TIME-DEPENDENT SYSTEMS OF GENERALIZED YOUNG MEASURES

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Abstract. In this paper some new tools for the study of evolution problems in the framework of Young measures are introduced. A suitable notion of time-dependent system of generalized Young measures is defined, which allows to extend the classical notions of total variation and absolute continuity with respect to time, as well as the notion of time derivative. The main results are a Helly type theorem for sequences of systems of generalized Young measures and a theorem about the existence of the time derivative for systems with bounded variation with respect to time.

1. **Introduction.** The notion of Young measure was introduced by L.C. Young in [29] to describe generalized solutions to minimum problems in the calculus of variations. Since then it has been applied to several problems in the calculus of variations, in control theory, in partial differential equations, and in mathematical economics. For the general theory of Young measures we refer to [3], [4], [7], [15, Chapters 2 and 3], [18], [23], [27], [28, Chapter IV], and [30]. Several applications are devoted to evolution problems (see, e.g., [12], [13], [21], [22], [24], and [25]).

In this paper we introduce some new tools in the theory of Young measures for the study of rate independent evolution problems. To describe the content of this paper, let us consider a problem defined on a time interval I, with space variable x in a compact metric space X, and state variable u in a finite dimensional Hilbert space Ξ. We assume that X is endowed with a given nonnegative Radon measure λ with $\operatorname{supp}\lambda = X$. Given a sequence $\boldsymbol{u}_k = \boldsymbol{u}_k(t,x)$ of functions from $I\times X$ to Ξ, satisfying suitable estimates, it is often possible to extract a subsequence converging, for every $t \in I$, to a Young measure $\boldsymbol{\mu}_t$, which encodes information on the statistics of the space oscillations of $\boldsymbol{u}_k(t,x)$ at time t.

To simplify the notation, the Young measure $\boldsymbol{\mu}_t$ will always be regarded as a measure on $X\times\Xi$, whose projection on X coincides with λ. In this introduction we will never consider the standard disintegration $(\boldsymbol{\mu}_t^x)_{x\in X}$, which is usual in the classical presentation of the theory (see Remark 3.5).

If we want to extend some natural notions, like total variation, absolute continuity, or time derivative, from the original context of time dependent functions to the generalized context of time-dependent Young measures, we need to know the joint oscillations of $\boldsymbol{u}_k(t_1,x),\dots,\boldsymbol{u}_k(t_m,x)$ for every finite sequence $t_1,\dots,t_m$ of

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times. These are described by the Young measure $\boldsymbol{\mu}_{t_1\dots t_m}$, with state space Ξ^m, generated by the sequence of Ξ^m-valued functions $(\boldsymbol{u}_k(t_1,x),\dots,\boldsymbol{u}_k(t_m,x))$. It is easy to see that $\boldsymbol{\mu}_{t_1\dots t_m}$ cannot be derived from the measures $\boldsymbol{\mu}_{t_1},\dots,\boldsymbol{\mu}_{t_m}$. Indeed, these measures give no information on the correlation between the oscillations at different times. The situation is similar to what happens in stochastic processes, where the knowledge of the distribution function of each single random variable is not enough to deduce their joint distribution.

This leads to the notion of system of Young measures, defined as a family $(\boldsymbol{\mu}_{t_1\dots t_m})$, where $t_1,\dots,t_m$ run over all finite sequences of elements of I, with $t_1<\dots<t_m$, and each $\boldsymbol{\mu}_{t_1\dots t_m}$ is a Young measure on X with values in Ξ^m. We assume that $(\boldsymbol{\mu}_{t_1\dots t_m})$ satisfies the following compatibility condition, which is always satisfied when $\boldsymbol{\mu}_{t_1\dots t_m}$ is generated by a sequence of time-dependent functions: if $\{s_1,\dots,s_n\}\subset\{t_1,\dots,t_m\}$ and $s_1<\dots<s_n$, then $\boldsymbol{\mu}_{s_1\dots s_n}$ coincides with the corresponding projection of $\boldsymbol{\mu}_{t_1\dots t_m}$.

The notions of total variation (Definition 8.1), time derivative (Definition 9.4), and absolute continuity (Definition 10.1) can be easily defined in the framework of systems of Young measures in such a way that they coincide with the standard notions in the case of time-dependent functions. The main result of the paper is a version of Helly's Theorem for systems of Young measures (Theorem 8.10): if $(\boldsymbol{\mu}^k_{t_1\dots t_m})$ has uniformly bounded variation, then there exist a system $(\boldsymbol{\mu}_{t_1\dots t_m})$ with bounded variation, a set $\Theta\subset I$, with $I\setminus\Theta$ at most countable, and a subsequence, still denoted $(\boldsymbol{\mu}^k_{t_1\dots t_m})$, such that $\boldsymbol{\mu}^k_{t_1\dots t_m}\rightharpoonup\boldsymbol{\mu}_{t_1\dots t_m}$ weakly* for every finite sequence $t_1,\dots,t_m\in\Theta$ with $t_1<\dots<t_m$.

Another important result provides the existence of the time derivative $\dot{\boldsymbol{\mu}}_t$ for almost every t whenever the family $(\boldsymbol{\mu}_{t_1\dots t_m})$ has bounded variation (Theorem 9.7). The variation can be expressed by an integral involving the time derivatives when $(\boldsymbol{\mu}_{t_1\dots t_m})$ is absolutely continuous (Theorem 10.4).

In the forthcoming papers [10] and [11] we will apply these results to deal with some quasistatic evolution problems with nonconvex energies, which arise in the study of plasticity with softening. Since in these applications the energy functionals have linear growth in some directions, we have to consider the case where the generating sequence $(\boldsymbol{u}_k(t,x))$ is bounded in $L^r_\lambda(X;\Xi)$ only for $r=1$. It is well known that in this case Young measures should be replaced by more general objects, which take into account concentrations at infinity (see [13]). In [1] and [14] this is done by considering a pair (μ^Y,μ^∞), where μ^Y is a Young measure on X with values in Ξ and μ^∞, called the varifold measure, is a measure supported on $X\times\Sigma_\Xi$, where Σ_Ξ denotes the unit sphere in Ξ. Other results on this subject are contained in [19], [20], and [17].

In the spirit of [13], we prefer to present these generalized Young measures in a different way, using homogeneous coordinates to describe the completion of Ξ obtained by adding a point at infinity for each direction. We replace the pair (μ^Y,μ^∞) by a single nonnegative measure μ on $X\times\Xi\times\mathbb{R}$ (Definition 3.9), acting only on continuous functions $f(x,\xi,\eta)$ which are positively homogeneous of degree one in (ξ,η). We assume that μ is supported on the set $\{\eta\geq 0\}$ and that the projection of $\eta\mu$ onto X coincides with λ. We show that, if λ is nonatomic, then the space $L^1_\lambda(X;\Xi)$ can be identified (Definition 3.1) with a dense subset of the space of generalized Young measures (Theorem 5.1).

Using this approach, we are able to prove the results on total variation and time derivatives for systems of Young measures in a context that is general enough for the applications considered in [10] and [11].

2. **A space of homogeneous functions and its dual.** If E is a locally compact space with a countable base and Ξ is a finite dimensional Hilbert space, $M_b(E;\Xi)$ denotes the space of bounded Radon measures on E with values in Ξ, endowed with the norm $\|\nu\| := |\nu|(E)$, where $|\nu|$ denotes the variation of ν. When $\Xi = \mathbb{R}$, the corresponding space will be denoted simply by $M_b(E)$. As usual, $M_b^+(E)$ denotes the cone of nonnegative bounded Radon measures on E. If $\nu \in M_b(E)$ and $f \in L^1_\nu(E;\Xi)$, the measure $f\,\nu \in M_b(E;\Xi)$ is defined by $(f\,\nu)(A) := \int_A f\,d\nu$ for every Borel set $A \subset E$.

By the Riesz Representation Theorem $M_b(E;\Xi)$ can be identified with the dual of $C_0(E;\Xi)$, the space of continuous functions $\varphi\colon E \to \Xi$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. The weak* topology of $M_b(E;\Xi)$ is defined using this duality.

Throughout the paper (X, d) is a given compact metric space and λ is a fixed nonnegative Radon measure on X with $\operatorname{supp}\lambda = X$. The symbol Ξ will denote any finite dimensional Hilbert space. The spaces $L^r(X;\Xi)$, $r \geq 1$, will always refer to the measure λ. If $\mu \in M_b(X;\Xi)$, μ^a and μ^s denote the absolutely continuous and the singular part of μ with respect to λ. Measures in $M_b(X;\Xi)$ which are absolutely continuous with respect to λ will always be identified with their densities, which belong to $L^1(X;\Xi)$. In this way $L^1(X;\Xi)$ is regarded as a subspace of $M_b(X;\Xi)$.

In order to define the notion of generalized Young measure on X with values in Ξ, it is convenient to introduce a space of homogeneous functions and to discuss some properties of its dual.

Definition 2.1. Let $C^{hom}(X\times\Xi)$ be the space of all continuous $f\colon X\times\Xi \to \mathbb{R}$ such that $\xi \mapsto f(x,\xi)$ is positively homogeneous of degree one on Ξ for every $x \in X$; i.e., $f(x, t\xi) = t f(x,\xi)$ for every $x \in X$, $\xi \in \Xi$, and $t \geq 0$. This space is endowed with the norm

$$\|f\|_{hom} := \max\{|f(x,\xi)| : x \in X,\ \xi \in \Sigma_\Xi\},$$

where $\Sigma_\Xi := \{\xi \in \Xi : |\xi| = 1\}$.

We introduce now two dense subspaces of $C^{hom}(X\times\Xi)$ that will be useful in the proof of some properties of generalized Young measures.

Definition 2.2. Let $C^{hom}_L(X\times\Xi)$ be the space of all $f \in C^{hom}(X\times\Xi)$ satisfying the following Lipschitz condition: there exists a constant $a \in \mathbb{R}$ such that

$$|f(x,\xi_1) - f(x,\xi_2)| \leq a\,|\xi_1 - \xi_2| \tag{2.1}$$

for every $x \in X$ and every $\xi_1, \xi_2 \in \Xi$.

Remark 2.3. If $f \in C^{hom}_L(X\times\Xi)$ and ω is the modulus of continuity of the restriction of f to $X\times\Sigma_\Xi$, then (2.1) and the homogeneity of f imply that

$$\begin{aligned} |f(x_1,\xi_1) - f(x_2,\xi_2)| &\leq |f(x_1,\xi_1) - f(x_1,\xi_2)| + |f(x_1,\xi_2) - f(x_2,\xi_2)| \leq \\ &\leq a\,|\xi_1 - \xi_2| + |\xi_2|\,\omega(d(x_1,x_2)). \end{aligned}$$

Exchanging the roles of ξ_1 and ξ_2 we obtain

$$|f(x_1,\xi_1) - f(x_2,\xi_2)| \leq a\,|\xi_1 - \xi_2| + \min\{|\xi_1|, |\xi_2|\}\,\omega(d(x_1,x_2))$$

for every $x_1, x_2 \in X$ and every $\xi_1, \xi_2 \in \Xi$.

Lemma 2.4. *The space* $C_L^{hom}(X\times\Xi)$ *is dense in* $C^{hom}(X\times\Xi)$.

Proof. Let us fix $f\in C^{hom}(X\times\Xi)$. For every $k>\|f\|_{hom}$ let us consider the Moreau-Yosida approximation $f_k\colon X\times\Xi\to\mathbb{R}$ defined by

$$f_k(x,\xi):=\min_{\xi'\in\Xi}\{f(x,\xi')+k\,|\xi'-\xi|\}\,.$$

Using the standard properties of Moreau-Yosida approximations it is easy to check that $f_k\in C_L^{hom}(X\times\Xi)$ (with constant k) and that the sequence f_k is nondecreasing and converges pointwise to f (see, e.g., [8, Remark 9.6 and Theorem 9.13]). By Dini's Theorem we conclude that $f_k\to f$ uniformly on $X\times\Sigma_\Xi$, hence $f_k\to f$ in $C^{hom}(X\times\Xi)$. □

Definition 2.5. Let $C_\triangle^{hom}(X\times\Xi)$ be the space of all $f\in C^{hom}(X\times\Xi)$ which satisfy the triangle inequality $f(x,\xi_1+\xi_2)\le f(x,\xi_1)+f(x,\xi_2)$ for every $x\in X$ and every ξ_1, $\xi_2\in\Xi$.

Remark 2.6. As $|f(x,\xi)|\le|\xi|\,\|f\|_{hom}$, each $f\in C_\triangle^{hom}(X\times\Xi)$ is Lipschitz continuous with respect to ξ and satisfies

$$|f(x,\xi_1)-f(x,\xi_2)|\le|\xi_1-\xi_2|\,\|f\|_{hom}$$

for every $x\in X$ and every ξ_1, $\xi_2\in\Xi$. Therefore $C_\triangle^{hom}(X\times\Xi)\subset C_L^{hom}(X\times\Xi)$.

Lemma 2.7. *The space of functions of the form* f_1-f_2, *with* f_1, $f_2\in C_\triangle^{hom}(X\times\Xi)$, *is dense in* $C^{hom}(X\times\Xi)$.

Proof. Thanks to the obvious density in $C^{hom}(X\times\Xi)$ of the space of functions $f\in C^{hom}(X\times\Xi)$ such that $f(x,\cdot)$ belongs to $C^2(\Xi\setminus\{0\})$ for every x, it is enough to prove that every such function can be written as $f=f_1-f_2$, with f_1, $f_2\in C_\triangle^{hom}(X\times\Xi)$. To this aim it suffices to show that there exists a constant $c:=c(f)$ such that $f_2(x,\xi):=c\,|\xi|-f(x,\xi)$ is convex in ξ for every $x\in X$. A simple calculation shows that the quadratic form corresponding to the Hessian matrix of f_2 with respect to ξ at a point (x,e), with $e\in\Sigma_\Xi$, is given by

$$D_\xi^2 f_2(x,e)\,\xi\cdot\xi=c\,|\xi|^2-c\,(\xi\cdot e)^2-D_\xi^2 f(x,e)\,\xi\cdot\xi\,. \tag{2.2}$$

By the Euler relation we have $D_\xi f(x,\xi)\,\xi=f(x,\xi)$. Taking the derivative with respect to ξ we obtain $D_\xi^2 f(x,\xi)\,\xi=0$ for every ξ, in particular $D_\xi^2 f(x,e)$ has an eigenvalue 0 with eigenvector e. This implies that there is a constant $b(x,e)$ such that $D_\xi^2 f(x,e)\,\xi\cdot\xi\le b(x,e)\,|\xi_e^\perp|^2$, where $\xi_e^\perp:=\xi-(\xi\cdot e)\,e$ is the component of ξ orthogonal to e. As $b(x,e)$ is bounded by the continuity of the second derivatives of f, and $|\xi_e^\perp|^2=|\xi|^2-(\xi\cdot e)^2$, by (2.2) there exists a constant c such that $D_\xi^2 f_2(x,e)$ is positive definite for every $x\in X$ and every $e\in\Sigma_\Xi$, hence $f_2(x,\xi)$ is convex with respect to ξ for every $x\in X$. □

Definition 2.8. The dual of the space $C^{hom}(X\times\Xi)$ is denoted by $M_*(X\times\Xi)$, and the corresponding dual norm by $\|\cdot\|_*$; the weak* topology of $M_*(X\times\Xi)$ is defined by using this duality. It is sometimes convenient to write the dummy variables explicitly and to use the notation $\langle f(x,\xi),\mu(x,\xi)\rangle$ for the duality product $\langle f,\mu\rangle$. The positive cone $M_*^+(X\times\Xi)$ is defined as the set of all $\mu\in M_*(X\times\Xi)$ such that

$$\langle f,\mu\rangle\ge 0\quad\text{for every } f\in C^{hom}(X\times\Xi) \text{ with } f\ge 0\,.$$

Remark 2.9. It is easy to see that for every $\mu\in M_*^+(X\times\Xi)$ we have

$$\|\mu\|_*=\langle|\xi|,\mu(x,\xi)\rangle\,.$$

Strictly speaking, the elements μ of $M_*(X{\times}\Xi)$ are not measures, because they act only on homogeneous functions. However, the notion of image of μ under a map ψ can be defined by duality, as in measure theory.

Definition 2.10. Let Ξ and Ξ' be two finite dimensional Hilbert spaces and let $\psi\colon X{\times}\Xi \to X{\times}\Xi'$ be a continuous map of the form $\psi(x,\xi) = (x, \varphi(x,\xi))$, with $\varphi\colon X{\times}\Xi \to \Xi'$ positively one-homogeneous in ξ. The *image* $\psi(\mu)$ of $\mu \in M_*(X{\times}\Xi)$ under ψ is defined as the element of $M_*(X{\times}\Xi')$ such that

$$\langle f, \psi(\mu)\rangle = \langle f \circ \psi, \mu\rangle = \langle f(x, \varphi(x,\xi)), \mu(x,\xi)\rangle$$

for every $f \in C^{hom}(X{\times}\Xi')$.

Similarly we can define the notion of support of $\mu \in M_*(X{\times}\Xi)$. We say that a subset C of $X{\times}\Xi$ is a Ξ-cone if $(x,\xi) \in C \Rightarrow (x, t\xi) \in C$ for every $t \geq 0$.

Definition 2.11. The support $\operatorname{supp}\mu$ of $\mu \in M_*(X{\times}\Xi)$ is defined as the smallest closed Ξ-cone $C \subset X{\times}\Xi$ such that $\langle f, \mu\rangle = 0$ for every $f \in C^{hom}(X{\times}\Xi)$ vanishing on C.

Remark 2.12. For every $\mu \in M_*(X{\times}\Xi)$ there exists a measure $\tilde\mu \in M_b(X{\times}\Xi)$ with compact support such that

$$\langle f, \mu\rangle = \int_{X\times\Xi} f\, d\tilde\mu \tag{2.3}$$

for every $f \in C^{hom}(X{\times}\Xi)$. A measure with this property can be constructed by considering the continuous linear map on $C(X{\times}\Sigma_\Xi)$ defined by

$$g \mapsto \langle |\xi| g(x, \xi/|\xi|), \mu(x,\xi)\rangle\,.$$

By the Riesz Representation Theorem there exists $\tilde\mu \in M_b(X{\times}\Sigma_\Xi)$ such that

$$\langle |\xi| g(x, \xi/|\xi|), \mu(x,\xi)\rangle = \int_{X\times\Sigma_\Xi} g\, d\tilde\mu$$

for every $g \in C(X{\times}\Sigma_\Xi)$. Regarding $\tilde\mu$ as a measure on $X{\times}\Xi$ supported by $X{\times}\Sigma_\Xi$, we obtain (2.3). This construction suggests that the measure $\tilde\mu$ satisfying (2.3) is not unique; indeed, we can repeat the same construction with Σ_Ξ replaced by any other concentric sphere. More in general, suppose that E is a compact subset of $X{\times}(\Xi \setminus \{0\})$ with the following property: for every $x \in X$ and every $\xi \in \Sigma_\Xi$ there exists a unique $t > 0$ such that $(x, t\xi) \in E$. Then there exists a unique measure $\check\mu \in M_b(E)$ such that

$$\langle f, \mu\rangle = \int_E f\, d\check\mu$$

for every $f \in C^{hom}(X{\times}\Xi)$. The measure $\check\mu$ clearly depends on E, as explained above in the case $E = X{\times}\Sigma_\Xi$.

For the applications it is convenient to extend some of the previous results to a suitable space of possibly discontinuous functions.

Definition 2.13. Given two finite dimensional Hilbert spaces Ξ and Ξ', define $B^{hom}_\infty(X{\times}\Xi; \Xi')$ as the space of Borel functions $f\colon X{\times}\Xi \to \Xi'$ such that

(a) for every $x \in X$ the function $\xi \mapsto f(x,\xi)$ is positively homogeneous of degree one on Ξ,
(b) there exists a constant $a \in \mathbb{R}$ such that $|f(x,\xi)| \leq a|\xi|$ for every $(x,\xi) \in X{\times}\Xi$.

The smallest constant a satisfying the previous inequality is denoted by $\|f\|_{hom}$. When $\Xi'=\mathbb{R}$, the corresponding space will be denoted simply by $B^{hom}_\infty(X\times\Xi)$.

Definition 2.14. For every $f\in B^{hom}_\infty(X\times\Xi)$ and every $\mu\in M_*(X\times\Xi)$ the duality product $\langle f,\mu\rangle$ is defined by

$$\langle f,\mu\rangle := \int_{X\times\Xi} f\,d\tilde\mu\,,$$

where $\tilde\mu$ is any measure satisfying the conditions of Remark 2.12. By homogeneity the value of $\langle f,\mu\rangle$ does not depend on the particular measure $\tilde\mu$ chosen in (2.3). The same definition (with values in $\mathbb{R}\cup\{+\infty\}$ this time) is adopted if $\mu\in M^+_*(X\times\Xi)$ and $f\colon X\times\Xi\to\mathbb{R}\cup\{+\infty\}$ is a Borel function such that $f(x,\xi)$ is positively homogeneous of degree one in ξ and $f(x,\xi)\ge -c|\xi|$ for some constant $c\ge 0$.

Let $\pi_X\colon X\times\Xi\to X$ be the projection onto X. We now define the image under π_X of the product $h\mu$ of an element μ of $M_*(X\times\Xi)$ by a homogeneous function h.

Definition 2.15. Let Ξ and Ξ' be two finite dimensional Hilbert spaces, let $\mu\in M_*(X\times\Xi)$, and let $h\in B^{hom}_\infty(X\times\Xi;\Xi')$. The measure $\pi_X(h\mu)$ is the element of $M_b(X;\Xi')$ such that

$$\int_X \varphi\cdot d\pi_X(h\mu) = \langle \varphi(x)\cdot h(x,\xi),\mu(x,\xi)\rangle \tag{2.4}$$

for every $\varphi\in C(X;\Xi')$, where the dot denotes the scalar product in Ξ'.

Remark 2.16. For every $\tilde\mu\in M_b(X\times\Xi)$ satisfying (2.3) we can consider the Radon measure $h\tilde\mu\in M_b(X\times\Xi;\Xi')$ having density h with respect to $\tilde\mu$. It is easy to check that the measure $\pi_X(h\mu)$ defined by (2.4) coincides with the image under π_X of the measure $h\tilde\mu$. Note that the measure $h\tilde\mu$ depends on the choice of $\tilde\mu$ satisfying (2.3), while, by (2.4), its projection $\pi_X(h\tilde\mu)$ does not.

Remark 2.17. It follows from the definition that we have the estimate

$$\|\pi_X(h\mu)\|\le \|h\|_{hom}\|\mu\|_*$$

for every $\mu\in M_*(X\times\Xi)$ and every $h\in B^{hom}_\infty(X\times\Xi;\Xi')$.

3. **Generalized Young measures.** As mentioned in the introduction, the notion of generalized Young measure is used to describe oscillation and concentration phenomena for sequences which are bounded in $L^r(X;\Xi)$ only for $r=1$. To study concentration phenomena, where the sequences tend to infinity along given directions in the space Ξ, it is useful to introduce homogeneous coordinates. This is done by replacing the space Ξ by $\Xi\times\mathbb{R}$, whose generic point is denoted by (ξ,η); the set of points with $\eta=1$ is identified with Ξ, while points with $\eta=0$ are interpreted as directions at infinity.

In our presentation the space of generalized Young measures will be a subset of the space $M^+_*(X\times\Xi\times\mathbb{R})$, where $\Xi\times\mathbb{R}$ plays the role of the Hilbert space Ξ of the previous section. Before describing this set, we first consider generalized Young measures associated with functions.

Definition 3.1. Given $u\in L^1(X;\Xi)$, the *generalized Young measure associated with* u is defined as the element δ_u of $M^+_*(X\times\Xi\times\mathbb{R})$ such that

$$\langle f,\delta_u\rangle = \int_X f(x,u(x),1)\,d\lambda(x)$$

for every $f \in C^{hom}(X{\times}\Xi{\times}\mathbb{R})$.

In the spirit of [16] and [26] we extend this definition to measures $p \in M_b(X;\Xi)$.

Definition 3.2. Given $p \in M_b(X;\Xi)$, the *generalized Young measure associated with* p is defined as the element δ_p of $M^+_*(X{\times}\Xi{\times}\mathbb{R})$ such that for every $f \in C^{hom}(X{\times}\Xi{\times}\mathbb{R})$

$$\langle f, \delta_p\rangle = \int_X f(x, \tfrac{dp}{d\sigma}(x), \tfrac{d\lambda}{d\sigma}(x))\, d\sigma(x)\,,$$

where σ is an arbitrary nonnegative Radon measure on X with $\lambda << \sigma$ and $p << \sigma$.

The homogeneity of f implies that the integral does not depend on σ and that the definitions coincide when $p = u \in L^1(X;\Xi)$. The norm of δ_p is given by the following lemma.

Lemma 3.3. *Let* $p \in M_b(X;\Xi)$. *Then*

$$\|\delta_p\|_* = \int_X \sqrt{1+|p^a|^2}\, d\lambda + |p^s|(X) \le \lambda(X) + |p|(X)\,.$$

Proof. Let us consider the Borel partition $X = X^a \cup X^s$ with $\lambda(X^s) = 0 = |p^s|(X^a)$ and let $\sigma := \lambda + |p^s|$, so that $\sigma = \lambda$ on X^a and $\sigma = |p^s|$ on X^s. By Remark 2.9 we have

$$\|\delta_p\|_* = \langle \sqrt{|\xi|^2+|\eta|^2}, \delta_p(x,\xi,\eta)\rangle = \int_X \sqrt{|\tfrac{dp}{d\sigma}|^2 + |\tfrac{d\lambda}{d\sigma}|^2}\, d\sigma =$$

$$= \int_{X^a} \sqrt{1+|p^a|^2}\, d\lambda + |p^s|(X^s) = \int_X \sqrt{1+|p^a|^2}\, d\lambda + |p^s|(X)\,,$$

which concludes the proof. □

We recall the definition of Young measure.

Definition 3.4. A *Young measure on* X *with values in* Ξ is a measure $\nu \in M^+_b(X{\times}\Xi)$ such that $\pi_X(\nu) = \lambda$. The space of Young measures on X with values in Ξ is denoted by $Y(X;\Xi)$. For every $r \ge 1$ let $Y^r(X;\Xi)$ be the space of all $\nu \in Y(X;\Xi)$ whose r*-moment*

$$\int_{X\times\Xi} |\xi|^r\, d\nu(x,\xi)$$

is finite.

Remark 3.5. By the Disintegration Theorem (see, e.g., [27, Appendix A2]) for every $\nu \in Y(X;\Xi)$ there exists a measurable family $(\nu^x)_{x\in X}$ of probability measures on Ξ such that

$$\int_{X\times\Xi} g(x,\xi)\, d\nu(x,\xi) = \int_X \Big(\int_\Xi g(x,\xi)\, d\nu^x(\xi)\Big)\, d\lambda(x)$$

for every bounded Borel function $g\colon X{\times}\Xi \to \mathbb{R}$. The probability measures ν^x are uniquely determined for λ-a.e. $x \in X$.

Definition 3.6. Given $\nu \in Y^1(X;\Xi)$, the *generalized Young measure associated with* ν is defined as the element $\overline{\nu}$ of $M^+_*(X{\times}\Xi{\times}\mathbb{R})$ such that

$$\langle f, \overline{\nu}\rangle := \int_{X\times\Xi} f(x,\xi,1)\, d\nu(x,\xi)$$

for every $f \in C^{hom}(X{\times}\Xi{\times}\mathbb{R})$.

Remark 3.7. It follows from Remark 2.9 that
$$\|\overline{\nu}\|_* = \int_{X\times\Xi} \sqrt{1+|\xi|^2}\, d\nu(x,\xi) \leq \lambda(X) + \int_{X\times\Xi} |\xi|\, d\nu(x,\xi)\,.$$

Remark 3.8. If $\mu = \delta_p$ for some $p \in M_b(X;\Xi)$, the following properties hold:
$$\operatorname{supp}\mu \subset X\times\Xi\times[0,+\infty)\,, \tag{3.1}$$
$$\pi_X(\eta\mu) = \lambda\,. \tag{3.2}$$
We will refer to (3.2) as the *projection property*. According to (2.4), it is equivalent to
$$\langle\varphi(x)\eta, \mu(x,\xi,\eta)\rangle = \int_X \varphi\, d\lambda \quad \text{for every } \varphi \in C(X)\,. \tag{3.3}$$
Properties (3.1) and (3.2) continue to hold if $\mu = \overline{\nu}$ for some $\nu \in Y^1(X;\Xi)$.

This motivates the following definition.

Definition 3.9. The space $GY(X;\Xi)$ of *generalized Young measures* on X with values in Ξ is defined as the set of all $\mu \in M_*^+(X\times\Xi\times\mathbb{R})$ satisfying (3.1) and (3.2). On $GY(X;\Xi)$ we consider the norm and the weak* topology induced by $M_*(X\times\Xi\times\mathbb{R})$.

Remark 3.10. By approximation we can prove that (3.3) holds for every $\mu \in GY(X;\Xi)$ and for every bounded Borel function $\varphi\colon X \to \mathbb{R}$.

The sequential compactness of every bounded subset of $GY(X;\Xi)$ is given by the following theorem.

Theorem 3.11. *Every bounded sequence in $GY(X;\Xi)$ has a subsequence which converges weakly* to an element of $GY(X;\Xi)$.*

Proof. Since $GY(X;\Xi)$ is closed in the weak* topology of $M_*(X\times\Xi\times\mathbb{R})$, the result follows from the Banach-Alaoglu Theorem. □

Remark 3.12. If μ_k is a sequence in $GY(X;\Xi)$ which converges weakly* to $\mu \in GY(X;\Xi)$, then $\|\mu_k\|_* \to \|\mu\|_*$ by Remark 2.9.

Remark 3.13. If $f\colon X\times\Xi\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ is lower semicontinuous, $(\xi,\eta) \mapsto f(x,\xi,\eta)$ is positively homogeneous of degree one, and $f(x.\xi,\eta) \geq -c\sqrt{|\xi|^2+|\eta|^2}$ for some constant $c \geq 0$, then
$$\langle f,\mu\rangle \leq \liminf_{k\to\infty} \langle f,\mu_k\rangle$$
for every sequence μ_k in $GY(X;\Xi)$ which converges weakly* to $\mu \in GY(X;\Xi)$, where $\langle f,\cdot\rangle$ is defined by (2.3). Indeed, any such f is the supremum of a family of functions in $C^{hom}(X\times\Xi\times\mathbb{R})$.

In the case of a generalized Young measure $\mu \in GY(X;\Xi)$ the duality product $\langle f,\mu\rangle$ can be defined for every f in the space $B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$ introduced by the following definition, which is slightly larger than the space $B^{hom}_{\infty}(X\times\Xi\times\mathbb{R})$ considered in the previous section.

Definition 3.14. Given two finite dimensional Hilbert spaces Ξ and Ξ', we consider the space $B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R};\Xi')$ of all Borel functions $f\colon X\times\Xi\times\mathbb{R} \to \Xi'$ such that

(a) for every $x \in X$ the function $(\xi,\eta) \mapsto f(x,\xi,\eta)$ is positively homogeneous of degree one on $\Xi\times\mathbb{R}$,

(b) there exist a constant $a \in \mathbb{R}$ and a function $b \in L^1(X)$ such that

$$|f(x,\xi,\eta)| \leq a|\xi| + b(x)|\eta| \tag{3.4}$$

for every $(x,\xi,\eta) \in X\times\Xi\times\mathbb{R}$.

When $\Xi' = \mathbb{R}$, the corresponding space will be denoted simply by $B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$.

Lemma 3.15. *Let $\mu \in GY(X;\Xi)$ and let $\tilde{\mu} \in M_b(X\times\Xi\times\mathbb{R})$ be a measure with compact support satisfying (2.3). Then every $f \in B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$ is $\tilde{\mu}$-integrable.*

Proof. Let us fix $f \in B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$. For every k let f_k be the function defined by $f_k(x,\xi,\eta) := f(x,\xi,\eta)$ if $|f(x,\xi,\eta)| \leq a|\xi| + k|\eta|$, and by $f_k(x,\xi,\eta) := 0$ otherwise. Then we have

$$\int_{X\times\Xi\times\mathbb{R}} |f_k|\, d\tilde{\mu} \leq \int_{X\times\Xi\times\mathbb{R}} [a|\xi| + (b(x)\wedge k)|\eta|]\, d\tilde{\mu} =$$
$$= \langle a|\xi| + (b(x)\wedge k)|\eta|, \mu(x,\xi,\eta)\rangle \leq \langle a|\xi|, \mu(x,\xi,\eta)\rangle + \int_X b\, d\lambda\,.$$

It follows from Fatou's Lemma that f is $\tilde{\mu}$-integrable. □

Definition 3.16. Given $f \in B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$ and $\mu \in GY(X;\Xi)$, the duality product $\langle f, \mu\rangle$ is defined by

$$\langle f, \mu\rangle := \int_{X\times\Xi\times\mathbb{R}} f\, d\tilde{\mu}\,, \tag{3.5}$$

where $\tilde{\mu} \in M_b(X\times\Xi\times\mathbb{R})$ is any measure with compact support satisfying (2.3), with Ξ replaced by $\Xi\times\mathbb{R}$.

Remark 3.17. The integral in (3.5) is well defined by Lemma 3.15. It is easy to see that the value of this integral does not depend on the choice of $\tilde{\mu}$ satisfying (2.3), and that

$$|\langle f, \mu\rangle| \leq a\, \|\mu\|_* + \|b\|_1\,,$$

where a and b satisfy (3.4) and $\|b\|_1$ denotes the L^1 norm of b.

We now consider the image of a generalized Young measure.

Definition 3.18. Let Ξ and Ξ' be two finite dimensional Hilbert spaces and let $\psi\colon X\times\Xi\times\mathbb{R} \to X\times\Xi'\times\mathbb{R}$ be a map of the form $\psi(x,\xi,\eta) = (x, \varphi(x,\xi,\eta), \eta)$, with $\varphi \in B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R};\Xi')$. The *image* $\psi(\mu)$ of $\mu \in GY(X;\Xi)$ under ψ is defined as the element of $GY(X;\Xi')$ such that

$$\langle f, \psi(\mu)\rangle = \langle f\circ\psi, \mu\rangle = \langle f(x, \varphi(x,\xi,\eta), \eta), \mu(x,\xi,\eta)\rangle \tag{3.6}$$

for every $f \in B^{hom}_{\infty,1}(X\times\Xi'\times\mathbb{R})$.

Remark 3.19. Under the assumptions of the definition the function $f\circ\psi$ belongs to $B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$, so that the duality product $\langle f\circ\psi, \mu\rangle$ is well defined. Moreover, by the particular form of the map ψ, the element of $M^+_*(X\times\Xi'\times\mathbb{R})$ defined by (3.6) satisfies (3.1) and (3.2), therefore it belongs to $GY(X;\Xi')$.

4. **Comparison with other presentations of the theory.** In this section we show that every $\mu \in GY(X;\Xi)$ can be represented by a unique Young measure-varifold pair (μ^Y, μ^∞), where $\mu^Y \in Y^1(X;\Xi)$ and $\mu^\infty \in M_b^+(X\times\Sigma_\Xi)$. To introduce this representation, we recall that $Y^1(X;\Xi)$ can be identified with a suitable subset of $GY(X;\Xi)$ (Definition 3.6). The following definition identifies the measures in $M_b^+(X\times\Sigma_\Xi)$ with particular elements of $M_*(X\times\Xi\times\mathbb{R})$.

Definition 4.1. For every $\nu \in M_b^+(X\times\Sigma_\Xi)$ let $\hat\nu$ be the element of $M_*(X\times\Xi\times\mathbb{R})$ defined by

$$\langle f, \hat\nu\rangle = \int_{X\times\Sigma_\Xi} f(x,\xi,0)\, d\nu(x,\xi)$$

for every $f \in C^{hom}(X\times\Xi\times\mathbb{R})$.

Remark 4.2. It follows from the definition that $\pi_X(\eta\hat\nu) = 0$ for every $\nu \in M_b^+(X\times\Sigma_\Xi)$. Since $\hat\nu$ does not satisfy the projection property (3.2), it does not belong to $GY(X;\Xi)$.

The main result of this section is the following theorem.

Theorem 4.3. *Let $\mu \in GY(X;\Xi)$. Then there exists a unique pair (μ^Y, μ^∞), with $\mu^Y \in Y^1(X;\Xi)$ and $\mu^\infty \in M_b^+(X\times\Sigma_\Xi)$, such that*

$$\mu = \overline{\mu}^Y + \hat\mu^\infty ,$$

which is equivalent to

$$\langle f, \mu\rangle = \int_{X\times\Xi} f(x,\xi,1)\, d\mu^Y(x,\xi) + \int_{X\times\Sigma_\Xi} f(x,\xi,0)\, d\mu^\infty(x,\xi) \tag{4.1}$$

for every $f \in B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$.

Remark 4.4. The converse of Theorem 4.3 is also true: if $\mu^Y \in Y^1(X;\Xi)$ and $\mu^\infty \in M_b(X\times\Sigma_\Xi)$, then formula (4.1) defines an element of $GY(X;\Xi)$.

Remark 4.5. Let $\lambda^\infty := \pi_X(\mu^\infty)$. Since $\lambda = \pi_X(\mu^Y)$, by the Disintegration Theorem (see, e.g., [27, Appendix A2]) there exist a measurable family $(\mu^{x,Y})_{x\in X}$ of probability measures on Ξ and a measurable family $(\mu^{x,\infty})_{x\in X}$ of probability measures on Σ_Ξ such that

$$\langle f, \mu\rangle = \int_{X\times\Xi} f(x,\xi,1)\, d\mu^Y(x,\xi) + \int_{X\times\Sigma_\Xi} f(x,\xi,0)\, d\mu^\infty(x,\xi) =$$
$$= \int_X \Big(\int_\Xi f(x,\xi,1)\, d\mu^{x,Y}(\xi)\Big)\, d\lambda(x) + \int_X \Big(\int_{\Sigma_\Xi} f(x,\xi,0)\, d\mu^{x,\infty}(\xi)\Big)\, d\lambda^\infty(x)$$

for every $f \in B^{hom}_{\infty,1}(X\times\Xi\times\mathbb{R})$.

Remark 4.6. Thanks to Remark 2.9, if we apply (4.1) to $f(x,\xi,\eta) = \sqrt{|\xi|^2 + |\eta|^2}$, we obtain

$$\|\mu\|_* = \int_{X\times\Xi} \sqrt{1+|\xi|^2}\, d\mu^Y(x,\xi) + \mu^\infty(X\times\Sigma_\Xi) \leq$$
$$\leq \lambda(X) + \int_{X\times\Xi} |\xi|\, d\mu^Y(x,\xi) + \mu^\infty(X\times\Sigma_\Xi).$$

Proof of Theorem 4.3. For every Borel function $g\colon X\times\Xi \to \mathbb{R}$ with

$$\kappa_g := \sup_{(x,\xi)\in X\times\Xi} \frac{|g(x,\xi)|}{\sqrt{1+|\xi|^2}} < +\infty , \tag{4.2}$$

we consider the Borel function $\varphi_g\colon X{\times}\Xi{\times}\mathbb{R}\to\mathbb{R}$ defined by

$$\varphi_g(x,\xi,\eta):=\begin{cases}\eta\, g(x,\xi/\eta) & \text{if } \eta>0\,,\\ 0 & \text{if } \eta\le 0\,.\end{cases}$$

Since $\varphi_g\in B^{hom}_\infty(X{\times}\Xi{\times}\mathbb{R})$, we can consider the duality product $\langle\varphi_g,\mu\rangle$. The function $g\mapsto\langle\varphi_g,\mu\rangle$ is linear, bounded, and positive on $C_0(X{\times}\Xi)$. By the Riesz Representation Theorem there exists $\mu^Y\in M^+_b(X{\times}\Xi)$ such that

$$\langle\varphi_g,\mu\rangle=\int_{X\times\Xi} g(x,\xi)\,d\mu^Y(x,\xi) \tag{4.3}$$

for every $g\in C_0(X{\times}\Xi)$. As $\|g\|_{hom}\le\kappa_g$, we have

$$|\int_{X\times\Xi} g(x,\xi)\,d\mu^Y(x,\xi)|\le\kappa_g\|\mu\|_*$$

for every $g\in C_0(X{\times}\Xi)$. By approximation we can prove that

$$\int_{X\times\Xi}|\xi|\,d\mu^Y(x,\xi)\le\|\mu\|_* \tag{4.4}$$

and that (4.3) holds for every Borel function $g\colon X{\times}\Xi\to\mathbb{R}$ satisfying (4.2). By (3.2) we have $\pi_X(\mu^Y)=\lambda$, which, together with (4.4), gives $\mu^Y\in Y^1(X;\Xi)$.

For every bounded Borel function $h\colon X{\times}\Sigma_\Xi\to\mathbb{R}$ we consider the Borel function $\psi_h\colon X{\times}\Xi{\times}\mathbb{R}\to\mathbb{R}$ defined by

$$\psi_h(x,\xi,\eta):=\begin{cases}|\xi|\,h(x,\xi/|\xi|) & \text{if } \eta=0\,,\xi\ne0\,,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\psi_h\in B^{hom}_\infty(X{\times}\Xi{\times}\mathbb{R})$, we can consider the duality product $\langle\psi_h,\mu\rangle$. The function $h\mapsto\langle\psi_h,\mu\rangle$ is linear, bounded, and positive on $C(X{\times}\Sigma_\Xi)$. By the Riesz Representation Theorem there exists $\mu^\infty\in M^+_b(X{\times}\Sigma_\Xi)$ such that

$$\langle\psi_h,\mu\rangle=\int_{X\times\Sigma_\Xi} h\,d\mu^\infty \tag{4.5}$$

for every $h\in C(X{\times}\Sigma_\Xi)$. By approximation we can prove that the previous equality holds for every bounded Borel function $h\colon X{\times}\Sigma_\Xi\to\mathbb{R}$.

Given any $f\in B^{hom}_\infty(X{\times}\Xi{\times}\mathbb{R})$, we consider the functions $g\colon X{\times}\Xi\to\mathbb{R}$ and $h\colon X{\times}\Sigma_\Xi\to\mathbb{R}$ defined by

$$g(x,\xi):=f(x,\xi,1)\,,\qquad h(x,\xi):=f(x,\xi,0)\,.$$

By homogeneity we have $f=\varphi_g+\psi_h$ on $X{\times}\Xi{\times}[0,+\infty)$. Then (4.1) follows from (3.1), (4.3), and (4.5). The result can be extended to $f\in B^{hom}_{\infty,1}(X{\times}\Xi{\times}\mathbb{R})$ by approximation.

The uniqueness of the pair (μ^Y,μ^∞) can be deduced from the fact that, if (4.1) is satisfied, then (4.3) holds for every $g\in C_0(X{\times}\Xi)$, while (4.5) holds for every $h\in C(X{\times}\Sigma_\Xi)$. ☐

Remark 4.7. It is easy to prove, by approximation, that (4.1) continues to hold when $f\colon X{\times}\Xi{\times}\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is a Borel function such that $f(x,\xi,\eta)$ is positively homogeneous of degree one in (ξ,η) and satisfies the inequality $f(x,\xi,\eta)\ge$

$-c\sqrt{|\xi|^2+|\eta|^2}$ for some constant $c\geq 0$. This allows to characterize the generalized Young measures associated with Young measures with finite r-moment, $r>1$, using the homogeneous functions $\{P_r\}\colon \Xi{\times}\mathbb{R}\to[0,+\infty]$ defined by

$$\{P_r\}(\xi,\eta):=\begin{cases}|\xi|^r/\eta^{r-1} & \text{if } \eta>0\,,\\ +\infty & \text{if } \eta\leq 0\,.\end{cases} \tag{4.6}$$

Indeed, for $r>1$ formula (4.1) implies that $\mu=\overline{\mu}^Y$ with $\mu^Y\in Y^r(X;\Xi)$ if and only if $\langle\{P_r\},\mu\rangle<+\infty$.

Let $\psi_0^\Xi\colon X{\times}\Xi{\times}\mathbb{R}\to X{\times}\Xi{\times}\mathbb{R}$ be the Borel map defined by

$$\psi_0^\Xi(x,\xi,\eta)=\begin{cases}(x,\xi,\eta) & \text{if } \eta\neq 0\,,\\ (x,0,0) & \text{if } \eta=0\,.\end{cases}$$

Note that ψ_0^Ξ satisfies the conditions of Definition 3.18.

For every $f\in B^{hom}_{\infty,1}(X{\times}\Xi{\times}\mathbb{R})$ we have

$$(f\circ\psi_0^\Xi)(x,\xi,\eta)=\begin{cases}f(x,\xi,\eta) & \text{if } \eta\neq 0\,,\\ 0 & \text{if } \eta=0\,.\end{cases}$$

From (4.1) it follows that for every $\mu\in GY(X;\Xi)$

$$\langle f,\overline{\mu}^Y\rangle=\langle f\circ\psi_0^\Xi,\overline{\mu}^Y\rangle=\langle f\circ\psi_0^\Xi,\mu\rangle\,,$$

hence $\overline{\mu}^Y=\psi_0^\Xi(\mu)$.

Lemma 4.8. *Let Ξ and Ξ' be two finite dimensional Hilbert spaces, let $\mu\in GY(X;\Xi)$, let $\psi\colon X{\times}\Xi{\times}\mathbb{R}\to X{\times}\Xi'{\times}\mathbb{R}$ be a map as in Definition 3.18, and let $\nu:=\psi(\mu)$. Then*

$$\overline{\nu}^Y=\psi(\overline{\mu}^Y)\,,\qquad \hat{\nu}^\infty=\psi(\hat{\mu}^\infty)\,.$$

Proof. The former equality follows from the fact that $\psi\circ\psi_0^\Xi=\psi_0^{\Xi'}\circ\psi$. The latter follows now from Theorem 4.3 by the linearity of the map $\mu\mapsto\psi(\mu)$. □

Combining the compactness property (Theorem 3.11) and the representation formula (Theorem 4.3) we recover the following result, originally proved in [1] (see Remark 4.5).

Theorem 4.9. *Let u_k be a bounded sequence in $L^1(X;\Xi)$. Then there exist a subsequence, still denoted u_k, a Young measure $\mu^Y\in Y^1(X;\Xi)$, and a measure $\mu^\infty\in M_b(X{\times}\Sigma_\Xi)$, such that*

$$\int_X g(x,u_k(x))\,d\lambda(x)\longrightarrow\int_{X\times\Xi}g(x,\xi)\,d\mu^Y(x,\xi)+\int_{X\times\Sigma_\Xi}g^\infty(x,\xi)\,d\mu^\infty(x,\xi) \tag{4.7}$$

for every continuous function $g\colon X{\times}\Xi\to\mathbb{R}$ such that for every $(x_0,\xi_0)\in X{\times}\Xi$ the limit

$$g^\infty(x_0,\xi_0):=\lim_{\substack{x\to x_0,\ \xi\to\xi_0\\ \eta\to 0^+}}\eta\, g(x,\xi/\eta)$$

exists and is finite.

Proof. Let us consider the sequence δ_{u_k} in $GY(X;\Xi)$ introduced in Definition 3.1. By Lemma 3.3 we have $\|\delta_{u_k}\|_*\leq\sup_j\|u_j\|_1+\lambda(X)<+\infty$. By Theorem 3.11 there exists a subsequence, still denoted u_k, such that δ_{u_k} converge weakly* to an element μ of $GY(X;\Xi)$. Let g be as in the statement of the theorem and let $f\colon X\times\Xi\times\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x,\xi,\eta) := \begin{cases} \eta\, g(x,\xi/\eta) & \text{if } \eta > 0\,, \\ g^\infty(x,\xi) & \text{if } \eta \le 0\,. \end{cases}$$

It is easy to check that f is continuous in (x,ξ,η) and homogeneous of degree one in (ξ,η). Therefore, the weak* convergence of δ_{u_k} to μ implies that

$$\int_X g(x,u_k(x))\,d\lambda(x) = \int_X f(x,u_k(x),1)\,d\lambda(x) \longrightarrow \langle f,\mu\rangle\,. \tag{4.8}$$

By Theorem 4.3, taking into account the definition of f, we obtain that there exists a pair (μ^Y,μ^∞), with $\mu^Y\in Y^1(X;\Xi)$ and $\mu^\infty\in M_b^+(X\times\Sigma_\Xi)$, such that

$$\langle f,\mu\rangle = \int_{X\times\Xi} g(x,\xi)\,d\mu^Y(x,\xi) + \int_{X\times\Sigma_\Xi} g^\infty(x,\xi)\,d\mu^\infty(x,\xi)\,. \tag{4.9}$$

The conclusion follows from (4.8) and (4.9). □

5. **A density result.** In this section we prove that, if λ is nonatomic, then the generalized Young measures of the form δ_u associated with functions $u\in L^1(X;\Xi)$ are dense in $GY(X;\Xi)$. The main result is the following approximation theorem.

Theorem 5.1. *Assume that λ is nonatomic and let $\mu\in GY(X;\Xi)$. Then there exists a sequence u_n in $L^1(X;\Xi)$ such that $\delta_{u_n}\rightharpoonup\mu$ weakly* in $GY(X;\Xi)$.*

Proof. We consider the decomposition

$$\mu = \overline{\mu}^Y + \hat{\mu}^\infty$$

of Theorem 4.3 and fix a sequence σ_n converging to 0 with $0<\sigma_n<\min\{1,\lambda(X)\}$. For every n we consider two countable partitions $\Xi=\bigcup_j B_j^{n,Y}$ and $\Sigma_\Xi=\bigcup_j B_j^{n,\infty}$, where the sets $B_j^{n,Y}$ and $B_j^{n,\infty}$ satisfy

$$\operatorname{diam} B_j^{n,Y}\le\sigma_n \qquad\text{and}\qquad \operatorname{diam} B_j^{n,\infty}\le\sigma_n\,. \tag{5.1}$$

Let $\lambda^Y := \pi_X(\sqrt{1+|\xi|^2}\mu^Y)$ and $\lambda^\infty := \pi_X(\mu^\infty)$; i.e.,

$$\lambda^Y(A) = \int_{A\times\Xi}\sqrt{1+|\xi|^2}\,d\mu^Y(x,\xi) \qquad\text{and}\qquad \lambda^\infty(A)=\mu^\infty(A\times\Sigma_\Xi)$$

for every Borel set $A\subset X$. As $\lambda=\pi_X(\mu^Y)$, the measure λ^Y is absolutely continuous with respect to λ.

Let us consider the partition $X=X^a\cup X^s$, with $\lambda(X^s)=0=\lambda^{\infty,s}(X^a)$, where $\lambda^{\infty,s}$ is the singular part of λ^∞ with respect to λ. To prove the theorem, for every n we will consider a new partition $X=X^{n,a}\cup X^{n,s}$, where $X^{n,a}$ and $X^{n,s}$ are suitable approximations of X^a and X^s such that $\lambda(X^{n,a})>0$ and $\lambda(X^{n,s})>0$. We will construct the approximating sequence u_n by defining it separately on $X^{n,a}$ and $X^{n,s}$.

Step 1. Definition of u_n on $X^{n,s}$. We begin by constructing $X^{n,s}$. For every n we can find a countable Borel partition $X^s=\bigcup_i X_i^{n,s}\cup N^{n,s}$, where each $X_i^{n,s}$ is closed,

$$\operatorname{diam} X_i^{n,s}\le\sigma_n/2\,, \qquad\text{and}\qquad \lambda^\infty(N^{n,s})=0\,. \tag{5.2}$$

In the following, given a subset $E \subset X$ and a radius $r > 0$, the r-neighbourhood of E will be denoted by

$$(E)_r := \{x \in X : \ d(x, E) < r\}\,.$$

Since $\lambda(X_i^{n,s}) = \lambda^Y(X_i^{n,s}) = 0$ and $\lambda((X_i^{n,s})_r) > 0$ for every $r > 0$, we can construct inductively a decreasing sequence r_i^n such that $0 < r_i^n \le \sigma_n/2$,

$$\lambda((X_i^{n,s})_{r_i^n}) \le \sigma_n \lambda^\infty(X_i^{n,s})\,, \tag{5.3}$$

$$\lambda^Y((X_i^{n,s})_{r_i^n}) \le 2^{-i}\sigma_n\,, \tag{5.4}$$

$$\lambda^\infty((X_i^{n,s})_{r_i^n} \setminus X^s) \le 2^{-i}\sigma_n\,, \tag{5.5}$$

$$\lambda((X_{i+1}^{n,s})_{r_{i+1}^n}) \le \tfrac{1}{3}\lambda((X_i^{n,s})_{r_i^n})\,. \tag{5.6}$$

We define

$$A_i^{n,s} := (X_i^{n,s})_{r_i^n} \setminus \textstyle\bigcup_{j>i} (X_j^{n,s})_{r_j^n}\,.$$

By (5.6) we have

$$\lambda(A_i^{n,s}) \ge \tfrac{1}{2}\lambda((X_i^{n,s})_{r_i^n}) > 0\,,$$

while (5.2), together with the inequality $0 < r_i^n \le \sigma_n/2$, yields

$$\operatorname{diam} A_i^{n,s} \le \sigma_n\,. \tag{5.7}$$

By (5.3) we have

$$0 < \lambda(A_i^{n,s}) \le \sigma_n \lambda^\infty(X_i^{n,s}) = \sigma_n \mu^\infty(X_i^{n,s} \times \Sigma_\Xi) = \sigma_n \textstyle\sum_j \mu^\infty(X_i^{n,s} \times B_j^{n,\infty})\,.$$

Since λ is nonatomic we can find a countable Borel partition $A_i^{n,s} = \bigcup_j A_{ij}^{n,s}$ such that

$$0 < \lambda(A_{ij}^{n,s}) \le \sigma_n \mu^\infty(X_i^{n,s} \times B_j^{n,\infty})\,. \tag{5.8}$$

For every n we define

$$X^{n,s} := \textstyle\bigcup_{ij} A_{ij}^{n,s} \cup X^s\,. \tag{5.9}$$

Note that by (5.4) and (5.5) we have

$$\lambda^Y(X^{n,s}) \le \sigma_n \qquad \text{and} \qquad \lambda^\infty(X^{n,s} \setminus X^s) \le \sigma_n\,. \tag{5.10}$$

We define

$$u_n(x) := c_{ij}^n \xi_j^{n,\infty} \quad \text{for } x \in A_{ij}^{n,s}\,, \tag{5.11}$$

where $\xi_j^{n,\infty}$ are arbitrary points of $B_j^{n,\infty}$ and

$$c_{ij}^n := \frac{\mu^\infty(X_i^{n,s} \times B_j^{n,\infty})}{\lambda(A_{ij}^{n,s})}\,. \tag{5.12}$$

By (5.8) we have that

$$c_{ij}^n \ge 1/\sigma_n\,. \tag{5.13}$$

By (5.12) and by (5.13) we have

$$\int_{X^{n,s}} \sqrt{1 + |u_n|^2}\, d\lambda \le \lambda^\infty(X^s)\sqrt{1 + \sigma_n^2}\,. \tag{5.14}$$

Step 2. Definition of u_n on $X^{n,a}$. We set

$$X^{n,a} := X \setminus X^{n,s}\,.$$

In order to define u_n on $X^{n,a}$ we consider a countable Borel partition $X^{n,a} = \bigcup_i A_i^{n,a}$, with $A_i^{n,a}$ satisfying

$$0 < \operatorname{diam} A_i^{n,a} \le \sigma_n\,. \tag{5.15}$$

As $X^{n,a} \subset X^a$ by (5.9), λ^∞ is absolutely continuous with respect to λ on $X^{n,a}$. Since λ is nonatomic, for every i we may choose $0 < \varepsilon_i^n \le \sigma_n$ and two disjoint Borel sets $A_i^{n,Y}$ and $A_i^{n,\infty}$ in such a way that $A_i^{n,a} = A_i^{n,Y} \cup A_i^{n,\infty}$ and

$$\lambda(A_i^{n,\infty}) = \varepsilon_i^n \lambda^\infty(A_i^{n,a}) \le \sigma_n \lambda(A_i^{n,a})\,. \tag{5.16}$$

Since λ is nonatomic and

$$\varepsilon_i^n \lambda^\infty(A_i^{n,a}) = \varepsilon_i^n \mu^\infty(A_i^{n,a} \times \Sigma_\Xi) = \textstyle\sum_j \varepsilon_i^n \mu^\infty(A_i^{n,a} \times B_j^{n,\infty})\,,$$

we can also find a countable Borel partition $A_i^{n,\infty} = \bigcup_j A_{ij}^{n,\infty}$ such that

$$\lambda(A_{ij}^{n,\infty}) = \varepsilon_i^n \mu^\infty(A_i^{n,a} \times B_j^{n,\infty})\,. \tag{5.17}$$

Note also that by (5.16) we have

$$\lambda(A_i^{n,Y}) = \lambda(A_i^{n,a}) - \lambda(A_i^{n,\infty}) \ge (1-\sigma_n)\,\lambda(A_i^{n,a})$$

and so there exists $0 < \delta_i^n \le \sigma_n$ such that $\lambda(A_i^{n,Y}) = (1-\delta_i^n)\lambda(A_i^{n,a})$. As $\lambda = \pi_X(\mu^Y)$, arguing as before we may find a countable Borel partition $A_i^{n,Y} = \bigcup_j A_{ij}^{n,Y}$ such that

$$\lambda(A_{ij}^{n,Y}) = (1-\delta_i^n)\mu^Y(A_i^{n,a} \times B_j^{n,Y})\,. \tag{5.18}$$

We are ready to define u_n on $X^{n,a}$ by setting

$$u_n(x) := \xi_j^{n,Y} \quad \text{for } x \in A_{ij}^{n,Y} \qquad \text{and} \qquad u_n(x) := \frac{1}{\varepsilon_i^n}\xi_j^{n,\infty} \quad \text{for } x \in A_{ij}^{n,\infty}\,, \tag{5.19}$$

where $\xi_j^{n,Y}$ are arbitrary points in $B_j^{n,Y}$ and $\xi_j^{n,\infty}$ are the points of $B_j^{n,\infty}$ chosen in (5.11). Using (5.17) and (5.18) it is easy to check that

$$\int_{X^{n,a}} \sqrt{1+|u_n|^2}\,d\lambda \le \sigma_n \lambda(X) + \lambda^Y(X) + \lambda^\infty(X^{n,a})\sqrt{1+\sigma_n^2}\,. \tag{5.20}$$

By (5.14) and (5.20) we have

$$\int_X \sqrt{1+|u_n|^2}\,d\lambda \le \sigma_n \lambda(X) + \lambda^Y(X) + \lambda^\infty(X)\sqrt{1+\sigma_n^2}\,, \tag{5.21}$$

which implies that u_n is bounded in $L^1(X;\Xi)$. It follows from Lemma 3.3 that $\|\delta_{u_n}\|_*$ is uniformly bounded.

Step 3. Proof of the convergence. Thanks to Lemma 2.4, to prove the weak* convergence of δ_{u_n} it is enough to show that

$$\langle f, \delta_{u_n}\rangle \to \langle f, \mu\rangle, \tag{5.22}$$

for every function $f \in C_L^{hom}(X \times \Xi \times \mathbb{R})$. Let us fix $f \in C_L^{hom}(X \times \Xi \times \mathbb{R})$. By Remark 2.3 there exist a constant $a \in \mathbb{R}$ and a continuous functon $\omega\colon [0+\infty) \to [0+\infty)$, with $\omega(0) = 0$, such that

$$\begin{aligned} |f(x_1,\xi_1,\eta_1) - f(x_2,\xi_2,\eta_2)| \le a\sqrt{|\xi_1-\xi_2|^2 + |\eta_1-\eta_2|^2} + \\ + \omega(d(x_1,x_2)) \min\{\sqrt{|\xi_1|^2+|\eta_1|^2}, \sqrt{|\xi_2|^2+|\eta_2|^2}\} \end{aligned} \tag{5.23}$$

for every $x_1, x_2 \in X$, $\xi_1, \xi_2 \in \Xi$, $\eta_1, \eta_2 \in \mathbb{R}$.

By definition we have

$$\langle f, \delta_{u_n}\rangle = \int_{X^{n,a}} f(x,u_n,1)\,d\lambda + \int_{X^{n,s}} f(x,u_n,1)\,d\lambda\,. \tag{5.24}$$

By (5.19) the first integral in the right-hand side can be written as

$$\begin{aligned}\int_{X^{n,a}} f(x,u_n,1)\,d\lambda = \\ = \sum_{ij}\int_{A^{n,Y}_{ij}} f(x,\xi^{n,Y}_j,1)\,d\lambda + \sum_{ij}\int_{A^{n,\infty}_{ij}} f(x,\xi^{n,\infty}_j/\varepsilon^n_i,1)\,d\lambda = \\ = \sum_{ij} f(x^{n,a}_i,\xi^{n,Y}_j,1)\lambda(A^{n,Y}_{ij}) + \sum_{ij} f(x^{n,a}_i,\xi^{n,\infty}_j/\varepsilon^n_i,1)\lambda(A^{n,\infty}_{ij}) + r^{a,1}_n,\end{aligned} \tag{5.25}$$

where $x^{n,a}_i$ are arbitrary points in $A^{n,a}_i$ and the remainder $r^{a,1}_n$ tends to 0 as a consequence of (5.15), (5.17), (5.18), and (5.23), which lead to the estimate

$$\begin{aligned}|r^{a,1}_n| \le \omega(\sigma_n)\sum_{ij}\sqrt{1+|\xi^{n,Y}_j|^2}\,\lambda(A^{n,Y}_{ij}) + \\ + \omega(\sigma_n)\sum_{ij}\frac{1}{\varepsilon^n_i}\sqrt{(\varepsilon^n_i)^2+|\xi^{n,\infty}_j|^2}\,\lambda(A^{n,\infty}_{ij}) \le \\ \le \sigma_n\,\omega(\sigma_n)\,\lambda(X^{n,a}) + \omega(\sigma_n)\,\lambda^Y(X^{n,a}) + \omega(\sigma_n)\,\lambda^\infty(X^{n,a})\sqrt{1+\sigma_n^2}\,.\end{aligned}$$

On the other hand by (5.17) and (5.18) we have

$$\begin{aligned}\sum_{ij} f(x^{n,a}_i,\xi^{n,Y}_j,1)\,\lambda(A^{n,Y}_{ij}) + \sum_{ij} f(x^{n,a}_i,\xi^{n,\infty}_j/\varepsilon^n_i,1)\,\lambda(A^{n,\infty}_{ij}) = \\ = \sum_{ij}(1-\delta^n_i)\,f(x^{n,a}_i,\xi^{n,Y}_j,1)\,\mu^Y(A^{n,a}_i\times B^{n,Y}_j) + \\ + \sum_{ij} f(x^{n,a}_i,\xi^{n,\infty}_j,\varepsilon^n_i)\,\mu^\infty(A^{n,a}_i\times B^{n,\infty}_j) = \\ = \int_{X\times\Xi} f(x,\xi,1)\,d\mu^Y(x,\xi) + \int_{X^a\times\Sigma_\Xi} f(x,\xi,0)\,d\mu^\infty(x,\xi) + r^{a,2}_n\,,\end{aligned} \tag{5.26}$$

where the remainder $r^{a,2}_n$ tends to 0 as a consequence of (5.1), (5.10), (5.15), and (5.23), which lead to the estimate

$$|r^{a,2}_n| \le \big(2a\sigma_n + \omega(\sigma_n)\big)\big(\lambda^Y(X) + \lambda^\infty(X^a)\sqrt{1+\sigma_n^2}\big) + 2a\sigma_n\,.$$

From (5.25) and (5.26) we obtain

$$\int_{X^{n,a}} f(x,u_n,1)\,d\lambda = \int_{X\times\Xi} f(x,\xi,1)\,d\mu^Y(x,\xi) + \int_{X^a\times\Sigma_\Xi} f(x,\xi,0)\,d\mu^\infty(x,\xi) + r^a_n\,, \tag{5.27}$$

where $r^a_n := r^{a,1}_n + r^{a,2}_n$ tends to 0.

By (5.11) and (5.12) the second integral in the right-hand side of (5.24) can be written as

$$\int_{X^{n,s}} f(x,u_n,1)\,d\lambda = \sum_{ij} f(x^{n,s}_i, c^n_{ij}\xi^{n,\infty}_j,1)\lambda(A^{n,s}_{ij}) + r^{s,1}_n\,, \tag{5.28}$$

where $x^{n,s}_i$ are arbitrary points in $X^{n,s}_i$ and the remainder $r^{s,1}_n$ tends to zero as a consequence of (5.2), (5.7), (5.12), (5.13), and (5.23), which lead to the estimate

$$|r^{s,1}_n| \le \omega(\sigma_n)\,\lambda^\infty(X^s)\sqrt{1+\sigma_n^2}\,.$$

On the other hand by (5.12)

$$\begin{aligned}&\sum_{ij} f(x_i^{n,s}, c_{ij}^n \xi_j^{n,\infty}, 1)\lambda(A_{ij}^{n,s}) = \\ &= \sum_{ij} f(x_i^{n,s}, \xi_j^{n,\infty}, 1/c_{ij}^n)\, \mu^\infty(X_i^{n,s} \times B_j^{n,\infty}) = \\ &= \int_{X^s \times \Sigma_\Xi} f(x,\xi,0)\, d\mu^\infty(x,\xi) + r_n^{s,2}\,,\end{aligned} \tag{5.29}$$

where the remainder $r_n^{s,2}$ tends to 0 as a consequence of (5.1), (5.2), (5.13), and (5.23), which lead to the estimate

$$|r_n^{s,2}| \le \big(a\sigma_n + \omega(\sigma_n)\sqrt{1+\sigma_n^2}\big)\lambda^\infty(X^s)\,.$$

From (5.28) and (5.29) we obtain

$$\int_{X^{n,s}} f(x,u_n,1)\, d\lambda = \int_{X^s \times \Sigma_\Xi} f(x,\xi,0)\, d\mu^\infty(x,\xi) + r_n^s\,, \tag{5.30}$$

where $r_n^s := r_n^{s,1} + r_n^{s,2}$ tends to 0.

From (4.1), (5.24), (5.27), and (5.30) we obtain (5.22), which concludes the proof of the theorem. ☐

Remark 5.2. If u_n is a sequence in $L^1(X;\Xi)$ such that $\delta_{u_n} \rightharpoonup \mu$ weakly* in $GY(X;\Xi)$, then

$$\int_X \sqrt{1+|u_n|^2}\, d\lambda \longrightarrow \|\mu\|_*$$

by Lemma 3.3 and Remark 3.12.

6. **The notion of barycentre.** In this section we study some properties of the barycentre of a generalized Young measure.

Definition 6.1. The *barycentre* of a generalized Young measure $\mu \in GY(X;\Xi)$ is the measure $\mathrm{bar}(\mu) \in M_b(X;\Xi)$ defined by

$$\mathrm{bar}(\mu) = \pi_X(\xi\,\mu)\,.$$

Remark 6.2. By Definition 2.15 a measure $p \in M_b(X;\Xi)$ coincides with $\mathrm{bar}(\mu)$ if and only if

$$\int_X \varphi \cdot dp = \langle \varphi(x)\cdot\xi, \mu(x,\xi,\eta)\rangle \tag{6.1}$$

for every $\varphi \in C(X;\Xi)$. By approximation we can prove that the same equality holds for every bounded Borel function $\varphi : X \to \Xi$.

Remark 6.3. Let $\mu = \overline{\mu}^Y + \hat{\mu}^\infty$ be the decomposition of Theorem 4.3, let $(\mu^{x,Y})_{x\in X}$ and $(\mu^{x,\infty})_{x\in X}$ be the families of probability measures introduced in Remark 4.5, and let $\lambda^\infty := \pi_X(\mu^\infty)$. We consider the functions $u^Y\colon \Omega \to \Xi$ and $u^\infty\colon \Omega \to \Xi$ defined by

$$u^Y(x) := \int_\Xi \xi\, d\mu^{x,Y}(\xi) \qquad \text{and} \qquad u^\infty(x) := \int_{\Sigma_\Xi} \xi\, d\mu^{x,\infty}(\xi)\,.$$

Then $\mathrm{bar}(\mu) = u^Y + u^\infty\lambda^\infty$. In particular, if $\mu = \overline{\mu}^Y$, then $\mathrm{bar}(\mu) = u^Y \in L^1(X;\Xi)$. Therefore, $\mathrm{bar}(\delta_u) = u$ for every $u \in L^1(X;\Xi)$. It follows immediately from Definition 3.2 and (6.1) that we have also $\mathrm{bar}(\delta_p) = p$ for every $p \in M_b(X;\Xi)$.

Remark 6.4. From Remark 2.17 we obtain

$$\|\mathrm{bar}(\mu)\| \le \|\mu\|_* .$$

If $\mu_k \rightharpoonup \mu$ weakly* in $GY(X;\Xi)$, then $\mathrm{bar}(\mu_k) \rightharpoonup \mathrm{bar}(\mu)$ weakly* in $M_b(X;\Xi)$. If $\mu = \overline{\mu}^Y$ with $\mu^Y \in Y^r(X;\Xi)$ for some $r > 1$, then Remark 6.3 implies that $\mathrm{bar}(\mu) \in L^r(X;\Xi)$ and

$$\|\mathrm{bar}(\mu)\|_r \le \Big(\int_{X\times\Xi} |\xi|^r \, d\mu^Y(x,\xi) \Big)^{1/r} = \langle \{P_r\}, \mu \rangle^{1/r} ,$$

where $\|\cdot\|_r$ denotes the norm in $L^r(X;\Xi)$ and $\{P_r\}$ is the homogeneous function defined in (4.6).

We now prove the *Jensen inequality* for generalized Young measures.

Theorem 6.5. *Let $f\colon X\times\Xi\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be a Borel function such that $(\xi,\eta) \mapsto f(x,\xi,\eta)$ is positively one-homogeneous, convex, and lower semicontinuous for every $x \in X$ and satisfies the inequality*

$$f(x,\xi,\eta) \ge -c\sqrt{|\xi|^2 + |\eta|^2}$$

for some constant c. Then

$$\langle f, \delta_{\mathrm{bar}(\mu)} \rangle \le \langle f, \mu \rangle \tag{6.2}$$

for every $\mu \in GY(X;\Xi)$.

Proof. Let us fix $\mu \in GY(X;\Xi)$, let $p := \mathrm{bar}(\mu)$, and let $\sigma \in M_b^+(X)$ be such that $p << \sigma$ and $\lambda << \sigma$. We consider an increasing sequence of functions f_k converging to f such that each f_k has the form

$$f_k(x,\xi,\eta) = \sup_{1\le i\le k} \{a_i(x)\cdot\xi + b_i(x)\eta\}$$

with $a_i\colon X \to \Xi$ and $b_i\colon X \to \mathbb{R}$ bounded σ-measurable functions (see, e.g., [6, Theorem 2.2.4]). For every k there exists a Borel partition $(B_i^k)_{1\le i\le k}$ such that

$$\int_X f_k(x, \tfrac{dp}{d\sigma}, \tfrac{d\lambda}{d\sigma})\, d\sigma = \sum_{i=1}^k \Big\{ \int_{B_i^k} a_i \cdot dp + \int_{B_i^k} b_i \, d\lambda \Big\} .$$

By (3.2) and (6.1) we obtain

$$\begin{aligned} \int_{B_i^k} a_i \cdot dp + \int_{B_i^k} b_i \, d\lambda &= \langle (a_i(x)\cdot\xi + b_i(x)\eta) 1_{B_i^k}(x), \mu(x,\xi,\eta) \rangle \le \\ &\le \langle f(x,\xi,\eta) 1_{B_i^k}(x), \mu(x,\xi,\eta) \rangle . \end{aligned}$$

Summing over i we get

$$\int_X f_k(x, \tfrac{dp}{d\sigma}, \tfrac{d\lambda}{d\sigma})\, d\sigma \le \langle f(x,\xi,\eta), \mu(x,\xi,\eta) \rangle ,$$

and taking the limit with respect to k gives inequality (6.2). □

Remark 6.6. Let $f\colon X\times\Xi\times\mathbb{R} \to [0,+\infty]$ be a Borel function such that $(\xi,\eta) \mapsto f(x,\xi,\eta)$ is positively one-homogeneous for every $x \in X$, and let $\overline{\mathrm{co}}\, f$ be the lower semicontinuous convex envelope of f with respect to (ξ,η). By applying (6.2) to $\overline{\mathrm{co}}\, f$ we obtain

$$\langle \overline{\mathrm{co}}\, f, \delta_{\mathrm{bar}(\mu)} \rangle \le \langle f, \mu \rangle$$

for every $\mu \in GY(X;\Xi)$.

The opposite inequality requires special conditions on f and μ, as shown in the following lemma, that will be used in [11].

Lemma 6.7. *Let $\mu \in GY(X;\Xi)$, let $f\colon X\times\Xi\times\mathbb{R} \to [0,+\infty]$ be a Borel function such that $(\xi,\eta) \mapsto f(x,\xi,\eta)$ is positively one-homogeneous for every $x \in X$, and let $\overline{\mathrm{co}}\, f$ be the lower semicontinuous convex envelope of f with respect to (ξ,η). Assume that $\langle f,\mu\rangle \le \langle \overline{\mathrm{co}}\, f, \delta_{\mathrm{bar}(\mu)}\rangle < +\infty$. Then $\operatorname{supp}\mu$ is contained in the closure of $\{f = \overline{\mathrm{co}}\, f\}$.*

Proof. Using the hypothesis and (6.2) we obtain

$$\langle \overline{\mathrm{co}}\, f, \delta_{\mathrm{bar}(\mu)}\rangle \le \langle \overline{\mathrm{co}}\, f, \mu\rangle \le \langle f,\mu\rangle \le \langle \overline{\mathrm{co}}\, f, \delta_{\mathrm{bar}(\mu)}\rangle\,,$$

hence, $\langle f - \overline{\mathrm{co}}\, f, \mu\rangle = 0$. Since $f - \overline{\mathrm{co}}\, f$ and μ are nonnegative, we conclude that $\operatorname{supp}\mu$ is contained in the closure of $\{f = \overline{\mathrm{co}}\, f\}$. □

7. **Compatible systems of generalized Young measures.** Let $A \subset \mathbb{R}$ and let $\boldsymbol{p}$ be a function from A into $M_b(X;\Xi)$. For every finite sequence $t_1 < t_2 < \cdots < t_m$ in A we consider the measure $(\boldsymbol{p}(t_1),\dots,\boldsymbol{p}(t_m)) \in M_b(X;\Xi^m)$ and the corresponding generalized Young measure

$$(\boldsymbol{\delta_p})_{t_1\dots t_m} := \delta_{(\boldsymbol{p}(t_1),\dots,\boldsymbol{p}(t_m))} \in GY(X;\Xi^m) \tag{7.1}$$

introduced in Definition 3.2, with Ξ replaced by Ξ^m. To describe an important property of this family of generalized Young measures it is convenient to introduce the following definition.

Definition 7.1. If $\{s_1,s_2,\dots,s_n\} \subset \{t_1,t_2,\dots,t_m\} \subset \mathbb{R}$, with $s_1 < s_2 < \cdots < s_n$ and $t_1 < t_2 < \cdots < t_m$, we define the projection $\pi^{t_1\dots t_m}_{s_1\dots s_n}\colon X\times\Xi^m\times\mathbb{R} \to X\times\Xi^n\times\mathbb{R}$ by

$$\pi^{t_1\dots t_m}_{s_1\dots s_n}(x,\xi_{t_1},\dots,\xi_{t_m},\eta) = (x,\xi_{s_1},\dots,\xi_{s_n},\eta)\,.$$

Remark 7.2. It is easy to see that the family of generalized Young measures $\boldsymbol{\delta_p}$, defined in (7.1), satisfies the *compatibility condition*

$$(\boldsymbol{\delta_p})_{s_1\dots s_n} = \pi^{t_1\dots t_m}_{s_1\dots s_n}((\boldsymbol{\delta_p})_{t_1\dots t_m})$$

whenever $\{s_1,s_2,\dots,s_n\}$ and $\{t_1,t_2,\dots,t_m\}$ are as in Definition 7.1.

This motivates the following definition.

Definition 7.3. A *compatible system of generalized Young measures* on X, with values in a finite dimensional Hilbert space Ξ and with time set $A \subset \mathbb{R}$, is a family $\boldsymbol{\mu} = (\boldsymbol{\mu}_{t_1\dots t_m})$ of generalized Young measures $\boldsymbol{\mu}_{t_1\dots t_m} \in GY(X;\Xi^m)$, with $t_1,\dots,t_m$ running over all finite sequences of elements of A with $t_1 < t_2 < \cdots < t_m$, such that the following compatibility condition holds:

$$\boldsymbol{\mu}_{s_1\dots s_n} = \pi^{t_1\dots t_m}_{s_1\dots s_n}(\boldsymbol{\mu}_{t_1\dots t_m})\,, \tag{7.2}$$

whenever $\{s_1,s_2,\dots,s_n\} \subset \{t_1,t_2,\dots,t_m\}$. The space of all such systems is denoted by $SGY(A,X;\Xi)$ and is equipped with the weakest topology such that the maps $\boldsymbol{\mu} \mapsto \boldsymbol{\mu}_{t_1\dots t_m}$ from $SGY(A,X;\Xi)$ into $GY(X;\Xi^m)$, endowed with the weak* topology, are continuous for every m and every finite sequence $t_1,\dots,t_m$ in A with $t_1 < t_2 < \cdots < t_m$. Although this topology is not induced by duality, we shall refer to it as the *weak* topology* of $SGY(A,X;\Xi)$.

Definition 7.4. Given a function $\boldsymbol{p}$ from $A \subset \mathbb{R}$ into $M_b(X;\Xi)$, the family $\boldsymbol{\delta_p}$ defined in (7.1) is called the *compatible system of generalized Young measures associated with $\boldsymbol{p}$*.

The compatibility condition (7.2) implies that the barycentre of $\boldsymbol{\mu}_{t_1\dots t_m}$ is completely determined by the barycentres of $\boldsymbol{\mu}_{t_1},\dots,\boldsymbol{\mu}_{t_m}$.

Proposition 7.5. *Let $\boldsymbol{\mu}\in SGY(A,X;\Xi)$. Then*

$$\mathrm{bar}(\boldsymbol{\mu}_{t_1\dots t_m})=(\mathrm{bar}(\boldsymbol{\mu}_{t_1}),\dots,\mathrm{bar}(\boldsymbol{\mu}_{t_m}))$$

for every finite sequence $t_1,\dots,t_m$ in A with $t_1<t_2<\dots<t_m$.

Proof. Let $(p_1,\dots,p_m):=\mathrm{bar}(\boldsymbol{\mu}_{t_1\dots t_m})$ and $q_i=\mathrm{bar}(\boldsymbol{\mu}_{t_i})$ for $i=1,\dots,m$. Using (6.1) for every $(\varphi_1,\dots,\varphi_m)\in C(X;\Xi^m)$ we have

$$\sum_{i=1}^m\int_X\varphi_i\cdot dp_i=\sum_{i=1}^m\langle\varphi_i(x)\cdot\xi_i,\boldsymbol{\mu}_{t_1\dots t_m}(x,\xi_1,\dots,\xi_m,\eta)\rangle\,, \tag{7.3}$$

$$\int_X\varphi_i\cdot dq_i=\langle\varphi_i(x)\cdot\xi_i,\boldsymbol{\mu}_{t_i}(x,\xi_i,\eta)\rangle \qquad \text{for } i=1,\dots,m\,. \tag{7.4}$$

The compatibility condition (7.2) implies that

$$\langle\varphi_i(x)\cdot\xi_i,\boldsymbol{\mu}_{t_1\dots t_m}(x,\xi_1,\dots,\xi_m,\eta)\rangle=\langle\varphi_i(x)\cdot\xi_i,\boldsymbol{\mu}_{t_i}(x,\xi_i,\eta)\rangle\,,$$

hence (7.3) and (7.4) yield

$$\sum_{i=1}^m\int_X\varphi_i\cdot dp_i=\sum_{i=1}^m\int_X\varphi_i\cdot dq_i$$

for every $(\varphi_1,\dots,\varphi_m)\in C(X;\Xi^m)$. This gives $p_i=q_i$ for $i=1,\dots,m$. □

The notion of left continuity, introduced in the next definition, is very useful in the applications.

Definition 7.6. A system $\boldsymbol{\mu}\in SGY(A,X;\Xi)$ is said to be *left continuous* if for every finite sequence $t_1,\dots,t_m$ in A with $t_1<\dots<t_m$ the following continuity property holds:

$$\boldsymbol{\mu}_{s_1\dots s_m}\rightharpoonup\boldsymbol{\mu}_{t_1\dots t_m}\quad\text{weakly}^*\text{ in } GY(X;\Xi^m) \tag{7.5}$$

as $s_i\to t_i$, with $s_i\in A$ and $s_i\le t_i$.

The following theorem proves the weak* compactness of subsets of $SGY(A,X;\Xi)$ defined by imposing bounds on the norms of $\boldsymbol{\mu}_t$ for every $t\in A$.

Theorem 7.7. *For every function $C\colon A\to[0,+\infty)$ the set*

$$\{\boldsymbol{\mu}\in SGY(A,X;\Xi):\ \|\boldsymbol{\mu}_t\|_*\le C(t)\ \textit{for every}\ t\in A\} \tag{7.6}$$

is weakly* compact in $SGY(A,X;\Xi)$.

To prove the theorem we need the following lemma which provides an estimate of the norm $\|\boldsymbol{\mu}_{t_1\dots t_m}\|_*$ in terms of the norms $\|\boldsymbol{\mu}_{t_i}\|_*$.

Lemma 7.8. *For every $\boldsymbol{\mu}\in SGY(A,X;\Xi)$ we have*

$$\|\boldsymbol{\mu}_{t_1\dots t_m}\|_*\le\sum_{i=1}^m\|\boldsymbol{\mu}_{t_i}\|_*$$

for every finite sequence $t_1,\dots,t_m$ in A with $t_1<t_2<\dots<t_m$.

Proof. By Remark 2.9 and by the compatibility condition (7.2) we have

$$\begin{aligned}\|\boldsymbol{\mu}_{t_1\dots t_m}\|_* &= \langle |(\xi_1,\dots,\xi_m,\eta)|, \boldsymbol{\mu}_{t_1\dots t_m}(x,\xi_1,\dots,\xi_m,\eta)\rangle \le \\ &\le \sum_{i=1}^m \langle |(\xi_i,\eta)|, \boldsymbol{\mu}_{t_1\dots t_m}(x,\xi_1,\dots,\xi_m,\eta)\rangle = \\ &= \sum_{i=1}^m \langle |(\xi_i,\eta)|, \boldsymbol{\mu}_{t_i}(x,\xi_i,\eta)\rangle = \sum_{i=1}^m \|\boldsymbol{\mu}_{t_i}\|_*\,,\end{aligned}$$

which concludes the proof. □

Proof of Theorem 7.7. By Lemma 7.8 for every function $C\colon A \to [0,+\infty)$ the set defined in (7.6) is contained in the set of all $\boldsymbol{\mu} \in SGY(A,X;\Xi)$ such that

$$\|\boldsymbol{\mu}_{t_1\dots t_m}\|_* \le \sum_{i=1}^m C(t_i)$$

for every finite sequence $t_1,\dots,t_m$ in A with $t_1 < t_2 < \dots < t_m$. As the topology in $SGY(A,X;\Xi)$ is induced by the product of the weak* topologies of the spaces $GY(X;\Xi^m)$ corresponding to the projections $\boldsymbol{\mu}_{t_1\dots t_m}$, the set (7.6) is compact in the weak* topology of $SGY(A,X;\Xi)$ by Tychonoff's Theorem. □

Remark 7.9. If $A = \{a_0,a_1,\dots,a_k\}$, with $a_0 < a_1 < \dots < a_k$, then for every $\mu \in GY(X;\Xi^{k+1})$ there exists a unique system $\boldsymbol{\mu}^A \in SGY(A,X;\Xi)$ such that $\boldsymbol{\mu}^A_{a_0\dots a_k} = \mu$. This system is defined by

$$\boldsymbol{\mu}^A_{t_1\dots t_m} = \pi^{a_0\dots a_k}_{t_1\dots t_m}(\mu)$$

for every $\{t_1,t_2,\dots,t_m\} \subset \{a_0,a_1,\dots,a_k\}$ with $t_1 < t_2 < \dots < t_m$.

The notion of piecewise constant interpolation will be useful in the application to evolution problems.

Definition 7.10. Let $A = \{a_0,a_1,\dots,a_k\}$, with $a_0 < a_1 < \dots < a_k$. For every $t_1,\dots,t_m$ in $[a_0,a_k]$ with $t_1 < t_2 < \dots < t_m$ let $\rho_{t_1\dots t_m}\colon X\times\Xi^{k+1}\times\mathbb{R} \to X\times\Xi^m\times\mathbb{R}$ be defined by

$$\rho_{t_1\dots t_m}(x,\xi_{a_0},\dots,\xi_{a_k},\eta) := (x,\xi_{t_1},\dots,\xi_{t_m},\eta)\,,$$

with $\xi_{t_i} = \xi_{a_j}$, where j is the largest index such that $a_j \le t_i$. For every $\mu \in GY(X;\Xi^{k+1})$ the *piecewise constant interpolation* $\boldsymbol{\mu}^{[A]}$ of μ is the element of $SGY([a_0,a_k],X;\Xi)$ defined by

$$\boldsymbol{\mu}^{[A]}_{t_1\dots t_m} := \rho_{t_1\dots t_m}(\mu) \tag{7.7}$$

for every $t_1,\dots,t_m$ in $[a_0,a_k]$ with $t_1 < t_2 < \dots < t_m$.

Remark 7.11. It is easy to check that $\rho_{s_1\dots s_n} = \pi^{t_1\dots t_m}_{s_1\dots s_n} \circ \rho_{t_1\dots t_m}$ whenever $\{s_1,s_2,\dots,s_n\} \subset \{t_1,t_2,\dots,t_m\} \subset [a_0,a_k]$, with $s_1 < s_2 < \dots < s_n$ and $t_1 < t_2 < \dots < t_m$. Therefore the family of generalized Young measures $(\boldsymbol{\mu}^{[A]}_{t_1\dots t_m})$ defined by (7.7) satisfies the compatibility condition (7.2).

8. **The notion of variation.** In this section we study the notion of variation on a time interval of a compatible system of generalized Young measures, and prove a compactness theorem which extends Helly's Theorem.

Definition 8.1. Given a set $A \subset \mathbb{R}$, the *variation* of $\boldsymbol{\mu} \in SGY(A, X; \Xi)$ on the time interval $[a,b]$, with $a,\ b \in A$, is defined as

$$\mathrm{Var}(\boldsymbol{\mu}; a, b) := \sup \sum_{i=1}^{k} \langle |\xi_i - \xi_{i-1}|, \boldsymbol{\mu}_{t_0 t_1 \dots t_k}(x, \xi_0, \dots, \xi_k, \eta) \rangle \,,$$

where the supremum is taken over all finite families $t_0, t_1, \dots, t_k$ in A such that $a = t_0 < t_1 < \dots < t_k = b$ (with the convention $\mathrm{Var}(\boldsymbol{\mu}; a, b) = 0$ if $a = b$).

Remark 8.2. If $\boldsymbol{\mu} = \boldsymbol{\delta_p}$ for some function $\boldsymbol{p}\colon A \to M_b(X; \Xi)$, then $\mathrm{Var}(\boldsymbol{\mu}; a, b)$ reduces to the variation of $\boldsymbol{p}$ on $[a,b] \cap A$.

Remark 8.3. Returning to the general case, the compatibility condition (7.2) yields

$$\mathrm{Var}(\boldsymbol{\mu}; a, b) = \sup \sum_{i=1}^{k} \langle |\xi_i - \xi_{i-1}|, \boldsymbol{\mu}_{t_{i-1} t_i}(x, \xi_{i-1}, \xi_i, \eta) \rangle \,,$$

where the supremum is taken over all finite families $t_0, t_1, \dots, t_k$ in A such that $a = t_0 < t_1 < \dots < t_k = b$.

Remark 8.4. If $t_1, t_2, t_3 \in A$ and $t_1 < t_2 < t_3$, by the compatibility condition (7.2) and by the triangle inequality we have

$$\begin{aligned} \langle |\xi_3 - \xi_1|, \boldsymbol{\mu}_{t_1 t_3}(x, \xi_1, \xi_3, \eta) \rangle &= \langle |\xi_3 - \xi_1|, \boldsymbol{\mu}_{t_1 t_2 t_3}(x, \xi_1, \xi_2, \xi_3, \eta) \rangle \le \\ \le \langle |\xi_3 - \xi_2|, \boldsymbol{\mu}_{t_1 t_2 t_3}(x, \xi_1, \xi_2, \xi_3, \eta) \rangle &+ \langle |\xi_2 - \xi_1|, \boldsymbol{\mu}_{t_1 t_2 t_3}(x, \xi_1, \xi_2, \xi_3, \eta) \rangle = \\ = \langle |\xi_3 - \xi_2|, \boldsymbol{\mu}_{t_2 t_3}(x, \xi_2, \xi_3, \eta) \rangle &+ \langle |\xi_2 - \xi_1|, \boldsymbol{\mu}_{t_1 t_2}(x, \xi_1, \xi_2, \eta) \rangle \,. \end{aligned}$$

Using this inequality it is easy to deduce from Remark 8.3 that

$$\mathrm{Var}(\boldsymbol{\mu}; a, c) = \mathrm{Var}(\boldsymbol{\mu}; a, b) + \mathrm{Var}(\boldsymbol{\mu}; b, c) \tag{8.1}$$

for every $a, b, c \in A$ with $a \le b \le c$. This implies in particular that the function $t \mapsto \mathrm{Var}(\boldsymbol{\mu}; a, t)$ is nondecreasing on $A \cap [a, +\infty)$.

Remark 8.5. If $A = \{a_0, \dots, a_k\} \subset \mathbb{R}$ is a finite set, with $a_0 < a_1 < \dots < a_k$, $\mu \in GY(X; \Xi^{k+1})$, and $\boldsymbol{\mu}^A \in SGY(A, X; \Xi)$ is the associated system defined in Remark 7.9, it follows from (8.1) that

$$\begin{aligned} \mathrm{Var}(\boldsymbol{\mu}^A; a_0, a_k) &= \sum_{i=1}^{k} \langle |\xi_i - \xi_{i-1}|, \boldsymbol{\mu}^A_{a_{i-1} a_i}(x, \xi_{i-1}, \xi_i, \eta) \rangle = \\ &= \sum_{i=1}^{k} \langle |\xi_i - \xi_{i-1}|, \mu(x, \xi_0, \dots, \xi_k, \eta) \rangle \,. \end{aligned}$$

It is easy to see that, if $\boldsymbol{\mu}^{[A]} \in SGY([a_0, a_k], X; \Xi)$ is the piecewise constant interpolation of μ defined by (7.7), then

$$\begin{aligned} \mathrm{Var}(\boldsymbol{\mu}^{[A]}; a_0, a_k) = \mathrm{Var}(\boldsymbol{\mu}^A; a_0, a_k) &= \sum_{i=1}^{k} \langle |\xi_i - \xi_{i-1}|, \boldsymbol{\mu}^A_{a_{i-1} a_i}(x, \xi_{i-1}, \xi_i, \eta) \rangle \\ &= \sum_{i=1}^{k} \langle |\xi_i - \xi_{i-1}|, \mu(x, \xi_0, \dots, \xi_k, \eta) \rangle \,. \end{aligned}$$

Definition 8.6. Let $h\colon \Xi \to [0,+\infty)$ be a positively one-homogeneous function satisfying the triangle inequality. Given a set $A \subset \mathbb{R}$, the *h-variation* of $\boldsymbol{\mu} \in SGY(A,X;\Xi)$ on the time interval $[a,b]$, with $a,\ b \in A$, is defined as

$$\mathrm{Var}_h(\boldsymbol{\mu};a,b) := \sup \sum_{i=1}^{k} \langle h(\xi_i - \xi_{i-1}), \boldsymbol{\mu}_{t_0 t_1 \ldots t_k}(x,\xi_0,\ldots,\xi_k,\eta)\rangle\,,$$

where the supremum is taken over all finite families $t_0,t_1,\ldots,t_k$ in A such that $a = t_0 < t_1 < \cdots < t_k = b$ (with the convention $\mathrm{Var}_h(\boldsymbol{\mu};a,b) = 0$ if $a = b$).

Remark 8.7. It is well known that every positively one-homogeneous function $h\colon \Xi \to [0,+\infty)$ satisfying the triangle inequality is continuous and satisfies an estimate of the form $h(\xi) \le c\,|\xi|$ for some constant c. It follows that $\mathrm{Var}_h(\boldsymbol{\mu};a,b) \le c\,\mathrm{Var}(\boldsymbol{\mu};a,b)$. It is easy to see that all properties of $\mathrm{Var}(\boldsymbol{\mu};a,b)$ proved so far can be extended to $\mathrm{Var}_h(\boldsymbol{\mu};a,b)$.

Using the compatibility condition it is easy to prove the following lemma.

Lemma 8.8. *Let $T > 0$ and let $\boldsymbol{\mu} \in SGY([0,T],X;\Xi)$ with $\mathrm{Var}(\boldsymbol{\mu};0,T) < +\infty$. For every $f \in C_L^{hom}(X{\times}\Xi{\times}\mathbb{R})$ the function $t \mapsto \langle f, \boldsymbol{\mu}_t\rangle$ has bounded variation on $[0,T]$.*

The proof is omitted, since it is similar to the proof of the following lemma, which will be used in Theorem 9.7.

Lemma 8.9. *Let $T > c > 0$ and let $\boldsymbol{\mu} \in SGY([0,T],X;\Xi)$ with $\mathrm{Var}(\boldsymbol{\mu};0,T) < +\infty$. For every $f \in C_L^{hom}(X{\times}\Xi^2{\times}\mathbb{R})$ the function $\Phi_c^f(t) := \langle f, \boldsymbol{\mu}_{t,t+c}\rangle$ has bounded variation on $[0,T-c]$.*

Proof. Let $V(t) := \mathrm{Var}(\boldsymbol{\mu};0,t)$ for every $t \in [0,T]$. Let us fix $f \in C_L^{hom}(X{\times}\Xi^2{\times}\mathbb{R})$ and let a be a constant satisfying (2.1). Let t_1,t_2 with $0 \le t_1 < t_1 + c < t_2 < t_2 + c \le T$. Using the compatibility condition (7.2) and (8.1), we obtain

$$\begin{aligned}
&|\Phi_c^f(t_2) - \Phi_c^f(t_1)| = \\
&= |\langle f(x,\xi_2,\xi_2',\eta) - f(x,\xi_1,\xi_1',\eta), \boldsymbol{\mu}_{t_1,t_1+c,t_2,t_2+c}(x,\xi_1,\xi_1',\xi_2,\xi_2',\eta)\rangle| \le \\
&\le a\langle |\xi_2 - \xi_1| + |\xi_2' - \xi_1'|, \boldsymbol{\mu}_{t_1,t_1+c,t_2,t_2+c}(x,\xi_1,\xi_1',\xi_2,\xi_2',\eta)\rangle = \\
&= a\langle |\xi_2 - \xi_1|, \boldsymbol{\mu}_{t_1,t_2}(x,\xi_1,\xi_2,\eta)\rangle + a\langle |\xi_2' - \xi_1'|, \boldsymbol{\mu}_{t_1+c,t_2+c}(x,\xi_1',\xi_2',\eta)\rangle \le \\
&\le V(t_2) - V(t_1) + V(t_2+c) - V(t_1+c)\,.
\end{aligned}$$

The same inequality can be proved if $0 \le t_1 < t_2 \le t_1 + c < t_2 + c \le T$. As V is nondecreasing, we conclude that the total variation of Φ_c^f on $[0,T-c]$ is less than or equal to $V(T-c) + V(T)$. □

The following result can be considered as a version of Helly's Theorem for compatible systems of generalized Young measures. Note that this is a sequential compactness result, in contrast with Theorem 7.7.

Theorem 8.10. *Let $T > 0$ and let $\boldsymbol{\mu}^k$ be a sequence in $SGY([0,T],X;\Xi)$ such that*

$$\sup_k \mathrm{Var}(\boldsymbol{\mu}^k;0,T) \le C\,, \tag{8.2}$$

$$\sup_k \|\boldsymbol{\mu}^k_{t_0}\|_* \le C_*\,, \tag{8.3}$$

for some $t_0 \in [0,T]$ *and some finite constants* C *and* C_**. Then there exist a subsequence, still denoted* $\boldsymbol{\mu}^k$*, a set* $\Theta \subset [0,T]$*, containing* 0 *and with* $[0,T]\setminus\Theta$ *at most countable, and a left continuous* $\boldsymbol{\mu} \in SGY([0,T],X;\Xi)$*, with*

$$\mathrm{Var}(\boldsymbol{\mu};0,T) \le C\,, \tag{8.4}$$

$$\|\boldsymbol{\mu}_t\|_* \le C_* + C \quad \text{for every } t \in [0,T]\,, \tag{8.5}$$

such that

$$\boldsymbol{\mu}^k_{t_1\dots t_m} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_m} \quad \text{weakly}^* \text{ in } GY(X;\Xi^m) \tag{8.6}$$

for every finite sequence $t_1,\dots,t_m$ *in* Θ *with* $0 \le t_1 < \dots < t_m \le T$*.*

Proof. The proof is divided in several steps.

Step 1. Boundedness of $\boldsymbol{\mu}^k_{t_1\dots t_m}$*.* We begin by proving that $\|\boldsymbol{\mu}^k_t\|_*$ is bounded uniformly with respect to $t \in [0,T]$ and k. Let us fix $t < t_0$. By the compatibility condition (7.2)

$$\begin{aligned}
&\langle |\xi|, \boldsymbol{\mu}^k_t(x,\xi,\eta)\rangle - \langle |\xi_0|, \boldsymbol{\mu}^k_{t_0}(x,\xi_0,\eta)\rangle = \\
&= \langle |\xi|, \boldsymbol{\mu}^k_{tt_0}(x,\xi,\xi_0,\eta)\rangle - \langle |\xi_0|, \boldsymbol{\mu}^k_{tt_0}(x,\xi,\xi_0,\eta)\rangle \le \\
&\le \langle |\xi-\xi_0|, \boldsymbol{\mu}^k_{tt_0}(x,\xi,\xi_0,\eta)\rangle \le \mathrm{Var}(\boldsymbol{\mu}^k;t,t_0) \le C\,.
\end{aligned}$$

Thanks to Remark 2.9, from (8.2) and (8.3) we obtain that

$$\sup_k \|\boldsymbol{\mu}^k_t\|_* \le C_* + C \tag{8.7}$$

for every $t \in [0,t_0)$. A similar argument proves (8.7) when $t \in [t_0,T]$.

By Lemma 7.8 and (8.7) we obtain

$$\|\boldsymbol{\mu}^k_{t_1\dots t_m}\|_* \le m(C_* + C) \tag{8.8}$$

for every finite sequence $t_1,\dots,t_m$ with $t_1 < \dots < t_m$.

Step 2. Choice of the subsequence. Let D be a countable dense subset of $[0,T]$ containing 0. By the compactness Theorem 3.11, using (8.8) and a diagonal argument, we can extract a subsequence, still denoted $\boldsymbol{\mu}^k$, such that, for every $s_1,\dots,s_m$ in D with $0 \le s_1 < \dots < s_m \le T$, the sequence $\boldsymbol{\mu}^k_{s_1\dots s_m}$ converges weakly* in $GY(X;\Xi^m)$.

Step 3. Choice of Θ*.* Let $V^k(t) := \mathrm{Var}(\boldsymbol{\mu}^k;0,t)$. Since V^k is nondecreasing, by (8.2) and by Helly's Theorem there exists a subsequence, still denoted V^k, such that, for every $t \in [0,T]$, $V^k(t) \to V(t)$, where V is a nondecreasing function on $[0,T]$ with values in $[0,C]$. Let

$$\Theta := \{0\} \cup \{t \in (0,T] : \lim_{s\to t-} V(s) = V(t)\}\,. \tag{8.9}$$

Step 4. Convergence and left continuity on Θ*.* Let us fix two finite sequences $t_1,\dots,t_m$ and $s_1,\dots,s_m$ in $[0,T]$ such that $0 \le s_1 < t_1 < \dots < s_m < t_m \le T$. We want to estimate the difference $\boldsymbol{\mu}^k_{t_1\dots t_m} - \boldsymbol{\mu}^k_{s_1\dots s_m}$. Let $f \in C^{hom}_L(X\times\Xi^m\times\mathbb{R})$. Then there exists a constant a such that

$$|f(x,\xi_{t_1},\dots,\xi_{t_m},\eta) - f(x,\xi_{s_1},\dots,\xi_{s_m},\eta)| \le a\sum_{i=1}^m |\xi_{t_i} - \xi_{s_i}|\,. \tag{8.10}$$

In order to estimate $|\langle f, \boldsymbol{\mu}^k_{t_1\dots t_m}\rangle - \langle f, \boldsymbol{\mu}^k_{s_1\dots s_m}\rangle|$, it is convenient to use the identities

$$\begin{aligned}
\langle f, \boldsymbol{\mu}^k_{t_1\dots t_m}\rangle &= \langle f(x,\xi_{t_1},\dots,\xi_{t_m},\eta), \boldsymbol{\mu}^k_{s_1t_1\dots s_mt_m}(x,\xi_{s_1},\xi_{t_1},\dots,\xi_{s_m},\xi_{t_m},\eta)\rangle\,,\\
\langle f, \boldsymbol{\mu}^k_{s_1\dots s_m}\rangle &= \langle f(x,\xi_{s_1},\dots,\xi_{s_m},\eta), \boldsymbol{\mu}^k_{s_1t_1\dots s_mt_m}(x,\xi_{s_1},\xi_{t_1},\dots,\xi_{s_m},\xi_{t_m},\eta)\rangle\,,
\end{aligned}$$

which follow from the compatibility condition (7.2). Taking into account (8.10), we then have

$$|\langle f, \boldsymbol{\mu}^k_{t_1\dots t_m}\rangle - \langle f, \boldsymbol{\mu}^k_{s_1\dots s_m}\rangle| \leq$$

$$\leq a\sum_{i=1}^m \langle |\xi_{t_i} - \xi_{s_i}|, \boldsymbol{\mu}^k_{s_1 t_1\dots s_m t_m}(x, \xi_{s_1}, \xi_{t_1}, \dots, \xi_{s_m}, \xi_{t_m}, \eta)\rangle =$$

$$= a\sum_{i=1}^m \langle |\xi_{t_i} - \xi_{s_i}|, \boldsymbol{\mu}^k_{s_i t_i}(x, \xi_{s_i}, \xi_{t_i}, \eta)\rangle\,,$$

which by (8.1) gives

$$|\langle f, \boldsymbol{\mu}^k_{t_1\dots t_m}\rangle - \langle f, \boldsymbol{\mu}^k_{s_1\dots s_m}\rangle| \leq a\sum_{i=1}^m (V^k(t_i) - V^k(s_i))\,. \tag{8.11}$$

A simple modification of the proof shows that (8.11) holds even if $0 = s_1 = t_1 < s_2 \leq t_2 < \dots < s_m \leq t_m \leq T$.

If $t_1, \dots, t_m \in \Theta$ with $0 \leq t_1 < \dots < t_m \leq T$, for every ε we can choose $s_1, \dots, s_m \in D$, with $0 \leq s_1 \leq t_1 < s_2 \leq t_2 < \dots < s_m \leq t_m \leq T$, such that $a\sum_i (V(t_i) - V(s_i)) < \varepsilon$. Using (8.11) we deduce that $|\langle f, \boldsymbol{\mu}^k_{t_1\dots t_m}\rangle - \langle f, \boldsymbol{\mu}^k_{s_1\dots s_m}\rangle| < \varepsilon$ for k large enough. As the sequence $\langle f, \boldsymbol{\mu}^k_{s_1\dots s_m}\rangle$ converges, we have $|\langle f, \boldsymbol{\mu}^k_{s_1\dots s_m}\rangle - \langle f, \boldsymbol{\mu}^{k'}_{s_1\dots s_m}\rangle| < \varepsilon$ for k, k' large enough. It follows that $|\langle f, \boldsymbol{\mu}^k_{t_1\dots t_m}\rangle - \langle f, \boldsymbol{\mu}^{k'}_{t_1\dots t_m}\rangle| < 3\varepsilon$ for k, k' large enough, hence $\langle f, \boldsymbol{\mu}^k_{t_1\dots t_m}\rangle$ is a Cauchy sequence for every $f \in C^{hom}_L(X \times \Xi^m \times \mathbb{R})$. By (8.8) we deduce from Lemma 2.4 that, for every $t_1, \dots, t_m$ in Θ with $0 \leq t_1 < \dots < t_m \leq T$, the sequence $\boldsymbol{\mu}^k_{t_1\dots t_m}$ converges weakly* to some element $\boldsymbol{\mu}_{t_1\dots t_m}$ of $GY(X; \Xi^m)$ satisfying

$$\|\boldsymbol{\mu}_{t_1\dots t_m}\|_* \leq m(C_* + C)\,. \tag{8.12}$$

We observe that, given $t_1, \dots, t_m$ and $s_1 \dots s_m$ in Θ, we can pass to the limit in (8.11) and obtain

$$|\langle f, \boldsymbol{\mu}_{t_1\dots t_m}\rangle - \langle f, \boldsymbol{\mu}_{s_1\dots s_m}\rangle| \leq a\sum_{i=1}^m (V(t_i) - V(s_i)) \tag{8.13}$$

for every f satisfying (8.10) and every pair of finite sequences $t_1, \dots, t_m$ and $s_1, \dots, s_m$ in Θ such that $s_1 \leq t_1 < s_2 \leq t_2 < \dots < s_m \leq t_m$. Using the definition (8.9) of Θ and Lemma 2.4, we deduce from (8.12) and (8.13) that, for every $t_1, \dots, t_m$ in Θ with $t_1 < \dots < t_m$, we have $\boldsymbol{\mu}_{s_1\dots s_m} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_m}$ weakly* in $GY(X; \Xi^m)$, as $s_i \to t_i$, $s_i \in \Theta$, and $s_i \leq t_i$.

Step 5. Extension to $[0, T]$. It remains to show that we can define $\boldsymbol{\mu}_{t_1\dots t_m}$ when some t_i does not belong to Θ, in such a way that the resulting system of generalized Young measures satisfies the compatibility conditions, inequalities (8.4) and (8.5), and the continuity property (7.5). To this purpose, it is enough to observe that, since V has a finite limit from the left at each point, we have

$$\lim_{k,k' \to \infty} \sum_{i=1}^m (V(s^k_i) - V(s^{k'}_i)) = 0$$

for every sequence $(s_1^k,\dots,s_m^k)$ in Θ^m with $s_i^k\to t_i$, $s_i^k\le t_i$. Indeed, if $t_i\notin\Theta$, we have

$$V(s_i^k)\to V^-(t_i):=\lim_{\substack{s\to t_i\\ s<t_i}}V(s)\,. \tag{8.14}$$

For these sequences $(s_1^k,\dots,s_m^k)$ we can deduce from (8.13) that $\langle f,\boldsymbol{\mu}_{s_1^k\dots s_m^k}\rangle$ satisfies a Cauchy condition for every f satisfying (8.10). By (8.12) we deduce from Lemma 2.4 the existence of the weak*-limit of $\boldsymbol{\mu}_{s_1\dots s_m}$ as $s_i\to t_i$, $s_i\in\Theta$, and $s_i\le t_i$. We take such a weak* limit as the definition of $\boldsymbol{\mu}_{t_1\dots t_m}$. Clearly $\boldsymbol{\mu}_{t_1\dots t_m}$ satisfies (8.12) and, by construction, from (8.13), we deduce that for every f satisfying (8.10) and every pair of finite sequences $t_1,\dots,t_m$ and $s_1,\dots,s_m$ in $[0,T]$, with $0\le s_1\le t_1<s_2\le t_2<\dots<s_m\le t_m\le T$, there holds

$$|\langle f,\boldsymbol{\mu}_{t_1\dots t_m}\rangle-\langle f,\boldsymbol{\mu}_{s_1\dots s_m}\rangle|\le a\sum_{i=1}^m(V^-(t_i)-V^-(s_i))\,, \tag{8.15}$$

where V^- is the left-continuous representative of V defined by (8.14). The continuity property (7.5) follows easily from (8.15) and from Lemma 2.4.

For every finite sequence $t_1,\dots,t_m$ in Θ with $t_1<\dots<t_m$ we have

$$\sum_{i=1}^m\langle|\xi_i-\xi_{i-1}|,\boldsymbol{\mu}^k_{t_{i-1}t_i}(x,\xi_{i-1},\xi_i,\eta)\rangle\le C\,.$$

Passing to the limit as $k\to\infty$, we obtain

$$\sum_{i=1}^m\langle|\xi_i-\xi_{i-1}|,\boldsymbol{\mu}_{t_{i-1}t_i}(x,\xi_{i-1},\xi_i,\eta)\rangle\le C$$

whenever $t_1,\dots,t_m\in\Theta$. This restriction can be removed by an approximation argument, and this proves (8.4).

The compatibility condition (7.2) for μ^k implies that

$$\langle f,\boldsymbol{\mu}^k_{s_1\dots s_n}\rangle=\langle f\circ\pi^{t_1\dots t_m}_{s_1\dots s_n},\boldsymbol{\mu}^k_{t_1\dots t_m}\rangle$$

for every $f\in C^{hom}(X{\times}\Xi^n{\times}\mathbb{R})$ and every pair of finite sequences $s_1,\dots,s_n$ and $t_1,\dots,t_m$ in $[0,T]$ with $s_1<\dots<s_n$, $t_1<\dots<t_m$, and $\{s_1,\dots,s_n\}\subset\{t_1,\dots,t_m\}$. Passing to the limit as $k\to\infty$, we obtain

$$\langle f,\boldsymbol{\mu}_{s_1\dots s_n}\rangle=\langle f\circ\pi^{t_1\dots t_m}_{s_1\dots s_h},\boldsymbol{\mu}_{t_1\dots t_m}\rangle\,,$$

whenever s_i and t_j belong to Θ. This restriction can be removed by an approximation argument, therefore $\boldsymbol{\mu}\in SGY([0,T],X;\Xi)$. □

Remark 8.11. By taking $\boldsymbol{\mu}^k=\boldsymbol{\mu}$ in Theorem 8.10 we obtain the following result: if $\boldsymbol{\mu}\in SGY([0,T],X;\Xi)$ and $\mathrm{Var}(\boldsymbol{\mu};0,T)<+\infty$, then there exist a left continuous $\boldsymbol{\nu}\in SGY([0,T],X;\Xi)$ and a set $\Theta\subset[0,T]$, containing 0 and with $[0,T]\backslash\Theta$ at most countable, such that $\mathrm{Var}(\boldsymbol{\nu};0,T)\le\mathrm{Var}(\boldsymbol{\mu};0,T)$ and $\boldsymbol{\mu}_{t_1\dots t_m}=\boldsymbol{\nu}_{t_1\dots t_m}$ for every finite sequence $t_1,\dots,t_m$ in Θ with $t_1<\dots<t_m$. A slight modification of the proof shows also that for every finite sequence $t_1,\dots,t_m$ in $[0,T]$ with $t_1<\dots<t_m$ we have $\boldsymbol{\mu}_{s_1\dots s_m}\rightharpoonup\boldsymbol{\nu}_{t_1\dots t_m}$ weakly* in $GY(X;\Xi^m)$ as $s_i\to t_i$ in $[0,T]$, with $s_i<t_i$.

We conclude this section by proving the lower semicontinuity of the h-variation.

Theorem 8.12. *Let $T>0$ and let $\boldsymbol{\mu}^k$ be a sequence in $SGY([0,T],X;\Xi)$. Suppose that there exist a dense set $D\subset[0,T]$, and a left continuous $\boldsymbol{\mu}\in SGY([0,T],X;\Xi)$ such that*

$$\boldsymbol{\mu}^k_{t_1\dots t_m}\rightharpoonup\boldsymbol{\mu}_{t_1\dots t_m}\quad \textit{weakly}^*\textit{ in } GY(X;\Xi^m)$$

for every finite sequence $t_1,\dots,t_m$ in D with $t_1<\dots<t_m$. Then

$$\mathrm{Var}_h(\boldsymbol{\mu};0,T)\le\liminf_{k\to\infty}\mathrm{Var}_h(\boldsymbol{\mu}^k;0,T)$$

for every positively one-homogeneous function $h\colon\Xi\to[0,+\infty)$ satisfying the triangle inequality.

Proof. Let us fix h. For every finite sequence $t_1,\dots,t_m$ in D with $t_1<\dots<t_m$ we have

$$\sum_{i=1}^m\langle h(\xi_i-\xi_{i-1}),\boldsymbol{\mu}^k_{t_{i-1}t_i}(x,\xi_{i-1},\xi_i,\eta)\rangle\le\mathrm{Var}_h(\boldsymbol{\mu}^k;0,T)\,.$$

Since h is continuous (Remark 8.7), passing to the limit as $k\to\infty$ we obtain

$$\sum_{i=1}^m\langle h(\xi_i-\xi_{i-1}),\boldsymbol{\mu}_{t_{i-1}t_i}(x,\xi_{i-1},\xi_i,\eta)\rangle\le\liminf_{k\to\infty}\mathrm{Var}_h(\boldsymbol{\mu}^k;0,T)$$

whenever $t_1,\dots,t_m\in D$. The same inequality can be proved when $t_1,\dots,t_m\in[0,T]$ by an approximation argument, thanks to left continuity. The conclusion is obtained by taking the supremum with respect to $t_1,\dots,t_m$. ☐

9. **Weak* derivatives of systems with bounded variation.** In this section we introduce the notion of weak* derivative of a compatible system of generalized Young measures on the time interval $[0,T]$, with $T>0$, and prove that, if $\mathrm{Var}(\mu;0,T)<+\infty$, then the weak* derivative exists at almost every $t\in[0,T]$.

Definition 9.1. Given $\boldsymbol{\mu}\in SGY([0,T],X;\Xi)$, the *difference quotient* of $\boldsymbol{\mu}$ between times t_1 and t_2, with $0\le t_1<t_2\le T$, is the element of $GY(X;\Xi)$ defined as the image

$$q_{t_1t_2}(\boldsymbol{\mu}_{t_1t_2})$$

of $\boldsymbol{\mu}_{t_1t_2}$ under the map $q_{t_1t_2}\colon X\times\Xi\times\Xi\times\mathbb{R}\to X\times\Xi\times\mathbb{R}$ defined by

$$q_{t_1t_2}(x,\xi_1,\xi_2,\eta)=(x,\tfrac{\xi_2-\xi_1}{t_2-t_1},\eta)\,.$$

Remark 9.2. It follows from Definition 3.18 that the difference quotient is characterized by the equality

$$\langle f(x,\xi,\eta),q_{t_1t_2}(\boldsymbol{\mu}_{t_1t_2})(x,\xi,\eta)\rangle=\langle f(x,\tfrac{\xi_2-\xi_1}{t_2-t_1},\eta),\boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle$$

for every $f\in C^{hom}(X\times\Xi\times\mathbb{R})$.

Remark 9.3. It follows from the definition of barycentre that

$$\mathrm{bar}(q_{t_1t_2}(\boldsymbol{\mu}_{t_1t_2}))=\frac{\mathrm{bar}(\boldsymbol{\mu}_{t_2})-\mathrm{bar}(\boldsymbol{\mu}_{t_1})}{t_2-t_1}\,. \tag{9.1}$$

In particular, if $\boldsymbol{\mu}=\boldsymbol{\delta_p}$ for some function $\boldsymbol{p}\colon[0,T]\to M_b(X;\Xi)$ (see (7.1)), then

$$q_{t_1t_2}(\boldsymbol{\mu}_{t_1t_2})=q_{t_1t_2}(\delta_{(\boldsymbol{p}(t_1),\boldsymbol{p}(t_2))})=\delta_{\frac{\boldsymbol{p}(t_2)-\boldsymbol{p}(t_1)}{t_2-t_1}} \tag{9.2}$$

(see Definition 3.2).

Definition 9.4. We say that $\boldsymbol{\mu} \in SGY([0,T],X;\Xi)$ has a *weak* derivative* $\dot{\boldsymbol{\mu}}_{t_0}$ at time $t_0 \in [0,T]$ if $q_{tt_0}(\boldsymbol{\mu}_{tt_0}) \rightharpoonup \dot{\boldsymbol{\mu}}_{t_0}$ weakly* in $GY(X;\Xi)$ as $t \to t_0^-$ and $q_{t_0t}(\boldsymbol{\mu}_{t_0t}) \rightharpoonup \dot{\boldsymbol{\mu}}_{t_0}$ weakly* in $GY(X;\Xi)$ as $t \to t_0^+$, which is equivalent to

$$\begin{aligned}\langle f(x,\xi_0,\eta), \dot{\boldsymbol{\mu}}_{t_0}(x,\xi_0,\eta)\rangle &= \lim_{t\to t_0^-} \langle f(x, \tfrac{\xi-\xi_0}{t-t_0}, \eta), \boldsymbol{\mu}_{tt_0}(x,\xi,\xi_0,\eta)\rangle = \\ &= \lim_{t\to t_0^+} \langle f(x, \tfrac{\xi-\xi_0}{t-t_0}, \eta), \boldsymbol{\mu}_{t_0t}(x,\xi_0,\xi,\eta)\rangle\end{aligned} \tag{9.3}$$

for every $f \in C^{hom}(X\times\Xi\times\mathbb{R})$.

Remark 9.5. It follows from (9.2) that, if $\boldsymbol{\mu} = \boldsymbol{\delta}_{\boldsymbol{p}}$ for some function $\boldsymbol{p}\colon [0,T] \to M_b(X;\Xi)$ and

$$\frac{\boldsymbol{p}(t)-\boldsymbol{p}(t_0)}{t-t_0} \to \dot{\boldsymbol{p}}(t_0)$$

strongly in $M_b(X;\Xi)$ as $t \to t_0$, then $\boldsymbol{\mu}$ has a weak* derivative at t_0 and

$$\dot{\boldsymbol{\mu}}_{t_0} = \delta_{\dot{\boldsymbol{p}}(t_0)}\,.$$

This is not true if

$$\frac{\boldsymbol{p}(t)-\boldsymbol{p}(t_0)}{t-t_0} \rightharpoonup \dot{\boldsymbol{p}}(t_0) \tag{9.4}$$

only in the weak* topology of $M_b(X;\Xi)$. However, using Remark 9.3, in this case we obtain

$$\mathrm{bar}(\dot{\boldsymbol{\mu}}_{t_0}) = \dot{\boldsymbol{p}}(t_0)\,,$$

if the weak* derivative of $\boldsymbol{\mu}_{t_1\ldots t_m} := \delta_{(\boldsymbol{p}(t_1),\ldots,\boldsymbol{p}(t_m))}$ exists at t_0.

An example where (9.4) holds but $\dot{\boldsymbol{\mu}}_{t_0} \neq \delta_{\dot{\boldsymbol{p}}(t_0)}$, can be constructed in the following way. Let $T=2$, $X=[-1,1]$, $\Xi=\mathbb{R}$, let λ be the Lebesgue measure, let $w\colon \mathbb{R}\to\mathbb{R}$ be the 2-periodic function defined by

$$w(x) := \begin{cases} 1 & \text{if } 2k \le x < 2k+1 \text{ for some } k\in\mathbb{Z}\,,\\ -1 & \text{if } 2k-1 \le x < 2k \text{ for some } k\in\mathbb{Z}\,.\end{cases}$$

For every $t\in[0,2]$ let $\boldsymbol{u}(t)\in L^1(X)$ be the function defined by

$$\boldsymbol{u}(t,x) := \begin{cases} (t-1)w(\frac{x}{t-1}) & \text{if } t\neq 1\,,\\ 0 & \text{if } t=1\,.\end{cases}$$

As $t\to 1$ we have

$$\frac{\boldsymbol{u}(t)-\boldsymbol{u}(1)}{t-1} \rightharpoonup 0 \qquad \text{weakly}^* \text{ in } M_b(X;\Xi)\,,$$

while

$$\delta_{\frac{\boldsymbol{u}(t)-\boldsymbol{u}(1)}{t-1}} \rightharpoonup \tfrac12\delta_1 + \tfrac12\delta_{-1} \qquad \text{weakly}^* \text{ in } GY(X;\Xi)\,,$$

which implies $\dot{\boldsymbol{\mu}}_1 = \frac12\delta_1 + \frac12\delta_{-1}$.

Remark 9.6. If $\boldsymbol{\mu}\in SGY([0,T],X;\Xi)$ has a weak* derivative $\dot{\boldsymbol{\mu}}_{t_0}$ at time $t_0 \in [0,T]$, then

$$\frac{\mathrm{bar}(\boldsymbol{\mu}_t)-\mathrm{bar}(\boldsymbol{\mu}_{t_0})}{t-t_0} \rightharpoonup \mathrm{bar}(\dot{\boldsymbol{\mu}}_{t_0})$$

weakly* in $M_b(X;\Xi)$ as $t\to t_0$. This follows from (9.1) and Remark 6.4.

The following theorem is the main result of this section.

Theorem 9.7. *Let* $T>0$ *and let* $\boldsymbol{\mu}\in SGY([0,T],X;\Xi)$ *with* $\mathrm{Var}(\boldsymbol{\mu};0,T)<+\infty$. *Then the weak** *derivative* $\dot{\boldsymbol{\mu}}_t$ *exists for a.e.* $t\in[0,T]$. *Moreover, for every* $f\in C^{hom}(X{\times}\Xi{\times}\mathbb{R})$ *the function* $t\mapsto\langle f,\dot{\boldsymbol{\mu}}_t\rangle$ *is integrable on* $[0,T]$. *Finally, if* $h\colon\Xi\to[0,+\infty)$ *is a positively one-homogeneous function satisfying the triangle inequality, then*

$$\int_a^b \langle h(\xi),\dot{\boldsymbol{\mu}}_t(x,\xi,\eta)\rangle\,dt\le \mathrm{Var}_h(\boldsymbol{\mu};a,b) \tag{9.5}$$

for every a, $b\in[0,T]$ *with* $a\le b$.

Proof. Some ideas of this proof are borrowed from the proof of [2, Theorem 4.1.1] on the existence of the metric derivative of a Lipschitz curve.

Step 1. Boundedness of the difference quotients. By Remark 2.9 and by (3.1) for every t_1, $t_2\in[0,T]$, with $t_1<t_2$, we have

$$\|q_{t_1t_2}(\boldsymbol{\mu}_{t_1t_2})\|_*\le \tfrac{1}{t_2-t_1}\langle|\xi_2-\xi_1|,\boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle+\langle\eta,\boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle\,.$$

Let $V\colon[0,T]\to[0,+\infty)$ be the nondecreasing function defined by

$$V(t):=\mathrm{Var}(\boldsymbol{\mu};0,t)\,. \tag{9.6}$$

By (3.2) and (8.1) we conclude that

$$\|q_{t_1t_2}(\boldsymbol{\mu}_{t_1t_2})\|_*\le \tfrac{V(t_2)-V(t_1)}{t_2-t_1}+\lambda(X)\,.$$

Let $t_0\in[0,T]$ be a point where the derivative of V exists. By the previous inequality we have that

$$\|q_{tt_0}(\boldsymbol{\mu}_{tt_0})\|_*\qquad\text{and}\qquad\|q_{t_0t}(\boldsymbol{\mu}_{t_0t})\|_*$$

are bounded uniformly with respect to t. By the separability of $C^{hom}(X{\times}\Xi{\times}\mathbb{R})$ there exists a countable dense subset $\mathcal{F}$ of the set $C^{hom}_{\triangle}(X{\times}\Xi{\times}\mathbb{R})$ introduced in Definition 2.5. Therefore, since $\mathcal{F}$ is dense in $C^{hom}(X{\times}\Xi{\times}\mathbb{R})$ (see Lemma 2.7), to prove the existence of the weak* derivative of $\boldsymbol{\mu}$ at t_0 it is enough to show that

$$\lim_{t\to t_0^-}\langle f(x,\tfrac{\xi-\xi_0}{t-t_0},\eta),\boldsymbol{\mu}_{tt_0}(x,\xi,\xi_0,\eta)\rangle=\lim_{t\to t_0^+}\langle f(x,\tfrac{\xi-\xi_0}{t-t_0},\eta),\boldsymbol{\mu}_{t_0t}(x,\xi_0,\xi,\eta)\rangle \tag{9.7}$$

for every $f\in\mathcal{F}$.

Step 2. Some auxiliary functions. In order to prove (9.7), let us fix $f\in\mathcal{F}$ and let τ_i be a countable dense sequence in $[0,T]$. For every i we define

$$\varphi_i^f(t):=\begin{cases}\langle f(x,\xi-\zeta_i,(t-\tau_i)\eta),\boldsymbol{\mu}_{t\tau_i}(x,\xi,\zeta_i,\eta)\rangle & \text{if } t<\tau_i\,,\\ 0 & \text{if } t=\tau_i\,,\\ \langle f(x,\xi-\zeta_i,(t-\tau_i)\eta),\boldsymbol{\mu}_{\tau_it}(x,\zeta_i,\xi,\eta)\rangle & \text{if } t>\tau_i\,.\end{cases} \tag{9.8}$$

Let us prove that φ_i^f has bounded variation. Let us fix t_1, $t_2\in[0,T]$, with $t_1<t_2$. We consider first the case $t_1<\tau_i<t_2$. By the compatibility condition (7.2) we have

$$|\varphi_i^f(t_2)-\varphi_i^f(t_1)|\le$$
$$\le\langle|f(x,\xi_2-\zeta_i,(t_2-\tau_i)\eta)-f(x,\xi_1-\zeta_i,(t_1-\tau_i)\eta)|,\boldsymbol{\mu}_{t_1\tau_it_2}(x,\zeta_i,\xi_1,\xi_2,\eta)\rangle\,.$$

Since, by Remark 2.6,

$$|f(x,\xi_2-\zeta_i,(t_2-\tau_i)\eta)-f(x,\xi_1-\zeta_i,(t_1-\tau_i)\eta)|\le(|\xi_2-\xi_1|+(t_2-t_1)|\eta|)\,\|f\|_{hom}\,,$$

using again (7.2) we obtain

$$|\varphi_i^f(t_2)-\varphi_i^f(t_1)|\le$$
$$\le\big(\langle|\xi_2-\xi_1|,\boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle+(t_2-t_1)\,\langle|\eta|,\boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle\big)\,\|f\|_{hom}\,.$$

The same inequality can be proved when $\tau_i\le t_1$ or $\tau_i\ge t_2$. By (3.1), (3.2), (8.1), and (9.6) we conclude that

$$|\varphi_i^f(t_2)-\varphi_i^f(t_1)|\le(V(t_2)-V(t_1)+(t_2-t_1)\lambda(X))\,\|f\|_{hom}\,. \tag{9.9}$$

We now prove that for every $t_1,\,t_2\in[0,T]$, with $t_1<t_2$, we have

$$\varphi_i^f(t_2)-\varphi_i^f(t_1)\le\langle f(x,\xi_2-\xi_1,(t_2-t_1)\eta),\boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle\,. \tag{9.10}$$

We consider first the case $t_1<\tau_i<t_2$. By (7.2) and (9.8) we have

$$\varphi_i^f(t_2)=\langle f(x,\xi_2-\xi_1+\xi_1-\zeta_i,(t_2-t_1+t_1-\tau_i)\eta),\boldsymbol{\mu}_{t_1\tau_it_2}(x,\xi_1,\zeta_i,\xi_2,\eta)\rangle\,.$$

From the triangle inequality and from (7.2) we get

$$\begin{aligned}\varphi_i^f(t_2)\le\langle f(x,\xi_2-\xi_1,(t_2-t_1)\eta),\boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle+\\+\langle f(x,\xi_1-\zeta_i,(t_1-\tau_i)\eta),\boldsymbol{\mu}_{t_1\tau_i}(x,\xi_1,\zeta_i,\eta)\rangle\,,\end{aligned} \tag{9.11}$$

which gives (9.10) by (9.8). The proof in the cases $\tau_i\le t_1$ and $\tau_i\ge t_2$ is similar.

Let $W\colon[0,T]\to\mathbb{R}$ be the increasing function defined by

$$W(t):=V(t)+t\,\lambda(X) \tag{9.12}$$

and let $\sigma\colon[0,W(T)]\to[0,T]$ be the nondecreasing function defined by

$$\sigma(s):=\inf\{t\in[0,T]:W(t)\ge s\}\,.$$

It is easy to see that

$$\sigma(W(t))=t\qquad\text{for every }t\in[0,T]\,. \tag{9.13}$$

As $W(t_2)-W(t_1)\ge(t_2-t_1)\lambda(X)$ for every $t_1<t_2$, we have

$$0\le\sigma(s_2)-\sigma(s_1)\le(s_2-s_1)/\lambda(X) \tag{9.14}$$

for every $s_1<s_2$, hence σ is Lipschitz continuous.

By (9.9) and (9.13) we have

$$|(\varphi_i^f\circ\sigma)(s_2)-(\varphi_i^f\circ\sigma)(s_1)|\le|s_2-s_1|\,\|f\|_{hom}$$

for every $s_1,s_2\in W([0,T])$. Therefore, there exists a function $\psi_i^f:[0,W(T)]\to\mathbb{R}$ such that $\psi_i^f(s)=(\varphi_i^f\circ\sigma)(s)$ for every $s\in W([0,T])$ and

$$|\psi_i^f(s_2)-\psi_i^f(s_1)|\le|s_2-s_1|\,\|f\|_{hom} \tag{9.15}$$

for every $s_1,s_2\in[0,W(T)]$. For every $s_0\in[0,W(T)]$ let

$$\dot\psi_i^f(s_0)=\limsup_{s\to s_0}\frac{\psi_i^f(s)-\psi_i^f(s_0)}{s-s_0}\,.$$

By (9.15) we have $|\dot\psi_i^f(s_0)|\le\|f\|_{hom}$, and by Lebesgue's Differentiation Theorem the limsup is a limit for a.e. $s_0\in[0,W(T)]$. Finally, let $\omega^f:[0,W(T)]\to\mathbb{R}$ be the function defined by

$$\omega^f(s):=\sup_i\dot\psi_i^f(s)\,. \tag{9.16}$$

By the bound on $\dot\psi_i^f$ we have

$$|\omega^f(s)|\le\|f\|_{hom} \tag{9.17}$$

for every $s \in [0, W(T)]$.

Step 3. The exceptional set. Let $\mathcal{L}^1$ be the Lebesgue measure on $\mathbb{R}$. By (9.14) and (9.15) there exists a measurable set $N \subset [0, W(T)]$, with $\mathcal{L}^1(N) = 0$, such that each point of $[0, W(T)] \setminus N$ is a Lebesgue point of ω^f for every $f \in \mathcal{F}$ and a differentiability point for ψ_i^f for every $f \in \mathcal{F}$ and for every i. Let N_W be the set of points of $[0, T]$ where the derivative $\dot{W}$ of W does not exist. By Lebesgue's Differentiation Theorem we have $\mathcal{L}^1(N_W) = 0$. Since σ is Lipschitz continuous and $W^{-1}(N) = \sigma(N \cap W([0, T]))$ by (9.13), we have that $\mathcal{L}^1(W^{-1}(N)) = 0$, hence

$$\mathcal{L}^1(N_W \cup W^{-1}(N)) = 0\,. \tag{9.18}$$

Step 4. Estimate from below. Let us fix $t_0 \notin N_W \cup W^{-1}(N)$, with $0 < t_0 < T$, and let $s_0 = W(t_0)$. As $\varphi_i^f(t) = \psi_i^f(W(t))$, from (9.10) we obtain

$$\psi_i^f(W(t_2)) - \psi_i^f(W(t_1)) \le \langle f(x, \xi_2 - \xi_1, (t_2 - t_1)\eta), \boldsymbol{\mu}_{t_1 t_2}(x, \xi_1, \xi_2, \eta)\rangle$$

for every $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. This implies

$$\dot{\psi}_i^f(W(t_0))\, \dot{W}(t_0) \le \liminf_{t \to t_0^-} \langle f(x, \tfrac{\xi - \xi_0}{t - t_0}, \eta), \boldsymbol{\mu}_{t t_0}(x, \xi, \xi_0, \eta)\rangle\,,$$

$$\dot{\psi}_i^f(W(t_0))\, \dot{W}(t_0) \le \liminf_{t \to t_0^+} \langle f(x, \tfrac{\xi - \xi_0}{t - t_0}, \eta), \boldsymbol{\mu}_{t_0 t}(x, \xi_0, \xi, \eta)\rangle$$

for every i, which by (9.16) gives

$$\omega^f(W(t_0))\, \dot{W}(t_0) \le \liminf_{t \to t_0^-} \langle f(x, \tfrac{\xi - \xi_0}{t - t_0}, \eta), \boldsymbol{\mu}_{t t_0}(x, \xi, \xi_0, \eta)\rangle\,, \tag{9.19}$$

$$\omega^f(W(t_0))\, \dot{W}(t_0) \le \liminf_{t \to t_0^+} \langle f(x, \tfrac{\xi - \xi_0}{t - t_0}, \eta), \boldsymbol{\mu}_{t_0 t}(x, \xi_0, \xi, \eta)\rangle\,. \tag{9.20}$$

Step 5. Estimate from above. To prove the opposite inequality we show that

$$\langle f(x, \xi_2 - \xi_1, (t_2 - t_1)\eta), \boldsymbol{\mu}_{t_1 t_2}(x, \xi_1, \xi_2, \eta)\rangle \le \int_{W(t_1)}^{W(t_2)} \omega^f(s)\, ds \tag{9.21}$$

for every t_1, t_2, with $0 < t_1 < t_2 < T$, such that W is continuous at t_1 or t_2. We prove (9.21) only when W is continuous at t_1, the other case being analogous. For every $\varepsilon > 0$ there exists i such that $\tau_i < t_1$ and

$$W(t_1) - W(\tau_i) < \varepsilon\,. \tag{9.22}$$

As ψ_i^f is Lipschitz, from the compatibility condition (7.2) we get

$$\begin{aligned} \int_{W(t_1)}^{W(t_2)} \omega^f(s)\, ds &\ge \int_{W(t_1)}^{W(t_2)} \dot{\psi}_i^f(s)\, ds = \psi_i^f(W(t_2)) - \psi_i^f(W(t_1)) = \\ = \varphi_i^f(t_2) - \varphi_i^f(t_1) &= \langle f(x, \xi_2 - \zeta_i, (t_2 - \tau_i)\eta), \boldsymbol{\mu}_{\tau_i t_1 t_2}(x, \zeta_i, \xi_1, \xi_2, \eta)\rangle - \\ &- \langle f(x, \xi_1 - \zeta_i, (t_1 - \tau_i)\eta), \boldsymbol{\mu}_{\tau_i t_1 t_2}(x, \zeta_i, \xi_1, \xi_2, \eta)\rangle\,. \end{aligned} \tag{9.23}$$

Using Remark 2.6 we obtain

$$f(x, \xi_2 - \zeta_i, (t_2 - \tau_i)\eta) \ge f(x, \xi_2 - \xi_1, (t_2 - t_1)\eta) - (|\xi_1 - \zeta_i| + (t_1 - \tau_i)|\eta|)\, \|f\|_{hom},$$

$$-f(x, \xi_1 - \zeta_i, (t_1 - \tau_i)\eta) \ge -(|\xi_1 - \zeta_i| + (t_1 - \tau_i)|\eta|)\, \|f\|_{hom}\,,$$

so that, using again (3.2) and (7.2), inequality (9.23) and the definition of W give

$$\int_{W(t_1)}^{W(t_2)} \omega^f(s)\,ds \geq \langle f(x,\xi_2-\xi_1,(t_2-t_1)\eta), \boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle - \\ - 2(W(t_1)-W(\tau_i))\|f\|_{hom}\,.$$

By (9.22) we conclude that

$$\int_{W(t_1)}^{W(t_2)} \omega^f(s)\,ds \geq \langle f(x,\xi_2-\xi_1,(t_2-t_1)\eta), \boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle - 2\varepsilon\|f\|_{hom}\,.$$

As $\varepsilon > 0$ is arbitrary, this proves (9.21).

Since W is differentiable at t_0 and $W(t_0)$ is a Lebesgue point of ω^f, inequality (9.21) implies

$$\limsup_{t\to t_0^-} \langle f(x,\tfrac{\xi-\xi_0}{t-t_0},\eta), \boldsymbol{\mu}_{tt_0}(x,\xi,\xi_0,\eta)\rangle \leq \omega^f(W(t_0))\,\dot W(t_0)\,,$$
$$\limsup_{t\to t_0^+} \langle f(x,\tfrac{\xi-\xi_0}{t-t_0},\eta), \boldsymbol{\mu}_{t_0t}(x,\xi_0,\xi,\eta)\rangle \leq \omega^f(W(t_0))\,\dot W(t_0)\,,$$

which, together with (9.19) and (9.20), give

$$\lim_{t\to t_0^-} \langle f(x,\tfrac{\xi-\xi_0}{t-t_0},\eta), \boldsymbol{\mu}_{tt_0}(x,\xi,\xi_0,\eta)\rangle = \omega^f(W(t_0))\,\dot W(t_0)\,,$$
$$\lim_{t\to t_0^+} \langle f(x,\tfrac{\xi-\xi_0}{t-t_0},\eta), \boldsymbol{\mu}_{t_0t}(x,\xi_0,\xi,\eta)\rangle = \omega^f(W(t_0))\,\dot W(t_0)\,.$$

By (9.18) this proves (9.7) and concludes the proof of the existence of the weak* derivative $\dot{\boldsymbol{\mu}}_{t_0}$ for a.e. $t_0\in[0,T]$. Moreover it shows that

$$\langle f(x,\xi,\eta), \dot{\boldsymbol{\mu}}_{t_0}(x,\xi,\eta)\rangle = \omega^f(W(t_0))\,\dot W(t_0) \tag{9.24}$$

for every $f\in\mathcal{F}$ and for a.e. $t_0\in[0,T]$.

Step 6. Integrability of $t\mapsto\langle f,\dot{\boldsymbol{\mu}}_t\rangle$. To prove the measurability of this function for every $f\in C^{hom}(X{\times}\Xi{\times}\mathbb{R})$, we fix a sequence ε_k of positive numbers converging to 0 and a function $f\in C^{hom}_L(X{\times}\Xi{\times}\mathbb{R})$. By Lemma 8.9 the function

$$t\mapsto \langle f(x,\tfrac{\xi'-\xi}{\varepsilon_k},\eta), \boldsymbol{\mu}_{t,t+\varepsilon_k}(x,\xi,\xi',\eta)\rangle$$

is measurable on $[0,T-\varepsilon_k]$. Since it converges to $t\mapsto\langle f,\dot{\boldsymbol{\mu}}_t\rangle$ for a.e. $t\in[0,T]$, we conclude that this function is measurable on $[0,T]$. The same property can be proved for an arbitrary $f\in C^{hom}(X{\times}\Xi{\times}\mathbb{R})$ by approximation, thanks to Lemma 2.4.

By (9.17) and (9.24) we have

$$|\langle f,\dot{\boldsymbol{\mu}}_t\rangle| \leq \dot W(t)\,\|f\|_{hom}$$

for every $f\in\mathcal{F}$. The same inequality holds for any $f\in C^{hom}(X{\times}\Xi{\times}\mathbb{R})$ by the density of $\mathcal{F}$ (see Lemma 2.7). Since $\dot W$ is integrable, this concludes the proof of the integrability of $t\mapsto\langle f,\dot{\boldsymbol{\mu}}_t\rangle$ on $[0,T]$.

Step 7. Estimate for $\mathrm{Var}_h(\boldsymbol{\mu};a,b)$. Let $h\colon\Xi\to[0,+\infty)$ be a positively one-homogeneous function satisfying the triangle inequality. Since the function $t\mapsto\mathrm{Var}_h(\boldsymbol{\mu};a,t)$ is nondecreasing on $[a,b]$, by Lebesgue Differentiation Theorem it is differentiable for a.e. $t\in[a,b]$ and

$$\int_a^b \frac{d}{dt}\mathrm{Var}_h(\boldsymbol{\mu};a,t)\,dt \leq \mathrm{Var}_h(\boldsymbol{\mu};a,b)\,. \tag{9.25}$$

Let $t_0 \in (a,b)$ be a point where $t \mapsto \mathrm{Var}_h(\boldsymbol{\mu}; a, t)$ is differentiable and the weak* derivative $\dot{\boldsymbol{\mu}}_{t_0}$ exists. By the definition of Var_h for every $t \in (t_0, b)$ we have

$$\mathrm{Var}_h(\boldsymbol{\mu}; a, t_0) + \langle h(\xi - \xi_0), \boldsymbol{\mu}_{t_0 t}(x, \xi_0, \xi, \eta)\rangle \leq \mathrm{Var}_h(\boldsymbol{\mu}; a, t)\,.$$

Since h is positively homogeneous of degree one, we obtain

$$\langle h(\tfrac{\xi - \xi_0}{t - t_0}), \boldsymbol{\mu}_{t_0 t}(x, \xi_0, \xi, \eta)\rangle \leq \frac{\mathrm{Var}_h(\boldsymbol{\mu}; a, t) - \mathrm{Var}_h(\boldsymbol{\mu}; a, t_0)}{t - t_0}\,.$$

From (9.3) we deduce that

$$\langle h(\xi), \dot{\boldsymbol{\mu}}_{t_0}(x, \xi, \eta)\rangle \leq \frac{d}{dt}\mathrm{Var}_h(\boldsymbol{\mu}; a, t)\Big|_{t=t_0}\,.$$

Since this inequality holds for a.e. $t_0 \in [a,b]$, from (9.25) we obtain

$$\int_a^b \langle h(\xi), \dot{\boldsymbol{\mu}}_t(x, \xi, \eta)\rangle\, dt \leq \mathrm{Var}_h(\boldsymbol{\mu}; a, b)\,,$$

which concludes the proof of (9.5). □

10. **Absolute continuity.** In this section we introduce the notion of absolutely continuous system of generalized Young measures on the time interval $[0,T]$, with $T > 0$, and prove that for these systems the h-variation can be computed using the weak* derivative by the formula

$$\mathrm{Var}_h(\boldsymbol{\mu}; a, b) = \int_a^b \langle h(\xi), \dot{\boldsymbol{\mu}}_t(x, \xi, \eta)\rangle\, dt \tag{10.1}$$

for every $a,\, b \in [0,T]$, with $a < b$.

Definition 10.1. We say that a compatible system of generalized Young measures $\boldsymbol{\mu} \in SGY([0,T], X; \Xi)$ is *absolutely continuous on* $[0,T]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^k \langle |\xi_2 - \xi_1|, \boldsymbol{\mu}_{a_i b_i}(x, \xi_1, \xi_2, \eta)\rangle \leq \varepsilon \tag{10.2}$$

for every finite family $(a_1, b_1), \ldots, (a_k, b_k)$ of nonoverlapping open intervals in $[0,T]$ with

$$\sum_{i=1}^k (b_i - a_i) \leq \delta\,.$$

Remark 10.2. It follows from Definition 3.2 that, if $\boldsymbol{\mu} = \boldsymbol{\delta_p}$ for some function $\boldsymbol{p}\colon [0,T] \to M_b(X;\Xi)$, then $\boldsymbol{\mu}$ is absolutely continuous on $[0,T]$ if and only if $\boldsymbol{p}$ is absolutely continuous on $[0,T]$ in the usual sense of functions with values in a Banach space.

If $\boldsymbol{u}$ is an absolutely continuous function from $[0,T]$ into $L^r(X;\Xi)$ for some $r > 1$, then the derivative $\dot{\boldsymbol{u}}(t)$, defined as the strong L^r limit of the difference quotients, exists at a.e. $t \in [0,T]$ (see, e.g., [5, Appendix]). By Remark 9.5 it follows that, if $\boldsymbol{\mu} = \boldsymbol{\delta_u}$, then $\dot{\boldsymbol{\mu}}_t = \delta_{\dot{\boldsymbol{u}}(t)}$ for a.e. $t \in [0,T]$, and (10.1) follows from the classical theory (see, e.g., [5, Appendix]).

If $\boldsymbol{p}$ is an absolutely continuous function with values in $M_b(X;\Xi)$, then the derivative $\dot{\boldsymbol{p}}(t)$, defined as the weak* limit of the difference quotients, exists at a.e. $t \in [0,T]$ (see [9, Appendix]). This is not enough to guarantee that $\dot{\boldsymbol{\mu}}_t = \delta_{\dot{\boldsymbol{p}}(t)}$ for a.e. $t \in [0,T]$ when $\boldsymbol{\mu} = \boldsymbol{\delta_p}$ (see Remark 9.5). Therefore, in this case (10.1) cannot be obtained directly from known results.

Remark 10.3. As in the classical case, one can see that, if $\boldsymbol{\mu} \in SGY([0,T],X;\Xi)$ is absolutely continuous on $[0,T]$, then $\mathrm{Var}(\boldsymbol{\mu};0,T) < +\infty$. In this case, if $V\colon [0,T] \to [0,+\infty)$ is the nondecreasing function defined by

$$V(t) := \mathrm{Var}(\boldsymbol{\mu};0,t)\,,$$

then for every $\varepsilon > 0$

$$\sum_{i=1}^{k} (V(b_i) - V(a_i)) \leq \varepsilon$$

for every finite family $(a_1,b_1),\dots,(a_k,b_k)$ of nonoverlapping open intervals in $[0,T]$ with

$$\sum_{i=1}^{k} (b_i - a_i) \leq \delta\,,$$

where δ is the constant in the definition of the absolute continuity of $\boldsymbol{\mu}$. In particular, V is absolutely continuous on $[0,T]$.

Theorem 10.4. *Suppose that* $\boldsymbol{\mu} \in SGY([0,T],X;\Xi)$ *is absolutely continuous on* $[0,T]$ *and that* $h\colon \Xi \to [0,+\infty)$ *is positively one-homogeneous and satisfies the triangle inequality. Then*

$$\mathrm{Var}_h(\boldsymbol{\mu};a,b) = \int_a^b \langle h(\xi), \dot{\boldsymbol{\mu}}_t(x,\xi,\eta)\rangle\, dt$$

for every a, $b \in [0,T]$ *with* $a \leq b$.

Proof. Let W be defined by (9.12). By Remark 10.3 W is absolutely continuous on $[0,T]$. By Remark 8.7 the function $f(x,\xi,\eta) := h(\xi)$ belongs to $C^{hom}_{\triangle}(X{\times}\Xi{\times}\mathbb{R})$. Therefore, we can add this function to the set $\mathcal{F}$ introduced in Step 1 of the proof of Theorem 9.7 and we can consider the corresponding function $\omega^h\colon [0,W(T)] \to \mathbb{R}$ defined by (9.16). By (9.21) and (9.24) we have

$$\langle h(\xi_2-\xi_1), \boldsymbol{\mu}_{t_1t_2}(x,\xi_1,\xi_2,\eta)\rangle \leq \int_{W(t_1)}^{W(t_2)} \omega^h(s)\, ds\,, \tag{10.3}$$

$$\langle h(\xi), \dot{\boldsymbol{\mu}}_t(x,\xi,\eta)\rangle = \omega^h(W(t))\, \dot{W}(t) \tag{10.4}$$

for every t_1, $t_2 \in [0,T]$, with $t_1 < t_2$, and for a.e. $t \in [0,T]$.

By the definition of $\mathrm{Var}_h(\boldsymbol{\mu};a,b)$, inequality (10.3) implies that

$$\mathrm{Var}_h(\boldsymbol{\mu};a,b) \leq \int_{W(a)}^{W(b)} \omega^h(s)\, ds \tag{10.5}$$

for every a, $b \in [0,T]$, with $a \leq b$. On the other hand, since W is absolutely continuous on $[0,T]$, we have

$$\int_{W(a)}^{W(b)} \omega^h(s)\, ds = \int_a^b \omega^h(W(t))\, W'(t)\, dt = \int_a^b \langle h(\xi), \dot{\boldsymbol{\mu}}_t(x,\xi,\eta)\rangle\, dt\,, \tag{10.6}$$

where the last equality follows from (10.4). The conclusion follows now from (9.5), (10.5), and (10.6). □

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