A MODEL FOR CRACK PROPAGATION BASED ON VISCOUS APPROXIMATION

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ABSTRACT. In the setting of antiplane linearized elasticity, we show the existence of quasistatic evolutions of cracks in brittle materials by using a vanishing viscosity approach, thus taking into account local minimization. The main feature of our model is that the path followed by the crack needs not be prescribed a priori: indeed, it is found as the limit (in the sense of Hausdorff convergence) of curves obtained by an incremental procedure. The result is based on a continuity property for the energy release rate in a suitable class of admissible cracks.

CONTENTS

INTRODUCTION

In this paper we present a vanishing viscosity approach to quasistatic evolution in brittle fracture. We adopt the setting of antiplane elasticity, where the domain is bidimensional (see (1.1)); moreover, we suppose that the crack is composed of a fixed number of noninteracting

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regular curves. The main issue in our work is that we do not assume to know a priori the crack path, which is selected through an energy criterion among curves belonging to a suitable class. The solution is obtained by an approximation method based on time discretization and viscous perturbation: this is possible because of the regularity requirements on the class of admissible curves, which ensure the continuity of the energy derivative with respect to the Hausdorff convergence of the approximate sets, as shown in [20]. Therefore, we are able to establish Griffith's principle, as well as the energy-dissipation balance; we also provide some information on the jumps of the crack length as a function of time, which are regarded as the limit of fast dynamic propagations. These properties were proven with a similar approach in [17, 18] under the hypothesis of a prescribed crack path, which is not needed in our work.

According to Griffith's theory [15], the stability of a crack depends on the competition of two types of energies:

- the bulk term, which is a quadratic function of the deformation gradient, in the context of linearized hyperelasticity;
- the surface term, proportional to the measure of the debonded crack (the proportionality constant being called *toughness*).

For the sake of simplicity, we suppose that no volume forces are imposed and the only time-dependent data are some Dirichlet conditions on the boundary of the body; on the lips of the crack we have a Neumann homogeneous condition, since there is no transmission of force. Hence we define the total energy as the sum of the bulk and of the crack part.

Griffith's criterion requires that

- the fracture process is irreversible, i.e., the crack is nondecreasing as a function of time;
- at every instant the configuration is a "stationary point" of the total energy among the cracks bigger than the current crack;
- the time variation of the total energy during the evolution balances the power of the external surface forces due to the Dirichlet conditions.

The properties of stationarity and energy balance are typical of a wide class of *rate-independent systems*: the quasistatic evolution of cracks fits in with this notion. We refer to [3, 21] for a deeper discussion on this subject.

Nevertheless, the application of the abstract mathematical theories is not straightforward, since there is no precise definition of critical point for the energy functional, which depends on a set. A possible way to avoid this difficulty is considering just *global minima* instead of all equilibria (see [14, 9, 5, 13, 7, 8]); under this restriction, the crack path can be found via the energy criterion, taking into account irregular sets (disconnected, branched, etc.). Unfortunately, the drawback of the global minimization is the phenomenon of jump in time between the energy wells: being the energy nonconvex (due to the dependence on the crack set), the crack passes through an energy barrier during the jump.

Therefore, it would be preferable to employ a procedure based on *local minimization*, since in this case the jumping in time happens later, without overtaking energy barriers (for a comparison between local and global solutions, we refer to [24] and to [17, Example 6.3]). To this aim, we follow the method of *vanishing viscosity*, which is related to the theory of *minimizing movements* (see [1, 2, 6, 12, 27, 22, 23, 28]); however, the general theorems do not apply directly to fracture mechanics. In the context of crack growth, some approaches based on local minimization were proposed in [29, 26, 17, 18] under the hypothesis of prescribed crack path. Instead, in [10] no prescriptions on the crack path are required, but there are some restrictions on the parameters that pass to the limit; moreover, in the setting of [10] it

is not possible to prove an energy equality. Finally, a different evolution of local minimizers without a priori assumptions on the crack path was introduced in [19], allowing the solution to overtake (only) small energy barriers.

The vanishing viscosity approach to fracture mechanics relies on the notion of *energy release rate*, a concept introduced in the earlier Griffith's works [15]. For simplicity, in the rest of the introduction we refer to the case where the crack is a single curve. The energy release rate is defined, for a prescribed curve, as the opposite of the derivative of the bulk energy when the crack length varies: hence, it estimates the gain in elastic energy got as the crack becomes longer. The criterion for the evolution is based on the interplay between the elastic energy released by the crack's increase and the energy dissipated in the process of crack's formation. In fact, Griffith's principle is equivalent to requiring that:

- the crack's growth is irreversible;
- the energy release rate never exceeds the fracture toughness;
- the crack can grow only when the energy release rate equals the toughness.

We refer to Section 2 for the details and to [3, 11, 16, 20, 24] for a deeper discussion.

In particular, it is possible to prove that the energy release rate at the crack tip is actually independent of the choice of the curve that extends the crack (among the possible continuations of class $C^{1,1}$): this allows us to study the evolution without knowing a priori the crack path. Indeed, we determine the crack through approximation by means of curves in a suitable class, satisfying some regularity conditions that ensure the Hausdorff compactness of the set of admissible curves and the convergence of the energy release rate.

More precisely, we consider an elastic body and suppose that its bidimensional section Ω is a bounded Lipschitz connected open set and presents already a cut Γ_0, a $C^{1,1}$ curve with one endpoint on the boundary of Ω. The admissible cracks depend on a parameter $\eta > 0$ sufficiently small, which will be fixed throughout the whole paper: they are the $C^{1,1}$ curves Γ containing Γ_0 such that, for every point in $\Gamma \backslash \Gamma_0$, the two open disks C_1, C_2 of radius η tangent to Γ at that point intersect neither the boundary $\partial \Omega$, nor the crack Γ itself (see Definition 1.1). This class of curves, denoted by $\mathcal{R}_\eta$, was introduced in [20], where we proved that it is closed under Hausdorff convergence, using the uniform bound on the curvatures (see Theorem 1.3 below).

We define a *quasistatic evolution* following an approximation scheme common in the variational approach to fracture mechanics and in the study of many other rate-independent processes [3, 21]. We fix a subdivision of the time interval and consider some incremental minimum problems in $\mathcal{R}_\eta$ at the discrete instants, adding a viscous term driven by a small parameter $\varepsilon > 0$; first we pass to the limit as the time step tends to zero, then we let the viscous parameter ε vanish (see Definitions 3.1 and 3.2). Thanks to the compactness of $\mathcal{R}_\eta$, we find an admissible evolution which satisfies Griffith's criterion (see Theorem 3.3 and Remark 3.4): this property is obtained in the limit because of the continuity of the energy release rate under the convergence of curves, proven in [20] (see Theorem 2.2 below). The proof is based on the method proposed in [17] in the one-dimensional case of a prescribed crack; for a comparison between the two approaches, see Section 3.

Hence by the hypothesis on $\mathcal{R}_\eta$, self-intersections are avoided during the evolution and the crack tip keeps at a distance $D > 0$ from $\partial \Omega$ (where D depends only on η and on the Lipschitz constant of Ω). Of course, these constraints are in general artificial: if at some instant the disks C_1, C_2 of (small) radius η tangent at the crack tip P are such that

$$(C_1 \cup C_2) \cap (\Gamma \cup \partial \Omega) = \varnothing \tag{0.1}$$

but

$$\left(\overline{C}_1 \cup \overline{C}_2\right) \cap \left(\Gamma \cup \partial\Omega\right) \setminus \{P\} \neq \varnothing \,, \tag{0.2}$$

then the crack could be forced to stop (depending on the geometry of the body). If (0.2) occurs, this might suggest that the applied boundary deformations are going to break the body in two parts, which is not compatible with our approach. Therefore, the evolution predicted by our model is significant as long as (0.2) does not occur.

Our results can be generalized to treat multiple cracks, under the assumption of noninteraction among them (guaranteed by suitable constraints on the class of admissible curves). In the case of many cracks, following the abstract finite-dimensional construction of [23], it is useful to define a reparametrization of the time interval such that the rescaled solutions do not exhibit jumps. This method allows us to compute the energy balance with an additional term that takes into account the instantaneous dissipation during the jumps. Furthermore, we present a differential characterization of the regime of quasistatic (rate-independent) motion versus the regime of jumping (see Remark 8.6). More precisely, in the reparametrized time scale the jumps correspond to rate-*dependent* motions, driven by an equation of viscous type (Theorem 8.7 and Remark 8.8). From the point of view of dynamics, these transitions model the brutal propagation of fracture.

The structure of the article is the following. In Section 1 we introduce the geometry of the body, the admissible curves (in the case of a single crack), the associated displacements, and the total energy. Section 2 contains the precise definition of energy release rate and the statement of the continuity result. In Section 3 we describe the approximation procedure that defines quasistatic evolutions and we state the main theorems for a single crack.

The sequel of the paper is devoted to the proof of the existence and of the properties of quasistatic evolutions: in Section 4 we study the incremental problems, in Section 5 we pass to the time-continuous limit, while in Section 6 we let the viscous parameter tend to zero.

The two final sections contain the case of a fixed number of different noninteracting curves, where a suitable dissipation distance must be chosen.

The main results of this paper are stated in Sections 3 and 8.

Notation. Throughout the paper, the symbol $\cdot$ denotes the scalar product in $\mathbb{R}^n$, $|\cdot|_2$ the corresponding Euclidean norm defined by $|v|_2 := \left(\sum_{h=1}^n v_h^2\right)^{\frac{1}{2}}$, and $\mathrm{dist}\,(\cdot,\cdot)$ the induced distance. We will consider also the norm $|\cdot|_1$ defined by $|v|_1 := \sum_{h=1}^n |v_h|$.

The symbol $\mathcal{L}^n$, with $n = 1, 2$, stands for the Lebesgue measure in $\mathbb{R}^n$, while the symbol $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

Fixed an open bounded subset $\Omega \subset \mathbb{R}^2$, the symbols $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle_2$ stand respectively for the norm and the duality in $L^2(\Omega)$ or $L^2(\Omega;\mathbb{R}^2)$ (it will be clear from the context which of the cases we refer to). The norm in $L^2([0,T])$ is denoted by $\|\cdot\|_{2,[0,T]}$.

Given a function $u \in H^1(\Omega\setminus\Gamma)$, where Γ is a closed subset of Ω, we will regard its gradient ∇u as an element of $L^2(\Omega;\mathbb{R}^2)$, by extending it to 0 on Γ (of course, this extension is not the distributional gradient of any extension of u).

Given a function $f \in BV([0,T])$, its time derivative $\mathrm{D}f$ is decomposed as

$$\mathrm{D}f = \widetilde{\mathrm{D}}f + \mathrm{D}^j f = \dot{f}\,\mathrm{d}\mathcal{L}^1 + \mathrm{D}^c f + \mathrm{D}^j f \,,$$

where $\dot{f}$ is the density of the absolutely continuous part of $\mathrm{D}f$ with respect to the Lebesgue measure and $\mathrm{D}^j f$ is concentrated in the jump set $J(f)$, which is at most countable.

1. ADMISSIBLE CRACKS AND DISPLACEMENTS

We consider a brittle body whose section is represented by a bounded connected open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary. The displacements are assumed to be real functions u of two variables, in the setting of *antiplane elasticity*: the cylinder $\Omega \times \mathbb{R}$ is subject to deformations of the type

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + u(x_1, x_2)). \tag{1.1}$$

For the sake of simplicity, we discuss first the case of a crack composed of a single connected curve; the general case will be analysed in Sections 7 and 8.

The set Ω contains an *initial crack*, represented by a closed nondegenerate arc of curve Γ_0 of class $C^{1,1}$. We suppose that Γ_0 has no self-intersections and is contained in Ω except for the initial point, which belongs to $\partial\Omega$. In order to employ the Poincaré inequality, we assume that $\Omega \backslash \Gamma_0$ is the union of two Lipschitz domains. We set $l_0 := \mathcal{H}^1(\Gamma_0) > 0$.

We are interested in studying the evolution of cracks which extend Γ_0 starting from its end point: more precisely, the cracks will be curves $\Gamma \supset \Gamma_0$ such that $\Gamma \backslash \Gamma_0 \subset\subset \Omega$. Throughout the paper, the length of these curves will be always measured starting from the initial point of Γ_0 lying on $\partial\Omega$.

We consider the class $\mathcal{R}_\eta$ of *admissible cracks* introduced in [20]: for the reader's convenience, we recall here its definition and its main properties.

Definition 1.1. Given $\eta > 0$, $\mathcal{R}_\eta$ denotes the set of closed arcs of curve Γ of class $C^{1,1}$, such that

  (a) $\Gamma \supset \Gamma_0$ and $\Gamma \backslash \Gamma_0 \subset\subset \Omega$;
  (b) for every point $x \in \Gamma \backslash \Gamma_0$ there exist two open balls $C_1, C_2 \subset \Omega$ of radius η, such that $(C_1 \cup C_2) \cap (\Gamma \cup \partial\Omega) = \emptyset$ and $\overline{C}_1 \cap \overline{C}_2 = \{x\}$.

Since Γ_0 is of class $C^{1,1}$ we can fix $\eta > 0$ so small that the curvature of Γ_0 is controlled from above by $\frac{1}{\eta}$ at a.e. point and the class $\mathcal{R}_\eta$ is not empty.

Notice that every curve $\Gamma \in \mathcal{R}_\eta$ has no self-intersections and its curvature is a.e. controlled from above. Under these assumptions, it is possible to see that $l_0 \leq \mathcal{H}^1(\Gamma) \leq L$ and $\mathrm{dist}\,(\Gamma \backslash \Gamma_0, \partial\Omega) \geq D$ for every $\Gamma \in \mathcal{R}_\eta$, where $L, D > 0$ depend only on η, Ω, and Γ_0.

Properties (a) and (b) guarantee the stability of $\mathcal{R}_\eta$ under Hausdorff convergence. To be more precise, we state the definition of convergence in the Hausdorff metric.

Definition 1.2. Given two compact subsets $\Gamma, \Gamma' \subset \overline{\Omega}$, their *Hausdorff distance* is given by

$$d_H(\Gamma'; \Gamma) := \max \left\{ \sup_{x \in \Gamma'} \mathrm{dist}\,(x, \Gamma), \sup_{x \in \Gamma} \mathrm{dist}\,(x, \Gamma') \right\},$$

with the conventions $d_H(x; \emptyset) = \mathrm{diam}\,(\Omega)$ and $\sup \emptyset = 0$. A sequence Γ_n of compact subsets of $\overline{\Omega}$ *converges to* Γ *in the Hausdorff metric* if $d_H(\Gamma_n; \Gamma) \to 0$.

Under this notion of convergence, the class of admissible cracks $\mathcal{R}_\eta$ is sequentially compact, as proven in [20, Proposition 2.9 and Remark 2.10].

Theorem 1.3. *Every sequence $\Gamma_n \in \mathcal{R}_\eta$ admits a limit $\Gamma_\infty \in \mathcal{R}_\eta$ in the Hausdorff metric (up to a subsequence). Moreover, $\mathcal{H}^1(\Gamma_n) \to \mathcal{H}^1(\Gamma)$ (along the subsequence).*

We suppose that the body has a perfectly elastic behaviour outside the cracked region and also that no force is transmitted across the cracks. Therefore, given $\Gamma \in \mathcal{R}_\eta$ the *displacement* $u \colon \Omega \backslash \Gamma \to \mathbb{R}$ is the unique minimum point of the elastic energy $\frac{1}{2} \|\nabla u\|_2^2$, under some imposed boundary conditions, which depend on time. For the sake of simplicity,

we assume that no external load is present; the case with added volume forces can be treated with minor modifications.

Let us fix $T > 0$ and $\psi \in AC([0,T]; H^1(\Omega \backslash \Gamma_0))$. Given $t \in [0,T]$ and $\Gamma \in \mathcal{R}_\eta$, $u(t; \Gamma)$ stands for the solution to the minimum problem

$$\min \left\{ \tfrac{1}{2} \|\nabla u\|_2^2 : u \in H^1(\Omega \backslash \Gamma), \ u = \psi(t) \text{ on } \partial \Omega \right\}. \tag{1.2}$$

The minimum *elastic energy* associated to the crack Γ and to the boundary displacement $\psi(t)$ is denoted by

$$\mathcal{E}(t; \Gamma) := \tfrac{1}{2} \|\nabla u(t; \Gamma)\|_2^2. \tag{1.3}$$

According to Griffith's theory [15], the energy spent to produce a crack is proportional to its length; we assume that the proportionality constant is 1. Hence, we define the *total energy* associated to a crack Γ at time t

$$\mathcal{F}(t; \Gamma) := \mathcal{E}(t; \Gamma) + \mathcal{H}^1(\Gamma). \tag{1.4}$$

Remark 1.4. Let $t_n \to t \in [0,T]$ and let $\Gamma_n \in \mathcal{R}_\eta$ be a sequence converging to Γ in the Hausdorff metric. Then by [9, Theorem 5.1] (see also [4, 20]), $\nabla u(t_n, \Gamma_n)$ converges to $\nabla u(t, \Gamma)$ strongly in $L^2(\Omega; \mathbb{R}^2)$, so $\mathcal{E}(t_n, \Gamma_n) \to \mathcal{E}(t, \Gamma)$.

2. Energy release rate

We now present the notion of *energy release rate*, which is the opposite of the derivative of the elastic energy as the crack grows. Given $t \in [0,T]$ and a curve $\Gamma \in \mathcal{R}_\eta$, we consider a curve Γ' of class $C^{1,1}$ (not necessarily in $\mathcal{R}_\eta$), such that $\Gamma' \supsetneq \Gamma$, and we denote by $\gamma \colon [0, l_{\Gamma'}] \to \overline{\Omega}$ its arc-length parametrization (chosen so that $\gamma(0) \in \partial \Omega$). Then the function

$$l \mapsto \mathcal{E}(t; \gamma([0,l]))$$

is derivable at $l = \mathcal{H}^1(\Gamma)$ and the derivative is independent of the choice of Γ'. Hence we define the *energy release rate* of Γ at time t as

$$G(t; \Gamma) := -\partial_l \mathcal{E}(t; \gamma([0,l]))\big|_{l = \mathcal{H}^1(\Gamma)}. \tag{2.1}$$

We refer to [20, Theorem 2.1] for the proof of the derivability and for the computation of the energy release rate in the case of a curve of class $C^{1,1}$. It turns out that $G(t; \Gamma)$ is nonnegative and proportional to the square of the *stress intensity factor*, which is a constant associated to the singularity of the displacement around the crack tip.

Remark 2.1. In [20, Remark 2.5] the energy release rate is characterized as a volume integral depending on the deformation gradient. Let V be a vector field of class $C^{0,1}$ with compact support in Ω. Assume that on Γ we have $V(\gamma(s)) = \zeta(\gamma(s))\dot\gamma(s)$, where $\gamma \colon [0, l_\Gamma] \to \overline{\Omega}$ is the arc-length parametrization of Γ (with $\gamma(0) \in \partial\Omega$) and ζ is a cut-off function, equal to one in a neighbourhood of $\gamma(l_\Gamma)$. Then

$$G(t; \Gamma) = \int_{\Omega \backslash \Gamma} \left[\frac{(\mathrm{D}_1 u)^2 - (\mathrm{D}_2 u)^2}{2} (\mathrm{D}_1 V^1 - \mathrm{D}_2 V^2) + \mathrm{D}_1 u \, \mathrm{D}_2 u \, (\mathrm{D}_2 V^1 + \mathrm{D}_1 V^2) \right] \mathrm{d}x,$$

where $u := u(t, \Gamma)$ (the integrand appearing in the last equation corresponds to the Eshelby or Hamilton tensor). In particular, one can see that the energy release rate is uniformly bounded by a constant depending only on $\mathcal{R}_\eta$ and ψ.

Using the previous integral formula, one can prove that the energy release rate is continuous with respect to the Hausdorff convergence in the class $\mathcal{R}_\eta$ and to time.

Theorem 2.2. *Let* $t_n \to t \in [0, T]$. *Let* $\Gamma_n \in \mathcal{R}_\eta$ *be a sequence converging to* Γ *in the Hausdorff metric. Then* $G(t_n; \Gamma_n) \to G(t; \Gamma)$.

Proof. This fact is a consequence of Theorem 1.3 and Remarks 1.4 and 2.1. We refer to [20, Theorem 2.12] for the proof in the case of a constant boundary datum. The case of time-dependent boundary conditions can be treated with elementary modifications, since $\psi \in AC([0, T]; H^1(\Omega \backslash \Gamma_0))$. $\qquad \square$

An integral representation of the energy release rate in terms of the Eshelby tensor for a crack with one kink was given in [25]. However, in this work we do not deal with this case.

Remark 2.3. The energy release rate is connected to the *unilateral slope* in the class $\mathcal{R}_\eta$ of the bulk energy $\mathcal{E}(t; \Gamma)$ associated to the boundary datum at time $t \in [0, T]$ and to the crack $\Gamma \in \mathcal{R}_\eta$:

$$|\partial_\Gamma \mathcal{E}|(t; \Gamma) := \limsup_{\substack{\Gamma' \to \Gamma \\ \Gamma' \in \mathcal{R}_\eta}} \frac{\left(\mathcal{E}(t; \Gamma) - \mathcal{E}(t; \Gamma')\right)^+}{d(\Gamma'; \Gamma)}, \qquad (2.2)$$

where the symbol $\to$ stands for the Hausdorff convergence, $(\cdot)^+$ denotes the positive part, and d is the dissipation distance defined for $\Gamma, \Gamma' \in \mathcal{R}_\eta$ by

$$d(\Gamma'; \Gamma) := \begin{cases} \mathcal{H}^1(\Gamma' \backslash \Gamma) & \text{if } \Gamma' \supset \Gamma, \\ +\infty & \text{otherwise.} \end{cases} \qquad (2.3)$$

If the set $\{\Gamma' \in \mathcal{R}_\eta \colon \Gamma' \supsetneq \Gamma\}$ is not empty, then it turns out that $G(t; \Gamma) = |\partial_\Gamma \mathcal{E}|(t; \Gamma)$. However, if $\{\Gamma' \in \mathcal{R}_\eta \colon \Gamma' \supsetneq \Gamma\} = \varnothing$, we have $|\partial_\Gamma \mathcal{E}|(t; \Gamma) = 0$, so the equality does not hold in general. Hence, the unilateral slope is only lower semicontinuous with respect to the Hausdorff convergence. For further discussions we refer to [2, 27].

3. Quasistatic evolution

We provide a notion of evolution based on the technique of *vanishing viscosity*. The solution is defined through a process of approximation based on time discretization: first we solve some incremental problems, then we pass to the limit as the time step vanishes. This method is common in the study of problems in fracture mechanics [3] and of many other rate-independent processes [21]. In order to enforce local minimality, the incremental problems are perturbed with a viscous parameter, which tends to zero more slowly than the time step. This procedure was employed in [2, 12, 22, 23] in an abstract setting and in [17] for the evolution of a crack with prescribed path.

For every $n \in \mathbb{N}$ we fix a *time discretization* $\{t_{n,i}\}_{0 \le i \le n}$ such that

$$0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = T \quad \text{and} \quad \lim_{n \to \infty} \max_{1 \le i \le n} (t_{n,i} - t_{n,i-1}) = 0. \qquad (3.1)$$

Fixed $\varepsilon > 0$, we define by induction a sequence of solutions to incremental minimum problems. Let $\Gamma_{\varepsilon,n,0} := \Gamma_0$; for $i \ge 1$ we set $\Gamma_{\varepsilon,n,i}$ to be a solution to

$$\min_{\Gamma \in \mathcal{R}_\eta} \left\{ \mathcal{E}(t_{n,i}; \Gamma) + \mathcal{H}^1(\Gamma \backslash \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2} \frac{d(\Gamma; \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}} \right\}, \qquad (3.2)$$

where $d(\cdot, \cdot)$ is the dissipation distance defined in (2.3). The existence of solutions to (3.2) will be proven in Proposition 4.1. Finally, we consider the piecewise constant interpolation

$$\Gamma_{\varepsilon,n}(0) := \Gamma_0, \quad \Gamma_{\varepsilon,n}(t) := \Gamma_{\varepsilon,n,i} \quad \text{for } t \in (t_{n,i-1}, t_{n,i}]. \qquad (3.3)$$

In (3.2) the problem of minimizing $\mathcal{F}(t; \Gamma) = \mathcal{E}(t; \Gamma) + \mathcal{H}^1(\Gamma)$ is perturbed by adding a term, driven by a small *viscosity parameter* $\varepsilon > 0$. Using this approximation, the evolution

resulting in the limit should follow the "local minimizers" of the energy (see [12, 22, 23, 28, 10, 17, 24] for discussions and applications). The passage to the limit is done in two steps: beforehands, we let $n \to \infty$ and find a *viscous solution*; afterwards, we obtain an *approximable quasistatic evolution* as the viscous parameter ε tends to zero. Notice that in the case of disconnected cracks, one has to choose a suitable extension of d: see Section 7.

Definition 3.1. Fixed $\varepsilon > 0$, a set function $t \mapsto \Gamma_\varepsilon(t) \in \mathcal{R}_\eta$ is a *viscous solution* if there exist a time discretization $\{t_{n,i}\}_{0 \le i \le n}$ satisfying (3.1) and a sequence of set functions $t \mapsto \Gamma_{\varepsilon,n}(t)$ such that $\Gamma_{\varepsilon,n}(0) = \Gamma_0$, $\Gamma_{\varepsilon,n}(t)$ is constant on every interval $(t_{n,i-1}, t_{n,i}]$, $\Gamma_{\varepsilon,n}(t_{n,i}) = \Gamma_{\varepsilon,n,i}$ solves (3.2) for $i \ge 1$, and $\Gamma_{\varepsilon,n}(t)$ converges to $\Gamma_\varepsilon(t)$ in the Hausdorff metric for every $t \in [0, T]$.

For the existence of viscous solutions, see Proposition 5.1.

Definition 3.2. An *approximable quasistatic evolution* is a set function $t \mapsto \Gamma(t) \in \mathcal{R}_\eta$ such that there are a sequence $\varepsilon_k \to 0$ and a sequence of viscous solutions $\Gamma_{\varepsilon_k}(t)$ with $\Gamma_{\varepsilon_k}(t) \to \Gamma(t)$ in the Hausdorff metric.

Given an approximable quasistatic evolution $t \mapsto \Gamma(t)$, we set

$$l(t) := \mathcal{H}^1(\Gamma(t)) \quad \text{and} \quad G(t) := G(t; \Gamma(t)). \tag{3.4}$$

Since $t \mapsto l(t)$ may show jumps, we set also

$$G^{(s)}(t) := G(t; \Gamma^{(s)}), \tag{3.5}$$

where $\Gamma^{(s)} \in \mathcal{R}_\eta$ is the curve of length s contained in $\Gamma(T)$.

We state here the existence and the main properties of approximable quasistatic evolutions, which will be proven in Section 6. The corresponding theorems for the case of many curves, which require more technicalities, will be presented in Sections 7 and 8.

Theorem 3.3. *There exists an approximable quasistatic evolution in the sense of Definition 3.2. Moreover, every approximable quasistatic evolution $t \mapsto \Gamma(t)$ satisfies the following properties*

(a) *$t \mapsto \Gamma(t)$ is nondecreasing;*
(b) *$G(t) \le 1$ for every $t \notin J(l)$;*
(c) *if $G(\bar{t}) < 1$ for some $\bar{t} \notin J(l)$, then l is locally constant around $\bar{t}$;*
(d) *for every $t \in [0, T]$ the following energy balance law holds:*

$$\mathcal{E}(t; \Gamma(t)) + l(t) - \mathcal{E}(0; \Gamma_0) - l_0 = \int_0^t \left\langle \nabla u(\tau; \Gamma(\tau)), \nabla \dot{\psi}(\tau) \right\rangle_2 \mathrm{d}\tau$$

$$- \int_{l(0)}^{l(0^+)} [G^{(s)}(0) - 1] \,\mathrm{d}s - \sum_{\tau \in J(l) \cap (0,t)} \int_{l(\tau^-)}^{l(\tau^+)} [G^{(s)}(\tau) - 1] \,\mathrm{d}s - \int_{l(t^-)}^{l(t)} [G^{(s)}(t) - 1] \,\mathrm{d}s \,.$$

The last theorem shows that approximable quasistatic evolutions fit in with the notion of BV solution studied in [22, 23]. The process satisfies the two fundamental properties of the variational approach to *rate-independent processes* [21]:

- *local stability*, in fact (b) and (c) ensure in a weak form that the configuration of the system is a critical point for the total energy;
- *energy-dissipation balance*, given by (d).

An alternative formula for the energy balance will be given in Section 8 in the case of many curves, using a rescaling of the time interval such that the reparametrized solutions do not exhibit jumps (see Theorem 8.4 and Remark 8.5).

Remark 3.4 (Griffith's criterion). Properties (a–c) guarantee that the crack growth follows Griffith's criterion: indeed, (c) implies the usual activation condition

$$[G(\cdot) - 1]\,\widetilde{\mathrm{D}}l = 0\,.$$

As for the jump points, in Remark 6.1 we will see the following property, opposite to (b): if $\bar{t} \in J(l)$, then $G^{(s)}(\bar{t}) \geq 1$ for every $s \in [l(\bar{t}^-), l(\bar{t}^+)]$. For the analogous property in the case of many curves, see part (d) of Theorem 8.1.

According to [17, Definition 2.2], a function that satisfies the properties of Remark 3.4 is called *local energetic solution*. In [17] the authors consider the case of a prescribed crack path of class C^2 and obtain a local energetic solution by employing the approximation scheme described above, based on time discretization and vanishing viscosity. Actually, in order to show our existence result, in the present work we modify the proof of [17], taking into account the convergence in the wider class of admissible cracks $\mathcal{R}_\eta$.

The existence of such local energetic solutions was also proven in [26] with a different method, inspired by the original formulation of Griffith's principle. In the context of prescribed crack path, the authors consider an incremental problem, searching at each discrete time the first point where the energy release rate is less than or equal to the toughness. Even if their approach does not employ the viscous approximation, the limit evolution coincides with the one found by [17], at least in the case of increasing loading.

We conclude the discussion with a comparison between our notion of approximable quasi-static evolution and the local energetic solution of [17]. Let $t \mapsto \Gamma(t)$ be an approximable quasistatic evolution in the sense of Definition 3.2; now, consider $\Gamma(T)$ as a prescribed crack path: then, $t \mapsto \Gamma(t)$ is also a solution in the sense of [17]. Viceversa, every local energetic solution in the sense of [17] satisfies the properties of Theorem 3.3.

Nevertheless, even prescribing the path $\Gamma(T)$ as above, the construction of [17, Theorem 5.1] may not give the same evolution $t \mapsto \Gamma(t)$ found in our setting, although we use a similar approximation. Indeed, in (3.2) we compare $\Gamma(t)$ with all possible cracks in the class $\mathcal{R}_\eta$, so the crack path $\Gamma(T)$ may be followed with different speed.

Since by (3.2) the crack is allowed to choose any admissible path in $\mathcal{R}_\eta$, our final evolution is "minimal" with respect to all possible configurations of $\mathcal{R}_\eta$. However, the properties of Theorem 3.3 do not give a full characterization of the solution found in our approximation scheme. The problem to establish a criterion for the minimality with respect to the admissible cracks remains still open.

Further comments on the behaviour during the jumps of the crack length will be given in Remark 8.8 in the case of several cracks.

4. The discrete-time problems

We now discuss the properties of the discrete solutions $\Gamma_{\varepsilon,n,i}$ introduced in the previous section. We recall that, given a time discretization $\{t_{n,i}\}_{0 \leq i \leq n}$ satisfying (3.1), $\Gamma_{\varepsilon,n,i}$ denotes a solution to (3.2), with $\Gamma_{\varepsilon,n,0} = \Gamma_0$. The piecewise constant interpolation is defined by $\Gamma_{\varepsilon,n}(0) = \Gamma_0$ and $\Gamma_{\varepsilon,n}(t) = \Gamma_{\varepsilon,n,i}$ for $t \in (t_{n,i-1}, t_{n,i}]$. We set also $u_{\varepsilon,n,i} := u(t_{n,i}; \Gamma_{\varepsilon,n,i})$.

Proposition 4.1. *There exists a solution to* (3.2).

Proof. We employ the direct method of the Calculus of Variations. Taking $\Gamma = \Gamma_{\varepsilon,n,i-1}$ one sees that the infimum is finite. Let $\Gamma_{\varepsilon,n,i}^k$ be a minimizing sequence for (3.2): by Theorem 1.3, $\Gamma_{\varepsilon,n,i}^k$ converges in the Hausdorff metric to a set $\Gamma_{\varepsilon,n,i} \subset \overline{\Omega} \in \mathcal{R}_\eta$, up to a subsequence (not relabelled). Since the infimum is finite, by definition of d we have $\Gamma_{\varepsilon,n,i}^k \supset \Gamma_{\varepsilon,n,i-1}$ for k

large; hence, $\Gamma_{\varepsilon,n,i-1}$ is contained in the limit, too. Now, let us consider the solutions to (1.2), namely $u^k_{\varepsilon,n,i} := u(t_{n,i}; \Gamma^k_{\varepsilon,n,i})$ and $u_{\varepsilon,n,i} := u(t_{n,i}; \Gamma_{\varepsilon,n,i})$: by Remark 1.4, $\nabla u^k_{\varepsilon,n,i}$ converges to $\nabla u_{\varepsilon,n,i}$ strongly in $L^2(\Omega; \mathbb{R}^2)$. Hence, $\mathcal{E}(t_{n,i}; \Gamma^k_{\varepsilon,n,i}) \to \mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i})$; since the Hausdorff measure is continuous in $\mathcal{R}_\eta$ (Theorem 1.3), $\Gamma_{\varepsilon,n,i}$ is a solution to (3.2). $\qquad\square$

Remark 4.2. Since the expression in (3.2) is finite, by definition of $d(\cdot, \cdot)$ the family $\{\Gamma_{\varepsilon,n,i}\}_{i=0,\dots,n}$ is nondecreasing.

We now provide some a priori bounds on the incremental solutions. Here, $\psi_{n,i} := \psi(t_{n,i})$.

Proposition 4.3. *For every ε, n, i, we have*

$$\mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i}) + \mathcal{H}^1(\Gamma_{\varepsilon,n,i}) + \frac{\varepsilon}{2} \sum_{j=1}^{i} \frac{d(\Gamma_{\varepsilon,n,j}; \Gamma_{\varepsilon,n,j-1})^2}{t_{n,j} - t_{n,j-1}} \tag{4.1}$$

$$\leq \mathcal{E}(0; \Gamma_0) + l_0 + \sum_{j=1}^{i} \langle \nabla u_{\varepsilon,n,j-1}, (\nabla\psi_{n,j} - \nabla\psi_{n,j-1}) \rangle_2 + \frac{1}{2} \sum_{j=1}^{i} \| \nabla\psi_{n,j} - \nabla\psi_{n,j-1} \|_2^2 \,.$$

Moreover, there exists a constant $C > 0$ independent of ε, n, and i, such that

$$\| \nabla u_{\varepsilon,n,i} \|_2 \leq C \tag{4.2}$$

and

$$\frac{\varepsilon}{2} \sum_{j=1}^{i} \frac{d(\Gamma_{\varepsilon,n,j}; \Gamma_{\varepsilon,n,j-1})^2}{t_{n,j} - t_{n,j-1}} \leq C \,. \tag{4.3}$$

Proof. Comparing $u_{\varepsilon,n,i}$ with $\psi_{n,i}$, by (1.2) we get $\mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i}) \leq \frac{1}{2} \| \nabla\psi_{n,i} \|_2^2$ and find (4.2) with $C = \max_{t \in [0,T]} \| \nabla\psi(t) \|_2$.

Comparing $\Gamma_{\varepsilon,n,i}$ and $\Gamma_{\varepsilon,n,i-1}$, by (3.2) we have

$$\mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i}) + \mathcal{H}^1(\Gamma_{\varepsilon,n,i} \backslash \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2} \frac{d(\Gamma_{\varepsilon,n,i}; \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}} \leq \mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i-1}) \,.$$

On the other side, using $u_{\varepsilon,n,i-1} + \psi_{n,i} - \psi_{n,i-1}$ as a competitor in (1.2) we find

$$\mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i-1}) \leq \mathcal{E}(t_{n,i-1}; \Gamma_{\varepsilon,n,i-1}) + \langle \nabla u_{\varepsilon,n,i-1}, \nabla\psi_{n,i} - \nabla\psi_{n,i-1} \rangle_2 + \frac{1}{2} \| \nabla\psi_{n,i} - \nabla\psi_{n,i-1} \|_2^2 \,.$$

Since by monotonicity $\mathcal{H}^1(\Gamma_{\varepsilon,n,i} \backslash \Gamma_{\varepsilon,n,i-1}) = \mathcal{H}^1(\Gamma_{\varepsilon,n,i}) - \mathcal{H}^1(\Gamma_{\varepsilon,n,i-1})$, summing up we obtain (4.1). Using (4.1) and (4.2) we get (4.3), changing the value of C. $\qquad\square$

Let us define the piecewise constant interpolations

$$\tau_n(t) := t_{n,i} \,, \quad u_{\varepsilon,n}(t) := u_{\varepsilon,n,i} \,, \quad l_{\varepsilon,n}(t) := \mathcal{H}^1(\Gamma_{\varepsilon,n,i}) \quad \text{for } t \in (t_{n,i-1}, t_{n,i}] \,.$$

Besides, following [1, 6, 17], we consider the piecewise affine interpolation

$$\hat{l}_{\varepsilon,n}(t) := p_{\varepsilon,n}(t)(t - t_{n,i-1}) + \mathcal{H}^1(\Gamma_{\varepsilon,n,i-1}) \quad \text{for } t \in (t_{n,i-1}, t_{n,i}] \,,$$

where

$$p_{\varepsilon,n}(t) := \frac{\mathcal{H}^1(\Gamma_{\varepsilon,n,i}) - \mathcal{H}^1(\Gamma_{\varepsilon,n,i-1})}{t_{n,i} - t_{n,i-1}} = \frac{d(\Gamma_{\varepsilon,n,i}; \Gamma_{\varepsilon,n,i-1})}{t_{n,i} - t_{n,i-1}} \,.$$

Finally, we set $G_{\varepsilon,n,i} := G(t_{n,i}; \Gamma_{\varepsilon,n,i})$ and $G_{\varepsilon,n}(t) := G_{\varepsilon,n,i}$ for $t \in (t_{n,i-1}, t_{n,i}]$. The next result is the equivalent of Griffith's criterion in the discrete framework.

Proposition 4.4. *For every ε, n, and t, we have*

(1) $p_{\varepsilon,n}(t) \geq 0$;
(2) $1 - G_{\varepsilon,n}(t) + \varepsilon\, p_{\varepsilon,n}(t) \geq 0$;
(3) $p_{\varepsilon,n}(t) \left[1 - G_{\varepsilon,n}(t) + \varepsilon\, p_{\varepsilon,n}(t) \right] = 0$.

Proof. The nonnegativity of $p_{\varepsilon,n}(t)$ is clear by the construction of $\Gamma_{\varepsilon,n}(t)$.

Let $\Gamma_{\varepsilon,n}^{(s)}$ be the curve of $\mathcal{R}_\eta$, contained in $\Gamma_{\varepsilon,n}(T)$, with length s. Assume that $t \in (t_{n,i-1}, t_{n,i}]$. For every $s \geq l_{\varepsilon,n,i-1}$ we compare $\Gamma_{\varepsilon,n,i}$ with $\Gamma_{\varepsilon,n}^{(s)}$ in (3.2) and get

$$\mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i}) + l_{\varepsilon,n,i} + \frac{\varepsilon}{2} \frac{(l_{\varepsilon,n,i} - l_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}} \leq \mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n}^{(s)}) + s + \frac{\varepsilon}{2} \frac{(s - l_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}. \quad (4.4)$$

If $s > l_{\varepsilon,n,i}$, dividing by $s - l_{\varepsilon,n,i}$ we obtain

$$1 + \frac{\mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n}^{(s)}) - \mathcal{E}(t_{n,i}; \Gamma_{\varepsilon,n,i})}{s - l_{\varepsilon,n,i}} + \frac{\varepsilon}{2} \frac{s + l_{\varepsilon,n,i} - 2\, l_{\varepsilon,n,i-1}}{t_{n,i} - t_{n,i-1}} \geq 0,$$

so, passing to the limit as $s \to l_{\varepsilon,n,i}$ we find (2).

If $p_{\varepsilon,n}(t) = 0$, (3) is obviously satisfied. Otherwise, we have $l_{\varepsilon,n,i} > l_{\varepsilon,n,i-1}$. We consider (4.4) with $l \in (l_{\varepsilon,n,i-1}, l_{\varepsilon,n,i})$; dividing by $l - l_{\varepsilon,n,i}$, passing to the limit as $l \to l_{\varepsilon,n,i}$, and recalling (2), we get (3). $\qquad\square$

5. Viscous solutions

Passing to the limit as $n \to \infty$ for ε fixed, we show in the next proposition the existence of viscous solutions and their properties. This first passage to the limit can be deduced by the abstract result of [27, Theorem 3.5] in an infinite-dimensional framework. For the reader's convenience we present the complete proof, which is simpler in our case.

Proposition 5.1. *For every $\varepsilon > 0$ there exists a subsequence (not relabelled) of $\Gamma_{\varepsilon,n}$ and a set function $t \mapsto \Gamma_\varepsilon(t) \in \mathcal{R}_\eta$ such that $\Gamma_{\varepsilon,n}(t)$ converges to $\Gamma_\varepsilon(t)$ in the Hausdorff metric for every $t \in [0, T]$.*

Moreover,

  (1) *Γ_ε is nondecreasing in time;*
  (2) *$l_{\varepsilon,n}(t) \to l_\varepsilon(t)$ and $\hat{l}_{\varepsilon,n}(t) \to l_\varepsilon(t)$ for every $t \in [0, T]$, where $l_\varepsilon(t) := \mathcal{H}^1(\Gamma_\varepsilon(t))$;*
  (3) *$\hat{l}_{\varepsilon,n} \rightharpoonup l_\varepsilon$ weakly in $H^1([0, T])$, $p_{\varepsilon,n} \rightharpoonup \dot{l}_\varepsilon$ weakly in $L^2([0, T])$, and $\varepsilon\|\dot{l}_\varepsilon\|_{2,[0,T]}^2$ is uniformly bounded in ε;*
  (4) *$\nabla u_{\varepsilon,n}(t) \to \nabla u_\varepsilon(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ for every $t \in [0, T]$, where $u_\varepsilon(t) := u(t; \Gamma_\varepsilon(t))$, and $\|\nabla u_\varepsilon(t)\|_2$ is uniformly bounded in ε and t;*
  (5) *$G_{\varepsilon,n}(t) \to G_\varepsilon(t)$ for every $t \in [0, T]$, where $G_\varepsilon(t) := G(t; \Gamma_\varepsilon(t))$;*
  (6) *for every $t \in [0, T]$, we have*

$$\mathcal{E}(t; \Gamma_\varepsilon(t)) + l_\varepsilon(t) + \frac{\varepsilon}{2} \int_0^t \dot{l}_\varepsilon(\tau)^2 \, \mathrm{d}\tau \leq \mathcal{E}(0; \Gamma_0) + l_0 + \int_0^t \left\langle \nabla u_\varepsilon(\tau), \nabla \dot{\psi}(\tau) \right\rangle_2 \mathrm{d}\tau. \quad (5.1)$$

Proof. Thanks to the Helly Theorem, we find a subsequence $\Gamma_{\varepsilon,n}$ and a nondecreasing set function $t \mapsto \Gamma_\varepsilon(t)$ such that $\Gamma_{\varepsilon,n}(t)$ converges in the Hausdorff metric to $\Gamma_\varepsilon(t)$ for every $t \in [0, T]$ (recall that in the class $\mathcal{R}_\eta$ the length of each curve is bounded). By Theorem 1.3 we have $\Gamma_\varepsilon(t) \in \mathcal{R}_\eta$ and $l_{\varepsilon,n}(t) \to l_\varepsilon(t)$ for every $t \in [0, T]$.

We now prove that the piecewise constant interpolations $l_{\varepsilon,n}$ and the piecewise affine interpolations $\hat{l}_{\varepsilon,n}$ have the same pointwise limit l_ε. Fix $t \in [0, T]$ and, for every n, fix i such that $t \in (t_{n,i-1}, t_{n,i}]$. By contradiction, let us assume that there is $\delta_\varepsilon(t) > 0$ with $l_{\varepsilon,n}(t) - \hat{l}_{\varepsilon,n}(t) > \delta_\varepsilon(t)$ for infinitely many indices n. Hence $p_{\varepsilon,n}(t)(t_{n,i} - t) > \delta_\varepsilon(t)$, so that $\frac{d(\Gamma_{\varepsilon,n,i}; \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}$ becomes arbitrarily large for $n \gg 1$, as one can easily see. On the contrary, by (4.3) this term is bounded uniformly in ε, n, and i. Since $l_{\varepsilon,n}(t) \geq \hat{l}_{\varepsilon,n}(t)$, we conclude that

$$\lim_{n \to \infty} \hat{l}_{\varepsilon,n}(t) = \lim_{n \to \infty} l_{\varepsilon,n}(t) = l_\varepsilon(t).$$

Furthermore, using (4.3) we have $\varepsilon \|p_{\varepsilon,n}\|_{2,[0,T]}^2 \leq C$ for a constant C independent of ε and n; therefore, we deduce that $\hat{l}_{\varepsilon,n} \rightharpoonup l_\varepsilon$ weakly in $H^1([0,T])$ (up to subsequences) and $\varepsilon \|\dot{l}_\varepsilon\|_{2,[0,T]}^2 \leq C$. For every $t \in [0,T]$, by Remark 1.4 $\nabla u_{\varepsilon,n}(t)$ converges to $\nabla u_\varepsilon(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$; by (4.2) $\|\nabla u_\varepsilon(t)\|_2 \leq C$ uniformly in ε. Using Theorem 2.2 we have the convergence of the energy release rates $G_{\varepsilon,n}(t)$ to $G_\varepsilon(t)$.

By (4.1) we get for every $t \in [0,T]$

$$\mathcal{E}(\tau_n(t); \Gamma_{\varepsilon,n}(t)) + l_{\varepsilon,n}(t) + \frac{\varepsilon}{2} \int_0^{\tau_n(t)} p_{\varepsilon,n}(\tau)^2 \, \mathrm{d}\tau$$

$$\leq \mathcal{E}(0; \Gamma_0) + l_0 + \int_0^{\tau_n(t)} \left\langle \nabla \underline{u}_{\varepsilon,n}(\tau), \nabla \dot{\psi}(\tau) \right\rangle_2 \mathrm{d}\tau + R_n \,,$$

where $R_n \to 0$ as $n \to \infty$ and $\underline{u}_{\varepsilon,n}$ denotes the right-continuous piecewise constant interpolation of $u_{\varepsilon,n,i}$, defined by $\underline{u}_{\varepsilon,n}(\tau) := u_{\varepsilon,n,i-1}$ for $\tau \in [t_{n,i-1}, t_{n,i})$. Arguing as before, it is possible to see that $\left\|\nabla \underline{u}_{\varepsilon,n}(t)\right\|_2 \leq C$ and $\nabla \underline{u}_{\varepsilon,n}(t) \to \nabla u_\varepsilon(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ (up to a further subsequence). Then, by (3) we can pass to the limit in the previous inequality to obtain (5.1) for every $t \in [0,T]$. $\qquad\square$

After the passage to the limit, we have the following result as a consequence of Proposition 4.4.

Proposition 5.2. *For every ε and a.e. t, we have*

(1) $\dot{l}_\varepsilon(t) \geq 0$;
(2) $1 - G_\varepsilon(t) + \varepsilon \dot{l}_\varepsilon(t) \geq 0$;
(3) $\dot{l}_\varepsilon(t) \left[1 - G_\varepsilon(t) + \varepsilon \dot{l}_\varepsilon(t)\right] = 0$.

Proof. As $l_\varepsilon \in H^1([0,T])$, its derivative $\dot{l}_\varepsilon(t)$ is a.e. defined, and nonnegative by the monotonicity.

By Proposition 5.1, $\hat{l}_{\varepsilon,n} \rightharpoonup l_\varepsilon$ weakly in $H^1([0,T])$ and $G_{\varepsilon,n}(t) \to G_\varepsilon(t)$. Moreover, $G_{\varepsilon,n}$ is uniformly bounded in $L^\infty([0,T])$ thanks to the integral representation formula of Remark 2.1; then, $G_{\varepsilon,n} \to G_\varepsilon$ strongly in $L^2([0,T])$.

By Proposition 4.4, we have

$$\int_0^T w(t) \left[1 - G_{\varepsilon,n}(t) + \varepsilon \, p_{\varepsilon,n}(t)\right] \mathrm{d}t \geq 0$$

for every $w \in L^2([0,T])$ such that $w \geq 0$. Passing to the limit, we find (2).

Analogously, passing to the limit in equality (3) of Proposition 4.4 and using the semicontinuity of $\|p_{\varepsilon,n}\|_{2,[0,T]}$, we obtain

$$\int_0^T \dot{l}_\varepsilon(t) \left[1 - G_\varepsilon(t) + \varepsilon \dot{l}_\varepsilon(t)\right] \mathrm{d}t \leq 0 \,.$$

On the other hand, we have just proven in (1) and (2) that the last integrand is a.e. nonnegative, so (3) holds. $\qquad\square$

The previous fact leads to the property of energy balance in the viscous setting.

Proposition 5.3. *The function $t \mapsto \mathcal{F}(t; \Gamma_\varepsilon(t)) = \mathcal{E}(t; \Gamma_\varepsilon(t)) + \mathcal{H}^1(\Gamma_\varepsilon(t))$ is absolutely continuous in $[0,T]$ and the following identity holds for every $t \in [0,T]$:*

$$\mathcal{E}(t; \Gamma_\varepsilon(t)) + l_\varepsilon(t) + \int_0^t \dot{l}_\varepsilon(\tau) \cdot (G_\varepsilon(\tau) - 1) \, \mathrm{d}\tau = \mathcal{E}(0; \Gamma_0) + l_0 + \int_0^t \left\langle \nabla u_\varepsilon(\tau), \nabla \dot{\psi}(\tau) \right\rangle_2 \mathrm{d}\tau \,. \quad (5.2)$$

Proof. The bulk energy is derivable with respect to the length along any fixed curve in $\mathcal{R}_\eta$ and its derivative is the opposite of the energy release rate (see (2.1) and [20, Theorem 2.1]). Moreover, the function $t \mapsto l_\varepsilon(t) = \mathcal{H}^1(\Gamma_\varepsilon(t))$ is of class $H^1([0,T])$. Therefore, the chain rule for $\mathcal{F}(t; \Gamma_\varepsilon(t)) = \mathcal{E}(t; \Gamma_\varepsilon(t)) + \mathcal{H}^1(\Gamma_\varepsilon(t))$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}(t; \Gamma_\varepsilon(t)) = \partial_t \mathcal{E}(t; \Gamma_\varepsilon(t)) + \dot{l}_\varepsilon(t) \left[1 - G_\varepsilon(t)\right]$$

Integrating, we conclude. $\qquad\square$

Remark 5.4. The energy balance (5.2) can be rewritten as

$$\mathcal{E}(t; \Gamma_\varepsilon(t)) + l_\varepsilon(t) + \varepsilon \int_0^t \dot{l}_\varepsilon(\tau)^2 \, \mathrm{d}\tau = \mathcal{E}(0; \Gamma_0) + l_0 + \int_0^t \left\langle \nabla u_\varepsilon(\tau), \nabla \dot{\psi}(\tau) \right\rangle_2 \mathrm{d}\tau \, ,$$

thanks to part (3) of Proposition 5.2.

6. The viscous limit

We now pass to the limit as the viscous parameter ε tends to zero. This allows us to prove the existence of quasistatic evolutions and their properties.

Proof of Theorem 3.3. We modify the proof of [17, Theorem 5.1] taking into account the convergence of cracks in the class $\mathcal{R}_\eta$.

Let $\varepsilon_k \to 0$ and let $t \mapsto \Gamma_{\varepsilon_k}(t)$ be a sequence of viscous solutions. Applying the Helly Theorem, we find a nondecreasing function $t \mapsto \Gamma(t)$ and a subsequence $\varepsilon_k \to 0$ (not relabelled) such that $\Gamma_{\varepsilon_k}(t)$ converges to $\Gamma(t)$ in the Hausdorff metric for every $t \in [0,T]$. In particular, for every $t \in [0,T]$ we have $\Gamma(t) \in \mathcal{R}_\eta$ and $l_{\varepsilon_k}(t) \to l(t) := \mathcal{H}^1(\Gamma(t))$ (see Theorem 1.3). Hence, (a) is proven. The derivative $\mathrm{D}l$ of the BV function $t \mapsto l(t)$ can be decomposed as

$$\mathrm{D}l = \widetilde{\mathrm{D}}l + \mathrm{D}^j l = \dot{l} \, \mathrm{d}\mathcal{L}^1 + \mathrm{D}^c l + \mathrm{D}^j l \, ,$$

where $\mathrm{D}^j l$ is concentrated in the jump set $J(l)$ and $\dot{l}$ is the density of the absolutely continuous part of $\mathrm{D}l$ with respect to the Lebesgue measure.

In order to prove (b), we use the following consequence of Proposition 5.2:

$$\int_0^T w(t) \left[1 - G_{\varepsilon_k}(t) + \varepsilon_k \dot{l}_{\varepsilon_k}(t)\right] \mathrm{d}t \geq 0 \tag{6.1}$$

for every $w \in L^2([0,T])$ such that $w \geq 0$. By part (3) of Proposition 5.1, we get $\varepsilon_k \dot{l}_{\varepsilon_k} \to 0$ strongly in $L^2([0,T])$. By Theorem 2.2 we have that $G_{\varepsilon_k}(t)$ converges pointwise to

$$G(t) := G(t; \Gamma(t))$$

(recall that $\Gamma(t) \subset \Gamma(T)$ for every t). Moreover this convergence is also strong in $L^2([0,T])$, because the energy release rate is uniformly bounded (see Remark 2.1). Passing to the limit in (6.1) and recalling that the energy release rate is a continuous function of the length, we obtain that $G(t) \leq 1$ if t does not belong to the (negligible) jump set $J(l)$. This shows (b).

As for (c), let $\bar{t} \notin J(l)$ be such that $G(\bar{t}) < 1$. We prove that there exists $\delta > 0$ (depending only on $\bar{t}$) such that, for every k large enough, $G_{\varepsilon_k}(t) < 1$ for every $t \in [\bar{t} - \delta, \bar{t} + \delta]$. By contradiction let us assume that $G_{\varepsilon_k}(t_k) \geq 1$ for a sequence $t_k \to \bar{t}$: then, by the continuity of the energy release rate (Theorem 2.2) we get $G(\bar{t}) \geq 1$, which contradicts our hypothesis. Hence, there exist $\delta > 0$ and $\bar{k} \in \mathbb{N}$ such that $G_{\varepsilon_k}(t) < 1$ in $[\bar{t} - \delta, \bar{t} + \delta]$ for $k \geq \bar{k}$. Finally, by part (3) of Proposition 5.2 we conclude that for $k \geq \bar{k}$ the function l_{ε_k} is constant in $[\bar{t} - \delta, \bar{t} + \delta]$ and so is l.

Let now $E(t; l) := \mathcal{E}(t; \gamma([0, l]))$, where γ is the arc-length parametrization of $\Gamma(T)$ (such that $\gamma(0) \in \partial\Omega$, as before). The chain rule for $E(t; l(t))$ then leads to

$$E(t; l(t^{-})) - E(0; l(0^{+})) = \int_{0}^{t} \partial_{t} E(\tau; l(\tau)) \, \mathrm{d}\tau + \int_{0}^{t} \partial_{l} E(\tau; l(\tau)) \, \mathrm{d}\widetilde{\mathrm{D}} l(\tau)$$
$$+ \sum_{\tau \in J(l) \cap (0, t)} \left[E(\tau; l(\tau^{+})) - E(\tau; l(\tau^{-})) \right].$$

As for the first summand, we have

$$\partial_{t} E(\tau; l(\tau)) = \left\langle \nabla u(\tau; \Gamma(\tau)), \nabla \dot{\psi}(\tau) \right\rangle_{2}.$$

On the other side, recalling (b), (c), and the definition of G, we see that

$$\partial_{l} E(\,\cdot\,; l(\cdot)) \, \widetilde{\mathrm{D}} l = -G(\cdot) \, \widetilde{\mathrm{D}} l = -\widetilde{\mathrm{D}} l.$$

Finally, for $\tau \in J(l)$ we have

$$E(\tau; l(\tau^{+})) - E(\tau; l(\tau^{-})) = \int_{l(\tau^{-})}^{l(\tau^{+})} [G^{(s)}(\tau) - 1] \, \mathrm{d}s,$$

where $G^{(s)}$ was defined in (3.5). Collecting all terms we obtain the energy balance law (d). $\qquad\square$

Remark 6.1. Through a change of variables it is possible to state some properties of the jump points of l; here we follow the proof of [17, Theorem 5.1]. Given $\bar{t} \in J(l)$, consider an interval $[a, b]$ with $l(\bar{t}^{-}) < a < b < l(\bar{t}^{+})$ (substitute $\bar{t}^{-}$ with 0 if $\bar{t} = 0$, and $\bar{t}^{+}$ with T if $\bar{t} = T$). By the continuity of $t \mapsto l_{\varepsilon}(t)$, for ε sufficiently small we can find $\underline{t}_{\varepsilon} < \bar{t}_{\varepsilon}$ such that $\underline{t}_{\varepsilon}, \bar{t}_{\varepsilon} \to \bar{t}$, $l_{\varepsilon}(\underline{t}_{\varepsilon}) = a$, and $l_{\varepsilon}(\bar{t}_{\varepsilon}) = b$ (if $\bar{t} = T$, recall that $l_{\varepsilon}(T) \to l(T)$). Since $\dot{l}_{\varepsilon}(t) [1 - G_{\varepsilon}(t)] = -\varepsilon \dot{l}_{\varepsilon}(t)^{2} \leq 0$ by part (3) of Proposition 5.2, we have

$$\int_{\underline{t}_{\varepsilon}}^{\bar{t}_{\varepsilon}} w(l_{\varepsilon}(t)) [1 - G_{\varepsilon}(t)] \, \dot{l}_{\varepsilon}(t) \, \mathrm{d}t \leq 0$$

for every $w \in L^{2}([0, L])$ such that $w \geq 0$. Changing variables we get

$$\int_{a}^{b} w(s) [1 - G_{\varepsilon}^{(s)}(t_{\varepsilon}(s))] \, \mathrm{d}s \leq 0,$$

where $t_{\varepsilon}(s) := \min\{t \in [\underline{t}_{\varepsilon}, \bar{t}_{\varepsilon}] : l_{\varepsilon}(t) = s\}$ and $G_{\varepsilon}^{(s)}(t) := G(t; \Gamma_{\varepsilon}^{(s)})$, with $\Gamma_{\varepsilon}^{(s)}$ the curve in $\mathcal{R}_{\eta}$ of length s, contained in $\Gamma_{\varepsilon}(T)$. The integrand passes to the limit because the energy release rate is continuous under Hausdorff convergence and uniformly bounded (Theorem 2.2 and Remark 2.1), so that

$$\int_{a}^{b} w(s) [1 - G^{(s)}(\bar{t})] \, \mathrm{d}s \leq 0$$

(see (3.5) for the notation). By the arbitrariness of a and b and by the continuity of $s \mapsto G^{(s)}(\bar{t})$, we conclude that

$$G^{(s)}(\bar{t}) \geq 1$$

for every $\bar{t} \in J(l)$ and every $s \in [l(\bar{t}^{-}), l(\bar{t}^{+})]$. This is the main feature of the *local energetic solution* proposed in [17].

For the analogous property in the case of many curves, see part (d) of Theorem 8.1.

7. THE CASE OF MANY CURVES

In this section we address the study of the evolution of multiple noninteracting cracks.

We assume that the initial crack Γ_0 is composed of a fixed number M of closed nondegenerate arcs of curve $\Gamma_0^1, \ldots, \Gamma_0^M$ of class $C^{1,1}$, without self-intersections. Each curve Γ_0^m, $m = 1, \ldots, M$, is contained in Ω, except at most for the initial point, which may belong either to $\partial\Omega$, or to Ω: in the latter case, it is supposed to coincide with the initial point of another curve Γ_0^h (with $h \neq m$), in such a way that the union $\Gamma_0^m \cup \Gamma_0^h$ is of class $C^{1,1}$. Hence, we consider also the case of cracks well contained in Ω, with two mobile tips. We assume that the curves $\Gamma_0^1, \ldots, \Gamma_0^M$ are disjoint, unless two initial points coincide. As before, we assume that $\Omega \backslash \Gamma_0$ is the finite union of Lipschitz domains. We set $l_0^m := \mathcal{H}^1(\Gamma_0^m) > 0$ for every m and $l_0 := (l_0^1, \ldots, l_0^M)$, so that $\mathcal{H}^1(\Gamma_0) = |l_0|_1 = \sum_{m=1}^M l_0^m$.

Definition 7.1. Given $\eta > 0$, $\mathcal{R}_\eta^M$ is the class of sets $\Gamma = \Gamma^1 \cup \cdots \cup \Gamma^M$, union of closed arcs of curve of class $C^{1,1}$, such that for every m and h with $m \neq h$

(a) $\Gamma^m \supset \Gamma_0^m$, $\Gamma^m \backslash \Gamma_0^m \subset\subset \Omega$, and $\Gamma^m \cap \Gamma^h = \Gamma_0^m \cap \Gamma_0^h$;
(b) for every point $x \in \Gamma^m \backslash \Gamma_0^m$ there exist two open balls $C_1, C_2 \subset \Omega$ of radius η, such that $(C_1 \cup C_2) \cap (\Gamma^m \cup \partial\Omega) = \emptyset$ and $\overline{C}_1 \cap \overline{C}_2 = \{x\}$;
(c) for every point $x \in \Gamma^m \backslash \Gamma_0^m$ the open ball C_3 of radius 2η centred at x satisfies $C_3 \cap \Gamma^h = \emptyset$.

As before, we fix η so small that for every $m = 1, \ldots, M$ the curvature of Γ_0^m is controlled from above by $\frac{1}{\eta}$ at a.e. point and the class $\mathcal{R}_\eta$ is not empty. We will always use the convention to write Γ^m for the part of the curve Γ containing Γ_0^m.

The sequential compactness of $\mathcal{R}_\eta^M$ with respect to the Hausdorff convergence can be obtained repeating the arguments of [20, Proposition 2.9]. Hence for every sequence $\Gamma_n \in \mathcal{R}_\eta^M$ we find a subsequence (not relabelled) and a curve $\Gamma \in \mathcal{R}_\eta^M$ such that Γ_n^m converges to Γ^m and $\mathcal{H}^1(\Gamma_n^m) \to \mathcal{H}^1(\Gamma^m)$ for every m.

To define the energy release rate, we choose some $C^{1,1}$ extensions $\Gamma^{m\prime}$ of Γ^m and denote by γ^m their arc-length parametrizations (where $\gamma^m(0)$ is the initial point of the curve). Then the m-th component of the energy release rate is

$$G^m(t; \Gamma) := -\partial_{l^m} \mathcal{E}(t; \gamma^1([0, l^1]) \cup \cdots \cup \gamma^M([0, l^M]))\big|_{l^1 = \mathcal{H}^1(\Gamma^1), \ldots, l^M = \mathcal{H}^1(\Gamma^M)}. \qquad (7.1)$$

We set $G(t; \Gamma) := (G^1(t; \Gamma), \ldots, G^M(t; \Gamma))$. All results and remarks stated in Section 2 hold also in this context.

In the case of many curves, the dissipation potential (2.3) must be extended in a suitable way in order to guarantee that the solutions are physically admissible (we follow [17, 23]). Given $\Gamma_1, \Gamma_2 \in \mathcal{R}_\eta^M$ with $\Gamma_1 = \Gamma_1^1 \cup \cdots \cup \Gamma_1^M$ and $\Gamma_2 = \Gamma_2^1 \cup \cdots \cup \Gamma_2^M$, we set

$$d(\Gamma_1; \Gamma_2) := \begin{cases} \left(\sum_{m=1}^M \mathcal{H}^1(\Gamma_1^m \backslash \Gamma_2^m)^2\right)^{\frac{1}{2}} & \text{if } \Gamma_1 \supset \Gamma_2, \\ +\infty & \text{otherwise.} \end{cases} \qquad (7.2)$$

The first part of the proof of the existence of evolutions is a simple generalization of what we saw in the previous sections: the main difference is the presence of two different "norms" in (3.2).

Fixed $\varepsilon > 0$ and a time discretization $\{t_{n,i}\}_{0 \leq i \leq n}$ such that (3.1) holds, we define inductively the approximate sequence $\Gamma_{\varepsilon,n,i} = \Gamma_{\varepsilon,n,i}^1 \cup \cdots \cup \Gamma_{\varepsilon,n,i}^M$ by solving (3.2) in $\mathcal{R}_\eta^M$; the existence of solutions to (3.2) can be shown as in Proposition 4.1. We consider the piecewise

constant interpolation as in (3.3) and we set for $t \in (t_{n,i-1}, t_{n,i}]$

$$l_{\varepsilon,n}^m(t) := \mathcal{H}^1(\Gamma_{\varepsilon,n,i}^m), \quad l_{\varepsilon,n}(t) := (l_{\varepsilon,n}^1(t), \dots, l_{\varepsilon,n}^M(t)),$$

$$\hat{l}_{\varepsilon,n}^m(t) := p_{\varepsilon,n}^m(t)(t - t_{n,i-1}) + \mathcal{H}^1(\Gamma_{\varepsilon,n,i-1}^m), \quad \hat{l}_{\varepsilon,n}(t) := (\hat{l}_{\varepsilon,n}^1(t), \dots, \hat{l}_{\varepsilon,n}^M(t)),$$

$$p_{\varepsilon,n}^m(t) := \frac{\mathcal{H}^1(\Gamma_{\varepsilon,n,i}^m) - \mathcal{H}^1(\Gamma_{\varepsilon,n,i-1}^m)}{t_{n,i} - t_{n,i-1}}, \quad p_{\varepsilon,n}(t) := (p_{\varepsilon,n}^1(t), \dots, p_{\varepsilon,n}^M(t)),$$

$$G_{\varepsilon,n}^m(t) := G^m(t_{n,i}; \Gamma_{\varepsilon,n,i}), \quad G_{\varepsilon,n}(t) = (G_{\varepsilon,n}^1(t), \dots, G_{\varepsilon,n}^M(t)).$$

Notice that

$$|p_{\varepsilon,n}(t)|_2 = \frac{d(\Gamma_{\varepsilon,n,i}; \Gamma_{\varepsilon,n,i-1})}{t_{n,i} - t_{n,i-1}}$$

and by Proposition 4.3 we get uniformly in ε and n

$$\varepsilon \int_0^T |p_{\varepsilon,n}(t)|_2^2 \, \mathrm{d}t \leq C.$$

Griffith's criterion reads componentwise.

Proposition 7.2. *For every m, ε, n, and t, we have*

(1) $p_{\varepsilon,n}^m(t) \geq 0$;
(2) $1 - G_{\varepsilon,n}^m(t) + \varepsilon\, p_{\varepsilon,n}^m(t) \geq 0$;
(3) $p_{\varepsilon,n}^m(t) \left[1 - G_{\varepsilon,n}^m(t) + \varepsilon\, p_{\varepsilon,n}^m(t)\right] = 0$.

Proof. It is sufficient to repeat the argument of Proposition 4.4 taking

$$\Gamma_{\varepsilon,n}^{(s)} := \Gamma_{\varepsilon,n,i}^1 \cup \cdots \cup \Gamma_{\varepsilon,n,i}^{m-1} \cup \Gamma_{\varepsilon,n}^{m,s} \cup \Gamma_{\varepsilon,n,i}^{m+1} \cup \cdots \cup \Gamma_{\varepsilon,n,i}^M,$$

where $\Gamma_{\varepsilon,n}^{m,s}$ is the curve of length s contained in $\Gamma_{\varepsilon,n}^m(T)$. $\square$

We can pass to the limit as $n \to \infty$ for $\varepsilon > 0$ fixed and find a subsequence of $\Gamma_{\varepsilon,n}(t)$ converging for every $t \in [0,T]$ in the Hausdorff metric to a set $\Gamma_\varepsilon(t) = \Gamma_\varepsilon^1(t) \cup \cdots \cup \Gamma_\varepsilon^M(t)$ in $\mathcal{R}_\eta^M$, nondecreasing in t. The map $t \mapsto \Gamma_\varepsilon(t)$ is a *viscous solution* (see Definition 3.1). As in Proposition 5.1, we have that $\hat{l}_{\varepsilon,n}^m$ converges pointwise and weakly in $H^1([0,T])$ to $l_\varepsilon^m(t) := \mathcal{H}^1(\Gamma_\varepsilon^m(t))$; moreover, by Remark 1.4 the corresponding deformation gradients converge strongly in $L^2(\Omega; \mathbb{R}^2)$ to the Jacobian of $u_\varepsilon(t) := u(t; \Gamma_\varepsilon(t))$, with $\|\nabla u_\varepsilon(t)\|_2 \leq C$ uniformly in ε and t. Therefore, for every $t \in [0,T]$

$$\mathcal{E}(t; \Gamma_\varepsilon(t)) + |l_\varepsilon(t)|_1 + \frac{\varepsilon}{2} \int_0^t \left|\dot{l}_\varepsilon(\tau)\right|_2^2 \, \mathrm{d}\tau \leq \mathcal{E}(0; \Gamma_0) + |l_0|_1 + \int_0^t \left\langle \nabla u_\varepsilon(\tau), \nabla \dot{\psi}(\tau) \right\rangle_2 \mathrm{d}\tau,$$

where

$$l_\varepsilon(t) := (l_\varepsilon^1(t), \dots, l_\varepsilon^M(t)) \quad \text{and} \quad \dot{l}_\varepsilon(t) := \left(\dot{l}_\varepsilon^1(t), \dots, \dot{l}_\varepsilon^M(t)\right).$$

We get also

$$\varepsilon \int_0^T \left|\dot{l}_\varepsilon(t)\right|_2^2 \, \mathrm{d}t \leq C \tag{7.3}$$

with C independent of ε.

Since by Theorem 2.2 the energy release rates $G_{\varepsilon,n}(t)$ converge to

$$G_\varepsilon(t) = \left(G_\varepsilon^1(t), \dots, G_\varepsilon^M(t)\right) := \left(G^1(t; \Gamma_\varepsilon(t)), \dots, G^M(t; \Gamma_\varepsilon(t))\right),$$

we obtain the viscous version of Griffith's criterion componentwise. Henceforth, the symbol $\mathbf{1}$ denotes the vector $(1, \dots, 1) \in \mathbb{R}^M$.

Proposition 7.3. *For every m, every ε, and a.e. t, we have*

(1) $\dot{l}_\varepsilon^m(t) \geq 0$;
(2) $1 - G_\varepsilon^m(t) + \varepsilon\, \dot{l}_\varepsilon^m(t) \geq 0$;

(3) $\dot{l}_\varepsilon^m(t)\left[1 - G_\varepsilon^m(t) + \varepsilon\, \dot{l}_\varepsilon^m(t)\right] = 0$.

Furthermore, we have the viscous energy balance

$$\mathcal{E}(t; \Gamma_\varepsilon(t)) + |l_\varepsilon(t)|_1 + \int_0^t \dot{l}_\varepsilon(\tau) \cdot (G_\varepsilon(\tau) - \mathbf{1})\, d\tau = \mathcal{E}(0; \Gamma_0) + |l_0|_1 + \int_0^t \Big\langle \nabla u_\varepsilon(\tau), \nabla \dot\psi(\tau) \Big\rangle_2 d\tau$$
(7.4)

for every $t \in [0, T]$.

Proof. Argue as in Propositions 5.2 and 5.3. $\square$

As in the proof of Theorem 3.3, for $\varepsilon \to 0$ we find a limit (up to subsequences) $\Gamma(t) = \Gamma^1(t) \cup \cdots \cup \Gamma^M(t)$ in $\mathcal{R}_\eta^M$, nondecreasing in t: the function $t \mapsto \Gamma(t)$ is said to be an *approximable quasistatic evolution* (see Definition 3.2). The limit length vector

$$l(t) = (l^1(t), \ldots, l^M(t)) := (\mathcal{H}^1(\Gamma^1(t)), \ldots, \mathcal{H}^1(\Gamma^M(t)))$$
(7.5)

has bounded variation as a function of time. The energy release rates $G_\varepsilon(t)$ converge (Theorem 2.2) to

$$G(t) = (G^1(t), \ldots, G^M(t)) := (G^1(t; \Gamma(t)), \ldots, G^M(t; \Gamma(t)))\,.$$
(7.6)

Finally, $\nabla u_\varepsilon(t)$ converges to $\nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ (Remark 1.4), where $u(t) := u(t; \Gamma(t))$. Repeating componentwise the arguments of the proof of Theorem 3.3, we obtain Griffith's criterion in the continuity points of $l(t)$.

Theorem 7.4 (Griffith's criterion). *Let* $t \mapsto \Gamma(t) = \Gamma^1(t) \cup \cdots \cup \Gamma^M(t)$ *be an approximable quasistatic evolution. Then the BV function* $t \mapsto l(t)$ *satisfies:*

(a) $\dot{l}^m(t) \geq 0$ *for every* $m = 1, \ldots, M$ *and for a.e.* $t \in [0, T]$;

(b) $G^m(t) \leq 1$ *for every* $m = 1, \ldots, M$ *and for every* $t \notin J(l^m)$;

(c) *if* $G^m(\bar t) < 1$ *for some* m *and* $\bar t \notin J(l^m)$, *then* l^m *is locally constant around* $\bar t$.

As before, (c) implies the usual form for the activation criterion

$$[G(\cdot) - \mathbf{1}] \cdot \widetilde{\mathrm{D}}l = 0\,,$$

where $\widetilde{\mathrm{D}}l := (\widetilde{\mathrm{D}}l^1, \ldots, \widetilde{\mathrm{D}}l^M)$.

However, differently from the case of a single curve, in the usual time scale it is difficult to state the properties of the jump points: in particular, the generalization of (d) in Theorem 3.3 is a nontrivial issue if many cracks are present. Therefore, we define a reparametrization that shall give some information on the behaviour during the jumps in time. Then we will be able to write the energy balance with the correct dissipative term during the jumps.

8. Parametrized solutions

We apply a change of variables that transforms the lengths in absolutely continuous functions of time. Since the crack paths have been determined in the previous section, we can follow the abstract construction of [23], where $\mathcal{R}_\eta^M$ is substituted by $\mathbb{R}^M$ (see also [17, Section 7]).

For $\varepsilon > 0$ and $0 \leq t \leq T$ we set

$$s_\varepsilon(t) := t + \mathcal{H}^1(\Gamma_\varepsilon(t) \backslash \Gamma_0) = t + |l_\varepsilon(t)|_1 - |l_0|_1 = t + \sum_{m=1}^M \left(l_\varepsilon^m(t) - l_0^m\right)\,.$$
(8.1)

By the properties of l_ε, s_ε is strictly increasing and $\dot{s}_\varepsilon(t) \geq 1$ for every ε and t. Therefore, we can define the inverse $s \mapsto \tilde{t}_\varepsilon(s)$ for $0 \leq s \leq S_\varepsilon := s_\varepsilon(T)$; it turns out that $\tilde{t}_\varepsilon$ is strictly

increasing and $0 < \tilde{t}'_\varepsilon(s) \leq 1$ for every ε and s, although $\tilde{t}'_\varepsilon(s)$ may approach 0 for some s as $\varepsilon \to 0$ (henceforth, the symbol $'$ denotes the derivative with respect to s).

For $0 \leq s \leq S_\varepsilon$ and $m = 1, \ldots, M$ we set

$$\tilde{l}^m_\varepsilon(s) := l^m_\varepsilon(\tilde{t}_\varepsilon(s)) \quad \text{and} \quad \tilde{l}_\varepsilon(s) := (\tilde{l}^1_\varepsilon(s), \ldots, \tilde{l}^M_\varepsilon(s)). \tag{8.2}$$

Since by (8.1) we have $s = \tilde{t}_\varepsilon(s) + \left|\tilde{l}_\varepsilon(s)\right|_1 - |l_0|_1$, deriving we obtain the relation

$$\tilde{t}'_\varepsilon(s) + \left|\tilde{l}'_\varepsilon(s)\right|_1 = 1 \tag{8.3}$$

for every ε and s, where $\tilde{l}'_\varepsilon(s) := ((\tilde{l}^1_\varepsilon)'(s), \ldots, (\tilde{l}^M_\varepsilon)'(s))$. Moreover, $0 \leq (\tilde{l}^m_\varepsilon)'(s) \leq 1$ for every m, ε, and s.

We define

$$\widetilde{\Gamma}_\varepsilon(s) := \Gamma_\varepsilon(\tilde{t}_\varepsilon(s)), \quad \widetilde{\Gamma}^m_\varepsilon(s) := \Gamma^m_\varepsilon(\tilde{t}_\varepsilon(s)),$$

$$\tilde{u}_\varepsilon(s) := u_\varepsilon(\tilde{t}_\varepsilon(s)),$$

$$\widetilde{G}_\varepsilon(s) = (\widetilde{G}^1_\varepsilon(s), \ldots, \widetilde{G}^M_\varepsilon(s)) := G_\varepsilon(\tilde{t}_\varepsilon(s)).$$

In particular, $\tilde{l}^m_\varepsilon(s) = \mathcal{H}^1(\widetilde{\Gamma}^m_\varepsilon(s))$.

Hence, the functions $\tilde{t}_\varepsilon$ and $\tilde{l}^m_\varepsilon$ are Lipschitz, then absolutely continuous; the change of variables in (8.1) may be substituted by any other transformation satisfying these properties. We set $\overline{S} := \sup_{\varepsilon > 0} S_\varepsilon$ (which is finite because of the bound on the length of the curves in $\mathcal{R}^M_\eta$) and extend $\tilde{t}_\varepsilon$, $\tilde{t}'_\varepsilon$, $\tilde{l}^m_\varepsilon$, and $(\tilde{l}^m_\varepsilon)'$ to $[0, \overline{S}]$ by setting $\tilde{t}'_\varepsilon(s) := \tilde{t}'_\varepsilon(S_\varepsilon)$ and $(\tilde{l}^m_\varepsilon)'(s) := (\tilde{l}^m_\varepsilon)'(S_\varepsilon)$ in $(S_\varepsilon, \overline{S}]$.

In the previous section we chose a subsequence ε_k such that $\Gamma_{\varepsilon_k}(t)$ converges to $\Gamma(t)$ in the Hausdorff metric and $l^m_{\varepsilon_k}(t)$ converges to $l^m(t)$ for every m and every t (where l was defined in (7.5)). Now, since $\tilde{t}_\varepsilon$ and $\tilde{l}^m_\varepsilon$ are bounded in $W^{1,\infty}([0, \overline{S}])$, we can choose a further subsequence (not relabelled) such that $\tilde{t}_{\varepsilon_k}$ and $\tilde{l}^m_{\varepsilon_k}$ converge weakly* in $W^{1,\infty}([0, \overline{S}])$ to some functions $\tilde{t}$ and $\tilde{l}^m$, respectively, and also S_ε converges to some $S > 0$. We set $\tilde{l}(s) := (\tilde{l}^1(s), \ldots, \tilde{l}^M(s))$ and $\tilde{l}'(s) := ((\tilde{l}^1)'(s), \ldots, (\tilde{l}^M)'(s))$. Passing to the limit in (8.3) we get

$$\tilde{t}'(s) + \left|\tilde{l}'(s)\right|_1 = 1 \tag{8.4}$$

for a.e. s.

For $s \in [0, S]$ we define $\widetilde{\Gamma}(s) = \widetilde{\Gamma}^1(s) \cup \cdots \cup \widetilde{\Gamma}^M(s)$ as the set in $\mathcal{R}^M_\eta$, contained in $\Gamma(T)$, such that, for every m, $\widetilde{\Gamma}^m(s) \supset \Gamma^m_0$ and

$$\mathcal{H}^1(\widetilde{\Gamma}^m(s)) = \tilde{l}^m(s). \tag{8.5}$$

We set also

$$\tilde{u}(s) := u(\tilde{t}(s); \widetilde{\Gamma}(s)) \tag{8.6}$$

and

$$\widetilde{G}(s) = (\widetilde{G}^1(s), \ldots, \widetilde{G}^M(s)) := G(\tilde{t}(s); \widetilde{\Gamma}(s)). \tag{8.7}$$

By Theorem 2.2 and by the continuity of $\tilde{t}$ and $\tilde{l}$, $s \mapsto \widetilde{G}(s)$ is continuous. According to [18, 23], the pair $(\tilde{t}, \widetilde{\Gamma})$ is said to be a *parametrized solution* of the quasistatic evolution problem.

Notice that, for every s, $\widetilde{\Gamma}(s)$ is the Hausdorff limit of $\widetilde{\Gamma}_{\varepsilon_k}(s)$, so that $\nabla \tilde{u}_{\varepsilon_k}(s)$ converge to $\nabla \tilde{u}(s)$ strongly in $L^2(\Omega; \mathbb{R}^2)$; moreover, $\|\nabla \tilde{u}(s)\|_2 \leq C$. As for the energy release rate, we extend $\widetilde{G}_\varepsilon$ and $\widetilde{G}$ to $[0, \overline{S}]$ by setting $\widetilde{G}_\varepsilon(s) = \widetilde{G}_\varepsilon(S_\varepsilon)$ in $(S_\varepsilon, \overline{S}]$ and $\widetilde{G}(s) = \widetilde{G}(S)$ in $(S, \overline{S}]$. We have $\widetilde{G}_{\varepsilon_k}(s) \to \widetilde{G}(s)$ pointwise; using Remark 2.1 as in the proof of Theorem 3.3, we have $\widetilde{G}_{\varepsilon_k} \to \widetilde{G}$ strongly in $L^2([0, \overline{S}]; \mathbb{R}^M)$.

Recalling that $\tilde{t}'_\varepsilon(s) > 0$, the viscous criterion of Proposition 7.3 reads in the new variables as

$$(\tilde{l}^m_\varepsilon)'(s) \geq 0 \,, \tag{8.8}$$

$$\tilde{t}'_\varepsilon(s) - \widetilde{G}^m_\varepsilon(s) \, \tilde{t}'_\varepsilon(s) + \varepsilon \, (\tilde{l}^m_\varepsilon)'(s) \geq 0 \,, \tag{8.9}$$

$$(\tilde{l}^m_\varepsilon)'(s) \, [\tilde{t}'_\varepsilon(s) - \widetilde{G}^m_\varepsilon(s) \, \tilde{t}'_\varepsilon(s) + \varepsilon \, (\tilde{l}^m_\varepsilon)'(s)] = 0 \tag{8.10}$$

for every m, every ε, and a.e. $s \in [0, S_\varepsilon]$. Moreover, (7.3) gives

$$
\begin{aligned}
\varepsilon \int_0^{S_\varepsilon} \left| \tilde{l}'_\varepsilon(s) \right|_2^2 \mathrm{d}s &= \varepsilon \int_0^{S_\varepsilon} \left| \dot{l}_\varepsilon(\tilde{t}_\varepsilon(s)) \right|_2^2 \tilde{t}'_\varepsilon(s)^2 \, \mathrm{d}s \\
&\leq \varepsilon \int_0^{S_\varepsilon} \left| \dot{l}_\varepsilon(\tilde{t}_\varepsilon(s)) \right|_2^2 \tilde{t}'_\varepsilon(s) \, \mathrm{d}s = \varepsilon \int_0^T \left| \dot{l}_\varepsilon(t) \right|_2^2 \mathrm{d}t \leq C
\end{aligned}
\tag{8.11}
$$

for C independent of ε. Hence, $\varepsilon \, \tilde{l}'_\varepsilon \, 1_{[0,S_\varepsilon]} \to 0$ strongly in $L^2([0,\overline{S}]; \mathbb{R}^M)$. Passing to the limit, we find the reparametrized version of Griffith's criterion, which extends Remark 3.4.

Theorem 8.1 (Griffith's criterion). *Let* $t \mapsto \Gamma(t) = \Gamma^1(t) \cup \cdots \cup \Gamma^M(t)$ *be an approximable quasistatic evolution and* $(\tilde{t}, \widetilde{\Gamma})$ *a corresponding parametrized solution; let* $\tilde{l}$ *and* $\widetilde{G}$ *be as in* (8.5) *and* (8.7). *Then the absolutely continuous functions* $\tilde{t}$ *and* $\tilde{l}$ *satisfy for a.e.* $s \in [0, S]$

    (a) $\tilde{t}'(s) \geq 0$ *and* $(\tilde{l}^m)'(s) \geq 0$ *for every* $m = 1, \ldots, M$;
    (b) *if* $\tilde{t}'(s) > 0$, *then* $\widetilde{G}^m(s) \leq 1$ *for every* $m = 1, \ldots, M$;
    (c) *if* $\tilde{t}'(s) > 0$ *and* $(\tilde{l}^m)'(s) > 0$ *for some* $m \in \{1, \ldots, M\}$, *then* $\widetilde{G}^m(s) = 1$;
    (d) *if* $\tilde{t}'(s) = 0$, *then there is* $m \in \{1, \ldots, M\}$ *such that* $(\tilde{l}^m)'(s) > 0$; *moreover, for every* m *with this property, we have* $\widetilde{G}^m(s) \geq 1$.

Remark 8.2. If (a), (b), and (8.4) hold, requiring (c) and (d) is equivalent to the following property: if $\widetilde{G}^m(\bar{s}) < 1$ for some m and $\bar{s}$, then $\tilde{l}^m$ is locally constant around $\bar{s}$ (see also the proof of point (c) of Theorem 3.3). In particular, for every s

$$\tilde{t}'(s) \, [\widetilde{G}(s) - \mathbf{1}] \cdot \tilde{l}'(s) = 0 \,.$$

Proof of Theorem 8.1. By monotonicity, $\tilde{t}'(s) \geq 0$ and $(\tilde{l}^m)'(s) \geq 0$ for a.e. s. Moreover, by (8.4) $\tilde{t}'(s)$ and $(\tilde{l}^m)'(s)$ cannot be both zero simultaneously.

Let $w \in L^2([0,\overline{S}])$ such that $w \geq 0$. By (8.9) we have for every m and ε

$$\int_0^{S_\varepsilon} w(t) \, [\tilde{t}'_\varepsilon(s) - \widetilde{G}^m_\varepsilon(s) \, \tilde{t}'_\varepsilon(s) + \varepsilon \, (\tilde{l}^m_\varepsilon)'(s)] \, \mathrm{d}s \geq 0 \,.$$

Since (up to subsequences) $\tilde{t}'_\varepsilon$ converges to $\tilde{t}'$ weakly* in $L^\infty([0,\overline{S}])$, $\widetilde{G}_\varepsilon \to \widetilde{G}$ strongly in $L^2([0,\overline{S}]; \mathbb{R}^M)$, and $\varepsilon \, \tilde{l}'_\varepsilon \, 1_{[0,S_\varepsilon]} \to 0$ strongly in $L^2([0,\overline{S}]; \mathbb{R}^M)$, then passing to the limit we get

$$\int_0^S w(t) \, \tilde{t}'(s) \, [1 - \widetilde{G}^m(s)] \, \mathrm{d}s \geq 0 \,,$$

which proves (b) by the arbitrarity of w.

For the proof of (c) and (d), we use Remark 8.2. Let us assume that $\widetilde{G}^m(\bar{s}) < 1$ for some m and $\bar{s}$. Arguing as in the proof of Theorem 3.3, we see that there exists $\delta > 0$ such that $\widetilde{G}^m_\varepsilon(s) < 1$ for every $t \in [\bar{s} - \delta, \bar{s} + \delta]$ and ε small enough. As $\tilde{t}_\varepsilon(s) > 0$ for every ε and s, by (8.10) $\tilde{l}^m_\varepsilon$ is locally constant in $[\bar{s} - \delta, \bar{s} + \delta]$ and so is $\tilde{l}^m$. $\qquad\square$

Remark 8.3. Equivalently, Theorem 8.1 states that for a.e. s

    • $\tilde{t}'(s) \geq 0$ and $(\tilde{l}^m)'(s) \geq 0$ for every $m = 1, \ldots, M$;
    • if $(\tilde{l}^m)'(s) = 0$ for every $m = 1, \ldots, M$, then $\tilde{t}'(s) > 0$;

- if $\tilde{t}'(s) > 0$, then $\widetilde{G}^m(s) \leq 1$ for every $m = 1, \ldots, M$;
- if $(\tilde{l}^m)'(s) > 0$ for some $m \in \{1, \ldots, M\}$, then $\widetilde{G}^m(s) \geq 1$.

We change variables in the viscous energy balance (7.4) to obtain for every $s \in [0, S_\varepsilon]$

$$
\mathcal{E}(\tilde{t}_\varepsilon(s); \widetilde{\Gamma}_\varepsilon(s)) + \left|\tilde{l}_\varepsilon(s)\right|_1 + \int_0^s \tilde{l}'_\varepsilon(\tau) \cdot (\widetilde{G}_\varepsilon(\tau) - \mathbf{1}) \, \mathrm{d}\tau
$$
$$
= \mathcal{E}(0; \Gamma_0) + |l_0|_1 + \int_0^s \left\langle \nabla \tilde{u}_\varepsilon(\tau), \nabla \tilde{\psi}'_\varepsilon(\tau) \right\rangle_2 \mathrm{d}\tau \,,
\tag{8.12}
$$

where $\tilde{\psi}_\varepsilon(s) := \psi(\tilde{t}_\varepsilon(s))$. Since all terms converge and $\psi \in AC([0, T]; H^1(\Omega \backslash \Gamma_0))$, passing to the limit we prove the energy balance in the reparametrized setting, with the dissipative term due to the jumps.

Theorem 8.4 (Energy balance). *Let $t \mapsto \Gamma(t) = \Gamma^1(t) \cup \cdots \cup \Gamma^M(t)$ be an approximable quasistatic evolution and $(\tilde{t}, \widetilde{\Gamma})$ a corresponding parametrized solution; let $\tilde{l}$, $\tilde{u}$, and $\widetilde{G}$ be as in (8.5)–(8.7); let $\tilde{\psi}(s) := \psi(\tilde{t}(s))$. Then for every $s \in [0, S]$*

$$
\mathcal{E}(\tilde{t}(s); \widetilde{\Gamma}(s)) + \left|\tilde{l}(s)\right|_1 + \int_0^s \tilde{l}'(\tau) \cdot (\widetilde{G}(\tau) - \mathbf{1}) \, \mathrm{d}\tau
$$
$$
= \mathcal{E}(0; \Gamma_0) + |l_0|_1 + \int_0^s \left\langle \nabla \tilde{u}(\tau), \nabla \tilde{\psi}'(\tau) \right\rangle_2 \mathrm{d}\tau \,.
\tag{8.13}
$$

Proof. Consider the left-hand side of (8.12): up to subsequences, $\tilde{t}_\varepsilon(s)$ converges to $\tilde{t}(s)$ and $\widetilde{\Gamma}_\varepsilon(s)$ tends to $\widetilde{\Gamma}(s)$ in the Hausdorff metric for every s, $\tilde{l}_\varepsilon$ converges to $\tilde{l}$ pointwise everywhere and weakly* in $W^{1,\infty}([0, \overline{S}]; \mathbb{R}^M)$, and $\widetilde{G}_\varepsilon$ tends to $\widetilde{G}$ strongly in $L^2([0, \overline{S}]; \mathbb{R}^M)$. As for the right-hand side, exploit the change of variables $\tilde{t}_\varepsilon$ under the integral. $\square$

Remark 8.5. By Remark 8.3, the integrand $\tilde{l}'(\tau) \cdot (\widetilde{G}(\tau) - \mathbf{1})$ in (8.13) is a.e. nonnegative and it is positive only if $\tilde{t}'(\tau) = 0$. Hence, this term represents the amount of energy dissipated "instantaneously" during the jumps in the original time scale t:

$$
\mathcal{E}(t; \Gamma(t)) + l(t) - \mathcal{E}(0; \Gamma_0) - l_0 = \int_0^t \left\langle \nabla u(\tau; \Gamma(\tau)), \nabla \dot{\psi}(\tau) \right\rangle_2 \mathrm{d}\tau
$$
$$
- \int_{\tilde{s}(0)}^{\tilde{s}(0^+)} \tilde{l}'(\tau) \cdot (\widetilde{G}(\tau) - \mathbf{1}) \, \mathrm{d}\tau - \sum_{\bar{t} \in J(l) \cap (0,t)} \int_{\tilde{s}(\bar{t}^-)}^{\tilde{s}(\bar{t}^+)} \tilde{l}'(\tau) \cdot (\widetilde{G}(\tau) - \mathbf{1}) \, \mathrm{d}\tau
\tag{8.14}
$$
$$
- \int_{\tilde{s}(t^-)}^{\tilde{s}(t)} \tilde{l}'(\tau) \cdot (\widetilde{G}(\tau) - \mathbf{1}) \, \mathrm{d}\tau \,,
$$

where $\tilde{s}$ is the inverse of $\tilde{t}$, well defined where $\tilde{t}$ is not constant.

In the following remark we make some comments about the mechanical interpretation of the notion of reparametrized solution.

Remark 8.6. Theorem 8.1 reflects the existence of three possible regimes:

- *Sticking:* $\tilde{t}'(s) > 0$ and $\tilde{l}'(s) = (0, \ldots, 0)$, i.e., there is no motion; in this case $\widetilde{G}^m(s) \leq 1$ for every m.
- *Sliding:* $\tilde{t}'(s) > 0$ and $\tilde{l}'(s) \neq (0, \ldots, 0)$, which corresponds to a rate-independent growth in the slow time scale; in this case $\widetilde{G}^m(s) = 1$ for every m such that $(\tilde{l}^m)'(s) \neq 0$.
- *Jumping:* $\tilde{t}'(s) = 0$ and $\tilde{l}'(s) \neq (0, \ldots, 0)$, i.e., some curves jump and their path is parametrized in the rescaled variable by $\tilde{l}$ as the time $\tilde{t}$ is frozen; in this case $\widetilde{G}^m(s) \geq 1$ for every m such that $(\tilde{l}^m)'(s) \neq 0$.

For more details, we refer to [17, 18, 22, 23, 24].

Using again (8.8)–(8.10), we obtain a law for the evolution in the rescaled variable during the regime of jumping ($\tilde{t}'(s) = 0$): in correspondence to the jumps in the slow time scale, the parametrized solution follows a viscous equation (see also [23, Corollary 5.5 and Remark 5.7]). Henceforth, the symbol $(v)^+$ stands for the vector whose components are the positive parts of the components of the vector v.

Theorem 8.7 (Viscous equation). *Let $t \mapsto \Gamma(t) = \Gamma^1(t) \cup \cdots \cup \Gamma^M(t)$ be an approximable quasistatic evolution and $(\tilde{t}, \widetilde{\Gamma})$ a corresponding parametrized solution; let $\tilde{l}$ and $\widetilde{G}$ be as in (8.5) and (8.7). Then there exists a continuous function $\lambda \colon [0, S] \to [0, +\infty)$ such that for a.e. s*

$$\lambda(s)\, \tilde{l}'(s) = \left(\widetilde{G}(s) - \mathbf{1}\right)^+ \quad and \quad \lambda(s)\, \tilde{t}'(s) = 0 \,. \tag{8.15}$$

Proof. By (8.10) we have for every m, every ε, and a.e. s

$$\varepsilon \left((\tilde{l}_\varepsilon^m)'(s)\right)^2 = \tilde{t}_\varepsilon'(s)\, (\tilde{l}_\varepsilon^m)'(s) \left(\widetilde{G}_\varepsilon^m(s) - 1\right)^+ \,.$$

With the aid of (8.9) we get

$$\varepsilon\, (\tilde{l}_\varepsilon^m)'(s) = \tilde{t}_\varepsilon'(s) \left(\widetilde{G}_\varepsilon^m(s) - 1\right)^+ \,.$$

Passing to the norms, from the last two equations we obtain respectively

$$\varepsilon \left|\tilde{l}_\varepsilon'(s)\right|_2^2 = \tilde{t}_\varepsilon'(s)\, \tilde{l}_\varepsilon'(s) \cdot \left(\widetilde{G}_\varepsilon(s) - \mathbf{1}\right)^+ \tag{8.16}$$

and

$$\sqrt{\varepsilon} \left|\tilde{l}_\varepsilon'(s)\right|_2 = \frac{1}{\sqrt{\varepsilon}}\, \tilde{t}_\varepsilon'(s) \left|\left(\widetilde{G}_\varepsilon(s) - \mathbf{1}\right)^+\right|_2 \,. \tag{8.17}$$

Using twice (8.17) we find

$$\begin{aligned}
\varepsilon \left|\tilde{l}_\varepsilon'(s)\right|_2^2 &= \frac{\varepsilon}{2} \left|\tilde{l}_\varepsilon'(s)\right|_2^2 + \frac{1}{2\varepsilon}\, \tilde{t}_\varepsilon'(s)^2 \left|\left(\widetilde{G}_\varepsilon(s) - \mathbf{1}\right)^+\right|_2^2 \\
&= \tilde{t}_\varepsilon'(s) \left|\tilde{l}_\varepsilon'(s)\right|_2 \left|\left(\widetilde{G}_\varepsilon(s) - \mathbf{1}\right)^+\right|_2 \,.
\end{aligned} \tag{8.18}$$

By (8.16) and (8.18) we deduce, recalling that $\tilde{t}_\varepsilon'(s) > 0$,

$$\left|\tilde{l}_\varepsilon'(s)\right|_2 \left|\left(\widetilde{G}_\varepsilon(s) - \mathbf{1}\right)^+\right|_2 = \tilde{l}_\varepsilon'(s) \cdot \left(\widetilde{G}_\varepsilon(s) - \mathbf{1}\right)^+ \,. \tag{8.19}$$

Then, we integrate (8.19) between two points $0 \le S_1 < S_2 \le \overline{S}$. Since (up to subsequences) $\tilde{l}_\varepsilon' \rightharpoonup \tilde{l}'$ weakly* in $L^\infty([0, \overline{S}]; \mathbb{R}^M)$ and $\widetilde{G}_\varepsilon \to \widetilde{G}$ strongly in $L^2([0, \overline{S}]; \mathbb{R}^M)$, the right-hand side converges. In the left-hand side we can apply the Ioffe lower semicontinuity theorem, so that

$$\int_{S_1}^{S_2} \left|\tilde{l}'(s)\right|_2 \left|\left(\widetilde{G}(s) - \mathbf{1}\right)^+\right|_2 \, \mathrm{d}s \le \int_{S_1}^{S_2} \tilde{l}'(s) \cdot \left(\widetilde{G}(s) - \mathbf{1}\right)^+ \, \mathrm{d}s \,.$$

Finally, the last inequality is actually an equality because of the Cauchy-Schwarz theorem, so $\tilde{l}'(s)$ and $(\widetilde{G}(s) - \mathbf{1})^+$ are proportional by arbitrarity of S_1, S_2. Therefore, using Theorem 8.1 we find for a.e. $s \in [0, S]$ a coefficient $\lambda(s) \ge 0$ such that (8.15) holds. If $\tilde{t}'(s) = 0$, by (8.4) we have $\lambda(s) = \left|(\widetilde{G}(s) - \mathbf{1})^+\right|_1$; since $s \mapsto \widetilde{G}(s)$ is continuous and $(\widetilde{G}(s) - \mathbf{1})^+ = 0$ when $\tilde{t}'(s) \ne 0$, it turns out that $s \mapsto \lambda(s)$ has a continuous representative. $\square$

In the following remark we show that, under further regularity assumptions, it is possible to write (8.15) in a simpler form during the unstable propagations (i.e., the reparametrized evolutions with $\widetilde{G} > 1$).

Remark 8.8. Consider an interval $[s_1, s_2] \subset [0, S]$, such that $(\widetilde{G}(s_1) - \mathbf{1})^+ = 0$, $(\widetilde{G}(s_2) - \mathbf{1})^+ = 0$, and $(\widetilde{G}(s) - \mathbf{1})^+ \neq 0$ in (s_1, s_2) (therefore $\tilde{t}'(s) = 0$ and $\lambda(s) > 0$, where λ is the function defined in Theorem 8.7). In this remark we assume that $s \mapsto \widetilde{G}(s)$ is Lipschitz.

Fixed $\bar{s} \in (s_1, s_2)$, we set for every $s \in (s_1, s_2)$

$$\hat{\theta}(s) := \int_{\bar{s}}^{s} \frac{1}{\lambda(\tau)} \, \mathrm{d}\tau \, ,$$

which is well defined, strictly increasing, and C^1, because $\frac{1}{\lambda(s)}$ is bounded in the compact subintervals of (s_1, s_2), positive, and continuous. Since $\widetilde{G}$ is Lipschitz and $\lambda(s) = \left| (\widetilde{G}(s) - \mathbf{1})^+ \right|_1$ by (8.4), we have $\lambda(s) \leq C |s - s_1|$ near s_1 and $\lambda(s) \leq C |s - s_2|$ near s_2 for some $C > 0$; therefore, $\hat{\theta}(s_1) = -\infty$ and $\hat{\theta}(s_2) = +\infty$. We will transform the viscous law (8.15) using the change of variables $\hat{\theta} \colon (s_1, s_2) \to (-\infty, +\infty)$ and obtaining an equation for $\hat{l}(\hat{\theta}(s)) := \tilde{l}(s)$.

Recall that at this stage the value of the boundary condition is fixed at $\psi(\tilde{t}(s_1))$, while the crack path $\Gamma(T)$ has already been determined (see Section 7). Hence, we are interested in characterizing the final value reached by the crack length after the unstable propagation. In order to emphasize the dependence on the length, let us introduce the following notation for the energy release rate along $\Gamma(T)$ (see also (7.1)):

$$\mathcal{G}(l) := G(\tilde{t}(s_1); \Gamma^{(l)}) \, ,$$

where $l = (l^1, \ldots, l^M)$ and $\Gamma^{(l)}$ is the set in $\mathcal{R}_\eta^M$, contained in $\Gamma(T)$, whose m-th component has length l^m (as before, the m-th connected component of a set in $\mathcal{R}_\eta^M$ is the one containing Γ_0^m). Notice that $\mathcal{G}(\hat{l}(\hat{\theta}(s))) = \widetilde{G}(s)$ in (s_1, s_2).

Employing the change of variables $\hat{\theta}$, the viscous law (8.15) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \hat{l}(\theta) = \left(\mathcal{G}(\hat{l}(\theta)) - \mathbf{1} \right)^+ \quad \text{for } \theta \in (-\infty, +\infty) \, . \tag{8.20}$$

This equation describes the unstable propagation between the stable configurations $\widetilde{\Gamma}(s_1)$ and $\widetilde{\Gamma}(s_2)$. Indeed, the possible values for $\tilde{l}(s_2)$ can be obtained as $\hat{l}(+\infty)$, where $\hat{l}$ is a solution of (8.20) such that $\hat{l}(-\infty) = \tilde{l}(s_1)$. The existence of such a solution is a consequence of our proof; on the other hand, the uniqueness would require some additional hypotheses on the function $\theta \mapsto \mathcal{G}(\hat{l}(\theta))$, that we do not investigate here. Notice that the right-hand side of (8.20) depends only on the geometry of the crack (found by incremental approximation), since the boundary datum remains constant during the unstable propagation.

In models of quasistatic evolution based on local minimization, the jumps in the slow time scale represent the limit of brutal propagations, i.e., fast dynamic motions through unstable states. When several cracks appear in the body, thanks to a suitable reparametrization of the time, it is possible to see that the evolution of their mutual positions is governed by the viscous equation (8.15) during the jumping regime. The reparametrized variables allow us to express, in the energy balance (8.14), the terms representing the energy dissipated "instantaneously" during each jump.

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