# VARIATIONAL MODELS FOR PEELING PROBLEMS

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Abstract. We study variational models for flexural beams and plates interacting with a rigid substrate through an adhesive layer. The general structure of the minimizers is investigated and some properties characterizing the behavior of the systems in dependence of the load and the material stiffnesses are discussed.

## Contents



## Introduction

Adhesion phenomena constitute a very challenging area of research from both the theoretical and applied points of view. The physics underlying the adhesion crucially depends on the properties at the microscale level of the material constituting the adhesive (usually polymers in the case of artificial adhesives) and the properties of the surfaces of the jointed bodies. In recent years the instances posed by the miniaturization technology, as in the case of microelectromechanical systems (MEMS) (see the review [16]) or, more in general by nanotechnology, as well as the understanding of the

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role played by adhesion and elasticity in biophysics (see [6], [10], [5]) have pointed out the need of rational accurate models capturing the essentials of the complex phenomenologies involved in the adhesion problems. At the macroscopic level the energetic of peeling off a deformable material body from a rigid (i.e. another material body assumed to be rigid) substrate incorporates an *adhesion* energy and an *elastic* energy. Indeed, starting from the essential analysis of J. Eriksen carried in [3, Chapter 7.3], the problem of peeling of a thin (extensible or inextensible) tape has been investigated in the context of continuum mechanics by using a variational formulation (see [13], [4], [2]) and, more recently also the dynamics of these phenomena has been investigated (see [12]). Much of these works assume a membrane like behavior for the elastic body and consider the adhesion energy as a linear function of the measure of the detached zone.

In this paper we study, in the classical setting of Calculus of Variations, the peeling problem for flexural elastic beams and plates. In particular, we consider a linear elastic behavior in the hypotheses of Bernoulli-Navier and Kirchhoff-Love for beams and plates, respectively and we assume the adhesion energy is modeled by a continuous monotone function of the measure of the detached zone. In the case of the elastic beams we get a detailed characterization of the structure of the minimizers which allows to study the quasistatic evolution of the system when a monotone increasing load is supposed to act. In the case of the elastic plates we get two key estimates which characterize the regimes of the system in which the plate is completely detached or completely bonded, in dependence of the value of the load. In both the problems it is seen that a crucial role is played by the growth exponent affecting the constitutive assumption characterizing the adhesive material, which can be thought as synthesizing the macroscopic counterpart of the peculiarities at the microscale. Indeed, different choices of such an exponent lead to different behaviors of the system.

## 1. Preliminaries and notation

In this paper $H^2(\Omega)$ will denote the Sobolev space of the functions $u : \Omega \to \mathbb{R}$ whose weak second partial derivatives are in $L^2(\Omega)$. Moreover, let $BV(\Omega)$ be the space of the functions with bounded variation, i.e. $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in $\Omega$ (see [1], [9]). Now we recall that the perimeter $P_\Omega(E)$ of a set $E$ relative to $\Omega$ is defined as the total variation of the measure

$D\mathbf{1}_{E\cap\Omega}$, where $\mathbf{1}_{E\cap\Omega}$ denotes the characteristic function of the set $E\cap\Omega$. Finally, we introduce the perimeter of $E$ relative to the closed set $\mathbb{R}^2\setminus\Omega$, namely

$$P_{\mathbb{R}^2\setminus\Omega}(E)=\inf\{P_G(E):G\text{ open},\mathbb{R}^2\setminus\Omega\subset G\}.$$

## 2. Peeling of elastic beams

In the following we shall consider the case of linearly elastic flexural beam, in the Bernoulli-Navier hypothesis, interacting with a rigid substrate through a thin adhesion layer. Let the beam occupy the interval $[0,L]$ in its reference configuration and let $u:[0,L]\to\mathbb{R}$ be the displacement function measuring the vertical deflection of the beam. The elastic potential energy is given by

$$W_e(u)=\frac{1}{2}\int_0^L k_b\,|u''|^2\,dx, \tag{1}$$

where $k_b=EI$ denotes the flexural rigidity of the beam, where $E$ is the Young modulus and $I$ is the inertia moment of the cross section. In order to model the energetic of the adhesion between the beam and the rigid substrate let us introduce the set

$$\Sigma_u=\{x\in[0,L]\ |u(x)>0\}$$

and let $|\Sigma_u|$ be the Lebesgue measure of $\Sigma_u$. We define the adhesion potential as

$$W_a(u)=\vartheta(L^{-1}|\Sigma_u|), \tag{2}$$

where $\vartheta:\mathbb{R}_+\to\mathbb{R}$ is assumed to be continuous and monotone increasing with $\vartheta(0)=0$. Let us assume the beam is subjected to a given load $f$ applied at the right end $x=L$. Then the load potential takes the form

$$W_l(u)=-fu(L). \tag{3}$$

Therefore the total thermodynamic potential is given by

$$\mathcal{F}(u)=W_e(u)+W_a(u)+W_l(u)$$

and so, according to the Gibbs Principle, stable equilibrium configurations of the beam are minimizers of the functional

$$\mathcal{F}(u)=\frac{1}{2}\int_0^L k_b\,|u''|^2\,dx+\vartheta(L^{-1}|\Sigma_u|)-fu(L) \tag{4}$$

among the functions $u \in \mathcal{A} = \{u \in H^2(0, L) \ |u(0) = 0, \ u(x) \geq 0\}$. It is worth noticing that if $f \leq 0$ then $u \equiv 0$ is the only trivial minimizer and therefore we will assume without restriction that $f > 0$.

We are going to apply the direct methods of the calculus of variations to the aim of establishing existence of solutions for this minimization problem.

**Lemma 2.1.** *(Lower semicontinuity of $W_a$) The functional $W_a$ defined in (2) is sequentially weakly lower semicontinuous in $H^2(]0, L[)$, i.e. if $\{u_k\}_{k\in\mathbb{N}}$ weakly converges to $u$ in $H^2(]0, L[)$, then*

$$W_a(u) \leq \liminf_{k\to\infty} W_a(u_k).$$

*Proof.* Let $u_k \rightharpoonup u$ in $H^2(]0, L[)$, with $u_k \geq 0$ for every $k \in \mathbb{N}$. Then $u \geq 0$ and $u_k \to u$ uniformly in $[0, L]$. Therefore we have that

$$\liminf_{k\to\infty} |\Sigma_{u_k}| \geq |\Sigma_u|.$$

Then, by virtue of the monotonicity and continuity properties of $\vartheta$, we get

$$\liminf_{k\to\infty} \vartheta(L^{-1}|\Sigma_{u_k}|) = \sup_{j\in\mathbb{N}} \inf_{k\geq j} \vartheta(L^{-1}|\Sigma_{u_k}|) \geq \sup_{j\in\mathbb{N}} \vartheta(\inf_{k\geq j} L^{-1}|\Sigma_{u_k}|) =$$

$$= \vartheta(\liminf_{k\to\infty} L^{-1}|\Sigma_{u_k}|) \geq \vartheta(L^{-1}|\Sigma_u|).$$

□

**Theorem 2.2.** *(Existence of minimizers) $\mathcal{F}$ admits a minimum in $\mathcal{A}$.*

*Proof.* Let us observe that $W_e$ and $W_l$ trivially satisfy the l.s.c. property. Then, by Lemma 2.1 we deduce the lower semicontinuity of $\mathcal{F}$ and so, a standard compactness argument allows to get the thesis. □

2.1. **Structure of the minimizers.** In this section we focus our attention on the essential properties of the solutions of the problem of minimizing the functional $\mathcal{F}$ given by (4) in the class $\mathcal{A}$, for a given value of the load $f$.

**Proposition 2.3.** *Let $u \in$ argmin $\mathcal{F}$, then there exists $\xi \in [0, L]$ such that $\Sigma_u = [\xi, L]$. In particular $u'(\xi) = 0$ and $u(x) = 0$ in $[0, \xi]$.*

*Proof.* Let us observe that, since $u$ is continuous, $\Sigma_u$ is open in $[0, L]$, moreover we claim that there is no interval $]\alpha, \beta[$, with $0 < \alpha < \beta < L$ such that $]\alpha, \beta[\subset \Sigma_u$. Indeed, if this were not true, we could take $\tilde{u} \in \mathcal{A}$ defined as follows

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in [0, L] \setminus ]\alpha, \beta[ \\ 0 & \text{if } x \in ]\alpha, \beta[, \end{cases}$$

to obtain the inequality $\mathcal{F}(\tilde{u}) < \mathcal{F}(u)$, which yields a contradiction since $u$ is a minimizer. Finally, $u'(\xi) = 0$ since $u \in H^2(]0, L[)$. □

The previous results allow to state that if $u \in \mathcal{A}$ minimizes $\mathcal{F}$ then it can be completely characterized in a form which depends on the detachment variable $\xi$, as we are going to prove in the following theorem.

**Theorem 2.4.** *Let $u \in$ argmin $\mathcal{F}$ then*

$$u(x) = \frac{f}{k_b}\left(x - \bar{\xi}\right)^2 \left(-\frac{x}{6} + \frac{L}{2} - \frac{\bar{\xi}}{3}\right), \tag{5}$$

*where $\bar{\xi}$ is a global minimizer of*

$$F(\xi) = \vartheta\left(1 - \frac{\xi}{L}\right) - \frac{f^2 L^3}{6k_b}\left(1 - \frac{\xi}{L}\right)^3. \tag{6}$$

*Proof.* If $u \in \text{argmin}\mathcal{F}$ then by Proposition 2.3, the total energy takes the form

$$\mathcal{F}(u) = \frac{1}{2}\int_{\xi}^{L} k_b \left|u''\right|^2 dx + \vartheta(1 - \frac{\xi}{L}) - f u(L). \tag{7}$$

Then $u$ minimizes

$$\frac{1}{2}\int_{\xi}^{L} k_b \left|v''\right|^2 dx - f v(L) \tag{8}$$

among all $v \in H^2(\xi, L)$ such that $v \geq 0,\ v(\xi) = v'(\xi) = 0$. By taking into account that $f > 0$, it is readily seen that any minimizer of (7) among all $v \in H^2(\xi, L)$ such that $v(\xi) = v'(\xi) = 0$ is greater or equal to zero and therefore standard first variation of (7) shows that

$$\begin{cases} v''''(x) = 0 & \forall x \in ]\xi, L[ \\ v(\xi) = v'(\xi) = 0, & \\ v''(L) = 0,\ v'''(L) = -\dfrac{f}{k_b}. & \end{cases} \tag{9}$$

It is easy to check that the unique solution $u$ of (9) is given by

$$u(x) = \frac{f}{k_b}\,(x-\xi)^2\left(-\frac{x}{6}+\frac{L}{2}-\frac{\xi}{3}\right). \tag{10}$$

By substituting (10) in (7) we obtain the minimum energy in dependence of the free boundary $\xi \in [0, L]$, that is

$$F(\xi) = \mathcal{F}(u) = \vartheta\left(1-\frac{\xi}{L}\right) - \frac{f^2L^3}{6k_b}\left(1-\frac{\xi}{L}\right)^3 \tag{11}$$

hence

$$\min \mathcal{F} = \min\{F(\xi) \mid \xi \in [0, L]\} \tag{12}$$

thus completing the proof. □

Then, a complete understanding of the analytical properties of the energy minimizers lead to study the minimization problem

$$\min\{F(\xi) \mid \xi \in [0, L]\}. \tag{13}$$

and the value of $\xi$ minimizing $F$ fully characterizes the detachment region $\Sigma_u = [\xi, L]$, for a given value of the load $f$.

For the sake of convenience and to simplify notations, we introduce the normalized variable $\zeta = \left(1-\frac{\xi}{L}\right)$ and so, after changing variable, (11) becomes

$$G(\zeta) = \vartheta(\zeta) - \frac{f^2L^3}{6k_b}\zeta^3, \tag{14}$$

with $\zeta \in [0, 1]$.

Since the uniqueness of $\zeta$ minimizing $G$ given by (14) depends on the constitutive choice of the adhesion potential $\vartheta$, we spend a few words concerning such uniqueness question. Obviously, strict convexity of $G$ leads to uniqueness of the minimum $\zeta$. Otherwise, different choices of $\vartheta$ lead to multiple minima. Indeed, if for instance we take

$$\vartheta(\zeta) = \frac{f^2L^3}{6k_b}\left(\zeta^3 + \zeta^2(\zeta-\frac{1}{2})^2\right),$$

which satisfies the assumptions of monotonicity and vanishing in zero, it is easy to see that the function $G$ in (14) has two different minima at $\zeta = 0$ and $\zeta = \frac{1}{2}$. Therefore, to the aim of studying stable equilibrium configurations we are lead to choice a suitable form of the adhesion potential. In particular, we shall assume a power like law for $\vartheta$ and, in the next section, we will focus the attention on suitable growth conditions to be satisfied by the function $\vartheta$, to the aim of distinguishing between stable and metastable configurations.

2.2. **Growth conditions and stability analysis.** We claim that different growth assumptions on $\vartheta$ give rise to different constitutive properties of the peeling model we are concerned . In particular, as we are going to show, we can characterize the stability of the equilibrium configurations in dependence of the growth properties of $\vartheta$.

**Proposition 2.5.** *Let $\alpha \in ]0,3]$, we assume that for every $\zeta \in [0,1]$*

$$\vartheta(\zeta) = c|\zeta|^\alpha,$$

*for some constant $c > 0$. We set $f_{cr} = \sqrt{\frac{6ck_b}{L^3}}$. Then, for $0 \le f < f_{cr}$, $G(\zeta)$ achieves the minimum at $\zeta = 0$, for $f \ge f_{cr}$ $G(\zeta)$ achieves the minimum at $\zeta = 1$.*

*Proof.* Let us take $\vartheta(\zeta) = c|\zeta|^\alpha$. We see that, for $0 < \alpha \le 1$,

$$G''(\zeta) = c\alpha(\alpha-1)\zeta^{\alpha-2} - \frac{f^2L^3}{k_b}\zeta \le 0 \quad \forall \zeta \in [0,1],$$

then $G(\zeta)$ is concave and we have that for $0 \le f \le f_{cr}$ it results $G(1) \ge 0$ and so the minimum point is $\zeta = 0$, since $G(0) = 0$. If $f > f_{cr}$, then $G(1) < 0$ and so the minimum point is $\zeta = 1$. If $1 < \alpha < 3$, $G(\zeta)$ has two critical pints, namely $\zeta = 0$ and $\zeta = \bar{\zeta}$, with

$$\bar{\zeta} = \left(\frac{2\alpha c k_b}{f^2L^3}\right)^{\frac{1}{3-\alpha}}, \ G'(\bar{\zeta}) = 0, \ G''(\bar{\zeta}) < 0.$$

Then, for $0 < f < f_{cr}$ $G(1) > 0$ and the minimum is at $\zeta = 0$, while for $f > f_{cr}$ $G(1) < 0$ and so the minimum is at $\zeta = 1$. Finally, if $\alpha = 3$ then $G''(\zeta) = \zeta(6c - \frac{f^2L^3}{k_b})$ and thus, by convexity argument, we get the thesis. $\square$

**Proposition 2.6.** *Let $\alpha > 3$, we assume that for every $\zeta \in [0,1]$*

$$\vartheta(\zeta) = c|\zeta|^\alpha,$$

*for some constant $c > 0$. We set $f_{cr}^\alpha = \sqrt{\frac{2\alpha c k_b}{L^3}}$. Then, for every $0 < f < f_{cr}^\alpha$ there exists a unique $\bar{\zeta} \in ]0,1[$ such that $G(\zeta)$ achieves the minimum at $\bar{\zeta}$. For every $f \ge f_{cr}^\alpha$, $G(\zeta)$ achieves the minimum at $\zeta = 1$.*

*Proof.* Let us see that

$$G'(\zeta) = \zeta^2 \left( \alpha c \zeta^{\alpha-3} - \frac{f^2 L^3}{2k_b} \right)$$

and so $G'(\zeta) = 0$ for $\zeta = 0$ and for $\zeta = \bar{\zeta}$, with

$$\bar{\zeta} = \left( \frac{f^2 L^3}{2\alpha c k_b} \right)^{\frac{1}{\alpha-3}}. \tag{15}$$

Clearly, it is easy to verify that $\bar{\zeta} \in ]0,1[$ for $0 < f < f_{cr}^{\alpha}$. Moreover, we have $G''(\bar{\zeta}) > 0$ and $G(\bar{\zeta}) < 0$ and therefore we have that$\bar{\zeta} \in ]0,1[$ is the absolute minimizer of $G$. Finally, for $f \geq f_{cr}^{\alpha}$, it results that $G(\zeta)$ is monotone decreasing in $[0,1]$ and so the minimum is at $\zeta = 1$ □

**Remark 2.7.** *It is easy to check that a slight generalization of Proposition 2.5 leads to state that for* $1 < \alpha < 3$*, if the adhesive has any constitutive function* $\vartheta$ *such that* $c_1\zeta^{\alpha} < \vartheta(\zeta) < c_2\zeta^{\alpha}$*, for some positive* $0 < c_1 < c_2$*, then for* $0 \leq f < \sqrt{\frac{6c_1 k_b}{L^3}}$ *the minimum of* $G$ *is attained at* $\zeta = 0$*.*

The above results allow to single out two different regimes of behavior of the mechanical system, according to the constitutive assumptions made for the adhesive material.In the case of slow growth condition, by Proposition 2.5 we have that the system exhibits a *catastrophe-like* behavior in correspondence of the critical value $f_{cr}$. Indeed, when $f = f_{cr}$, the beam can be complete attached or completely detached from the rigid support, the two configuration being energetically equivalent. In other words, we have a bifurcation of the equilibrium and no partial detachment is allowed.In the case of fast growth condition, Proposition 2.6 tells us that, for every value of the force $f$, the minimizer $u$ of (4) is a stable equilibrium configuration of the beam and the free boundary $\xi$ is a continuous and increasing function of $f$. Furthermore, by looking at the equation (15) we notice that $\bar{\zeta} = O(f^{\frac{2}{\alpha-3}})$,while a simple computation gives, for the maximum elastic displacement, the relation

$$u_{max} = \frac{f(L-\xi)^3}{3k_b} = O(f).$$

Therefore, for $\alpha$ sufficiently large, we argue that, if an equilibrium configuration has $u \neq 0$ and $\xi \neq 1$ and we let $f$ increase quasistatically, then the elastic maximum displacement $u_{max}$ becomes very large, before the variation of $\xi$ becomes appreciable. Since beyond certain values of the maximum displacement the material fails to behave elastically, we are lead to study constitutive models which take into account phenomena beyond the elastic range and this will be the subject of [8] where we will consider an elastic-plastic energetic model for the beam. Finally, we remark that a more complex behavior of the system, regarding the mechanical stability, can be captured by assuming for $\vartheta$ varying growth law like, for instance, those of hardening or softening type.

2.3. **General load conditions.** In this section we assume the beam is loaded with a vertical force and an end moment. Before studying the combined effect, we now assume that the beam is subjected to a moment $M$ acting at the end $x = L$, then we have

$$W_l(u) = -Mu'(L).$$

Analogously to the previous section, we can prove the following result.

**Theorem 2.8.** *Let $u \in$ argmin $\mathcal{F}$, then*

$$u(x) = \frac{M}{2k_b}(x - \bar{\xi})^2, \tag{16}$$

*where $\bar{\xi}$ is a global minimizer of*

$$F(u) = \vartheta(1 - \frac{\xi}{L}) - \frac{M^2 L}{2k_b}\left(1 - \frac{\xi}{L}\right). \tag{17}$$

*Proof.* If $u \in \operatorname{argmin} \mathcal{F}$, then

$$\mathcal{F}(u) = \frac{1}{2}\int_{\xi}^{L} k_b\,|u''|^2\,dx + \vartheta(1 - \frac{\xi}{L}) - Mu'(L).$$

Therefore, arguing as in the proof of Theorem 2.4, we can say that $u$ satisfies the following equations.

$$\begin{cases} u''''(x) = 0 & \forall x \in ]\xi, L[ \\ u(\xi) = u'(\xi) = 0, & \\ u''(L) = \dfrac{M}{k_b}, & \\ u'''(L) = 0. & \end{cases} \tag{18}$$

The solution of (18) is

$$u(x) = \frac{M}{2k_b}(x-\xi)^2, \tag{19}$$

and so the minimum the energy in dependence of the free boundary $\xi \in [0, L]$ is given by

$$F(\xi) = \mathcal{F}(u) = \vartheta(1-\frac{\xi}{L}) - \frac{M^2 L}{2k_b}\left(1-\frac{\xi}{L}\right)$$

and, finally, minimizing with respect to $\xi$ we get the thesis. □

The same analysis carried in in Section 2.2 leads to study the function $G : [0, 1] \to \mathbb{R}$ given by

$$G(\zeta) = \vartheta(\zeta) - \frac{M^2 L}{2k_b}\zeta \tag{20}$$

obtained as (14). As in Section 2.2 we assume that $\vartheta(\zeta) = c|\zeta|^\alpha$ for some $c > 0$ and we discuss the stability conditions by varying $\alpha$. In this case we have that for $\alpha \in ]0, 1]$ the minimum of $G$ is achieved at $\zeta = 0$ or $\zeta = 1$ and so the system exhibits a catastrophe-like behavior. Specifically we have that for $0 < \alpha < 1$ $G$ has the minimum at $\zeta = 0$, while for $\alpha = 1$ the minimum is $\zeta = 0$ if $M \le M_{cr} = \sqrt{\frac{2ck_b}{L}}$ and $\zeta = 1$ if $M \ge M_{cr}$. Obviously in correspondence of $M = M_{cr}$ we have a bifurcation point. If $\alpha > 1$ then the minimum of $G$ is $\zeta = 0$ for every value of $M$.

Finally, when the load condition is given by assigning both the force and the moment at $x = L$, the previous results allow to conclude that the stability analysis falls in the range of Section 2.2 and so, if the adhesive is modeled by the constitutive law $\vartheta(\zeta) = |\zeta|^\alpha$, the critical threshold is $\alpha = 3$. Indeed, roughly speaking, the moment offers a low order contribution, as we have just observed.

2.4. **Local Minimizers.** The stability analysis carried out in the previous section involves only global minimizers of $\mathcal{F}$. Therefore, a careful analysis of a quasistatic evolution of the mechanical system requires to detect the presence of local minimizers, as we are going to study in this section.

**Definition 2.9.** *We say that $u \in \mathcal{A}$ is a local minimizer of $\mathcal{F}$ if there exists $\delta > 0$ such that for every $v \in \mathcal{A}$, $\|v-u\|_{\mathcal{A}} < \delta$ we have $\mathcal{F}(u) \le \mathcal{F}(v)$.*

The main result of this section is the following

**Theorem 2.10.** *A function $u \in \mathcal{A}$ is a local minimizer of $\mathcal{F}$ if either $u \equiv 0$ or*

$$u(x) = \frac{f}{6k_b}(3L - 2\bar{\xi} - x)(x - \bar{\xi})^2 \mathbf{1}_{(\bar{\xi},L]} \tag{21}$$

*where $\bar{\xi} \in [0, L)$ is a local minimizer of the function*

$$\xi \to \vartheta(1 - \frac{\xi}{L}) - \frac{f^2 L^3}{6k_b}(1 - \frac{\xi}{L})^3. \tag{22}$$

The proof of the previous theorem needs some preliminary lemmas.

**Lemma 2.11.** *Assume that $u \in \mathcal{A}$ is a local minimizer of $\mathcal{F}$ such that $u \not\equiv 0$. Then there exists a unique $\xi \in [0, L)$ such that $\Sigma_u = (\xi, L)$.*

*Proof.* Since $u \in H^2(0, L)$ then $u' \equiv 0$ on the set $\Sigma_u$; moreover continuity of $u$ yields that $\Sigma_u$ is a countable union of pairwise disjoint open intervals, namely

$$\Sigma_u = \cup_{j=1}^{\infty} (a_j, b_j) \tag{23}$$

and $u(a_j) = u(b_j) = u'(a_j) = u'(b_j) = 0$. Suppose now that $b_j \neq L$ for some $j$ and choose $\varepsilon > 0$ such that $\varepsilon\|u\|_{\mathcal{A}} \leq \delta$. Then by taking $v = u - \varepsilon u \mathbf{1}_{(a_j,b_j)}$ we have $\|v - u\| \leq \delta$ and $\mathcal{F}(v) < \mathcal{F}(u)$, a contradiction. □

**Lemma 2.12.** *Assume that $u \in \mathcal{A}$ is a local minimizer of $\mathcal{F}$. Then either $u \equiv 0$ or $u(L) > 0$.*

*Proof.* If $u$ is a local minimizer then

$$0 \leq \mathcal{F}(u + \varepsilon u) - \mathcal{F}(u) = (\varepsilon k_b + \varepsilon^2 \frac{k_b}{2}) \int_0^L |u''|^2 - \varepsilon f u(L) \tag{24}$$

and arbitrariness of $\varepsilon$ yields

$$k_b \int_0^L |u''|^2 = f u(L). \tag{25}$$

If $u(L) = 0$ then $u'' \equiv 0$ and by recalling the definition of $\mathcal{A}$ we get $u \equiv 0$. □

**Lemma 2.13.** *Assume that $u \in \mathcal{A}$ is a local minimizer of $\mathcal{F}$ and that $u \not\equiv 0$. Then*

$$u(x) = \frac{f}{6k_b}(3L - 2\xi - x)(x - \xi)^2 \mathbf{1}_{(\xi,L]} \tag{26}$$

*for a suitable $\xi \in [0, L)$.*

*Proof.* If $u$ is a local minimizer and $u \not\equiv 0$ then by Lemma 2.11 there exists a unique $\xi \in [0, L)$ such that $\Sigma_u = (\xi, L)$. Let now $(a, b) \subset\subset (\xi, L)$ and $\psi \in C_0^2(a, b)$ : then $u + \varepsilon\psi > 0$ in $(a, b)$ for $|\varepsilon|$ small enough and $|\Sigma_{u+\varepsilon\psi}| = |\Sigma_u|$. Since $u$ is a local minimizer we have

$$\mathcal{F}(u + \varepsilon\psi) - \mathcal{F}(u) = \frac{k_b}{2} \int_a^b 2\varepsilon\psi'' u'' + \varepsilon^2 |\psi''|^2 \geq 0 \tag{27}$$

and the arbitrariness of $\varepsilon$ yields

$$\int_a^b \psi'' u'' = 0 \tag{28}$$

hence $u'''' = 0$ in $(a, b)$ and again being $(a, b)$ arbitrarily chosen in $(\xi, L)$ we have $u'''' = 0$ in the whole $(\xi, L)$. Since $u \in H^2(0, L)$ and $u \equiv 0$ in $[0, \xi]$ then $u(x) = (x - \xi)^2(\alpha x + \beta)$. By taking into account Lemma 2.12 we have that $u(L) > 0$ and then for every $a > \xi$ and for every $\psi \in C^2([a, L])$ such that $\psi(a) = \psi'(a) = 0$ we have $u + \varepsilon\psi > 0$ when $|\varepsilon|$ is small enough. This implies that

$$\int_a^L \psi'' u'' - f\psi(L) = 0 \tag{29}$$

that is $u''(L) = 0$ and $u'''(L) = \dfrac{-f}{k_b}$. Therefore

$$u(x) = \frac{f}{6k_b}(3L - 2\xi - x)(x - \xi)^2. \tag{30}$$

□

We are now in a position to prove Theorem 2.10. Indeed, by using formula (30) it is readily seen that when $\varepsilon$ is suitably chosen then $\|u_\varepsilon - u\| < \delta$, where we have set

$$u_\varepsilon(x) = \frac{f}{6k_b}(3L - 2\bar{\xi} - 2\varepsilon - x)(x - \bar{\xi} - \varepsilon)^2. \tag{31}$$

Hence $u$ will be a local minimizer if and only if $\bar{\xi}$ is a local minimizer of the function

$$\xi \to \vartheta(1 - \frac{\xi}{L}) - \frac{f^2 L^3}{6k_b}(1 - \frac{\xi}{L})^3. \tag{32}$$

## 3. Peeling of elastic plates

In this section we shall consider the case of an elastic plate interacting with a rigid substrate through a thin adhesive layer. Then, let $\Omega \subset \mathbb{R}^2$ be an open bounded set, representing the reference configuration of a Kirchhoff-Love plate. Assume that $|\Omega| = 1$, $\partial\Omega \in \mathrm{Lip}$ and $\Gamma_i \subset \partial\Omega,\ i = 1, 2$ are two measurable subsets of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\mathcal{H}^1(\Gamma_i),\ \mathcal{H}^1(\partial\Omega\backslash\Gamma_i) > 0$. The elastic energy of the plate is

$$J_e(u) = \int_\Omega k_p \left(|\Delta u|^2 - 2(1-\nu)\mathrm{det}D^2u\right)\ dx,$$

where $k_p$ is the flexural rigidity and $\nu$ is the Poisson ratio. We assume the plate is loaded by a force distribution $f \in L^\infty(\Gamma_2)$ acting perpendicular to the plane of the plate, then the load potential is

$$J_l(u) = -\int_{\Gamma_2} fu\ d\mathcal{H}^1.$$

Let $\vartheta : \mathbb{R}_+ \to \mathbb{R}$ be a continuous and increasing function with $\vartheta(0) = 0$, the adhesion potential is

$$J_a(u) = \vartheta(|\Sigma_u|).$$

We denote with $\nu$ the unit outer normal vector to $\partial\Omega$ and we introduce the function space

$$\mathcal{A} = \{v \in H^2(\Omega) \mid\ \ v \geq 0,\ v = \frac{\partial v}{\partial \nu} = 0, \text{in } \Gamma_1\}. \tag{33}$$

Let us consider a general formulation of the problem, so let Sym be the space of real symmetric $2 \times 2$ matrices and consider a quadratic form $W : \mathrm{Sym} \to [0, +\infty)$ such that for a suitable $\delta > 0$

$$W(A) \geq \delta \|A\|^2\ \ \forall A \in \mathrm{Sym}. \tag{34}$$

We define for every $v \in H^2(\Omega)$ the functional

$$\mathcal{F}(v) = \begin{cases} \displaystyle\int_\Omega W(D^2v)\,dx - \int_{\Gamma_2} fv\,d\mathcal{H}^1 + \vartheta(|\Sigma_v|) \text{ if } v \in \mathcal{A} \\ \\ +\infty \text{ otherwise in } H^2(\Omega). \end{cases} \tag{35}$$

Then we are interested in the minimization of the functional $\mathcal{F}$. As in Lemma 2.1 we can prove that $\mathcal{F}$ is lower semicontinuous with respect to the weak topology of $H^2$ and by using (34) it is readily seen that a minimizer of $\mathcal{F}$ always exists. In this section we want to investigate qualitative properties of such minimizers, in particular we are interested, as pursued in the case of the elastic beam, in establishing stability like conditions involving thresholds of the load $f$ characterizing equilibrium states with the the plate completely bonded to the substrate and completely detached and this will be obtained in Theorem 3.4 and Theorem 3.5 below. It is worth to notice the role played by the exponent $\alpha = 2$, whereas in the case of the beam the critical value was $\alpha = 3$. As in the case of the elastic beam we have that this analysis depends on the constitutive behavior of the adhesive layer through the growth assumptions made for $\vartheta$.

To this aim we proceed by proving some preliminary results. The next lemma establishes the so called *compliance identity* enjoyed by the minimizers of $\mathcal{F}$.

**Lemma 3.1.** *Let $u \in$ argmin $\mathcal{F}$ then*

$$2\int_\Omega W(D^2u)\,dx = \int_{\Gamma_2} fu\,d\mathcal{H}^1. \tag{36}$$

*Proof.* Let $0 < |t| < 1$ then $(1+t)u \in \mathcal{A}$ and

$$\begin{aligned} 0 \le \mathcal{F}((1+t)u) - \mathcal{F}(u) = (1+t)^2 \int_\Omega W(D^2u)\,dx + \\ -\int_\Omega W(D^2u)\,dx - t\int_{\Gamma_2} fu\,d\mathcal{H}^1 = \\ = 2t\int_\Omega W(D^2u)\,dx + t^2\int_\Omega W(D^2u)\,dx - t\int_{\Gamma_2} fu\,d\mathcal{H}^1. \end{aligned} \tag{37}$$

Hence

$$\frac{t}{|t|}\left(2\int_\Omega W(D^2u)\,dx - \int_{\Gamma_2} fu\,d\mathcal{H}^1\right) \ge -|t|\int_\Omega W(D^2u)\,dx, \tag{38}$$

that is

$$\left|2\int_\Omega W(D^2u)\,dx - \int_{\Gamma_2} fu\,d\mathcal{H}^1\right| \le |t|\int_\Omega W(D^2u)\,dx \tag{39}$$

and so the thesis follows by letting $t \to 0$. □

We define for every $s \in [0, \mathcal{H}^1(\partial\Omega))$ the function

$$\rho(s) = \sup\left\{\frac{P_{\mathbb{R}^2\setminus\Omega}(E)}{P_\Omega(E)} :\ E \subset \Omega,\ P_\Omega(E) > 0, P_{\mathbb{R}^2\setminus\Omega}(E) \le s\right\} \tag{40}$$

and we recall the following result whose proof can be found in [9].

**Proposition 3.2.** *Let* $w \in BV(\Omega)$ *with* $\mathcal{H}^1(\{x \in \partial\Omega : \ w(x) \neq 0\}) \leq s$*; then*

$$\int_{\partial\Omega} |w| \, d\mathcal{H}^1 \leq \rho(s) \int_{\Omega} |Dw|. \tag{41}$$

*Moreover the constant* $\zeta(s)$ *is as best as possible.*

Based on the previous results we are going to prove the following statement.

**Lemma 3.3.** *Let* $u \in argmin \mathcal{F}$ *and let*

$$\Lambda = \frac{\Gamma(2)^{\frac{1}{2}}}{4\sqrt{\pi}} \rho(\mathcal{H}^1(\partial\Omega \setminus \Gamma_1))(1 + \rho(\mathcal{H}^1(\partial\Omega \setminus \Gamma_1))). \tag{42}$$

*Then*

$$\|D^2 u\|_{L^2(\Omega)} \leq \frac{\Lambda}{\delta} |\Sigma_u| \|f\|_\infty. \tag{43}$$

*Proof.* By (34) and Lemma 3.1 we get via Hölder inequality

$$\int_\Omega |D^2 u|^2 \, dx \leq \frac{1}{2\delta} \|f\|_\infty \int_{\partial\Omega} |u| \, d\mathcal{H}^1 \tag{44}$$

and, since $u \in \mathcal{A}$, then

$$\mathcal{H}^1(\{x \in \partial\Omega : \ u(x) \neq 0\}) \leq \mathcal{H}^1(\partial\Omega \setminus \Gamma_1) < \mathcal{H}^1(\partial\Omega).$$

Hence, by applying Proposition 3.2, we obtain by (44)

$$\begin{aligned} \int_\Omega |D^2 u|^2 \, dx &\leq \frac{1}{2\delta} \|f\|_\infty \rho(\mathcal{H}^1(\partial\Omega \setminus \Gamma_1)) \int_{\Sigma_u} |\nabla u| \, dx \leq \\ &\leq \frac{1}{2\delta} |\Sigma_u|^{\frac{1}{2}} \|f\|_\infty \rho(\mathcal{H}^1(\partial\Omega \setminus \Gamma_1)) \left( \int_{\Sigma_u} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned} \tag{45}$$

A well known result about best constants in Sobolev inequalities ( see again [9, p.189]) yields

$$\|\nabla u\|_{L^2(\Omega)} \leq \frac{\Gamma(2)^{\frac{1}{2}}}{2\sqrt{\pi}} \left( \|D^2 u\|_{L^1(\Omega)} + \|\nabla u\|_{L^1(\partial\Omega)} \right) \tag{46}$$

and applying again Proposition 3.2 and Hölder inequality we obtain from (45) and (46)

$$\begin{aligned}\int_\Omega |D^2u|^2\,dx &\le \frac{1}{2\delta}|\Sigma_u|^{\frac12}\|f\|_\infty \rho(\mathcal{H}^1(\partial\Omega\setminus\Gamma_1))\|\nabla u\|_{L^2(\Omega)} \le \\ &\le \frac{\Gamma(2)^{\frac12}}{4\delta\sqrt{\pi}}|\Sigma_u|^{\frac12}\|f\|_\infty \rho(\mathcal{H}^1(\partial\Omega\setminus\Gamma_1))\left(\|D^2u\|_{L^1(\Omega)}+\|\nabla u\|_{L^1(\partial\Omega)}\right) \le \\ &\le \frac{\Lambda}{\delta}|\Sigma_u|^{\frac12}\|f\|_\infty\|D^2u\|_{L^1(\Omega)} = \frac{\Lambda}{\delta}|\Sigma_u|^{\frac12}\|f\|_\infty\|D^2u\|_{L^1(\Sigma_u)} \le \\ &\le \frac{\Lambda}{\delta}|\Sigma_u|\|f\|_\infty\|D^2u\|_{L^2(\Omega)}.\end{aligned} \tag{47}$$

Therefore, in view of (42), (47) yields

$$\|D^2u\|_{L^2(\Omega)} \le \frac{\Lambda}{\delta}|\Sigma_u|\|f\|_\infty \tag{48}$$

□

We are now in a position to state and prove the main results of this section. More precisely, in the following two theorems we shall obtain precise estimates on the force $f$ and on the growth exponent $\alpha$ characterizing stable equilibrium configurations of the plate.

**Theorem 3.4.** *Assume that $\vartheta(\tau) \ge \lambda\tau^2$ for some $\lambda > 0$. Then there exists $K > 0$ such that if*

$$\|f\|_\infty < K \tag{49}$$

*then $u \in argmin\mathcal{F}$ if and only if $u \equiv 0$ a.e..*

*Proof.* Assume $u \in \text{argmin}\mathcal{F}$. Since $W$ is a quadratic form there exists a suitable $\gamma > 0$ such that $W(A) \le \gamma\|A\|^2$ for every $A \in \text{Sym}$. Then by Lemma 3.1 and minimality, we get

$$\begin{aligned}\mathcal{F}(u) &= -\int_\Omega W(D^2u)\,dx + \vartheta(|\Sigma_u|) \ge \\ &\ge -\gamma\int_\Omega |D^2u|^2\,dx + \vartheta(|\Sigma_u|).\end{aligned} \tag{50}$$

Let

$$\|f\|_\infty < K = \frac{\delta}{\Lambda}\sqrt{\frac{\lambda}{\gamma}}$$

then, by virtue of Lemma 3.3, we obtain

$$\begin{aligned} 0 = \mathcal{F}(0) \geq \mathcal{F}(u) \geq \\ |\Sigma_u|^2 \left(-\gamma\frac{\Lambda^2}{\delta^2}\|f\|_\infty^2 + \lambda\right) \end{aligned} \tag{51}$$

which easily implies $|\Sigma_u| = 0$ and so we get $u \equiv 0$ a.e.. □

**Theorem 3.5.** *Assume that $\vartheta(\tau) \geq \lambda\tau^\alpha$ with $0 < \alpha < 2$. Let $u \in$ argmin$\mathcal{F}$ and let $K = \frac{\delta}{\Lambda}\sqrt{\frac{\lambda}{\gamma}}$. Then the following implications hold true.*

i) *If $\|f\|_\infty < K$ then $|\Sigma_u| = 0$.*

ii) *If $\|f\|_\infty = K$ then either $|\Sigma_u| = 0$ or $|\Sigma_u| = 1$.*

iii) *If $|\Sigma_u| = 1$ then $\|f\|_\infty \geq K$.*

*Proof.* We argue as in the previous proof. So, let $u \in \text{argmin}\mathcal{F}$, since $W$ is a quadratic form there exists a suitable $\gamma > 0$ such that $W(A) \leq \gamma\|A\|^2$ for every $A \in \text{Sym}$. Then by Lemma 3.1 and minimality, we get

$$\begin{aligned} \mathcal{F}(u) = -\int_\Omega W(D^2u)\,dx + \vartheta(|\Sigma_u|) \geq \\ \geq -\gamma\int_\Omega |D^2u|^2\,dx + \vartheta(|\Sigma_u|). \end{aligned} \tag{52}$$

Thus, by virtue of Lemma 3.3, we obtain

$$0 = \mathcal{F}(0) \geq \mathcal{F}(u) \geq \lambda\left(-\frac{1}{K^2}|\Sigma_u|^2\|f\|_\infty^2 + |\Sigma_u|^\alpha\right) \tag{53}$$

and thus

$$\lambda|\Sigma_u|^2\left(-\frac{1}{K^2}\|f\|_\infty^2 + |\Sigma_u|^{\alpha-2}\right) \leq 0 \tag{54}$$

To prove the first assertion let us see that if $|\Sigma_u| \neq 0$, since $0 < \alpha < 2$ and $|\Sigma_u| \leq 1$, then (54) implies $\|f\|_\infty \geq K$. The second assertion follows by considering that if $\|f\|_\infty = K$, then (54) becomes

$$\lambda|\Sigma_u|^2 \left(-1 + |\Sigma_u|^{\alpha-2}\right) \leq 0,$$

from which we get the thesis. Finally, by substituting $|\Sigma_u| = 1$ in (54), we get

$$\lambda \left(-\frac{1}{K^2}\|f\|^2_\infty + 1\right) \leq 0,$$

which proves the last statement. □

We point out that the condition $\|f\|_\infty = K$ defines a bifurcation point in correspondence of which the material system exhibits a catastrophe like behavior. At this value of the load, both the states of plate completely bonded and plate completely detached are admissible. Therefore, by considering a monotone increasing variation of the load starting from the value $K$, we can expect two possible scenarios. That is, if $|\Sigma_u| = 0$ the system will expend energy in deforming the plate and in breaking the adhesive, whereas if $|\Sigma_u| = 1$ the only further energy will be expended in the elastic deformation of the plate.

Finally, let us notice that $K = \frac{\delta}{\Lambda}\sqrt{\frac{\lambda}{\gamma}}$ plays the role of a threshold on the force $f$. Likewise the case of elastic beam studied in Section 2, where the analogous parameter is denoted by $f_{cr}$, $K$ depends on the elastic stiffness of the plate through the ratio $\frac{\delta}{\sqrt{\gamma}}$, on the stiffness of the adhesive layer through $\sqrt{\lambda}$ and on a geometric parameter $\Lambda$.

## References


[1] L. Ambrosio, N. Fusco & D. Pallara,*Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford, 2000.

[2] K.T. Andrews, L. Chapman, J.R. Fernandez, M. Fisackerly, L. Vanerian, T. VanHouten, *A membrane in adhesive contact*, SIAM J. Appl. Math. (2003) **64**, 152-169.

[3] J.L. Ericksen, *Introduction to the Thermodynamics of solids*, Chapman & Hall, 1991.

[4] M. Frémond, *Contact with adhesion*, Chapter IV of *Topics in Nonsmooth Mechanics*, Birkhäuser Verlag (1988).

[5] A. Ghatak, L. Mahadevan, J.Y. Chung, M.K. Chaudhury, V. Shenoy *Peeling from a biomimetically patterned thin elastic film*, Proc. Royal Soc. London A (2004) **460**, 2725-2735.

[6] N. Israelachvili, *Intramolecular and Surface Forces*, Academic Press, New York (1992).

[7] K.L. Johnson, *Adhesion and friction between a smooth elastic spherical asperity and a smooth surface*, Proc. Royal Soc. London A (1997) **453**, 163-179.
[8] F. Maddalena & D. Percivale, *Forthcoming*.
[9] V. G. Maz'ja, *Sobolev Spaces*, Springer-Verlag, 1985.
[10] X. Oyharcabal & T. Frisch, *Peeling off an elastica from a smooth atractive substrate*, Physical Review E **71**, pp.036611-6 (2005).
[11] D. Percivale & F. Tomarelli, *Smooth and creased equilibria for elastic-plastic plates and beams*,Preprint.
[12] N. Pede, P. Podio-Guidugli, G. Tomassetti, *Balancing the force that drives the peeling of an adhesive tape*, Preprint.
[13] P. Podio-Guidugli, *Peeling tapes*, in *Mechanics of Material Forces*, ed. by P. Steinmann and G. Maugin, Springer-Verlag (2005).
[14] P. Villaggio, *Qualitative methods in elasticity*, Noordhoff International Publishing, 1977.
[15] Y. Wei & J.W. Hutchinson, *Interface strenght, work of adhesion and plasticity in the peel test*,Int. Journ. of Fracture, **93**, (1998), 315-333.
[16] Y.-P. Zhao, L.S. Wang & T.X. Yu, *Mechanics of adhesion in MEMS – a review*, J. Adhesion Sci. Technol., Vol. 17, No. 4, pp. 519-546 (2003).

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