Notes on viscosity solution for partial differential equations

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Abstract

In this notes we describe the basic ideas and techniques used to prove existence and uniqueness results for viscosity solution of elliptic partial differential equations. The aim is to give a simple description of the basic ideas and motivations behind the definition of viscosity solution, thus the description is sometimes lacking in details and precision. The first part is devoted to the introduction of the definition of viscosity solution. In the second one we discuss the first order case, giving an example of application and a detailed description of the technique used to prove comparison results. The third, and last, part is devoted to the discussion of general stability results, the Perron's method to prove existence and some remarks on the proof of the comparison result for the second order case.

Introduction

These notes are the result of a series of seminars given at the Dublin City University. The aim was to describe the main ideas on viscosity solution for partial differential equations. We started with a general plan and we went on following the interest and the curiosity of the participants. In fact, some remarks or further explanations are just answers to precise questions. As a result these notes may lack in precise statements, complete and detailed proofs, and results "ready to use", but are more orientated to the description of some ideas and techniques used to prove existence and uniqueness results for viscosity solutions of partial differential equations. The main reference is "Viscosity solution: a primer" by M.G. Crandall (see [3]), from which everything on the second order case is taken, while for the first order case we basically refereed to [2].

These notes are divided in 3 parts.

In the first one we describe the equation we consider and we try to explain the basic motivations and ideas behind the definition of viscosity solution. The definition is given

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and, as main application and example, an optimal control problem (the infinite horizon discounted regulator problem) is described.

Section 2 is completely devoted to the first order case. First, we explicitly prove that the value function of the optimal control problem is a viscosity solution of a first order partial differential equations (Subsection 2.1). Then, in Subsection 2.2, we describe the technique used to prove the uniqueness result. This is done "step by step" and the assumptions needed are described "following the proof".

In Section 3 we start by discussing an equivalent definition of viscosity solution for both the first and second order case which is very useful for some proofs. Subsection 3.2 is devoted to the description of the stability results. Two very general results are given (Theorem 3.7 and Theorem 3.10), and, to explain the technique more clearly, the computation is explicitly exploited in the first order case (see Remark 3.12). In the last part we discuss a general existence and uniqueness result for a Dirichlet problem (Theorem 3.18). In Subsection 3.3 we describe the so-called Perron's method to obtain existence results and we see how one of the main ingredients is a comparison result which is discussed in Subsection 3.4.

As last remark I would like to say that these notes are strictly related to the arguments we discussed, of course they are lacking in more recent developments and in many aspects of the problem. For all this we refer to the bibliography and the references therein.

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1 The definition of viscosity solution.

Let Ω be an open subset of $\mathbb{R}^N$, we consider the following second order fully nonlinear partial differential equation,

$$F(x,u,Du,D^2u)=0 \quad x \in \Omega, \tag{1.1}$$

where the unknown is a function $u:\Omega\to\mathbb{R}$, $Du=(u_{x_1},u_{x_2},...,u_{x_N})$ is the gradient vector and $D^2u=(u_{x_i,x_j})$ is the Hessian matrix. Thus, $F:\Omega\times\mathbb{R}\times\mathbb{R}^N\times S(N)\to\mathbb{R}$ where $S(N)$ is the set of real symmetric $N\times N$ matrices. We called this equation fully non linear because we will not require linearity in any variable. The main assumptions on the function F will be the following.

Definition 1.1 *A function $F:\Omega\times\mathbb{R}\times\mathbb{R}^N\times S(N)\to\mathbb{R}$ is* degenerate elliptic *if $F(x,r,p,X)\le F(x,r,p,Y)$ for each $X,Y\in S(N)$ such that $Y\le X$, (i.e. $Y\eta\cdot\eta\le X\eta\cdot\eta$ for all $\eta\in\mathbb{R}^N$).*

Definition 1.2 *A function $F:\Omega\times\mathbb{R}\times\mathbb{R}^N\times S(N)\to\mathbb{R}$ is* proper *if $F(x,r,p,X)\le F(x,s,p,Y)$ for each $X,Y\in S(N)$ such that $Y\le X$, and for each $r,s\in\mathbb{R}$ such that $r\le s$.*

Let us see now some basic examples.

Example 1: Hamilton-Jacobi equations.
The function F might be first order, i.e. $F(x,r,p,X) = H(x,r,p)$. This is a (very) degenerate elliptic function and it is proper if it is non decreasing w.r.t. r. The equation

$$H(x,u,Du) = 0 \quad x \in \Omega,$$

is usually called Hamilton-Jacobi equation. We will refer to this as the *first order case*, and, although they are subcases of the second order case, we will discuss separately some existence and uniqueness results.
Observe that the classical Burger's equation (i.e. $u_t + u\, u_x = 0$) is not nondecreasing w.r.t. u, so we will not consider it.

Example 2: The linear case.
Of course we do not want to rule out the linear case. Indeed, $F(x,r,p,X) = -Trace(X)$ is a degenerate elliptic function which gives us the classical Laplace equation $-\Delta u(x) = 0$. More generally, if we consider

$$F(x,r,p,X) = -Trace(A(x)X) + b(x)\cdot p + c(x)r - f(x),$$

where the matrix $A(x) = (a_{ij}(x))$ is symmetric and $A(x) \geq 0$, b is a vector in $\mathbb{R}^N$, $c(x) \geq 0$ and $f(x)$ is a given function, we will obtain the linear equation,

$$-\sum_{i,j=1}^{N} a_{ij}(x)u_{x_i,x_j}(x) + \sum_{i=1}^{N} b_i(x)u_{x_i}(x) + c(x)u(x) - f(x) = 0 \quad x \in \Omega.$$

Remark 1.3 We remark that the class of proper functions is very rich, in fact if F and G are proper, so is $\mu F + \lambda G$ for $\mu, \lambda \geq 0$. Moreover if A and B are two index sets, and $F_{\alpha,\beta}$ is proper for $\alpha \in A$ and $\beta \in B$, so is $F = \sup_{\alpha\in A}\inf_{\beta\in B} F_{\alpha,\beta}$.

1.1 On the need for non smooth solution.

In what follows we will say that u is a *classical solution* of (1.1) if it is twice differentiable and satisfies the equation pointwise. In order to explain one of the reason why people looked for nonsmooth solution we will describe now an optimal control problem. For all you want to know about optimal control and viscosity solution of Hamilton-Jacobi equations, we refer to [2] and the references therein.

The infinite horizon discounted regulator problem.
We consider a control system governed by the *state equation*

$$\begin{cases} y'(t) = f(y(t), \alpha(t)) & t > 0 \\ y(0) = x \end{cases} \tag{1.2}$$

where α is the *control*. More precisely, A, the *control space*, is a closed bounded subsbset of $\mathbb{R}^M$ and α is a measurable function of $t \in [0, +\infty[$ with values in A, (we will denote this set of functions with $\mathcal{A}$). We assume that for any choice of the control α and of the initial position x the *dynamic* $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ is such that there exists a unique solution of (1.2), $y_x(t, \alpha)$. We will consider the following *cost functional*

$$J(x, \alpha) = \int_0^\infty l(y_x(t, \alpha), \alpha(t))e^{-\lambda t} dt \tag{1.3}$$

where $l : \mathbb{R}^N \times A \to \mathbb{R}$ is the *running cost*, and $\lambda > 0$ is a *discount factor*.

We want to minimize the cost functional over the control $\alpha \in \mathcal{A}$. To do that we begin defining the *value function*

$$v(x) = \inf_{\alpha \in \mathcal{A}} \{J(x, \alpha)\}. \tag{1.4}$$

The aim of the theory is to define a partial differential equation solved by the value function. To do so, one first prove the *Dynamic Programming Principle*: for each time T

$$v(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^T l(y_x(t, \alpha), \alpha(t))e^{-\lambda t} dt + v(y_x(T, \alpha))e^{-\lambda T} \right\}. \tag{1.5}$$

(For a proof see, for instance, Proposition 2.5 in [2].) This principle express the intuitive remark that the minimum cost is achieved if one behaves as follows.
(a) Let the system evolve for a small amount of time choosing an arbitrary control $\alpha(\cdot)$ on the interval $[0, T]$. ($\Rightarrow y_x(t, \alpha)$, for $t \in [0, T]$).
(b) Pay the corresponding cost ($\Rightarrow \int_0^T l(y_x(t, \alpha), \alpha(t))e^{-\lambda t} dt$).
(c) Pay what remains to pay after time T with the best possible control
($\Rightarrow v(y_x(T, \alpha))e^{-\lambda T}$).
(d) Minimize the sum of these two costs over all the possible controls ($\Rightarrow$ (1.5)).

We want to derive now an infinitesimal version of the Dynamic Programming Principle, the so called *Hamilton-Jacobi* equation. We will assume v regular, (at least $C^1(\mathbb{R}^N)$). Take $\alpha(\cdot) = \alpha$, by (1.5) we have for each $T > 0$,

$$v(x) \le \int_0^T l(y_x(t, \alpha), \alpha)e^{-\lambda t} dt + v(y_x(T, \alpha))e^{-\lambda T}.$$

Thus,

$$\frac{v(x)(1 - e^{-\lambda T})}{T} + \frac{v(x) - v(y_x(T, \alpha))}{T} e^{-\lambda T} \le \frac{1}{T} \int_0^T l(y_x(t, \alpha), \alpha)e^{-\lambda t} dt,$$

which, letting $T \to 0$ and recalling (1.2), leads us to

$$\lambda v(x) - Dv(x) \cdot f(x, \alpha) - l(x, \alpha) \le 0$$

for all $\alpha \in \mathbb{R}^M$. Taking the supremum over $\alpha \in A$ we obtain

$$\lambda v(x) + \sup_{\alpha \in A} \{-Dv(x) \cdot f(x, \alpha) - l(x, \alpha)\} \le 0.$$

A similar calculation gives us the reverse inequality, so we can conclude that *if v is regular*, v solves in the classical sense the following Hamilton-Jacobi equation

$$\lambda v(x) + \sup_{\alpha \in A}\{-Dv(x) \cdot f(x,\alpha) - l(x,\alpha)\} = 0, \quad x \in \mathbb{R}^N. \tag{1.6}$$

We are going to show now by a simple example that the assumption v regular is too restrictive. Consider $N = 1$, $A = \{-1, 1\}$, $f(x,\alpha) = \alpha$ and $l(x,\alpha) = l(x)$ with $l(x)$ an even smooth function such that $l \equiv 0$ for $|x| > R$, $\max_{x \in \mathbb{R}} l(x) = l(0) > 0$ and $xl'(x) < 0$ for $|x| < R$.
A calculation shows that

$$v(x) = \begin{cases} \int_0^\infty l(x-t)e^{-\lambda t}dt & x < 0 \\ \int_0^\infty l(-t)e^{-\lambda t}dt = \int_0^\infty l(t)e^{-\lambda t}dt & x = 0 \\ \int_0^\infty l(x+t)e^{-\lambda t}dt & x > 0, \end{cases}$$

thus

$$v'_+(0) := \lim_{x \to 0^+} \frac{v(x) - v(0)}{x} = \int_0^\infty l'(t)e^{-\lambda t}dt,$$

and

$$v'_-(0) := \lim_{x \to 0^-} \frac{v(x) - v(0)}{x} = \int_0^\infty l'(-t)e^{-\lambda t}dt.$$

Since $l'(-x) = -l'(x)$ then $v'_+(0) \neq v'_-(0)$. Even if the dynamic and the running cost are regular the value function v is not differentiable at $x = 0$!
Remark also that the Hamilton Jacobi equation for this example is

$$\lambda v(x) + |v'(x)| - l(x) = 0$$

and has not classical meaning for $x = 0$ since v is not differentiable.

We are going to discuss now different definitions one can give for the solution of the equation (1.1) and introduce the concept of viscosity solution. In Section 2.1 we will prove that the value function (1.4) is in fact a viscosity solution of (1.6).

1.2 From generalized solution to viscosity solution.

To overcame the problem of defining a solution not regular for the equation (1.1) Kruzkow introduced in the '60 the idea of *generalized solutions*, i.e. solutions which satisfies the equation almost everywhere. This is a powerful idea and a lot of result have been obtained under different set of hypotheses. (For a complete description see [6] and references therein.) We want to see now, thought a simple example why a different definition has been introduced.
Consider the equation $|u'| = 1$ in $(-1, 1)$ with boundary conditions $u(-1) = u(1) = 0$. (Take $N = 1$, $\Omega = (-1, 1)$, $F(x, r, p, X) = |p| - 1$ in the general framework.) Clearly there are not classical solutions, but the function $u(x) = 1 - |x|$ is a generalized solution. The point is that one can built infinitely many generalized solutions (it is enough to alternate

segments with slope 1 to segments with slope -1), moreover one can construct a sequence of generalized solutions which converge to $u \equiv 0$ that is not a generalized solution. From an application point of view this *luck of uniqueness and stability* is an important problem. This also pushed to the definition of viscosity solution we are going to introduce now.

First consider the following formulation of the **Maximum principle**.

Theorem 1.4 *Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ be proper. A function $u \in C^2(\Omega)$ is a classical solution of*

$$F(x, u(x), Du(x), D^2u(x)) = 0 \quad x \in \Omega, \tag{1.7}$$

if and only if

(a) for all $\phi \in C^2(\Omega)$, if x_0 is a local maximum of $u - \phi$ then

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0, \tag{1.8}$$

(b) for all $\phi \in C^2(\Omega)$, if x_0 is a local minimum of $u - \phi$ then

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0. \tag{1.9}$$

Proof. If $u \in C^2(\Omega)$ is a classical solution of (1.7), $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ is a local maximum of $u - \phi$ then $Du(x_0) = D\phi(x_0)$ and $D^2u(x_0) \leq D^2\phi(x_0)$. Since F is proper we have

$$0 = F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \geq F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0))$$

which is (1.8). One obtains (1.9) in the same way.
Conversely, if u is $C^2(\Omega)$ we can take $\phi = u$ in (a) and (b) so that each $x_0 \in \Omega$ is both a local maximum and a local minimum of $u - \phi \equiv 0$. Thus (1.8) and (1.9) hold for each $x_0 \in \Omega$. We can conclude that $F(x_0, u(x_0), Du(x_0), D^2u(x_0)) = 0$ for all $x_0 \in \Omega$, thus u is a classical solution of (1.7). □

Observe that in (a) and (b) there is no need to ask regularity for the solution u (the derivation is performed on ϕ!). In fact the basic idea in the definition of viscosity solution is to use (a) and (b) to *define* the solution and then to study the existence, uniqueness, stability and regularity of the solutions so defined.

More precisely the definition of viscosity solution for

$$F(x, u(x), Du(x), D^2u(x)) = 0 \quad x \in \Omega, \tag{1.10}$$

is the following.

Definition 1.5 *Let Ω be an open subset of $\mathbb{R}^N$, F be proper and $u : \Omega \to \mathbb{R}$.*

(a) u is a viscosity subsolution *of (1.10) in Ω if it is upper semicontinuous and for each $\phi \in C^2(\Omega)$ and local maximum point $x_0 \in \Omega$ of $u - \phi$ we have*

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0. \tag{1.11}$$

(b) u is a viscosity supersolution *of (1.10) in* Ω *if it is lower semicontinuous and for each* $\phi \in C^2(\Omega)$ *and local minimum point* $x_0 \in \Omega$ *of* $u - \phi$ *we have*

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0. \tag{1.12}$$

u is a viscosity solution *of (1.10) in* Ω *if it is both viscosity subsolution and supersolution (hence continuous) of (1.10).*

This definition was first introduced in 1981 by M.G. Crandall and P.L.Lions. The first two papers in which appeared are [4] and [5].

Remark 1.6 We remark that, thanks to the Maximum principle, if u is a classical solution then is a viscosity solution.

We end this paragraph by showing that if we consider again the equation $|u'| = 1$ in $(-1, 1)$ with boundary conditions $u(-1) = u(1) = 0$, we can prove that $u(x) = 1 - |x|$ is a viscosity solution. Let $\phi \in C^1$, and x_0 be a local maximum of $1 - |x| - \phi(x)$. We can always assume $1 - |x_0| - \phi(x_0) = 0$, thus, we have $\phi(x) - \phi(x_0) \geq |x| - |x_0|$ for all $x \in (-1, 1)$. By calculation one obtain the following, if $x_0 > 0$ then $\phi'(x_0) = -1$, if $x_0 < 0$ then $\phi'(x_0) = 1$, and $-1 \leq \phi'(0) \leq 1$, so in any cases, $|\phi'(x_0)| \leq 1$ which is, u is a viscosity subsolution. Similarly one can prove that u is a viscosity supersolution, and thus a viscosity solution. In Section 2.2 we will see that u is also unique.

1.3 Further remarks on the origin of the definition.

We consider now a *stochastic optimal control problem.* (Compare with the infinite horizon discounted regulator problem.) The state equation is the following stochastic differential equation

$$\begin{cases} dy = f(y(t), \alpha(t))dt + \sqrt{2\varepsilon}\, dw(t) \\ y(0) = x \end{cases} \tag{1.13}$$

where $\varepsilon > 0$ and w is the N-dimensional standard Browinian motion. The value function will be

$$v^\varepsilon(x) = \inf_\alpha \; E_x \left(\int_0^\infty l(y_x^\varepsilon(t, \alpha), \alpha(t)) e^{-\lambda t} dt \right) \tag{1.14}$$

where E_x denotes the expectation and the infimum is taken on the class of progressive measurable functions α with values on A. Under suitable condition on the data v^ε happens to be a smooth (at least C^2) solution of

$$-\varepsilon \Delta v^\varepsilon(x) + \lambda v^\varepsilon(x) + \sup_{\alpha \in A} \{-Dv^\varepsilon(x) \cdot f(x, \alpha) - l(x, \alpha)\} = 0, \quad x \in \mathbb{R}^N. \tag{1.15}$$

A natural question arises: if $\varepsilon \to 0$ does v^ε tends to a function v solution (in some sense) of the limit equation

$$\lambda v(x) + \sup_{\alpha \in A} \{-Dv(x) \cdot f(x, \alpha) - l(x, \alpha)\} = 0, \quad x \in \mathbb{R}^N. \tag{1.16}$$

The question is not so easy because the regularizing effect of the term $\varepsilon\Delta u$ vanishes as $\varepsilon \to 0$ and we end up with an equation that we have seen easily having non regular solutions. Of course the answer is that v is a viscosity solution of (1.16). This is very important because can be thought as a way to define "weak" solution of the limit equation, and is actually the motivation for the terminology "viscosity solution" used in the original paper of M.G. Crandall and P.L.Lions. (See [4].)

We conclude this section roughly giving the idea of the proof (for subsolution). Assume $v^\varepsilon \in C^2(\mathbb{R}^N)$ and v^ε converging to a continuous function v locally uniformly as ε tends to 0. Let $\phi \in C^2(\mathbb{R}^N)$ and x_0 be a local maximum of $v - \phi$. By uniform convergence $v^\varepsilon - \phi$ attains a local maximum at some point x_0^ε and $x_0^\varepsilon \to x_0$ as ε tends to 0. So by elementary calculus, $D(v_\varepsilon - \phi)(x_0^\varepsilon) = 0$ and $-\Delta(v^\varepsilon - \phi)(x_0^\varepsilon) \geq 0$. Moreover, v^ε is a classical solution, so

$$
\begin{aligned}
0 = -\varepsilon\Delta v^\varepsilon(x_0^\varepsilon) + \lambda v^\varepsilon(x_0^\varepsilon) + \sup_{\alpha\in A}\{-Dv^\varepsilon(x_0^\varepsilon)\cdot f(x_0^\varepsilon,\alpha) - l(x_0^\varepsilon,\alpha)\} \geq \\
\geq -\varepsilon\Delta v^\varepsilon(x_0^\varepsilon) + \lambda v^\varepsilon(x_0^\varepsilon) - Dv^\varepsilon(x_0^\varepsilon)\cdot f(x_0^\varepsilon,\alpha) - l(x_0^\varepsilon,\alpha) \geq \\
\geq -\varepsilon\Delta \phi(x_0^\varepsilon) + \lambda v^\varepsilon(x_0^\varepsilon) - D\phi(x_0^\varepsilon)\cdot f(x_0^\varepsilon,\alpha) - l(x_0^\varepsilon,\alpha).
\end{aligned}
$$

Assuming enough regularity on the data f and l we can let $\varepsilon \to 0$ in the last inequality and obtain, for each $\alpha \in A$

$$
-0\Delta\phi(x_0) + \lambda v(x_0) - D\phi(x_0)\cdot f(x_0,\alpha) - l(x_0,\alpha) \leq 0,
$$

so

$$
\lambda v(x_0) + \sup_{\alpha\in A}\{-D\phi(x_0)\cdot f(x_0,\alpha) - l(x_0,\alpha)\} \leq 0,
$$

that is, by definition, v is a viscosity subsolution of (1.16).

2 The first order case.

2.1 An existence result.

When one deals with a first order equation the basic way to prove existence of viscosity solution is to consider v as the value function of an optimal control problem, and to prove directly that is a viscosity solution of the Hamilton-Jacobi equation. Of course this works when one knows the optimal control problem related to the equation he is considering, otherwise we will see in Section 3.3 a general technique to prove existence result for the second order case, so, as a particular case, for the first order.

We consider again the infinite horizon discounted regulator problem. We recall that by definition the value function is

$$
v(x) = \inf_{\alpha\in\mathcal{A}}\left\{\int_0^\infty l(y_x(t,\alpha),\alpha(t))e^{-\lambda t}dt\right\} \tag{2.1}
$$

and that the following Dynamic Programming Principle holds: for each time $T > 0$

$$
v(x) = \inf_{\alpha\in\mathcal{A}}\left\{\int_0^T l(y_x(t,\alpha),\alpha(t))e^{-\lambda t}dt + v(y_x(T,\alpha))e^{-\lambda T}\right\}. \tag{2.2}
$$

We are going to prove now that v is a viscosity subsolution of the Hamilton-Jacobi equation

$$\lambda v(x) + \sup_{\alpha \in A}\{-Dv(x) \cdot f(x,\alpha) - l(x,\alpha)\} = 0, \quad x \in \mathbb{R}^N. \tag{2.3}$$

(For a complete proof see Proposition 2.8, Chapter III, in [2].) We assume:

(Ha) A is a compact subset of $\mathbb{R}^M$.

(Hf) f is continuous and $x \to f(x,\alpha)$ is Lipscitz continuous uniformly w.r.t. $\alpha \in A$, i.e. there exists $C > 0$ such that

$$|f(x,\alpha) - f(y,\alpha)| \leq C|x-y| \quad \forall x, y \in \mathbb{R}^N, \forall \alpha \in A.$$

(Hl) l is continuous with a modulus of continuity $\omega_l : [0, +\infty[\to [0, +\infty[$ independent of $\alpha \in A$, i.e.

$$|l(x,\alpha) - l(y,\alpha)| \leq \omega_l(|x-y|) \quad \forall x, y \in \mathbb{R}^N, \forall \alpha \in A.$$

We first observe that under these assumptions one can prove that the value function v is continuous in $\mathbb{R}^N$(for a proof see Proposition 2.1, Chapter III, in [2]).

Let $\phi \in C^1(\mathbb{R}^N)$ and x be a local maximum point of $v - \phi$, there exists then an $r > 0$ such that $v(x) - \phi(x) \geq v(z) - \phi(z)$ for all $z \in B(x,r)$. Fix $\alpha \in A$, for regularity of the state equation (1.2), there exists a time T_0 such that $y_x(t,\alpha) \in B(x,r)$, for all $t \leq T_0$. Thus

$$v(x) - v(y_x(t,\alpha)) \geq \phi(x) - \phi(y_x(t,\alpha)) \quad \forall t \leq T_0.$$

Now, applying also the Dynamic Programming Principle (with $T = t$) we have

$$\phi(x) - \phi(y_x(t,\alpha)) \leq v(x) - v(y_x(t,\alpha)) \leq \int_0^t l(y_x(s,\alpha),\alpha)e^{-\lambda s}ds + v(y_x(t,\alpha))(e^{-\lambda t} - 1).$$

So,

$$\frac{\phi(x) - \phi(y_x(t,\alpha))}{t} \leq \frac{1}{t}\int_0^t l(y_x(s,\alpha),\alpha)e^{-\lambda s}ds + \frac{v(y_x(t,\alpha))(e^{-\lambda t} - 1)}{t}.$$

Letting $t \to 0$ we obtain (also thanks to (1.2), assumptions (Hl), (Hf) and the continuity of v)

$$-D\phi(x) \cdot f(x,\alpha) \leq l(x,\alpha) - \lambda v(x),$$

and finally

$$\lambda v(x) + \sup_{\alpha \in A}\{-D\phi(x) \cdot f(x,\alpha) - l(x,\alpha)\} \leq 0$$

that is, v is a subsolution of (2.3).

Remark 2.1 Observe how this proof is similar to the one we performed in Subsection 1.1 to derive the Hamilton-Jacobi equation. There we assumed v regular and we calculated her derivatives, while now we did not to assume more then mere continuity on v and, thanks to the definition of viscosity solution, we could perform the derivatives on the test function ϕ.

2.2 The technique for the uniqueness result.

In this section we want to give the main ideas of the technique used to prove uniqueness result for the following Diriclet problem.

$$\begin{cases} H(x,u,Du) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We will assume H continuous and proper on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, $g : \partial\Omega \to \mathbb{R}^N$ continuous and Ω bounded. (Of course more general result can be obtained, see for example [1], [2] and references therein.) We first give the definition of viscosity solution for problem (2.4).

Definition 2.2

(a) u is a viscosity subsolution *of (1.10) in $\bar{\Omega}$ if : it is upper semicontinuous, for each $\phi \in C^1(\Omega)$ and local maximum point $x_0 \in \Omega$ of $u - \phi$ we have*

$$H(x_0, u(x_0), D\phi(x_0)) \leq 0 \tag{2.5}$$

and satisfies $u \leq g$ on $\partial\Omega$.

(b) u is a viscosity supersolution *of (1.10) in $\bar{\Omega}$ if : it is lower semicontinuous, for each $\phi \in C^1(\Omega)$ and local minimum point $x_0 \in \Omega$ of $u - \phi$ we have*

$$H(x_0, u(x_0), D\phi(x_0)) \geq 0 \tag{2.6}$$

and satisfies $u \geq g$ on $\partial\Omega$.

u is a viscosity solution *of (1.10) in $\bar{\Omega}$ if it is both viscosity subsolution and supersolution (hence continuous) of (1.10).*

Remark 2.3 Remark that in general without further condition on H a uniqueness result does not hold. Indeed, consider Ω as the unit ball, fix $w \in C^1(\bar{\Omega})$ any function which vanishes on $\partial\Omega$ but does not vanishes identically, take $H(x,u,p) = |p|^2 - |Dw(x)|^2$, and $g \equiv 0$. Then, w and $-w$ are both classical solution (hence viscosity solution) but $w \neq -w$.

We have a uniqueness result once we have a *comparison result, i.e. if u is a subsolution of (2.4) and v is a supersolution of (2.4), then $u \leq v$.* In fact, suppose that u_1 and u_2 are two viscosity solution of (2.4), they are both viscosity subsolution and supersolution, so by the comparison result we have $u_1 \leq u_2$ and $u_2 \leq u_1$, thus $u_1 = u_2$.

We are going to discuss now the idea of the proof of a comparison result and the hypotheses we have to ask on the function H at the same time.

We assume

(HC) u is a viscosity subsolution of (2.4) and v is a viscosity supersolution of (2.4)

and we want to prove that $u(x) \leq v(x)$ for all $x \in \Omega$.

The idea of the proof is the following.

Let $\hat{x}$ be an interior maximum of $u(x)-v(x)$, and suppose that u and v are smooth at point $\hat{x}$. Since u and v are respectively a sub and a supersolution we have

$$H(\hat{x},u(\hat{x}),Du(\hat{x}))\leq 0 \quad \text{and} \quad H(\hat{x},v(\hat{x}),Dv(\hat{x}))\geq 0,$$

thus

$$H(\hat{x},u(\hat{x}),Du(\hat{x}))-H(\hat{x},v(\hat{x}),Dv(\hat{x}))\leq 0.$$

If, by the assumption one has on H, this implies $u(\hat{x})-v(\hat{x})\leq 0$, the proof is concluded. In fact, since $\hat{x}$ is a maximum, we have $u(x)-v(x)\leq u(\hat{x})-v(\hat{x})\leq 0$ thus $u(x)\leq v(x)$ for all $x\in\Omega$.

The first problem we have to overcame is the lack of regularity of u and v at point $\hat{x}$. The idea is to "double the variable". More precisely, we choose a test function $\varphi:\Omega\times\Omega\to\mathbb{R}$ for which $u(x)-v(y)-\varphi(x,y)$ has a maximum at point $(\hat{x},\hat{y})\in\Omega\times\Omega$. Thus, $\hat{x}$ is a maximum of $x\to u(x)-\varphi(x,\hat{y})$, and $\hat{y}$ is a minimum of $y\to v(y)-(-\varphi(\hat{x},y))$. We apply now the definition of viscosity subsolution and viscosity supersolution respectively, and we get

$$H(\hat{x},u(\hat{x}),D_x\varphi(\hat{x},\hat{y}))\leq 0 \quad \text{and} \quad H(\hat{y},v(\hat{y}),-D_y\varphi(\hat{x},\hat{y}))\geq 0,$$

where $D_x\varphi$ denotes the vector of the first N partial derivatives of $\varphi(x,y)$, and $D_y\varphi$ the vector of the remaining N, i.e. $D\varphi=(\varphi_{x_1},.....\varphi_{x_N},\varphi_{y_1},.....\varphi_{y_N})=(D_x\varphi,D_y\varphi)$. Thus

$$H(\hat{x},u(\hat{x}),D_x\varphi(\hat{x},\hat{y}))-H(\hat{y},v(\hat{y}),-D_y\varphi(\hat{x},\hat{y}))\leq 0. \tag{2.7}$$

The main idea to conclude is to choose the test function φ is such a way that $D_x\varphi(\hat{x},\hat{y})=-D_y\varphi(\hat{x},\hat{y})$, and to use the hypotheses on H.

To be more precise we need the following Lemma. (For the proof see Lemma 4.1, pag.11 in [3]).

Lemma 2.4 *Suppose $O\subset\mathbb{R}^N$, and $w,\Psi:O\to\mathbb{R}$ be such that $\Psi\geq 0$, $-\Psi$ and w are upper semicontinuous. Let $N=\{z\in O:\Psi(z)=0\}\neq\emptyset$, $\sup_{z\in O}\{w(z)-\Psi(z)\}<+\infty$, and define for $\varepsilon\leq 1$, $M_\varepsilon:=\sup_{z\in O}\{w(z)-\dfrac{\Psi(z)}{\varepsilon}\}$. If $z_\varepsilon\in O$ is such that*

$$\lim_{\varepsilon\to 0}\left(M_\varepsilon-w(z_\varepsilon)-\frac{\Psi(z_\varepsilon)}{\varepsilon}\right)=0$$

then

$$\lim_{\varepsilon\to 0}\frac{\Psi(z_\varepsilon)}{\varepsilon}=0.$$

Moreover, if $\hat{z}$ is a cluster point of z_ε as $\varepsilon\to 0$ then $\hat{z}\in N$ and $w(z)\leq w(\hat{z})$ for all $z\in N$.

Fix $\varepsilon > 0$, we consider the test function $\varphi(x,y) = \frac{1}{2\varepsilon}|x-y|^2$. (Observe that this is a good choice because $D_x\varphi(x,y) = -D_y\varphi(x,y) = \frac{1}{\varepsilon}(x-y)$). We set

$$\Phi(x,y) := u(x) - v(y) - \frac{1}{2\varepsilon}|x-y|^2 \quad \text{in } \bar{\Omega}\times\bar{\Omega}. \tag{2.8}$$

Since Φ is upper semicontinuous on $\bar{\Omega}\times\bar{\Omega}$, which is compact, there exists a maximum point $(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \in \bar{\Omega}\times\bar{\Omega}$, moreover, there exists a point $(\hat{x},\hat{y}) \in \bar{\Omega}\times\bar{\Omega}$ such that $\lim_{\varepsilon\to 0}(\hat{x}_\varepsilon, \hat{y}_\varepsilon) = (\hat{x},\hat{y})$. We now apply Lemma 2.4 with $O = \Omega\times\Omega$, $w(x,y) = u(x)-v(y)$ (is upper semicontinuous), $\Psi(x,y) = \frac{1}{2}|x-y|^2$ (is continuous). Thus

$$N = \{(x,y) \in \Omega\times\Omega : \frac{|x-y|}{2} = 0\} \neq \emptyset \ , \sup_{(x,y)\in\Omega\times\Omega}\{u(x) - v(y) - \frac{|x-y|^2}{2}\} < +\infty,$$

and taking $z_\varepsilon = (\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ we have $M_\varepsilon - w(z_\varepsilon) - \frac{\Psi(z_\varepsilon)}{\varepsilon} = 0$ for every $\varepsilon \leq 1$. Thus, Lemma 2.4 applies and we get $\lim_{\varepsilon\to 0}\dfrac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{2\varepsilon} = 0$. Moreover, since $\lim_{\varepsilon\to 0}(\hat{x}_\varepsilon, \hat{y}_\varepsilon) = (\hat{x},\hat{y})$, we have $(\hat{x},\hat{y}) \in N$, so $(\hat{x},\hat{y}) = (\hat{x},\hat{x})$.
Summarizing, if $(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \in \bar{\Omega}\times\bar{\Omega}$ is a maximum of Φ, (given in (2.8)) we have

$$\lim_{\varepsilon\to 0}\frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{2\varepsilon} = 0 \quad \text{and } \lim_{\varepsilon\to 0}(\hat{x}_\varepsilon, \hat{y}_\varepsilon) = (\hat{x},\hat{x}). \tag{2.9}$$

Observe now that by definition

$$u(x) - v(x) \leq \Phi(x,x) \leq \Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \quad \forall x \in \Omega.$$

So, if we prove

$$\limsup_{\varepsilon\to 0}\Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) = 0 \tag{2.10}$$

we have $u(x) - v(x) \leq 0$ for each $x \in \Omega$ which is our conclusion.

We have to consider two cases.

(1) $\hat{x} \in \partial\Omega$.

If $\hat{x} \in \partial\Omega$, by assumption (HC), and Definition 2.2, we have $u(\hat{x}) \leq g(\hat{x}) \leq v(\hat{x})$, so $u(\hat{x}) \leq v(\hat{x})$. Moreover, by (2.8), the upper semicontinuity of $u - v$, and (2.9), we have

$$\limsup_{\varepsilon\to 0}\Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq \limsup_{\varepsilon\to 0}\,(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)) \leq u(\hat{x}) - v(\hat{x}).$$

So,

$$\limsup_{\varepsilon\to 0}\Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq u(\hat{x}) - v(\hat{x}) \leq 0,$$

which is (2.10) and ends the proof.

(2) $\hat{x} \notin \partial\Omega$.

If $\hat{x} \notin \partial\Omega$, $(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ must lie in Ω for small ε. So $(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ is an interior maximum of $\Phi(x, y)$ and we can work as we have done to obtain (2.7) and get

$$H\left(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) - H\left(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) \leq 0. \tag{2.11}$$

We are going to discuss now how we can deduce (2.10) from (2.11) under different set of hypotheses on H.

(H1) $H(x, r, p) = r + G(p) - f(x)$, with f a continuous function on $\bar{\Omega}$.

Is H has this form, (2.11) becomes

$$u(\hat{x}_\varepsilon) - f(\hat{x}_\varepsilon) \leq v(\hat{y}_\varepsilon) - f(\hat{y}_\varepsilon).$$

Thus, also by (2.8),

$$\limsup_{\varepsilon \to 0} \Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq \limsup_{\varepsilon \to 0} \left(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)\right) \leq \limsup_{\varepsilon \to 0} \left(f(\hat{x}_\varepsilon) - f(\hat{y}_\varepsilon)\right) = 0.$$

So, (2.10) is fulfilled and the proof completed.

(H2) $H(x, r, p) = G(r, p) - f(x)$. Moreover, there exists $\gamma > 0$ such that

$$\gamma(r - s) \leq G(r, p) - G(s, p) \quad \forall r, s \in \mathbb{R}, \forall p \in \mathbb{R}^N,$$

and f is a continuous function on $\bar{\Omega}$.

Estimate (2.11) becomes

$$G\left(u(\hat{x}_\varepsilon), \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) - f(\hat{x}_\varepsilon) - G\left(v(\hat{y}_\varepsilon), \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) + f(\hat{y}_\varepsilon) \leq 0.$$

So, by assumption **(H2)**,

$$\gamma\left(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)\right) \leq f(\hat{x}_\varepsilon) - f(\hat{y}_\varepsilon),$$

then, also thanks to (2.8),

$$\limsup_{\varepsilon \to 0} \Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq \limsup_{\varepsilon \to 0} \left(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)\right) \leq \frac{1}{\gamma} \limsup_{\varepsilon \to 0} \left(f(\hat{x}_\varepsilon) - f(\hat{y}_\varepsilon)\right) = 0,$$

(2.10) is fulfilled and the proof completed.

(H3) $H(x, r, p) = \lambda r + \mathcal{H}(x, p)$, where $\lambda > 0$. Moreover, there exists $\omega : \mathbb{R} \to [0, +\infty[$ a continuous function such that $\lim_{r \to 0^+} \omega(r) = 0$, for which

$$|\mathcal{H}(x, p) - \mathcal{H}(y, p)| \leq \omega(|x - y|(1 + |p|)) \quad \forall x, y \in \Omega, \quad \forall p \in \mathbb{R}^N.$$

Estimate (2.11) becomes

$$\lambda u(\hat{x}_\varepsilon) + \mathcal{H}\left(\hat{x}_\varepsilon, \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) - \lambda v(\hat{y}_\varepsilon) - \mathcal{H}\left(\hat{y}_\varepsilon, \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) \leq 0.$$

So, by assumption **(H3)**,

$$\lambda\left(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)\right) \leq \mathcal{H}\left(\hat{y}_\varepsilon, \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) - \mathcal{H}\left(\hat{x}_\varepsilon, \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2}{\varepsilon}\right) \leq$$
$$\leq \omega\left(|\hat{x}_\varepsilon - \hat{y}_\varepsilon|\left(1 + \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|}{\varepsilon}\right)\right).$$

Then

$$\limsup_{\varepsilon\to 0} \Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq \limsup_{\varepsilon\to 0}\left(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)\right) \leq \frac{1}{\lambda}\limsup_{\varepsilon\to 0} \omega\left(|\hat{x}_\varepsilon - \hat{y}_\varepsilon|\left(1 + \frac{|\hat{x}_\varepsilon - \hat{y}_\varepsilon|}{\varepsilon}\right)\right) = 0,$$

thanks to **(H3)** and (2.9). Thus (2.10) is fulfilled and the proof completed.

We end this section by considering again the Hamilton-Jacobi equation related to the infinite horizon optimal control problem (2.3), i.e.

$$\lambda v(x) + \sup_{\alpha\in A}\{-Dv(x)\cdot f(x,\alpha) - l(x,\alpha)\} = 0, \quad x \in \mathbb{R}^N.$$

If we set $\mathcal{H}(x,p) := \sup_{\alpha\in A}\{p\cdot f(x,\alpha) - l(x,\alpha)\}$ and we assume (Ha), (Hf) and (Hl) we prove now that (H3) is fulfilled. This remark and the existence result we have proved in the previous section gives us an existence and uniqueness result for viscosity solution of the Hamilton-Jacobi equation (2.3).

Since A is compact and f and l are continuous, there exist $\alpha_x \in A$ and $\alpha_y \in A$ such that $\mathcal{H}(x,p) = -p\cdot f(x,\alpha_x) - l(x,\alpha_x)$ and $\mathcal{H}(y,p) = -p\cdot f(y,\alpha_y) - l(y,\alpha_y)$, respectively. Moreover, by definition,

$$-\mathcal{H}(x,p) \leq p\cdot f(x,\alpha) + l(x,\alpha) \quad \forall\alpha\in A,$$

and

$$-\mathcal{H}(y,p) \leq p\cdot f(y,\alpha) + l(y,\alpha) \quad \forall\alpha\in A.$$

Thus, fix $x, y \in \Omega$ and $p \in \mathbb{R}^N$, we have

$$\mathcal{H}(x,p) - \mathcal{H}(y,p) \leq -p\cdot f(x,\alpha_x) - l(x,\alpha_x) + p\cdot f(y,\alpha_y) + l(y,\alpha_y) \leq$$
$$\leq |p|C|x-y| + \omega_l(|x-y|),$$

and

$$\mathcal{H}(y,p) - \mathcal{H}(x,p) \leq -p\cdot f(y,\alpha_y) - l(y,\alpha_y) + p\cdot f(x,\alpha_x) + l(x,\alpha_x) \leq$$
$$\leq |p|C|x-y| + \omega_l(|x-y|).$$

So

$$|\mathcal{H}(x,p) - \mathcal{H}(y,p)| \leq |p|C|x-y| + \omega_l(|x-y|)$$

and **(H3)** is fulfilled.

3 The second order case.

3.1 An equivalent definition of viscosity solution.

In this section we want to describe an equivalent definition of viscosity solution that we will use to prove the second order results. In order to make it more clear we start giving this definition for the first order case.

Let us consider on open subset Ω of $\mathbb{R}^N$ and a continuous function $u : \Omega \to \mathbb{R}$. Fix $\hat{x} \in \Omega$ we define the *superdifferential of* u *at point* $\hat{x}$ as

$$D^+u(\hat{x}) := \left\{ p \in \mathbb{R}^N \ : \ \limsup_{x \to \hat{x},\, x \in \Omega} \frac{u(x) - u(\hat{x}) - p \cdot (x - \hat{x})}{|\hat{x} - x|} \leq 0 \right\} \tag{3.1}$$

and the *subdifferential of* u *at point* $\hat{x}$ as

$$D^-u(\hat{x}) := \left\{ p \in \mathbb{R}^N \ : \ \liminf_{x \to \hat{x},\, x \in \Omega} \frac{u(x) - u(\hat{x}) - p \cdot (x - \hat{x})}{|\hat{x} - x|} \geq 0 \right\}. \tag{3.2}$$

The main motivation for introducing these two sets is given by the following Lemma.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be a continuous function. Then*

(a) $p \in D^+u(\hat{x})$ if and only if there exists $\phi \in C^1(\Omega)$ such that $D\phi = p$ and $u - p$ has a local maximum at point $\hat{x}$;

(b) $p \in D^-u(\hat{x})$ if and only if there exists $\phi \in C^1(\Omega)$ such that $D\phi = p$ and $u - p$ has a local minimum at point $\hat{x}$.

Proof. We prove (a). To prove (b), since $D^-u(\hat{x}) = -[D^+(-u)(\hat{x})]$, just apply the same arguments to $-u$.
If $p \in D^+u(\hat{x})$, there exists $\delta > 0$ such that

$$u(x) \leq u(\hat{x}) + p \cdot (x - \hat{x}) + \sigma(|\hat{x} - x|)|\hat{x} - x| \quad \forall x \in B(0, \delta)$$

where $\sigma : [0, \infty[\to \mathbb{R}$ is a continuous increasing function such that $\sigma(0) = 0$. Define

$$\rho(r) = \int_0^r \sigma(t)dt$$

then the thesis is fulfilled with $\phi(x) := u(\hat{x}) + p \cdot (x - \hat{x}) + \rho(2|x - \hat{x}|)$. Indeed, since $\rho(2r) \geq \sigma(r)r$ for all $r \in \mathbb{R}$ and $\phi(\hat{x}) = u(\hat{x})$, fix $x \in \Omega$ we have

$$u(x) - \phi(x) = u(x) - u(\hat{x}) - p \cdot (x - \hat{x}) - \rho(2|x - \hat{x}|) \leq$$
$$\leq \sigma(|x - \hat{x}|)|x - \hat{x}| - \rho(2|x - \hat{x}|) \leq 0 = u(\hat{x}) - \phi(\hat{x})$$

thus $\hat{x}$ is a maximum of $u - \phi$. Moreover we have $D\phi(\hat{x}) = p$.
Conversely, if $\hat{x}$ is a maximum of $u - \phi$ and $p = D\phi(\hat{x})$, we have

$$\limsup_{x \to \hat{x},\, x \in \Omega} \frac{u(x) - u(\hat{x}) - p \cdot (x - \hat{x})}{|x - \hat{x}|} \leq \limsup_{x \to \hat{x},\, x \in \Omega} \frac{\phi(x) - \phi(\hat{x}) - D\phi(\hat{x}) \cdot (x - \hat{x})}{|x - \hat{x}|} = 0$$

thus $p \in D^+u(\hat{x})$. □

From this lemma it follows directly that the following is an equivalent definition of viscosity solution.

Definition 3.2

(a) *A function $u \in C(\Omega)$, is a viscosity subsolution of $H(x, u, Du) = 0$ in Ω if*

$$H(x, u(x), p) \leq 0 \quad \forall x \in \Omega, \quad \forall p \in D^+u(x).$$

(b) *A function $u \in C(\Omega)$, is a viscosity supersolution of $H(x, u, Du) = 0$ in Ω if*

$$H(x, u(x), p) \geq 0 \quad \forall x \in \Omega, \quad \forall p \in D^-u(x).$$

This definition is more easy to use in some situation. For example, let us prove that if $u \in C(\Omega)$ is a viscosity solution of $H(x, u, Du) = 0$, then $H(x, u(x), Du(x)) = 0$ at any point x where u is differentiable. Indeed, if u is differentiable at point x then $Du(x) = \{D^+u(x)\} = \{D^-u(x)\}$ (see Lemma 3.3 below). Hence, by Definition 3.2, we have $H(x, u(x), Du(x)) \leq 0$ and $H(x, u(x), Du(x)) \geq 0$, so $H(x, u(x), Du(x)) = 0$.

For the sake of completeness let us recall some basic properties of the sub and super differential sets. (For the proof see Lemma 1.8, Chapter II in [2].)

Lemma 3.3 *Let $u \in C(\Omega)$ and $\hat{x} \in \Omega$. Then,*

(a) *$D^+u(\hat{x})$ and $D^-u(\hat{x})$ are closed convex (possibly empty) subset of $\mathbb{R}^N$.*

(b) *If u is differentiable at $\hat{x}$ then $Du(\hat{x}) = \{D^+u(\hat{x})\} = \{D^-u(\hat{x})\}$.*

(c) *If for some $\hat{x}$, both $D^+u(\hat{x})$ and $D^-u(\hat{x})$ are non empty then $D^+u(\hat{x}) = D^-u(\hat{x}) = \{Du(\hat{x})\}$.*

(d) *The sets $A^+ = \{x \in \Omega : D^+u(x) \neq \emptyset\}$ and $A^- = \{x \in \Omega : D^-u(x) \neq \emptyset\}$ are non empty.*

With the same idea let us discuss now the equivalent definition for the second order case. First, let us observe the following. If u is upper semicontinuous, $\varphi \in C^2(\Omega)$, and $u - \varphi$ has a maximum at $\hat{x}$ we have, as $x \to \hat{x}$ by Taylor's expansion

$$u(x) \leq u(\hat{x}) + \varphi(x) - \varphi(\hat{x}) =$$

$$= u(\hat{x}) + p \cdot (x - \hat{x}) + \frac{1}{2} X(x - \hat{x}) \cdot (x - \hat{x}) + o(|x - \hat{x}|^2)$$

where $p = D\varphi(\hat{x})$ and $X = D^2\varphi(\hat{x})$. Moreover, one can prove that the viceversa is also true, more precisely, if there exist $(p, X) \in \mathbb{R}^N \times S(N)$ such that

$$u(x) \leq u(\hat{x}) + p \cdot (x - \hat{x}) + \frac{1}{2} X(x - \hat{x}) \cdot (x - \hat{x}) + o(|x - \hat{x}|^2) \tag{3.3}$$

for $x \in \Omega$ and as $x \to \hat{x}$, then there exists a function φ such that $u - \varphi$ has a maximum at point $\hat{x}$, $D\varphi(\hat{x}) = p$ and $D^2\varphi(\hat{x}) = X$.

Thus, the idea it is clearly to use (3.3) to give an equivalent definition of viscosity solution for the second order case. First, let us define two important sets. (These are the natural generalizations of the sub and super differential).

The *superjet of* u *at point* $\hat{x}$ is the following set

$$J^{2,+}u(\hat{x}) := \Big\{(u(\hat{x}), p, X) \in \mathbb{R} \times \mathbb{R}^N \times S(N) \text{ such that } u(x) \leq u(\hat{x}) + p \cdot (x - \hat{x}) + \tfrac{1}{2} X(x-\hat{x}) \cdot (x - \hat{x}) + o(|x-\hat{x}|^2) \text{ for } x \in \Omega \text{ and as } x \to \hat{x}\Big\}\,.$$

The *subjet of* u *at point* $\hat{x}$ is the following set

$$J^{2,-}u(\hat{x}) := \Big\{(u(\hat{x}), p, X) \in \mathbb{R} \times \mathbb{R}^N \times S(N) \text{ such that } u(x) \geq u(\hat{x}) + p \cdot (x - \hat{x}) + \tfrac{1}{2} X(x-\hat{x}) \cdot (x - \hat{x}) + o(|x-\hat{x}|^2) \text{ for } x \in \Omega \text{ and as } x \to \hat{x}\Big\}\,.$$

Moreover, it is useful to define the closure of those sets. Precisely,

$$\bar{J}^{2,+}u(\hat{x}) := \Big\{(r, p, X) \in \mathbb{R} \times \mathbb{R}^N \times S(N) \text{ such that there exist } x_n \in \Omega \text{ and } (u(x_n), p_n, X_n) \in J^{2,+}u(x_n) \text{ such that } x_n \to x \text{ and } (u(x_n), p_n, X_n) \to (r, p, X)\Big\}\,.$$

$$\bar{J}^{2,-}u(\hat{x}) := \Big\{(r, p, X) \in \mathbb{R} \times \mathbb{R}^N \times S(N) \text{ such that there exist } x_n \in \Omega \text{ and } (u(x_n), p_n, X_n) \in J^{2,-}u(x_n) \text{ such that } x_n \to x \text{ and } (u(x_n), p_n, X_n) \to (r, p, X)\Big\}\,.$$

We are finally ready to give our equivalent definition of viscosity solution.

Definition 3.4 *Let* Ω *be an open subset of* $\mathbb{R}^N$*, let* $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N)$ *be proper.*

(a) A function $u : \Omega \to \mathbb{R}$*, is a viscosity subsolution of* $F(x, u, Du, D^2u) = 0$ *in* Ω *if and only if* u *is upper semicontinuous and*

$$F(x, u(x), p, X) \leq 0 \quad \forall x \in \Omega, \quad \forall (u(x), p, X) \in J^{2,+}u(x).$$

Moreover, if F *is continuous (or only lower semicontinuous), since the inequality* $F \leq 0$ *persists under taking limits, it is equivalent to say*

$$F(x, r, p, X) \leq 0 \quad \forall x \in \Omega, \quad \forall (r, p, X) \in \bar{J}^{2,+}u(x).$$

(b) A function $u : \Omega \to \mathbb{R}$*, is a viscosity supersolution of* $F(x, u, Du, D^2u) = 0$ *in* Ω *if and only if* u *is lower semicontinuous and*

$$F(x, u(x), p, X) \geq 0 \quad \forall x \in \Omega, \quad \forall (u(x), p, X) \in J^{2,-}u(x).$$

Moreover, if F *is continuous (or only upper semicontinuous), since the inequality* $F \geq 0$ *persists under taking limits, it is equivalent to say*

$$F(x, r, p, X) \geq 0 \quad \forall x \in \Omega, \quad \forall (r, p, X) \in \bar{J}^{2,-}u(x).$$

3.2 Stability of the notion.

We want to prove now that the definition of viscosity solution is stable. We will describe two results. The first one says that if u_n is a viscosity subsolution of $F_n = 0$, $u_n \to u$ and $F_n \to F$ in a suitable sense, then u is a viscosity subsolution of $F = 0$. The second one tells us that if $\mathcal{F}$ is a collection of subsolutions of $F = 0$, then $(\sup_{u\in\mathcal{F}} u)^*$ is another subsolution. We recall that the *upper semicontinuous envelope* of u is given by

$$u^*(x) := \lim_{r\downarrow 0} \ \sup\{u(y) \ : \ y \in \Omega, \ |x-y| \leq r\},$$

while the *lower semicontinuous envelope* of u is

$$u_*(x) := \lim_{r\downarrow 0} \ \inf\{u(y) \ : \ y \in \Omega, \ |x-y| \leq r\}.$$

Remark 3.5 In the following we will give some results in a more general framework of a locally compact subset $\mathcal{O}$ of $\mathbb{R}^N$. The definition of the "jets" can be given in the same way, in fact if $\mathcal{O}$ is locally compact, $\varphi \in C^2(\mathcal{O})$ means that φ is the restriction of a twice differentiable function defined in a neighborhood of $\mathcal{O}$, and then the definition follows similarly. If $\mathcal{O}$ is a locally compact subset of $\mathbb{R}^N$ we will denote the superjet and the subjet with $J^{2,+}_{\mathcal{O}}u(x)$ and $J^{2,-}_{\mathcal{O}}u(x)$, respectively.

The basic technical result we will use is the following. (For the proof see Proposition 8.1 pag. 20 in [3].)

Proposition 3.6 *Let $\mathcal{O} \subset \mathbb{R}^N$ be locally compact. $U : \mathcal{O} \to \mathbb{R}$ be upper semicontinuous, $z \in \mathcal{O}$ and $(U(z), p, X) \in J^{2,+}_{\mathcal{O}}U(z)$.*

Suppose that there exists a sequence u_n of upper semicontinuous functions on $\mathcal{O}$ such that
(i) $\exists x_n \in \mathcal{O}$ such that $(x_n, u(x_n)) \to (z, U(z))$.
(ii) $\forall z_n \in \mathcal{O}$ such that $z_n \to x \in \mathcal{O}$ then $\limsup_{n\to\infty} u_n(z_n) \leq U(x)$.

Then, there exists $\hat{x}_n \in \mathcal{O}$, $(u_n(\hat{x}_n), p_n, X_n) \in J^{2,+}_{\mathcal{O}}(u_n(\hat{x}_n))$ such that

$$(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \to (z, U(z), p, X).$$

We are now ready to prove our first stability result.

Theorem 3.7 *Let Ω be an open subset of $\mathbb{R}^N$, and let F be proper and continuous. Let $\mathcal{F}$ be a nonempty collection of viscosity subsolutions of $F = 0$ in Ω.*

If $U(x) := \sup_{u\in\mathcal{F}} u(x)$ and U^* is finite on Ω then U^* is a viscosity subsolution of $F = 0$ in Ω.

Proof. Suppose $z \in \Omega$ and $(U^*(z), p, X) \in J^{2,+}U^*(z)$, by Definition 3.4 the thesis is $F(z, U^*(z), p, X) \leq 0$. The idea is to apply Proposition 3.6 with $U = U^*$. Indeed, one can prove that there exist a sequence $u_n \in \mathcal{F}$ and a sequence x_n such that $(x_n, u_n(x_n)) \to (z, U^*(z))$ and for each sequence $z_n \in \Omega$ such that $z_n \to x$ one has

$\limsup_{n\to\infty} u_n(z_n) \le U^*(x)$. Thus Proposition 3.6 apply and tells us that there exists a sequence $\hat{x}_n \in \Omega$ such that $(u_n(\hat{x}_n), p_n, X_n) \in J^{2,+}u_n(\hat{x}_n)$, so, since u_n is a subsolution, we obtain $F(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \le 0$. Moreover, always thanks to Proposition 3.6, we know that $(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \to (z, U^*(z), p, X)$, thus by the continuity of F we have $F(z, U^*(z), p, X) \le 0$ and we conclude. □

In order to describe the second stability result in all his generality we need a definition. Given a sequence of functions u_n on $\mathcal{O}$, we want to define the smallest function U such that if a sequence $x_n \in \mathcal{O}$ is such that $x_n \to x$ then $\limsup_{n\to\infty} u_n(x_n) \le U(x)$. This function is denoted by $\limsup_{n\to\infty}{}^* u_n$ and defined as follows.

Definition 3.8 *Let $u_n : \Omega \to \mathbb{R}$, we define*

$$\limsup_{n\to\infty}{}^* u_n(x) := \lim_{m\to\infty} \sup \left\{ u_n(y) \;:\; n \ge m, y \in \mathcal{O}, |y - x| \le \frac{1}{m} \right\}.$$

Moreover,

$$\liminf_{n\to\infty}{}_* u_n(x) := -\limsup_{n\to\infty}{}^* (-u_n(x))$$

Note that $U := \limsup_{n\to\infty}{}^* u_n$ can be characterized by the following two properties.

(I) For each $z \in \mathcal{O}$ there exists a sequence $x_n \in \mathcal{O}$ such that $(x_n, u_n(x_n)) \to (z, U(z))$.

(II) For each sequence $x_n \in \mathcal{O}$ such that $x_n \to z$ we have $\limsup_{n\to\infty} u_n(x_n) \le U(z)$.

Remark 3.9 If $u_n \equiv U$ then, $U(z) = \limsup_{n\to\infty}{}^* u_n(x) = U^*(z)$.
Moreover, the $\limsup_{n\to\infty}{}^* u_n(x)$ is an upper semicontinuous function.

Let us now prove the second stability result.

Theorem 3.10 *Let Ω be an open subset of $\mathbb{R}^N$, and F_n be a sequence of proper functions on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N)$. For each $n \in \mathbb{N}$ let u_n be a viscosity subsolution of $F_n = 0$ on Ω, moreover let $U = \limsup_{n\to\infty}{}^* u_n$ and F be such that*

$$F \le \liminf_{n\to\infty}{}_* F_n. \tag{3.4}$$

If U is finite, then is a viscosity subsolution of $F = 0$. In particular, if $u_n \to U$ and $F_n \to F$ locally uniformly then U is a viscosity subsolution of $F = 0$.

Proof. Fix $z \in \Omega$ and $(U(z), p, X) \in J^{2,+}U(z)$, by Definition 3.4 we have to prove that $F(z, U(z), p, X) \le 0$. Now, by the characterization of U given in **(I)** and **(II)**, we have that U fulfills assumptions (i) and (ii) in Proposition 3.6. Then there exist $\hat{x}_n \in \Omega$, and $(u_n(\hat{x}_n), p_n, X_n) \in J^{2,+}u_n(\hat{x}_n)$ such that $(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \to (z, U(z), p, X)$. Since u_n is a viscosity subsolution of $F_n = 0$ we have

$$F_n(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \le 0,$$

then, by assumption (3.4), we have

$$F(z, U(z), p, X) \le \liminf_{n\to\infty} F_n(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \le 0$$

and we conclude. □

Remark 3.11 Everything is still true is one consider viscosity supersolutions instead of viscosity subsolutions with all the reversed inequalities!

Remark 3.12 Of course Theorems 3.7 and 3.10 apply in particular to the first order equation $H(x, u, Du) = 0$. To make the technique more clear we briefly describe now the proof of a stability result for a sequence $H_n(x, u, Du) = 0$. More precisely, let u_n be a viscosity subsolution of $H_n = 0$ and assume that $u_n \to u$ and $H_n \to H$ locally uniformly. We want to prove directly that u is a viscosity subsolution of $H = 0$. If $\hat{x}$ is a maximum of $u - \varphi$ our thesis is $H(\hat{x}, u(\hat{x}), D\varphi(\hat{x})) \le 0$. By local uniform convergence if $\hat{x}$ is a maximum of $u - \varphi$ we have that, if $\hat{x}_n$ is a maximum of $u_n - \varphi$ then $\hat{x}_n \to \hat{x}$, $u_n(\hat{x}_n) \to u(\hat{x})$ and $D\varphi(\hat{x}_n) \to D\varphi(\hat{x})$. (Compare with Proposition 3.6!) Thus, since u_n is a viscosity subsolution of $H_n = 0$, $H_n(\hat{x}_n, u(\hat{x}_n), D\varphi(\hat{x}_n)) \le 0$, and letting $n \to \infty$ we conclude, also by local uniformly convergence of $H_n \to H$, that $H(\hat{x}, u(\hat{x}), D\varphi(\hat{x})) \le 0$.

3.3 Existence via Perron's method.

We consider now the Dirichlet problem

$$(DP) \qquad \begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We will assume F continuous and proper on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N)$, $g : \partial\Omega \to \mathbb{R}^N$ continuous and Ω an open subset of $\mathbb{R}^N$. We first give the definition of viscosity solution for problem (DP).

Definition 3.13

(a) *u is a* viscosity subsolution *of (DP) in $\bar{\Omega}$ if it is a viscosity subsolution of $F(x, u, Du, D^2u) = 0$ in Ω and satisfies $u \le 0$ on $\partial\Omega$.*

(b) *u is a* viscosity supersolution *of (DP) in $\bar{\Omega}$ if it is a viscosity supersolution of $F(x, u, Du, D^2u) = 0$ in Ω and satisfies $u \ge 0$ on $\partial\Omega$.*

u is a viscosity solution *of (DP) in $\bar{\Omega}$ if it is both viscosity subsolution and supersolution of (DP).*

We are going to state now an existence result. This theorem is due to Ishii and is a good example of the so called Perron's method.

Theorem 3.14 *Let F be continuous in all variables and proper. Assume that*

(I) *a comparison result holds for (DP); i.e. if w is a viscosity subsolution of (DP) and v is a viscosity supersolution of (DP) then $w \le v$.*

(II) *There exist a viscosity subsolution* $\underline{u}$ *and a viscosity supersolution* $\overline{u}$ *of* (DP) *which satisfy the boundary condition* $\underline{u}_*(x) = \overline{u}^*(x) = 0$ *on* $\partial\Omega$*.*

Then

$$W(x) := \sup\{w(x) \ : \ \underline{u}(x) \leq w(x) \leq \overline{u}(x) \ \textit{and } w \textit{ is a viscosity subsolution of } (DP)\}$$

is a solution of (DP)*.*

Note that the idea is that if a comparison principle holds for (DP) and there exists a viscosity solution of (DP) then the latest has to be the maximal viscosity subsolution. Indeed, if u is a viscosity solution it is a viscosity supersolution, so, by comparison, any other viscosity subsolution w is less that equal to u.

Before proving this theorem we observe that one of the main ingredients is the stability result 3.7. In fact it tells us that it make sense to consider as subsolution the supremum of viscosity subsolutions. The second important ingredient is the technical Lemma that follows. The idea of the latter is that if u is a viscosity subsolution but not a viscosity solution then there is "room" for a viscosity subsolution greater that equal to u. Precisely.

Lemma 3.15 *Let* Ω *be an open subset of* $\mathbb{R}^N$ *and* $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ *be proper. Let* u *be a viscosity subsolution of* $F = 0$ *in* Ω*. If* u_* *fails to be a viscosity supersolution at some point* $\hat{x}$*, (i.e. there exists* $(u(\hat{x}), p, X) \in J^{2,-}u_*(\hat{x})$ *such that* $F(\hat{x}, u(\hat{x}), p, X) < 0$*), then for any small* $k > 0$ *there is a viscosity subsolution* U_k *of* $F = 0$ *in* Ω *satisfying*

$$\begin{cases} U_k(x) \geq u(x) \ \textit{and} \ \sup_\Omega(U_k - u) > 0 \\ U_k(x) = u(x) \ \textit{for } x \in \Omega \ \textit{and} \ |x - \hat{x}| \geq k. \end{cases} \tag{3.5}$$

(For the proof see Lemma 9.1 pag. 24 in [3].)

Proof of Theorem 3.14. By definition W is such that $\underline{u}_* \leq W_* \leq W \leq W^* \leq \overline{u}^*$, so $W_* = W = W^* = 0$ on $\partial\Omega$, thus the boundary condition is fulfilled.
By the stability result 3.7, W^* is a viscosity subsolution of $F = 0$, so by comparison we have $\underline{u} \leq W \leq W^* \leq \overline{u}$. Thus W^* is a viscosity subsolution such that $\underline{u} \leq W^* \leq \overline{u}$; but, since W is the maximal one, $W \geq W^*$. This implies $W = W^*$ and tells us that W is a viscosity subsolution.
Let us now prove by contradiction that W_* is a viscosity supersolution. If not by Lemma 3.15 there exists a viscosity subsolution W_k such that $W_k \geq W$ and $W_k = 0$ on $\partial\Omega$ (for a sufficient small k). Moreover, by comparison, $W_k \leq \overline{u}$, thus $\underline{u} \leq W_k \leq \overline{u}$, but then by definition of W, $W_k \leq W$ and this is not possible.
Summarising W_* is a viscosity supersolution and W is a viscosity subsolution, thus by comparison $W \leq W^*$, which implies $W = W_* = W^*$ is continuous and is a viscosity solution, this leads us to the conclusion. □

By the structure of Theorem 3.14 it is clear that in order to really have obtained an existence (and uniqueness) result for the viscosity solution of (DP) one needs to discuss hypotheses **(I)** and **(II)**. The assumption we need to obtain the comparison result and the technique used to prove it will be described in the next section. Here we just mention a case in which **(II)** is also fulfilled. If F is decreasing with respect to X at an at least linear rate and it is Lipschitz continuous in X and p as well, then one can construct explicit functions $\underline{u}$ and $\overline{u}$ that fulfill **(II)**. (For all the details and the construction see pag.26 in [3] or Section 4 in [7].) The precise assumptions one has to ask are the following.

(A) There exist $0 < \lambda \leq \Lambda$ such that

$$F(x, r, p, X + Z) \leq F(x, r, p, X) - \lambda Trace(Z)$$

and

$$|F(x, r, p, X) - F(x, r, p, Y)| \leq \Lambda \parallel X - Y \parallel$$

for $X, Y, Z \in S(N)$, $Z \geq 0$. (Where $\parallel X \parallel = \sum_{\mu \in eig(X)} |\mu|$).

(B) The exists $\gamma > 0$ such that

$$|F(x, u, p, X) - F(x, u, q, X)| \leq \gamma |p - q|$$

for $x \in \Omega$, $u \in \mathbb{R}$, $p, q \in \mathbb{R}^N$ and $X \in S(N)$.

3.4 A comparison result.

In this section we want to discuss a comparison result for the following particular Dirichlet problem. Let Ω be an open subset of $\mathbb{R}^N$ and $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be degenerate elliptic.

$$(DP)_I \qquad \begin{cases} u + F(Du, D^2u) = f(x) & \text{in } \Omega \\ u(x) = \psi(x) & \text{on } \partial\Omega. \end{cases}$$

First, let us see what happens if we try to apply the same technique we used for the first order case (See Section 2.2). Let u and v be respectively a viscosity subsolution and supersolution of problem $(DP)_I$, and let $(\hat{x}, \hat{y})$ be a local maximum of $\Phi(x, y) = u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon}$. Since $\hat{x}$ is a local maximum of $x \to u(x) - \frac{|x-y|^2}{2\varepsilon}$ and $\hat{y}$ is a local minimum of $y \to v(y) + \frac{|x-y|^2}{2\varepsilon}$, by definition of viscosity sub and supersolution we have

$$u(\hat{x}) + F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, \frac{1}{\varepsilon} I\right) \leq f(\hat{x}) \quad \text{and} \quad v(\hat{y}) + F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, -\frac{1}{\varepsilon} I\right) \geq f(\hat{y})$$

where I denotes the identity in any dimension. Thus, we easily have obtained

$$u(\hat{x}) - v(\hat{y}) + F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, \frac{1}{\varepsilon} I\right) - F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, -\frac{1}{\varepsilon} I\right) \leq f(\hat{x}) - f(\hat{y}). \tag{3.6}$$

Is it useful?

Following the discussion of (2.11) in Section 2.2 it is clear that (3.6) is useful if $F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, \frac{1}{\varepsilon} I\right) - F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, -\frac{1}{\varepsilon} I\right) \geq 0$. But, since $\frac{1}{\varepsilon} I \geq -\frac{1}{\varepsilon} I$ and F is degenerate elliptic, we have $F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, \frac{1}{\varepsilon} I\right) - F\left(\frac{\hat{x} - \hat{y}}{\varepsilon}, -\frac{1}{\varepsilon} I\right) \leq 0$....!

Let us remark now that the full Hessian of Φ gives us

$$H\Phi(\hat{x},\hat{y}) = \begin{pmatrix} D^2u(\hat{x}) - \dfrac{I}{\varepsilon} & \dfrac{I}{\varepsilon} \\ \\ \dfrac{I}{\varepsilon} & D^2v(\hat{y}) + \dfrac{I}{\varepsilon} \end{pmatrix}$$

thus since $(\hat{x},\hat{y})$ is a local maximum,

$$\begin{pmatrix} D^2u(\hat{x}) & 0 \\ \\ 0 & D^2v(\hat{y}) \end{pmatrix} \leq \begin{pmatrix} -\dfrac{I}{\varepsilon} & \dfrac{I}{\varepsilon} \\ \\ \dfrac{I}{\varepsilon} & \dfrac{I}{\varepsilon} \end{pmatrix}$$

which implies, $D^2u(\hat{x}) \leq D^2v(\hat{y})$. So, by the degenerate ellipticity of F this condition implies $F(D^2v(\hat{y})) \leq F(D^2u(\hat{x}))$, which is in the right direction. Summarizing, the point is that we have to use the full information given by the fact that $(\hat{x},\hat{y})$ is a maximum on the second order term. More precisely, we state now the technical result we will need in the proof. (For the proof see Section 11, pag.31 in [3]).

Theorem 3.16 Theorem on Sums.

Let $\mathcal{O}$ be a locally compact subset of $\mathbb{R}^N$. Let $u, -v : \mathcal{O} \to \mathbb{R}$ be upper semicontinuous and φ be twice continuously differentiable in a neighborhood of $\mathcal{O} \times \mathcal{O}$.
Set $w(x,y) = u(x) - v(y)$ for $x, y \in \mathcal{O}$ and suppose $(\hat{x},\hat{y}) \in \mathcal{O} \times \mathcal{O}$ is a local maximum of $w(x,y) - \varphi(x,y)$ relative to $\mathcal{O} \times \mathcal{O}$.

Then for each $k > 0$ with $kD^2\varphi(\hat{x},\hat{y}) < I$ there exist $X, Y \in S(N)$ such that

$$(u(\hat{x}), D_x\varphi(\hat{x},\hat{y}), X) \in \bar{J}^{2,+}_{\mathcal{O}} u(\hat{x}), \qquad (u(\hat{y}), -D_y\varphi(\hat{x},\hat{y}), Y) \in \bar{J}^{2,-}_{\mathcal{O}} v(\hat{y}),$$

and the block diagonal matrix with entries $X, -Y$ satisfies

$$-\frac{1}{k} I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \left(I - kD^2\varphi(\hat{x},\hat{y})\right)^{-1} D^2\varphi(\hat{x},\hat{y}).$$

We are now ready to prove our comparison result.

Theorem 3.17 *Let Ω be a bounded open subset of $\mathbb{R}^N$, $f \in C(\Omega)$ and $F(p,X)$ be continuous and degenerate elliptic. Let $u, v : \bar{\Omega} \to \mathbb{R}$ be upper semicontinuous and lower semicontinuous respectively, u be a viscosity subsolution of $u + F(Du, D^2u) = f$ and v be a viscosity supersolution of $u + F(Du, D^2u) = f$ in Ω, such that $u \leq v$ on $\partial\Omega$.*

Then $u \leq v$ in Ω.

Proof. Set $\Phi := u(x) - v(y) - \dfrac{1}{2\varepsilon}|x-y|^2$ and consider $(\hat{x},\hat{y})$ a maximum over $\bar{\Omega} \times \bar{\Omega}$. Let A be the Hessian matrix of $\frac{1}{2\varepsilon}|x-y|^2$ evaluated at point $(\hat{x},\hat{y})$, i.e.

$$A = \begin{pmatrix} \dfrac{I}{\varepsilon} & -\dfrac{I}{\varepsilon} \\ -\dfrac{I}{\varepsilon} & \dfrac{I}{\varepsilon} \end{pmatrix}$$

and observe that $A \leq \frac{2}{\varepsilon} I$. Since our aim is to apply Theorem 3.16 with $A = D^2\varphi(\hat{x},\hat{y})$ we have to choose k such that $kA < 1$, which is $k < \frac{\varepsilon}{2}$. Thus, Theorem on Sums guarantees us that there exist $X, Y \in S(N)$ such that $(u(\hat{x}), D_x\varphi(\hat{x},\hat{y}), X) \in \bar{J}^{2,+}u(\hat{x})$ and $(u(\hat{y}), -D_y\varphi(\hat{x},\hat{y}), Y) \in \bar{J}^{2,-}v(\hat{y})$. Moreover, since $D_x\varphi(\hat{x},\hat{y}) = -D_y\varphi(\hat{x},\hat{y}) = \frac{\hat{x}-\hat{y}}{\varepsilon}$, and u and v are a viscosity subsolution and a viscosity supersolution respectively, we have

$$u(\hat{x}) + F\left(\frac{\hat{x}-\hat{y}}{\varepsilon}, X\right) \leq f(\hat{x}) \quad \text{and} \quad v(\hat{y}) + F\left(\frac{\hat{x}-\hat{y}}{\varepsilon}, Y\right) \geq f(\hat{y}). \tag{3.7}$$

Theorem 3.16 gives us also the following inequality.

$$-\frac{1}{k} I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \left(I - kD^2\varphi(\hat{x},\hat{y})\right)^{-1} D^2\varphi(\hat{x},\hat{y}) \leq \frac{1}{\varepsilon - 2k} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and since that last matrix annihilates any vector, we obtain $X \leq Y$, thus by the degenerate ellipticity of F we have

$$F\left(\frac{\hat{x}-\hat{y}}{\varepsilon}, X\right) \geq F\left(\frac{\hat{x}-\hat{y}}{\varepsilon}, Y\right).$$

Inequalities (3.7) gives us then

$$u(\hat{x}) - v(\hat{y}) \leq f(\hat{x}) - f(\hat{y})$$

and we conclude as in the first order case. (Compare with Section 2.2.) □

This result can be generalized to the Dirichlet problem (DP) provided that one makes the following assumption on F.

(C) There exist $\gamma > 0$ such that

$$\gamma(r - s) \leq F(x, r, p, X) - F(x, s, p, X)$$

for $r \geq s$, $(x, p, X) \in \bar{\Omega} \times \mathbb{R}^N \times S(N)$.

(D) There exists a continuous function $\omega : [0, \infty] \to [0, \infty]$ which satisfies $\lim_{t \to 0^+} \omega(t) = 0$ such that

$$F\left(y, r, \frac{x-y}{\varepsilon}, Y\right) - F\left(x, r, \frac{x-y}{\varepsilon}, X\right) \leq \omega(|x-y|(1 + \frac{|x-y|}{\varepsilon}))$$

whenever $x, y \in \Omega, r \in \mathbb{R}, X, Y \in S(N)$ and

$$\frac{3}{\varepsilon} I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

We conclude by finally giving a complete result.

Theorem 3.18 *Let Ω be an open subset of $\mathbb{R}^N$, and let $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ be continuous in all variables and proper. Assume* (**A**), (**B**), (**C**) *and* (**D**). *Then there exists a unique viscosity solution of* (DP).

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