A note on Alexsandrov type theorem for k-convex functions

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Abstract

In this note we show that k-convex functions on $\mathbb{R}^n$ are twice differentiable almost everywhere for every positive integer $k > n/2$. This generalizes the classical Alexsandrov's theorem for convex functions.

1 Introduction

A classical result of Alexsandrov [1] asserts that convex functions in $\mathbb{R}^n$ are twice differentiable *a.e.*, (see also [3], [8] for more modern treatments). It is well known that Sobolev functions $u \in W^{2,p}$, for $p > n/2$ are twice differentiable a.e.. The following weaker notion of convexity known as k-convexity was introduced by Trudinger and Wang [12, 13]. Let $\Omega \subset \mathbb{R}^n$ be an open set and $C^2(\Omega)$ be the class of continuously twice differentiable functions on Ω. For $k = 1, 2, \ldots, n$ and a function $u \in C^2(\Omega)$, the k-Hessian operator, F_k, is defined by

$$F_k[u] := S_k(\lambda(\nabla^2 u)), \qquad (1.1)$$

where $\nabla^2 u = (\partial_{ij} u)$ denotes the Hessian matrix of the second derivatives of u, $\lambda(A) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ the vector of eigenvalues of an $n \times n$ matrix

$A \in \mathbb{M}^{n\times n}$ and $S_k(\lambda)$ is the k-th elementary symmetric function on $\mathbb{R}^n$, given by

$$S_k(\lambda) := \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}\,. \tag{1.2}$$

Alternatively we may write

$$F_k[u] = [\nabla^2 u]_k \tag{1.3}$$

where $[A]_k$ denotes the sum of $k\times k$ principal minors of an $n\times n$ matrix A, which may also be called the k-trace of A. The study of k-Hessian operators was initiated by Caffarelli, Nirenberg and Spruck [2] and Ivochkina [6] with further developed by Trudinger and Wang [10, 12, 13, 14, 15].

A function $u \in C^2(\Omega)$ is called *k-convex* in Ω if $F_j[u] \geq 0$ in Ω for $j = 1, 2 \ldots, k$; that is, the eigenvalues $\lambda(\nabla^2 u)$ of the Hessian $\nabla^2 u$ of u lie in the closed convex cone given by

$$\Gamma_k := \{\lambda \in \mathbb{R}^n \ :\ S_j(\lambda) \geq 0,\ j = 1, 2, \ldots, k\}\,. \tag{1.4}$$

(see [2] and [13] for the basic properties of Γ_k.) We notice that $F_1[u] = \Delta u$, is the Laplacian operator and 1-convex functions are subharmonic. When $k = n$, $F_n[u] = \det(\nabla^2 u)$, the Monge-Ampére operator and n-convex functions are convex. To extend the definition of k-convexity for non-smooth functions we adopt a *viscosity* definition as in [13]. *An upper semi-continuous function $u : \Omega \to [-\infty,\infty)$ ($u \not\equiv -\infty$ on any connected component of Ω) is called k-convex if $F_j[q] \geq 0$, in Ω for $j = 1, 2, \ldots, k$, for every quadratic polynomial q for which the difference $u-q$ has a finite local maximum in Ω.* Henceforth, we shall denote the class of k-convex functions in Ω by $\Phi^k(\Omega)$. When $k = 1$ the above definition is equivalent to the usual definition of subharmonic function, see, for example (Section 3.2, [5]) or (Section 2.4, [7]). Thus $\Phi^1(\Omega)$ is the class of subharmonic functions in Ω. We notice that $\Phi^k(\Omega) \subset \Phi^1(\Omega) \subset L^1_{\text{loc}}(\Omega)$ for $k = 1, 2, \ldots, n$, and a function $u \in \Phi^n(\Omega)$ if and only if it is convex on each component of Ω. Among other results Trudinger and Wang [13] (Lemma 2.2) proved that $u \in \Phi^k(\Omega)$ if and only if

$$\int_\Omega u(x) \left(\sum_{i,j}^n a^{ij}\partial_{ij}\phi(x)\right) dx \geq 0 \tag{1.5}$$

for all smooth compactly supported functions $\phi \geq 0$, and for all constant $n\times n$ symmetric matrices $A = (a^{ij})$ with eigenvalues $\lambda(A) \in \Gamma_k^*$, where Γ_k^* is dual cone defined by

$$\Gamma_k^* := \{\lambda \in \mathbb{R}^n \ :\ \langle\lambda,\mu\rangle \geq 0 \ \text{ for all }\ \mu \in \Gamma_k\}\,. \tag{1.6}$$

In this note we prove the following Alexsandrov type theorem for k-convex functions.

Theorem 1.1. *Let $k > n/2$, $n \geq 2$ and $u : \mathbb{R}^n \to [-\infty, \infty)$ ($u \not\equiv -\infty$ on any component of $\mathbb{R}^n$), be a k-convex function. Then u is twice differentiable almost everywhere. More precisely, we have the Taylor's series expansion for $\mathcal{L}^n$ x a.e.,*

$$\left| u(y) - u(x) - \langle \nabla u(x) \, , \, y - x \rangle - \frac{1}{2} \langle \nabla^2 u(x)(y - x) \, , \, y - x \rangle \right| = o(|y - x|^2) \, , \tag{1.7}$$

as $y \to x$.

In Section 3 (see, Theorem 3.2.), we also prove that the absolute continuous part of the k-Hessian measure (see, [12, 13]) $\mu_k[u]$, associated to a k-convex function for $k > n/2$ is represented by $F_k[u]$. For the Monge-Ampére measure $\mu[u]$ associated to a convex function u, such result is obtained in [16].

To conclude this introduction we note that it is equivalent to assume only $F_k[q] \geq 0$, in the definition of k-convexity [13]. Moreover Γ_k may also be characterized as the closure of the positivity set of S_k containing the positive cone Γ_n, [2].

2 Notations and preliminary results

Throughout the text we use following standard notations. $|\cdot|$ and $\langle \cdot, \cdot \rangle$ will stand for Euclidean norm and inner product in $\mathbb{R}^n$, and $B(x, r)$ will denote the open ball in $\mathbb{R}^n$ of radius r centered at x. For measurable $E \subset \mathbb{R}^n$, $\mathcal{L}^n(\mathrm{E})$ will denote its Lebesgue measure. For a smooth function u, the gradient and Hessian of u are denoted by $\nabla u = (\partial_1 u, \cdots, \partial_n u)$ and $\nabla^2 u = (\partial_{ij} u)_{1 \leq i,j \leq n}$ respectively. For a locally integrable function f, the distributional gradient and Hessian are denoted by $Df = (D_1 f, \cdots, D_n f)$ and $D^2 u = (D_{ij} u)_{1 \leq i,j \leq n}$ respectively.

For the convenience of the readers, we cite the following Hölder and gradient estimates for k-convex functions, and the weak continuity result for k-Hessian measures, [12, 13].

Theorem 2.1. (Theorem 2.7, [13]) *For $k > n/2$, $\Phi^k(\Omega) \subset C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha := 2 - n/k$ and for any $\Omega' \subset\subset \Omega$, $u \in \Phi^k(\Omega)$, there exists $C > 0$, depending* *only on n and k such that*

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}} d_{x,y}^{n+\alpha}\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C\int_{\Omega'}|u| \,, \tag{2.1}$$

where $d_x := \operatorname{dist}(x\,,\partial\Omega')$ *and* $d_{x,y} := \min\{d_x\,,d_y\}$.

Theorem 2.2. (Theorem 4.1, [13]) *For* $k=1,\ldots,n,$ *and* $0<q<\dfrac{nk}{n-k}$, *the space of k-convex functions* $\Phi^k(\Omega)$ *lie in the local Sobolev space* $W^{1,q}_{\mathrm{loc}}(\Omega)$. *Moreover, for any* $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ *and* $u\in\Phi^k(\Omega)$ *there exists* $C>0$, *depending on n, k, q, Ω' and Ω'', such that*

$$\left(\int_{\Omega'}|Du|^q\right)^{1/q} \leq C\int_{\Omega''}|u|\,. \tag{2.2}$$

Theorem 2.3. [Theorem 1.1, [13]] *For any* $u\in\Phi^k(\Omega)$, *there exists a Borel measure* $\mu_k[u]$ *in* Ω *such that*

(i) $\mu_k[u](V)=\int_V F_k[u](x)\,dx$ *for any Borel set* $V\subset\Omega$, *if* $u\in C^2(\Omega)$ *and*

(ii) *if* $(u_m)_{m\geq 1}$ *is a sequence in* $\Phi^k(\Omega)$ *converges in* $L^1_{\mathrm{loc}}(\Omega)$ *to a function* $u\in\Phi^k(\Omega)$, *the sequence of Borel measures* $(\mu_k[u_m])_{m\geq1}$ *converges weakly to* $\mu_k[u]$.

Let us recall the definition of the dual cones, [11]

$$\Gamma_k^* := \{\lambda\in\mathbb{R}^n \;:\; \langle\lambda,\mu\rangle\geq 0 \;\text{ for all }\; \mu\in\Gamma_k\}\,,$$

which are also closed convex cones in $\mathbb{R}^n$. We notice that $\Gamma_j^*\subset\Gamma_k^*$ for $j\leq k$ with $\Gamma_n^*=\Gamma_n=\{\lambda\in\mathbb{R}^n \;:\; \lambda_i\geq 0,\;\; j=1,2,\ldots,n\}$, Γ_1^* is the ray given by

$$\Gamma_1^* = \{\,t(1,\cdots,1)\;:\; t\geq 0\}\,,$$

and Γ_2^* has the following interesting characterization,

$$\Gamma_2^* = \left\{\lambda\in\Gamma_n \;:\; |\lambda|^2 \leq \frac{1}{n-1}\left(\sum_{i=1}^n\lambda_i\right)^2\right\}. \tag{2.3}$$

We use this explicit representation of Γ_2^* to establish that the distributional derivatives $D_{ij}u$ of the k-convex function u are signed Borel measures for $k\geq 2$, (see also [13]).

Theorem 2.4. *Let $2 \leq k \leq n$ and $u : \mathbb{R}^n \to [-\infty, \infty)$, be a k-convex function. Then there exist signed Borel measures $\mu^{ij} = \mu^{ji}$ such that*

$$\int_{\mathbb{R}^n} u(x)\, \partial_{ij}\phi(x)\, dx = \int_{\mathbb{R}^n} \phi(x)\, d\mu^{ij}(x) \quad \text{for } i,\, j = 1, 2, \ldots, n\,, \tag{2.4}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Proof. Let $k \geq 2$ and $u \in \Phi^k(\mathbb{R}^n)$. Since $\Phi^k(\mathbb{R}^n) \subset \Phi^2(\mathbb{R}^n)$ for $k \geq 2$, it is enough to prove the theorem for $k = 2$. Let u be a 2-convex function in $\mathbb{R}^n$. For $A \in \mathbb{S}^{n\times n}$, the space of $n \times n$ symmetric matrices, define the distribution $T_A : C_c^2(\mathbb{R}^n) \to \mathbb{R}$, by

$$T_A(\phi) := \int_{\mathbb{R}^n} u(x) \sum_{i,j}^{n} a^{ij} \partial_{ij}\phi(x)\, dx$$

By (1.5), $T_A(\phi) \geq 0$ for $A \in \mathbb{S}^{n\times n}$ with eigenvalues $\lambda(A) \in \Gamma_2^*$, and $\phi \geq 0$. Therefore, by Riesz representation (see, for example Theorem 2.14 in [9] or Theorem 1, Section 1.8 in [3]), there exist a Borel measure μ^A in $\mathbb{R}^n$, such that

$$T_A(\phi) = \int_{\mathbb{R}^n} \phi \sum_{i,j}^{n} a^{ij} D_{ij}u\, dx = \int_{\mathbb{R}^n} \phi\, d\mu^A\,, \tag{2.5}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$ and all $n \times n$ symmetric matrices A with $\lambda(A) \in \Gamma_2^*$. In order to prove the second order distributional derivatives $D_{ij}u$ of u to be signed Borel measures, we need to make special choices for the matrix A. By taking $A = I_n$, the identity matrix, $\lambda(A) \in \Gamma_1^* \subset \Gamma_2^*$, we obtain a Borel measure μ^{I_n} such that

$$\int_{\mathbb{R}^n} \phi \sum_{i=1}^{n} D_{ii}u\, dx = \int_{\mathbb{R}^n} \phi\, d\mu^{I_n}\,, \tag{2.6}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$. Therefore, the trace of the distributional Hessian D^2u, is a Borel measure. For each $i = 1, \ldots, n$, let A_i be the diagonal matrix with all entries 1 but the i-th diagonal entry being 0. Then by the characterization of Γ_2^* in (2.3), it follows that $\lambda(A_i) \in \Gamma_2^*$. Hence there exist a Borel measure μ^i in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \phi \sum_{j\neq i}^{n} D_{jj}u\, dx = \int_{\mathbb{R}^n} \phi\, d\mu^i\,, \tag{2.7}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$. From (2.6) and (2.7) it follows that, the diagonal entries $D_{ii}u = \mu^{I_n} - \mu^i := \mu^{ii}$ are signed Borel measure and

$$\int_{\mathbb{R}^n} u\, \partial_{ij}\phi\, dx = \int_{\mathbb{R}^n} \phi\, d\mu^{ii}\,, \tag{2.8}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$. Let $\{e_1, \dots, e_n\}$ be the standard orthonormal basis in $\mathbb{R}^n$ and for a, $b \in \mathbb{R}^n$, $a \otimes b := (a^i b^j)$, denotes the $n \times n$ rank-one matrix. For $0 < t < 1$ and $i \neq j$, let us define $A_{ij} := I_n + t[\, e_i \otimes e_j + e_j \otimes e_i]$. By a straight forward calculation, it is easy to see that the vector of eigenvalues $\lambda(A_{ij}) = (1-t, 1+t, 1, \cdots, 1) \in \Gamma_2^*$, for $0 < t < (n/2(n-1))^{1/2}$. Note that for this choice of A_{ij}

$$\sum_{k,l=1}^{n} a^{kl} \partial_{kl} \phi = \sum_{k=1}^{n} \partial_{kk} \phi + 2t\, \partial_{ij} \phi \,.$$

Thus for $i \neq j$, (2.5) and (2.6) yields

$$\begin{aligned}
\int_{\mathbb{R}^n} u\, \partial_{ij} \phi \, dx &= \frac{1}{2t} \left[\int_{\mathbb{R}^n} u \sum_{k,l=1}^{n} a^{kl} \partial_{kl} \phi \, dx - \int_{\mathbb{R}^n} u \sum_{k=1}^{n} \partial_{kk} \phi \, dx \right] \\
&= \frac{1}{2t} \left[\int_{\mathbb{R}^n} \phi \, d\mu^{A_{ij}} - \int_{\mathbb{R}^n} \phi \, d\mu^{I_n} \right] \\
&= \int_{\mathbb{R}^n} \phi \, d\mu^{ij} \,, \qquad (2.9)
\end{aligned}$$

where

$$\mu^{ij} := \frac{1}{2t} \left(\mu^{A_{ij}} - \mu^{I_n} \right) = \frac{1}{2t} \left(\mu^{A_{ij}} - \sum_{k=1}^{n} \mu^{kk} \right) .$$

Therefore $D_{ij} u = \mu^{ij}$, are signed Borel measures and satisfies the identity (2.4). □

A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to have *locally bounded variation* in $\mathbb{R}^n$ if for each bounded open subset Ω' of $\mathbb{R}^n$,

$$\sup \left\{ \int_{\Omega'} f \operatorname{div} \phi \, dx \;:\; \phi \in C_c^1(\Omega'; \mathbb{R}^n), \; |\phi(x)| \leq 1 \text{ for all } x \in \Omega' \right\} \, < \, \infty \,.$$

We use the notation $BV_{\text{loc}}(\mathbb{R}^n)$, to denote the space of such functions. For the theory of *functions of bounded variation* readers are referred to [4, 17, 3].

Theorem 2.5. *Let $n \geq 2$, $k > n/2$ and $u : \mathbb{R}^n \to [-\infty, \infty)$, be a k-convex function. Then u is differentiable a.e. $\mathcal{L}^n$ and $\dfrac{\partial u}{\partial x_i} \in BV_{\text{loc}}(\mathbb{R}^n)$, for all $i = 1, \dots, n$.*

Proof. Observe that for $k > n/2$, we can take $n < q < \dfrac{nk}{n-k}$ and by the gradient estimate (2.2), we conclude that k-convex functions are differentiable

$\mathcal{L}^n$ a.e. x. Let $\Omega' \subset\subset \mathbb{R}^n$, $\phi = (\phi^1, \cdots, \phi^n) \in C_c^1(\Omega'; \mathbb{R}^n)$ such that $|\phi(x)| \le 1$ for $x \in \Omega'$. Then by integration by parts and the identity (2.4), we have for $i = 1, \ldots, n$,

$$\begin{aligned} \int_{\Omega'} \frac{\partial u}{\partial x_i} \operatorname{div} \phi \, dx &= -\sum_{j=1}^{n} \int_{\Omega'} u \frac{\partial^2 \phi^j}{\partial x_i \partial x_j} \, dx \\ &= -\sum_{j=1}^{n} \int_{\Omega'} \phi^j \, d\mu^{ij} \\ &\le \sum_{j=1}^{n} |\mu^{ij}|(\Omega') < \infty \,, \end{aligned}$$

where $|\mu^{ij}|$ is the total variation of the Radon measure μ^{ij}. This proves the theorem. □

3 Twice differentiability

Let u be a k-convex function, $k \ge 2$, then by the Theorem 2.4, we have $D^2 u = (\mu^{ij})_{i,j}$, where μ^{ij} are Radon measures. By Lebesgue's Decomposition Theorem, we may write

$$\mu^{ij} = \mu^{ij}_{\mathrm{ac}} + \mu^{ij}_{\mathrm{s}} \quad \text{ for } \ i, j = 1, \cdots, n \,,$$

where μ^{ij}_{ac} is absolutely continuous with respect to $\mathcal{L}^n$ and μ^{ij}_{s} is supported on a set with Lebesgue measure zero. Let u_{ij} be the density of the absolutely continuous part, i.e., $d\mu^{ij}_{\mathrm{ac}} = u_{ij} \, dx$, $u_{ij} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Set $u_{ij} := \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, $\nabla^2 u := \left(\dfrac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i,j} = (u_{ij})_{i,j} \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and $[D^2 u]_s := (\mu^{ij}_{\mathrm{s}})_{i,j}$. Thus the vector valued Radon measure $D^2 u$ can be decomposed as $D^2 u = [D^2 u]_{\mathrm{ac}} + [D^2 u]_s$, where $d[D^2 u]_{\mathrm{ac}} = \nabla^2 u \, dx$. Now we are in a position to prove the theorem 1.1. To carry out the proof, we use a similar approach to Evans and Gariepy, see, Section 6.4, in [3].

Proof of Theorem 1.1. Let $n \ge 2$ and u be a k-convex function on $\mathbb{R}^n$, $k > n/2$. Then by Theorem 2.4, and Theorem 2.5, we have for $\mathcal{L}^n$ a.e. x

$$\lim_{r \to 0} \fint_{B(x,r)} |\nabla u(y) - \nabla u(x)| \, dy \; = \; 0 \,, \tag{3.1}$$

$$\lim_{r \to 0} \fint_{B(x,r)} |\nabla^2 u(y) - \nabla^2 u(x)| \, dy \; = \; 0 \tag{3.2}$$

and

$$\lim_{r\to 0} \frac{|[D^2u]_s|(B(x,r))}{r^n} = 0\,. \tag{3.3}$$

where $⨍_E f\,dx$ we denote the mean value $(\mathcal{L}^n(E))^{-1}\int_E f\,dx$. Fix a point x for which (3.1)-(3.3) holds. Without loss generality we may assume $x = 0$. Then following similar calculations as in the proof of Theorem 1, Section 6.4 in [3], we obtain,

$$⨍_{B(r)} \left| u(y) - u(0) - \langle \nabla u(0), y\rangle - \frac{1}{2}\left\langle \nabla^2 u(0)y, y\right\rangle \right| dy = o(r^2)\,, \tag{3.4}$$

as $r \to 0$. In order to establish

$$\sup_{B(r/2)} \left| u(y) - u(0) - \langle \nabla u(0), y\rangle - \frac{1}{2}\left\langle \nabla^2 u(0)y, y\right\rangle \right| = o(r^2) \quad \text{as } r \to 0\,, \tag{3.5}$$

we need the following lemma.

Lemma 3.1. *Let* $h(y) := u(y) - u(0) - \langle \nabla u(0), y\rangle - \frac{1}{2}\left\langle \nabla^2 u(0)y, y\right\rangle$. *Then there exists a constant* $C > 0$ *depending only on* n, k *and* $|\nabla^2 u(0)|$, *such that for any* $0 < r < 1$

$$\sup_{\substack{y,z\in B(r)\\ y\neq z}} \frac{|h(y)-h(z)|}{|y-z|^\alpha} \le \frac{C}{r^\alpha}⨍_{B(2r)} |h(y)|\,dy + Cr^{2-\alpha}\,, \tag{3.6}$$

where $\alpha := (2 - n/k)$.

Proof. Let $\Lambda := |\nabla^2 u(0)|$ and define $g(y) := h(y) + \frac{\Lambda}{2}|y|^2$. Since $\frac{\Lambda}{2}|y|^2 - u(0) - \langle \nabla u(0), y\rangle - \frac{1}{2}\left\langle \nabla^2 u(0)y, y\right\rangle$ is convex and sum of two k-convex functions are k-convex (follows from (1.4)), we conclude that g is k-convex. Applying the Hölder estimate in (2.1) for g with $\Omega' = B(2r)$, there exists $C := C(n,k) > 0$, such that

$$\begin{aligned}
r^{n+\alpha} \sup_{\substack{y,z\in B(r)\\ y\neq z}} \frac{|g(y)-g(z)|}{|y-z|^\alpha} &= \operatorname{dist}(B(r), \partial B(2r))^{n+\alpha} \sup_{\substack{y,z\in B(r)\\ y\neq z}} \frac{|g(y)-g(z)|}{|y-z|^\alpha} \\
&\le \sup_{\substack{y,z\in B(2r)\\ y\neq z}} d_{y,z}^{n+\alpha} \frac{|g(y)-g(z)|}{|y-z|^\alpha} \\
&\le C \int_{B(2r)} |g(y)|\,dy \\
&\le C \int_{B(2r)} |h(y)|\,dy + Cr^{n+2}\,,
\end{aligned} \tag{3.7}$$

where $d_{y,z} := \min\{\mathrm{dist}(y, \partial B(2r)), \mathrm{dist}(z, \partial B(2r))\}$. Therefore the estimate (3.6) for h follows from the estimate (3.7) and the definition of g. $\square$

Proof of Theorem 1.1. (ctd.) To prove (3.5), take $0 < \epsilon, \delta < 1$, such that $\delta^{1/n} \leq 1/2$. Then there exists r_0 depending on ϵ and δ, sufficiently small, such that, for $0 < r < r_0$

$$\begin{aligned}\mathcal{L}^n\left\{z \in B(r) \ : \ |h(z)| \geq \epsilon r^2\right\} &\leq \frac{1}{\epsilon r^2}\int_{B(r)} |h(z)|\,dz \\ &= o(r^n) \qquad \text{by} \quad (3.4) \\ &< \delta \mathcal{L}^n(B(r)) \end{aligned} \tag{3.8}$$

Set $\sigma := \delta^{1/n} r$. Then for each $y \in B(r/2)$ there exists $z \in B(r)$ such that

$$|h(z)| \leq \epsilon r^2 \qquad \text{and} \qquad |y - z| \leq \sigma.$$

Hence for each $y \in B(r/2)$, we obtain by (3.4) and (3.6),

$$\begin{aligned}|h(y)| &\leq |h(z)| + |h(y) - h(z)| \\ &\leq \epsilon r^2 + C|y - z|^\alpha \left(\frac{1}{r^\alpha} ⨍_{B(2r)} |h(y)|\,dy + r^{2-\alpha}\right) \\ &\leq \epsilon r^2 + C\delta^{\alpha/n} r^\alpha \left(\frac{1}{r^\alpha} ⨍_{B(2r)} |h(y)|\,dy + r^{2-\alpha}\right) \\ &\leq \epsilon r^2 + C\delta^{\alpha/n} \left(⨍_{B(2r)} |h(y)|\,dy + r^2\right) \\ &= r^2\left(\epsilon + C\delta^{\alpha/n}\right) + o(r^2) \qquad \text{as} \quad r \to 0\end{aligned}$$

By choosing δ such that, $C\delta^{\alpha/n} = \epsilon$, we have for sufficiently small $\epsilon > 0$ and $0 < r < r_0$,

$$\sup_{B(r/2)} |h(y)| \leq 2\epsilon r^2 + o(r^2)\,.$$

Hence

$$\sup_{B(r/2)} \left| u(y) - u(0) - \langle \nabla u(0), y\rangle - \frac{1}{2}\left\langle \nabla^2 u(0) y, y\right\rangle \right| \, dy = o(r^2) \ \text{ as } r \to 0\,.$$

This proves (1.7) for $x = 0$ and hence u is twice differentiable at $x = 0$. Therefore u is twice differentiable at every x and satisfies (1.7), for which (3.1)-(3.3) holds. This proves the theorem. $\square$

Let u be a k-convex function and $\mu_k[u]$ be the associated k-Hessian measure. Then $\mu_k[u]$ can be decomposed as the sum of a regular part $\mu_k^{\mathrm{ac}}[u]$ and a singular part $\mu_k^s[u]$. As an application of the Theorem 1.1, we prove the following theorem.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \Phi^k(\Omega)$, $k > n/2$. Then the absolute continuous part of $\mu_k[u]$ is represented by the k-Hessian operator $F_k[u]$. That is*

$$\mu_k^{\mathrm{ac}}[u] = F_k[u]\,dx\,. \tag{3.9}$$

Proof. Let u be a k-convex function, $k > n/2$ and u_ϵ be the mollification of u. Then by (1.5) and properties of mollification (see, for example Theorem 1, Section 4.2 in [3]) it follows that $u_\epsilon \in \Phi^k(\Omega) \cap C^\infty(\Omega)$. Since u is twice differentiable a.e. (by Theorem 1.1) and $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$ (by Theorem 2.5), we conclude that $\nabla^2 u_\epsilon \to \nabla^2 u$ in L^1_{loc}. Let $\mu_k[u_\epsilon]$ and $\mu_k[u]$ are the Hessian measures associated to the functions u_ϵ and u respectively. Then the by weak continuity Theorem 2.3 (Theorem 1.1, [13]), $\mu_k[u_\epsilon]$ converges to $\mu_k[u]$ in measure and $\mu_k[u_\epsilon] = F_k[u_\epsilon]\,dx$. It follows that for any compact set $E \subset \Omega$,

$$\mu_k[u](E) \geq \limsup_{\epsilon\to 0} \mu_k[u_\epsilon](E) = \limsup_{\epsilon\to 0} \int_E F_k[u_\epsilon]\,. \tag{3.10}$$

Since $F_k[u_\epsilon] \geq 0$ and $F_k[u_\epsilon](x) \to F_k[u](x)$ a.e., by Fatou's lemma, for every relatively compact measurable subset E of Ω, we have

$$\int_E F_k[u] \leq \liminf_{\epsilon\to 0} \int_E F_k[u_\epsilon]\,. \tag{3.11}$$

Therefore by Theorem 3.1, [13], it follows that $F_k[u] \in L^1_{\mathrm{loc}}(\Omega)$. Let $\mu_k[u] = \mu_k^{\mathrm{ac}}[u] + \mu_k^{\mathrm{s}}[u]$, where $\mu_k^{\mathrm{ac}}[u] = h\,dx$, $h \in L^1_{\mathrm{loc}}(\Omega)$ and $\mu_k^{\mathrm{s}}[u]$ is the singular part supported on a set of Lebesgue measure zero. We would like to prove that $h(x) = F_k[u](x)$ $\mathcal{L}^n$ a.e. x. By taking $E := \overline{B}(x,r)$, from (3.10) and (3.11), we obtain

$$⨍_{\overline{B}(x,r)} F_k[u]\,dy \leq \frac{\mu_k[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))} = ⨍_{\overline{B}(x,r)} h\,dy + \frac{\mu_k^{\mathrm{s}}[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))}\,. \tag{3.12}$$

Hence by letting $\epsilon \to 0$, we obtain

$$F_k[u](x) \leq h(x) \quad \mathcal{L}^n \text{ a.e. } x. \tag{3.13}$$

To prove the reverse inequality, let us recall that h is the density of the absolute continuous part of the measure $\mu_k[u]$, that is for $\mathcal{L}^n$ a.e. x

$$h(x) = \lim_{r\to 0} \frac{\mu_k^{\mathrm{ac}}[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))} = \lim_{r\to 0} \frac{\mu_k[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))}\,. \tag{3.14}$$

Since $\mu_k^s[u]$ is supported on a set of Lebesgue measure zero,

$$\mu_k^s[u](\partial B(x,r)) = 0, \quad \mathcal{L}^1 \text{ a.e. } r > 0.$$

Therefore by the weak continuity of $\mu_k[u_\epsilon]$ (see, for example Theorem 1, Section 1.9 [3]), we conclude that

$$\lim_{\epsilon \to 0} \mu_k[u_\epsilon](B(x,r)) = \mu_k[u](B(x,r)), \quad \mathcal{L}^1 \text{ a.e. } r > 0. \tag{3.15}$$

Let $\delta > 0$, then for $\epsilon < \epsilon' = \epsilon(\delta)$ and for $\mathcal{L}^1$ a.e. $r > 0$, $\mathcal{L}^n$ a.e. x

$$\begin{aligned} h(x) &\leq \lim_{r\to 0} \frac{(1+\delta)\mu_k[u_\epsilon](B(x,r))}{\mathcal{L}^n(B(x,r))} \\ &= (1+\delta) \lim_{r\to 0} \fint_{B(x,r)} F_k[u_\epsilon]\, dy \\ &= (1+\delta) F_k[u_\epsilon](x) \end{aligned} \tag{3.16}$$

By letting $\epsilon \to 0$ and finally $\delta \to 0$, we obtain

$$h(x) \leq F_k[u](x), \mathcal{L}^n \text{ a.e. } x.$$

This proves the theorem. □

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