] \in \operatorname{sign}(M - u) \quad \mathcal{H}^{m-1} - a.e. \text{ in } \partial B_R,$$

where ν is the outward normal to B_R and where for a and b real numbers, by $a \in \operatorname{sign}(b)$ we mean that if $b \neq 0$, $a = \operatorname{sign}(b)$ and if $b = 0$, $a \in [0, 1]$.

Adapting very slightly the proof of [3, Th. 5.16], we get the following comparison principle:

Proposition 3.2 (Comparison). *Let $g_1 \geq g_2$ and $\varphi_1 \geq \varphi_2$ then the minimizers u_i with $i = 1, 2$ of*

$$\min_{BV_\gamma(B_R) \cap L^2_\gamma(\mathbb{R}^m)} \int_{B_R} F(D_\gamma u) d\gamma + \frac{1}{2} \int_{B_R} |u - g_i|^2 d\gamma + \int_{\partial B_R} |u - \varphi_i| \gamma d\mathcal{H}^{m-1}$$

verify $u_1 \geq u_2$.

With this comparison property in hands, we can prove that for M large enough, the minimizer of (6) makes vertical contact angle with the boundary of B_R . In the following, we will say that a function v is a supersolution of (6) if there exists $\tilde{g} \geq g$ and $\varphi \geq M$ such that v minimizes the functional (6) with \tilde{g} instead of g and φ instead of M . By Proposition 3.2 this implies that v is larger than the minimizer of (6).

Proposition 3.3 (vertical contact angle). *If $C \geq \frac{m}{\varepsilon r} + \frac{R}{\varepsilon} + r + |g|_{L^\infty(B_R)}$, then*

$$v(x) = \begin{cases} C - \sqrt{r^2 - |x - x_0|^2} & \text{if } x \in B_r(x_0) \\ M & \text{otherwise} \end{cases}$$

is a supersolution of (6) if $B_r(x_0) \subset B_R$. Then for $M > C$, the minimizer of (6) has vertical contact angle with ∂B_R .

Proof. We must show that for C large enough,

$$-\operatorname{div}_\gamma(\nabla F(\nabla v)) + v - g \geq 0.$$

A direct computation shows that in $B_r(x_0)$ we have $\nabla v = \frac{x - x_0}{\sqrt{r^2 - |x - x_0|^2}}$ thus

$$\nabla F(\nabla v) = \frac{x - x_0}{\sqrt{\varepsilon^2 r^2 + (1 - \varepsilon^2)|x - x_0|^2}}.$$

From this we get that

$$\begin{aligned} -\operatorname{div}_\gamma(\nabla F(\nabla v)) + v - g &\geq -\frac{m}{\varepsilon r} + \frac{x - x_0}{\sqrt{\varepsilon^2 r^2 + (1 - \varepsilon^2)|x - x_0|^2}} \cdot x + C \\ &\quad - \sqrt{r^2 - |x - x_0|^2} - |g|_{L^\infty(B_R)} \\ &\geq -\frac{m}{\varepsilon r} - \frac{|x - x_0|}{\sqrt{\varepsilon^2 r^2 + (1 - \varepsilon^2)|x - x_0|^2}} |x| + C \\ &\quad - r - |g|_{L^\infty(B_R)} \\ &\geq -\frac{m}{\varepsilon r} - \frac{R}{\varepsilon} + C - r - |g|_{L^\infty(B_R)}. \end{aligned}$$

Thus if $C \geq \frac{m}{\varepsilon r} + \frac{R}{\varepsilon} + r + |g|_{L^\infty(B_R)}$ then v is a super-solution.

If $M > C$, then considering balls of radius r such that $\partial B_r \cap \partial B_R$ is reduced to a point, by the comparison Proposition 3.2, if u minimizes (6) then $M > C \geq v \geq u$ and thus by Theorem 3.1 we have

$$\left[\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \cdot \nu \right] = 1 \quad \mathcal{H}^{m-1} - a.e. \text{ on } \partial B_R,$$

which is the vertical contact angle condition. \square

The interior regularity of minimizers of (6) easily follows by a result of Giaquinta, Modica and Soucek [6].

Proposition 3.4. *Let g be a $\mathcal{C}^\alpha(B_R)$ function then the minimizer of (6) is $\mathcal{C}^{2,\alpha}(B_R)$.*

Proof. By Theorem 3.3 of [6] we have that minimizers of

$$\min_{BV_\gamma(B_R)} \int_{B_R} F(D_\gamma u) + \int_{B_R} G(x, u) d\gamma + \int_{\partial B_R} |u - M| \gamma d\mathcal{H}^{m-1}$$

are locally Lipschitz if $G(x, u)$ verifies the following hypotheses:

- $\left| \frac{\partial G}{\partial u} \right| + \left| \frac{\partial \nabla_x G}{\partial u} \right| \leq C.$
- $\frac{\partial^2 G}{\partial u^2} \geq 0.$

Originally we have $G(x, u) = \frac{1}{2}|u - g(x)|^2$ which does not verify exactly the hypotheses. However if we set $\tilde{G}(x, u) = \Psi(u) - g(x)u + \frac{1}{2}g(x)^2$ where $\Psi(u) = \frac{1}{2}u^2$ if $|u| \leq C$ and Ψ convex, \mathcal{C}^2 with linear growth at infinity then \tilde{G} verifies the condition mentioned above. The Euler-Lagrange equation verified by the minimizers with \tilde{G} instead of G is

$$-\operatorname{div}_\gamma \left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \right) + \frac{\partial \Psi}{\partial u} = g. \quad (7)$$

Now we can apply Theorem 3.3 of [6] to find that solutions of (7) are locally Lipschitz. Exactly as in Proposition 3.2 the comparison principle holds for this equation and thus M (respectively $-M$) is a supersolution (respectively a subsolution). This implies that if $C \geq M$ solutions of (7) are also solutions of (6) which are thus locally Lipschitz. By classical regularity theory for elliptic equations (see [7]) this implies that the solutions are indeed $\mathcal{C}^{2,\alpha}(B_R)$. \square

Remark 3.5. This proposition in particular applies for g convex since convex functions are locally Lipschitz.

Having only interior regularity it is not possible to directly apply the results of Korevaar [10] which need continuity up to the boundary. The idea will be to use a geometric version of Korevaar's argument to get the convexity of the minimizers.

For simplicity, in this part of the proof we focus on the case $\varepsilon = 1$. By rescaling, the general case of $\varepsilon \neq 1$ can be easily recovered (the Gaussian measure γ is not invariant by this scaling but it does not matter). Consider now the set (see Figure 1)

$$U = \{(x, t) \in \overline{B_R} \times [-M; M] / t \geq u(x)\}. \quad (8)$$

The aim is to show that U is a convex set. First we need to show that U is regular. For this we follow an idea of Giusti (see [8] and [9]) showing that the complement of U is a solution of a certain obstacle problem.

For F a set of finite perimeter in \mathbb{R}^{m+1} let $\tilde{P}(F)$ be defined by

$$\tilde{P}(F) = \int_{\partial^* F} \gamma(x) d\mathcal{H}^m(x, t).$$

\tilde{P} is thus the perimeter associated to the measure $\mu(x, t) = \gamma(x) dx dt$. Let now $H(x, t) = (t - g(x))\gamma(x)$ then we have the following:

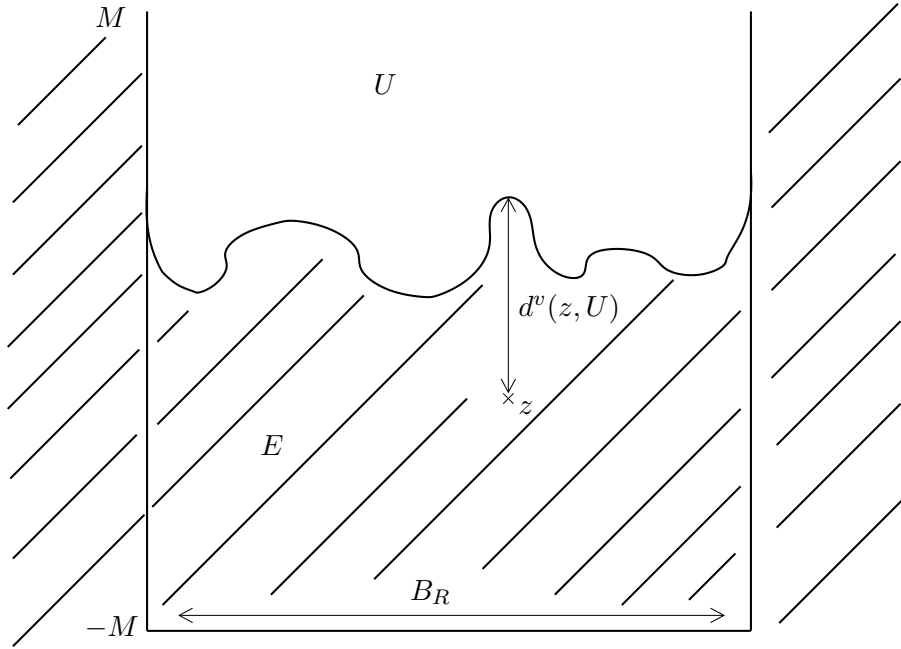


Figure 1:

Proposition 3.6. *Let E be the complement of U in $\mathbb{R}^m \times [-M; M]$ then E is a minimizer of*

$$\tilde{P}(F) + \int_F H(x, t) \, dx \, dt \quad (9)$$

among all sets containing $B_R^c \times [-M; M]$. As a consequence ∂E is C^1 .

Proof. Let us define the field

$$z(x, t) = \begin{cases} \left(-\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \frac{1}{\sqrt{1 + |\nabla u|^2}} \right) & (x, t) \in B_R \times]-M; M[\\ -\nu^{B_R}(x) & (x, t) \in \partial B_R \times]-M; M[\end{cases} .$$

Then z is a X_2 vector field in $B_R \times]-M; M[$ satisfying $|z|_{\mathbb{R}^{m+1}} = 1$ and $[z \cdot \nu^E] = 1$ where ν^E is the outward normal to E . Moreover if $z = (z', z_{m+1})$ with $z' \in \mathbb{R}^m$ and $z_{m+1} \in \mathbb{R}$ then setting by a slight abuse of notations

$$\operatorname{div}_\gamma z = \operatorname{div}_\gamma z' + \frac{\partial z_{m+1}}{\partial t}$$

we have $\operatorname{div}_\gamma z = g - u$. Hence if $F \Delta E \subset B_R \times]-M; M[$, as $t < u(x)$ in E ,

$$\begin{aligned} \int_{E \setminus F} (\operatorname{div}_\gamma z) d\mu &= \int_{E \setminus F} (g - u) d\mu \leq \int_{E \setminus F} (g(x) - t) d\mu \\ &= - \int_{E \setminus F} H dx dt = \int_{E \cap F} H dx dt - \int_E H dx dt. \end{aligned}$$

On the other hand we have:

$$\int_{E \setminus F} (\operatorname{div}_\gamma z) d\mu = \int_{\partial^*(E \setminus F)} [z \cdot \nu^{E \setminus F}] \gamma d\mathcal{H}^m(x, t)$$

But $E \setminus F = E \cap F^c$ and since

$$\nu^E = \nu^{F^c} \quad \mathcal{H}^m - a.e. \text{ on } \partial^* E \cap \partial^* F$$

and $\mathcal{H}^m(\mathbb{R}^{m+1} \setminus (E^{(0)} \cup E^{(1)} \cup \partial^* E)) = 0$,

$$\partial^*(E \cap F^c) = J_{E, F^c} \cup \left(\partial^* E \cap (F^c)^{(1)} \right) \cup \left(\partial^* F^c \cap E^{(1)} \right)$$

where $J_{E, F^c} = \{x \in \partial^* E \cap \partial^* F^c / \nu^E = \nu^{F^c}\}$. Moreover we have:

$$\nu^{E \setminus F} = \begin{cases} \nu^E & \text{in } \partial^* E \cap (F^c)^{(1)} \\ \nu^{F^c} = -\nu^F & \text{in } \partial^* F^c \cap E^{(1)} \\ \nu^E = -\nu^F & \text{in } J_{E, F^c} \end{cases}.$$

From this we find

$$\begin{aligned} \int_{E \setminus F} (\operatorname{div}_\gamma z) d\mu &= \int_{\partial^* E \cap F^{(0)}} \gamma d\mathcal{H}^m - \int_{\partial^* F \cap E^{(1)}} \nu^F \cdot z \gamma d\mathcal{H}^m + \int_{J_{E, F^c}} [z \cdot \nu^E] \gamma d\mathcal{H}^m \\ &\geq \int_{\partial^* E \cap F^{(0)}} \gamma d\mathcal{H}^m - \int_{\partial^* F \cap E^{(1)}} \gamma d\mathcal{H}^m + \int_{J_{E, F^c}} [z \cdot \nu^E] \gamma d\mathcal{H}^m. \end{aligned}$$

We thus find:

$$\int_{E \cap F} H dx dt - \int_E H dx dt \geq \int_{\partial^* E \cap F^{(0)}} \gamma d\mathcal{H}^m - \int_{\partial^* F \cap E^{(1)}} \gamma d\mathcal{H}^m + \int_{J_{E, F^c}} [z \cdot \nu^E] \gamma d\mathcal{H}^m.$$

Similarly, studying what happens on $F \setminus E$ we get:

$$\int_{E \cap F} H dx dt - \int_F H dx dt \leq \int_{\partial^* F \cap E^{(0)}} \gamma d\mathcal{H}^m - \int_{\partial^* E \cap F^{(1)}} \gamma d\mathcal{H}^m + \int_{J_{F, E^c}} [z \cdot \nu^F] \gamma d\mathcal{H}^m.$$

Summing these two inequalities and using that $\int_{J_{E,F^c}} [z \cdot \nu^E] \gamma d\mathcal{H}^m = \int_{J_{F,E^c}} [z \cdot \nu^F] \gamma d\mathcal{H}^m$ we have:

$$\begin{aligned} \int_{\partial^* F \cap (E^{(0)} \cup E^{(1)})} \gamma(x) d\mathcal{H}^m(x, t) + \int_F H(x, t) dx dt &\geq \\ \int_{\partial^* E \cap (F^{(0)} \cup F^{(1)})} \gamma(x) d\mathcal{H}^m(x, t) + \int_E H(x, t) dx dt. \end{aligned}$$

Adding to this equality $\int_{\partial^* E \cap \partial^* F} \gamma(x) d\mathcal{H}^m(x, t)$ and using that $\mathcal{H}^m((A^{(1)} \cup A^{(0)} \cup \partial^* A)^c) = 0$ for every set of finite perimeter $A \subset \mathbb{R}^{m+1}$, we find as desired that

$$\int_{\partial^* F} \gamma(x) d\mathcal{H}^m(x, t) + \int_F H(x, t) dx dt \geq \int_{\partial^* E} \gamma(x) d\mathcal{H}^m(x, t) + \int_E H(x, t) dx dt.$$

The regularity of ∂E follows from an old paper of Miranda [11, Teorema 2] which states that E is smooth in the neighborhood of the obstacle. Since u is smooth by Proposition 3.4, E is smooth everywhere. We point out that in the paper cited above, the results are written for the classical perimeter without curvature terms. However, the argument is based on a blow-up procedure under which our functional reduces to the classical perimeter. \square

We can now prove the convexity of U .

Proposition 3.7. *The set U is convex thus u is also convex.*

Proof. Let us define for every $z = (x, t) \in \overline{B}_R \times [-M; M]$ the vertical distance of z to U by

$$d^v(z, U) = \inf (|t - t'| / (x, t') \in U).$$

The function d^v is continuous since ∂U is a \mathcal{C}^1 surface by Proposition 3.6. U is a compact set thus the function

$$C(\lambda, z, z') = d^v(\lambda z + (1 - \lambda)z', U) \quad \text{for } (\lambda, z, z') \in [0; 1] \times U \times U$$

attains its maximum. If this maximum is zero then U is convex and we are done. Assume on the contrary that this maximum is positive.

By the vertical contact angle condition we can assume that this maximum is attained at points z and z' in the interior of $\overline{B}_R \times [-M; M]$. Moreover, if $z = (x, t) \in U$, by decreasing t (which increases C), we can assume that $t = u(x)$. Analogously we can assume that $z' = (x', u(x'))$. Then we find

$$C(\lambda, z, z') = u(\lambda x + (1 - \lambda)x') - \lambda u(x) - (1 - \lambda)u(x').$$

We are thus in the situation of applying Korevaar's concavity maximum principle [10, Theorem 1.3] to conclude. We briefly recall the argument for the reader's convenience. As (λ, z, z') is a point of maximum, the gradient in x and in x' is zero and thus

$$\nabla u(\lambda x + (1 - \lambda)x') = \nabla u(x) = \nabla u(x').$$

As the second derivative of $C(\lambda, (x + \tau, u(x + \tau)), (x' + \tau, u(x' + \tau)))$ is nonpositive in zero for every direction $\tau \in \mathbb{R}^m$ we get

$$D^2u(\lambda x + (1 - \lambda)x') - \lambda D^2u(x) - (1 - \lambda)D^2u(x') \leq 0. \quad (10)$$

Let $x'' = \lambda x + (1 - \lambda)x'$ then u satisfies at the points x , x' and x'' the equation

$$-Tr(D^2F(\nabla u)D^2u) + \nabla u \cdot x + u = g$$

where we remind that $F(p) = \sqrt{1 + |p|^2}$. Since F is a convex function, D^2F is a positive symmetric matrix and since $\nabla u(x) = \nabla u(x') = \nabla u(x'')$, using (10) we get

$$C(\lambda, x, x') = u(x'') - \lambda u(x') - (1 - \lambda)u(x'') \leq g(x'') - \lambda g(x') - (1 - \lambda)g(x'') \leq 0$$

which gives a contradiction. \square

We now finally turn to the proof of our main result:

Theorem 3.8. *Let $g \in L^2_\gamma(\mathbb{R}^m)$ be a convex function and u be the minimizer of*

$$\min_{BV_\gamma \cap L^2_\gamma(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_\gamma u| + \frac{1}{2} \int_{\mathbb{R}^m} |u - g|^2 d\gamma;$$

then u is a convex function.

Proof. By Proposition 3.3 we see that if u_R is the minimizer of (6) then it is convex. Arguing as in [5, Theorem 4.1], we see that u_R converges locally uniformly to u_ε , the minimizer of (5), when R goes to infinity. Analogously, we can let ε goes to zero and get that u_ε converges locally uniformly to u the solution of (4) which is thus convex. \square

Let us also notice that along the same lines we can prove the following result:

Theorem 3.9. *Let g be a convex $L^2_\gamma(\mathbb{R}^m)$ function; then the minimizer of*

$$\min_{u \in H^1_\gamma(\mathbb{R}^m)} \int_{\mathbb{R}^m} \frac{|\nabla u|^2}{2} d\gamma + \frac{1}{2} \int_{\mathbb{R}^m} |u - g|^2 d\gamma$$

is convex.

Proof. Let

$$J_\lambda(u) = \lambda^2 \int_{\mathbb{R}^m} \left[\sqrt{1 + \frac{|\nabla u|^2}{\lambda^2}} - 1 \right] d\gamma + \frac{1}{2} |u - g|^2 d\gamma;$$

then u_λ minimizes J_λ if and only if it minimizes

$$\int_{\mathbb{R}^m} \sqrt{\lambda^2 + |\nabla u|^2} d\gamma + \frac{1}{2\lambda} \int_{\mathbb{R}^m} |u - g|^2 d\gamma.$$

Thus u_λ is convex. Using that for any p ,

$$\lim_{\lambda \rightarrow \infty} \lambda^2 \left[\sqrt{1 + \frac{|p|^2}{\lambda^2}} - 1 \right] = \frac{|p|^2}{2};$$

we get the conclusion. □

Remark 3.10. When trying to extend the previous method for more general functionals, a difficulty arise due to the lack of boundary regularity of the minimizers. More precisely, when reasoning as in Proposition 3.6, these functionals give rise to anisotropic perimeters, for which it is not known if the minimizers of the corresponding obstacle problem are smooth in a neighborhood of the obstacle.

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