A p-LAPLACIAN APPROXIMATION FOR SOME MASS OPTIMIZATION PROBLEMS

GUY BOUCHITTÉ[1] GIUSEPPE BUTTAZZO[2] LUIGI DE PASCALE[3]

Date: May 29, 2002.
[1]Département de Mathématiques, Université de Toulon et du Var, BP 132, 83957 LA GARDE Cedex (France).
E-mail: `bouchitte@univ-tln.fr`
[2]Dipartimento di Matematica, Università di Pisa, Via Buonarroti 2, 56127 PISA (Italy).
E-mail: `buttazzo@dm.unipi.it`
[3]Dipartimento di Matematica Applicata, Università di Pisa, Via Bonanno 25/B, 56126 PISA (Italy).
E-mail: `depascal@dm.unipi.it`

Abstract. We show that the problem of finding the best mass distribution, both in the case of conductivity and elasticity cases, can be approximated by means of solutions of a p-Laplace equation, as $p \to +\infty$. This seems to provide a selection criterion when the optimal solutions are nonunique.

Mathematics Subject Classification (2000): 49J45, 49L25

Keywords: Shape optimization, p-Laplacian, Mass transportation problems, Monge-Kantorovich PDE.

1. Introduction

In some recent papers [1], [2] Bouchitté , Buttazzo and Seppecher considered a rather general formulation for the problem of finding the optimal distribution of a given amount of material in order to obtain the most performant conductor (in the case of a scalar state function u, the temperature of the system) or elastic body (in the case of mechanical structures). The existence of an optimal solution was shown to occur in the class of nonnegative measures, under the sole assumption that the data (the heat sources in the scalar case, the forces in the case of elasticity) are bounded measures, possibly concentrated on sets of lower dimension. The precise formulation of the mass optimization problem will be recalled in Section 2.

In [1] (see also [2]) the problem was shown to be equivalent to a system of PDE of Monge-Kantorovich type which, in the scalar case, is in turn equivalent to a problem of optimal mass transportation with a cost proportional to the Euclidean distance, of the form extensively illustrated in [3] and in [4].

Due to the general framework considered, when the data are singular measures, the uniqueness of an optimal solution may fail. Then it becomes important, also from a numerical point of view, to find a criterion which, by means of an approximation procedure, selects in a canonical way the *"best solution"* among all the ones that solve the optimization problem.

Here we propose the approximation of the Monge-Kantorovich PDE through the p-Laplace equation, as $p \to +\infty$. This idea was already used in [3] in the case of data regular enough, to show that in this case the optimal mass distribution density is actually a L^∞ function. However, as it has been shown recently in [5] (see also [6]), data which are absolutely continuous with respect to the Lebesgue measure always produce optimal mass densities which are unique and Lebesgue integrable. All these results are obtained in the scalar case, thanks to the equivalence with the mass transportation problem, while nothing is known in the case of elasticity.

We show that for all right-hand sides which are measures, and in both the scalar as well as the vector case, the approximation with p-power energies always produces a solution to the mass optimization problem. The numerical counterpart of these results has recently been developed by Golay and Seppecher in [7].

In the final section we give some examples and raise the question whether the solution determined by approximation with p-power energies is unique or not.

2. The mass optimization problem

In the following we always assume that Ω is a bounded connected open subset of $\mathbb{R}^N$ with a smooth boundary and Σ is a closed subset of $\overline{\Omega}$. The notations $|z|$ and $z \cdot z'$ respectively denote the euclidean norm and the scalar product of vectors in $\mathbb{R}^N$. The same notation will be kept when z, z' are matrices in $\mathbb{R}^{N^2}$. We also consider a given right-hand side f which we assume to be a bounded measure (real valued in the scalar case and $\mathbb{R}^N$-valued in the case of elasticity), and a stored energy density described by a function $j(z)$. The customary choice for the scalar case is

$$j(z) = \frac{1}{2}|z|^2$$

whereas in the case of linear elasticity j is given by

$$j(z) = \beta|z^{sym}|^2 + \frac{\alpha}{2}|tr(z^{sym})|^2$$

being α and β the so called Lamé constants, and where z^{sym} denotes the symmetric part of the matrix z. More generally, we consider a stored energy density function j of the form:

$$j(z) = \frac{1}{p}(\rho(z))^p \ ,$$

where ρ satisfies conditions a), b), c), d) below:

a) ρ is convex;

b) ρ is positively 1-homogeneous;

c) $\rho(z) = \rho(z^{sym})$ for all z;

d) there exist two positive constants k_1 and k_2 such that

$$k_1|z^{sym}| \leq \rho(z) \leq k_2|z^{sym}| \qquad \forall z \in \mathbb{R}^{N \times N}.$$

From now on, unless differently specified, we refer to the case of elasticity; it is intended that similar statements hold for the scalar case as well.

Mass distributions in $\overline{\Omega}$ will be described by nonnegative measures μ with support in $\overline{\Omega}$. For such a measure we consider the energy

$$\mathcal{E}(\mu) = \inf\Big\{ \int j(Du)\,d\mu - \langle f, u\rangle \ : \ u \in \mathcal{U}(\Sigma)\Big\}, \tag{1}$$

where $\mathcal{U}(\Sigma)$ denotes the class of smooth displacements $u : \mathbb{R}^N \to \mathbb{R}^N$ vanishing on Σ, and the corresponding compliance $\mathcal{C}(\mu) = -\mathcal{E}(\mu)$. By standard duality argument, we can also write the compliance $\mathcal{C}(\mu)$ in the form

$$\mathcal{C}(\mu) = \inf\Big\{ \int j^*(\sigma)\,d\mu \ : \ \sigma \in L^{p'}_\mu(\mathbb{R}^N;\mathbb{R}^{N^2}_{\rm sym}),\ -\operatorname{div}(\sigma\mu) = f \text{ in } \mathbb{R}^N \setminus \Sigma\Big\}. \tag{2}$$

Remark 2.1. In (2) the condition $-\operatorname{div}\sigma\mu = f$ is intended in the sense of distributions on the open set $\mathbb{R}^N \setminus \Sigma$. Then it contains implicitely interface conditions on $\partial\Omega \setminus \Sigma$; for instance if μ is the Lebesgue measure on Ω and the field σ is regular enough, then the normal trace $\sigma\nu_\Omega$ (being ν_Ω the exterior unit normal on $\partial\Omega$) vanishes on $\partial\Omega \setminus \Sigma$.

The mass optimization problem can now be written as the minimization problem for $\mathcal{C}(\mu)$ once the total mass of μ and a bound on its support region are prescribed:

$$\inf\Big\{\mathcal{C}(\mu) \ : \ \int d\mu = m,\ \operatorname{spt}\mu \subset \overline{\Omega}\Big\}. \tag{3}$$

A key result in [1] (cf. Theorem 2.3) states that the infimum in (3) is directly related with the following gradient constraint optimization problem:

$$I = I(f,\Omega,\Sigma) := \sup\big\{\langle f,u\rangle \ : \ u \in \mathcal{U}(\Sigma)\ ,\ \rho(Du) \le 1 \text{ on } \Omega\big\}.$$

Indeed we have

$$\inf(3) \ = \ \frac{I^{p'}}{p' m^{1/(p-1)}}\ .$$

Moreover the infimum in (3) is actually a minimum provided $I(f,\Omega,\Sigma)$ is finite, which (see [1], Proposition 3.1) turns out to be equivalent to the following compatibility condition:

$$\langle f,u\rangle \ = 0 \quad \forall u \in \mathcal{R}_\Sigma = \{r(x) = Ax + b :\ b \in \mathbb{R}^N, A \in \mathbb{R}^{N^2}_{\rm skew}, r = 0 \text{ on } \Sigma\}, \tag{4}$$

where $\mathbb{R}^{N^2}_{\rm skew}$ denotes the family of skew-symmetric $N \times N$ matrices. In the scalar case, the previous condition changes so that either Σ is not empty or that the source term is balanced, i.e. $\int df = 0$.

In order to relax the maximization problem above, we introduce the class $\mathrm{Lip}_{1,\rho}(\Omega,\Sigma)$ as the closure in $C(\overline{\Omega};\mathbb{R}^N)$ of the set $\{u \in \mathcal{U}(\Sigma) \ : \ \rho(Du) \leq 1 \text{ on } \Omega\}$. A characterization of $\mathrm{Lip}_{1,\rho}(\Omega,\Sigma)$ is given in the next section. It is then easy to check that under condition (4)

$$I = \max\left\{\langle f, u\rangle \ : \ u \in \mathrm{Lip}_{1,\rho}(\Omega,\Sigma)\right\}. \tag{5}$$

The dual formulation of (5) involves measure valued stress vector fields and by standard convex analysis, we find (see [1]) that I agrees with the quantity

$$\min\left\{\int \rho^0(\lambda) \ : \ \lambda \in \mathcal{M}(\Omega, \mathbb{R}^{N\times N}), \ -\operatorname{div}(\lambda) = f \text{ in } \mathbb{R}^N \setminus \Sigma\right\} \tag{6}$$

where ρ^0 is the polar function associated to ρ:

$$\rho^0(z) = \sup\left\{z \cdot \xi \ : \ \rho(\xi) \leq 1\right\},$$

and where the integral is intended in the sense of convex 1-homogeneous functionals on measures.

A second key result (see [1], Theorem 2.3) states the following implication:

$$\lambda \quad \text{solution of (6)} \quad \Rightarrow \quad \mu = \frac{m}{I}\,\rho^0(\lambda) \quad \text{solution of (3)}\ . \tag{7}$$

As a consequence, the solutions of the mass optimization problem (3) *do not depend on the exponent* p. Then it seems natural to send p to infinity interpreting (5) as the limit problem associated with the strain problem (1). This procedure is the aim of Section 4.

To complete the picture, we now give the system of optimality conditions which allows us to characterize the solutions of the different problems described above. In [1] it is proved that if μ solves the mass optimization problem (3) and if u solves (5), then (6) has a solution of the form $\lambda = \sigma\mu$, $\sigma \in L^1_\mu(\mathbb{R}^N, \mathbb{R}^{N\times N})$ and the triple $(u, m\sigma/I, I\mu/m)$ satisfies the following system (that we call Monge-Kantorovich equation). Let us stress that in (8), both u and μ are unknown.

$$\begin{cases} i) \quad -\operatorname{div}(\sigma\mu) = f \quad \text{on } \mathbb{R}^N \setminus \Sigma \\ ii) \quad \sigma \in \partial j_\mu\big(x, e_\mu(u)\big) \quad \mu\text{-a.e. on } \mathbb{R}^N \\ iii) \quad u \in \mathrm{Lip}_{1,\rho}(\Omega, \Sigma) \\ iv) \quad j_\mu\big(x, e_\mu(u)\big) = 1/p \quad \mu\text{-a.e. on } \mathbb{R}^N \\ v) \quad \mu(\Sigma) = 0 \ (\operatorname{spt}\mu \subset \overline{\Omega}) \ . \end{cases} \tag{8}$$

Here the tangential strain $e_\mu(u)$ and the integrand j_μ are defined in a precise way which is related to some notion of tangent space to measure μ which will be recalled in Section 3.

On the other hand, up to a rescaling, the converse also holds (see [1], Theorem 3.9): if a triple (u, σ, μ) solves the Monge-Kantorovich equation (8), then u is a solution of problem (5), the measure $m\mu/I$ is a solution of the mass optimization problem (3) and $\sigma\mu$ is optimal for (6) yielding that $I\sigma/m$ is a solution of the stress problem (2) related to optimal measure $m\mu/I$.

Let us point out that in the scalar isotropic case, conditions ii) and iv) above become simpler and Monge-Kantorovich equation can be rewritten as follows (see [8])

$$\begin{cases} i) \quad -\operatorname{div}(\nabla_\mu u\, \mu) = f \quad \text{on } \mathbb{R}^N \setminus \Sigma \\ ii) \quad u \in \mathrm{Lip}_1(\Omega, \Sigma) \\ iii) \quad |\nabla_\mu u| = 1, \quad \mu\text{-a.e. on } \mathbb{R}^N \\ iv) \quad \mu(\Sigma) = 0 \ (\operatorname{spt}\mu \subset \overline{\Omega}) \ . \end{cases} \tag{9}$$

3. Some preliminary results

In order to exploit equation (8) it is convenient to give a simpler characterization of the space $\mathrm{Lip}_{1,\rho}(\Omega, \Sigma)$. We notice that in the scalar case when $\rho(z) = |z|$ (or more generally when $\rho(z) \geq |z|$) , every function in $\mathrm{Lip}_{1,\rho}(\Omega, \Sigma)$ is Lipschitz continuous; this is no more true in the case of elasticity, because of the lack of Korn inequality in $W^{1,\infty}$ (see [9]). The following space introduced in [9] will be useful:

$$U^\infty(\Omega; \mathbb{R}^N) = \big\{ u \in L^\infty(\Omega; \mathbb{R}^N) \ : \ e(u) \in L^\infty(\Omega, \mathbb{R}^{N\times N}) \big\}. \tag{10}$$

Thanks to the Korn inequalities it turns out that $U^\infty(\Omega;\mathbb{R}^N)$ is the intersection of the spaces $\{u \in W^{1,s}(\Omega;\mathbb{R}^N) \ : \ e(u) \in L^\infty(\Omega,\mathbb{R}^{N\times N})\}$ for $1 \le s < +\infty$.

Theorem 3.1. *We have*

$$\mathrm{Lip}_{1,\rho}(\Omega,\Sigma) = \big\{u \in U^\infty(\Omega;\mathbb{R}^N) \ : \ u = 0 \text{ on } \Sigma, \ \rho(e(u)) \le 1 \text{ a.e. on } \Omega\big\}.$$

Proof. The proof will be divided in several steps.

Step 1. Let $u \in U^\infty$ be such that $u = 0$ on Σ and $\rho(e(u)) \le 1$ a.e., then $u_n = (1 - 1/n)u$ is still in U^∞, $u_n = 0$ on Σ and $\rho(e(u_n)) \le 1 - 1/n$. Moreover $u_n \to u$ uniformly. Then to prove the inclusion $\supseteq$ it is enough to prove that each function v in U^∞, with $v = 0$ on Σ and $\rho(e(v)) \le c < 1$ can be approximated uniformly by a sequence v_n in $\mathcal{D}(\mathbb{R}^N;\mathbb{R}^N)$ such that $v_n = 0$ on Σ and $\rho(e(v_n)) \le 1$ in Ω.

Step 2. Let v in U^∞ with $v = 0$ on Σ. Fix $\delta > 0$, we now show that there exists $v_\delta \in \mathcal{D}(\mathbb{R}^N;\mathbb{R}^N)$ such that

$$|v - v_\delta| \le \delta \tag{11}$$

$$\big|\|\rho(e(v))\|_\infty - \|\rho(e(v_\delta))\|_\infty\big| \le 3\delta \tag{12}$$

$$v_\delta = 0 \text{ on } \Sigma. \tag{13}$$

By using Step 1, the inclusion $\supseteq$ will then follow.
Since v vanishes on Σ, we have $|\rho(e(v))\|_{L^\infty(\Omega)} = |\rho(e(v))\|_{L^\infty(\Omega\setminus\Sigma)}$ and therefore we may choose a suitable $r > 0$ such that the open set

$$\Omega_0 = \big\{x \in \Omega \ : \ d(x,\Sigma) > 1/r\big\}$$

satisfies

$$\|\rho(e(v))\|_{L^\infty(\Omega_0)} \ge \|\rho(e(v))\|_{L^\infty(\Omega)} - \delta. \tag{14}$$

Then we define the open sets Ω_j, A_j as follows:

$$\Omega_j = \big\{x \in \Omega \ : \ d(x,\Sigma) > 1/(j+r)\big\}\ , \quad A_1 = \Omega_2\ , \quad A_j = \Omega_{j+1} \setminus \overline{\Omega}_{j-1} \quad (j > 1).$$

Clearly $\Omega \setminus \Sigma = \bigcup_j \Omega_j$, $\Omega_j \subset \overline{\Omega_j} \subset \Omega_{j+1}$ and the sets A_j constitute a covering of $\Omega \setminus \Sigma$. Now we consider a partition of unity ϕ_j related to this covering and select a sequence of mollifiers g_{ε_j} with g_{ε_j} compactly supported in the ball $B(0, \varepsilon_j)$ where ε_j is chosen decreasing in such a way that:

$$\begin{aligned}
&\Omega_0 + B(0,\varepsilon_1) \subset A_1,\\
&A_2 + B(0,\varepsilon_2) \subset \Omega_0^c \cap A_3,\\
&A_j + B(0,\varepsilon_j) \subset A_{j-1} \cup A_j \cup A_{j+1}, \quad (\text{for } j \geq 3).
\end{aligned}$$

In addition, since by the continuity of $v\phi_j$, $g_\varepsilon * (\phi_j v)$ converges uniformly to $\phi_j v$ on $\overline{\Omega}$, we can choose ε_j such that:

$$\begin{aligned}
&\|g_{\varepsilon_j} * (\phi_j v) - (\phi_j v)\|_\infty \leq \delta 2^{-j},\\
&\|g_{\varepsilon_j} * \phi_j - \phi_j\|_\infty \leq \delta 2^{-j},\\
&\rho\big(g_{\varepsilon_j} * (\nabla\phi_j \otimes^{sym} v) - (\nabla\phi_j \otimes^{sym} v)\big) \leq \delta 2^{-j},
\end{aligned}$$

where the last choice is allowed by the fact that $\rho(z) \leq k|z|$ for some constant k. Moreover we can choose ε_1 so small that

$$\big|\|\rho\big(g_{\varepsilon_1} * (\phi_1 e(v))\big)\|_{L^\infty(\Omega_0)} - \|\rho(\phi_1 e(v))\|_{L^\infty(\Omega_0)}\big| \leq \delta.$$

Now we consider

$$v_\delta = \sum_{j=1}^{\infty} g_{\varepsilon_j} * (\phi_j v).$$

The function v_δ is in $\mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$ and vanishes on Σ (because each term is zero on Σ). We have, thanks to the previous construction,

$$|v - v_\delta| \leq \sum_{j=1}^{\infty} |g_{\varepsilon_j} * (\phi_j v) - \phi_j v| \leq \delta;$$

moreover $e(v_\delta) = \sum_j g_{\varepsilon_j} * (\nabla\phi_j \otimes^{sym} v + e(v)\phi_j)$ and since $\sum_j \nabla\phi_j = 0$ we obtain

$$e(v_\delta) = \sum_{j=1}^{\infty} \big(g_{\varepsilon_j} * (\nabla\phi_j \otimes^{sym} v) - \nabla\phi_j \otimes^{sym} v\big) + \sum_{j=1}^{\infty} g_{\varepsilon_j} * (e(v)\phi_j).$$

Then, since ρ is convex, sub-additive and 1-homogeneous we obtain

$$\begin{aligned} \rho(e(v_\delta)) &\le \sum_{j=1}^{\infty} \rho\big(g_{\varepsilon_j} * (\nabla\phi_j \otimes^{sym} v) - \nabla\phi_j \otimes^{sym} v\big) + \sum_{j=1}^{\infty} \rho(g_{\varepsilon_j} * (e(v)\phi_j)) \\ &\le \sum_{j=1}^{\infty} \delta 2^{-j} + \delta + \rho(e(v)) \le 2\delta + \rho(e(v)). \end{aligned} \tag{15}$$

On the other hand, if we consider a point $x \in \Omega_0$, we have $v_\delta(x) = g_{\varepsilon_1} * (\phi_1 v)(x)$ and $e(v_\delta)(x) = (g_{\varepsilon_1} * (\phi_1 e(v)))(x)$; therefore

$$\|\rho(e(v_\delta))\|_{L^\infty(\Omega_0)} \ge \|\rho(e(v))\|_{L^\infty(\Omega_0)} - 2\delta,$$

which, together with (14) and (15) concludes the second step.

Step 3. It remains to show the inclusion $\subseteq$. This is a consequence of the Korn inequalities in $W^{1,s}$ for $1 < s < +\infty$. Indeed, let $u \in \mathrm{Lip}_{1,\rho}(\Omega;\Sigma)$ and let u_n be a sequence in $\mathcal{D}(\mathbb{R}^N;\mathbb{R}^N)$ converging to u uniformly, and such that $\rho(e(u_n)) \le 1$ and $u_n = 0$ on Σ. If $s < +\infty$, using the Korn inequality and the growth conditions on ρ we have:

$$\int_\Omega |Du_n|^s\,dx \le C_s \int_\Omega (\rho(e(u_n))^s\,dx \le C_s|\Omega|\ .$$

We infer that u_n is weakly precompact in $W^{1,s}$, and therefore $u \in W^{1,s}$ and $e(u_n) \to e(u)$ weakly in L^s. Since ρ^s is a convex continuous integrand the following lower semicontinuity inequality holds

$$\Big(\int_\Omega \big(\rho(e(u))\big)^s\Big)^{1/s} \le \liminf_{n\to\infty} \Big(\int_\Omega \big(\rho(e(u_n))\big)^s\Big)^{1/s} \le |\Omega|^{1/s}\ .$$

As this is valid for every $s > 1$, the conclusion follows by sending s to infinity.

The proof of Theorem 3.1 is now complete. □

In the scalar isotropic case, the characterization of $\mathrm{Lip}_1(\Omega;\Sigma)$ becomes simpler:

Theorem 3.2. *In the scalar case we have*

$$\mathrm{Lip}_1(\Omega;\Sigma)=\big\{u\in W^{1,\infty}(\Omega)\ :\ u=0\ on\ \Sigma,\ |Du|\le 1\ a.e.\ on\ \Omega\big\}.$$

Proof. The proof of the inclusion $\subseteq$ is similar to the third step of the previous theorem. To prove the opposite inclusion, we consider an element $u\in W^{1,\infty}(\Omega)$ such that $u=0$ on Σ and $|Du|\le 1$ a.e.. Let d_Σ denote the distance function from Σ and define, for every $\varepsilon>0$, the function $v_\varepsilon=\alpha_\varepsilon u$ where α_ε is the following cut-off function:

$$\alpha_\varepsilon(x):=\begin{cases}0 & if\ d_\Sigma(x)\le\varepsilon,\\ \dfrac{d_\Sigma(x)-\varepsilon}{d_\Sigma(x)} & if\ d_\Sigma(x)\in[\varepsilon,\sqrt{\varepsilon}],\\ (1-\sqrt{\varepsilon}) & if\ d_\Sigma(x)\ge\sqrt{\varepsilon}\ .\end{cases}$$

Since u vanishes on Σ we clearly have $u(x)\le d_\Sigma(x)$. Then it is easy to check that v_ε vanishes in an ε-tubular neighborhood of Σ, converges uniformly to u as $\varepsilon\to 0$ and that the norm of the gradient of v_ε remains bounded by 1. Mollifying v_ε will provide the requested approximating sequence for u. $\square$

Here we recall some features of calculus of variations with respect to a measure which are needed to give a precise meaning to equations (8) and (9). These notions have been introduced in [2] and further details can also be found in [1], [10], [11], [8], [12], where this theory has been fruitfully applied. Given $\mu\in\mathcal{M}(\Omega)$, we consider the space of *tangent* vector fields

$$X_\mu=\big\{\varphi\in L^1_\mu\ :\ -\mathrm{div}(\varphi\mu)\in\mathcal{M}(\Omega)\big\}\ ,$$

where the divergence operator is intended in the sense of distributions, that is,

$$\langle\mathrm{div}(\varphi\mu),\psi\rangle:=-\int_\Omega\varphi\cdot\nabla\psi\,d\mu\qquad\forall\psi\in\mathcal{C}_0^\infty(\mathbb{R}^N).$$

Then we locally define the tangent space to μ for μ-a.e. $x\in\Omega$ as:

$$T_\mu(x)=\mu-ess\bigcup\big\{\varphi(x)\ :\ \varphi\in X_\mu\big\}.$$

The μ essential union is defined as a μ-measurable closed multifunction such that:

i) $\varphi \in X_\mu \Rightarrow \varphi(x) \in T_\mu(x)$ for μ-a.e. $x \in \Omega$;

ii) between all the multifunction with the previous property the μ essential union is minimal with respect to the inclusion μ-a.e..

We notice that in the case $\mu = \mathcal{H}^k \llcorner S$, with S a k-dimensional Lipschitz manifold in $\mathbb{R}^N$ $(k \leq n)$, then T_μ coincides μ-a.e. with the usual tangent bundle T_S given by differential geometry.

Once we have the notion of tangent space to μ it is natural to define the notion of μ-tangential gradient of a smooth scalar function u as:

$$\nabla_\mu u(x) = P_{T_\mu(x)}(\nabla u(x)) \quad \mu\text{-a.e.}$$

where $P_{T_\mu(x)}$ denotes the orthogonal projection on $T_\mu(x)$. It turns out that the operator $\nabla_\mu(x)$ is closable in L^p_μ and this allows to define the Sobolev space $W^{1,p}_\mu$ as the domain of the unique closed extension (see [2]). For our purpose an important issue is that, under the condition $\operatorname{spt}\mu \subset \overline{\Omega}$, every element of $\mathrm{Lip}_1(\Omega, \Sigma)$ belongs to $W^{1,p}_\mu$ (for every p) and therefore the equation iii) in (9) is meaningful.

Similarly to what done in the scalar case, for μ-a.e. x, we define the linear space of matrices

$$M_\mu(x) := \mu - ess \bigcup \left\{ \Phi(x) \ : \ \Phi \in X_\mu(\mathbb{R}^N, \mathbb{R}^{N^2}_{\text{sym}}) \right\} , \tag{16}$$

where

$$X_\mu(\mathbb{R}^N, \mathbb{R}^{N^2}_{\text{sym}}) := \left\{ \Phi \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^{N^2}_{\text{sym}}) \ : \ \operatorname{div}(\Phi\mu) \in L^2_\mu(\Omega, \mathbb{R}^N) \right\} . \tag{17}$$

For every smooth vector field ψ on Ω, the μ-tangential strain $e_\mu(\psi)$ can be defined by $P_{M_\mu}(e(\psi))$, being P_{M_μ} the orthogonal projector from $\mathbb{R}^{N^2}_{\text{sym}}$ onto M_μ. In particular, one can show that, if μ is the Lebesgue measure over an open subset of $\mathbb{R}^N$, then $e_\mu(\psi)$ coincides with the usual strain tensor $e(\psi)$, while, if μ is the Hausdorff measure $\mathcal{H}^k$ over a k-dimensional smooth manifold in $\mathbb{R}^N$, then $e_\mu(\psi) = P_{T_\mu}(e(\psi))P_{T_\mu}$.

The operator $e_\mu : L^p_\mu(\mathbb{R}^N; \mathbb{R}^N) \to L^p_\mu(\mathbb{R}^N, \mathbb{R}^{N^2}_{\text{sym}})$, defined either on $\mathcal{C}^\infty_0(\mathbb{R}^N; \mathbb{R}^N)$ or on the space of smooth fields compactly supported in $\mathbb{R}^N \setminus \Sigma$ turns out to be closable with respect to the norm of L^p_μ and the domain of its unique closed extension gives a Banach space $\mathcal{D}^p_\mu(\Omega; \mathbb{R}^N)$

of μ-admissible displacements. and in the latter case the Banach space $\mathcal{D}_\mu^{p,0}$ of μ-admissible displacements vanishing on Σ. Similarly to the scalar case, as we have the inclusion of $\mathrm{Lip}_{1,\rho}(\Omega,\Sigma)$ in $\mathcal{D}_\mu^{p,0}$, the expression $e_\mu(u)$ in the equation ii) of (8) makes sense where the integrand $j_\mu(x,z)$ is defined by

$$j_\mu(x,z) = \inf\{j(x,z+\xi) \ : \ \xi \in (M_\mu(x))^\perp\}. \tag{18}$$

This new integrand j_μ which depends on x through the geometry of μ plays a crucial role in the relaxation process of strain problem (1). Briefly if j satisfies the assumptions a),c),d) of Section 2, then the relaxation in the weak-L^p_μ topology of the functional defined on smooth functions by $\int_\Omega j(x,Du)\,d\mu$ is given by

$$\int_\Omega j_\mu(x,\nabla_\mu u)\,d\mu \quad \text{if } u \in \mathcal{D}^p_\mu(\mathbb{R}^N;\mathbb{R}^N) \text{ (resp. } \mathcal{D}^{p,0}_\mu(\mathbb{R}^N;\mathbb{R}^N)), \quad +\infty \text{ otherwise}$$

($\mathcal{D}_\mu^{p,0}$ if one starts with functions u vanishing on Σ).

We finally will need some extension of the notion of convergence in sense of Young measures and some related results which can be found in [11]. This kind of results can be also found in the framework of the theory of varifolds for which we refer to [13].

Lemma 3.3. *Let $\{\mu_h\} \subset \mathcal{M}$, with $\mu_h \rightharpoonup \mu$. Let d be a positive integer and let $f_h \in (L^p_{\mu_h})^d$ with $\|f_h\|_{(L^p_{\mu_h})^d} \le M$ for some $p \in [1,+\infty[$. Then the sequence $\{f_h\mu_h\}$ is bounded in $\mathcal{M}^d$, and the weak limit of any convergent subsequence is absolutely continuous with respect to μ, with a density $f \in (L^p_\mu)^d$. Moreover for every convex and lower semicontinuous function $\Phi : \mathbb{R}^d \to [0,+\infty[$ we have:*

$$\liminf_{h\to\infty} \int \Phi(f_h)d\mu_h \ge \int \Phi(f)d\mu.$$

Definition 3.4. *Let $\mu_h, \mu \in \mathcal{M}^+$, with $\mu_h \rightharpoonup \mu$ and let $f_h, f : \mathbb{R}^N \to \mathbb{R}^d$ be respectively μ_h and μ integrable. We say that f_h weakly converges to f with respect to the pair (μ_h,μ) and we write $f_h \overset{w(\mu_h,\mu)}{\rightarrow} f$, if $f_h\mu_h \rightharpoonup f\mu$ in $\mathcal{M}^d$.*

When $f_h \stackrel{w(\mu_h,\mu)}{\to} f$ it is natural to consider the sequence $\nu_h \in Y(\mathbb{R}^N, \mu_h; \mathbb{R}^d)$ of Young measures associated to f_h, which describe the oscillation of f_h with respect to the measure μ_h, that is

$$\langle \nu_h, \varphi \rangle = \int \varphi(x, f_h(x))\, d\mu_h(x) \qquad \forall \varphi \in C_0(\mathbb{R}^N \times \mathbb{R}^d).$$

Since $\{\nu_h\}$ is a bounded sequence, by possibly extracting a subsequence we may assume that it converges to some measure ν on the product space $\mathbb{R}^N \times \mathbb{R}^d$ whose marginal on the first factor is μ, and the disintegration of ν yields

$$\langle \nu, \varphi \rangle = \int \langle \varphi(x, \cdot), \nu_x \rangle\, d\mu(x) \qquad \forall \varphi \in C_0(\mathbb{R}^N \times \mathbb{R}^d).$$

This justifies the following definition.

Definition 3.5. *We say that $\{f_h\}$ weakly converges to the family of probabilities (ν_x) in the sense of Young measures with respect to (μ_h, μ), and we write $f_h \stackrel{Y(\mu_h,\mu)}{\to} (\nu_x)$, if*

$$\int \varphi(x, f_h(x))\, d\mu_h(x) \to \int \langle \varphi(x, \cdot), \nu_x \rangle\, d\mu(x) \qquad \forall \varphi \in C_0(\mathbb{R}^N \times \mathbb{R}^d).$$

Proposition 3.6. *Let $f_h \in (L^1_{\mu_h})^d$, with $f_h \stackrel{Y(\mu_h,\mu)}{\to} (\nu_x)$. Then*

i) *For every nonnegative lower semicontinuous function φ on $\mathbb{R}^N \times \mathbb{R}^d$ we have*

$$\liminf_{h\to\infty} \int \varphi(x, f_h(x))\, d\mu_h(x) \geq \int \langle \varphi(x, \cdot), \nu_x \rangle\, d\mu(x) \ .$$

ii) *Assume in addition that $\{f_h\}$ is uniformly bounded in the $(L^p_{\mu_h})^d$-norm, for some $p > 1$ and that $\mu_h \to \mu$ tightly (i.e. $\mu_h(\mathbb{R}^N) \to \mu(\mathbb{R}^N)$). Then, for any continuous function φ on $\mathbb{R}^N \times \mathbb{R}^d$ satisfying a growth condition of the type $|\varphi(x,z)| \leq C(1 + |z|^q)$ with $q < p$, we have*

$$\lim_{h\to\infty} \int \varphi(x, f_h(x))\, d\mu_h(x) = \int \langle \varphi(x, \cdot), \nu_x \rangle\, d\mu(x).$$

By applying Proposition 3.6 ii) with $\varphi(x,z) = z \cdot \Psi(x)$ with $\Psi \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^d)$ we find that $f_h \stackrel{w(\mu_h,\mu)}{\to} f$ where $f(x) = [\nu_x]$ ($[\nu_x] := \int \xi d\nu_x(\xi)$ denotes the first momentum of ν_x). In fact the family $\{\nu_x\}$ describes the oscillations of $\{f_h\}$. Claiming that ν_x is a Dirac mass at $f(x)$ means that no oscillations occur; this situation motivates the following definition:

Definition 3.7. *Let $\mu_h, \mu \in \mathcal{M}^+$, with $\mu_h \rightharpoonup \mu$ and let $f_h, f : \mathbb{R}^N \to \mathbb{R}^d$ be respectively μ_h and μ integrable. We say that f_h strongly converges to f with respect to the pair (μ_h, μ) and we write $f_h \overset{s(\mu_h,\mu)}{\rightharpoonup} f$, if*

$$\lim_{h\to\infty} \int \varphi(x, f_h(x))\, d\mu_h = \int \varphi(x, f(x))\, d\mu, \qquad \forall \varphi \in C_0(\mathbb{R}^N \times \mathbb{R}^d).$$

In Lemma 3.8 below, we give a particular case of strong convergence which will be used in Section 4.

Lemma 3.8. *Let $\{f_p : \ p \in \mathbb{N}\}$ be a sequence of measurable non negative functions such that $\int_\Omega |f_p|^p dx \ \leq C$. Set $\mu_p = |f_p|^{p-2}$ and assume that $\mu_p \rightharpoonup \mu$. Then the following implication holds*

$$\lim_{p\to\infty} \int_\Omega |f_p|^p dx = \lim_{p\to\infty} \int_\Omega |f_p|^{p-1} dx \quad \Rightarrow \quad f_p \overset{s(\mu_p,\mu)}{\rightharpoonup} 1 \ . \tag{19}$$

Proof. Possibly passing to a subsequence, we may assume that $f_p \overset{Y(\mu_p,\mu)}{\rightharpoonup} \nu_x$. Set for every $t \in [0,2]$:

$$h(x,t) := \int |z|^t \nu_x \ .$$

Since $\int |f_p|^2 d\mu_p \leq C$, by Proposition 3.6 i) applied with $\varphi(x,z) = |z|^p$ we have

$$+\infty > \liminf_{p\to\infty} \int_\Omega |f_p|^p dx = \liminf_{p\to\infty} \int_\Omega |f_p|^2 d\mu_p \geq \int_{\overline{\Omega}} h(x,2) d\mu \ .$$

Therefore $h(x,\cdot)$ is continuous on $[0,2]$ for μ-a.e. $x \in \Omega$. In addition as Ω is bounded, the convergence of μ_h to μ is tight on $\overline{\Omega}$ and applying Proposition 3.6 ii) we obtain that for every $s \in [0,2[$ and for every $\Psi \in C^0(\overline{\Omega})$ we have:

$$\lim_{p\to\infty} \int_\Omega |f_p|^{p-2+s} \Psi \, dx = \lim_{p\to\infty} \int_\Omega |f_p|^s \, \Psi \, d\mu_p = \int_{\overline{\Omega}} h(x,s)\, \Psi(x)\, d\mu. \tag{20}$$

In the previous equalities we may substitute Ψ with the characteristic function of any ball B such that $\mu(\partial B) = 0$ and then, by Hölder inequality, we obtain that for every $0 \le t_1 < t_2 < 2$:

$$\begin{aligned}\int_B h(x,t_2) d\mu &= \lim_{p\to\infty} \int_B |f_p|^{p-2+t_2}\, dx \\ &\ge \lim_{p\to\infty} (\int_B |f_p|^{p-2+t_1}\, dx)^{\frac{p-2+t_2}{p-2+t_1}} \\ &= \int_B h(x,t_1)\, d\mu.\end{aligned}$$

Then we deduce the monotonicity property $h(x,t_1) \le h(x,t_2)$, which can be extended to $t_2 = 2$ using the continuity of $h(x,\cdot)$ on $[0,2]$. This implies: $1 = h(x,0) \le h(x,1) \le h(x,2)$. Now thanks to (20), the equality in the right hand member of (19) implies that

$$\int_{\overline{\Omega}} h(x,1)\, d\mu = \int_{\overline{\Omega}} h(x,2)\, d\mu \ ,$$

hence there exists a suitable $f \in L^2_\mu$ which satisfies, for every $t \in [1,2]$:

$$h(x,t) \ = \ f(x) \qquad \mu \text{ a.e. } x \in \overline{\Omega}.$$

In particular we have that $1 \le f(x)$ μ-a.e.. On the other hand, by Jensen inequality and since $f(x)$ is the barycenter of ν_x:

$$f(x) = h(x,2) = \int |z|^2 \nu_x(dz) \ge (f(x))^2 \ .$$

Therefore $f(x) = 1$ μ a.e. yielding that the previous Jensen inequality becomes an equality. Thus ν_x is the Dirac mass at $f(x)$. □

4. The approximation result

Our approximation procedure for the mass optimization problem focuses on problem (5) introduced in Section 2. From now on, the function ρ is fixed so that it fulfills conditions a),b),c), d) of Section 2.

For every $p > N$, we consider the p-power version of strain problem (1), i.e.

$$\alpha_p := \inf \left\{ \frac{1}{p} \int_\Omega \rho(e(u))^p dx - \langle f,u\rangle : \quad u \in W^{1,p}(\Omega,\mathbb{R}^N) \ , \ u = 0 \text{ on } \Sigma \right\}, \tag{21}$$

expecting that, for p tending to infinity, we will have $\alpha_p \to \alpha$ where

$$\alpha := -I(f,\Omega,\Sigma) = \inf\{\langle -f,u\rangle : \ u \in Lip_{1,\rho}(\Omega,\Sigma)\}\,. \tag{22}$$

Through this section we will assume that the compatibility condition (4) holds ensuring that α is finite; this assumption is needed to have the existence of an optimal mass distribution and then it is not restrictive for our purpose. Under (4) it is easy to check also, using Korn inequality in $W^{1,p}$ (see Lemma 4.4 below) and embedding in the space of continuous functions for $p > N$, that (21) has at least one minimizer u_p. Moreover by standard convex analysis methods, taking into account the growth conditions, we can express α_p in terms of the dual stress problem

$$\alpha_p = -\min\{\frac{1}{p'}\int_\Omega (\rho^0(\sigma))^{p'}dx \ : \ \sigma \in L^{p'}(\Omega,\mathbb{R}^{N\times N}),\ -\operatorname{div}\sigma = f \ in\ \Omega\setminus\Sigma\}. \tag{23}$$

Before stating the main result we complete the picture with the following lemma.

Lemma 4.1. *Let u_p be a minimizer of (21). Then any solution of (23) can be written in the form $\sigma_p = \xi_p(\rho(e(u_p)))^{p-2}$ where ξ_p is an element of $\partial\rho(e(u_p))$. Moreover the following relationship holds:*

$$\int_\Omega \rho(e(u_p))^p dx = \langle f,u_p\rangle = \frac{p}{1-p}\alpha_p. \tag{24}$$

Proof. To prove the first part of this lemma we just notice that any solution σ_p of (23) satisfies the equality

$$\frac{1}{p'}\int_\Omega \rho^0(\sigma_p)^{p'}dx + \frac{1}{p}\int_\Omega \rho(e(u_p))^p dx = \langle f,u_p\rangle = \int_\Omega \sigma_p\cdot e(u_p)dx,$$

from which, by Young inequality, we derive that

$$\rho(e(u_p))\rho^0(\sigma_p) = \sigma_p\cdot e(u_p), \qquad \rho^0(\sigma_p) = \rho(e(u_p))^{p-1}\ . \tag{25}$$

Now if we write $\sigma_p = \xi_p\rho(e(u_p))^{p-2}$, we deduce from (25) that

$$\rho^0(\xi_p) = \rho(e(u_p)), \quad \rho^0(\xi_p)\rho(e(u_p)) = \xi_p\cdot e(u_p)\ . \tag{26}$$

Therefore $\xi_p \in \partial\rho(e(u_p))$. In addition we deduce from (26) that

$$\sigma_p \cdot e(u_p) = \xi_p \cdot e(u_p)\rho(e(u_p))^{p-2} = \rho(e(u_p))^p \ .$$

Then, since $-\operatorname{div}\sigma_p = f$ holds on $\Omega \setminus \Sigma$, we obtain after integrating by parts

$$\langle f, u_p \rangle = \int_\Omega \rho(e(u_p))^p dx \ .$$

The second statement now follows since u_p is a solution of (21), and we have

$$\alpha_p = \frac{1}{p}\int_\Omega \rho(e(u_p))^p dx - \langle f, u_p \rangle.$$

$\square$

The aim of this section is to prove the following result.

Theorem 4.2. *Assume (4) and let $\{(u_p, \sigma_p)\}$ be a sequence of minimizers for (21) and (23). Set:*

$$\mu_p := \rho(e(u_p))^{p-2}, \quad \sigma_p = \xi_p\, \mu_p \ .$$

Then there exist $u \in Lip_{1,\rho}(\Omega, \Sigma)$, $\mu \in \mathcal{M}(\overline{\Omega})$ and suitable rigid displacements $r_p \in \mathcal{R}_\Sigma$, such that, possibly extracting a subsequence which we still denote by u_p:

(i) *$u_p - r_p \to u$ uniformly;*

(ii) *$\mu_p \rightharpoonup \mu$ in $\mathcal{M}(\overline{\Omega})$;*

(iii) *$\xi_p \mu_p \rightharpoonup \xi\mu$ in $\mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$;*

(iv) *u is a minimizer of (5) and (u, ξ, μ) is a solution of (8).*

Moreover if the set $\{\rho^0 \leq 1\}$ is strictly convex, then the strong convergence $\xi_p \overset{s(\mu_p,\mu)}{\to} \xi$ holds in the sense of Definition 3.7.

Remark 4.3. In the scalar case of conductivity, Theorem 4.2 can be reformulated as follows: assume that $\Sigma = \emptyset$ (or that $\int f = 0$) and let u_p be the unique minimizer (up to an additive constant) of:

$$-\Delta_p u_p = f \quad \text{on } \Omega \setminus \Sigma, \quad u_p = 0 \quad \text{on } \Sigma, \quad \frac{\partial u_p}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Sigma \ .$$

Then by possibly extracting a subsequence, we have:

$$u_p \to u \quad \text{uniformly}, \quad |\nabla u_p|^{p-2} \rightharpoonup \mu \quad \text{in } \mathcal{M}(\overline{\Omega}), \quad |\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \nabla_\mu \mu \quad \text{in } \mathcal{M}(\overline{\Omega}, \mathbb{R}^N) \ ,$$

where (u, μ) solves (9). In addition there is the strong convergence $\nabla u_p \overset{s(\mu_p,\mu)}{\to} \nabla_\mu$.

The proof will be obtained after several lemmas and propositions.

Lemma 4.4 (Korn inequality). *Let $q \in (1, +\infty)$; then there exists a constant C_q such that for every $v \in W^{1,q}(\Omega, \mathbb{R}^N)$:*

$$\inf \left\{ \|v - r\|_{W^{1,q}(\Omega,\mathbb{R}^N)} \ , \ r \in \mathcal{R}_\Sigma \right\} \ \leq \ C_q \ \|e(v)\|_{L^q(\Omega,\mathbb{R}^{N^2})} \ . \tag{27}$$

Proof. The conclusion follows by contradiction considering a sequence $\{v_h\}$ such that $dist(v_h, \mathcal{R}_\Sigma) = 1$, $\|e(v_h)\|_{L^q} \to 0$ and applying the classical Korn inequality in $W^{1,q}(\Omega, \mathbb{R}^N)$. □

Remark 4.5. The fact that Korn inequality fails if $q = +\infty$ (or $q = 1$) implies that the optimal constant C_q in (27) tends to infinity as $q \to \infty$ (or as $q \to 1$). In the scalar case (27) is valid replacing $e(v)$ by ∇v and $\mathcal{R}_\Sigma$ by the null space if $\Sigma \neq \emptyset$ (or by constant functions if $\Sigma \neq \emptyset$ and Ω is connected). It is straightforward from Poincaré inequality that the constant C_q is uniformly bounded in this case.

Proposition 4.6 (Γ-convergence). *Let us consider the following functionals on $C^0(\overline{\Omega}, \mathbb{R}^N)$:*

$$\begin{aligned} F_p(v) &= \frac{1}{p}\int_\Omega (\rho(Dv))^p dx \ \text{ if } v \in W^{1,p}(\Omega, \mathbb{R}^N) \ , \ v = 0 \text{ on } \Sigma \quad (= +\infty \text{ otherwise}) \ , \\ F_\infty(v) &= 0 \quad \text{if } v \in Lip_{1,\rho}(\Omega, \Sigma) \quad (= +\infty \ \textit{otherwise}) \ . \end{aligned}$$

Then F_p $\Gamma-$converges to F_∞ with respect to the uniform convergence of continuous functions.

Proof. As the $\Gamma-$limsup inequality is obvious we only prove the $\Gamma-$liminf inequality. Let $v_p \to v$ uniformly in Ω and assume without loss of generality that $\{F_p(v_p)\} \leq C$. Noticing that for every $t > 0$ the function $s \mapsto (t^s - 1)/s$ is monotone increasing on $(0, +\infty)$, we infer that

$$q \leq p \quad \Rightarrow \quad F_q(w) \ \leq F_p(w) + (\frac{1}{q} - \frac{1}{p})|\Omega|, \quad w \in W^{1,p}(\Omega, \mathbb{R}^N) \ . \tag{28}$$

Therefore, for every $q > 1$, we have $\sup_{p \geq q} F_q(v_p) < +\infty$. Then from growth conditions and Lemma 4.4 we deduce that v_p belongs to $W^{1,q}$ for p large enough and that $v_p \rightharpoonup v$ in $W^{1,q}$. Thanks to the semicontinuity of the convex functional

$$w \mapsto \int_\Omega (\rho(e(w)))^q dx$$

we obtain

$$\left(\int_\Omega (\rho(e(v)))^q dx\right)^{1/q} \leq |\Omega|^{1/q} C^{1/q}, \qquad \forall q > 1. \tag{29}$$

Passing to the limit in (29) as $q \to \infty$ we obtain $e(v) \in L^\infty$ and $\rho(e(v)) \leq 1$. Since clearly $v = 0$ on Σ, the conclusion follows from Theorem 3.1. □

Proposition 4.7 (Boundedness). *Let $\{u_p\}$ be a sequence of minimizers for (21). Then there exists a constant C such that :*

$$\int_\Omega (\rho(e(u_p)))^p dx \leq C. \tag{30}$$

Proof. By (24) it is enough to prove that α_p has a uniform lower bound. We know that α_q is finite for every $q > N$ and that by (28) we have $\alpha_q \leq \alpha_p + (\frac{1}{q} - \frac{1}{p})|\Omega|$. The conclusion then follows. □

Corollary 4.8. *For each fixed $q \in [1, \infty[$ the family $\{e(u_p)\}$ is bounded in $L^q(\Omega, \mathbb{R}^{N^2})$ and there exist rigid motions $\{r_p\}$ in $\mathcal{R}_\Sigma$ such that the family $\{u_p - r_p\}$ is bounded in $W^{1,q}(\Omega, \mathbb{R}^N)$ and then it is precompact in $C(\overline{\Omega}, \mathbb{R}^N)$.*

Proof. It follows from Proposition 4.7 and from Lemma 4.4. □

Proof (*of Theorem* 4.2). By (30), (26) and the growth condition on ρ, we have that

$$\sup_p \int_\Omega (1 + |\xi_p|^2) d\mu_p < +\infty \ . \tag{31}$$

Taking into account Corollary 4.8 and Lemma 3.3 , we can then assume that (up to subsequences):

$$\mu_p \rightharpoonup \mu \qquad \text{in } \mathcal{M}(\Omega) \tag{32}$$

$$\xi_p\,\mu_p \rightharpoonup \xi\,\mu \quad \text{in } \mathcal{M}(\Omega, \mathbb{R}^{N\times N}) \tag{33}$$

$$u_p - r_p \to u \quad \text{uniformly on } \Omega\ , \tag{34}$$

where ξ, r_p are suitable elements in $L^2_\mu(\Omega, \mathbb{R}^{N\times N})$ and in $\mathcal{R}_\Sigma$ respectively.

We have to show that the triple (u, ξ, μ) is a solution of equation (8) or equivalently (see Section 2) that:

i) u is a solution of (5)

ii) $\rho^0(\xi) = 1$ μ a.e. and $\lambda := \xi\mu$ is a solution of (6).

Indeed under ii) we deduce from (7) that $\tilde{\mu} := m\mu/I$ $(= m\rho^0(\lambda)/I)$ is optimal for (3) and λ can be written as $\lambda = \sigma\tilde{\mu}$ with $\sigma = I\xi/m$.

Proof of claim i). It is a straightforward consequence of (34) and of the Γ-convergence result of Proposition 4.6. In addition we have the convergence of minima, i.e.

$$\alpha_p \ \to\ \alpha \quad (\alpha = -I(f, \Omega, \Sigma))\ . \tag{35}$$

Proof of claim ii). By (23) and by using the inequality $\frac{s^{p'}}{p'} \geq s - \frac{1}{p}$ $(s > 0)$, we obtain

$$-\alpha_p = \frac{1}{p'}\int_\Omega (\rho^0(\sigma_p))^{p'}\,dx \geq \int_\Omega \rho^0(\sigma_p)\,dx \ - \frac{1}{p}\ .$$

Thus passing to the limit as $p \to \infty$, with the help of (35), (33) and by using Lemma 3.3, we deduce

$$-\alpha \geq \liminf_p \int_\Omega \rho^0(\sigma_p)\,dx = \liminf_p \int_\Omega \rho^0(\xi_p)\,d\mu_p \geq \int_{\overline{\Omega}} \rho^0(\xi)\,d\mu\ . \tag{36}$$

Clearly $\lambda = \xi\mu$ satisfies $-\operatorname{div}\lambda = f$ in $\Omega\setminus\Sigma$ and so problem (6) whose infimum is $-\alpha$ $(= I)$ admits λ as a solution. In turn the inequalities in (36) become equalities and we obtain

$$-\alpha \ = \lim_{p\to\infty} \int_\Omega \ \rho^0(\xi_p)\,d\mu_p \ = \ \int_{\overline{\Omega}} \rho^0(\xi)\,d\mu\ . \tag{37}$$

Let us now prove that $\rho^0(\xi) = 1$ holds μ a.e.. This is a consequence of Lemma 3.8 which we apply to the sequence $f_p := \rho^0(\xi_p)$. Indeed by (26) we have $f_p = \rho^0(\xi_p) = \rho(e(u_p))$, hence $\mu_p = f_p^{p-2}$ and $\int_\Omega |f_p|^p dx < +\infty$ by (30). Moreover, by (24) and (37), we have

$$\lim_{p\to\infty} \int_\Omega |f_p|^2 \, d\mu_p = \lim_{p\to\infty} \int_\Omega (\rho^0(\xi_p))^2 \, d\mu_p = \lim_{p\to\infty} \int_\Omega \rho^0(\xi_p) \, d\mu_p = \lim_{p\to\infty} \int_\Omega |f_p| \, d\mu_p \ . \tag{38}$$

We derive from (19) that $\rho^0(\xi_p)$ converges (strongly) to 1. Applying the assertion i) of Proposition 3.6 with $\varphi(x,z) = \theta(x)\rho^0(z)$ where θ is a nonnegative test function and applying Jensen inequality we infer that $\rho^0(\xi) \leq 1$ holds μ a.e.. The claim ii) now follows since by (37) we have $-\alpha = \mu(\overline{\Omega}) = \int \rho^0(\xi) \, d\mu$

To conclude the proof apply again assertion i) of Proposition 3.6 with $\varphi(x,z) = (\rho^0)^2(z)$. Then the equalities in (38) imply that $(\rho^0)^2$ satisfies Jensen equality with respect to the family of Young measures $\{\nu_x\}$ generated by $\{\xi_p\}$. If the set $\{\rho^0 \leq 1\}$ is strictly convex, the strict convexity of $(\rho^0)^2$ yields that $\nu_x = \delta_{\xi(x)}$, that is the strong convergence of $\{\xi_p\}$. $\square$

5. Examples

The main feature of Theorem 4.2 is the convergence of the measures we denoted by μ_p to a solution μ of the shape optimization problem (up to a normalization factor). We remark that in many cases the mass optimization problem we are considering has many solutions (see for example [1]). An important issue discovered in [1] is that, in the scalar case, such optimal solutions can be written explicitely in terms of optimal transport plans in the following way: we define $d_{\Omega,\Sigma}$ as follows

$$d_{\Omega,\Sigma}(x,y) = \min\{d_\Omega(x,y), d_\Omega(x,\Sigma) + d_\Omega(y,\Sigma)\},$$

where d_Ω denotes the geodesic distance in Ω. Let γ be an optimal transport plan for the following problem:

$$\min_{P_\Sigma(f^+,f^-)} \int_{\Omega\times\Omega} d_{\Omega,\Sigma}(x,y) d\gamma(x,y). \tag{39}$$

For each pair $(x,y) \in spt(\gamma)$ let $[x,y]$ denote the geodesic line joining x and y in $\overline{\Omega}$. If for every Borel set $A \subset \Omega$ we define

$$\mu(A) := \int_{\overline{\Omega}\times\overline{\Omega}} \mathcal{H}^1(A \cap [x,y])d\gamma(x,y), \tag{40}$$

then μ is an optimal mass distribution (see [1], Theorem 4.6) and for each solution u of (5) the triple (u, Du, μ) is a solution of the scalar version of (8). Moreover in [6] it was proved (under the assumption Ω convex and $\Sigma = \emptyset$) that for all the optimal mass distribution μ there exists a suitable optimal plan γ such that μ is the optimal measure associated to γ through formula (40). Still in the same paper [6] it is also proved that if in addition $f \in L^1$ then there is only one optimal mass distribution μ. As we will see, in many situations guessing an optimal plan is much simpler than guessing the optimal mass distributions. Here we would like to discuss some cases in which it is possible to prove that the whole sequence μ_p converges to a particular solution of the limit problem.

Example 5.1. Consider the one-dimensional case: $N = 1$, $\Omega := [-1,1]$ and $f = dx \llcorner (-1,0) - dx \llcorner (0,1)$. In this case the p-Laplacian equation is reduced to

$$-(|u_p'|^{p-2}u_p')' = sgn\ x \quad on\ ]-1,1[\tag{41}$$

whose unique solution is given by

$$u_p = \frac{p}{p-1} sgn\ x\ (1-(1-|x|)^{p/(p-1)}). \tag{42}$$

Then clearly the unique limit of $|u_p'|^{p-2}dx$ is given by $\mu = (1-|x|)dx$ which is also the unique solution of the mass optimization problem.

Example 5.2. Consider $\Omega = \mathbb{R}^2$ and, $f = \{\delta_{(-1,1)} + \delta_{(1,-1)} - \delta_{(1,1)} - \delta_{(-1,-1)}\}$. In this case formula (40) implies that all the solutions μ of the limit problem are supported on the boundary of the unitary square of $\mathbb{R}^2$ and can be written as follows. We denote by A the segment $[(-1,1),(1,1)]$, by B the segment $[(1,1),(1,-1)]$, by C segment $[(1,-1),(-1,-1)]$ and finally by D the segment $[(-1,-1),(-1,1)]$. Then given two numbers α and β in the interval $[0,1]$ we can write an optimal measure μ as follows:

$$\mu = \alpha\mathcal{H}^1 \llcorner A + (1-\alpha)\mathcal{H}^1 \llcorner D + \beta\mathcal{H}^1 \llcorner B + (1-\beta)\mathcal{H}^1 \llcorner C. \tag{43}$$

However the symmetry of the data implies that the solutions of the approximating problems u_p are invariant with respect to the action of π rotations and of symmetry with respect to the lines $y = x$ and $y = -x$. It then follows that the solution μ selected by the $p-$Laplacian is invariant with respect to the same symmetries and therefore is unique and given by:

$$\mu = \frac{1}{2}\mathcal{H}^1 \llcorner A + \frac{1}{2}\mathcal{H}^1 \llcorner D + \frac{1}{2}\mathcal{H}^1 \llcorner B + \frac{1}{2}\mathcal{H}^1 \llcorner C. \tag{44}$$

Example 5.3. Here we consider the Dirichlet problem: $N = 2$, $\Omega = B(0,1)$, $\Sigma = \partial\Omega$ and $f = \delta_{(0,0)}$. Here the transport plans are of the form $\gamma = \delta_{0,0} \otimes \alpha$, α being a probability measure on Σ. Since the distance from $(0,0)$ is constant on Σ, they are all optimal and then according to formula (40) produce the following optimal mass distribution μ given by:

$$\mu(A) = \int_\Sigma \mathcal{H}^1(S_x \cap A)\, d\alpha(x), \tag{45}$$

where S_x denotes the segment $[0,x]$. Making the probability measure α vary in formula (45) we are led to an infinite family of solutions of the mass optimization problem. However the invariance of the data with respect to the action of the group of the rotation of $\mathbb{R}^N$ gives the invariance of (u_p, μ_p) with respect to the action of the same group. It follows that among all the solution given by formula (45), the sequence μ_p selects the one given by the uniform probability $\alpha = \frac{1}{2\pi}\mathcal{H}^1 \llcorner \Sigma$.

Remark 5.4. In the same setting of the previous example we could consider a Dirichlet condition assigned only on a subset Γ of the boundary $\partial\Omega$. In this case the approximating solutions u_p and μ_p are not radial anymore, indeed the radial solutions do not satisfy the Neumann homogeneous conditions (required by the minimization problem) on the remaining part of the boundary $\partial\Omega \setminus \Gamma$. This very simple variant of Example 5.3 particularizes very well the open problem of understanding if the whole sequence μ_p converges to some μ (which is of course an optimal mass distribution), and then of finding a selection criterium in order to identify this particular solution μ .

References

[1] G. Bouchitté, G. Buttazzo: Characterization of optimal shapes and masses through Monge-Kantorovich equation, J. Eur. Math.Soc., **3** (2001), 139-168.

[2] G. Bouchitté, G. Buttazzo, P. Seppecher: Energy with Respect to a Measure and Applications to Low Dimensional Structures, Calc. Var., **5** (1997), 37-54.

[3] L. C. Evans, W. Gangbo: *Differential Equations methods for the Monge-Kantorovich Mass Transfer Problem*, Mem. Amer.Math.Soc. **137**, Providence (1999).

[4] L.C. Evans: *Partial Differential Equations and Monge–Kantorovich Mass Transfer*, Current Developments in Mathematics 1997 (Cambridge, MA), 65–126, Int. Press, Boston, MA, 1999.

[5] M. Feldman, R.J McCann: *Monge's transport problem on a Riemannian manifold* , Trans. Amer. Math. Soc., **354** (2002), 1667–1697.

[6] L. Ambrosio: Lecture Notes on Optimal Transport Problems, Lectures given in Madeira (Portugal), 2-9 July 2000, preprint available on: http://cvgmt.sns.it/people/ambrosio/ .

[7] F. Golay, P. Seppecher: *Locking materials and the topology of optimal shapes*, Eur. J. Mech. A Solids, **20** (4) (2001), 631–644.

[8] G. Bouchitté, G. Buttazzo, P. Seppecher: Shape optimization solutions via Monge-Kantorovich equation, C.R. Acad. Sci. Paris, **324-I** (1997), 1185-1191.

[9] F. Demengel: Déplacements á déformations bornées et champs de contrainte mesures, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **12** (1985), 243–318.

[10] G. Bouchitté, G. Buttazzo, I. Fragalá: Mean curvature of a measure and related variational problems, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **XXV** (4) (1997), 179–196.

[11] G. Bouchitté, G. Buttazzo, I. Fragalá: Convergence of Sobolev spaces on varying manifolds, J. Geom. Anal., **11** (2001), 399–422.

[12] I.Fragalá, C. Mantegazza: *On some notion of tangent space to a measure*, Proc. R. Soc. Edinb., Sect. A, Math., **129** (2) (1999), 331–342.

[13] J.E. Hutchinson: *Second fundamental form for varifolds and the existence of surfaces minimizing curvature*, Indiana Univ. Math. J., **35** (1986), 45–71.

[14] G.Allaire, R.V.Kohn: Optimal design for minimum weight and compliance in plane stress using extremal microstructures, Europ. J. Mech. A/Solids, **12** (6) (1993), 839-878.

[15] T. Bhattacharya, E. Di Benedetto, J. Manfredi: Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems, Rend. Sem. Mat. Univ. Politec. Torino, 1989, Special Issue, (1991), 15–68.

[16] G. Dal Maso: *An introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and their Applications. **8**. Birkhäuser, Basel (1993).

[17] L. De Pascale, A. Pratelli: Regularity properties for Monge transport density and for solutions of some shape optimization problem, Calc. Var. and P.D.E., **14** (2002), 249–274.

[18] I. Ekeland, R. Temam: *Convex Analysis and Variational Problems*, North-Holland Publishing Company-Amsterdam (1976).